Improved and perfect actions in discrete gravity

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We consider the notion of improved and perfect actions within Regge calculus. These actions are constructed in such a way that they—although being defined on a triangulation—reproduce the continuum dynamics exactly, and therefore capture the gauge symmetries of general relativity. We construct the perfect action in three dimensions with a cosmological constant, and in four dimensions for one simplex. We conclude with a discussion about Regge calculus with curved simplices, which arises naturally in this context.

I. INTRODUCTION

In general relativity (GR) the notion of diffeomorphism invariance, resulting from Einstein's covariance principle, is of ultimate importance [1]. In particular, its correct implementation on the quantum level is a challenging task for every candidate quantum gravity theory.

Prior to quantizing a classical field theory, it is usually first discretized, since discrete systems generically have finitely many degrees of freedom. These are usually easier to quantize than the infinitely many degrees of freedom of field theories. A natural discretization of general relativity is achieved by Regge calculus, where the smooth space-time is replaced by a simplicial complex, and the metric information is contained in the edge lengths and deficit angles around the hinges [2,3]. A similar discretization is used within the spin foam quantization approach, where the variables of the first order Plebanski formulation of GR are discretized on Regge triangulations, prior to quantization [4].

It is an important question what happens with the diffeomorphism invariance of general relativity in these discretized gravity theories (see [5] and references therein).

Discretizing a theory often breaks symmetries, such as in QCD, where the introduction of a lattice breaks e.g. rotation invariance. Another example is reparametrization invariant one-dimensional (1D) systems, where the discretization scheme generically breaks the reparametrization invariance [6]. The latter example resembles the situation in GR in many ways [7].

In a canonical formulation the problem becomes even more apparent, where the symmetries turn into constraints, and it is notoriously difficult to implement them correctly in the discretized quantum theory (see also [8] for a discussion). Even in quantum gravity theories like loop quantum gravity, which are inherently set up to capture the full continuum of physics, the discretized nature of the constituents, i.e. the graphs, makes the implementation of the constraint algebra rather nontrivial [5,9].

In general, breaking of symmetries is, however, not ultimately tied to the discretization, but rather the approximation involved, i.e. by replacing spatial derivatives with differential quotients between neighboring lattice points.

For instance in lattice QCD one ideally would want to construct a Lagrangian which, although describing a theory on the lattice, still encodes the symmetries of the continuum theory [10,11]. A lattice action which reproduces the same dynamics as the continuum theory and therefore also reflects the symmetries of the continuum limit is termed *perfect action* in that context. That perfect actions exist for asymptotically free theories follows from Wilson's theory of renormalization group flow [12]. Although for actual problems at hand the perfect actions are very hard to compute, the (numerical) computation of improved actions, i.e. actions that capture the continuum symmetries much better than the actual naïve lattice discretization, is an important task. These actions are widely sought for in order to suppress lattice artifacts in numerical calculations [13].

In this article we investigate the question of how improved and perfect actions within the context of discretizations of general relativity, in particular, Regge calculus, can be constructed. We will start with reviewing one-dimensional reparametrization invariant systems and their discretization in Sec. II. These systems exhibit a gauge symmetry which mimics the diffeomorphism symmetry of GR in many respects. This symmetry is broken in the naïve discretization of those systems, and we will have a look at how one can construct improved and perfect actions for them. In particular we will see how the perfect actions restore the gauge invariance of the continuum limit within the discretized setting. Part of this section will follow [6].

In Secs. III and IVA we will focus on Regge calculus with a cosmological constant in three and four dimensions. Whereas Regge calculus in 3D with $\Lambda = 0$ exhibits a well-known vertex displacement symmetry which is a result of the discrete Bianchi identities [14,15], this symmetry is broken for $\Lambda \neq 0$. We show how to construct improved

actions in this case and analytically compute the perfect action, which regains the vertex displacement symmetry and hence reflects the dynamics and the symmetry of the continuum, albeit formulated on a Regge triangulation.

We also formulate improved actions for Regge calculus in 4D, and investigate some properties of its continuum limit, i.e. the corresponding perfect action. In particular we are able to show that the perfect action from the Regge action, and the one obtained by using simplices of constant curvature instead of internally flat ones, coincide. In the language of renormalization group flow this demonstrates that the two actions one started with lie in the same universality class.

We will, in particular, comment about the conclusions one can draw from these findings for the corresponding quantum theories.

II. DISCRETIZED ACTIONS IN 1D

In this section we will discuss theories arising from discretizations of systems with one-dimensional reparametrization invariance, that is, invariance under redefinitions of the time variable. As we will see under discretization the exact reparametrization invariance is typically lost similar to the diffeomorphism invariance in the Regge action. However for the examples we consider in this section there is a procedure to obtain a discrete action with an exact reparametrization invariance. This procedure resembles the "blocking from the continuum" construction in lattice QCD, where a lattice action is constructed which has the exact symmetries of the continuum action [13]. In some parts of our discussion we will follow [6].

We start from a regular Lagrangian $L(q, \dot{q})$ where q denotes the configuration variable. We assume that the dynamics determined by L leads to a unique solution $q(t)$ for given boundary values $q(t_i)$, $q(t_f)$ if t_i and t_f are sufficiently close together.

From this we construct a reparametrization invariant action by adding the time variable t to the configuration variables and use s as an (auxiliary) evolution parameter instead. If we define

$$\tilde{L}(t, q, t', q') := L\left(q, \frac{q'}{t'}\right)t', \qquad (2.1)$$

where a prime denotes differentiation with respect to s, it is then straightforward to verify that

$$S = \int_{s_i}^{s_f} \tilde{L}(t, q, t', q')ds \qquad (2.2)$$

is indeed invariant under reparametrizations $\tilde{s} = f(s)$ of the evolution parameter and the induced change $\tilde{t}(\tilde{s}) = t(f^{-1}(\tilde{s}))$, $\tilde{q}(\tilde{s}) = q(f^{-1}(\tilde{s}))$ of the evolution paths.

The Euler-Lagrange equations for (2.2) for the variables t, q are given by

$$\left(\frac{\partial\tilde{L}}{\partial q} - \frac{d}{ds}\frac{\partial\tilde{L}}{\partial q'}\right) = \frac{\partial L}{\partial q}t' - \frac{d}{ds}\frac{\partial L}{\partial\dot{q}} = \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial\dot{q}}\right)t', \qquad (2.3)$$

$$\left(\frac{\partial\tilde{L}}{\partial t} - \frac{d}{ds}\frac{\partial\tilde{L}}{\partial t'}\right) = -\frac{d}{ds}L + \frac{\partial L}{\partial q}q' + \frac{\partial L}{\partial\dot{q}}\frac{d}{ds}\frac{dq'}{dt'}$$

$$= \left(-\frac{dL}{dt} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial\dot{q}}\ddot{q}\right)t', \qquad (2.4)$$

where $\frac{\partial L}{\partial q}$ and $\frac{\partial L}{\partial\dot{q}}$ denote the derivative of L with respect to its first and its second entry, respectively. Note that (2.3) is equivalent to the Euler-Lagrange equations for L, and (2.4) is satisfied identically due to the chain rule. So this is just a reformulation of the dynamics determined by L via introduction of a gauge degree of freedom. The nonuniqueness of the solutions $t(s)$, $q(s)$ directly corresponds to the reparametrization independence of the dynamical system defined by the Lagrangian $\tilde{L}$.

A naïve discretization of the action (2.2) is given by

$$S_d = \sum_{n=0}^{N-1}(t_{n+1} - t_n)L_n \qquad (2.5)$$

with

$$L_n := L\left(q_n, \frac{q_{n+1} - q_n}{t_{n+1} - t_n}\right). \qquad (2.6)$$

The dynamics of this discretized system is obtained by looking for stationary variations of (2.5) with respect to the t_n, q_n. The equations of motion are

$$0 = \frac{\partial S_d}{\partial q_n} = \partial_q L_n(t_{n+1} - t_n) + \partial_{\dot{q}}L_{n-1} - \partial_{\dot{q}}L_n, \qquad (2.7)$$

$$0 = \frac{\partial S_d}{\partial t_n}$$

$$= L_{n-1} - L_n + \partial_{\dot{q}}L_n\frac{q_{n+1} - q_n}{t_{n+1} - t_n} - \partial_{\dot{q}}L_{n-1}\frac{q_n - q_{n-1}}{t_n - t_{n-1}}, \qquad (2.8)$$

where $\partial_q L_n$ denotes the derivative $\frac{\partial L}{\partial q}$ evaluated at $q = q_n$, $\dot{q}_n = \frac{q_{n+1} - q_n}{t_{n+1} - t_n}$. Similarly $\partial_{\dot{q}}L_n$ is the derivative of L with respect to its second entry evaluated at $(q_n, \dot{q}_n)$. With the product rule $A_{n+1}B_{n+1} - A_n b_n = A_{n+1}(B_{n+1} - B_n) + (A_{n+1} - A_n)B_n$, Eq. (2.8) for the t_n can, using (2.7), be rewritten as

$$0 = -\frac{L_n - L_{n-1}}{t_{n+1} - t_n} + \partial_q L_n\frac{q_{n+1} - q_n}{t_{n+1} - t_n}$$

$$+ \partial_{\dot{q}}L_{n-1}\frac{1}{t_{n+1} - t_n}\left(\frac{q_{n+1} - q_n}{t_{n+1} - t_n} - \frac{q_n - q_{n-1}}{t_n - t_{n-1}}\right). \qquad (2.9)$$

In the continuum limit (2.9) converges to

$$- \frac{dL}{dt} + \frac{\partial L}{\partial q}\dot{q} + \frac{\partial L}{\partial \dot{q}}\ddot{q} \qquad (2.10)$$

which vanishes identically, and is equivalent to (2.4). In the discrete case however, the equations of motion (2.9) for the t_n do not vanish in general. So the equations for the t_n are nontrivial, and have to be solved along with the q_n. Since the equations (2.9) only couple t_n at most two steps apart from each other, the discrete system is of second order and generically imposing boundary values t_0, q_0, t_N, q_N uniquely determines a solution. As a consequence, the discrete system defined by the action (2.5) does not capture the reparametrization invariance of the continuum dynamics defined by (2.2). One can show that this is directly linked to the failure of energy conservation within the the discrete system [6–8].

The loss of reparametrization invariance is, however, not ultimately tied to the discretization itself, but rather to the approximation (2.6). If, however, one can find a discrete action that exactly reproduces the continuum dynamics, one can regain the reparametrization freedom. Such actions are termed *perfect actions* e.g. in lattice gauge theory.[1] In the following section we will show how to construct a perfect action for the 1D systems discussed above, in order to restore reparametrization invariance.

A. Regaining reparametrization invariance

For the type of discretized actions we discussed so far one can always define a discrete action which displays exact reparametrization invariance. This so-called perfect action reflects the gauge freedom of the continuous system, which results in a nonuniqueness of the solution $\{t_n, q_n\}$. The idea is that the discrete system should exactly reproduce the dynamics of the continuous system, determined by the continuum Lagrange function $L(q, \dot{q})$.

We define the perfect action as follows: For t_n, q_n, and for each $n = 0, \dots N - 1$, solve the continuum Euler-Lagrange equations for $t^{(n)}(s)$, $q^{(n)}(s)$, $s \in [0, 1]$ with boundary values

$$
\begin{aligned}
t^{(n)}(0) &= t_n, & q^{(n)}(0) &= q_n, \\
t^{(n)}(1) &= t_{n+1}, & q^{(n)}(1) &= q_{n+1}.
\end{aligned}
\qquad (2.11)
$$

Denote the value of the action S on that solution, which is nothing but the Hamilton-Jacobi functional, by $S_{\mathrm{HJ}}^{(n)}$ and define

$$
\begin{aligned}
S_e &:= \sum_{n=0}^{N-1} S_{\mathrm{HJ}}^{(n)}(t_n, q_n, t_{n+1}, q_{n+1}) \\
&= \sum_{n=0}^{N-1} \int_0^1 ds \tilde{L}(t^{(n)}(s), q^{(n)}(s), t^{(n)\prime}(s), q^{(n)\prime}(s)), \quad (2.12)
\end{aligned}
$$

where $t^{(n)\prime}$ and $q^{(n)\prime}$ denote the derivatives of $t^{(n)}$ and $q^{(n)}$ with respect to the curve parameter s, respectively, and $\tilde{L}$ is given by (2.1).

The discrete action S_e defined in (2.12) is exactly reparametrization invariant, as the following theorem shows.

Theorem: For each solution $\{t_n, q_n\}$ of the equations of motion determined by the action (2.12) and each sequence $\{s_n\}$, there is a solution $t(s)$, $q(s)$ of the equations of motion (2.3) and (2.4) with $t(s_n) = t_n$, $q(s_n) = q_n$. Furthermore, for every such solution $t(s)$, $q(s)$ and each $s_0 < s_1 \dots < s_N$, $\{t(s_n), q(s_n)\}$ is a solution to the equations of motion determined by (2.12).

Proof: A detailed proof of this can be found in [6].

Since the continuous system with the Lagrangian $\tilde{L}$ is reparametrization invariant, the solutions of (2.3) and (2.4) are highly nonunique. Therefore, also the boundary value problem for the action (2.12) has a vast amount of different solutions for the same boundary conditions. This non-uniqueness directly corresponds to the reparametrization invariance of the action (2.2), and hence the discrete action S_e exactly captures this invariance. In particular, the t_n, q_n are underdetermined. Given the uniqueness of solutions to the dynamics determined by the deparametrized system with Lagrangian $L(q, \dot{q})$—the q_n are uniquely determined by the t_n, which by themselves can be chosen arbitrarily.[2] It follows that there is one gauge degree of freedom per vertex. Note that the $q_n(t_n)$ are Dirac observables in the sense of [16].

We have seen that the discrete action (2.12) exactly mimics the continuum dynamics of the system and therefore exhibits exact reparametrization invariance, unlike the system defined by the naïve discretization (2.5). Note that, as the discretization becomes very fine, one can expect the system to be approximately reparametrization invariant in the sense of [7]. The Hessian of S_d at the solution will contain a large number of eigenvalues[3] approaching zero in the continuum limit, when reparametrization invariance is restored.

B. Improving the discrete action S_d

The perfect action S_e contains the Hamilton-Jacobi functional of the system defined by the Lagrangian L, which might in general be hard to compute, or even unknown. In the following we present a procedure to construct sequences of improved actions, which converge to the perfect action, and which satisfy the constraints in an approximate way [7].

In order to improve the action S_d, which is a naïve discretization of the action (2.2) on the discretized interval $\{t_n\}$, one needs to refine the interval by $t_n = \tilde{t}_{nM} < \tilde{t}_{nM+1} < \dots < \tilde{t}_{nM+(M-1)} < \tilde{t}_{(n+1)M} = t_{n+1}$. Fix $\{t_n, q_n\}$,

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[1]Where, however, the broken symmetry in question is usually global Poincaré symmetry, and not gauge symmetries.

[2]As long as $t_n < t_{n+1}$ for all n, i.e. the t_n are a growing sequence.
[3]Namely one per (inner) vertex.

and for each interval $[t_n, t_{n+1}]$ solve the discrete equations of motion or the $\tilde{t}_k$, $\tilde{q}_k$, given by the naïve discretization of the action S; i.e. find an extremum of the action

$$S_d^{(n)} = \sum_{k=Mn}^{Mn+M-1} L\left(\tilde{q}_k, \frac{\tilde{q}_{k+1} - \tilde{q}_k}{\tilde{t}_{k+1} - \tilde{t}_k}\right)(\tilde{t}_{k+1} - \tilde{t}_k) \qquad (2.13)$$

with the boundary conditions

$$\tilde{t}_{Mn} = t_n, \qquad \tilde{q}_{Mn} = q_n,$$
$$\tilde{t}_{M(n+1)} = t_{n+1}, \qquad \tilde{q}_{M(n+1)} = q_{n+1}.$$

Denote the value of $S_d^{(n)}$ on the solution by $S_*^{(n)}$. Then the action

$$S_* := \sum_{n=0}^{N-1} S_*^{(n)} \qquad (2.14)$$

is clearly a function of the chosen t_n, q_n. It is more complicated than the naïve discretization (2.5).

Since for very fine subdivision the $\tilde{t}_k$, $\tilde{q}_k$ converge to a solution $t(s)$, $q(s)$ of the continuum dynamics given by $\tilde{L}$, it is easy to see that—in the limit of very fine discretization $\tilde{t}_k$—each of the contributions $S_*^{(n)}$ converges to its continuous counterpart, i.e.

$$\lim_{M \to \infty} S_*^{(n)} = S_{\mathrm{HJ}}^{(n)}(t_n, q_n, t_{n+1}, q_{n+1})$$
$$= \int_{s_n}^{s_{n+1}} ds L(t(s), q(s)). \qquad (2.15)$$

Therefore S_* converges to the exact discrete action (2.12).

The naïvely discretized action (2.5) approximates the exact discrete action (2.12) by replacing, for each interval $[s_n, s_{n+1}]$, the integral over the Lagrangian by a Riemann sum involving only two points. The improvement within the action S_* lies in the fact that the Riemann sum used to approximate the integral relies on many more intermediate points, therefore delivering a better approximation.

In order to compute the improved actions, only the solutions to the naïvely discretized action S_d for a refined discretization is involved, making the computation possibly more feasible, if the continuum system is not at hand, or too difficult to solve. Furthermore, the improved action S_* can be made an arbitrarily good approximation to the exact discrete action S_e, by using a very fine discretization, or by iterating the process, i.e. computing $S_*, (S_*)_*, ((S_*)_*)_*, \ldots$, which leads to the same limit S_e. It can therefore be used to compute S_e recursively, which can therefore be seen as the "perfect limit" of the S_*. We will use this strategy in order to investigate the perfect action in Regge gravity later on.

Note that although S_* still does not retain the full reparametrization invariance of S_e, it is closer to it than the naïvely discretized action S_d, in the sense that the constraints are satisfied to a greater accuracy.[4]

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[4] See [7] for details on approximate constraints.

III. REGGE CALCULUS

In the previous system we have seen that one-dimensional reparametrization invariant systems usually lose that invariance after discretization. This is also true for higher-dimensional field theories: Classical GR, as a theory of metrics on a differential manifold, is reparametrization invariant, due to the principle of covariance [1]. It is this invariance which makes it very difficult to compute (or interpret the physics of) solutions. It also is connected to many obstacles for quantizing the theory [5,8,17,18].

Regge calculus provides a discretization of GR, by triangulating the manifold, and replacing curvature expressions with deficit angles around 2-codimensional subsimplices [2]. Just as the discretization of one-dimensional systems replaces the search for smooth solutions to the equations by piecewise linear ones, Regge calculus replaces smooth curved metrics by piecewise linear flat ones.

A. Continuous preliminaries

The Einstein-Hilbert action in D dimensions with cosmological constant Λ is given by

$$S_{\mathrm{EH}} = \frac{1}{8\pi} \int_{\mathcal{M}} d^D x \sqrt{|g|}\left(\Lambda - \frac{1}{2}R\right)$$

leading to the equations of motion

$$8\pi \frac{\partial S_{\mathrm{EH}}}{\partial g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = 0. \qquad (3.1)$$

There is a special solution to (3.1), which will be very important later on. The Riemann tensor of a space of constant (sectional) curvature κ has the property

$$R_{\mu\nu\sigma\rho} = \kappa(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) \qquad (3.2)$$

leading to $R_{\mu\nu} = \kappa(D-1)g_{\mu\nu}$ and $R = \kappa D(D-1)$. Therefore the metric satisfying (3.2) satisfies the equations (3.1) for

$$\Lambda = \frac{(D-1)(D-2)}{2}\kappa. \qquad (3.3)$$

B. Discrete action

In Regge calculus, the smooth manifold $\mathcal{M}$ is replaced by a triangulated manifold $\mathcal{T}$, the D simplices of which are internally flat [2]. The Riemann curvature in this now arises as nontrivial parallel transport resulting from the nontrivial way of gluing the simplices together. The curvature is therefore naturally associated to the $D-2$ simplices H (also called "hinges") in the complex.

The Ricci scalar for such a manifold is 2 times the deficit angle at a $D-2$ simplex. If the triangulation $\mathcal{T}$ has a boundary $\partial \mathcal{T}$ (which is a triangulated $D-1$-dimensional manifold), then the action has a contribution from the extrinsic curvature in the boundary and the Ricci curvature

in the bulk $\mathcal{T}^{\circ} := \mathcal{T} \setminus \partial\mathcal{T}$, and reads[5] [19]

$$S_{\mathcal{T}} = \sum_{h \in \mathcal{T}^{\circ}} F_h \epsilon_h - \Lambda \sum_{\sigma \subset \mathcal{T}^{\circ}} V_\sigma + \sum_{h \in \partial\mathcal{T}} F_h \psi_h. \qquad (3.4)$$

The sum goes over all $(D-2)$ simplices h in the bulk and the boundary separately. The associated angles are

$$\epsilon_h = 2\pi - \sum_{\sigma \supset h} \theta_h^\sigma \quad \text{for } h \in \mathcal{T}^{\circ}, \qquad (3.5)$$

$$\psi_h = \pi - \sum_{\sigma \supset h} \theta_h^\sigma \quad \text{for } h \in \partial\mathcal{T}, \qquad (3.6)$$

where the θ_h^σ is the interior dihedral angle in the D-simplex σ associated to the $D-2$-subsimplex $h \subset \sigma$. For these angles within a flat simplex σ the so-called Schlaefli identity reads

$$\sum_{h \subset \sigma} F_h \frac{\partial \theta_h^\sigma}{\partial l_e} = 0 \quad \text{for all } 1 \text{ simplices(``edges'') } e \subset \sigma.$$

$$(3.7)$$

The dynamical variables are taken to be the lengths l_e of the edges $e \in \mathcal{T}^{\circ}$ in the bulk. For those edges the equations of motion can be computed with (3.7) to be

$$\sum_{h \supset e} \frac{\partial F_h}{\partial l_e} \epsilon_h - \Lambda \sum_{\sigma \supset e} \frac{\partial V_\sigma}{\partial l_e} = 0. \qquad (3.8)$$

Instead of piecewise linear flat simplices, one can build up the triangulation with simplices of constant (sectional) curvature κ (see the Appendix). The Regge action for such a triangulation $\mathcal{T}$ with cosmological constant Λ is a sum of the overall curvature of the manifold, having a contribution from the deficit angles at the $D-2$ dimensional subsimplices, the constant curvature of the tetrahedra, and the term with the cosmological constant. For Λ and κ having the relation (3.3), this leads to

$$S_{\mathcal{T}}^{(\kappa)} = \sum_{h \subset \mathcal{T}^{\circ}} F_h^{(\kappa)} \epsilon_h^{(\kappa)} + (D-1)\kappa \sum_{\sigma \subset \mathcal{T}^{\circ}} V_\sigma^{(\kappa)}$$

$$+ \sum_{h \in \partial\mathcal{T}} F_h^{(\kappa)} \psi_h^{(\kappa)}. \qquad (3.9)$$

where $F_h^{(\kappa)}$ denotes the $D-2$-dimensional volume of the $D-2$ simplex $h \subset \sigma$. Furthermore $\epsilon_h^{(\kappa)}$ and $\psi_h^{(\kappa)}$ denote deficit angle and exterior angle in the curved simplices analogously to (3.5) and (3.6).

The Schlaefli identity (A5) for curved simplices leads to the equation

$$\frac{\partial S_{\mathcal{T}}^{(\kappa)}}{\partial l_e} = \sum_{h \supset e} \frac{\partial F_h^{(\kappa)}}{\partial l_e} \epsilon_h^{(\kappa)} = 0. \qquad (3.10)$$

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[5]Up to a factor of 8π, which we ignore from now on.

C. Gauge invariance in Regge calculus

Analogously to our observations in the last section, the reparametrization invariance of general relativity is lost in Regge calculus, in the following sense: For a given set of boundary lengths, the solutions to Regge's equations (3.8) are generically unique, i.e. completely determined by the boundary data. The only exceptions to this are the cases in which the discrete dynamics exactly reproduces the continuum dynamics.

In 3D with $\Lambda = 0$ the Regge equations (3.8) are simply the vanishing of the deficit angles $\epsilon_e = 0$, the solution of which is a triangulation of a locally flat space-time. This is also the solution to 3D GR with vanishing cosmological constant. In higher dimensions there is, among other solutions, also $\epsilon_h = 0$, which can readily be seen to solve (3.8) for $\Lambda = 0$. Again, this coincides with locally flat space-time which is also one (among many solutions) of GR for $D > 3$.

In all of these cases the solutions possess a vertex displacement symmetry and an invariance under Pachner moves, which in 3D can e.g. be seen as a result of the second Bianchi identities [14,15]. As a result, the bulk lengths l_e, $e \in \mathcal{T}^{\circ}$ are not uniquely determined by the boundary lengths l_e, $e \in \partial\mathcal{T}$; rather the vertex displacement symmetry results in D gauge degrees of freedom per vertex.

Apart from these special cases, where the discrete dynamics exactly reproduces the continuum dynamics, the boundary data fix uniquely the lengths of the edges in the interior of the triangulation [7].[6] That is, translating a vertex in a solution does not lead to another solution (as it does for 3D Regge calculus with $\Lambda = 0$) and the Hamilton-Jacobi functional, i.e. the action evaluated on a solution, is not invariant under Pachner moves of the bulk. This is analogous to the situation in one dimension, where the reparametrization invariance (which in discretized gravity would amount to an invariance under change of triangulation) is lost in the naïve discretization.[7]

IV. IMPROVED AND PERFECT ACTION IN 3D

Since lattice gauge theory is not diffeomorphism invariant, the symmetries that are broken by discretization are not its local gauge symmetries, which are of a different

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[6]Apart from discrete ambiguities, which we ignore for the time being [7,20].

[7]The exceptions for this, e.g. 3D with $\Lambda = 0$, can be compared to the free particle in one dimension, where the continuum solutions are linear dependencies between the t_n and the q_n. In fact, the naïve discretization (2.5) already coincides with the perfect action (2.12) for this case, and the solutions are not uniquely determined by the boundary data t_0, t_N, q_0, q_N. Rather, the Hessian has as many zero eigenvalues as inner vertices, and the solutions t_n, q_n are nonunique in the sense of Sec. II A, which is a reflection of the gauge symmetry of the continuum limit in this case.
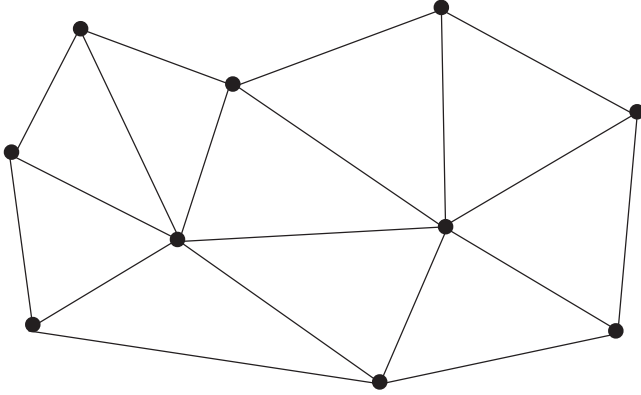
FIG. 1. Coarse triangulation $\mathcal{T}$ consisting of edged E, triangles T and tetrahedra Σ.

nature than in GR, but Poincaré invariance. The methods to construct improved and perfect actions in QCD can therefore not be directly transferred to GR. We therefore attempt to generalize the way this is done for one-dimensional systems, encountered in Sec. II, to the case of Regge calculus.

In one dimension the interval, on which the continuous theory is defined, is divided into smaller intervals as a result of the discretization, and in order to define the improved action (2.14) the interiors of these intervals are then further refined. The discrete equations are then solved for the refined lattice, subject to boundary conditions which relate them to values on the coarse lattice. Therefore, since in Regge calculus space-time is split into simplices via a triangulation, we will refine this triangulation further into smaller simplices in order to improve the action. Note that in more than one dimension the boundary of a triangulation and between single simplices is nontrivial, and it needs to be refined as well.
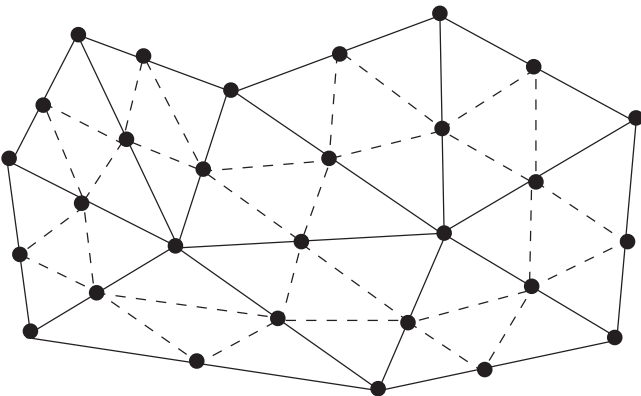
A. Refinement of the Regge action

We will first demonstrate the procedure for $D = 3$ to show the general idea, before we turn to the case of higher dimensions (in particular $D = 4$, which is the case of most interest to us) in Sec. V.

Consider a three-dimensional triangulation $\mathcal{T}$, consisting of edges E, triangles T and tetrahedra Σ (see Fig. 1), possibly with a boundary $\partial \mathcal{T}$. The Regge action $S_{\mathcal{T}}$ is given by (3.4), and is a function of the edge lengths L_E. Now subdivide $\mathcal{T}$ into a finer triangulation τ, consisting of edges e, triangles t and tetrahedra σ, as in Fig. 2. Similarly to the definition of the improved action in 1D, we solve the Regge equations for the edge lengths l_e subject to the conditions

$$\sum_{e \subset E} l_e = L_E \qquad (4.1)$$

and define the improved action $S_{\mathcal{T},\tau}$ as the value of the Regge action S_τ on a solution of the equations for l_e subject to (4.1). We add the constraint (4.1) via Lagrange multipliers; i.e. we have to vary the action

$$S_\tau = \sum_e l_e \varphi_e - \Lambda \sum_\sigma V_\sigma + \sum_E \alpha_E \left(L_E - \sum_{e \subset E} l_e \right), \quad (4.2)$$

where we have defined $\varphi_e := \psi_e$ for $e \in \partial \tau$ and $\varphi_e := \epsilon_e$ for $e \in \tau^\circ$, to unify notation. The equations of motion are then given by deriving (4.2) with respect to the l_e and α_E; i.e. one gets

$$\frac{\partial S_\tau}{\partial l_e} = \varphi_e - \Lambda \sum_{\sigma \supset e} \frac{\partial V_\sigma}{\partial l_e} - \sum_{E \supset e} \alpha_E = 0, \qquad (4.3)$$

$$\frac{\partial S_\tau}{\partial \alpha_E} = L_E - \sum_{e \subset E} l_e = 0. \qquad (4.4)$$

The improved action is then defined as the value of S_τ on a solution of (4.3) and (4.4), i.e.

$$S_{\mathcal{T},\tau} := S_{\tau | \frac{\partial S_\tau}{\partial l_e} = \frac{\partial S_\tau}{\partial \alpha_E} = 0}. \qquad (4.5)$$

Note that the improved action $S_{\mathcal{T},\tau}$ depends on the "large" lengths L_E for $E \in \mathcal{T}$, but incorporates the dynamics of the finer triangulation τ. A quick calculation using Euler's theorem (A6) shows that the l_e, α_E satisfy

$$0 = \sum_e l_e \frac{\partial S_\tau}{\partial l_e} = \sum_e l_e \varphi_e - 3\Lambda \sum_\sigma V_\sigma - \sum_E \alpha_E \sum_{e \subset E} l_e. \qquad (4.6)$$

So with (4.4) the improved action can be put into the form

$$S_{\mathcal{T},\tau} = \sum_E L_E \alpha_E + 2\Lambda \sum_\Sigma V_\Sigma, \qquad (4.7)$$

where we have defined $V_\Sigma := \sum_{\sigma \subset \Sigma} V_\sigma$. Note that the α_E, V_Σ are complicated functions of the L_E, which have to be determined by the equations of motion (4.3) and (4.4).

FIG. 2. Fine triangulation τ consisting of edged e, triangles t and tetrahedra σ.

Nevertheless, one can derive the equations of motion by varying $S_{\mathcal{T},\tau}$ with respect to the L_E. This can be achieved by changing the $L_E \to L_E + \delta L_E$, and assuming that the solutions for the l_e and α_E also change only slightly to $l_e \to l_e + \delta l_e$, $\alpha_E \to \alpha_E + \delta \alpha_E$. Therefore the value of $S_{\mathcal{T},\tau}$ changes by

$$
\begin{aligned}
\delta S_{\mathcal{T},\tau} &= \sum_e \frac{\partial S_\tau}{\partial l_e} \delta l_e + \sum_E \frac{\partial S_\tau}{\partial \alpha_E} \delta \alpha_E \\
&\quad + \sum_E \frac{\partial S_\tau}{\partial L_E} \delta L_E|_{\frac{\partial S_\tau}{\partial l_e} = \frac{\partial S_\tau}{\partial \alpha_E} = 0} \\
&= \alpha_E \delta L_E.
\end{aligned}
\tag{4.8}
$$

Therefore the equations for the L_E determined by the improved action $S_{\mathcal{T},\tau}$ are the vanishing of the Lagrange multipliers, i.e.

$$
\frac{\partial S_{\mathcal{T},\tau}}{\partial L_E} = \alpha_E \stackrel{!}{=} 0 \quad \text{for } E \in \mathcal{T}^\circ,
\tag{4.9}
$$

which, together with (4.3), are equivalent to the Regge equations for the l_e.[8]

B. Perfect action in 3D

There is a similarity between the improved action (4.7) and the Regge action with curved simplices (3.9), as well as the respective resulting equations of motion (3.10) and (4.9). The similarity becomes more apparent if we define

$$
\Theta_E^\Sigma := \sum_{\sigma \supset e, \sigma \in \Sigma} \left(\theta_e^\sigma - \Lambda \frac{\partial V_\sigma}{\partial l_e} \right)
\tag{4.10}
$$

for some[9] $e \subset E$; then we have $\alpha_E = 2\pi - \sum_{\Sigma \supset E} \Theta_E^\Sigma$ for $E \subset \mathcal{T}^\circ$ being in the bulk and $\alpha_E = \pi - \sum_{\Sigma \supset E} \Theta_E^\Sigma$ for $E \subset \partial\mathcal{T}$ being in the boundary. For every edge $E \subset \mathcal{T}^\circ$ in the bulk, the equations of motion determined by $S_{\mathcal{T},\tau}$ therefore are

$$
2\pi - \sum_{\Sigma \supset E} \Theta_E^\Sigma = 0.
\tag{4.11}
$$

Note that, despite the formal similarity, the Θ_E^Σ are not quite the interior dihedral angles at the edges E in the tetrahedra Σ. It is, however, not hard to show that they become so in the perfect limit, i.e. the limit of infinitely fine subdivision, which we denote by $\tau \to \infty$. If the triangulation τ is such that its simplices σ are regular, i.e. their edge lengths l_e after solving (4.3) and (4.4) are all of the same

---

[8]To be precise, it is equivalent to $\frac{\partial S_\tau}{\partial l_e} = 0$ for all bulk edges $e \in \tau^\circ$, all boundary edges $e \in \partial\tau$ which are *not* a subedge of an edge in $\partial\mathcal{T}$, i.e. $e \not\subset E$, and (4.3) for all $e \in \partial\tau$ which are, i.e., also $e \subset E \in \partial\mathcal{T}$. This is equivalent to the Regge equations for the e in the finer triangulation τ, plus the vanishing of the canonical momenta on the boundary triangles $T \in \partial\mathcal{T}$.
[9]The Eqs. (4.3) guarantee that this choice does not depend on the actual $e \subset E$.

small order of magnitude λ, then the term in (4.10) containing the derivative of the volume scales like $O(\lambda^2)$, as compared to the θ_e^σ, which scale as $O(1)$, and therefore dominate the expression. (Also note that $\theta_e^\sigma - \kappa \frac{\partial V_\sigma}{\partial l_e}$ is the first order Taylor expansion in κ of the dihedral angle in a curved tetrahedron.) We conclude that, in the perfect limit $\tau \to \infty$, the Θ_E^Σ indeed converge to the sum of the interior angles at $e \subset E$ in $\sigma \subset \Sigma$, i.e. to the interior angle at E in Σ. Note that this interior angle is the same everywhere "on" E, since it is independent of $e \subset E$. Even more, we can demonstrate the perfect limit of V_Σ (as a function of the L_E) by e.g. considering a triangulation τ consisting of only one tetrahedron $|\mathcal{T}| = \Sigma$. Then the variations (4.9) of $S_{\mathcal{T},\tau}$ with respect to one of the L_E, $E = 1, \ldots, 6$ is equivalent to

$$
\sum_{E=1}^{6} L_E \frac{\partial \Theta_E}{\partial L_E} = 2\Lambda \frac{\partial V_\Sigma}{\partial L_E},
\tag{4.12}
$$

which, in the perfect limit, is exactly the Schlaefli identity for curved tetrahedra (A5) which related the interior angles of curved dihedral angles and volumes on tetrahedra of constant curvature $\kappa = \Lambda$. In the perfect limit, the formal similarity becomes an equality, and we conclude that the perfect action in 3D is given by

$$
S_{\mathcal{T},*} := \lim_{\tau \to \infty} S_{\mathcal{T},\tau} = S_{\mathcal{T}}^{(\kappa)},
\tag{4.13}
$$

i.e. coincides with the Regge action for constantly curved tetrahedra with curvature $\kappa = \Lambda$. It is quite easy to show that this action has three gauge degrees of freedom per vertex, unlike $S_{\mathcal{T}}$, since the equations of motion—given by the perfect limit of (4.11)—are equivalent to the vanishing of all deficit angles $\epsilon_E = 0$ for interior edges $E \in \mathcal{T}^\circ$, which results in the triangulation of a manifold of constant sectional curvature $\kappa = \Lambda$. This not only reproduces exactly the continuum dynamics of 3D GR with cosmological constant Λ, but also possesses the exact vertex displacement symmetry as 3D Regge calculus with flat simplices exhibits for $\Lambda = 0$. Furthermore, it is invariant under refinement of triangulation $\mathcal{T}$, as it should be by construction.

We conclude that in 3D, the gauge symmetry of *GR* containing 3 gauge degrees of freedom per vertex, which is broken for $\Lambda \neq 0$, is restored in the perfect limit. The Regge action for constantly curved tetrahedra arises naturally as perfect action in this context. It should be noted that the Regge action (3.4) with flat simplices arises naturally as first order approximation, by the following argument: By investigating the scaling property of the curved Regge action (3.9), e.g. by considering (A7), one can easily see that a scaling of the edge lengths $l_e \to \lambda l_e$ can be absorbed into a scaling of the curvature $\kappa \to \lambda^2 \kappa$. Expanding the curved functions $\theta_e^{(\kappa),\sigma}$, $V_\sigma^{(\kappa)}$ into linear order in κ, one obtains, by using the identities (A4) and (A5), that

$$S_\tau^{(\kappa)} = S_\tau + O(\kappa^2), \tag{4.14}$$

where S_τ is the Regge action (3.4) for flat simplices with cosmological constant $\Lambda = \kappa$.

V. HIGHER DIMENSIONS

We now consider the concept of improved and perfect actions for dimensions $D > 3$, where of course the case of ultimate interest is $D = 4$. Nevertheless, since the arising procedures are generic for arbitrary higher dimensions, we shall treat the problem for arbitrary dimension D, and comment about the implications for $D = 4$ in the end.

The general concept for defining the improved action $S_{\mathcal{T},\tau}$ for $D > 3$ is similar to $D = 3$. We start with a triangulation $\mathcal{T}$ consisting of simplices Σ, hinges H and edges E. Now subdivide $\mathcal{T}$ into a finer triangulation τ, consisting of D simplices σ, hinges h and edges e. Note that some of the hinges h are contained in the "larger" hinges H. The action for the finer triangulation S_τ is a function of the edge lengths l_e. It turns out that the most convenient generalization of the condition (4.1) to $D > 3$ is not to keep the edge lengths L_E fixed, but rather the $D - 2$ volumes F_H, i.e. to constrain the variation of the Regge action for τ by

$$\sum_{h \subset H} f_h = F_H, \tag{5.1}$$

where f_h is the $D - 2$ volume of the hinge h. In other words, we vary

$$S_\tau = \sum_h f_h \varphi_h - \Lambda \sum_\sigma V_\sigma + \sum_H \alpha_H \left(F_H - \sum_{h \subset H} f_h \right) \tag{5.2}$$

with respect to l_e and α_H, where the Lagrange multipliers α_H have been introduced in order to enforce (5.1), and φ_h denotes the deficit angle ϵ_h for $h \in \tau^\circ$ being in the bulk, and the extrinsic curvature angle ψ_h for $h \in \partial\tau$ in the boundary. The improved action is—similar as for $D = 3$—defined as

$$S_{\mathcal{T},\tau} := S_\tau \big|_{\frac{\partial S_\tau}{\partial l_e} = \frac{\partial S_\tau}{\partial \alpha_H} = 0}, \tag{5.3}$$

and is naturally a function of the F_H (e.g. the areas of the triangles for $D = 4$). The resulting equations for the l_e, α_H are, using the Schlaefli identity (3.7),

$$\frac{\partial S_\tau}{\partial l_e} = \sum_{h \supset e} \frac{\partial f_h}{\partial l_e} \varphi_h - \Lambda \sum_{\sigma \supset e} \frac{\partial V_\sigma}{\partial l_e} - \sum_{h \supset e} \sum_{H \supset h} \alpha_H \frac{\partial f_h}{\partial l_e} = 0, \tag{5.4}$$

$$\frac{\partial S_\tau}{\partial \alpha_H} = F_H - \sum_{h \subset H} f_h = 0. \tag{5.5}$$

Using Euler's theorem (A6) we get

$$0 = \sum_e l_e \frac{\partial S_\tau}{\partial l_e}$$

$$= (D - 2) \sum_h f_h \varphi_h - D\Lambda \sum_\sigma V_\sigma - (D - 2) \sum_H \alpha_H \sum_{h \subset H} f_h, \tag{5.6}$$

which, inserted into (5.2) together with (5.5), results in the improved action

$$S_* = \sum_H F_H \alpha_H + \frac{2}{D - 2} \Lambda \sum_\Sigma V_\Sigma, \tag{5.7}$$

where we have defined $V_\Sigma := \sum_{\sigma \subset \Sigma} V_\sigma$. Note the similarity between the improved action (5.7) and the Regge action (3.9) for simplices of constant curvature κ, for κ and Λ being related by (3.3).

The improved action (5.7) is a function of the F_H via the α_H and V_Σ, which will depend on the F_H in a complicated manner to be determined by solving Eqs. (5.4) and (5.5). Nevertheless, we can derive the equations for the F_H determined by the improved action. For this we consider the same set of equations, just with slightly changed parameters $F_H + \delta F_H$. It can be expected that the solutions for l_e, α_H will also change just slightly via

$$l_e \to l_e + \delta l_e, \qquad \alpha_H \to \alpha_H + \delta \alpha_H.$$

Then $S_{\mathcal{T},\tau}$ changes slightly via

$$\delta S_{\mathcal{T},\tau} = \sum_e \frac{\partial S_\tau}{\partial l_e} \delta l_e + \sum_H \frac{\partial S_\tau}{\partial \alpha_H} \delta \alpha_H + \sum_H \frac{\partial S_\tau}{\partial F_H} \delta F_H, \tag{5.8}$$

and evaluating (5.8) on a solution results in

$$\frac{\partial S_{\mathcal{T},\tau}}{\partial F_H} = \alpha_H = 0. \tag{5.9}$$

A. Improving the curved Regge action

It is instructive to repeat the calculation with curved simplices. We start from the action (3.9) and impose the constraints via Lagrange multipliers $\alpha_H^{(\kappa)}$. In other words, we have to vary the action

$$S_\tau^{(\kappa)} = \sum_h f_h^{(\kappa)} \varphi_h^{(\kappa)} + (D - 1)\kappa \sum_\sigma V_\sigma^{(\kappa)}$$

$$+ \sum_H \alpha_H^{(\kappa)} \left(F_H - \sum_{h \subset H} f_h^{(\kappa)} \right), \tag{5.10}$$

where the superscript (κ) denotes the volume of hinges and simplices of constant curvature κ. Again, $\varphi_h^{(\kappa)}$ is shorthand for $\epsilon_h^{(\kappa)}$ whenever $h \in \tau^\circ$ is a hinge in the bulk, and $\psi_h^{(\kappa)}$, whenever $h \in \partial\tau$ is in the boundary. With the Schlaefli identity (A5) for simplices of constant curvature, the resulting equations for the l_e are

$$\frac{\partial S_\tau^{(\kappa)}}{\partial l_e} = \sum_{h \supset e} \frac{\partial f_h^{(\kappa)}}{\partial l_e} \varphi_h^{(\kappa)} - \sum_{h \supset e} \sum_{H \supset h} \alpha_H^{(\kappa)} \frac{\partial f_h^{(\kappa)}}{\partial l_e} = 0, \quad (5.11)$$

$$\frac{\partial S_\tau^{(\kappa)}}{\partial \alpha_H^{(\kappa)}} = F_H - \sum_{h \subset H} f_h^{(\kappa)} = 0. \quad (5.12)$$

With the geometric identity (A9) for simplices of constant curvature κ, we get

$$0 = \sum_e l_e \frac{\partial S_\tau^{(\kappa)}}{\partial l_e}$$

$$= (D-2) \sum_h f_h^{(\kappa)} \varphi_h^{(\kappa)} + 2\kappa \sum_h \frac{\partial f_h^{(\kappa)}}{\partial \kappa} \varphi_h^{(\kappa)}$$

$$- (D-2) \sum_H \sum_{h \subset H} \alpha_H^{(\kappa)} f_h^{(\kappa)} - 2\kappa \sum_H \sum_{h \subset H} \alpha_H^{(\kappa)} \frac{\partial f_h^{(\kappa)}}{\partial \kappa}$$

$$(5.13)$$

which results in the improved action

$$S_{\mathcal{T},\tau}^{(\kappa)} = \sum_H F_H \alpha_H^{(\kappa)} + (D-1)\kappa \sum_\Sigma V_\Sigma^{(\kappa)}$$

$$+ \frac{2}{D-2} \kappa \sum_h \frac{\partial f_h^{(\kappa)}}{\partial \kappa} \varphi_h^{(\kappa)}$$

$$+ \frac{2}{D-2} \kappa \sum_H \sum_{h \subset H} \alpha_H^{(\kappa)} \frac{\partial f_h^{(\kappa)}}{\partial \kappa}, \quad (5.14)$$

where we have defined $V_\Sigma^{(\kappa)} := \sum_{\sigma \subset \Sigma} V_\sigma^{(\kappa)}$. By a similar reasoning as in the case with flat simplices, the equations for the improved action is easily obtained to be

$$\frac{\partial S_{\mathcal{T},\tau}^{(\kappa)}}{\partial F_H} = \alpha_H^{(\kappa)} = 0. \quad (5.15)$$

B. Perfect actions with flat and curved simplices

If we consider the improved actions (5.7) and (5.14), which result from refining the triangulations with flat and curved simplices, respectively, we see that their expressions seem to be quite different. However, in performing the continuum limit for both actions, we will demonstrate that they both converge to the same perfect action, when Λ and κ satisfy the relation (3.3). In order to do this, we show that—as functions of the lengths F_H—both perfect limits satisfy the same ordinary differential equation with respect to Λ (or, equivalently, κ). We do this by considering the ordinary differential equations that the two improved actions (5.7) and (5.14) satisfy, and show that in the continuum limit they converge to each other.

We first vary the improved action $S_{\mathcal{T},\tau}$ for flat simplices with respect to Λ, by solving the equations of motion again with $\Lambda \to \Lambda + \delta\Lambda$, and assume the resulting solutions l_e, α_H also change only slightly by $l_e \to l_e + \delta l_e$ and $\alpha_H \to$

$\alpha_H + \delta\alpha_H$. The change of the action is therefore

$$\delta S_{\mathcal{T},\tau} = \sum_e \frac{\partial S_\tau}{\partial l_e} \delta l_e + \sum_H \frac{\partial S_\tau}{\partial \alpha_H} \delta\alpha_H + \frac{\partial S_\tau}{\partial \Lambda} \delta\Lambda$$

$$= -\sum_\Sigma V_\Sigma \delta\Lambda, \quad (5.16)$$

where the Regge equations have been used. With (5.7) this results in

$$S_{\mathcal{T},\tau} + \frac{2}{D-2} \Lambda \frac{\partial S_{\mathcal{T},\tau}}{\partial \Lambda} = \sum_H F_H \alpha_H. \quad (5.17)$$

The same calculation for the improved action (5.14) with curved simplices is more involved, since the constituents depend explicitly on κ. Since $S_{\mathcal{T},\tau}^{(\kappa)}$ is the value of $S_\tau^{(\kappa)}$ evaluated on a solution, varying $S_{\mathcal{T},\tau}^{(\kappa)}$ with resect to κ is equivalent to varying $S_\tau^{(\kappa)}$, and inserting the solutions for l_e, $\alpha_H^{(\kappa)}$ afterwards [since the variations of l_e, $\alpha_H^{(\kappa)}$ vanish on solutions, by definition]. We have

$$\frac{\partial S_{\mathcal{T},\tau}^{(\kappa)}}{\partial \kappa} = \frac{\partial S_\tau^{(\kappa)}}{\partial \kappa}$$

$$= \sum_h \frac{\partial f_h^{(\kappa)}}{\partial \kappa} \varphi_h^{(\kappa)} + \sum_h f_h^{(\kappa)} \frac{\partial \epsilon_h^{(\kappa)}}{\partial \kappa} + (D-1) \sum_\sigma V_\sigma^{(\kappa)}$$

$$+ (D-1)\kappa \sum_\sigma \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa} - \sum_H \sum_{h \subset H} \alpha_H^{(\kappa)} \frac{\partial f_h^{(\kappa)}}{\partial \kappa}.$$

$$(5.18)$$

With (A4) and the Schlaefli identity (A5), we have

$$\sum_h f_h^{(\kappa)} \frac{\partial \varphi_h^{(\kappa)}}{\partial \kappa} = -\frac{D-1}{2} \sum_\sigma \sum_{e \subset \sigma} l_e \frac{\partial V_\sigma^{(\kappa)}}{\partial l_e} \quad (5.19)$$

which results in

$$S_\tau^{(\kappa)} + \frac{2}{D-2} \kappa \frac{\partial S_\tau^{(\kappa)}}{\partial \kappa} = \sum_h \sum_{e \subset h} l_e \frac{\partial f_h^{(\kappa)}}{\partial l_e} \varphi_h^{(\kappa)} + \sum_H \alpha_H^{(\kappa)} F_H$$

$$- \sum_{h \subset H} \sum_{e \subset h} l_e \frac{\partial f_h^{(\kappa)}}{\partial l_e} \quad (5.20)$$

which, evaluated on a solution to (5.11) and (5.12), results in

$$S_{\mathcal{T},\tau}^{(\kappa)} + \frac{2}{D-2} \kappa \frac{\partial S_{\mathcal{T},\tau}^{(\kappa)}}{\partial \kappa} = \sum_H \alpha_H^{(\kappa)} F_H. \quad (5.21)$$

Note the similarity to (5.17).

The solutions for the α_H, $\alpha_H^{(\kappa)}$ in fact converge to each other in the perfect limit. In order to show this, we assume that l_e, α_H satisfy Eqs. (5.4) and (5.5), and $l_e + \delta l_e$, $\alpha_H + \delta\alpha_H$ satisfy Eqs. (5.11) and (5.12). We consider the limit of very fine triangulations τ—in particular we assume that both solutions are sufficiently close to a solution to the

Einstein equations—this, in particular, means that the scale over which the curvature changes is much larger than l_e or $l_e + \delta l_e$. For curved simplices the limit of small edge lengths coincides with the limit of small curvature. Expanding curved quantities in κ results in

$$V_\sigma^{(\kappa)} = V_\sigma + \kappa \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} + O(\kappa^2) \qquad (5.22)$$

In the Appendix it is proved that the term linear in κ is of order $O(l_e^{D+2})$. Furthermore, for a dihedral angle $\theta_h^{\sigma(\kappa)}$ one has, using (A4) and the Schlaefli identity (A5),

$$\sum_{h \subset \sigma} f_h^{(\kappa)} \theta_h^{\sigma(\kappa)} = \sum_{h \subset \sigma} f_h \theta_h^\sigma + \frac{D(D-1)}{2} \kappa V_\sigma$$
$$+ \kappa \sum_{h \subset \sigma} \frac{\partial f_h^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} \theta_h^\sigma + O(\kappa^2), \qquad (5.23)$$

where quantities without superscript are volumes and angles in flat simplices. As a result, we get

$$S_\tau^{(\kappa)} = S_\tau + \kappa \sum_h \frac{\partial f_h^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} \xi_h + O(\kappa^2), \qquad (5.24)$$

where

$$\xi_h := \begin{cases} \varphi_h - \sum_{H \supset h} \alpha_H & \text{for } h \subset H \\ \varphi_h & \text{for } h \not\subset H. \end{cases} \qquad (5.25)$$

Because of the Regge equations (5.4)

$$\sum_{h \supset e} \frac{\partial f_h}{\partial l_e} \xi_h = \Lambda \sum_{\sigma \supset e} \frac{\partial V_\sigma}{\partial l_e} \qquad (5.26)$$

and due to the assumed regularity of the triangulation τ, where the edge lengths are all of the order of magnitude of some lengths l, one has that $\xi_h \sim l^2$. In the limit of very fine τ, both l_e and $l_e + \delta l_e$ can be expected to tend to zero, so we can expand (5.11) in δl_e and compare it with l_e. We get

$$\sum_{e'} \frac{\partial^2 S_\tau}{\partial l_e \partial l_{e'}} \delta l_{e'} + \sum_H \frac{\partial^2 S_\tau}{\partial l_e \partial \alpha_H} \delta \alpha_H$$
$$+ \kappa \frac{\partial}{\partial l_e} \sum_h \frac{\partial f_h^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} \xi_h + O(l^{D+3}). \qquad (5.27)$$

Since $\frac{\partial^2 S_\tau}{\partial l_e \partial l_{e'}} \sim l^{D-2}$ and $\frac{\partial^2 S_\tau}{\partial l_e \partial \alpha_H} \sim l$ for $e \subset H$, we get that

$$\delta l_{e'} \sim l^3, \qquad \delta \alpha_H \sim l^D.$$

Hence the perfect limit $\tau \to \infty$ corresponds to the limit $l \to 0$. Therefore $\alpha_H^{(\kappa)} = \alpha_H + \delta \alpha_H$ converges to α_H in the continuum limit.

Furthermore, the perfect actions $S_{T,*}$ and $S_{T,*}^{(\kappa)}$ obviously coincide for $\kappa = \Lambda = 0$. So not only do they satisfy the same ordinary differential equation with respect to $\Lambda =$

$(D-1)(D-2)\kappa/2$, which is first order, they also coincide for one value. Therefore, they must coincide as functions of the F_H, and we conclude

$$S_{T,*} = \lim_{\tau \to \infty} S_{T,\tau} = \lim_{\tau \to \infty} S_{T,\tau}^{(\kappa)} = S_{T,*}^{(\kappa)}. \qquad (5.28)$$

C. Constantly curved subsector

For $D > 3$ it is nontrivial to compute the perfect limit of the improved action $S_{T,\tau}$ given by (5.7), since the α_H do not necessarily, unlike in $D = 3$, have to have the interpretation of deficit angles at the hinges H in that limit. In general, it will be quite complicated to compute the α_H. However, there is a special case in which one can compute the perfect action $S_{T,*}$, which is when the F_H satisfy the following requirement:

Let T be a triangulation of a manifold $|T| = M$ with constant curvature κ with constantly curved simplices Σ, such that there are vanishing deficit angles. If the $D-2$-dimensional hinges H have a volume F_H, then the value of the perfect action $S_{\tau,*}$ on that configuration F_H is given by

$$S_{T,*}(F_H) = \sum_{H \subset \partial T} F_H \left(\pi - \sum_{\Sigma \supset H} \theta_H^{(\kappa)\Sigma} \right) + (D-1)\kappa V_M, \qquad (5.29)$$

where $\theta_H^{(\kappa)\Sigma}$ is the dihedral angle in the curved simplex Σ at the hinge H, and V_M is the volume of the manifold M. This can be seen as follows. In the last section we have shown that the Regge action with curved simplices and the flat simplices leads to the same perfect action $S_{T,*}$ if Λ and κ are related by (3.3). Therefore we can use curved simplices instead of flat ones in our triangulation T. However, curved simplices can be glued together with vanishing deficit angles $\epsilon_H^{(\kappa)} = 0$ to form the manifold M, since M has constant sectional curvature κ. There are in fact infinitely many ways to do this, which can all be related by Pachner moves that do not change the boundary ∂T. For all of these possibilities, the geometry satisfies trivially the Regge equations (3.10), because the deficit angles all vanish. Moreover, the constraints (5.1) are satisfied by definition. The value of the Regge action S_T does not actually depend on the exact triangulation T; it is only dependent on the boundary data, i.e. the F_H for $H \in \partial T$ and the extrinsic dihedral angles. The action (3.9) evaluated on $\epsilon_H^{(\kappa)} = 0$ gives exactly (5.29). Since it is invariant under refinement of the triangulation, it is by definition the perfect action. Moreover, it is invariant under Pachner moves, and invariant under variations of the F_H which are a result of vertex displacements, since these only change the F_H in the triangulation, but do not change the geometry, which is that of constant curvature. Thus, in this special case we regain D gauge degrees of freedom per vertex (the vertex displacements), which reflects the diffeo-

morphism symmetry of lapse and shift from the continuum theory.

Note that in this case (5.14) shows that $\alpha_H \equiv \epsilon_H^{(\kappa)}$ for $H \in \mathcal{T}^\circ$. Moreover, for the special case of $\mathcal{T}$ consisting of one simplex Σ, we can, in a similar derivation as for $D = 3$, show that

$$\sum_H F_H \frac{\partial \alpha_H}{\partial F_{H'}} = (D-1)\kappa \frac{\partial V_\Sigma}{\partial F_{H'}}, \qquad (5.30)$$

which—since the $\frac{D(D-1)}{2} \times \frac{D(D-1)}{2}$ matrix $\partial F_H / \partial L_E$ is invertible[10]—is equivalent to the Schlaefli identity within curved simplices (A5).

In general, the F_H that are the arguments of the improved and the perfect action will not satisfy the requirement that there exists a triangulation of curved simplices that can be glued together with vanishing deficit angles.[11] In these cases α_H will have a much more complicated interpretation, and will be much harder to compute. In the case above where we have computed the perfect action, however, we have recovered the perfect action to reproduce a manifold with constant curvature κ, which is a solution of the continuum theory of GR, which exists in all dimensions D, as we have shown in Sec. III A. For $D > 3$, the sector of solutions is much larger, however, and contains many more solutions.

VI. SUMMARY AND CONCLUSION

We have investigated the concept of improved and perfect actions in Regge calculus, where the reparametrization invariance of general relativity is usually broken.

Discretizations of theories with symmetries usually lose that symmetry, e.g. in lattice gauge theory, where Poincaré-invariance is broken by introduction of a lattice. The motivation for our analysis was that the concept of improved and perfect actions is used in order to regain the symmetry within the lattice formulation. The QCD Lagrangian is not diffeomorphism invariant, however, and the techniques for lattice QCD are therefore not directly applicable to Regge gravity.

It is well-known that one-dimensional systems with reparametrization invariance lose that symmetry upon discretization, and there is a procedure to construct improved and perfect actions in this case in order to arrive at discrete actions which retain exact reparametrization invariance

[6]. We have reviewed this in Sec. II, and have proposed a procedure to construct improved and perfect (classical) actions for discretized, reparametrization invariant field theories, in particular, Regge calculus in Sec. IV A. We have applied this scheme to Regge gravity in arbitrary dimensions D.

We have done this by considering improved actions $S_{\mathcal{T},\tau}$ which are defined on a triangulation $\mathcal{T}$, which however incorporate the dynamics of the refined triangulation τ, i.e. is closer to the actual continuum dynamics. In the canonical formulation this leads also to a better approximation of the constraints in the sense of [7]. It seems that this is a useful setting in which to think about renormalization group flow in a diffeomorphism-invariant context. Since the actual scales in the theory have to be determined dynamically, they are not available to label a cutoff for the theory, and, in particular, to compare them for different labels. However, one can investigate the difference of the dynamics which are discretized on two triangulations $\mathcal{T}$ and τ, the continuum limit being better and better approximated the larger the difference between the two, i.e. the finer τ is compared to $\mathcal{T}$.[12]

We have shown that the perfect action or 3D Regge calculus for $\Lambda \neq 0$ can be computed explicitly.[13] It can be obtained by replacing the flat tetrahedra by tetrahedra of constant curvature $\kappa = \Lambda$. This leads to the action (4.13) which exactly reproduces the continuum dynamics of 3D GR with cosmological constant, i.e. vanishing deficit angles $\epsilon_e^{(\kappa)} = 0$, leading to space-time with constant local curvature. As a consequence, the thus obtained perfect action leads to a similar vertex displacement symmetry than one finds in 3D Regge calculus for $\Lambda = 0$.

Since for $D > 3$ the continuum theory possesses local degrees of freedom, the perfect action is much harder to construct in this case. Nevertheless, we could show that the Regge actions with flat simplices, and that with simplices of constant curvature $\kappa = \Lambda/3$ lie in the same universality class, i.e. lead to the same perfect action $S_{\mathcal{T},*}$. Moreover, we were able to express the perfect action in terms of the continuum limit of the Lagrange multipliers α_H and the volumes of the simplices. For the subsector of constantly curved solutions, which exists in GR for all dimensions, the α_H can in fact be computed to be the deficit (or, in case of boundary hinges $H \subset \partial\mathcal{T}$, extrinsic) angles in constantly curved simplices, where the curvature and the cosmological constant are related by (3.3). For this subsector of solutions, the perfect action possesses the vertex displacement symmetry, which lead to D gauge degrees of freedom per vertex. It therefore captures the gauge symmetry of

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[10]Apart from discretely many cases, see e.g. [21].

[11]It might not be possible to glue constantly curved simplices with these F_H together at all—although for each separate simplex the relation between the L_E and the F_H can be inverted, and the resulting geometries of neighboring simplices might be incompatible. One can suspect that the geometry described will not be that of constantly curved simplices, but rather of objects which are topological simplices, but have a geometry which satisfies Einstein's equations in D dimensions with a cosmological constant Λ (of which the constantly curved ones are a special case).

[12]Since the triangulations τ form a partially ordered set, it might be that—in mathematical terms—the renormalization group flow in this context has to be treated with the convergence of filters, rather than sequences.

[13]For $\Lambda = 0$ the Regge action is already perfect.

lapse and shift, since it reproduces exactly the continuum dynamics (of constant curvature).

In this work we did not obtain explicitly an improved action which takes into account propagating degrees of freedom. This would correspond to integrating out higher frequency gravitons and their interactions and finding an effective action. We expect this to be a very complicated task leading to a nonlocal action. However, it is a promising one with possible contacts to other quantum gravity approaches [22]. As a first step one can consider an expansion around flat space and define an action that takes into account the lowest nonlinear dynamics of the gravitons [23]. As the perfect action is by construction triangulation independent, this could be also helpful for understanding how to obtain triangulation independent models.

The κ-curved simplices, which appear in the improved and perfect actions, can be useful for the construction of quantum gravity models for several reasons.

(i) Using the perfect action $S_{\mathcal{T}}^{(\kappa)}$ given by (4.13) instead of the Regge action (3.4) is a more appropriate description for the problem at hand, since for 3D the perfect action correctly reflects the finite number of degrees of freedom of the continuum theory. These are not directly visible if one uses flat tetrahedra, since for $\Lambda \neq 0$ the corresponding Regge equations lead to a unique solution for the edge lengths. So no gauge freedom is apparent in this description. The edge lengths can therefore be mistaken to be physical degrees of freedom. The perfect action however is not only invariant under further refinement of the triangulation, it also shows that the edge lengths in themselves are not physical, but rather are a gauge artefact introduced by a choice of triangulation.

Not does this show that in construction of quantized models of 3D Regge calculus with $\Lambda \neq 0$ the perfect action $S_{\mathcal{T}}^{(\kappa)}$ might be more suitable than $S_{\mathcal{T}}$, in a broader context it shows how in discretized gravity theories it can be difficult to tell physical from gauge degrees of freedom.

This is, in particular, important in 4D, where the solutions to the Regge equations (even for $\Lambda = 0$) are generically unique. This is usually taken as proof that the diffeomorphism symmetry of GR has been successfully divided out, and one is only working with gauge-invariant quantities (i.e. the edge lengths), since the gauge symmetry of GR, apparent in the nonuniqueness of solutions to the boundary value problem, vanishes in the discrete theory. However, in light of the analysis of [7] and this article, one might consider that not all of the configuration variables of Regge calculus might be in fact physical. Rather, by constructing a perfect action for discretized gravity, which reflects the continuum dynamics and hence the gauge symmetries of

GR, one might get more insight into which of the degrees of freedom are actually physical, and which are gauge. This is, in particular, important in attempts to quantize discrete gravity theories by using Regge triangulations, as happens in spin foams.

We therefore suggest that it might be valuable to study how gauge degrees of freedom are regained in the continuum limit, and think that the improved and perfect actions presented in this article can be helpful in this pursuit.

(ii) In particular the usage of simplices with constant curvature might be useful for first order formulations in Regge calculus, and the questions of constraints in this context [5,7,8]. Furthermore, an area-angle formulation [24] with simplices of constant (nonzero) curvature might be more viable than in the flat case, since e.g. in 4D the 10 dihedral angles of a 4 simplex determine its geometry completely, not just its conformal structure as with flat simplices. These variables not only are appropriate for spin foam models but seem also to be useful to obtain a canonical formulation [25]. See [26] for formulations based on different sets of basic variables and a first order formulation involving κ-curved simplices. Curved simplices have been proposed in [27], but no action has been proposed there. In general quantum gravity models with a positive cosmological constant are better behaved in the infrared and can even serve as regulators for models without a cosmological constant. Therefore it seems useful to investigate the construction of spin foam models with a cosmological constant.

(iii) The Turaev-Viro invariant [28] for 3 manifolds reproduces in the semiclassical limit the geometry of constantly curved simplices for $\Lambda > 0$ [29]. The construction of corresponding spin foam models for $\Lambda < 0$, which is still elusive, could benefit from the formalism presented here by starting a quantization of the perfect action (4.13) for $\Lambda < 0$. In general we note that for the 3D, $\Lambda > 0$ case a quantization having the perfect action as a limit is available (namely, the Turaev-Viro models), whereas a similar quantization based on the nonperfect action is missing. In the canonical formulation one has to worry about complicated factor ordering ambiguities [30] in addition to an anomalous constraint algebra. In contrast a quantization based on κ-curved simplices could avoid these issues.

In general it would be interesting to see whether a similar procedure for reobtaining gauge symmetries (and triangulation independence) as presented here for the classical theory works also for the quantum theory. The Ponzano-Regge with an added cosmological term and the Tuarev-Viro model would be an interesting example [31]. See also [32] where spatial

diffeomorphism symmetry has been reobtained in the continuum limit for a symmetry reduced model.

APPENDIX: CURVED SIMPLICES

In the following, let σ denote a D-dimensional simplex of constant curvature κ. Denote its D-dimensional volume by $V_\sigma^{(\kappa)}$. A hinge h is a $D-2$-dimensional subsimplex (which is again a simplex of constant curvature κ), and we denote its $D-2$-dimensional volume by $F_h^{(\kappa)}$. For a hinge $h \subset \sigma$ denote the interior deficit angle between the two $D-1$-dimensional subsimplices of σ meeting at h by $\theta_h^{(\kappa)}$.

The simplex σ is completely determined by the lengths of its $N := \frac{D(D+1)}{2}$ edges (the 1 simplices). All of the above are regarded as functions of their lengths $L_1, \dots, L_N$.

If we numerate the vertices of σ from 1 to $D+1$, we specify a subsimplex by $(ij\dots k)$ if it is spanned by the vertices with the numbers $i, j \dots, k$, and by $[ij\dots k]$ if it is spanned by all vertices *except* $i, j, \dots, k$. In this notation an edge can be denoted as $e = (ij)$, and its dual hinge by $h = [ij]$.

Denote the geodesic lengths of the edges (ij) by $L_{(ij)}$. Then the $(D+1) \times (D+1)$ matrix G with entries

$$G_{ij} = c_\kappa(L_{(ij)}), \qquad (A1)$$

where the function $c_\kappa(x)$ is defined by

$$c_\kappa(x) := \begin{cases} \cos(\sqrt{\kappa}x) & \kappa > 0 \\ \cosh(\sqrt{-\kappa}x) & \kappa < 0, \end{cases}$$

is called the Gram matrix of the simplex. We denote by G^{ij} the inverse of G_{ik}. Then the interior dihedral angle $\theta_{[ij]}$ opposite of the edge (ij) is given by [33]

$$\cos\theta_{[ij]}^{(\kappa)} = -\frac{G^{ij}}{\sqrt{G^{ii}}\sqrt{G^{jj}}}. \qquad (A2)$$

Hence, for any hinge h the exterior angle $\theta_h^{(\kappa)}$, regarded as a function of the lengths $L_1, \dots, L_N$, exhibits the scaling behavior[14]

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[14]The formulas presented here hold for $\kappa > 0$. For $\kappa < 0$ analogous formulas can be deduced.

$$\theta_h^{(\kappa)}(L_1, \dots, L_N) = \theta_h^{(1)}(\sqrt{\kappa}L_1, \dots, \sqrt{\kappa}L_N). \qquad (A3)$$

As a result we have

$$\frac{\partial}{\partial\kappa}\theta_h^{(\kappa)} = \frac{1}{2\kappa}\sum_{e\subset\sigma}L_e\frac{\partial\theta_h^{(\kappa)}}{\partial L_e}. \qquad (A4)$$

Furthermore, the geometric quantities in curved simplices satisfy the Schlaefli identity

$$\sum_{h\subset\sigma}F_h^{(\kappa)}\frac{\partial\theta_h^{\sigma(\kappa)}}{\partial L_e} = (D-1)\kappa\frac{\partial V_\sigma^{(\kappa)}}{\partial L_e} \quad \text{for all edges } e \subset \sigma. \qquad (A5)$$

In this section we derive a generalization of Euler's theorem

$$\frac{1}{D}\sum_{e\subset\sigma}L_e\frac{\partial V_\sigma}{\partial L_e} = V_\sigma \qquad (A6)$$

to curved tetrahedra.

Lemma A.1. For a simplex σ of dimension D and constant curvature we have

$$V_\sigma^{(\kappa)}(sL_1, \dots, sL_N) = s^D V_\sigma^{(\kappa s^2)}(L_1, \dots, L_N). \qquad (A7)$$

Proof: This can in fact be seen easily for $\kappa > 0$, where the simplex is a subset of a D-dimensional sphere of radius $R = 1/\sqrt{\kappa}$. If the radius is scaled by s, as well as all the edge lengths, the volume of the sphere is scaled by s^D, hence also the volume of the simplex. For $\kappa < 0$ a similar reasoning for hyperbolic spheres applies. The formula (A7) follows.

Corollary A.1. For any D-dimensional simplex σ of constant curvature κ we have

$$\frac{\partial}{\partial s}V_\sigma^{(\kappa)}(L_1, \dots, L_N) = s^{D-1}\sum_{e\subset\sigma}L_e\frac{\partial}{\partial L_e}V_\sigma^{(\kappa s^2)}(L_1, \dots, L_N). \qquad (A8)$$

Proof: By explicit calculation,

$$\frac{\partial}{\partial s}V_\sigma^{(\kappa)}(sL_1, \dots, sL_N) = \sum_{e\subset\sigma}L_e\frac{\partial}{\partial(sL_e)}V_\sigma^{(\kappa)}(sL_1, \dots, sL_N)$$

$$= \frac{1}{s}\sum_{e\subset\sigma}L_e\frac{\partial}{\partial L_e}V_\sigma^{(\kappa)}(sL_1, \dots, sL_N)$$

$$= s^{D-1}\sum_{e\subset\sigma}L_e\frac{\partial}{\partial L_e}V_\sigma^{(\kappa s^2)}(L_1, \dots, L_N).$$

This was the claim.

Another important identity is the following generalization of Euler's formula to curved simplices:

Lemma A.2. If σ is a D-dimensional simplex of constant curvature κ, then

$$\frac{1}{D}\sum_{e\subset\sigma}L_e\frac{\partial V_\sigma^{(\kappa)}}{\partial L_e} = V_\sigma^{(\kappa)} + \frac{2}{D}\kappa\frac{\partial V_\sigma^{(\kappa)}}{\partial\kappa}. \qquad (A9)$$

Proof: We prove this by induction over D, and first note that it is trivially true for $D = 1$. The case $D = 2$ can be shown explicitly by recalling the formula for the area of a spherical (or hyperbolical) triangle t

$$V_t^{(\kappa)} = \frac{\theta_1^{(\kappa)} + \theta_2^{(\kappa)} + \theta_3^{(\kappa)} - \pi}{\kappa}, \qquad \text{(A10)}$$

where the $\theta_i^{(\kappa)}$ are the interior angles of t. Since they are interior dihedral angles of curved simplices, they satisfy the relations (A4). This leads to

$$\frac{\partial V_t^{(\kappa)}}{\partial \kappa} = -\frac{V_t^{(\kappa)}}{\kappa} + \frac{1}{2\kappa} \sum_{i=1}^{3} L_i \frac{\partial V_t^{(\kappa)}}{\partial L_i}. \qquad \text{(A11)}$$

This shows (A9) for $D = 2$. We now show that the formula is true for D if it is true for $D - 2$. We begin with Schlaefli's formula [34]

$$(D - 1)\kappa dV_\sigma^{(\kappa)} = \sum_{h \subset \sigma} F_h^{(\kappa)} d\theta_h^{(\kappa)}. \qquad \text{(A12)}$$

As a consequence, we have [whenever a function appears without arguments, it is supposed to be taken at the point $(L_1, \ldots, L_N)$]:

$$(D - 1)\kappa V_\sigma^{(\kappa)} = \int_0^1 ds \sum_{h \subset \sigma} F_h^{(\kappa)}(sL_1, \ldots, sL_n)$$

$$\times \frac{\partial}{\partial s} \theta_h^{(\kappa)}(sL_1, \ldots, sL_N)$$

$$= \sum_{h \subset \sigma} F_h^{(\kappa)} \theta_h^{(\kappa)}$$

$$- \int_0^1 ds \sum_{h \subset \sigma} \theta^{(\kappa s^2)}(L_1, \ldots, L_N)$$

$$\times \frac{\partial}{\partial s} F_h^{(\kappa)}(sL_1, \ldots sL_N).$$

Remembering that each hinge h is a $D - 2$-dimensional simplex of constant curvature κ, we conclude with (A8) that

$$S^{(\kappa)} := -\sum_{h \subset \sigma} F_h^{(\kappa)} \theta_h^{(\kappa)} + (D - 1)\kappa V_\sigma^{(\kappa)}$$

$$= -\int_0^1 ds s^{D-3} \sum_{h \subset \sigma} \theta_h^{(\kappa s^2)} \sum_{e \subset h} L_e \frac{\partial F_h^{(\kappa s^2)}}{\partial L_e}$$

$$= -\frac{1}{2} \kappa^{-((D-2)/2)} \int_0^\kappa dy y^{(D-4)/2} \sum_{h \subset \sigma} \theta_h^{(y)} \sum_{e \subset h} L_e \frac{\partial F_h^{(y)}}{\partial L_e}, \qquad \text{(A13)}$$

where we have used a change of variable $y = \kappa s^2$.

We now derive the two different ways (A13) of writing $S^{(\kappa)}$ with respect to κ. The first one gives us

$$\frac{\partial S^{(\kappa)}}{\partial \kappa} = \frac{\partial}{\partial \kappa} \left(-\sum_{h \subset \sigma} F_h^{(\kappa)} \theta_h^{(\kappa)} + (D - 1)\kappa V_\sigma^{(\kappa)} \right)$$

$$= -\sum_{h \subset \sigma} \frac{F_h^{(\kappa)}}{\partial \kappa} \theta_h^{(\kappa)} - \sum_{h \subset \sigma} F_h^{(\kappa)} \frac{\partial \theta_h^{(\kappa)}}{\partial \kappa}$$

$$+ (D - 1)V_\sigma^{(\kappa)} + (D - 1)\kappa \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}. \qquad \text{(A14)}$$

Note that with (A4) and (A12) we have

$$\sum_{h \subset \sigma} F_h^{(\kappa)} \frac{\partial \theta_h^{(\kappa)}}{\partial \kappa} = \frac{1}{2\kappa} \sum_{h \subset \sigma} F_h^{(\kappa)} \sum_{e \subset \sigma} L_e \frac{\partial V_\sigma^{(\kappa)}}{\partial L_e}$$

$$= \frac{D - 1}{2} \sum_{e \subset \sigma} L_e \frac{\partial V_\sigma^{(\kappa)}}{\partial L_e}. \qquad \text{(A15)}$$

Now we use the induction hypothesis, which means that (A9) in particular holds for h, i.e.

$$\frac{\partial F_h^{(\kappa)}}{\partial \kappa} = \frac{1}{2\kappa} \sum_{e \subset h} L_e \frac{\partial F_h^{(\kappa)}}{\partial L_e} - \frac{D - 2}{2\kappa} F_h^{(\kappa)}. \qquad \text{(A16)}$$

Inserting (A15) and (A16) into (A14) we arrive at

$$\frac{\partial S^{(\kappa)}}{\partial \kappa} = -\frac{1}{2\kappa} \sum_{h \subset \sigma} \theta_h^{(\kappa)} \sum_{e \subset h} L_e \frac{\partial F_h^{(\kappa)}}{\partial L_e} + \frac{D - 2}{2\kappa} \sum_{h \subset \sigma} \theta_h^{(\kappa)} F_h^{(\kappa)}$$

$$- \frac{D - 1}{2} \sum_{e \subset \sigma} L_e \frac{\partial V_\sigma^{(\kappa)}}{\partial L_e} + (D - 1)V_\sigma^{(\kappa)}$$

$$+ (D - 1)\kappa \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}. \qquad \text{(A17)}$$

On the other hand, by (A13) we have

$$\frac{\partial S^{(\kappa)}}{\partial \kappa} = \frac{\partial}{\partial \kappa} \left(-\frac{1}{2} \kappa^{-((D-2)/2)} \int_0^\kappa dy y^{(D-4)/2} \right.$$

$$\left. \times \sum_{h \subset \sigma} \theta_h^{(y)} \sum_{e \subset h} L_e \frac{\partial F_h^{(y)}}{\partial L_e} \right)$$

$$= -\frac{D - 2}{2\kappa} S^{(\kappa)} - \frac{1}{2\kappa} \sum_{h \subset \sigma} \theta_h^{(\kappa)} \sum_{e \subset h} L_e \frac{\partial F_h^{(\kappa)}}{\partial L_e}$$

$$= \frac{D - 2}{2\kappa} \sum_{h \subset \sigma} \theta_h^{(\kappa)} F_h^{(\kappa)} - \frac{(D - 1)(D - 2)}{2} V_\sigma^\kappa$$

$$- \frac{1}{2\kappa} \sum_{h \subset \sigma} \theta_h^{(\kappa)} \sum_{e \subset h} L_e \frac{\partial F_h^{(\kappa)}}{\partial L_e}. \qquad \text{(A18)}$$

By comparing (A17) and (A18) we arrive at

$$-\frac{D - 1}{2} \sum_{e \subset \sigma} L_e \frac{\partial V_\sigma^{(\kappa)}}{\partial L_e} + (D - 1)V_\sigma^{(\kappa)} + (D - 1)\kappa \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}$$

$$= -\frac{(D - 1)(D - 2)}{2} V_\sigma^{(\kappa)} \qquad \text{(A19)}$$

which is equivalent to (A9).

There is an important corollary: Deriving (A9) with respect to κ and setting $\kappa = 0$, one can see that

$$(D+2)\frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} = \sum_{e \subset \sigma} l_e \frac{\partial}{\partial l_e} \frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0}, \qquad (A20)$$

which, by Euler's theorem, shows that $\frac{\partial V_\sigma^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0}$ is a homogenous function of the edge lengths l_e of degree $D + 2$. An

explicit example for this is e.g. $D = 2$, where one can, with (A10), show that

$$\frac{\partial a_t^{(\kappa)}}{\partial \kappa}\Big|_{\kappa=0} = \frac{1}{24}\sum_{e \subset t} l_e^2 a_t^{(\kappa=0)}, \qquad (A21)$$

which is indeed homogenous of degree 4.

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