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Construction of infrared finite observables in $\mathcal{N} = 4$ super Yang-Mills theory

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In this paper we give all the details of the calculation that we presented in our previous paper [58], where the infrared structure of the maximally helicity violating gluon amplitudes in the planar limit for $\mathcal{N} = 4$ super Yang-Mills theory was considered in the next-to-leading order of perturbation theory. Explicit cancellation of the infrared divergencies in properly defined inclusive cross sections is demonstrated first in a toy model example of "conformal QED" and then in the real $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. We give the full-length details both for the calculation of the real emission and for the diagrams with splitting in initial and final states. The finite parts for some inclusive differential cross sections are presented in an analytical form. In general, contrary to the virtual corrections, they do not reveal any simple structure. An example of the finite part containing just the log functions is presented. The dependence of inclusive cross section on the external scale related to the definition of asymptotic states is discussed.

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I. INTRODUCTION

In recent years remarkable progress in understanding the structure of the planar limit[1] of the $\mathcal{N} = 4$ SYM (supersymmetric Yang-Mills) theory has been achieved. In the planar limit this theory seems to be integrable at the quantum level and its possible solution would be the first example of a solvable nontrivial four-dimensional quantum field theory. The objects which were in the spotlight starting from the AdS/CFT (anti–de Sitter/conformal field theory) correspondence [1] were the local operators, namely, the spectrum of their anomalous dimensions. They were calculated on the one hand from the field theory approach [2] and, on the other hand, as energy levels of a string in the classical background [3,4] revealing a remarkable coincidence. This coincidence being part of the general conjecture suggests the way toward solution of the model at the quantum level.

A. Scattering amplitudes at weak coupling

Other quantities of interest are the so-called MHV[2] scattering amplitudes. It was realized long ago that in the planar limit the pure non-Abelian gauge theories do have a truly simple structure [5]. In papers [6–8], the powerful tool for calculating the loop expansion for these amplitudes was suggested which allows one to calculate the loop contributions to the amplitudes without calculating the usual Feynman diagrams, the number of which grows exponentially with the growth of the order of perturbation theory. Even greater simplification occurs in the case of the $\mathcal{N} = 4$ SYM theory, where the loop expansion takes extremely simple form in comparison with a less supersymmetric case [9].

To see the hidden symmetries of the MHV amplitudes, it is useful to consider the color-ordered amplitude defined through the group structure decomposition

$$\mathcal{A}_n^{(l\text{-loop})} = g^{n-2}\left(\frac{g^2N_c}{16\pi^2}\right)^l \sum_{\text{perm}} \text{Tr}(T^{a_{\rho(1)}} \dots T^{a_{\rho(n)}}) \times A_n^{(l)}(p_{\rho(1)}, \dots, p_{\rho(n)}), \tag{1.1}$$

where $\mathcal{A}_n$ is the physical amplitude, A_n are the partial color-ordered amplitudes, $T^{a(i)}$ are the generators of the gauge group $SU(N_c)$, $a_{\rho(i)}$ is the color index of the $\rho(i)$-th external particle, and $p_{\rho(i)}$ is its momentum.

To be more precise, it was found that these amplitudes revealed the iterative structure which was first established in two loops [10] and then confirmed at the three-loop level by Bern, Dixon, and Smirnov (BDS), who formulated the ansatz [11] for the all-loop n-point MHV amplitudes:

$$\mathcal{M}_n \equiv \frac{A_n}{A_n^{\text{tree}}} = 1 + \sum_{L=1}^{\infty}\left(\frac{g^2N_c}{16\pi^2}\right)^L M_n^{(L)}(\epsilon) = \exp\left[\sum_{l=1}^{\infty}\left(\frac{g^2N_c}{16\pi^2}\right)^l (f^{(l)}(\epsilon)M_n^{(1)}(l\epsilon) + C^{(l)} + E_n^{(l)}(\epsilon))\right], \tag{1.2}$$

where $E_n^{(l)}$ vanishes as $\epsilon \to 0$, $C^{(l)}$ are some finite constants, and $M_n^{(1)}(l\epsilon)$ is the $l\epsilon$-regulated one-loop n-point amplitude.

[1]Defined as $g \to 0$; $N_c \to \infty$; $\lambda = g^2N_c$ *fixed*.
[2]MHV (maximally helicity violating) amplitudes are the amplitudes where all particles are treated as outgoing and the net helicity is equal to $n - 4$, where n is the number of particles. For gluon amplitudes MHV amplitudes are defined as the amplitudes in which all but two gluons have positive helicities.

It is not surprising that the IR-divergent parts of the amplitudes factorize and exponentiate [12]. What is less obvious is that it is also true for the finite part

$$\mathcal{M}_n(\epsilon) = \exp\Bigg[-\frac{1}{8}\sum_{l=1}^{\infty}\left(\frac{g^2N_c}{16\pi^2}\right)^l\left(\frac{\gamma_{\rm cusp}^{(l)}}{(l\epsilon)^2}+\frac{2G_0^{(l)}}{l\epsilon}\right) \times \sum_{i=1}^{n}\left(\frac{\mu^2}{-s_{i,i+1}}\right)^{l\epsilon} + \frac{1}{4}\sum_{l=1}^{\infty}\left(\frac{g^2N_c}{16\pi^2}\right)^l\gamma_{\rm cusp}^{(l)}F_n^{(1)}(0) + C(g)\Bigg], \tag{1.3}$$

where $\gamma_{\rm cusp}(g)=\sum_l(\frac{g^2N_c}{16\pi^2})^l\gamma_{\rm cusp}^{(l)}$ is the so-called cusp anomalous dimension [13] and $G_0(g)=\sum_l(\frac{g^2N_c}{16\pi^2})^lG_0^{(l)}$ is the second function (dependent on the IR regularization) which defines the IR structure of the amplitude.

According to the BDS ansatz, the finite part of the amplitude is defined by the cusp anomalous dimension and a function of kinematic parameters specified at one loop. For a four-gluon amplitude one has

$$F_4^{(1)}(0) = \frac{1}{2}\log^2\left(\frac{-t}{s}\right) + 4\zeta_2. \tag{1.4}$$

The cusp anomalous dimension is a function of the gauge coupling, for which four terms of the weak coupling expansion [2] and three terms of the strong coupling expansion [3,4,14] are known. Integrability from the both sides of the AdS/CFT correspondence leads to the all-order integral equation [15] solution which, being expanded in the coupling, reproduces both series [16].

For $n = 4, 5$ the BDS ansatz goes through all checks, namely, the amplitudes were calculated up to four loops for four gluons [2] for divergent terms (see also [17] for checking at order $1/\epsilon$) up to two loops for five gluons [18] and up to three loops in [19]. However, starting from $n = 6$, it fails. The first indication of the problem was the strong coupling calculation in the limit $n \to \infty$ [20] where the authors compute the value of the amplitude for a particular kinematic configuration for a large number of gluons and find that the result disagrees with the exact value of the amplitude from the BDS formula. The second indication came also from this duality, namely, from the comparison of hexagonal lightlike Wilson loop and finite part of the BDS ansatz for the six-gluon amplitude. It was found that the two expressions differ by a nontrivial function of the three (dual) conformally invariant variables [21]. The third indication appeared in [22] where the analytical structure of the BDS ansatz was analyzed and starting from $n = 6$ the Regge limit factorization of the amplitude in some physical regions failed. Finally, it was shown by explicit two-loop calculation [23] that the BDS ansatz is not true and it needs to be modified by some unknown finite function, which is an open and intriguing problem. However, from the two-loop calculation for the six-point amplitude [23] and hexagonal lightlike Wilson loop [24], it was shown that the gluon amplitude/Wilson loop duality [25] is still valid.

B. Strong coupling dual of amplitudes, lightlike Wilson loops, and dual conformal invariance

In [26], the authors defined the prescription for calculating the amplitudes at strong coupling. It happens that in leading order the amplitude is given by the lightlike Wilson loop living on the boundary of dual AdS space

$$\mathcal{M}_n \sim \exp[-S_{\rm cl}^E] = \exp\left[\frac{\sqrt{\lambda}}{2\pi}({\rm area})_{\rm cl}\right], \tag{1.5}$$

where $S_{\rm cl}^E$ denotes the classical action of classical solution of the string world-sheet equations in Euclidean space-time, which is proportional to the area of the string world sheet.

After this in [25] it was conjectured that duality between lightlike Wilson loops and MHV scattering amplitudes is valid at any coupling, which was proved for n-point MHV amplitudes at one loop [27] and for $n = 6$ at two loops [24] (for more details and references, see the review [28]).

Because of the cusps the lightlike Wilson loop is UV divergent; however, this divergency is under control, namely, one can write the divergent factor in all orders in the coupling governed by two functions, one of them being the cusp anomalous dimension mentioned above. This allows one to define the finite parts for both the Wilson loop with n cusps and the n-point MHV amplitude which, according to the Drummond-Korchemsky-Sokatchev (DKS) conjecture [25], are equal to each other:

$${\rm Fin}[\log\mathcal{M}_n] = {\rm Fin}[\log\mathcal{W}_n]. \tag{1.6}$$

In [29] the notion of dual superconformal symmetry was introduced, which is conformal invariance acting in momentum space. What is important is that this symmetry has a non-Lagrangian nature. After this in [30,31] the fermionic T-duality was suggested which maps the dual superconformal symmetry of the original theory to the ordinary superconformal symmetry of the dual model.

For a Wilson loop the conformal invariance is broken due to the cusps, but one can write the anomalous Ward identities which allows one to find the finite parts of the Wilson loop with $n = 4$ and $n = 5$ cusps exactly [32]:

$$\sum_{i=1}^{n}(2x_i^\nu x_i\partial_i - x_i^2\partial_i^\nu){\rm Fin}[\log\mathcal{W}_n] = \frac{1}{2}\gamma_{\rm cusp}\sum_{i=1}^{n}\log\frac{x_{i,i+2}^2}{x_{i-1,i+1}^2}x_{i,i+1}^\nu, \tag{1.7}$$

where the connection between the momentum space and its dual $x_{i,i+1}^\mu = x_i^\mu - x_{i+1}^\mu = p_i^\mu$ is used. This equation uniquely fixes the finite parts of the Wilson loop with $n = 4$ and $n = 5$ cusps; however, starting from $n = 6$ more input is needed since the finite part of the Wilson loop in this case can be a function of the three conformal invariant variables. Hopefully, one can find hidden symmetries which fix the finite part for any n [33].

It is not clear how to derive this duality from the field theory point of view, and also how to extend it to the NMHV case.[3] At one loop, one can show that finite part of the so-called two mass easy box which governs the finite function of MHV amplitudes could be directly mapped to Wilson loop diagrams through a simple change of variables in the space of Feynman parameters and also through the connection between scalar integrals in different dimensions [34].

C. Infrared-safe observables

While all the UV divergences in $\mathcal{N} = 4$ SYM are absent in scattering amplitudes, the IR ones remain and are supposed to be canceled in properly defined quantities. By themselves the divergent amplitudes have no sense. Regularized expressions act like some kind of scaffolding which has to be removed to obtain eventual physical observables. It is these quantities that are the aim of our calculation. And though the Kinoshita-Lee-Nauenberg [35] theorem in principle tells us how to construct such quantities, explicit realization of this procedure is not simple and one can think of various possibilities. The well-known example is a successful application to observables in QED [36]. The other suggestion is to consider the so-called energy flow functions defined in terms of the energy-momentum tensor correlators introduced earlier (see for example [37]) and considered in the weak coupling regime in [38,39] and recently in the strong coupling regime in [40]. From our side we concentrated on inclusive cross sections in hope that they reveal some factorization properties discovered in the regularized amplitudes. Similar questions were discussed in [41], where the inclusive cross sections like the IR-safe observables based on on-shell form factors in $\mathcal{N} = 4$ SYM were constructed.

To perform the procedure of cancellation of the IR divergences, one should have in mind that in conformal theory all the masses are zero and one has additional collinear divergences which need special care. In this work we employ the method developed in the QCD parton model [42–46]. It includes two main ingredients in the cancellation of infrared divergencies coming from the loops: emission of additional soft real quanta and redefinition of the asymptotic states resulting in the splitting terms governed by the kernels of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi (DGLAP) equations [47,48]. The latter ones take care of the collinear divergences.

Typical observables in QCD parton model calculations are inclusive jet cross sections, where the total energy of scattered partons is not fixed since they are considered to be parts of the scattered hadrons. In [43], the algorithm for extracting divergences was developed which allows one to cancel divergences and apply numerical methods for calculation of the finite part. In our paper we choose as our observables the inclusive cross sections with fixed initial energy and get an analytical expression for the finite part of the differential cross section. We do not assume any confinement and consider the scattering of the single parton based "coherent" states,[4] being the asymptotic states of conformal field theory.

There are some attempts to deal with the divergences for the amplitudes themselves. For example in [49] a deformation of the free superconformal representation by contributions which change the number of external legs was proposed which looks similar to the procedure that we apply below considering the inclusive cross sections. Acting along the same lines the authors of [50] have observed that the holomorphic anomaly [51] gives an extra modification of superconformal algebra for the tree-level scattering amplitudes. They argued that superconformal symmetry survives regularization and introduced a new holomorphic anomaly friendly regularization to deal with the IR divergences.

The paper is organized as follows. In Sec. II, we consider general issues concerning the construction of the infrared-finite observables in the massless QFT. We discuss the IR and collinear divergencies for the scattering amplitudes and the ways of their cancellation based on the Kinoshita-Lee-Nauenberg theorem. We introduce the notion of the measurement functions and discuss their properties. Then the concept of the splitting functions and splitting counterterms is outlined. We define the IR-finite inclusive cross sections which are the subject of calculations in the subsequent sections.

Section III is devoted to the demonstration of the techniques discussed above in practice. In a toy model of "conformal QED" we consider the α_s correction to the massless electron-quark scattering. We show how the IR and collinear divergences cancel and calculate analytically the remaining finite part of the differential cross section. Because of absence of the identical particles in the final state this example turns out to be much simpler than gluon scattering in $\mathcal{N} = 4$ SYM and serves as a good warm-up exercise before going to $\mathcal{N} = 4$ SYM.

In Sec. IV, we calculate the leading order perturbation theory (PT) correction to the gluon-gluon scattering inclusive cross section. It includes the one-loop contribution to the $2 \to 2$ scattering differential cross section, the tree-level $2 \to 3$ scattering with the integration over the phase space of the fifth gluon, and an account of the splitting of

[3] NMHV (next-to-maximally helicity violating) amplitudes are the amplitudes where all particles are treated as outgoing and the net helicity is equal to $n - 6$, where n is the number of particles.

[4] The squared perturbative amplitudes used in our calculation are summed over colors, so in this sense they are colorless and there is no contradiction with statements that cancellation of IR divergences occurs only for colorless objects.

the initial and final states. We consider also the amplitudes with creation of pairs of the matter fields from the $\mathcal{N} = 4$ supermultiplet.

In Sec. V, using the results of Sec. IV we present the infrared-finite results for the differential inclusive cross section in $\mathcal{N} = 4$ SYM theory for different physical setups.

Section VI contains discussion and concluding remarks. In appendixes we present the technical details of our calculations.

II. CONSTRUCTION OF INFRARED-SAFE OBSERVABLES

The tree-level matrix elements are finite and well defined in perturbation theory. Divergences appear when integrating over virtual loops or over phase space of real particles. So the first step is to choose a proper quantity which is finite in the lowest order of PT prior to calculation of radiative corrections. For example, the total elastic 2×2 cross section is divergent, but the differential cross section is well defined. The choice of a proper quantity is performed by imposing conditions on the phase space. This can be achieved by introducing the concept of the *measurement function* $\mathcal{S}_n$, where n is the number of particles in the final state. It defines which physical quantity we are measuring. Typical examples are: a total cross section, a differential cross section, an n-jet cross section, etc. In the case of $2 \times n$ scattering the differential cross section is given by

$$\frac{d\sigma_{2\to n}}{d\Omega} = \frac{1}{J}\int |M_{2+n}|^2 d\phi_n \mathcal{S}_n,$$

where $\mathcal{S}_n$ is the measurement function and the n-particle phase space $d\phi_n$ is given by

$$d\phi_n = \prod_{k=3}^{n+2} \delta^+(p_k^2)\frac{d^D p_k}{(2\pi)^{D-1}}(2\pi)^D \times \delta^D(p_1 + p_2 - p_3 - \cdots - p_{n+2}). \tag{2.1}$$

Here J is the flux factor, p_1, p_2 are the momenta of the incoming particles, $p_3, \ldots, p_{n+2}$ are the momenta of the outgoing ones, M_{2+n} is the matrix element of the corresponding process, and we use the dimensional regularization with $D = 4 - 2\epsilon$.

Then, for example, choosing the measurement function to be

$$\mathcal{S}_2 = \delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}),$$

one singles out the standard differential cross section for the scattering of a third particle on a certain solid angle Ω_{13} for the $2 \to 2$ process

$$\frac{d\sigma_{2\to 2}}{d\Omega_{13}} = \int |M_4|^2 d\phi_2 \mathcal{S}_2. \tag{2.2}$$

If one wants to construct the IR finite quantity then, according to the Kinoshita-Lee-Nauenberg [35] theorem, it is not sufficient to consider the process with the fixed number of final particles. One has to include the processes of the same order of the perturbation theory with emission of extra soft quanta and integrate over their momenta. This leads to the notion of inclusive cross section when one fixes some particles and integrates over all the others allowed by the conservation laws.

When the number of particles increases, one has to specify the measurable quantity in a more accurate way and to distinguish the particle(s) in the final state. Thus, one can introduce the energy and angular resolution for the detector and cut the phase space so that the soft quanta with total energy below the threshold as well as all the particles within the given solid angle are included. This procedure requires the corresponding measurement functions and works well in QED but introduces explicit dependence on the energy and angular cutoff, thus violating conformal invariance.

We adopt here a different attitude without introducing any cutoffs but rather considering the inclusive cross section with the emission of particles with all possible momenta allowed by kinematics. Having identical particles in the final state one has to specify which particles are detected by introducing some measurement function. For instance, one can detect the given particle scattering on a given angle while integrating over the phase space of the other particles. As will be clear later in this case, due to collinear divergences one still cannot avoid introducing some scale related to the definition of the asymptotic states of a theory. Below we show how it works in particular examples.

To have the cancellation of all the IR divergences, according to the analysis of Ellis, Kunszt, and Soper [43], the measurement functions for the processes with a different number of external particles have to obey the following conditions:

$$\mathcal{S}_{n+1}(\ldots, \lambda\vec{p}, \ldots) = \mathcal{S}_n(\ldots\ldots), \qquad \lambda \to 0, \tag{2.3}$$

which reflects the insensitivity to the soft quanta, and

$$\mathcal{S}_{n+1}(\ldots, \lambda\vec{p}, \ldots, (1-\lambda)\vec{p}, \ldots) = \mathcal{S}_n(\ldots, \vec{p}, \ldots). \tag{2.4}$$

Here $0 \leq \lambda \leq 1$. This condition expresses our insensibility to collinear quanta.

It should be pointed out that in case of identical particles one has an additional problem when calculating the differential cross sections: one has to specify the scattering angles and to choose the detectable particle. This requirement imposes further conditions on the phase space as will be shown below when considering the gluon scattering.

The additional divergences appearing in the massless case which come from the integration over angles rather than the modulus of momentum, as in the case of the IR divergences, are related to the collinearity of momenta of two particles. For this reason they are called *the collinear divergences*. To get the cancellation of all divergences, the observed cross section should include, besides the main process and emission of the soft quanta, the process of emission of collinear particles with kinematically allowed absolute values of momenta. As we will see below, the leading IR divergences coming from the cross section of the processes with the virtual loop correction and from the real emission of the soft quanta cancel. However, the total cancellation of divergencies does not happen. The remaining divergences in the form of a single pole have the collinear nature. For the cancellation of the remaining pole one has to properly define the initial (and final) states. The reason is that a massless particle can emit a collinear one which carries part of the initial momentum and, in this case, it is impossible to distinguish one particle propagating with the speed of light from the two flying parallel. This is the common problem for any theory containing the interacting massless particles.

To deal with this problem, let us consider a particle in the initial state and introduce the notion of distribution of a particle with respect to the fraction of the carried momentum z: $q(z)$. Then the zero-order distribution corresponds to $q(z) = \delta(1-z)$ and the emission of a collinear particle leads to the splitting: the particle i carries the fraction of momentum equal to z, while the collinear particle $j - (1-z)$. The probability of this event is given by the so-called *splitting functions* $P_{ij}(z)$ [48]. In case of a particle in a final state, this corresponds to the fragmentation into a pair of particles i and j. In the lowest order of perturbation theory the distribution can be written in the form

$$q_i\left(z, \frac{Q_f^2}{\mu^2}\right) = \delta(1-z) + \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\sum_j P_{ij}(z), \qquad (2.5)$$

where the scale Q_f^2, sometimes called factorization scale, defines the measure of collinearity of the emitted particles, i.e., it refers to the definition of the initial state. In fact, in the massless case one cannot define the initial state that contains just one particle; it exists together with the set of collinear particles forming a coherent state.

This leads to the additional terms in the cross section

$$\frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz \sum_j P_{ij}(z) d\sigma_j^{\mathrm{Born}}(zp_1, p_2, p_3, p_4) + (p_1 \leftrightarrow p_2), \qquad (2.6)$$

referred to hereafter as the initial splitting contributions or collinear counterterms.

The same is true for the final states. The corresponding final state collinear counterterms are

$$\frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dx \sum_j P_{ij}(x) d\sigma_j^{\mathrm{Born}}(p_1, p_2, p_3, p_4) + (p_3 \leftrightarrow p_4). \qquad (2.7)$$

Summarizing all the contributions, we come to the following set of IR-safe observables that we consider here:

$$\begin{aligned} d\sigma_{\mathrm{obs}}^{\mathrm{incl}} &= \sum_{n=2}^{\infty}\int_0^1 dz_1 q_1\left(z_1, \frac{Q_f^2}{\mu^2}\right)\int_0^1 dz_2 q_2\left(z_2, \frac{Q_f^2}{\mu^2}\right) \\ &\quad \times \prod_{i=3}^{n+2}\int_0^1 dz_i q_i\left(z_i, \frac{Q_f^2}{\mu^2}\right) \\ &\quad \times d\sigma^{2\to n}(z_1 p_1, z_2 p_2, \ldots) S_n(\{z\}) \\ &= g^4 N_c^4 \sum_{L=0}^{\infty}\left(\frac{g^2 N_c}{16\pi^2}\right)^L d\sigma_L^{\mathrm{Finite}}(s, t, u, Q_f^2), \end{aligned} \qquad (2.8)$$

where p_1, p_2 are the momenta of the initial particles, p_i are the momenta of the final particles, $\mathcal{S}_n$ are the measurement functions which define the measurable quantity, and q_i are the initial and final state distributions.

The above expression looks like the parton model cross section. The difference is that in the parton model one uses both the parton distributions inside hadrons and fragmentation functions for final-state hadrons while here it belongs to the definition of the asymptotic states.

III. TOY MODEL: "CONFORMAL QED"

To illustrate the main ideas of the previous chapter, we study first a toy model example. Let us consider the electron-quark scattering and put all the masses equal to zero. We will be interested in the radiative corrections in the first order with respect to the strong coupling α_s. The corresponding diagrams are shown in Fig. 1. We define the initial electron and quark momenta as $p_e = p_1$ and $p_q = p_2$ and the final ones as $p_{e'} = p_3$ and $p_{q'} = p_4$, respectively.

In the chosen process the UV divergences cancel due to the Ward identities so we are left only with the IR ones. To handle them, we use dimensional regularization. This situation exactly imitates the four-dimensional CFT's like the $\mathcal{N} = 4$ SYM theory.

Define the measurement function in the following way:

$$\mathcal{S}_2 = \delta_{\pm,h_3}\delta^{D-2}(\Omega_{\mathrm{Det}} - \Omega_{13}),$$

where $\delta_{\pm,h_3}$ means that we detect the third particle with any helicity, i.e. we are interested in the unpolarized differential cross section. Here $d\Omega_{13} = d\phi_{13} d\cos(\theta_{13})$,[5] θ_{13} is the scattering angle of the particle with momentum $\mathbf{p}_3$ with respect to the particle with momentum $\mathbf{p}_1$ in the center-of-mass frame. In the leading order (LO) we have

[5]If to be more accurate in dimensional regularization we have $d\Omega_{13}^{D-2} = d\phi_{13}\sin(\phi_{13})^{-2\epsilon} d\cos(\theta_{13})\sin(\theta_{13})^{-2\epsilon}$, $D = 4 - 2\epsilon$.

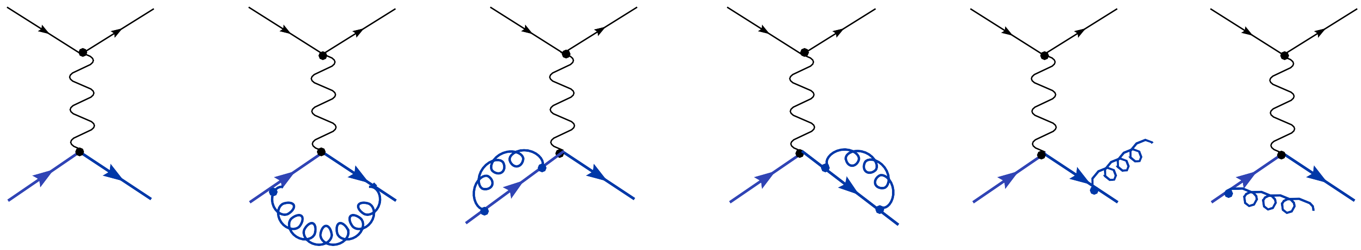

FIG. 1 (color online). The process of electron-quark scattering in the first order in α_s: (a) the Born diagram, (b)–(d) the corrections due to the virtual gluons, (e)–(f) the corrections due to the real gluons.

the well-known textbook formula [52]

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\text{Born}} = \frac{\alpha^2}{2E^2}\left(\frac{s^2+u^2}{t^2}-\epsilon\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{3.1}$$

where E is the total energy of initial particles in the center-of-mass frame, and s, t, u are the standard Mandelstam variables. In the c.m. frame $s = E^2$, $t = -E^2/2(1-c)$, $u = -E^2/2(1+c)$, $c = \cos\theta_{13}$.

The one-loop correction coming from the diagrams with virtual gluon, Fig. 1(b)–1(d), has the form

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\text{1-loop}} = \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\text{Born}} \times\left[-2C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\right)\right]. \tag{3.2}$$

In order to avoid the transcendental numbers, we used the helpful definition of the angular measure in the space of $4-2\epsilon$ dimensions and multiplied the standard expression by $\Gamma(1-\epsilon)/(4\pi)^{\epsilon}$. Then the constants like γ_E, $\log(4\pi)$, and $\zeta(2)$ disappear from the intermediate expressions. Because of the cancellation of divergences in the final expressions, this redefinition does not influence the answer.

Now, following the general prescription, we have to calculate the diagrams with emission of real gluons, Fig. 1(e) and 1(f). For the measurement function of these processes we take $\mathcal{S}_3 = \mathcal{S}_2$, then all requirements on $\mathcal{S}_2$ and $\mathcal{S}_3$ are satisfied trivially. Besides the squares of each of the diagrams one should also take into account the interference term. After contracting all the indices the phase integral takes the form [we denote the quark momentum as $p_4 - k$ and the gluon momentum as k and keep the standard notation for the Mandelstam variables $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_2 - p_3)^2$]

$$\begin{aligned}
\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)_{\text{Born}} &= \frac{1}{2\pi E^2}\int d^D p_3\,\delta^+(p_3^2)\int\frac{d^D k}{(2\pi)^D}\,\delta^+(k^2)\delta^+((p_4-k)^2)\mathcal{S}_3|M|^2_{p_4=p_1+p_2-p_3},\\
|M|^2 &= \frac{e^4g^2}{4}8\frac{M_0+\epsilon M_1+\epsilon^2 M_2}{t(s+t+u)},\\
M_0 &= 4s-8p_1k-4p_2k+\frac{-8(p_1k)^2+4(2s+t)p_1k-(3s^2+t^2+u^2+2st)}{p_2k},\\
M_1 &= -4(s+u)+8p_1k+8p_2k+\frac{8(p_1k)^2-4(s+t+u)p_1k+2(s+t+u)^2-2(u+s)t}{p_2k},\\
M_2 &= 4(s+t+u)-4p_2k-\frac{(s+t+u)^2}{p_2k} = -\frac{(s+t+u-2p_2k)^2}{p_2k}.
\end{aligned} \tag{3.3}$$

It is useful to pass to the spherical coordinates and use the c.m. frame. After the integration over the phase volume the result can be represented as

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)_{\text{Born}} = \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\text{Born}}\left[2C_F\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(\frac{2}{\epsilon^2}+\frac{3}{\epsilon}+8\right)\right] + C_F\frac{\alpha^2}{E^2}\frac{\alpha_s}{4\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(\frac{f_1}{\epsilon}+f_2\right)+O(\epsilon), \tag{3.4}$$

where the functions f_1 and f_2 in the c.m. frame are

$$f_1 = -2\frac{(1-c)(c^3+5c^2-3c+5)\log(\frac{1-c}{2})-(c-1)^2(c+1)(c-11)/4}{(1-c)^2(1+c)^2}, \tag{3.5}$$

$$f_2 = -\frac{1}{(1-c)^2(1+c)^2}\left[(1-c)(c^3+5c^2-3c+5)\log^2\left(\frac{1-c}{2}\right)+\frac{1}{2}(1-c)(3c^3+15c^2+77c-31)\log\left(\frac{1-c}{2}\right)\right.$$
$$\left.+(1+c)^2(c^2+2c+5)\pi^2-12(9c^2+2c+5)\mathrm{Li}_2\left(\frac{1+c}{2}\right)+\frac{1}{2}(1-c)(1+c)(5c^2-42c-23)\right]. \quad (3.6)$$

As one can see from comparison of the cross sections of the processes with virtual (3.2) and real gluons (3.4), in the sum the virtual part *completely* cancels and the second-order pole disappears. However, the total cancellation of divergences does not happen. The remaining divergences in the form of a single pole have a collinear nature. As was already mentioned, for their cancellation one has to define properly the initial states.

Introducing the distribution function for initial quark state one gets the additional contribution [53] to the cross section which looks like[6]

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\mathrm{Split}} = \frac{1}{\epsilon}\frac{\alpha_s}{2\pi}\int_0^1 dz P_{qq}(z)\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon} \times\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}(p_1, zp_2)\right)_{\mathrm{Born}}, \quad (3.7)$$

where the Born cross section is given by (3.1) with the replacement of the initial quark momentum p_2 by $p_2 z$. This means that one should keep the total energy but replace the Mandelstam variables s, t, u according to Eq. (4.38) (see below). One should also keep the ϵ term which gives contributions to the finite part. The splitting function $P_{qq}(z)$ [48] here is

$$P_{qq}(z) = C_F\left(\frac{1+z^2}{(1-z)_+}+\frac{3}{2}\delta(1-z)\right). \quad (3.8)$$

Notice also the change of the momenta conservation condition which now looks like $p_1 + zp_2 - p_3 - p_4 = 0$. This gives an additional factor of $4z^{-2\epsilon}/(1+z-c(1-z))^{2(1-\epsilon)}$.

One might also have a contribution from the final-state counterterm; however, since in this case, according to (2.7), the cross section does not depend on the fraction z, one has to integrate only the splitting function $P_{qq}(z)$. And this integral equals zero due to the requirement of conservation of the number of quarks. Therefore, one has no contribution from the final-state splitting. It will not be the case for the gluon scattering cross section considered below.

The factorization scale Q_f^2 is an arbitrary quantity associated with the quark distribution function which may depend on z. It is quite natural to choose the factorization scale equal to the characteristic scale of the process of interest. Thus, in our case this choice corresponds to $Q_f^2 = -\hat{t}$, where $\hat{t}$ is the Mandelstam parameter t for the process where p_2 is replaced by $p_2 z$. One has $\hat{t} = t\frac{2z}{(z+1)-c(1-z)}$ [see Eq. (4.38) below]. Substituting this value of Q_f^2 into (3.7) leads to the following result:

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\mathrm{Split}} = C_F\frac{\alpha^2}{2E^2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{-t}\right)^{\epsilon}\left(-\frac{f_1}{\epsilon}+f_3\right), \quad (3.9)$$

where

$$f_3 = -\frac{1}{(1-c^2)^2}\left[2(1-c)(c^3+c^2-33c+7)\times\log\left(\frac{1-c}{2}\right)+12(9c^2+2c+5)\mathrm{Li}_2\left(\frac{1+c}{2}\right)-(1+c)^2(c^2+2c+5)\pi^2-\frac{1}{2}(1-c)(1+c)(11c^2-19)\right]. \quad (3.10)$$

Gathering all pieces together we finally obtain the IR-finite answer in the next-to-leading order (NLO) of PT:

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)_{\text{IR-safe}} = \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\mathrm{Born}}+\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\text{1-loop}}+\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)_{\mathrm{Born}}+\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)_{\mathrm{Split}}$$
$$= \frac{\alpha^2}{2E^2}\left\{\frac{c^2+2c+5}{(1-c)^2}-\frac{\alpha_s}{2\pi}\frac{C_F}{(1-c)(1+c)^2}\left[(c^3+5c^2-3c+5)\log^2\left(\frac{1-c}{2}\right)\right.\right.$$
$$\left.\left.+\frac{1}{2}(7c^3+19c^2-55c-3)\log\left(\frac{1-c}{2}\right)-(1+c)(3c^2+21c+2)\right]\right\}. \quad (3.11)$$

This expression is the final answer for the cross section of the physical process of the electron-quark scattering where the initial and the final state include the soft and collinear gluons. It includes also the definition of the initial state and can be recalculated for the alternative choice of the factorization scale similar to what happens to the ultraviolet scale which defines the coupling constant. Thus, we practically deal with the scattering not of individual particles but rather with coherent states with a fixed total momentum. This process contrary to the scattering of

[6]We put here $(\frac{\mu^2}{Q_f^2})^{\epsilon}$ inside the integration over z since in general one may consider Q_f to be a function of z.

individual massless quanta has a physical meaning. The drawback is the dependence on Q_f which reflects the definition of the asymptotic state. This dependence explicitly violates the conformal invariance.

IV. CALCULATION OF THE INCLUSIVE CROSS SECTIONS IN $\mathcal{N} = 4$ SYM THEORY

Consider now the gluon scattering in the $\mathcal{N} = 4$ SYM theory. Our aim is to evaluate the NLO correction to the inclusive differential polarized cross section in the weak coupling limit in the planar limit in analytical form and to trace the cancellation of the IR divergences.

We start with the tree level $2 \to 2$ MHV scattering amplitude with two incoming positively polarized gluons and two outgoing positively polarized gluons and consider the differential cross section $d\sigma_{2\to 2}(g^+g^+ \to g^+g^+)/d\Omega$ as a function of the scattering solid angle. Treating all the particles as outgoing this amplitude is denoted as $(- - ++)$ MHV amplitude. At tree level the differential cross section is given by

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{(\mathrm{tree})} = \frac{1}{J}\int d\phi_2 |\mathcal{M}_4^{(\mathrm{tree})}|^2 \mathcal{S}_2, \tag{4.1}$$

where J is a flux factor, in our case $J = s$, and the phase volume of the two-particle process (we use the **FDH** version of the dimensional reduction; see [54] for details) is

$$d\phi_2 = \frac{d^D p_3 \delta^+(p_3^2)}{(2\pi)^{D-1}} \frac{d^D p_4 \delta^+(p_4^2)}{(2\pi)^{D-1}} (2\pi)^D \times \delta^D(p_1 + p_2 - p_3 - p_4), \tag{4.2}$$

and $\mathcal{S}_n$ ($n = 2$) in this particular case is

$$\mathcal{S}_2 = \delta_{+,h_3} \delta^{D-2}(\Omega_{\mathrm{Det}} - \Omega_{13}), \tag{4.3}$$

where $\delta^{D-2}(\Omega_{\mathrm{Det}} - \Omega_{13})$ means that our observable is the differential cross section and δ_{+,h_3} indicates that we detect a particle with positive helicity.

The squared matrix element is obtained from the color-ordered amplitudes via summation

$$|\mathcal{M}_4^{(\mathrm{tree})}|^2 = g^4 N_c^2 (N_c^2 - 1) \times \sum_{\sigma \in P_3} |A_4^{(\mathrm{tree})}(p_1, p_{\sigma(2)}, p_{\sigma(3)}, p_{\sigma(4)})|^2, \tag{4.4}$$

where P_n is the set of the permutations of n objects ($n = 3$ in this case), so that in our case [5,55] (see also Appendix A for details)

$$|\mathcal{M}_4^{(\mathrm{tree})(--++)}|^2 = g^4 N_c^2 (N_c^2 - 1) \times \sum_{\sigma \in P_3} \frac{s_{12}^4}{s_{1\sigma(2)} s_{\sigma(2)\sigma(3)} s_{\sigma(3)\sigma(4)} s_{\sigma(4)1}}, \tag{4.5}$$

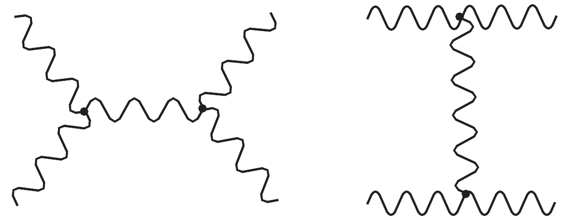

FIG. 2. Tree-level diagrams for the color-ordered MHV amplitudes.

where we use the notation $s_{ij} = (p_i + p_j)^2$. The corresponding Feynman diagrams are shown in Fig. 2.

Within the dimensional regularization (reduction) the cross section in the planar limit looks like

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{(\mathrm{tree})} = \frac{\alpha^2 N_c^2}{2E^2}\left(\frac{s^2}{t^2} + \frac{s^2}{u^2} + \frac{s^4}{t^2 u^2}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{4.6}$$

where s, t, u are the Mandelstam variables, E is the total energy in the center-of-mass frame, and $\alpha = g^2 N_c / 4\pi$. So in the center-of-mass frame the cross section can be rewritten as

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{(\mathrm{tree})} = \frac{\alpha^2 N_c^2}{E^2} \frac{4(3 + c^2)}{(1 - c^2)^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{4.7}$$

where $c = \cos\theta_{13}$. The next step is to calculate the NLO corrections.

A. Virtual part

To get the one-loop contribution to the differential cross section, one has to consider the diagrams shown in Fig. 3. We use the already known one-loop contribution to the color-ordered amplitude [9]

$$M_4^{(1\text{-loop})}(\epsilon) = A_4^{(1\text{-loop})}/A_4^{(\mathrm{tree})} = -\frac{1}{2} s t I_4^{(1\text{-loop})}(s, t),$$

where $I_4^{(1\text{-loop})}(s, t)$ is the scalar box diagram

$$I_4^{(1\text{-loop})}(s, t) = \frac{2}{st} \frac{\Gamma(1-\epsilon)^2}{\Gamma(1-2\epsilon)}\left[-\frac{1}{\epsilon^2}\left(\left(\frac{\mu^2}{s}\right)^{\epsilon} + \left(\frac{\mu^2}{-t}\right)^{\epsilon}\right) + \frac{1}{2}\log^2\left(\frac{s}{-t}\right) + \frac{\pi^2}{3}\right] + \mathcal{O}(\epsilon).$$

The square of the matrix element summed over colors

$$|\mathcal{M}_4^{(1\text{-loop})}|^2 = \sum_{\mathrm{colors}} (\mathcal{A}_4^{(\mathrm{tree})} \mathcal{A}_4^{(1\text{-loop})*} + \mathrm{c.c.})$$

has the form

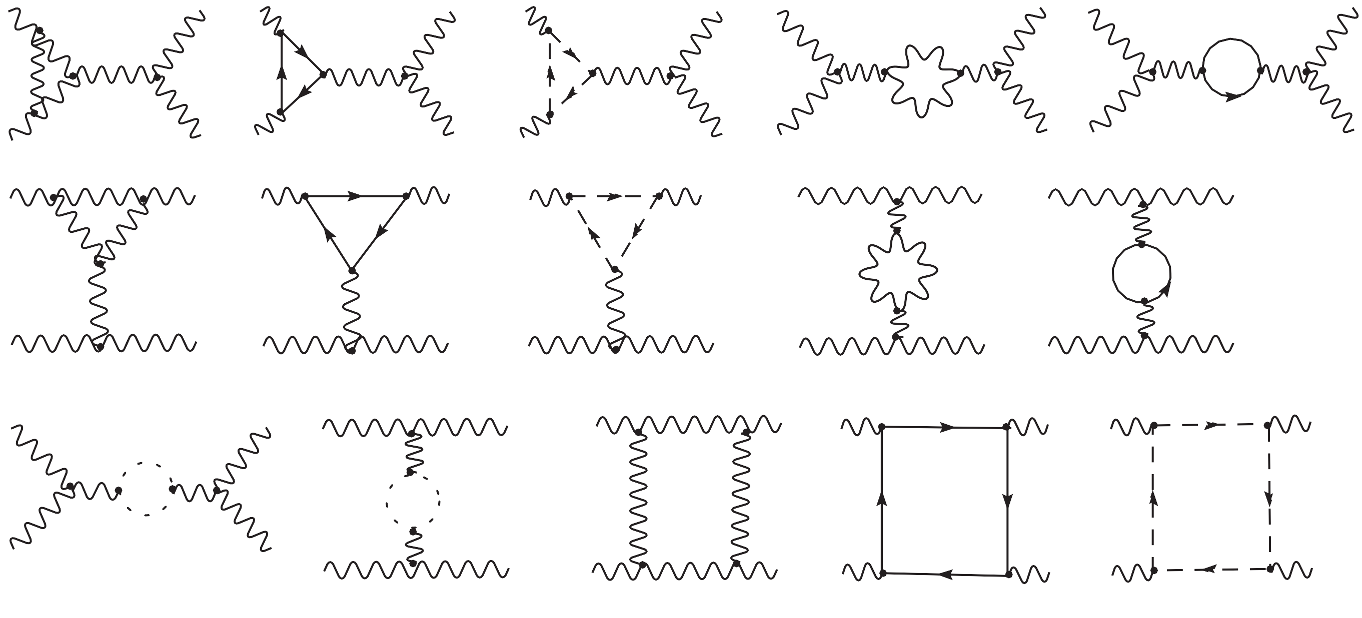

FIG. 3. The one-loop diagrams for the color-ordered MHV amplitude in the $\mathcal{N} = 4$ SYM theory. Particles running inside the loop include all the members of the $\mathcal{N} = 4$ supermultiplet. The solid and dashed lines correspond to the fermion and scalar particles, respectively.

$$|\mathcal{M}_4^{(1\text{-loop})(--++)}|^2 = -g^4 N_c^2(N_c^2-1)\left(\frac{g^2N_c}{16\pi^2}\right)\times\left[\frac{s^4}{s^2t^2}stI_4^{(1\text{-loop})}(s,t)+\frac{s^4}{s^2u^2}suI_4^{(1\text{-loop})}(s,u)-\frac{s^4}{t^2u^2}tuI_4^{(1\text{-loop})}(-t,u)\right], \tag{4.8}$$

which gives the one-loop contribution to the cross section in the planar limit

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(--++)}_{\text{virt}} = \frac{\alpha^2N_c^2}{2E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left\{\frac{\alpha}{4\pi}\frac{s^4}{s^2t^2u^2}\left[-\frac{8}{\epsilon^2}\left(\left(\left(\frac{\mu^2}{-t}\right)^{\epsilon}+\left(\frac{\mu^2}{-u}\right)^{\epsilon}\right)s^2+\left(\left(\frac{\mu^2}{s}\right)^{\epsilon}+\left(\frac{\mu^2}{-t}\right)^{\epsilon}\right)u^2+\left(\left(\frac{\mu^2}{s}\right)^{\epsilon}+\left(\frac{\mu^2}{-u}\right)^{\epsilon}\right)t^2\right)\right.\right.$$
$$\left.\left.+\frac{16}{3}\pi^2(s^2+t^2+u^2)+4\left(u^2\log^2\left(\frac{s}{-t}\right)+t^2\log^2\left(\frac{s}{-u}\right)+s^2\log^2\left(\frac{t}{u}\right)\right)\right]\right\}. \tag{4.9}$$

Rewriting this expression in the center-of-mass frame we have

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(--++)}_{\text{virt}} = \frac{\alpha^2N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}4\left\{\frac{\alpha}{4\pi}\left[-\frac{16}{\epsilon^2}\frac{3+c^2}{(1-c^2)^2}+\frac{4}{\epsilon}\left(\frac{5+2c+c^2}{(1-c^2)^2}\log\left(\frac{1-c}{2}\right)+\frac{5-2c+c^2}{(1-c^2)^2}\log\left(\frac{1+c}{2}\right)\right)\right.\right.$$
$$\left.\left.+\frac{16(3+c^2)\pi^2}{3(1-c^2)^2}-\frac{16}{(1-c^2)^2}\log\left(\frac{1-c}{2}\right)\log\left(\frac{1+c}{2}\right)\right]\right\}. \tag{4.10}$$

It should be stressed that because of the conformal invariance of the $\mathcal{N} = 4$ SYM theory at the quantum level, there are no UV divergences in (4.10) and all divergences have the IR soft or collinear nature. They have to be canceled in properly defined observables. Note also the simplicity of the finite part which is a consequence of symmetries of $\mathcal{N} = 4$ SYM and the fact that all the terms in (4.10) have the same transcendentality [15,56].

B. Real emission

The next step, as in the toy model considered above, is the calculation of the amplitude with three outgoing particles. Here we have to define the process we are interested in. There are several possibilities.

(1) Three gluons with positive helicities: $g^+g^+ \to g^+g^+g^+$. This is the MHV amplitude.
(2) Two gluons with positive helicities and the third one with negative helicity: $g^+g^+ \to g^+g^+g^-$.[7] This is the anti-MHV amplitude.

[7]There is also a $g^+g^+ \to g^+g^-g^+$ helicity configuration. The partial amplitudes for both cases where the additional gluon with negative helicity is the second or the third gluon in the final state are equal. We will use the $(--++-)$ notation for both of them.

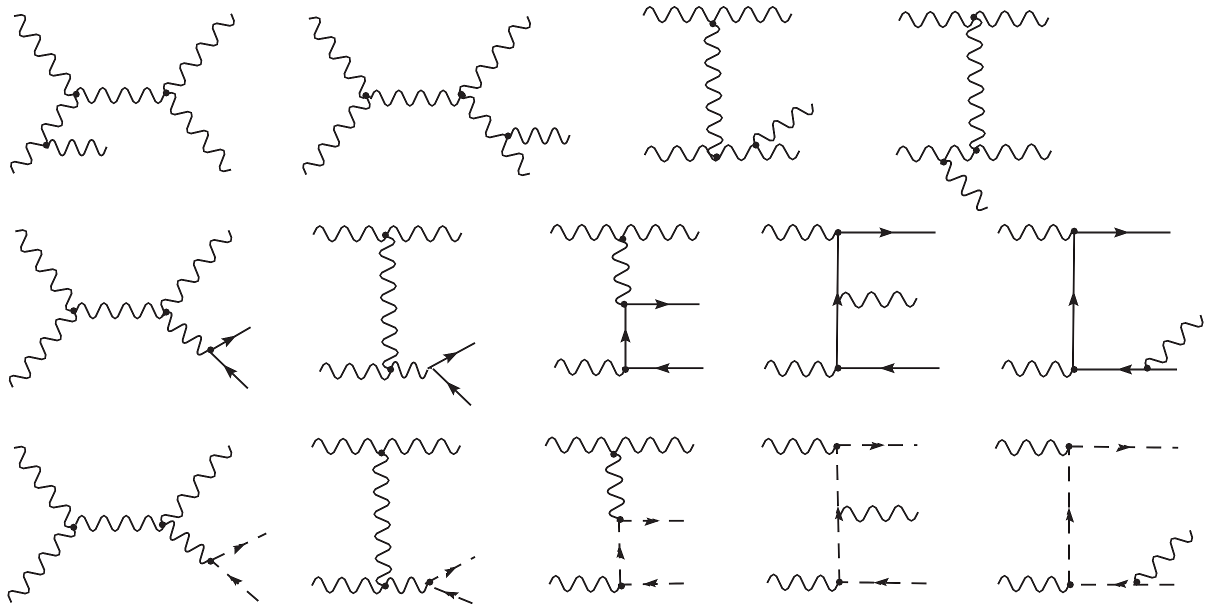

FIG. 4. The tree diagrams with three outgoing particles for the color-ordered amplitudes. Permutations are not shown.

(3) One of three final particles is the gluon with positive helicity and the rest is the quark-antiquark pair[8]: $g^+g^+ \to g^+q^-\bar{q}^+$ or $g^+g^+ \to g^+q^+\bar{q}^-$. This is an anti-MHV amplitude.
(4) One of three final particles is the gluon with positive helicity and the rest are two scalars: $g^+g^+ \to g^+\Lambda\Lambda$. This is an anti-MHV amplitude.

The corresponding diagrams are shown in Fig. 4.

If one fixes one gluon with positive helicity scattered at angle θ and sums over all the other particles then all the processes mentioned above contribute. In the case when one fixes two gluons with positive helicity and looks for the rest, only the first two options are allowed.

The cross section of these processes can be written as

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)_{\rm Real} = \frac{1}{J}\int d\phi_3 |\mathcal{M}_5^{(\rm tree)}|^2 \mathcal{S}_3, \qquad (4.11)$$

where $d\phi_3$ is the three-particle phase volume

$$d\phi_3 = \frac{d^D p_3 \delta^+(p_3^2)}{(2\pi)^{D-1}} \frac{d^D p_4 \delta^+(p_4^2)}{(2\pi)^{D-1}} \frac{d^D p_5 \delta^+(p_5^2)}{(2\pi)^{D-1}} \times (2\pi)^D \delta^D(p_1 + p_2 - p_3 - p_4 - p_5), \qquad (4.12)$$

and $\mathcal{S}_3$ is the measurement function which constrains the phase space and defines the particular observable.

The squared matrix element is expressed through the amplitudes as before:

$$|\mathcal{M}_5^{(\rm tree)}(p_1, \ldots, p_5)|^2 = g^6 N_c^3 (N_c^2 - 1) \sum_{\sigma \in P_4} |A_5^{(\rm tree)}(p_1, p_{\sigma(1)}, \ldots, p_{\sigma(4)})|^2. \qquad (4.13)$$

For the processes mentioned above one has the following expressions for the matrix elements[9]:

(1)

$$|\mathcal{M}_5^{(\rm tree)(--+++)}|^2 = g^6 N_c^3 (N_c^2 - 1) \sum_{\sigma \in P_4} \frac{s_{12}^4}{s_{1\sigma(1)} s_{\sigma(1)\sigma(2)} s_{\sigma(2)\sigma(3)} s_{\sigma(3)\sigma(4)} s_{\sigma(4)1}}. \qquad (4.14)$$

Since there are three identical gluons with positive helicity in the final state one has to define which ones are detected. In case of one detectable particle, one can choose the fastest one; in case of two, the two fastest ones. The measurement function for detecting only one gluon with momentum p_3 with positive helicity can be written as

$$\mathcal{S}_3^{(--+++),1} = \delta_{+,h_3} \Theta(p_3^0 > p_4^0) \Theta(p_3^0 > p_5^0) \delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}); \qquad (4.15)$$

[8]The $\mathcal{N} = 4$ supermultiplet consists of a gluon g, four fermions (“quarks”) q^A and six real scalars Λ^{AB}; A and B are $SU(4)_R$ indices, Λ is an antisymmetric tensor. It is implied that all squared amplitudes with quarks and scalars are summed over these indices.

[9]It is implied that all squared amplitudes with quarks and scalars are summed over $SU(4)_R$ indices.

and for detecting of two gluons with positive helicities as

$$\mathcal{S}_3^{(--+++),2} = \delta_{+,h_3}\delta_{+,h_4}\Theta(p_3^0 > p_5^0)\Theta(p_4^0 > p_5^0)\delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}), \tag{4.16}$$

where we detect the third and the fourth gluons. Analogous measurement function would appear if we would like to detect the third and the fifth gluons.

(2)

$$|\mathcal{M}_5^{(\rm tree)(--++-)}|^2 = g^6 N_c^3(N_c^2 - 1)\sum_{\sigma\in P_4}\frac{s_{34}^4}{s_{1\sigma(1)}s_{\sigma(1)\sigma(2)}s_{\sigma(2)\sigma(3)}s_{\sigma(3)\sigma(4)}s_{\sigma(4)1}}; \tag{4.17}$$

The measurement function for detecting one gluon with positive helicity and momentum p_3 is given by

$$\mathcal{S}_3^{(--+++),1} = \delta_{+,h_3}\Theta(p_3^0 > p_4^0))\delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}); \tag{4.18}$$

and for detecting of two gluons with positive helicity by

$$\mathcal{S}_3^{(--++-),2} = \delta_{+,h_3}\delta_{+,h_{4(5)}}\delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}). \tag{4.19}$$

(3)

$$|\mathcal{M}_5^{(\rm tree)(--+q\bar{q})}|^2 = g^6 N_c^3(N_c^2 - 1)\sum_{\sigma\in P_4}\frac{s_{34}s_{35}(s_{34}^2 + s_{35}^2)}{s_{1\sigma(1)}s_{\sigma(1)\sigma(2)}s_{\sigma(2)\sigma(3)}s_{\sigma(3)\sigma(4)}s_{\sigma(4)1}}; \tag{4.20}$$

The measurement function in this case is simple since we have only one gluon in the final state

$$\mathcal{S}_3^{(--+q\bar{q})} = \delta_{+,h_3}\delta^{D-2}(\Omega_{\rm Det} - \Omega_{13}). \tag{4.21}$$

(4)

$$|\mathcal{M}_5^{(\rm tree)(--+\Lambda\Lambda)}|^2 = g^6 N_c^3(N_c^2 - 1)\sum_{\sigma\in P_4}\frac{s_{34}^2 s_{35}^2}{s_{1\sigma(1)}s_{\sigma(1)\sigma(2)}s_{\sigma(2)\sigma(3)}s_{\sigma(3)\sigma(4)}s_{\sigma(4)1}}. \tag{4.22}$$

The measurement function is given by the same formula (4.21) as in the previous case.

One can check that the measurement functions written above satisfy the IR and collinear limit conditions (2.3) and (2.4). Indeed, one has

(1) $p_5 \to 0$, $|\mathbf{p}_4| = |\mathbf{p}_3|$:

$$S_3(p_3, p_4, 0) \to \Theta(p_3^0 - p_4^0)\delta^{D-2}(\Omega - \Omega_3),$$

(2) $\mathbf{p}_3 = -\mathbf{P}$, $\mathbf{p}_4 = x\mathbf{P}$, $\mathbf{p}_5 = (1-x)\mathbf{P}$:

$$S_3(p_3, p_4, p_5) \to \Theta(1-x)\Theta(x)\delta^{D-2}(\Omega - \Omega_3).$$

The latter θ functions give $0 < x < 1$ restricting the fraction of momenta in a natural way.

Choosing the fastest momentum one has to have in mind the conservation of momentum and energy

$$\mathbf{p}_3 + \mathbf{p}_4 + \mathbf{p}_5 = 0, \qquad |\mathbf{p}_3| + |\mathbf{p}_4| + |\mathbf{p}_5| = E.$$

This means that the three momenta form a triangle with the perimeter equal to E. Hence the requirement that, say, the third particle is the fastest one means that $p_3^0 > E/3$. Therefore, to simplify the integration, in what follows we choose the universal measurement function

$$\mathcal{S}_3(p_3, p_4, p_5) = \Theta\left(p_3^0 - \frac{1-\delta}{2}E\right)\delta^{D-2}(\Omega_{\rm Det} - \Omega_3), \tag{4.23}$$

where we take $\delta = 1/3$ in the case of identical particles and $\delta = 1$ in the other cases. Thus, the registration of one fastest gluon corresponds to $\delta = 1/3$ for the MHV and anti-MHV amplitudes and $\delta = 1$ for the matter-antimatter amplitude, while the registration of two fastest gluons corresponds to $\delta = 1/3$ for the MHV amplitude and $\delta = 1$ for the anti-MHV amplitude.[10] In what follows we keep the value of δ arbitrarily and show that the IR and collinear divergences cancel in observables for any value of δ. We omit the details of the calculation, which can be found in Appendix B, and present here only the divergent parts of the calculated objects. All the finite parts can be found in Appendix D.

With these definitions the contributions to the $2 \to 3$ cross sections from the amplitudes that are listed above are

[10]These are not precisely the needed requirements but are pretty close to them. Fulfillment of the exact requirements of the fastest particles is technically more involved but does not change the picture.

(1) Real emission (MHV)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\text{Real}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{\alpha}{\pi}\left\{\frac{8}{\epsilon^2}\frac{(3+c^2)}{(1-c^2)^2}+\frac{1}{\epsilon}\left[\frac{2}{(1+c)^2}\log\left(\frac{1-c}{2}\right)+\frac{2}{(1-c)^2}\log\left(\frac{1+c}{2}\right)\right.\right.$$
$$\left.\left.+\frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}+\frac{12(3+c^2)}{(1-c^2)^2}\log\left(\frac{1-\delta}{\delta}\right)\right]+\text{finite part}\right\}; \quad (4.24)$$

notice the singularity in the limit $\delta \to 1$.

(2) Real emission (anti-MHV)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\text{Real}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{\alpha}{\pi}\left\{\frac{1}{\epsilon^2}\frac{8(3+c^2)}{(1-c^2)^2}+\frac{1}{\epsilon}\left[-\frac{12(c^2+3)\log\delta}{(1-c^2)^2}+\frac{64(12c^2+17)}{3(1-c^2)^3}\right.\right.$$
$$+\frac{2\delta}{(1-c^2)^2}\left(\frac{2}{3}(5+3c^2)\delta^2-(c^2+19)\delta+2(5c^2+43)\right)$$
$$+\left(\frac{2(3c^2-24c+85)}{(1-c)(1+c)^3}\log\left(\frac{1-c}{2}\right)-\frac{8(c^2-6c+21)}{(1-c)(1+c)^3}\log\left(\frac{1+\delta-(1-\delta)c}{2}\right)\right.$$
$$-\frac{32(c^2-4c+7)}{(1+c)^3(1-c)(1+\delta-c(1-\delta))}+\frac{32(2-c)}{(1+c)^3(1+\delta-c(1-\delta))^2}$$
$$\left.\left.\left.-\frac{64(1-c)}{3(1+c)^3(1+\delta-c(1-\delta))^3}+(c\leftrightarrow -c)\right)\right]+\text{finite part}\right\}. \quad (4.25)$$

Contrary to the MHV case the limit $\delta \to 1$ is regular here and greatly simplifies the final result.

(3) Fermions (for four fermions in adjoint representation of $SU(N_c)$)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+q\bar{q})}_{\text{Real}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{\alpha}{\pi}\left\{-\frac{16}{\epsilon}\left[\frac{(79+25c^2)}{3(1-c^2)^2}+\frac{2(3-c)^2}{(1-c)(1+c)^3}\log\left(\frac{1-c}{2}\right)+\frac{2(3+c)^2}{(1-c)^3(1+c)}\log\left(\frac{1+c}{2}\right)\right]\right.$$
$$\left.+\text{finite part}\right\}. \quad (4.26)$$

(4) Scalars (for six scalars in adjoint representation of $SU(N_c)$)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+\Lambda\Lambda)}_{\text{Real}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{2\epsilon}\frac{\alpha}{\pi}\left\{-\frac{8}{\epsilon}\left[-\frac{2(10+7c^2)}{(1-c^2)^2}-\frac{3(5-c)}{(1+c)^3}\log\left(\frac{1-c}{2}\right)-\frac{3(5+c)}{(1-c)^3}\log\left(\frac{1+c}{2}\right)\right]\right.$$
$$\left.+\text{finite part}\right\}. \quad (4.27)$$

In the last two expressions we chose the parameter $\delta = 1$ since there are no identical particles in these cases and there is no need to restrict the phase space. Note also the absence of the second-order pole in ϵ which means that there is no IR soft divergency here but only a collinear one.

C. Splitting

Now we have to deal with an additional $1/\epsilon$ pole coming from the collinear divergences. As one can see from the toy model example, taking into account emission of additional quanta in the initial and final states allows one to cancel the IR divergences (double poles in ϵ) but leaves the single poles originating from collinear ones. Indeed, as it has been discussed earlier, the asymptotic states (both the initial and final ones) are not well defined since a massless quantum can split into two parallel ones indistinguishable from the original. To take this into account, we introduce the distribution of the initial and final particle (gluon or any other member of the $\mathcal{N} = 4$ SYM supermultiplet) with respect to the fraction of the carried momentum z: $q_i(z, Q_f^2/\mu^2)$. Also, one has to keep in mind that the particles in this case are polarized. The corresponding Feynman diagrams are shown in Fig. 5.

Additional contributions from collinear particles in the initial or final states to the inclusive gluon cross section (the collinear counterterms) have the following form, respectively:

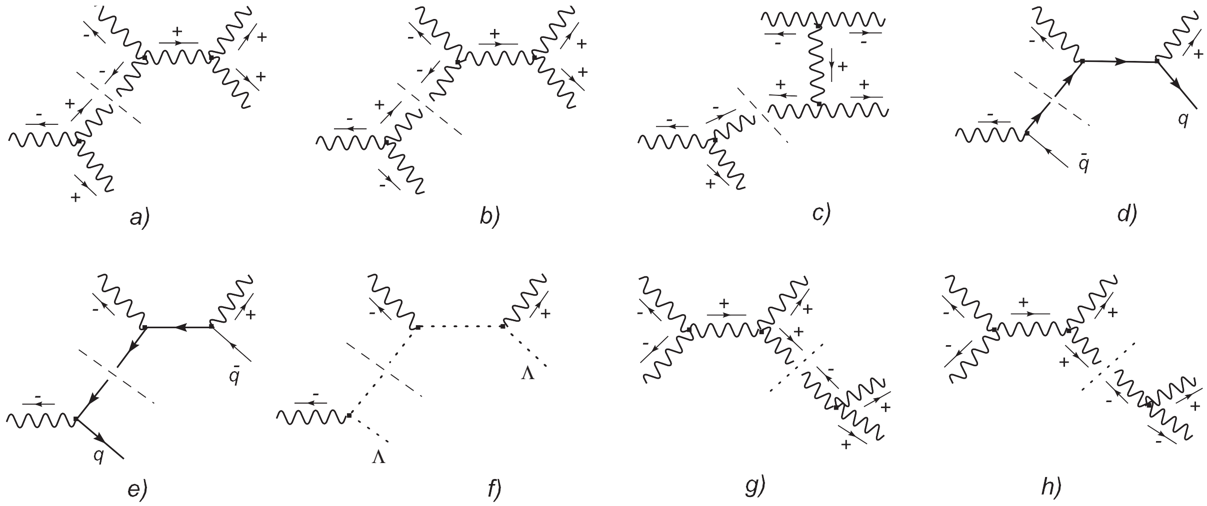

FIG. 5. The initial and final particle splitting diagrams: (a) the initial MHV amplitude, (b)–(c) the initial anti-MHV amplitudes, (d)–(f) the initial matter amplitudes, (g)–(h) the final MHV and anti-MHV amplitudes. Permutations are not shown.

$$d\sigma_{2\to 2}^{\mathrm{spl,init}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\sum_{i,j=1,2;i\neq j}\int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z)d\sigma_{2\to 2}(zp_i, p_j, p_3, p_4)\mathcal{S}_2^{\mathrm{spl,init}}(z), \tag{4.28}$$

$$d\sigma_{2\to 2}^{\mathrm{spl,fin}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon} d\sigma_{2\to 2}(p_1, p_2, p_3, p_4)\int_0^1 dz \sum_{l=g,q,\Lambda} P_{gl}(z)\mathcal{S}_2^{\mathrm{spl,fin}}(z). \tag{4.29}$$

Having particles with different helicities, we have the following set of collinear counterterms. (We use here slightly different notation for the splitting functions indicating explicitly all three particles like $P^{\mathrm{init}}_{\mathrm{fin}_1,\mathrm{fin}_2}(z)$ to avoid confusion.)

(1) Initial state splitting MHV amplitude $(- - + + +)$

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+++)}_{\mathrm{InSplit}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz 2P^{g^-}_{g^+g^+}(z)\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}(zp_1, p_2, p_3, p_4)\mathcal{S}^{(--+++)}_{2,\mathrm{init}}(z) + (p_1 \leftrightarrow p_2). \tag{4.30}$$

Final state splitting MHV amplitude $(- - + + +)$

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+++)}_{\mathrm{FnSplit}} = 2\frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}(p_1, p_2, p_3, p_4)\int_0^1 dz P^{g^-}_{g^+g^+}(z)\mathcal{S}^{(--+++)}_{2,\mathrm{fin}}(z). \tag{4.31}$$

(2) Initial state splitting anti-MHV amplitude $(- - + + -)$

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{InSplit}} = \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz 2P^{g^-}_{g^-g^+}(z)\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(-++-)}(zp_1, p_2, p_3, p_4)\mathcal{S}^{(--++-)}_{2,\mathrm{init}}(z) + \frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz 2P^{g^-}_{g^+g^-}(z)\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}(zp_1, p_2, p_3, p_4)\mathcal{S}^{(--++-)}_{2,\mathrm{init}}(z) + (p_1 \leftrightarrow p_2). \tag{4.32}$$

Final state splitting anti-MHV amplitude $(- - + + -)$

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{FnSplit}} = 2\frac{\alpha}{2\pi}\frac{1}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}(p_1, p_2, p_3, p_4)\int_0^1 dz P^{g^-}_{g^+g^-}(z)\mathcal{S}^{(--++-)}_{2,\mathrm{fin}}(z). \tag{4.33}$$

One has also the collinear counterterms containing the other members of the $\mathcal{N} = 4$ supermultiplet.

(3) Initial state splitting into a fermion-antifermion pair

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+q\bar{q})}_{\rm InSplit} = \frac{\alpha}{2\pi}\frac{n_f}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz\bigg[2P^{g^-}_{\bar{q}^-q^+}(z)\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(q-+\bar{q})}(zp_1,p_2,p_3,p_4)\mathcal{S}^{(--+q\bar{q})}_{2,\rm init}(z) + 2P^{g^-}_{q^+\bar{q}^-}(z) \times\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(\bar{q}-+q)}(zp_1,p_2,p_3,p_4)\mathcal{S}^{(--+q\bar{q})}_{2,\rm init}(z)\bigg] + (p_1\leftrightarrow p_2). \quad (4.34)$$

(4) Initial state splitting into a scalar pair

$$\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+\Lambda\Lambda)}_{\rm InSplit} = \frac{\alpha}{2\pi}\frac{n_s}{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\int_0^1 dz 2P^{g^-}_{\Lambda\Lambda}(z)\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(\Lambda-+\Lambda)}(zp_1,p_2,p_3,p_4)\mathcal{S}^{(--+\Lambda\Lambda)}_{2,\rm init}(z) + (p_1\leftrightarrow p_2), \quad (4.35)$$

where n_f and n_s is the number of fermions and scalars, respectively. One should put $n_f = 4$ and $n_s = 6$ in our case.

The explicit form of the Born cross sections and the splitting functions $P^i_{jk}(z)$ can be found in Appendixes A and C, respectively. Notice that when changing the momentum $p_i \to zp_i$ one has to modify the Mandelstam variables according to Eq. (4.38) and take into account the additional factor from the phase space in full analogy with the QED case (see the comment after Eq. (3.7)).

Note that there is no final state splitting counterterms for fermions and scalars. The reason is that one has to take into account only those final splittings where the original state (gluon in our case) survives with momentum multiplied by fraction z.

The measurement functions here are the same as in the case of real emission but depend now on fraction z and restrict the integration region over z. They take the form

$$\mathcal{S}^{\rm spl,1}_2(z) = \delta_{+,h_3}\delta^{D-2}(\Omega-\Omega_{13})\Theta(z-z_{\min}) \quad (4.36)$$

for detecting of one gluon and

$$\mathcal{S}^{\rm spl,2}_2(z) = \delta_{+,h_3}\delta_{+,h_{4,5}}\delta^{D-2}(\Omega-\Omega_{13})\Theta(z-z_{\min}) \quad (4.37)$$

for detecting of two gluons.

The values of $z_{\min}$ can be calculated from the requirement $p_3^0 > (1-\delta)E/2$ in different kinematics. Indeed, for the initial splitting process one has to change the momentum of the ingoing particle, for example, p_1 to zp_1 which gives in the c.m. frame

$$s\to zs, \qquad t\to \frac{2z}{1+z-c(1-z)}t,$$
$$u\to\frac{2z^2}{1+z-c(1-z)}u, \qquad p_3^0\to\frac{2z}{1+z-c(1-z)}\frac{E}{2}. \quad (4.38)$$

At the same time, for the final splitting one has to substitute $p_3^0\to z\frac{E}{2}$. This leads to the values of $z_{\min}$, respectively,

$$z^{\rm in}_{\min} = \frac{(1-\delta)(1-c)}{1+\delta-c(1-\delta)}, \qquad z^{\rm fin}_{\min} = (1-\delta). \quad (4.39)$$

Taking into account the splitting of the initial states and the fragmentation of the final states we get the following contribution to the inclusive cross sections:

(1) The initial and final splitting for the MHV amplitude

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\rm InSplit} = \frac{\alpha^2N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\bigg\{\frac{1}{\epsilon}\bigg[-\frac{4(c^2+3)}{(1-c^2)^2}\left(\log\frac{1-c}{2}+\log\frac{1+c}{2}\right)-\frac{8(c^2+3)}{(1-c^2)^2}\log\frac{1-\delta}{\delta} - \frac{16\delta(2\delta-3)}{(1-c^2)^2(1-\delta)^2}\bigg] + \text{finite part}\bigg\}, \quad (4.40)$$

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\rm FnSplit} = \frac{\alpha^2N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\bigg\{-\frac{1}{\epsilon}\frac{4(c^2+3)}{(1-c^2)^2}\log\frac{1-\delta}{\delta}\bigg\}. \quad (4.41)$$

(2) The initial and final splitting for the anti-MHV amplitude

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{InSplit}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\left\{\frac{1}{\epsilon}\left[\frac{8(c^2+3)}{(1-c^2)^2}\log\delta - \frac{64(12c^2+17)}{3(1-c^2)^3} - \frac{4\delta}{(1-c^2)^2}\left(\frac{2}{3}(1+c^2)\delta^2 + (c^2-5)\delta + 2(c^2+17)\right) + \left(\frac{4(c^3-15c^2+51c-45)}{(1-c)^2(1+c)^3}\log\frac{1-c}{2} + \frac{8(c^2-6c+21)}{(1-c)(1+c)^3}\log\frac{1+\delta-c(1-\delta)}{2} + \frac{32(c^2-4c+7)}{(1+c)^3(1-c)(1+\delta-c(1-\delta))} - \frac{32(2-c)}{(1+c)^3(1+\delta-c(1-\delta))^2} + \frac{64(1-c)}{3(1+c)^3(1+\delta-c(1-\delta))^3} + (c\leftrightarrow -c)\right)\right] + \text{finite part}\right\}, \tag{4.42}$$

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{FnSplit}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\left\{\frac{1}{\epsilon}\frac{4(c^2+3)}{(1-c^2)^2}\left[\log\delta - \delta\left(\frac{1}{3}\delta^2 - \frac{3}{2}\delta + 3\right)\right]\right\}. \tag{4.43}$$

(3) The initial splitting for the fermion final states ($\delta = 1$)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+q\bar{q})}_{\mathrm{InSplit}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\left\{\frac{16}{\epsilon}\left[\frac{(79+25c^2)}{3(1-c^2)^2} + \frac{2(3-c)^2}{(1-c)(1+c)^3}\log\left(\frac{1-c}{2}\right) + \frac{2(3+c)^2}{(1-c)^3(1+c)}\log\left(\frac{1+c}{2}\right)\right] + \text{finite part}\right\}. \tag{4.44}$$

(4) The initial splitting for the scalar final states ($\delta = 1$)

$$\left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+\Lambda\Lambda)}_{\mathrm{InSplit}} = \frac{\alpha^2 N_c^2}{E^2}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{\mu^2}{Q_f^2}\right)^{\epsilon}\frac{\alpha}{\pi}\left\{\frac{8}{\epsilon}\left[-\frac{2(10+7c^2)}{(1-c^2)^2} - \frac{3(5-c)}{(1+c)^3}\log\left(\frac{1-c}{2}\right) - \frac{3(5+c)}{(1-c)^3}\log\left(\frac{1+c}{2}\right)\right] + \text{finite part}\right\}. \tag{4.45}$$

V. IR-SAFE OBSERVABLES IN $\mathcal{N} = 4$ SYM

In the NLO there are two sets of amplitudes, namely, the MHV and anti-MHV amplitudes which contribute to the observables. The leading order four-gluon amplitude is both MHV and anti-MHV and we split it into two parts. Then one can construct three types of infrared finite quantities in the NLO of perturbation theory, namely,

(i) Pure gluonic MHV amplitude

$$A^{\mathrm{MHV}} = \frac{1}{2}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{\mathrm{Virt}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+++)}_{\mathrm{Real}} + \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+++)}_{\mathrm{InSplit}} + \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+++)}_{\mathrm{FnSplit}}. \tag{5.1}$$

(ii) Pure gluonic anti-MHV amplitude

$$B^{\text{anti-MHV}} = \frac{1}{2}\left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++)}_{\mathrm{Virt}} + \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{Real}} + \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{InSplit}} + \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--++-)}_{\mathrm{FnSplit}}. \tag{5.2}$$

(iii) Anti-MHV amplitude with fermions or scalars forming the full $\mathcal{N} = 4$ supermultiplet

$$C^{\mathrm{Matter}} = \left(\frac{d\sigma_{2\to 3}}{d\Omega_{13}}\right)^{(--+,q\bar{q}+\Lambda\Lambda)}_{\mathrm{Real}} + \left(\frac{d\sigma_{2\to 2}}{d\Omega_{13}}\right)^{(--+,q\bar{q}+\Lambda\Lambda)}_{\mathrm{InSplit}}. \tag{5.3}$$

We would like to stress once more that in each expression

(5.1), (5.2), and (5.3) *all IR divergencies cancel* for arbitrary δ and only the finite part is left.

Two comments are in order. First, this decomposition is valid only in the leading order in α. In the next orders the inclusive cross section requires extra emitted particles that take us away from the class of the MHV amplitudes. It is not clear whether in this case one has separate IR-safe sets or everything is mixed together and only the total cross section is finite. In the latter case one probably faces the complication that the non-MHV amplitudes are not known to possess a simple structure as the MHV ones, though the origin of this simplicity is unclear. The second comment concerns the contribution of the matter fields. In the leading order we singled it out in the class C. At the same time, in general, there is their contribution to the virtual part and to the splitting one via the gluon distribution function. They contain the $1/\epsilon$ terms. However, the matter field contribution to the virtual part is proportional to the tree-level 2×2 cross section with the coefficient $(\frac{2}{3}n_f + \frac{1}{6}n_s)$ [57] and the contribution to the splitting function comes with the β function, i.e. with the same coefficient but with the opposite sign. Thus, these contributions have the same structure and completely cancel each other. So, in the leading order our separation becomes possible.

Defining now the physical condition for the observation we get several infrared-safe inclusive cross sections

(i) Registration of *two fastest* gluons of positive helicity

$$A^{\mathrm{MHV}}|_{\delta=1/3} + B^{\text{anti-MHV}}|_{\delta=1}. \tag{5.4}$$

(ii) Registration of *one fastest* gluon of positive helicity

$$A^{\mathrm{MHV}}|_{\delta=1/3} + B^{\text{anti-MHV}}|_{\delta=1/3} + C^{\mathrm{Matter}}|_{\delta=1}. \tag{5.5}$$

(iii) Anti-MHV cross section

$$B^{\text{anti-MHV}}|_{\delta=1} + C^{\mathrm{Matter}}|_{\delta=1}. \tag{5.6}$$

Relative simplicity of the virtual contribution (4.10) which contains logarithms and no other special functions suggests a similar structure of the real part. However, this is not the case. While the singular terms are simple enough and cancel completely, the finite parts are usually cumbersome and contain polylogarithms. The only expression where they cancel corresponds to the $\delta = 1$ case which is possible only for the last set of observables, namely, for the anti-MHV cross section (5.6). Choosing the factorization scale to be $Q_f = E$ we get

$$\left(\frac{d\sigma}{d\Omega_{13}}\right)_{\text{anti-MHV}} = \frac{4\alpha^2 N_c^2}{E^2}\left\{\frac{3+c^2}{(1-c^2)^2} - \frac{\alpha}{4\pi}\left[2\frac{(c^4+2c^3+4c^2+6c+19)\log^2(\frac{1-c}{2})}{(1-c)^2(1+c)^4} + 2\frac{(c^4-2c^3+4c^2-6c+19)\log^2(\frac{1+c}{2})}{(1-c)^4(1+c)^2} - 8\frac{(c^2+1)\log(\frac{1+c}{2})\log(\frac{1-c}{2})}{(1-c^2)^2} + \frac{6\pi^2(3c^2+13)-5(61c^2+99)}{9(1-c^2)^2} - 2\frac{(11c^3-31c^2-47c-133)\log(\frac{1-c}{2})}{3(1+c)^3(1-c)^2} + 2\frac{(11c^3+31c^2-47c+133)\log(\frac{1+c}{2})}{3(1-c)^3(1+c)^2}\right]\right\}. \tag{5.7}$$

One can see that even this expression does not repeat the form of the Born amplitude and does not have any simple structure. While the dependence on the parameter μ of dimensional reduction is completely canceled, the finite answer, as in the toy model example, depends on the factorization scale. This dependence comes from the asymptotic states which violate conformal invariance of the Lagrangian. This dependence seems to be unavoidable and reflects the act of measurement. Construction of observables which do not contain any external scale remains an open question.

VI. DISCUSSION

Remarkable factorization properties of the MHV amplitudes accumulated in the BDS ansatz (with the so-far unknown modification) and duality with the string amplitudes via the AdS/CFT correspondence seem to suggest the way to get the exact solution of the $\mathcal{N} = 4$ SYM theory. However, "to solve the model" might have a different meaning. Calculation of divergences and understanding of their structure is very useful but surely not enough—it is the finite part that we are really looking for. The knowledge of the S-matrix would be the final goal, though the definition of the S-matrix in conformal theory is a problem. Even in the absence of the UV divergences there are severe IR problems and matrix elements do not exist after removal of regularization.

The purpose of this paper is to present all the details of the calculation with explicit cancellation of the infrared divergencies in properly defined cross sections in the planar limit for $\mathcal{N} = 4$ SYM. The main results were summarized in our short letter [58]. We do obtain IR-safe observables in the weak coupling regime in the next-to-leading order of PT which are calculated analytically. The same procedure can also be applied to $\mathcal{N} = 8$ supergravity [59].

Unfortunately, our calculation has demonstrated that the simple structure of the amplitudes governed by the cusp anomalous dimension has been totally washed out by complexity of the real emission matrix elements integrated over the phase space. This means that either the $\mathcal{N} = 4$ SYM theory does not allow such a simple factorizable solution or that we considered the inappropriate observables that do not bear the impact of the $\mathcal{N} = 4$ symmetry. One can obviously see the presence of $\mathcal{N} = 4$ supermultiplet in the virtual part but not in the real emission. It would be of great importance to find such quantities.

Another unfortunate feature of inclusive cross sections is the dependence on the factorization scale. The experience of QCD, which is very similar to the $\mathcal{N} = 4$ SYM theory from the point of view of the IR problems, tells us that in inclusive cross sections the IR divergences cancel and one has finite physical observables. However, in QCD one has confinement and considers the scattering of the bound states (hadrons, glueballs) rather than the individual particles. In this case, one usually factorizes the hard part from the soft part introducing the factorization scale. The dependence on this scale is canceled between the hard and soft parts contrary to our case where only the hard part is present. But in QCD one also has an additional scale. The parton distributions are defined experimentally at some scale Q_0 and the dependence on this scale is left. This dependence is governed by the same DGLAP equations as the dependence on the factorization scale, so from this point of view the situation in QCD is not better than in our case.

In both cases, one has to introduce some parton distributions which are the functions of a fraction of momenta and, in higher orders, of momenta transferred. This leads to the appearance of an additional scale which breaks the conformal invariance. One might think of some observables where this scale dependence is canceled, like the ratio of some cross sections, etc. We have not found such quantities so far, though the construction of such truly conformal observables is of great interest. Probably, they might have the desired simple structure.

There is an interesting duality between the MHV amplitudes and the Wilson loop, between the weak and the strong coupling regime [25,26,29]. Perhaps, it would be possible, using the AdS/CFT correspondence, to construct the IR-safe observables in the strong coupling limit (similarly to what we did here) and to shed some light on the true calculable objects in conformal theories.

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APPENDIX A. COMPUTATION OF PARTIAL AMPLITUDES

To calculate the cross section we need the squared matrix elements summed over helicities and color. They can be expressed in terms of the corresponding partial amplitudes [5]

$$\begin{aligned}|\mathcal{M}_n(p_1,\dots,p_n)|^2 &= g^{2n-4}\left(\frac{g^2N_c}{16\pi^2}\right)^{2l}\sum_{\text{colors}}|\mathcal{A}_n^{(l\text{-loop})}| \\ &= g^{2n-4}N_c^{n-2}(N_c^2-1)\left(\frac{g^2N_c}{16\pi^2}\right)^{2l} \\ &\quad\times\sum_{\sigma\in P_{n-1}}|A_n^{(l\text{-loop})}(p_1,p_{\sigma(2)},\dots,p_{\sigma(n)})|^2.\end{aligned} \tag{A1}$$

For the massless partial helicity amplitudes it is convenient to use the so-called spinor helicity formalism initially introduced in [60–62] (for a review see [63]). In this formalism the on-shell momenta of every i-th external massless particle $p_\mu^{(i)}p^{(i)\mu} = 0$ is represented in terms of a pair of massless commuting spinors $\lambda_a^{(i)}$ and $\bar{\lambda}_{\dot{a}}^{(i)}$ of positive and negative chirality in the following way:

$$p_\mu^{(i)} \to p_{a\dot{a}}^{(i)} = p_\mu^{(i)}(\sigma^\mu)_{a\dot{a}} = \lambda_a^{(i)}\bar{\lambda}_{\dot{a}}^{(i)}. \tag{A2}$$

The spinor inner product is defined by:

$$\begin{aligned}\epsilon^{ab}\lambda_a^{(i)}\lambda_b^{(j)} &= \langle\lambda^{(i)}\lambda^{(j)}\rangle \doteq \langle ij\rangle, \\ \epsilon^{\dot{a}\dot{b}}\bar{\lambda}_{\dot{a}}^{(i)}\bar{\lambda}_{\dot{b}}^{(j)} &= [\bar{\lambda}^{(i)}\bar{\lambda}^{(j)}] \doteq [ij],\end{aligned} \tag{A3}$$

thus the complex conjugation of the product is

$$(\langle ij\rangle)^* = [ij]. \tag{A4}$$

The scalar product of the two lightlike momenta can be represented in terms of these products as

$$p^{\mu(i)}p_\mu^{(j)} = \frac{1}{2}\langle ij\rangle[ij], \tag{A5}$$

or equivalently

$$\langle ij\rangle[ij] = s_{ij}, \tag{A}$$

where the standard notation $(p_i + p_j)^2 = s_{ij}$ is used.

All the tree-level partial MHV amplitudes can be combined into a single $\mathcal{N} = 4$ supersymmetric expression, first suggested by Nair [64],

$$\mathcal{Z}_n^{\mathcal{N}=4\text{MHV}} = \delta^8\Big(\sum_{i=1}^n \lambda_i^a \eta_i^{(A)}\Big)\frac{1}{\prod_{i=1}^n\langle i, i+1\rangle}, \tag{A7}$$

where $\eta_i^{(A)}$ are the Grassmannian coordinates, $A = 1, \ldots, 4$ is the $SU(4)_R$ fundamental representation index. $\mathcal{Z}_n^{\mathcal{N}=4\text{MHV}}$ is invariant under $SU(4)_R$ transformations of $\eta_i^{(A)}$ and under the cyclic permutations of momentum labels i. In the product $\prod_{i=1}^n\langle i, i+1\rangle$ one has to identify $i+n$ with i. The Grassmannian-valued delta function is defined in the usual way:

$$\delta^8\Big(\sum_{i=1}^n \lambda_i^a \eta_i^{(A)}\Big) = \prod_{A=1}^4 \frac{1}{2}\Big(\sum_{i=1}^n \lambda_i^a \eta_i^{(A)}\Big)\Big(\sum_{k=1}^n \lambda_{ka} \eta_k^{(A)}\Big) = \frac{1}{16}\prod_{A=1}^4 \sum_{i,k=1}^n \langle ik\rangle(\eta_i^{(A)}\eta_k^{(A)}). \tag{A8}$$

So one can rewrite $\mathcal{Z}_n^{\mathcal{N}=4\text{MHV}}$ as

$$\mathcal{Z}_n^{\mathcal{N}=4\text{MHV}} = \frac{1}{16}\sum_{i,\ldots,c=1}^n \langle ik\rangle\langle lm\rangle\langle ab\rangle\langle dc\rangle \times (\eta_i^{(1)}\eta_k^{(1)}\eta_l^{(2)}\eta_m^{(2)}\eta_a^{(3)}\eta_b^{(3)}\eta_d^{(4)}\eta_c^{(4)})\frac{1}{\mathcal{P}_n}, \tag{A9}$$

where

$$\mathcal{P}_n = \prod_{i=1}^n \langle i, i+1\rangle. \tag{A10}$$

Using the Taylor expansion of $\mathcal{Z}_n^{\mathcal{N}=4\text{MHV}}$ in powers of $\eta^{(A)}$ one gets the sum of $(\frac{n(n-1)}{2})^4$ terms each involving a product of eight distinct $\eta_i^{(A)}$. One can identify the coefficient of the product of eight η's in each term in the expansion with a particular tree component partial amplitude. It is very useful to define the following differential operators with the self-explanatory notation:

$$\begin{aligned}
\hat{g}^+(i) &= 1,\\
\hat{g}^-(i) &= \frac{1}{4!}\epsilon^{ABCD}\frac{\partial^4}{\partial\eta_i^{(A)}\partial\eta_i^{(B)}\partial\eta_i^{(C)}\partial\eta_i^{(D)}} = \frac{\partial^4}{\partial\eta_i^{(1)}\partial\eta_i^{(2)}\partial\eta_i^{(3)}\partial\eta_i^{(4)}},\\
\hat{q}^+(i)^A &= \frac{\partial}{\partial\eta_i^{(A)}},\\
\hat{q}^-(i)_A &= -\frac{1}{3!}\epsilon_{ABCD}\frac{\partial^3}{\partial\eta_i^{(B)}\partial\eta_i^{(C)}\partial\eta_i^{(D)}},\\
\hat{\Lambda}(i)^{AB} &= \frac{\partial^2}{\partial\eta_i^{(A)}\partial\eta_i^{(B)}},\\
\hat{\Lambda}(i)_{CD} &= \frac{1}{2!}\epsilon^{ABCD}\frac{\partial^2}{\partial\eta_i^{(A)}\partial\eta_i^{(B)}}.
\end{aligned} \tag{A11}$$

Taking various combinations of products of these operators one can construct a set of eighth order differential operators. These eighth order differential operators act as projectors on the component partial amplitudes: $\hat{q}^+(i)^A$ corresponds to the fermion $q^{A,+}$ of the $\mathcal{N} = 4$ supermultiplet, $\hat{q}_A^-$ to $\bar{q}_A^-$, $\hat{\Lambda}^{AB}(i)$ to Λ^{AB}, and $\hat{\Lambda}_{AB}(i)$ to Λ_{AB}.

For example, the Parke-Taylor n-gluon amplitude can be written as:

$$A_n^{(\text{tree})}(g^-g^-g^+\ldots g^+) = \hat{g}^-(1)\hat{g}^-(2)\hat{g}^+(3)\ldots\hat{g}^+(n) \times \mathcal{Z}_n^{\mathcal{N}=4\text{MHV}} = \langle 12\rangle^4\frac{1}{\mathcal{P}_n}, \tag{A12}$$

and the squared partial amplitude $|A_n^{(\text{tree})}(g^-g^-g^+\ldots g^+)|^2$ then takes the simple form (it is implemented that momenta are ordered as $p_1, p_2, p_3, \ldots, p_n$)

$$|A_n^{(\text{tree})}(g^-g^-g^+\ldots g^+)|^2 = A_n^{(\text{tree})}(g^-g^-g^+\ldots g^+) \times A_n^{(\text{tree})}(g^-g^-g^+\ldots g^+)^* = \frac{\langle 12\rangle^4[12]^4}{\mathcal{P}_n\mathcal{P}_n^*} = \frac{s_{12}^4}{s_{12}s_{23}\ldots s_{n1}}. \tag{A13}$$

To extract from (A1) some specific helicity configuration for the MHV amplitude, one has to sum over the permutations only in the denominator of (A13) [55]. So, for example, for the Parke-Taylor n-gluon amplitude one has

$$|M_n^{(\text{tree})}(g^-g^-g^+\ldots g^+)|^2 = g^{2n-4}N_c^n s_{12}^4 \sum_{\sigma\in P_{n-1}}\frac{1}{s_{1\sigma(2)}s_{\sigma(2)\sigma(3)}\ldots s_{\sigma(n)\sigma(1)}}. \tag{A14}$$

The anti-MHV amplitudes also needed for our computation can be obtained from the corresponding conjugated MHV amplitudes. For example the anti-MHV amplitude

$A_5(g^-g^-g^+g^-g^+)$ can be obtained from the MHV amplitude $A_5(g^+g^+g^-g^+g^-)$ by making a complex conjugation.

Below we present the list of four- and five-point tree amplitudes which are relevant to our calculation. The four-point amplitudes are

$$A_4^{(\mathrm{tree})}(g^-g^-g^+g^+) = \hat{g}^-(1)\hat{g}^-(2)\hat{g}^+(3)\hat{g}^+(4)Z_4^{\mathcal{N}=4\mathrm{MHV}} = \langle 12\rangle^4\frac{1}{\mathcal{P}_4}, \tag{A15}$$

$$A_4^{(\mathrm{tree})}(g^-g^+g^-g^+) = g^-(1)g^+(2)g^-(3)g^+(4)Z_4^{\mathcal{N}=4\mathrm{MHV}} = \langle 13\rangle^4\frac{1}{\mathcal{P}_4}, \tag{A16}$$

$$A_4^{(\mathrm{tree})}(g^-q^Ag^+\bar{q}_A) = \hat{g}^-(1)\hat{q}^{A,+}(2)\hat{g}^+(3)\hat{q}_A^-(4)Z_4^{\mathcal{N}=4\mathrm{MHV}} = \langle 12\rangle\langle 14\rangle^3\frac{1}{\mathcal{P}_4}, \tag{A17}$$

$$A_4^{(\mathrm{tree})}(g^-\bar{q}_Ag^+q^A) = \hat{g}^-(1)\hat{q}_A^-(2)\hat{g}^+(3)\hat{q}^{A,+}(4)Z_4^{\mathcal{N}=4\mathrm{MHV}} = \langle 14\rangle\langle 12\rangle^3\frac{1}{\mathcal{P}_4}, \tag{A18}$$

$$A_4^{(\mathrm{tree})}(g^-\Lambda^{\mathrm{AB}}g^+\Lambda_{\mathrm{AB}}) = \hat{g}^-(1)\hat{\Lambda}^{\mathrm{AB}}(2)\hat{g}^+(3)\times\hat{\Lambda}_{\mathrm{AB}}(4)Z_4^{\mathcal{N}=4\mathrm{MHV}} = \frac{\langle 12\rangle^2\langle 14\rangle^2}{\mathcal{P}_4}. \tag{A19}$$

For the computation of the real emission we need the five-point tree amplitudes

$$A_5^{(\mathrm{tree})}(g^-g^-g^+g^+g^+) = \hat{g}^-(1)\hat{g}^-(2)\hat{g}^+(3)\hat{g}^+(4)\times\hat{g}^+(5)Z_5^{\mathcal{N}=4\mathrm{MHV}} = \langle 12\rangle^4\frac{1}{\mathcal{P}_5}, \tag{A20}$$

$$A_5^{(\mathrm{tree})}(g^-g^-g^+g^-g^+) = (\hat{g}^+(1)\hat{g}^+(2)\hat{g}^-(3)\hat{g}^+(4)\times\hat{g}^-(5)Z_5^{\mathcal{N}=4\mathrm{MHV}})^* = [35]^4\frac{1}{\mathcal{P}_5^*}, \tag{A21}$$

$$A_5^{(\mathrm{tree})}(g^-g^-g^+g^+g^-) = (\hat{g}^+(1)\hat{g}^+(2)\hat{g}^-(3)\hat{g}^-(4)\hat{g}^+(5)\times Z_5^{\mathcal{N}=4\mathrm{MHV}})^* = [35]^4\frac{1}{\mathcal{P}_5^*}, \tag{A22}$$

$$A_5^{(\mathrm{tree})}(g^-g^-g^+q^A\bar{q}_A) = (\hat{g}^+(1)\hat{g}^+(2)\hat{g}^-(3)\hat{q}^{A,+}(4)\hat{q}_A^-(5)\times Z_5^{\mathcal{N}=4\mathrm{MHV}})^* = \frac{[34]^3[35]}{\mathcal{P}_5^*}, \tag{A23}$$

$$A_5^{(\mathrm{tree})}(g^-g^-g^+\bar{q}_Aq^A) = (\hat{g}^+(1)\hat{g}^+(2)\hat{g}^-(3)\hat{q}_A^-(4)\hat{q}^{A,+}(5)\times Z_5^{\mathcal{N}=4\mathrm{MHV}})^* = \frac{[34][35]^3}{\mathcal{P}_5^*}, \tag{A24}$$

$$A_5^{(\mathrm{tree})}(g^-g^-g^+\Lambda^{\mathrm{AB}}\Lambda_{\mathrm{AB}}) = (\hat{g}^+(1)\hat{g}^+(2)\hat{g}^-(3)\hat{\Lambda}^{\mathrm{AB}}(4)\times\hat{\Lambda}_{\mathrm{AB}}(5)Z_5^{\mathcal{N}=4\mathrm{MHV}})^* = \frac{[34]^2[35]^2}{\mathcal{P}_5^*}. \tag{A25}$$

We also provide the list of the Born cross sections used in Sec. IV:

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(--++)}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}s^2\left(\frac{s^2+t^2+u^2}{t^2u^2}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{A26}$$

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(-+-+)}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}t^2\left(\frac{s^2+t^2+u^2}{s^2u^2}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{A27}$$

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(-++-)}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}u^2\left(\frac{s^2+t^2+u^2}{t^2s^2}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{A28}$$

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(-q+\bar{q})}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}|u|\left(\frac{s^2+t^2+u^2}{t^2s}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{A29}$$

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(-\bar{q}+q)}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}s\left(\frac{s^2+t^2+u^2}{t^2|u|}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}, \tag{A30}$$

$$\left(\frac{d\sigma_{2\to2}}{d\Omega_{13}}\right)^{(-\Lambda+\Lambda)}_{(\mathrm{tree})} = \frac{\alpha^2N_c^2}{2E^2}\left(\frac{s^2+t^2+u^2}{t^2}\right)\left(\frac{\mu^2}{s}\right)^{\epsilon}. \tag{A31}$$

These cross sections are written down for the set of momenta (p_1, p_2, p_3, p_4) with the conservation law $p_1 + p_2 = p_3 + p_4$. In the case of initial splitting, according to (4.28), one should use the cross sections calculated for the set (zp_1, p_2, p_3, p_4) with a new conservation law $zp_1 + p_2 = p_3 + p_4$. To get them, one should substitute the modified values for the Mandelstam variables (4.38) into (A26)–(A31) and multiply the cross sections by the factor $4/(1+z-c(1-z))^2$ which comes from the modified delta function $\delta^D(zp_1 + p_2 - p_3 - p_4)$. The same procedure but with the replacement $c \leftrightarrow -c$ refers to the $p_1 \leftrightarrow p_2$ case.

APPENDIX B. CALCULATION OF PHASE SPACE INTEGRALS

Consider the structure of the matrix elements in detail. First of all it is convenient to rewrite the standard three-particle phase space

$$d\phi_3 = \delta^+(p_3^2)\frac{d^D p_3}{(2\pi)^{D-1}}\delta^+(p_4^2)\frac{d^D p_4}{(2\pi)^{D-1}}\delta^+(p_5^2)\frac{d^D p_5}{(2\pi)^{D-1}} \times (2\pi)^D\delta^D(p_1+p_2-p_3-p_4-p_5) \tag{B1}$$

in the following form:

$$d\phi_3 = \delta^+(p_3^2)\frac{d^D p_3}{(2\pi)^{D-1}}\delta^+((p_4-k)^2)\frac{d^D p_4}{(2\pi)^{D-1}}\delta^+(k^2) \times \frac{d^D k}{(2\pi)^{D-1}}(2\pi)^D\delta^D(p_1+p_2-p_3-p_4). \tag{B2}$$

The integral we are interested in is

$$\int |\mathcal{M}_5|^2\mathcal{S}_3(p_3, k, p_4-k)d\phi_3, \tag{B3}$$

where the matrix element $|\mathcal{M}_5|^2$ for the five-point amplitude consists of 12 terms with identical numerator but different denominators. The typical integrand looks like

$$I = \frac{2s_{12}^4}{s_{13}s_{25}s_{35}s_{24}s_{14}} = \frac{2((p_1+p_2)^2)^4}{(p_1-p_3)^2(p_2-k)^2(-p_3-k)^2(p_2-[p_4-k])^2(p_1-[p_4-k])^2}. \tag{B4}$$

Our strategy is to use the on-shell conditions to simplify all the terms in the sum so that the integral over $d^D k$ can be calculated exactly. For the remaining integrals we evaluate the necessary terms of the ϵ expansion.

Taking into account the conservation of the momentum $p_1+p_2=p_3+p_4$ and the on-shell conditions

$$p_1^2=0, \qquad p_2^2=0, \qquad p_3^2=0, \qquad k^2=0, \qquad (p_4-k)^2=0,$$

one can rewrite the integrand (B4) as

$$I = \frac{(p_1,p_2)^4}{(p_1,p_3)(p_2,k)(p_3,k)(p_2,p_4-k)(p_1,p_4-k)}, \tag{B5}$$

where we use the notation (p, k) = pk for the scalar product.

The next step is to use the partial fraction with respect to k

$$\begin{aligned}
I &= \frac{(p_1,p_2)^4}{(p_1,p_3)(p_3,k)(p_1,p_4-k)}\frac{1}{(p_2,p_4)}\left(\frac{1}{(p_2,k)}+\frac{1}{(p_2,p_4-k)}\right)\\
&= \frac{(p_1,p_2)^4}{(p_1,p_3)(p_2,p_4)}\frac{1}{(p_3,k)(p_2,k)(p_1,p_4-k)} + \frac{(p_1,p_2)^4}{(p_1,p_3)(p_2,p_4)}\frac{1}{(p_3,k)(p_2,p_4-k)(p_1,p_4-k)}\\
&= \frac{(p_1,p_2)^4}{(p_1,p_3)(p_2,p_4)}\frac{1}{(p_1,p_4)-(p_4,p_4)/2}\left(\frac{1}{(p_3,k)(p_2,k)}-\frac{1}{(p_3,k)(p_1,p_4-k)}+\frac{1}{(p_2,k)(p_1,p_4-k)}\right)\\
&\quad + \frac{(p_1,p_2)^4}{(p_1,p_3)(p_2,p_4)}\frac{1}{(p_1,p_4)+(p_2,p_4)-(p_4,p_4)/2}\left(\frac{1}{(p_3,k)(p_2,p_4-k)}+\frac{1}{(p_3,k)(p_1,p_4-k)}\right.\\
&\quad \left. + \frac{1}{(p_2,p_4-k)(p_1,p_4-k)}\right)
\end{aligned} \tag{B6}$$

so that one gets at most two brackets with momentum k in the denominator.

In the case of momentum k in the numerator, this procedure also works but with some variation. For example, one has

$$J = \frac{(p_1, k)}{(p_1, p_2)(p_1, p_3)(p_3, p_4 - k)(p_2, k)(k, p_4 - k)} = \frac{1}{(p_1, p_2)(p_1, p_3)(p_4^2/2)} \frac{(p_3, k) + p_4^2/2 - (p_2, k)}{(p_2, k)(p_3, p_4 - k)}$$
$$= \frac{1}{(p_1, p_2)(p_1, p_3)(p_4^2/2)} \frac{(p_3, p_4) - (p_3, p_4 - k) + p_4^2/2 - (p_2, k)}{(p_2, k)(p_3, p_4 - k)}$$
$$= \frac{(p_3, p_4) + p_4^2/2}{(p_1, p_2)(p_1, p_3)(p_4^2/2)(p_2, k)(p_3, p_4 - k)} - \frac{1}{(p_1, p_2)(p_1, p_3)(p_4^2/2)(p_2, k)} - \frac{1}{(p_1, p_2)(p_1, p_3)(p_4^2/2)(p_3, p_4 - k)}. \tag{B7}$$

Since we usually have $(p_i, k)^4$ in the numerator this procedure has to be applied several times. This way we increase the number of terms in the integrand but drastically simplify the integration.

The resulting integrals over k have the standard form

$$\int d^D k \delta^+((p_4 - k)^2)\delta^+(k^2)\Upsilon_i, \tag{B8}$$

where

$$\Upsilon_1 = \frac{1}{(p_i, k)^a (p_j, k)^b}, \qquad \Upsilon_2 = \frac{1}{(p_i, k)^a (p_j, p_4 - k)^b},$$
$$\Upsilon_3 = \frac{1}{(p_i, p_4 - k)^a (p_j, p_4 - k)^b},$$

and can be calculated by the method of unitarity. They correspond to the box-type diagrams and one can perform the cuts and then take the imaginary part. For example, the integral

$$\int \frac{d^D k \delta^+(k^2)\delta^+((p_4 - k)^2)}{(p_1 + k)^2 (p_2 - p_4 + k)^2}, \tag{B9}$$

where $p_1^2 = p_2^2 = p_3^2 = 0$ and $p_4^2 \neq 0$ can be obtained from the box diagram shown in Fig. 6.

For the first time this integral was calculated by van Neerven [65] and the answer is given by

$$\int d^D k \delta^+((p_4 - k)^2)\delta^+(k^2)\Upsilon_i = \frac{(p_4^2/4)^{-\epsilon}}{(p_i p_4)^a (p_j p_4)^b} \Theta(p_4^2) 2\pi 2^{-a-b} \frac{\Gamma(D-3)\Gamma(D/2 - 1 - a)\Gamma(D/2 - 1 - b)}{\Gamma^2(D/2 - 1)\Gamma(D - 2 - a - b)} \times {}_2F_1\left(a, b; \frac{D}{2} - 1 | \tilde{\Upsilon}_i\right), \tag{B10}$$

where

$$\tilde{\Upsilon}_1 = 1 - \frac{(p_i, p_j)(p_4, p_4)}{2(p_i, p_4)(p_j, p_4)}, \qquad \tilde{\Upsilon}_2 = \frac{(p_i, p_j)(p_4, p_4)}{2(p_i, p_4)(p_j, p_4)},$$
$$\tilde{\Upsilon}_3 = 1 - \frac{(p_i, p_j)(p_4, p_4)}{2(p_i, p_4)(p_j, p_4)}.$$

Removing the integral over $d^D p_4$ with the help of the delta function we are left with the last integration over $d^D p_3$. Using $\delta^+(p_3^2)$ one can take the integral over p_3^0 and going to the spherical coordinates $d^{D-1}\mathbf{p}_3 = |\mathbf{p}_3|^{D-2} d|\mathbf{p}_3| d\Omega_{13}$ arrive to the single integration over the modulus of $|\mathbf{p}_3|$.

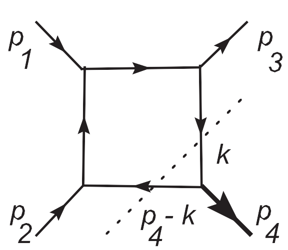

FIG. 6. The box diagram corresponding to the integral (B9).

Here we face the problem of singularity at $|\mathbf{p}_3| = 0$. It comes from the delta function in the integration over p_3^0 and in some cases is not compensated by the matrix element. For two matrix elements corresponding to the MHV $(g^+ g^+ \to g^+ g^+ g^+)$ and anti-MHV $(g^+ g^+ \to g^+ g^+ g^-)$ amplitudes, the first case is singular while the second is not. However, as we explained earlier, we cut the integral over $|\mathbf{p}_3|$ at $(1 - \delta)E/2$ and no singularity appears.

Let us now turn to the calculation of the last integral. Since p_3 is a dimensionful parameter, it is appropriate to change the variable to a dimensionless one using

$$|\mathbf{p}_3| = \frac{E}{2}(1 - x). \tag{B11}$$

Then the integral over x goes from 0 to δ.

The typical integral to be calculated is of the form

$$\int_0^\delta dx \frac{x^{\alpha-1}(1-x)^{\beta-1}}{1 + px} {}_2F_1(1, -\epsilon; 1 - \epsilon; qx^m(1-x)^n), \tag{B12}$$

where m and n take the values

$$(m, n) = \{(1, 0), (0, 2), (1, -2), (-1, 2)\}.$$

For our purposes we need to calculate this integral to the order $\mathcal{O}(\epsilon)$. The source of divergence is the singularity at $x = 0$. When $\delta \neq 1$, one can expand the hypergeometric function in ϵ up to the order ϵ and then calculate the integral. Then for the configurations (1, 0), (0, 2), and $(1, -2)$ the calculation is straightforward while for the case of $(-1, 2)$ one first makes the transformation of the argument of the hypergeometric function from z to $1/z$

$$ {}_2F_1(a, b; c|z) = \frac{\Gamma(c)\Gamma(-a+b)}{\Gamma(c-a)\Gamma(b)}\, {}_2F_1\left(a, a-c+1; a-b+1\Big|\frac{1}{z}\right)(-z)^{-a} + \frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\, {}_2F_1\left(b, -c+b+1; -a+b+1\Big|\frac{1}{z}\right)(-z)^{-b} \tag{B13}$$

and then apply the expansion.

For example, consider the integral

$$I_2 = \int_0^\delta dx x^{-1-\epsilon}(1-x)^{-2\epsilon}\, {}_2F_1(1, -\epsilon; 1-\epsilon; qx^{-1}(1-x)^2). \tag{B14}$$

After applying the transformation (B13), it is reduced to the following form:

$$I_2 = \int_0^\delta dx\left(\Gamma(1-\epsilon)\Gamma(1+\epsilon)\left(\frac{q(1-x)^2}{x}\right)^\epsilon + \frac{x\Gamma(-1-\epsilon)\Gamma(1-\epsilon)\,{}_2F_1(1, 1+\epsilon; 2+\epsilon; -\frac{x}{q(1-x)^2})}{q(1-x)^2\Gamma(-\epsilon)^2}\right). \tag{B15}$$

Performing the expansion over ϵ one gets to the order of $\mathcal{O}(\epsilon)$

$$\begin{aligned} I_2 = {} & -\frac{1}{2\epsilon} + \left(\log\delta - \frac{\log q}{2}\right) + \Bigg[-\log^2\delta + \log q \log\delta - \log((\delta-1)^2 q)\log\delta - \log\left(\frac{2(\delta-1)q - \sqrt{1-4q}+1}{2q}\right)\log\delta \\ & - \log\left(\frac{2(\delta-1)q + \sqrt{1-4q}+1}{2q}\right)\log\delta + \log(q\delta^2 - 2q\delta + \delta + q)\log\delta - \frac{\log^2 q}{4} - 2\mathrm{Li}_2(1-\delta) - \frac{\pi^2}{12} \\ & - \log\left(\frac{-2q+\sqrt{1-4q}+1}{2q}\right)\log\left(-\frac{2q}{-2q+\sqrt{1-4q}+1}\right) - \log\left(\frac{2q}{2q+\sqrt{1-4q}-1}\right)\log\left(-\frac{2q+\sqrt{1-4q}-1}{2q}\right) \\ & + \log\left(\frac{2\delta q}{2q+\sqrt{1-4q}-1}\right)\log\left(\frac{2(\delta-1)q - \sqrt{1-4q}+1}{2q}\right) + \log\left(-\frac{2\delta q}{-2q+\sqrt{1-4q}+1}\right) \\ & \times \log\left(\frac{2(\delta-1)q + \sqrt{1-4q}+1}{2q}\right) + \mathrm{Li}_2\left(\frac{2(\delta-1)q+\sqrt{1-4q}+1}{-2q+\sqrt{1-4q}+1}\right) + \mathrm{Li}_2\left(\frac{-2\delta q + 2q + \sqrt{1-4q}-1}{2q+\sqrt{1-4q}-1}\right)\Bigg]\epsilon. \end{aligned}$$

Note the singularity when $\delta \to 1$ in this expression.

The case of $\delta = 1$ is more tricky. Here one has the overlapping of two singularities. The argument of the hypergeometric function goes to the edge of the circle of convergence and it is convenient to use the integral representation

$${}_2F_1(a, b, c, z) = \frac{\Gamma(c)\Gamma(b)}{\Gamma(c-b)}\int_0^1 dt t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}. \tag{B16}$$

As a result, one has a two-fold integral

$$\frac{\Gamma(c)\Gamma(b)}{\Gamma(c-b)}\int_0^1 dx dt t^{b-1}(1-t)^{c-b-1}x^{\alpha-1}(1-x)^{\beta-1} \times (1 - tqx^m(1-x)^n)^{-a}, \tag{B17}$$

where parameters a, b, and c take the values $a = 1$, $b = -\epsilon$, $c = 1 - \epsilon$. Choosing particular values of α, β, m, n one can observe the overlapping divergencies. Consider, for example, the integral

$$\int_0^1 dx dt t^{-1-\epsilon}x^{-1-\epsilon}(1-x)^{-2\epsilon}\frac{1}{1 - qt\frac{(1-x)^2}{x}} = \int_0^1 dx dt t^{-1-\epsilon}x^{-\epsilon}(1-x)^{-2\epsilon}\frac{1}{x - qt(1-x)^2}, \tag{B18}$$

where we see in the last term that the denominator equals zero at $t = 0$ and $x = 0$. The divergence, which occurs in this case, is the overlapping IR divergence and to handle it we use the following trick: we insert in the integral a unity

$$1 = \Theta(x - t) + \Theta(t - x),$$

which splits the integral into two parts. The first θ function gives

$$\int_0^1 dxdzz^{-1-\epsilon}x^{-1-2\epsilon}(1-x)^{-2\epsilon}\frac{1}{1-qz(1-x)^2}, \quad \text{(B19)}$$

while the other leads to

$$\int_0^1 dzdtt^{-1-2\epsilon}z^{-\epsilon}(1-zt)^{-2\epsilon}\frac{1}{z-q(1-zt)^2}. \quad \text{(B20)}$$

The calculation now is straightforward. One has to extract a few terms of the ϵ expansion.

For example, the first three terms of the ϵ expansion for the integral (B14) when $\delta = 1$ are

$$\int_0^1 dxx^{-1-\epsilon}(1-x)^{-2\epsilon}{}_2F_1(1,-\epsilon;1-\epsilon;qx^{-1}(1-x)^2)$$
$$= -\frac{1}{\epsilon} - \mathrm{Li}_2\Big(\frac{2}{\sqrt{\frac{1}{q}-4}\sqrt{\frac{1}{q}}-\frac{1}{q}+2}\Big)\epsilon$$
$$- \mathrm{Li}_2\Big(-\frac{2}{\sqrt{\frac{1}{q}-4}\sqrt{\frac{1}{q}}+\frac{1}{q}-2}\Big)\epsilon + \frac{2\pi^2\epsilon}{3} + O(\epsilon^2). \quad \text{(B21)}$$

APPENDIX C: SPLITTING FUNCTIONS

The splitting functions P_{ij} which we use to calculate the splitting contribution to the cross section can be obtained from the collinear limit of the color-ordered tree-level partial amplitudes. Suppose one has an n-point partial tree amplitude in $0 \le \mathcal{N} \le 4$ supersymmetric gauge theory

$$A_n^{(\mathrm{tree})}(p_{a(1)}^{\lambda_1}, \ldots, p_{a(i)}^{\lambda_i}, \ldots, p_{a(n)}^{\lambda_n}),$$

where $a(i)$ is the color index of i-th particle and λ_i is its helicity.

It can be shown [5,6] that the MHV amplitudes have the following universal behavior in the collinear limit when momenta of two particles i and $i+1$ become collinear $i\|i+1$:

$$A_n^{\mathrm{tree}}(\ldots, p_a^{\lambda_i}, p_b^{\lambda_{i+1}}, \ldots) \stackrel{i\|i+1}{\longrightarrow} \sum_{\lambda,c} \mathrm{split}_{-\lambda}(a^{\lambda_i}, b^{\lambda_{i+1}}, z) \times A_{n-1}^{\mathrm{tree}}(\ldots, p_c^{\lambda}, \ldots), \quad \text{(C1)}$$

where the two momenta satisfy

$$p_i = zp, \qquad p_{i+1} = (1-z)p,$$

p being some arbitrary momentum. The sum goes over all possible helicities and particle types for which A_{n-1}^{tree} is nonvanishing. The function $\mathrm{split}_{-\lambda}(a^{\lambda_i}, b^{\lambda_{i+1}}, z)$ depends on p and z. Notice the flip of helicity in $\mathrm{split}_{-\lambda}(a^{\lambda_i}, b^{\lambda_{i+1}}, z)$ which comes from considering all particles as outgoing ones.

Then the polarized version of the splitting function P_{ij} can be obtained from $\mathrm{split}_{-\lambda}$, up to the terms proportional to $\delta(1-z)$ by means of

$$P^{c^{-\lambda}}_{a^{\lambda_i}\,b^{\lambda_{i+1}}} = (p_i + p_{i+1})^2|\mathrm{split}_{-\lambda}(a^{\lambda_i}, b^{\lambda_{i+1}}, z)|^2 \quad \text{(C2)}$$

and corresponds to the process $c \to i,\ i+1$ when the particle c with momentum p and helicity λ splits into collinear particles i and $i+1$ with momenta zp and $(1-z)p$ and helicities λ_i and λ_{i+1}, respectively.

For example, the splitting function $P^{g^-}_{g^+g^+}$ can be obtained from the partial amplitude $A_5^{\mathrm{tree}}(g^-g^-g^+g^+g^+)$ taking the limit $4\|5$ $(p_4 = zp,\ p_5 = (1-z)p)$ in

$$A_5^{\mathrm{tree}}(g^-g^-g^+g^+g^+) = \frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 34\rangle\langle 45\rangle\langle 51\rangle}. \quad \text{(C3)}$$

One has

$$A_5^{\mathrm{tree}}(g^-g^-g^+g^+g^+) \stackrel{4\|5}{\longrightarrow} \frac{1}{\langle 45\rangle}\frac{1}{\sqrt{z(1-z)}}\frac{\langle 12\rangle^4}{\langle 12\rangle\langle 23\rangle\langle 3p\rangle\langle p1\rangle}.$$

Thus, the only one term in the sum (C1) survives and A_{n-1}^{tree} in this case is $A_4^{\mathrm{tree}}(g^-g^-g^+g^+)$. This gives

$$\mathrm{split}_-(g^+, g^+, z) = \frac{1}{\langle 45\rangle}\frac{1}{\sqrt{z(1-z)}}, \quad \text{(C4)}$$

so that, according to (C2),

$$P^{g^-}_{g^+g^+} = \frac{1}{z} + \frac{1}{(1-z)_+}. \quad \text{(C5)}$$

All the splitting functions necessary for our computation can be obtained in a similar fashion. They look like

$$P^{g^-}_{g^+g^+} = \frac{1}{z} + \frac{1}{(1-z)_+}, \qquad P^{g^-}_{g^+g^-} = \frac{z^3}{(1-z)_+},$$
$$P^{g^-}_{g^-g^+} = \frac{(1-z)^3}{z}, \qquad P^{g^-}_{q^+\bar{q}^-} = z^2, \quad \text{(C6)}$$
$$P^{g^-}_{\bar{q}^-q^+} = (1-z)^2, \qquad P^{g^-}_{\Lambda\Lambda} = z(1-z).$$

The contributions proportional to $\delta(1-z)$ are calculated separately from the requirement of conservation of momenta and are absent in our case since they are proportional to the β function which vanishes in the $\mathcal{N} = 4$ SYM theory.

The "plus" prescription in the expression $\frac{1}{(1-z)_+}$ in (C6) should be understood in the usual way:

$$\int_0^1 dz\frac{f(z)}{(1-z)_+} = \int_0^1 dz\frac{f(z)-f(1)}{(1-z)}. \quad \text{(C7)}$$

When $f(z)$ contains the theta function like in the splitting counterterm

$$f(z) = \Theta(z - z_{\mathrm{min}})g(z),$$

one has

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{\Theta(z-z_{\min})g(z) - g(1)}{(1-z)}$$
$$= \int_{z_{\min}}^1 dz \frac{g(z)-g(1)}{(1-z)} - \int_0^{z_{\min}} dz \frac{g(1)}{1-z}$$
$$= \int_{z_{\min}}^1 dz \frac{g(z)}{(1-z)_+} + \log(1-z_{\min})g(1). \quad \text{(C8)}$$

The splitting function $P_{qq}(z)$ (3.8) used in our toy model example can be obtained from the polarized splitting functions

$$P_{\bar{q}^- g^-}^{q^+} = \frac{1}{(1-z)_+}, \qquad P_{\bar{q}^- g^+}^{q^+} = \frac{z^2}{(1-z)_+} \quad \text{(C9)}$$

by summation over helicities. The term proportional to $\delta(1-z)$ is obtained from the requirement of conservation of the number of quarks

$$\int_0^1 dz q(z, Q_f^2/\mu^2) = 1 \Rightarrow \int_0^1 dz P_{qq}(z) = 0.$$

APPENDIX D: FINITE PARTS OF AMPLITUDES

In general all finite parts have the following structure:

$$\text{finite part } = \frac{1}{(1-c^2)^2}[f_{\text{Sym}}(c,\delta) + (f_{\text{Asym}}(c,\delta) + f_{\text{Asym}}(-c,\delta))],$$

where the functions f_{Sym} and f_{Asym} contain log[11] and polylog functions of c and δ. Below we present the expressions for $f_{\text{Sym}}(c,\delta)$ and $f_{\text{Asym}}(c,\delta)$.

$((\frac{d\sigma_{2\to3}}{d\Omega_{13}})^{(--+++)}_{\text{Real}})_{\text{fin}}$, *general* δ.

$$f_{\text{Sym}}^{(--+++)}(c,\delta) = \frac{\mathcal{S}_1 + \mathcal{S}_2 L(1-\delta) + \mathcal{S}_3 L(\delta)}{(1-\delta)^2} - 4(13+3c^2)L(\delta)L(1-\delta) + 10(3+c^2)L^2(\delta) - 4(5+c^2)L\left(\frac{1-c}{2}\right)L\left(\frac{1+c}{2}\right) - 16(3+c^2)L^2(1-\delta) - 4(9c^2+35)\text{Li}_2(\delta), \quad \text{(D1)}$$

where

$$\mathcal{S}_1 = \frac{4}{3}(3+c^2)\pi^2(1-\delta)^2 + 32(4-3\delta)\delta, \qquad \mathcal{S}_2 = 4(3c^2(1-\delta)^2 + 37 - 26\delta + 5\delta^2),$$
$$\mathcal{S}_3 = -4\delta(c^2(\delta-1) + 11\delta - 15).$$

$$f_{\text{Asym}}^{(--+++)}(c,\delta) = \frac{1}{(1-\delta)^2}\left(\mathcal{A}_1 L\left(\frac{1-c}{2}\right) + \mathcal{A}_2 L\left(\frac{1+\delta-c(1-\delta)}{2}\right)\right) - 2(-1+4c+c^2)L^2\left(\frac{1-c}{2}\right) - 8(3+2c+c^2)L(1-\delta)L\left(\frac{1-c}{2}\right) + 16cL(\delta)L\left(\frac{1-c}{2}\right) + 4(1+c)^2 L\left(\frac{1-c}{2}\right)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) + 4(1+c)^2 L(1-\delta)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - 4(1+c)^2 L(\delta)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) + 8(1+c)\text{Li}_2\left(\frac{1-c}{2}\right) + 4(5+2c+c^2)\text{Li}_2\left(-\frac{\delta(1-c)}{1+c}\right) - 4(5+2c+c^2)\text{Li}_2\left(\frac{(1-\delta)(1-c)}{2}\right), \quad \text{(D2)}$$

where

$$\mathcal{A}_1 = 4(c^2(1-\delta)^2 - 6c(1-\delta)^2 + 5 + 2\delta - 3\delta^2), \qquad \mathcal{A}_2 = -4(c^2(1-\delta)^2 - 2c(3-4\delta+\delta^2) + 5 + 2\delta - 3\delta^2).$$

$((\frac{d\sigma_{2\to3}}{d\Omega_{13}})^{(--++-)}_{\text{Real}})_{\text{fin}}$, *general* δ.

$$f_{\text{Sym}}^{(--++-)}(c,\delta) = \mathcal{S}_1 + \mathcal{S}_2 L(\delta) + \mathcal{S}_3 L(1-\delta) + 10(c^2+3)L^2(\delta) - 2(37+18c^2+c^4)\text{Li}_2(\delta) + \frac{8(6+9c^2+c^4)}{(1-c^2)}L\left(\frac{1-c}{2}\right)L\left(\frac{1+c}{2}\right), \quad \text{(D3)}$$

[11]To make the expressions more compact we use L for the logarithms.

where

$$\mathcal{S}_1 = \frac{8(3c^2+5)\delta^3 + 3(7c^2-95)\delta^2 + 6(67c^2+513)\delta}{9} + \frac{2}{3}(c^2+3)\pi^2 + \frac{64(11c^2+7)}{3(c^2-1)} + \frac{32(c^3+12c^2+19c+9)}{3(1-c)(1+\delta+c(1-\delta))} - \frac{32(2c^3-12c^2+19c-9)}{3(1+c)(1+\delta-c(1-\delta))} - \frac{32(c^3+4c^2+5c+2)}{3(1-c)(1+\delta+c(1-\delta))^2} + \frac{32(c^3-4c^2+5c-2)}{3(1+c)(1+\delta-c(1-\delta))^2},$$

$$\mathcal{S}_2 = -\frac{16}{3}(c^2+1)\delta^3 + 2(c^2+19)\delta^2 - \frac{32(c^4+6c^2-5)\delta}{c^2-1} + \frac{64(12c^2+17)}{3(c^2-1)} + \frac{32(c^3+5c^2+11c+7)}{(1-c)(1+\delta+c(1-\delta))} - \frac{32(c^3-5c^2+11c-7)}{(1+c)(1+\delta-c(1-\delta))} - \frac{32(c^3+4c^2+5c+2)}{(1-c)(1+\delta+c(1-\delta))^2} + \frac{32(c^3-4c^2+5c-2)}{(1+c)(1+\delta-c(1-\delta))^2} + \frac{64(c^3+3c^2+3c+1)}{3(1-c)(1+\delta+c(1-\delta))^3} - \frac{64(c^3-3c^2+3c-1)}{3(1+c)(1+\delta-c(1-\delta))^3},$$

$$\mathcal{S}_3 = -\frac{8}{3}(3c^2+5)\delta^3 - (c^4+2c^2-75)\delta^2 - \frac{32(c^4+8c^2-11)\delta}{c^2-1} + \frac{3c^6+251c^4+2953c^2+313}{3(c^2-1)} - \frac{64(c^3-5c^2+11c-7)}{(1+c)(1+\delta-c(1-\delta))} + \frac{64(c^3+5c^2+11c+7)}{(1-c)(1+\delta+c(1-\delta))} + \frac{64(c^3-4c^2+5c-2)}{(1+c)(1+\delta-c(1-\delta))^2} - \frac{64(c^3+4c^2+5c+2)}{(1-c)(1+\delta+c(1-\delta))^2} + \frac{128(1+c)^3}{3(1-c)(1+\delta+c(1-\delta))^3} + \frac{128(1-c)^3}{3(1+c)(1+\delta-c(1-\delta))^3}.$$

$$f_{\rm Asym}^{(--++-)}(c,\delta) = \frac{8(c^4-6c^3+24c^2+6c-17)}{(1+c)^2}L(\delta)L\left(\frac{1-c}{2}\right) + \frac{4(3+c^2)^2}{(1+c)^2}L\left(\frac{1-c}{2}\right)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - \frac{4(7+c^2)(1-c)}{(1+c)}L\left(\frac{1+c}{2}\right)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - \frac{8(c^4-12c^3+34c^2+12c-43)}{(1+c)^2}L(1-\delta)L \times\left(\frac{1+\delta-c(1-\delta)}{2}\right) - \frac{2(c^4-2c^3+8c^2-6c+15)}{(1+c)^2}L^2\left(\frac{1-c}{2}\right) + \mathcal{A}_1 L\left(\frac{1-c}{2}\right) + \mathcal{A}_2 L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - \frac{4(3c^4-12c^3+46c^2+12c-33)}{(1+c)^2}L(\delta)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) + \frac{(c^6-2c^5+3c^4-76c^3-153c^2+14c+149)}{(1-c)^2}{\rm Li}_2\left(-\frac{1-c}{1+c}\delta\right) + \frac{8(c^4+12c^3+34c^2-12c-43)}{(1-c)^2}\left({\rm Li}_2\left(\frac{1-c}{2}\right) - {\rm Li}_2\left(\frac{(1-\delta)(1-c)}{2}\right)\right), \quad \text{(D4)}$$

where

$$\mathcal{A}_1 = -\frac{1}{6(1+c)}(2597 + 240\delta - 105\delta^2 + 24\delta^3 + 3c^5(\delta^2-1) + 3c^4(3+\delta^2) - 2c^3(111 - 24\delta + 9\delta^2 - 4\delta^3) + 2c^2(489 + 120\delta - 69\delta^2 + 20\delta^3) - c(2655 - 240\delta + 225\delta^2 - 56\delta^3)),$$

$$\mathcal{A}_2 = \frac{1}{6(1+c)}(3(\delta^2-1)c^5 + 3(\delta^2+3)c^4 + 2(8\delta^3 - 3\delta^2 + 36\delta - 111)c^3 + 6(8\delta^3 - 17\delta^2 + 28\delta + 163)c^2 + 3(16\delta^3 - 63\delta^2 + 104\delta - 885)c + 16\delta^3 - 93\delta^2 + 216\delta + 2597).$$

In the case $\delta = 1$ one gets major simplifications:

$$f_{\rm Sym}^{(--++-)}(c,1) = \frac{2257 - 93\pi^2 - 3c^4\pi^2 + c^2(303 - 48\pi^2)}{9} + \frac{8(6+9c^2+c^4)}{1-c^2}L\left(\frac{1-c}{2}\right)L\left(\frac{1+c}{2}\right), \quad \text{(D5)}$$

$$f_{\rm Asym}^{(--++-)}(c,1) = \frac{\mathcal{A}_1}{1+c}L\left(\frac{1-c}{2}\right) + \frac{\mathcal{A}_2}{(1+c)^2}L^2\left(\frac{1-c}{2}\right) + \frac{\mathcal{A}_3}{1-c}{\rm Li}_2\left(\frac{1-c}{2}\right), \quad \text{(D6)}$$

where

$$\mathcal{A}_1 = -\frac{2}{3}(3c^4 - 46c^3 + 280c^2 - 646c + 689), \qquad \mathcal{A}_2 = -\frac{1}{2}(c^6 + 2c^5 + 7c^4 + 68c^3 - 121c^2 - 38c + 209),$$
$$\mathcal{A}_3 = c^5 - c^4 - 6c^3 - 178c^2 - 603c - 493.$$

$((\frac{d\sigma_{2\to 3}}{d\Omega_{13}})^{(--+q\bar{q})}_{\text{Real}})_{\text{fin}}$, $\delta = 1$.

$$f^{(--+q\bar{q})}_{\text{Sym}}(c,1) = -\frac{32(c^2+1)^2}{(1-c^2)}L\left(\frac{1-c}{2}\right)L\left(\frac{1+c}{2}\right) - \frac{4\pi^2(1-c^4)+132(c^2+3)}{3}, \tag{D7}$$

$$f^{(--+q\bar{q})}_{\text{Asym}}(c,1) = \frac{\mathcal{A}_1}{(1+c)}L\left(\frac{1-c}{2}\right) + \frac{\mathcal{A}_2}{(1+c)}L^2\left(\frac{1-c}{2}\right) + \frac{\mathcal{A}_3}{(1-c)}\text{Li}_2\left(\frac{1-c}{2}\right), \tag{D8}$$

where

$$\mathcal{A}_1 = \frac{8}{3}(3c^4 - 44c^3 + 222c^2 - 450c + 277), \qquad \mathcal{A}_2 = -2(c^4 + 2c^3 - 2c^2 + 50c - 67)(1-c),$$
$$\mathcal{A}_3 = -4(c^4 - 2c^3 - 18c^2 - 146c - 211)(1+c).$$

$((\frac{d\sigma_{2\to 3}}{d\Omega_{13}})^{(--+\Lambda\Lambda)}_{\text{Real}})_{\text{fin}}$, $\delta = 1$.

$$f^{(--+\Lambda\Lambda)}_{\text{Sym}}(c,1) = -24(c^2+1)L\left(\frac{1-c}{2}\right)L\left(\frac{1+c}{2}\right) + 6(11c^2-3) - \pi^2(1-c^2)^2, \tag{D9}$$

$$f^{(--+\Lambda\Lambda)}_{\text{Asym}}(c,1) = -\frac{3(c+5)(c^2-2c+9)(1-c)^2}{2(1+c)}L^2\left(\frac{1-c}{2}\right) - \frac{2(3c^4 - 47c^3 + 213c^2 - 369c + 184)}{1+c}L\left(\frac{1-c}{2}\right) + \frac{3(c^3 - 3c^2 - 17c - 125)(1+c)^2}{(1-c)}\text{Li}_2\left(\frac{1-c}{2}\right). \tag{D10}$$

$((\frac{d\sigma_{2\to 3}}{d\Omega_{13}})^{(--+++)}_{\text{In Split}})_{\text{fin}}$, *general* δ.

$$f^{(--+++)}_{\text{Sym}}(c,\delta) = 16\frac{\delta(3\delta-4)}{(1-\delta)^2} + \frac{32(\delta-2)}{(1-\delta)^2}L(1-\delta) + 8(3+c^2)L^2(1-\delta), \tag{D11}$$

$$f^{(--+++)}_{\text{Asym}}(c,\delta) = 4(c^2+3)\left(L^2\left(\frac{1-c}{2}\right) - 2L\left(\frac{1-c}{2}\right)L\left(\frac{1+\delta-c(1-\delta)}{2}\right) + L^2\left(\frac{1+\delta-c(1-\delta)}{2}\right) + \text{Li}_2\left(\frac{1-c}{2}\right) - 2\text{Li}_2\left(-\frac{(1-c)(1-\delta)}{2}\right) + 2\text{Li}_2\left(-\frac{(1-c)\delta}{1+c}\right) + 2\text{Li}_2\left(\frac{2\delta}{1+\delta-c(1-\delta)}\right)\right). \tag{D12}$$

$((\frac{d\sigma_{2\to 3}}{d\Omega_{13}})^{(--++-)}_{\text{In Split}})_{\text{fin}}$, *general* δ.

$$f^{(--++-)}_{\text{Sym}}(c,\delta) = \mathcal{S}_1 L(1-\delta) + \mathcal{S}_2, \tag{D13}$$

where

$$\mathcal{S}_1 = \frac{16}{3}(c^2+1)\delta^3 + 8(c^2-5)\delta^2 + 16(c^2+17)\delta - \frac{8(27c^4+378c^2+59)}{3(c^2-1)} - \frac{64(c^3+5c^2+11c+7)}{(1-c)(1+\delta+c(1-\delta))} + \frac{64(c^3-5c^2+11c-7)}{(1+c)(1+\delta-c(1-\delta))} + \frac{64(c^3+4c^2+5c+2)}{(1-c)(1+\delta+c(1-\delta)^2} - \frac{64(c^3-4c^2+5c-2)}{(1+c)(1+\delta-c(1-\delta))^2} - \frac{128(c^3+3c^2+3c+1)}{3(1-c)(1+\delta+c(1-\delta))^3} + \frac{128(c^3-3c^2+3c-1)}{3(1+c)(1+\delta-c(1-\delta))^3},$$

$$\mathcal{S}_2 = -\frac{16}{9}(c^2+1)\delta^3 - \frac{4}{3}(5c^2-13)\delta^2 - \frac{8}{3}(11c^2+89)\delta - \frac{256(2c^2+1)}{3(c^2-1)} - \frac{32(2c^3+9c^2+12c+5)}{3(1-c)(1+\delta+c(1-\delta))} + \frac{32(2c^3-9c^2+12c-5)}{3(1+c)(1+\delta-c(1-\delta))} + \frac{32(c^3+3c^2+3c+1)}{3(1-c)(1+\delta+c(1-\delta))^2} - \frac{32(c^3-3c^2+3c-1)}{3(1+c)(1+\delta-c(1-\delta))^2},$$

$$f_{\text{Asym}}^{(--++-)}(c,\delta) = \frac{16(1-c)(4c^2-17c+37)}{3(1+c)}\left(L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - L\left(\frac{1-c}{2}\right)\right) + 4(3+c^2)\left(L\left(\frac{1+\delta-c(1-\delta)}{2}\right) - L\left(\frac{1-c}{2}\right)\right)^2 + \frac{8(c^3-15c^2+51c-45)}{1+c}L(1-\delta)L \times \left(\frac{1+\delta-c(1-\delta)}{2}\right) + 8(3+c^2)\left(\mathrm{Li}_2\left(-\frac{1-c}{1+c}\delta\right) - \mathrm{Li}_2\left(\frac{2\delta}{1+\delta-c(1-\delta)}\right)\right) + \frac{8(c^3+15c^2+51c+45)}{1-c}\left(\mathrm{Li}_2\left(\frac{1-c}{2}\right) - \mathrm{Li}_2\left(\frac{1}{2}(1-c)(1-\delta)\right)\right). \tag{D15}$$

In the case $\delta = 1$ one gets major simplifications:

$$f_{\text{Sym}}^{(--++-)}(c,1) = \frac{4}{9}((6\pi^2-49)c^2 + 18\pi^2 - 415), \tag{D16}$$

$$f_{\text{Asym}}^{(--++-)}(c,1) = \frac{16(1-c)(4c^2-17c+37)}{3(1+c)}L\left(\frac{1-c}{2}\right) + \frac{16(1+c)(c^2+6c+21)}{1-c}\mathrm{Li}_2\left(\frac{1-c}{2}\right). \tag{D17}$$

$\left(\left(\frac{d\sigma_{2\to3}}{d\Omega_{13}}\right)_{\text{In Split}}^{(--+q\bar{q})}\right)_{\text{fin}}$, $\delta = 1$

$$f_{\text{Sym}}^{(--+q\bar{q})}(c,1) = \frac{16}{3}(9c^2+23), \tag{D18}$$

$$f_{\text{Asym}}^{(--+q\bar{q})}(c,1) = -\frac{64(4c^2-17c+19)(1-c)}{3(1+c)}L\left(\frac{1-c}{2}\right) - \frac{64(c+3)^2(1+c)}{(1-c)}\mathrm{Li}_2\left(\frac{1-c}{2}\right). \tag{D19}$$

$\left(\left(\frac{d\sigma_{2\to3}}{d\Omega_{13}}\right)_{\text{In Split}}^{(--+\Lambda\Lambda)}\right)_{\text{fin}}$, $\delta = 1$

$$f_{\text{Sym}}^{(--+\Lambda\Lambda)}(c,1) = -16(3c^2-1), \tag{D20}$$

$$f_{\text{Asym}}^{(--+\Lambda\Lambda)}(c,1) = \frac{16(13-4c)(1-c)^2}{(1+c)}L\left(\frac{1-c}{2}\right) + \frac{48(c+5)(1+c)^2}{(1-c)}\mathrm{Li}_2\left(\frac{1-c}{2}\right). \tag{D21}$$

[1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998); Int. J. Theor. Phys. **38**, 1113 (1999); S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998); E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).

[2] Z. Bern, M. Czakon, L. J. Dixon, D. A. Kosower, and V. A. Smirnov, Phys. Rev. D **75**, 085010 (2007); F. Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **75**, 105011 (2007).

[3] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Nucl. Phys. **B636**, 99 (2002).
[4] S. Frolov and A. A. Tseytlin, J. High Energy Phys. 06 (2002) 007.
[5] S. J. Parke and T. R. Taylor, Phys. Rev. Lett. **56**, 2459 (1986); F. A. Berends and W. T. Giele, Nucl. Phys. **B306**, 759 (1988).
[6] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B425**, 217 (1994).
[7] Z. Bern, L. J. Dixon, D. C. Dunbar, and D. A. Kosower, Nucl. Phys. **B435**, 59 (1995).
[8] Z. Bern, L. J. Dixon, and D. A. Kosower, Annu. Rev. Nucl. Part. Sci. **46**, 109 (1996).
[9] M. B. Green, J. H. Schwarz, and L. Brink, Nucl. Phys. **B198**, 474 (1982).
[10] C. Anastasiou, Z. Bern, L. J. Dixon, and D. A. Kosower, Phys. Rev. Lett. **91**, 251602 (2003).
[11] Z. Bern, L. J. Dixon, and V. A. Smirnov, Phys. Rev. D **72**, 085001 (2005).
[12] A. H. Mueller, Phys. Rev. D **20**, 2037 (1979); J. C. Collins, Phys. Rev. D **22**, 1478 (1980); A. Sen, Phys. Rev. D **24**, 3281 (1981); G. P. Korchemsky, Phys. Lett. B **217**, 330 (1989); L. Magnea and G. Sterman, Phys. Rev. D **42**, 4222 (1990); G. P. Korchemsky, Phys. Lett. B **220**, 629 (1989).
[13] S. V. Ivanov, G. P. Korchemsky, and A. V. Radyushkin, Yad. Fiz. **44**, 230 (1986) [Sov. J. Nucl. Phys. **44**, 145 (1986)].G. P. Korchemsky and A. V. Radyushkin, Phys. Lett. B **171**, 459 (1986).
[14] R. Roiban and A. A. Tseytlin, J. High Energy Phys. 11 (2007) 016.
[15] B. Eden and M. Staudacher, J. Stat. Mech. 11 (2006) P014; N. Beisert, B. Eden, and M. Staudacher, J. Stat. Mech. 01 (2007) P021.
[16] B. Basso, G. P. Korchemsky, and J. Kotanski, Phys. Rev. Lett. **100**, 091601 (2008).
[17] F. Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **76**, 106004 (2007).
[18] Cachazo, M. Spradlin, and A. Volovich, Phys. Rev. D **74**, 045020 (2006); Z. Bern, M. Czakon, D. A. Kosower, R. Roiban, and V. A. Smirnov, Phys. Rev. Lett. **97**, 181601 (2006).
[19] M. Spradlin, A. Volovich, and C. Wen, Phys. Rev. D **78**, 085025 (2008).
[20] L. F. Alday and J. Maldacena, J. High Energy Phys. 11 (2007) 068.
[21] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Phys. Lett. B **662**, 456 (2008).
[22] J. Bartels, L. N. Lipatov, and A. S. Vera, Phys. Rev. D **80**, 045002 (2009).
[23] Z. Bern, L. J. Dixon, D. A. Kosower, R. Roiban, M. Spradlin, C. Vergu, and A. Volovich, Phys. Rev. D **78**, 045007 (2008).
[24] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B815**, 142 (2009).
[25] J. M. Drummond, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 385 (2008).
[26] L. F. Alday and J. M. Maldacena, J. High Energy Phys. 06 (2007) 064.
[27] A. Brandhuber, P. Heslop, and G. Travaglini, Nucl. Phys. **B794**, 231 (2008).
[28] L. F. Alday and R. Roiban, Phys. Rep. **468**, 153 (2008).
[29] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B795**, 52 (2008).
[30] N. Berkovits and J. Maldacena, J. High Energy Phys. 09 (2008) 062.
[31] R. Ricci, A. A. Tseytlin, and M. Wolf, J. High Energy Phys. 12 (2007) 082; N. Beisert, R. Ricci, A. A. Tseytlin, and M. Wolf, Phys. Rev. D **78**, 126004 (2008).
[32] J. M. Drummond, J. Henn, G. P. Korchemsky, and E. Sokatchev, Nucl. Phys. **B828**, 317 (2010).
[33] J. M. Drummond, J. M. Henn, and J. Plefka, J. High Energy Phys. 05 (2009) 046; A. Gorsky, AIP Conf. Proc. **1134**, 202 (2009); Phys. Rev. D **80**, 125002 (2009).
[34] A. Gorsky and A. Zhiboedov, J. Phys. A **42**, 355214 (2009).
[35] T. Kinoshita, J. Math. Phys. (N.Y.) **3**, 650 (1962); T. D. Lee and M. Nauenberg, Phys. Rev. **133**, B1549 (1964).
[36] S. Weinberg, Phys. Rev. **140**, B516 (1965).
[37] C. L. Basham, L. S. Brown, S. D. Ellis, and S. T. Love, Phys. Rev. Lett. **41**, 1585 (1978); Phys. Rev. D **19**, 2018 (1979).
[38] N. A. Sveshnikov and F. V. Tkachov, Phys. Lett. B **382**, 403 (1996); M. Testa, J. High Energy Phys. 09 (1998) 006.
[39] G. P. Korchemsky, G. Oderda, and G. Sterman, arXiv:hep-ph/9708346; G. P. Korchemsky and G. Sterman, Nucl. Phys. **B555**, 335 (1999); A. V. Belitsky, G. P. Korchemsky, and G. Sterman, Phys. Lett. B **515**, 297 (2001).
[40] D. M. Hofman and J. Maldacena, J. High Energy Phys. 05 (2008) 012.
[41] W. L. van Neerven, Z. Phys. C **30**, 595 (1986).
[42] S. D. Ellis, Z. Kunszt, and D. E. Soper, Phys. Rev. D **40**, 2188 (1989).
[43] S. D. Ellis, Z. Kunszt, and D. E. Soper, Phys. Rev. Lett. **64**, 2121 (1990).
[44] Z. Kunszt and D. E. Soper, Phys. Rev. D **46**, 192 (1992).
[45] S. Frixione, Z. Kunszt, and A. Signer, Nucl. Phys. **B467**, 399 (1996).
[46] S. Catani and M. H. Seymour, Nucl. Phys. **B485**, 291 (1997).
[47] V. N. Gribov and L. N. Lipatov, Yad. Fiz. **15**, 781 (1972) [Sov. J. Nucl. Phys. **15**, 438 (1972)].
[48] G. Altarelli and G. Parisi, Nucl. Phys. **B126**, 298 (1977).
[49] T. Bargheer, N. Beisert, W. Galleas, F. Loebbert, and T. McLoughlin, J. High Energy Phys. 11 (2009) 056.
[50] A. Sever and P. Vieira, arXiv:0908.2437.
[51] F. Cachazo, P. Svrcek, and E. Witten, J. High Energy Phys. 10 (2004) 077.
[52] T. Muta, World Sci. Lect. Notes Phys. **5**, 1 (1987).
[53] G. Altarelli, R. K. Ellis, and G. Martinelli, Nucl. Phys. **B157**, 461 (1979).
[54] Z. Bern and D. A. Kosower, Nucl. Phys. **B379**, 451 (1992).
[55] M. L. Mangano and S. J. Parke, Phys. Rep. **200**, 301 (1991).
[56] A. V. Kotikov and L. N. Lipatov, Nucl. Phys. **B769**, 217 (2007).
[57] Z. Kunszt, A. Signer, and Z. Trocsanyi, Nucl. Phys. **B411**, 397 (1994).
[58] L. V. Bork, D. I. Kazakov, G. S. Vartanov, and A. V. Zhiboedov, Phys. Lett. B **681**, 296 (2009).

[59] L. V. Bork, D. I. Kazakov, G. S. Vartanov, and A. V. Zhiboedov in *Proceedings to the Conference "Gauge Fields: Yesterday, Today, Tomorrow," Moscow, 2010* (to be published).
[60] F. A. Berends, R. Kleiss, P. De Causmaecker, R. Gastmans, and T. T. Wu, Phys. Lett. **103B**, 124 (1981); P. De Causmaecker, R. Gastmans, W. Troost, and T. T. Wu, Nucl. Phys. **B206**, 53 (1982); R. Kleiss and W. J. Stirling, Nucl. Phys. **B262**, 235 (1985); R. Gastmans and T. T. Wu, *The Ubiquitous Photon: Helicity Method for QED and QCD* (Clarendon, Oxford, 1990).
[61] Z. Xu, D. H. Zhang, and L. Chang, Nucl. Phys. **B291**, 392 (1987).
[62] J. F. Gunion and Z. Kunszt, Phys. Lett. **161B**, 333 (1985).
[63] L. J. Dixon, arXiv:hep-ph/9601359.
[64] V. P. Nair, Phys. Lett. B **214**, 215 (1988).
[65] W. L. van Neerven, Nucl. Phys. **B268**, 453 (1986); W. Beenakker, H. Kuijf, W. L. van Neerven, and J. Smith, Phys. Rev. D **40**, 54 (1989).