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Effective action for the Einstein–Maxwell theory at order RF^4

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Abstract

We use a recently derived integral representation of the one-loop effective action in the Einstein–Maxwell theory for an explicit calculation of the part of the effective action containing the information on the low energy limit of the five-point amplitudes involving one graviton, four photons and either a scalar or spinor loop. All available identities are used to get the result in a relatively compact form.

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1. Introduction

In recent years, much effort has been devoted to the study of the structure of graviton amplitudes. This was largely due to developments in string theory, which led to the prediction that such amplitudes should be much more closely related to gauge theory amplitudes than one would suspect by comparing the Lagrangians or Feynman rules of gravitational and gauge theories. Specifically, the Kawai–Lewellen–Tye (KLT) relations in string theory imply that graviton amplitudes should be 'squares' of gauge theory amplitudes [1–5]. String theory was also instrumental in providing guiding principles to develop new powerful techniques for the computation of graviton amplitudes [6–9]. Additional motivation comes from the possible finiteness of $N = 8$ supergravity (see [10] and references therein).

This work was largely confined to the case of massless on-shell amplitudes, for which particularly efficient computation methods are available. Relatively little work seems to have been done on amplitudes involving the interaction of gravitons with massive matter. At tree level, there are some classic results on amplitudes involving gravitons [11, 12]. More recently, the tree-level Compton-type amplitudes involving gravitons and spin zero, half and one particles were computed [13] to verify another remarkable factorization property [14] of the graviton–graviton scattering amplitudes in terms of the photonic Compton amplitudes.

However, we are not aware of results on graviton amplitudes involving a massive loop, other than the cases of the graviton propagator [15, 16] and of photon–graviton conversion [17, 18]. We believe that new insight into the structural relations between photon and graviton amplitudes might be obtained by studying the N-graviton amplitudes involving a massive loop, and more generally the mixed one-loop graviton–photon amplitudes. Generally, massive one-loop N-point amplitudes are significantly more difficult to compute than massless ones; on the other hand, their large mass limit is quite accessible through the effective action. For the prototypical case, the QED N-photon amplitude, the information on the large mass limit is contained in the Euler–Heisenberg Lagrangian ('EHL') [19]. We recall the standard proper time representation of this effective Lagrangian:

$$\mathcal{L}_{\rm spin} = -\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}T}{T^3}\,\mathrm{e}^{-m^2T}\left[\frac{(eaT)(ebT)}{\tanh(eaT)\tan(ebT)} - \frac{e^2}{3}(a^2-b^2)T^2 - 1\right]. \tag{1.1}$$

Here T is the proper time of the loop fermion, m is its mass and a, b are the two Maxwell field invariants, related to $\mathbf{E}$, $\mathbf{B}$ by $a^2 - b^2 = B^2 - E^2$, $ab = \mathbf{E}\cdot\mathbf{B}$. The analogous representation for scalar QED is due to Weisskopf [20].

After expanding the EHL in powers of the field invariants, it is straightforward to obtain the large mass limit of the N-photon amplitudes from the terms in this expansion involving N powers of the field. This limit is, of course, also the limit of low photon energies. The result of this procedure can be expressed quite concisely [21]:

$$\begin{aligned}\Gamma^{(EH)}_{\rm spin}\left[\varepsilon_1^+;\ldots;\varepsilon_K^+;\varepsilon_{K+1}^-;\ldots;\varepsilon_N^-\right] &= -\frac{m^4}{8\pi^2}\left(\frac{2ie}{m^2}\right)^N(N-3)!\\ &\quad\times\sum_{k=0}^{K}\sum_{l=0}^{N-K}(-1)^{N-K-l}\frac{\mathcal{B}_{k+l}\mathcal{B}_{N-k-l}}{k!l!(K-k)!(N-K-l)!}\chi_K^+\chi_{N-K}^-.\end{aligned} \tag{1.2}$$

Here the superscripts $\pm$ refer to circular polarizations, and $\mathcal{B}_k$ are Bernoulli numbers. The invariants $\chi_K^\pm$ are written, in standard spinor helicity notation, as

$$\begin{aligned}\chi_K^+ &= \frac{\left(\frac{K}{2}\right)!}{2^{\frac{K}{2}}}\{[12]^2[34]^2\cdots[(K-1)K]^2 + \text{all permutations}\},\\ \chi_{N-K}^- &= \frac{\left(\frac{N-K}{2}\right)!}{2^{\frac{N-K}{2}}}\{\langle(K+1)(K+2)\rangle^2\langle(K+3)(K+4)\rangle^2\cdots\langle(N-1)N\rangle^2 + \text{all permutations}\}.\end{aligned} \tag{1.3}$$

A very similar formula results for the scalar loop case [21]. For the case of the 'maximally helicity-violating' (MHV) amplitudes, which have all '+' or all '−' helicities, equation (1.2) and its scalar analogue have been generalized to the two-loop level [22]. A recently discovered correspondence of effective actions points to a relation between scalar loop MHV photon amplitudes in $2n$ dimensions and spinor loop graviton amplitudes in $4n$ dimensions [23].

One of the long-term goals of the present line of work is to obtain a generalization of (1.2) to the case of the mixed N-photon/M-graviton amplitudes. As a first step, in [24] the EHL (1.1) and its scalar analogue were generalized to the case relevant for the case of the amplitudes involving N photons and just one graviton. This corresponded to calculating the one-loop effective action in the scalar and spinor Einstein–Maxwell theory, to all orders in the electromagnetic field strength, and to leading order in the curvature, also including terms where the curvature tensor gets replaced by two covariant derivatives. These integral representations are given below in section 2 for easy reference. Although they contain the full information on the low energy limit of the N-photon/one-graviton amplitudes, it is, contrary to the Euler–Heisenberg case, a nontrivial task to expand them out in powers of the field invariants and extract the explicit form of those amplitudes. In [24] this was done at the F^2 level, as a check of consistency with previous results in the literature. In particular, the F^2 part for the spinor loop was shown to coincide, up to total derivative terms, with the effective Lagrangian obtained first by Drummond and Hathrell [25]:

$$\mathcal{L}^{(\mathrm{DH})}_{\mathrm{spin}} = \frac{1}{180(4\pi)^2 m^2}\big(5RF^2_{\mu\nu} - 26R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha} + 2R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} + 24(\nabla^\alpha F_{\alpha\mu})^2\big) \tag{1.4}$$

(here and in the following we will absorb the electric charge e into the field strength tensor F).

In this paper, we present the next order in the expansion of the effective Lagrangians obtained in [24] in powers of the field strength, i.e. the terms of order RF^4 (there are no order RF^3 terms for parity reasons). The explicit form of these Lagrangians is given in section 3, in a form made as compact as possible by the use of the gauge and gravitational Bianchi identities.

2. Gravitational Euler–Heisenberg Lagrangians to order *R*

In [24] Euler–Heisenberg-type integral representations were obtained for the scalar and spinor loop effective Lagrangians in the approximation discussed above. For the spinor loop, the result reads

$$\begin{aligned}
\mathcal{L}^{R}_{\mathrm{spin}} = -\frac{1}{8\pi^2}\int_0^\infty \frac{\mathrm{d}T}{T^3}\,\mathrm{e}^{-m^2T}\det{}^{-1/2}\left[\frac{\tan(FT)}{FT}\right]\Bigg\{&1 + \frac{\mathrm{i}T^2}{8}F_{\mu\nu;\alpha\beta}\,\mathcal{G}^{\alpha\beta}_{B11}\big(\dot{\mathcal{G}}^{\mu\nu}_{B11} - 2\,\mathcal{G}^{\mu\nu}_{F11}\big)\\
&+ \frac{\mathrm{i}T^2}{8}(F_{\mu\nu;\beta\alpha} + F_{\mu\nu;\alpha\beta})\dot{\mathcal{G}}^{\mu\beta}_{B11}\mathcal{G}^{\nu\alpha}_{B11} + \frac{T}{3}R_{\alpha\beta}\,\mathcal{G}^{\alpha\beta}_{B11}\\
&- \frac{\mathrm{i}T^2}{24}F_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu}\,\big(\dot{\mathcal{G}}^{\nu\mu}_{B11}\,\mathcal{G}^{\alpha\beta}_{B11} + \dot{\mathcal{G}}^{\alpha\mu}_{B11}\,\mathcal{G}^{\nu\beta}_{B11} + \dot{\mathcal{G}}^{\beta\mu}_{B11}\,\mathcal{G}^{\nu\alpha}_{B11} + 4\,\mathcal{G}^{\mu\nu}_{F11}\,\mathcal{G}^{\alpha\beta}_{B11}\big)\\
&+ \frac{T}{12}R_{\mu\alpha\beta\nu}\big(\dot{\mathcal{G}}^{\mu\alpha}_{B11}\dot{\mathcal{G}}^{\beta\nu}_{B11} + \dot{\mathcal{G}}^{\mu\beta}_{B11}\dot{\mathcal{G}}^{\alpha\nu}_{B11} + \big(\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2g^{\mu\nu}\delta(0)\big)\mathcal{G}^{\alpha\beta}_{B11}\\
&+ \dot{\mathcal{G}}^{\alpha\beta}_{B11}\,\mathcal{G}^{\mu\nu}_{F11} + \dot{\mathcal{G}}^{\nu\beta}_{B11}\,\mathcal{G}^{\mu\alpha}_{F11} - \mathcal{G}^{\alpha\beta}_{B11}\,\big(\dot{\mathcal{G}}^{\mu\nu}_{F11} - 2g^{\mu\nu}\delta(0)\big)\big) - \frac{1}{6}T^3F_{\alpha\beta;\gamma}\,F_{\mu\nu;\delta}\\
&\times\int_0^1 \mathrm{d}\tau_1\left(\dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\dot{\mathcal{G}}^{\beta\mu}_{B12}\,\mathcal{G}^{\gamma\delta}_{B12} + \dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\mathcal{G}^{\beta\delta}_{B12}\,\dot{\mathcal{G}}^{\gamma\mu}_{B12} + \frac{3}{2}\,\mathcal{G}^{\gamma\delta}_{B12}\,\mathcal{G}^{\alpha\mu}_{F12}\,\mathcal{G}^{\beta\nu}_{F12}\right)\Bigg\}.
\end{aligned} \tag{2.1}$$

Here the determinant factor $\det^{-1/2}\big[\frac{\tan(FT)}{FT}\big]$ by itself would just reproduce the (unrenormalized) Euler–Heisenberg Lagrangian (1.1). The integrand involves the worldline Green's functions in a constant field, as well as their derivatives. Those Green's functions can be written as

$$\begin{aligned}
\mathcal{G}_{B12} &\equiv \mathcal{G}_B(\tau_1,\tau_2) = \frac{1}{2\mathcal{Z}^2}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-\mathrm{i}\mathcal{Z}\dot{G}_{B12}} + \mathrm{i}\mathcal{Z}\dot{G}_{B12} - 1\right),\\
\dot{\mathcal{G}}_{B12} &\equiv \frac{\partial}{\partial\tau_1}\mathcal{G}_B(\tau_1,\tau_2) = \frac{\mathrm{i}}{\mathcal{Z}}\left(\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-\mathrm{i}\mathcal{Z}\dot{G}_{B12}} - 1\right),\\
\ddot{\mathcal{G}}_{B12} &\equiv \frac{\partial^2}{\partial\tau_1^2}\mathcal{G}_B(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) - 2\frac{\mathcal{Z}}{\sin(\mathcal{Z})}\,\mathrm{e}^{-\mathrm{i}\mathcal{Z}\dot{G}_{B12}},\\
\mathcal{G}_{F12} &\equiv \mathcal{G}_F(\tau_1,\tau_2) = G_{F12}\frac{\mathrm{e}^{-\mathrm{i}\mathcal{Z}\dot{G}_{B12}}}{\cos(\mathcal{Z})},\\
\dot{\mathcal{G}}_{F12} &\equiv \frac{\partial}{\partial\tau_1}\mathcal{G}_F(\tau_1,\tau_2) = 2\delta(\tau_1-\tau_2) + 2\mathrm{i}G_{F12}\frac{\mathcal{Z}}{\cos(\mathcal{Z})}\,\mathrm{e}^{-\mathrm{i}\mathcal{Z}\dot{G}_{B12}},
\end{aligned} \tag{2.2}$$

with $\dot{G}_{B12} = \mathrm{sign}(\tau_1 - \tau_2) - 2(\tau_1 - \tau_2)$, $G_{F12} = \mathrm{sign}(\tau_1 - \tau_2)$. The right-hand sides of equations (2.2) are to be understood as power series in the matrix $\mathcal{Z}_{\mu\nu} := TF_{\mu\nu}(x_0)$, where the indices are raised and lowered with $g_{\mu\nu}(x_0)$. We remark that the explicit $\delta(0)$'s in (2.1) subtract other $\delta(0)$'s contained in the coincidence limits $\ddot{\mathcal{G}}_{B11}$ and $\dot{\mathcal{G}}_{F11}$ [24]. The Green's functions in (2.1) with two different indices (i.e. which are not coincidence limits) are understood to have $\tau_2 = 0$.

For the case of a scalar in the loop, the result is somewhat simpler:

$$
\begin{aligned}
\mathcal{L}^R_{\mathrm{scal}} = {}& \frac{1}{16\pi^2}\int_0^\infty \frac{\mathrm{d}T}{T^3}\,\mathrm{e}^{-m^2T}\det{}^{-1/2}\left[\frac{\sin(FT)}{FT}\right] \\
& \times \Bigg\{ 1 - T\bar{\xi}R + \frac{T}{3}\mathcal{G}^{\alpha\beta}_{B11}R_{\alpha\beta} + \frac{\mathrm{i}T^2}{8}F_{\mu\nu;\alpha\beta}\,\dot{\mathcal{G}}^{\mu\nu}_{B11}\,\mathcal{G}^{\alpha\beta}_{B11} \\
& + \frac{\mathrm{i}}{8}T^2(F_{\mu\nu;\beta\alpha} + F_{\mu\nu;\alpha\beta})\dot{\mathcal{G}}^{\mu\beta}_{B11}\mathcal{G}^{\nu\alpha}_{B11} \\
& - \frac{\mathrm{i}T^2}{24}F_{\lambda\nu}R^{\lambda}{}_{\alpha\beta\mu}\left(\dot{\mathcal{G}}^{\nu\mu}_{B11}\,\mathcal{G}^{\alpha\beta}_{B11} + \dot{\mathcal{G}}^{\alpha\mu}_{B11}\,\mathcal{G}^{\nu\beta}_{B11} + \dot{\mathcal{G}}^{\beta\mu}_{B11}\,\mathcal{G}^{\nu\alpha}_{B11}\right) \\
& + \frac{T}{12}R_{\mu\alpha\beta\nu}\left(\dot{\mathcal{G}}^{\mu\alpha}_{B11}\dot{\mathcal{G}}^{\beta\nu}_{B11} + \dot{\mathcal{G}}^{\mu\beta}_{B11}\dot{\mathcal{G}}^{\alpha\nu}_{B11} + \left(\ddot{\mathcal{G}}^{\mu\nu}_{B11} - 2g^{\mu\nu}\delta(0)\right)\mathcal{G}^{\alpha\beta}_{B11}\right) \\
& - \frac{T^3}{6}F_{\alpha\beta;\gamma}F_{\mu\nu;\delta}\int_0^1 \mathrm{d}\tau_1\left(\dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\dot{\mathcal{G}}^{\beta\mu}_{B12}\,\mathcal{G}^{\gamma\delta}_{B12} + \dot{\mathcal{G}}^{\alpha\nu}_{B12}\,\mathcal{G}^{\beta\delta}_{B12}\,\dot{\mathcal{G}}^{\gamma\mu}_{B12}\right)\Bigg\}.
\end{aligned}
\tag{2.3}
$$

Here $\bar{\xi} = \xi - \frac{1}{4}$ where ξ parametrizes the coupling of the loop scalar to the scalar curvature (see appendix A for our conventions). In the last term it is again understood that $\tau_2 = 0$.

3. Effective Lagrangians at order RF^4

To obtain the effective Lagrangians at a given order $O(F^n)$ from the integral representations (2.1), (2.3), first one needs to expand the worldline Green's functions to the required order. Adequate formulae for an arbitrary order have been given in appendix B of [24]; here in appendix B we write down this expansion explicitly to the order required for the present calculation. The integrals are then elementary, and can be easily done using MATHEMATICA. However, the form of the result is still highly redundant, and can be considerably reduced by an application of the gauge and gravitational Bianchi identities. This is by far the most laborious step of the procedure (we have found the program MathTensor very useful for this task). We believe that the results given below are in the most compact form which can be achieved by the use of these identities (further reduction may be possible by the addition of total derivative terms, but we have not attempted this here). We include also the order $O(F^2)$ terms for easy reference (although not the pure Euler–Heisenberg terms). Our conventions are given in appendix A, where we also collect some useful formulae:

$$
\begin{aligned}
\mathcal{L}^{R(4)}_{\mathrm{scal}} = {}& \frac{1}{16\pi^2}\frac{1}{m^2}\left[\frac{1}{12}\left(\bar{\xi} + \frac{1}{12}\right)R(F_{\mu\nu})^2 + \frac{1}{180}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\right. \\
& \left. - \frac{1}{72}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta} - \frac{1}{180}(\nabla_\alpha F_{\mu\nu})^2 - \frac{1}{72}F_{\mu\nu}\Box F^{\mu\nu}\right] \\
& + \frac{1}{16\pi^2}\frac{1}{m^6}\left[-\frac{1}{144}\left(\bar{\xi} + \frac{1}{12}\right)R(F_{\mu\nu})^4 - \frac{1}{180}\left(\bar{\xi} + \frac{1}{12}\right)R\,\mathrm{tr}[F^4]\right.
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{945}R_{\alpha\beta}(F^4)^{\alpha\beta}+\frac{1}{1080}R_{\alpha\beta}(F^2)^{\alpha\beta}(F_{\gamma\delta})^2+\frac{1}{540}R_{\alpha\mu\beta\nu}(F^2)^{\alpha\beta}(F^2)^{\mu\nu}\\
&-\frac{1}{360}R_{\alpha\mu\beta\nu}(F^3)^{\alpha\mu}F^{\beta\nu}+\frac{1}{432}R_{\alpha\mu\beta\nu}F^{\alpha\mu}F^{\beta\nu}(F_{\gamma\delta})^2\\
&-\frac{1}{540}(F^3)^{\mu\nu}\Box F_{\mu\nu}+\frac{1}{432}F^{\mu\nu}\Box F_{\mu\nu}(F_{\gamma\delta})^2-\frac{1}{1080}F_{\mu\nu;\alpha\beta}(F^2)^{\alpha\beta}F^{\mu\nu}\\
&+\frac{1}{540}F_{\mu\nu;\alpha\beta}(F^2)^{\alpha\nu}F^{\beta\mu}+\frac{1}{1080}(F_{\alpha\beta;\gamma})^2(F_{\mu\nu})^2\\
&+\frac{1}{1890}F_{\alpha\beta;\gamma}F_{\mu\nu;}{}^{\gamma}F^{\alpha\mu}F^{\beta\nu}+\frac{1}{1890}F_{\alpha\beta;\gamma}F_{\mu}{}^{\alpha}{}_{;\delta}F^{\beta\mu}F^{\gamma\delta}\\
&\left.+\frac{2}{945}F_{\alpha\beta;}{}^{\mu}F_{\mu}{}^{\alpha}{}_{;\delta}(F^2)^{\beta\delta}-\frac{1}{1890}F_{\alpha\beta;\gamma}F_{\mu}{}^{\beta;\gamma}(F^2)^{\alpha\mu}\right],
\end{aligned}
\tag{3.1}
$$

$$
\begin{aligned}
\mathcal{L}^{R(4)}_{\text{spin}}=&-\frac{1}{8\pi^2}\frac{1}{m^2}\left[-\frac{1}{72}R(F_{\mu\nu})^2+\frac{1}{180}R_{\mu\nu}F^{\mu\alpha}F^{\nu}{}_{\alpha}\right.\\
&\left.+\frac{1}{36}R_{\mu\nu\alpha\beta}F^{\mu\nu}F^{\alpha\beta}-\frac{1}{180}(\nabla_\alpha F_{\mu\nu})^2+\frac{1}{36}F_{\mu\nu}\Box F^{\mu\nu}\right]\\
&-\frac{1}{8\pi^2}\frac{1}{m^6}\left[-\frac{1}{432}R(F_{\mu\nu})^4+\frac{7}{1080}R\,\mathrm{tr}[F^4]\right.\\
&-\frac{1}{945}R_{\alpha\beta}(F^4)^{\alpha\beta}-\frac{1}{540}R_{\alpha\beta}(F^2)^{\alpha\beta}(F_{\gamma\delta})^2+\frac{1}{540}R_{\alpha\mu\beta\nu}(F^2)^{\alpha\beta}(F^2)^{\mu\nu}\\
&+\frac{11}{360}R_{\alpha\mu\beta\nu}(F^3)^{\alpha\mu}F^{\beta\nu}+\frac{1}{108}R_{\alpha\mu\beta\nu}F^{\alpha\mu}F^{\beta\nu}(F_{\gamma\delta})^2\\
&-\frac{11}{945}F_{\alpha\beta;\gamma}F_{\mu}{}^{\beta;\gamma}(F^2)^{\alpha\mu}+\frac{2}{945}F_{\alpha\beta;}{}^{\mu}F_{\mu}{}^{\alpha}{}_{;\delta}(F^2)^{\beta\delta}\\
&+\frac{7}{270}(F^3)^{\mu\nu}\Box F_{\mu\nu}+\frac{1}{108}F^{\mu\nu}\Box F_{\mu\nu}(F_{\gamma\delta})^2+\frac{1}{216}F_{\mu\nu;\alpha\beta}(F^2)^{\alpha\beta}F^{\mu\nu}\\
&+\frac{1}{540}F_{\mu\nu;\alpha\beta}(F^2)^{\alpha\nu}F^{\beta\mu}-\frac{1}{540}(F_{\alpha\beta;\gamma})^2(F_{\mu\nu})^2\\
&\left.-\frac{2}{189}F_{\alpha\beta;\gamma}F_{\mu\nu;}{}^{\gamma}F^{\alpha\mu}F^{\beta\nu}-\frac{2}{189}F_{\alpha\beta;\gamma}F_{\mu}{}^{\alpha}{}_{;\delta}F^{\beta\mu}F^{\gamma\delta}\right].
\end{aligned}
\tag{3.2}
$$

We remark that the effective Lagrangians (3.1), (3.2) are equivalent to but not identical with what one would obtain by the more standard heat kernel method; both methods in general yield effective Lagrangians which differ by total derivative terms. To obtain the heat kernel form of the effective Lagrangians in the worldline formalism is possible; however, one has to use a different set of worldline Green's functions which, contrary to the Green's functions (2.2), are not proper-time invariant. For the calculations presented here this would already imply a significant increase in technical difficulty. See [24] for a discussion of these issues.

4. Conclusions

To summarize, the effective Lagrangians (3.1), (3.2) constitute the natural generalization of the Drummond–Hathrell Lagrangian (1.4) to the order $O(F^4)$ level, but still at linear order in the curvature, in the Einstein–Maxwell theory. They contain the full information on the one-loop amplitude involving four photons and one graviton, with a massive scalar or spinor in the loop, in the limit where all photon and graviton energies are small compared to the loop particle mass. In future work, we hope to elaborate these amplitudes in an explicit form, as a first step towards generalizing the N-photon amplitudes (1.2) to the full N-photon/M-graviton case.

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Appendix A. Conventions and useful formulae

In our conventions, the Einstein–Maxwell theory is described by

$$\Gamma[g,A] = \int \mathrm{d}^D x\,\sqrt{g}\left(\frac{1}{\kappa^2}R - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \tag{A.1}$$

where the metric $g_{\mu\nu}$ has signature $(-,+,+,\ldots,+)$, $g = |\det g_{\mu\nu}|$, and $\kappa^2 = 16\pi G_N$. We use the following conventions for the curvature tensors:

$$[\nabla_\mu,\nabla_\nu]V^\lambda = R_{\mu\nu}{}^\lambda{}_\rho V^\rho, \qquad R_{\mu\nu} = R_{\lambda\mu}{}^\lambda{}_\nu, \qquad R = R^\mu{}_\mu > 0 \text{ on spheres,}$$
$$[\nabla_\mu,\nabla_\nu]\phi = \mathrm{i}F_{\mu\nu}\phi, \tag{A.2}$$

where V^μ is an uncharged vector and ϕ is a charged scalar. The one-loop effective action for the scalar loop is defined by

$$\Gamma[g,A] = \ln \mathrm{Det}^{-1}(-\Box_A + m^2 + \xi R) \tag{A.3}$$

where $\Box_A$ is the gauge and gravitational covariant Laplacian for scalar fields. The parameter ξ describes an additional non-minimal coupling to the scalar curvature R. For the (Dirac) spinor loop, we define it by

$$\Gamma[g,A] = \ln \mathrm{Det}(\not\nabla + m) \tag{A.4}$$

where

$$\not\nabla = \gamma^a e_a{}^\mu \nabla_\mu, \qquad \nabla_\mu = \partial_\mu + \mathrm{i}eA_\mu + \tfrac{1}{4}\omega_{\mu ab}\gamma^a\gamma^b \tag{A.5}$$

with $e_\mu{}^a$ the vielbein and $e = \det e_\mu{}^a$, $\omega_{\mu ab}$ the spin connection.

The following identities have been used for simplifying the effective Lagrangians (3.1), (3.2):

$$F_{\mu\alpha;\beta}\,F^{\mu\beta;\alpha} = \tfrac{1}{2}F_{\mu\beta;\alpha}\,F^{\mu\beta;\alpha}, \tag{A.6}$$

$$F_\mu{}^\alpha\,F^{\mu\beta}{}_{;\alpha\beta} = \tfrac{1}{2}F_{\mu\nu}\Box F^{\mu\nu}, \tag{A.7}$$

$$F_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\alpha\nu\beta} = \tfrac{1}{2}F_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\nu\alpha\beta}, \tag{A.8}$$

$$(F^3)_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\alpha\nu\beta} = \tfrac{1}{2}(F^3)_{\mu\nu}\,F_{\alpha\beta}\,R^{\mu\nu\alpha\beta}, \tag{A.9}$$

$$F_{\alpha\beta;\mu}\,F^{\alpha\beta}{}_{;\nu}\,(F^2)^{\mu\nu} = -2F_{\alpha\beta;}{}^\mu\,F_\mu{}^\alpha{}_{;\nu}\,(F^2)^{\beta\nu}, \tag{A.10}$$

$$F_{\alpha\beta;\gamma}\,F_{\mu\nu;}{}^\beta\,F^{\alpha\nu}\,F^{\gamma\mu} = -\tfrac{1}{2}F_{\alpha\beta;\gamma}\,F_{\mu\nu;}{}^\gamma\,F^{\alpha\mu}\,F^{\beta\nu}, \tag{A.11}$$

$$F_{\alpha\beta;}{}^\mu\,F_{\mu\nu;\gamma}\,F^{\alpha\nu}\,F^{\beta\gamma} = -\tfrac{1}{2}F_{\alpha\beta;\gamma}\,F_{\mu\nu;}{}^\gamma\,F^{\alpha\mu}\,F^{\beta\nu}, \tag{A.12}$$

$$F_\mu{}^\alpha F^{\mu\beta}{}_{;\alpha\beta} + F_\mu{}^\alpha F^{\mu\beta}{}_{;\beta\alpha} = \tfrac{1}{2} F_{\mu\nu} F_{\alpha\beta} R^{\mu\nu\alpha\beta} + (F^2)^{\alpha\beta} R_{\alpha\beta} + F_{\mu\nu} \Box F^{\mu\nu}, \tag{A.13}$$

$$F_{\alpha\beta;}{}^\mu F_{\mu\nu;}{}^\beta (F^2)^{\alpha\nu} = -F_{\alpha\beta;}{}^\mu F_\mu{}^\alpha{}_{;\gamma} (F^2)^{\beta\gamma} - F_{\alpha\beta;\gamma} F_\mu{}^{\beta;\gamma} (F^2)^{\alpha\mu}, \tag{A.14}$$

$$F_{\alpha\beta;\gamma} F_{\mu\nu;}{}^\beta F^{\alpha\nu} F^{\gamma\mu} = F_{\alpha\beta;\gamma} F_\mu{}^\alpha{}_{;\eta} F^{\beta\mu} F^{\gamma\eta} - F_{\alpha\beta;\gamma} F_\mu{}^\alpha{}_{;\eta} F^{\beta\eta} F^{\gamma\mu}, \tag{A.15}$$

$$F_{\alpha\beta;\gamma} F_\mu{}^\alpha{}_{;\eta} F^{\beta\eta} F^{\gamma\mu} = \tfrac{1}{2} F_{\alpha\beta;\gamma} F_{\mu\nu;}{}^\gamma F^{\alpha\mu} F^{\beta\nu} + F_{\alpha\beta;\gamma} F_\mu{}^\alpha{}_{;\eta} F^{\beta\mu} F^{\gamma\eta}, \tag{A.16}$$

$$F_{\mu\nu;\beta}{}^\nu F^{\beta\mu} + F_{\mu\nu;}{}^\nu{}_\beta F^{\beta\mu} = -\tfrac{1}{2} F_{\mu\nu} F_{\alpha\beta} R^{\mu\nu\alpha\beta} - (F^2)^{\alpha\beta} R_{\alpha\beta} - F_{\mu\nu} \Box F^{\mu\nu}, \tag{A.17}$$

$$\begin{aligned}(F^2)^{\alpha\nu} F^{\beta\mu} F_{\mu\nu;\beta\alpha} - (F^2)^{\alpha\nu} F^{\beta\mu} F_{\mu\nu;\alpha\beta} \\ = \tfrac{1}{2}(F^3)^{\mu\nu} F^{\alpha\beta} R_{\mu\nu\alpha\beta} + (F^2)^{\mu\alpha} (F^2)^{\nu\beta} R_{\mu\nu\alpha\beta},\end{aligned} \tag{A.18}$$

$$\begin{aligned}F_{\mu\nu;\beta}{}^\nu (F^3)^{\beta\mu} + F_{\mu\nu;}{}^\nu{}_\beta (F^3)^{\beta\mu} \\ = -\tfrac{1}{2}(F^3)_{\mu\nu} F_{\alpha\beta} R^{\mu\nu\alpha\beta} - (F^4)^{\alpha\beta} R_{\alpha\beta} - (F^3)_{\mu\nu} \Box F^{\mu\nu}.\end{aligned} \tag{A.19}$$

Identities (A.6)–(A.19) are simple consequences of the Bianchi identities

$$\nabla_\alpha F_{\beta\gamma} + \nabla_\beta F_{\gamma\alpha} + \nabla_\gamma F_{\alpha\beta} = 0, \tag{A.20}$$

$$R_{\alpha\beta\gamma\delta} + R_{\beta\gamma\alpha\delta} + R_{\gamma\alpha\beta\delta} = 0. \tag{A.21}$$

Appendix B. Expansion of the field-dependent worldline Green's functions

In this appendix we give the expansion of the constant field worldline Green's functions $\mathcal{G}_B, \dot{\mathcal{G}}_B, \ddot{\mathcal{G}}_B, \mathcal{G}_F, \dot{\mathcal{G}}_F$ to the order $O(F^4)$ required for the present computation. Defining

$$\bar{G}_{B12} := |\tau_1 - \tau_2| - (\tau_1 - \tau_2)^2 \tag{B.1}$$

those expansions can be written as

$$\begin{aligned}\mathcal{G}_{B12} &= \bar{G}_{B12} - \frac{1}{6} - \frac{\mathrm{i}}{3}\dot{G}_{B12}\bar{G}_{B12}\mathcal{Z} + \left(\frac{1}{3}\bar{G}_{B12}^2 - \frac{1}{90}\right)\mathcal{Z}^2 - \frac{\mathrm{i}}{15}\bar{G}_{B12}\dot{G}_{B12}\left(\bar{G}_{B12} + \frac{1}{3}\right)\mathcal{Z}^3 \\ &\quad + \frac{1}{45}\left(2\bar{G}_{B12}^2\left(\bar{G}_{B12} + \frac{1}{2}\right) - \frac{1}{21}\right)\mathcal{Z}^4 + \mathcal{O}(\mathcal{Z}^5),\end{aligned} \tag{B.2}$$

$$\begin{aligned}\dot{\mathcal{G}}_{B12} &= \dot{G}_{B12} + 2\mathrm{i}\left(\bar{G}_{B12} - \frac{1}{6}\right)\mathcal{Z} + \frac{2}{3}\dot{G}_{B12}\bar{G}_{B12}\mathcal{Z}^2 + \mathrm{i}\left(\frac{2}{3}\bar{G}_{B12}^2 - \frac{1}{45}\right)\mathcal{Z}^3 \\ &\quad + \frac{2}{15}\bar{G}_{B12}\dot{G}_{B12}\left(\bar{G}_{B12} + \frac{1}{3}\right)\mathcal{Z}^4 + \mathcal{O}(\mathcal{Z}^5),\end{aligned} \tag{B.3}$$

$$\begin{aligned}\ddot{\mathcal{G}}_{B12} &= 2\delta_{12} - 2 + 2\mathrm{i}\,\dot{G}_{B12}\mathcal{Z} - 4\left(\bar{G}_{B12} - \frac{1}{6}\right)\mathcal{Z}^2 + \frac{4}{3}\mathrm{i}\bar{G}_{B12}\dot{G}_{B12}\mathcal{Z}^3 \\ &\quad - \left(\frac{4}{3}\bar{G}_{B12}^2 - \frac{2}{45}\right)\mathcal{Z}^4 + \mathcal{O}(\mathcal{Z}^5),\end{aligned} \tag{B.4}$$

$$\begin{aligned}\mathcal{G}_{F12} &= G_{F12} - \mathrm{i}\,G_{F12}\dot{G}_{B12}\mathcal{Z} + 2G_{F12}\bar{G}_{B12}\mathcal{Z}^2 - \frac{1}{3}\mathrm{i}\,G_{F12}\dot{G}_{B12}(2\bar{G}_{B12} + 1)\mathcal{Z}^3 \\ &\quad + \frac{2}{3}G_{F12}\bar{G}_{B12}\left(\bar{G}_{B12} + 1\right)\mathcal{Z}^4 + \mathcal{O}(\mathcal{Z}^5),\end{aligned} \tag{B.5}$$

$$\begin{aligned}\dot{\mathcal{G}}_{F12} &= 2\delta_{12} + 2\mathrm{i}\,G_{F12}\mathcal{Z} + 2\,G_{F12}\dot{G}_{B12}\mathcal{Z}^2 + 4\mathrm{i}\,G_{F12}\bar{G}_{B12}\mathcal{Z}^3 \\ &\quad + \frac{2}{3}G_{F12}\dot{G}_{B12}(2\bar{G}_{B12} + 1)\mathcal{Z}^4 + \mathcal{O}(\mathcal{Z}^5).\end{aligned} \tag{B.6}$$

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