Late-time attractor for the cubic nonlinear wave equation

Nikodem Szpak[a]

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Golm 14476, Germany

We apply our recently developed scaling technique for obtaining late-time asymptotics to the cubic nonlinear wave equation and explain the appearance and approach to the two-parameter attractor found recently by Bizon and Zenginoglu. © *2010 American Institute of Physics.* [doi:10.1063/1.3470957]

I. INTRODUCTION

In Ref. 1, Bizoń and Zenginoğlu conjectured and presented some analytical and numerical evidence that the spherically symmetric cubic nonlinear wave equation in three dimensions,

$$\Box u = u^3 \quad (\Box \equiv \partial_t^2 - \Delta),$$
(1)

has a universal two-parameter attractor for the late-time asymptotics,

$$u_{a,b}(t,r) = \frac{\sqrt{2}}{t + a + b[(t+a)^2 - r^2]},$$
(2)

for a big family of initial data, being itself an exact solution of (1). The rate of approach to attractor (2) with suitably chosen a, b is t^{-4} for a fixed r. The aim of this article is to prove this conjecture by deriving a precise late-time asymptotics and comparing it with the form of attractor (2). Recently, in Ref. 2, we have developed a method for obtaining late-time asymptotics from the scaling properties of a given wave equation. It allows us to show that, at least for small initial data, the generic late-time asymptotics of solutions to (1) takes the form

$$u(t,r) = \varepsilon u_0(t,r) + \frac{A_0}{t^2 - r^2} + \frac{A_1 t}{(t^2 - r^2)^2} + \frac{A_2(3t^2 + r^2)}{(t^2 - r^2)^3} + \dots,$$
(3)

where A_i are given in terms of the initial data. This expansion coincides with attractor (2) when $A_0 = \sqrt{2}/b$ and $A_1 = \sqrt{2}/(2ab+1)$ up to the term of order $t^2(t^2-r^2)^{-3}$. Hence, as long as A_0, A_1 are nonzero, there exist unique parameters a, b, such that the solution u asymptotically approaches the two-parameter attractor (2). The rate of approach is determined by the failure of (2) to reproduce the third term in the generic asymptotics (3) which contains a third independent parameter A_2.

Such precise asymptotic analysis has only become possible with the recent developments in the perturbation theory. Crucial are the first rigorous works[3,4] allowing for establishing a link between the late-time asymptotics and the small initial data (see also references therein for earlier nonrigorous but important works, e.g., by Bizoń *et al.*). They build up on earlier decay estimates for small data by John[5] and Asakura.[6] Later, in Ref. 2, an equivalent technique based on scaling has been introduced which simplifies the asymptotic calculations of higher order terms. In it, the initial value problem for a class of nonlinear wave equations is considered which we restrict here for the sake of simplicity to

---

[a]Electronic mail: nikodem.szpak@uni-due.de.

51, 082901-1

$$\Box u = u^p, \tag{4}$$

with integer $p \geq 3$ and small initial data,

$$u(0,r) = \varepsilon f(r), \quad \partial_t u(0,r) = \varepsilon g(r), \tag{5}$$

being smooth functions of compact support in three spatial dimensions restricted to spherical symmetry. Then, for small ε, we have $u \in \mathcal{C}^\infty(\mathbb{R}_+^{1+1})$.

The main idea of deriving asymptotics from scaling is based on the observation of Lindblad[7] that in the limit $\varepsilon \to 0$ the solutions u_ε of (4) and (5) tend, under suitable scaling, to some nontrivial u_* which satisfies a linear wave equation with a distributional source. This equation can be solved exactly. For small but finite values of ε, the solutions u_ε are near to u_* in a suitable sense with a uniform error bound, such that u_* determines the late-time asymptotics of u_ε.

A straightforward generalization of Theorem 1 of Ref. 2 leads to

$$u(t,r) = \varepsilon u_0(t,r) + \varepsilon^p \sum_{k=0}^n \frac{B_{p,k}}{r} \left[\frac{1}{(t-r)^{p+k-2}} - \frac{1}{(t+r)^{p+k-2}} \right]$$
$$+ O\left(\frac{\varepsilon^p}{\langle t+r \rangle \langle t-r \rangle^{p+n-2}} \right) + O\left(\frac{\varepsilon^{p+\lambda_0}}{\langle t+r \rangle \langle t-r \rangle^{p-2}} \right) \tag{6}$$

for $\varepsilon \to 0$, $t-r > 1/\varepsilon^a$, a given scaling parameter $0 < a < p(p-1)/(p+1)$ and any non-negative integer n. u_0 solves corresponding linear problems (7) and (8) (see below) and $B_{p,k}$ are determined by the initial data and are defined below.

The error terms mean that the asymptotics holds with respect to weighted-L^∞ norms (cf. Ref. 2). Here, they are restricted to the region $t-r > 1/\varepsilon^a$ and imply a uniform convergence there, as $\varepsilon \to 0$. The first error term describes correction entering at the same nonlinear order (the same power of ε) as the leading terms but having faster decay in time, while the second error term, with $\lambda_0 := (p-1)(1-a) + a[(p-1)^2 - 2]/p$, stays for corrections with the same decay in time as the leading terms but entering with higher powers of ε.

For $p=3$ and $n=2$, we essentially obtain asymptotic expansion (3). The only point which requires some additional but straightforward work is to show that all expressions appearing in the second error term are actually of the same functional form as those already present in the leading asymptotics multiplied by higher powers of ε. Then they do not change the character of the asymptotic expansion but only alter the constants, thus leading to (3) with $A_k(\varepsilon) = B_{3,k} \varepsilon^3 + O(\varepsilon^4)$.

In Ref. 1, Sec. 5.1.2, also nongeneric solutions with faster late-time decay than this given by (2) have been found numerically. This can be explained by the observation that the initial data can be chosen such that at late-times $A_0 = 0$ or $A_0 = A_1 = 0$, etc., thus leading to faster decay in (3).

II. THE SCALING TECHNIQUE

Here, we briefly introduce the method of scaling developed in Ref. 2 and extend it to calculate higher order terms appearing in (6). All technical details and proofs can be found or easily adapted from Ref. 2. The scaling method has an advantage over the standard perturbation theory in generating simpler-to-solve effective equations for the higher order asymptotic terms.

In the first step we solve a corresponding linear equation with removed scale factor ε,

$$\Box u_0 = 0, \tag{7}$$

$$u_0(0,r) = f(r), \quad \partial_t u_0(0,r) = g(r). \tag{8}$$

Its solution can be written in the form

$$u_0(t,r) = \frac{h(t-r) - h(t+r)}{r}, \tag{9}$$

where

$$h(x) := -\frac{x}{2}f(x) - \frac{1}{2}\int_x^\infty yg(y)dy \tag{10}$$

has compact support [the functions $f(r), g(r)$ have been symmetrically continued to negative r]. Next, we subtract the linear solution from the nonlinear one by introducing

$$w(t,r) := u(t,r) - \varepsilon u_0(t,r), \tag{11}$$

which satisfies

$$\Box w = (w + \varepsilon u_0)^p. \tag{12}$$

Now, we scale this function to

$$W_\varepsilon(t,r) := \varepsilon^{-b}w(\varepsilon^{-a}t, \varepsilon^{-a}r), \tag{13}$$

with $b = p + a(p-1)$ and some $a > 0$ to be chosen later. It satisfies

$$
\begin{aligned}
\Box W_\varepsilon(t,r) &= \varepsilon^{-a}[\varepsilon^{(p-1)+a(p-2)}W_\varepsilon(t,r) + \varepsilon^{-a}u_0(\varepsilon^{-a}t, \varepsilon^{-a}r)]^p \\
&= \varepsilon^{-a}[\varepsilon^{-a}u_0(\varepsilon^{-a}t, \varepsilon^{-a}r)]^p \\
&\quad + \sum_{k=1}^p \binom{p}{k}\varepsilon^{-a}[\varepsilon^{-a}u_0(\varepsilon^{-a}t, \varepsilon^{-a}r)]^{p-k}\varepsilon^{[(p-1)+a(p-2)]k}W_\varepsilon^k(t,r).
\end{aligned}
\tag{14}
$$

For this equation, we want to consider the limit $\varepsilon \to 0$.

Let us recall the following fact of the distributional calculus: any smooth function $H(x)$ of compact support can be squeezed to the delta distribution under appropriate scaling as $\varepsilon \to 0$. The corrections can be written as a sum over derivatives of the delta,

$$\varepsilon^{-1}H(\varepsilon^{-1}x) \approx C_0\delta(x) + \varepsilon C_1\delta'(x) + \dots, \tag{15}$$

that, in the precise (distributional) sense, means

$$\lim_{\varepsilon \to 0}\left[\int \frac{1}{\varepsilon^{n+1}}H\left(\frac{x}{\varepsilon}\right)g(x)dx - \sum_{k=0}^{n-1}\frac{1}{\varepsilon^{n-k}}C_k g^{(k)}(0)\right] = C_n g^{(n)}(0), \tag{16}$$

where $g \in \mathcal{C}_0^\infty$ is a test function, n is any non-negative integer, and $C_k := \int x^k H(x)dx$.

Having this in mind we observe that the first term in (14), by use of representation (9), will have a distributional limit (in the above sense)

$$\varepsilon^{-a}[\varepsilon^{-a}u_0(\varepsilon^{-a}t, \varepsilon^{-a}r)]^q \approx C_{q,0}\frac{\delta(t-r) - \delta(t+r)}{r^q} + \varepsilon^a C_{q,1}\frac{\delta'(t-r) - \delta'(t+r)}{r^q} + \dots, \tag{17}$$

where

$$C_{q,i} := \int x^i h^q(x)dx. \tag{18}$$

The terms $\delta^{(k)}(t+r)$ will further play no role since their support is outside the region of our interest $t+r > 0$.

The sum in (14) will be treated as an error term. The expansion

$$W_\varepsilon = W^{(0)} + \varepsilon^a W^{(1)} + \cdots + \varepsilon^{na} W^{(n)} + \varepsilon^{(n+1)a}\hat{W} + \widetilde{W}_\varepsilon \tag{19}$$

allows us to write the following limiting equations:

$$\Box W^{(k)}(t,r) = C_{p,k}\frac{\delta^{(k)}(t-r)}{r^p} \tag{20}$$

for $k = 0,\ldots,n$ which can be solved exactly as

$$W^{(k)}(t,r) = B_{p,k}\frac{\Theta(t-r)}{r}\left[\frac{1}{(t-r)^{p+k-2}} - \frac{1}{(t+r)^{p+k-2}}\right] \tag{21}$$

with $B_{p,k} := 2^{p+k-3} C_{p,k}/(p+k-2)$.

The first error term $\hat{W}$ comes from the truncation of expansion [(17)](#) of $\varepsilon^{-a}[\varepsilon^{-a}u_0(\varepsilon^{-a}t,\varepsilon^{-a}r)]^p$ into deltas and can be bound by

$$\langle t+r\rangle\langle t-r\rangle^{p+n-2}|\hat{W}(t,r)| \le C, \tag{22}$$

while the second error term $\widetilde{W}_\varepsilon$ satisfies

$$\Box\widetilde{W}_\varepsilon = \sum_{k=1}^{p}\binom{p}{k}\varepsilon^{-a}[\varepsilon^{-a}u_0(\varepsilon^{-a}t,\varepsilon^{-a}r)]^{p-k}\varepsilon^{[(p-1)+a(p-2)]k}W_\varepsilon^k(t,r). \tag{23}$$

In Ref. [2](#) we have shown that it can be bound,

$$\langle t+r\rangle\langle t-r\rangle^{p-2}|\widetilde{W}_\varepsilon(t,r)| = O(\varepsilon^{(p-1)(1-a)+a[(p-1)^2-2]/p}), \tag{24}$$

uniformly for all $t-r>1$. It vanishes in the limit $\varepsilon \to 0$ for $0 < a < p(p-1)/(p+1)$.

It allows to write the asymptotics of u,

$$\begin{aligned}
u(t,r) &= \varepsilon u_0(t,r) + \varepsilon^{p+a(p-1)}W_\varepsilon(\varepsilon^a t,\varepsilon^a r)\\
&= \varepsilon u_0(t,r) + \varepsilon^p\sum_{k=0}^{n}B_{p,k}\frac{\Theta(t-r)}{r}\left[\frac{1}{(t-r)^{p+k-2}} - \frac{1}{(t+r)^{p+k-2}}\right]\\
&\quad + O\left(\frac{\varepsilon^p}{\langle t+r\rangle\langle t-r\rangle^{p+n-2}}\right) + O\left(\frac{\varepsilon^{p+\lambda_0}}{\langle t+r\rangle\langle t-r\rangle^{p-2}}\right),
\end{aligned} \tag{25}$$

which holds uniformly in $t-r > \varepsilon^{-a}$ for $\varepsilon \to 0$, a given scaling parameter $0 < a < p(p-1)/(p+1)$, any non-negative integer n, and $\lambda_0 := (p-1)(1-a) + a[(p-1)^2-2]/p$.

This procedure can be made rigorous as has been demonstrated in Ref. [2](#).

[1] P. Bizon and A. Zenginoglu, Nonlinearity **22**, 2473 (2009); e-print arXiv:0811.3966 [math.AP].
[2] N. Szpak, Comm. in PDE e-print arXiv:0907.4287 [math-ph] (accepted for publication).
[3] N. Szpak, Journal of Hyperbolic Differential Equations (JHDE) **5**, 741 (2008); e-print arXiv:0710.1782 [math-ph].
[4] N. Szpak, P. Bizon, T. Chmaj, and A. Rostworowski, Journal of Hyperbolic Differential Equations (JHDE) **6**, 107 (2009); e-print arXiv:0712.0493 [math-ph].
[5] F. John, Manuscr. Math. **28**, 235 (1979).
[6] F. Asakura, Commun. Partial Differ. Equ. **11**, 1459 (1986).
[7] H. Lindblad, Commun. Partial Differ. Equ. **15**, 757 (1990).