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Refined Partition Functions for Open Superstrings with 4, 8 and 16 Supercharges

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Abstract: We analyse the perturbative massive open string spectrum of even dimensional superstring compactifications with four, eight and sixteen supercharges. In each of such cases, we focus on universal states that exist independently on the internal geometry and other compatification details. We analytically compute refined partition functions that count these states at each mass level. Such refined partition functions are written in a super-Poincaré covariant form, providing information on how supermultiplets transform under the little group and the R symmetry. Various asymptotic limits of the partition functions and their associated quantities, such as the leading and subleading Regge trajectories, are studied empirically and analytically. In the phenomenologically relevant case of four supercharges, the partition function can be cast into the most compact form and the asymptotic formula in the large spin limit is derived explicitly.

Contents

1 Introduction

The purpose of this article is to compute the super Poincaré covariant perturbative open superstring spectrum which is completely universal to all compactifications to 4, 6 or 8 spacetime dimensions with 4, 8 or 16 supercharges. The number of such models is, of course, enormous and generic representatives have their own characteristic spectrum. Nevertheless, for each given number of supersymmetries (SUSYs), one can identify a set of physical states that exist *independently* on the internal geometry and any other compatification details. In this sense, one of the main aims of this work is to focus on universal statements about the spectrum in scenarios with various (even) numbers of spacetime dimensions and supercharges. The basic quantity we compute in different contexts is the number of model independent super Poincaré multiplets of given Lorentz and R symmetry quantum numbers on each mass level of the superstring.

The existence of 4, 8 and 16 supercharges is compatible with various spacetime dimensions. Theories with a fixed number of supercharges are related to each other through dimensional reduction. Note that the minimum number of supercharges existing in 4, 6 and 10 dimensions is 4, 8 and 16, labelled by $\mathcal{N}_{4d} = 1$, $\mathcal{N}_{6d} = (1, 0)$ and $\mathcal{N}_{10d} = 1$ respectively. From each of such theories, one can therefore obtain theories with the same amount of SUSYs in lower dimensions via toroidal compactifications which preserve all the SUSYs [1]. In this paper, Kaluza-Klein and winding modes are neglected, as these depend on compactification details. Thus, determining the lower dimensional spectra becomes a group theoretical problem of branching the associated Lorentz and R symmetry groups. The following Figure 1 gives an overview of the supersymmetric theories for which we will work out the model independent subset of the open superstring spectrum.

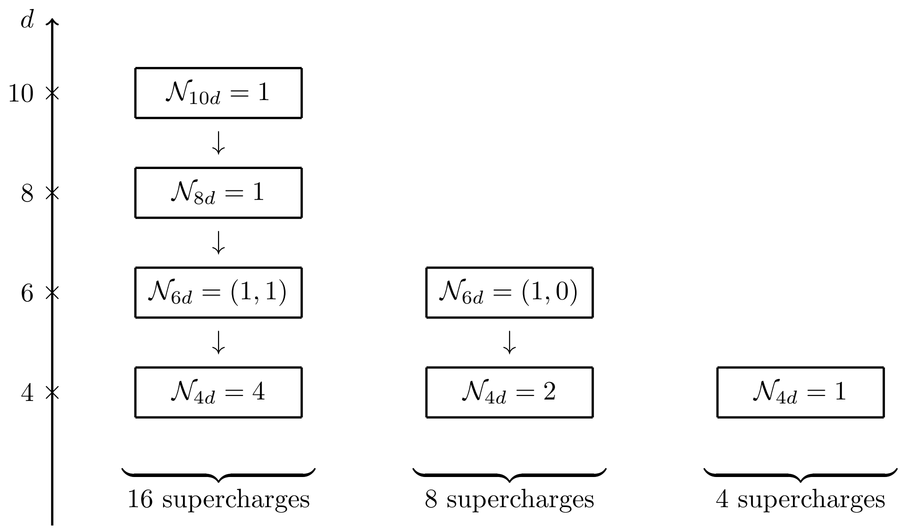

Figure 1: Classes of superstring compactifications for which we will discuss the universal particle content. The arrows within the columns represent dimensional reduction on a T^2 torus.

In this paper we are providing for the first time a complete investigation of the universal massive open string states of higher spin within supersymmetric compactifications of the open Type I superstring. In particular, we will compute the partition functions of the universal open string spectra for all type I compactifications with 4, 8 and 16 preserved supercharges. Spectra of the associated Type IIA/B closed superstring theories with twice as many preserved supercharges can be easily inferred from our open string results through a double copy of the open string Hilbert space, that is why they will not be explicitly addressed in this paper.[1] Four dimensional superstrings subject to $\mathcal{N}_{4d} = 1$ super Poincaré invariance are especially worth to be studied, since $\mathcal{N}_{4d} = 1$ compactifications with broken supersymmetry are expected to provide phenomenologically interesting string solutions at low energies with the spectrum of certain extensions of the supersymmetric Standard Model (see *e.g.* [2] for a stable low energy open string vacuum, the Standard Model^{++} with two Higgs fields). In addition to the light states, the knowledge of the universal massive string spectrum is also important in order to compute string scattering amplitudes of massive open string states in $\mathcal{N}_{4d} = 1$ string compactifications [3]. This task is particularly relevant, if the string scale is low compared to the Planck mass, as it is true in large volume compactifications.

Refined partition functions

A convenient way to study the spectrum of string states is to compute a *partition function* that counts such states with respect to their mass levels. Since the string states transform under representations of super-Poincaré algebra, such a counting can be done in a representation theoretic way, namely the partition function can be written in terms of an infinite power series such that each power keeps track of the mass level and the coefficient of each term in the series comprises irreducible characters of the super-Poincaré algebra. In this way, the symmetry of the problem is manifest in the partition function and the characters contain information on how a supermultiplet transform under the little group and the R symmetry. Moreover, knowing a partition function is equivalent to knowing how many times a given representation appears at each mass level – also known as the multiplicity. Hence, given a representation of super-Poincaré algebra, our aim is to compute its *multiplicity generating function*, a power series such that each power keeps track of the mass level and each coefficient are the multiplicity of this particular representation.

Such a way of counting of string states was already performed explicitly in [4, 5] for the case of uncompactified (ten-dimensional) string theories. It has also been extensively applied to the study of moduli spaces of supersymmetric gauge theories [6–14]; in such a context the partition function is also known as the Hilbert series.

One can also view the partition function we are considering as a trace over the space of physical states. In the trace, we grade the states according to their mass levels and global charges, but *not* their spacetime fermion numbers. The variables used in keeping track of these levels and charges are called fugacities. The fugacities for the global charges are indeed the ones that appear in the character of a representation of the super-Poincaré algebra. In general, the partition function is therefore a multivariate function. We call the insertion of global fugacities into the trace so as to make the global symmetry manifest a *refinement*, and we refer to the corresponding partition function as a *refined partition function*. On the other hand, in order to compute the total number of states at each mass level, one can set the fugacities in the characters to unity. This amounts to computing the dimension of the corresponding representation, and we call the resulting partition function an *unrefined* one.

The term 'refinement' as for the insertion of the aforementioned types of fugacities has also been used recently in various papers on elliptic genera and loop amplitudes. There are various

[1]For the case of heterotic string compactifications, the charged matter fields originate from closed strings, and hence a priori one expects a different pattern of massive string states. In order to match the heterotic-type I massive string spectrum via heterotic-type I string duality also non-perturbative states are needed.

'synonyms' that have been adopted in the literature, *e.g.* McKay-Thompson series [15], twining characters [16, 17] and twisted elliptic genera [18–20]. We emphasise that, on the contrary to elliptic genera or other types of characters that are used in loop amplitude computations, the states that we trace over are not graded with a minus sign for spacetime fermions[2]. As a result, the partition functions we are considering in this paper do not exhibit a modular invariant property[3].

Open string states also carry Chan-Paton factors. The massless states and their massive excitations that arise from open strings with both endpoints attached to a stack of D branes transform in the adjoint representation of the Chan-Paton gauge group. Their character can therefore be obtained by multiplying the character discussed here by the character of the adjoint representation[4]. The massive states corresponding to unoriented strings, on the other hand, transform in various representations according to the gauge symmetry (see e.g. Page 294 of [21] for further details), and the character can be computed by multiplying an appropriate character of the gauge group to our existing character at a given mass level. All partition functions computed in this paper allow for a straightforward inclusion of the Chan-Paton contributions; hence, we shall not discuss Chan-Paton factors in the subsequent.

The number of universal open string states

To give a first idea of the orders of magnitude governing the number of universal open string states at individual mass levels, the following Table 1 summarizes their numbers at low levels ≤ 9 in scenarios with 4, 8 and 16 supercharges, respectively. They are obtained by expanding the associated *unrefined partition functions*. For the cases of 4, 8 and 16 supercharges, the exact generating functions are respectively given by (4.8), (5.5), (6.6) and their asymptotics at large mass levels are respectively given by (4.17), (5.12), (6.10). Roughly speaking, the number of states increases exponentially with respect to the square root of the mass level. The plot is depicted in Figure 2.

Stable patterns and Regge trajectories

For any number of dimensions and supercharges, we can examine the multiplicities of a supermultiplet transforming under a given super-Poincaré representation. In four spacetime dimensions, such a representation contains an $SO(3)$ spin quantum number; this is *half* of the $SO(3)$ Dynkin label. For dimensions $d > 4$, we refer to the first $SO(d-1)$ Dynkin label as 'spin' in slight abuse of terminology. This allows for the generalised notion of spin in higher dimensions. It is interesting to study the multiplicities associated with large spin quantum numbers, *i.e.* a large spin limit.

There are certain crucial asymptotic patterns that universally appear for families of supermultiplets, regardless of the number of dimensions and supercharges. In particular, there are certain sets of numbers that repeatedly appear at various mass levels when spins are sufficiently large (and other quantum numbers are kept fixed). As an example, it is convenient to consider table 3 where

[2]To illustrate this point, let us look at the first mass level for a 10d theory with 16 supercharges, there are 256 states in total (see Table 1). This number comes from (a) 44 spin two degrees of freedom and 84 three-form degrees of freedom constituting the spacetime bosonic states, and (b) 128 spin 3/2 degrees of freedom constituting the spacetime fermonic states. If we had included the grading with a minus sign for spacetime fermions into the trace, we would have a zero here.

[3]To illustrate this point, we compare the unrefined partition functions presented in (5.3.37) of [21] and (9.1.14), (9.1.15) of [22]. The former is the partition function we are interested in and it is clear that such a partition function does not possess a modular invariant property. On the other hand, observe in the latter that if the grading with a minus sign for spacetime fermions is introduced in the trace, the contributions from the fermionic and bosonic excited states precisely cancel in the unrefined partition function, as exemplified in the preceding footnote.

[4]Furthermore, compactifications with intersecting D branes give rise to model dependent excitations of open strings that end on different stacks of D-branes [23]. These non-universal states beyond the scope of this work transform in the bifundamental representation.

$\alpha' m^2$	# states for 4 supercharges	# states for 8 supercharges	# states for 16 supercharges
0	4	8	16
1	24	80	256
2	104	512	2.304
3	384	2.576	15.360
4	1.240	11.008	84.224
5	3.648	41.792	400.896
6	9.992	144.784	1.711.104
7	25.792	465.856	6.690.816
8	63.392	1.409.792	24.332.544
9	149.464	4.050.112	83.219.712

Table 1. The number of model independent open string states in compactifications with 4, 8 and 16 supercharges, respectively, up to mass level $\alpha' m^2 = 9$.

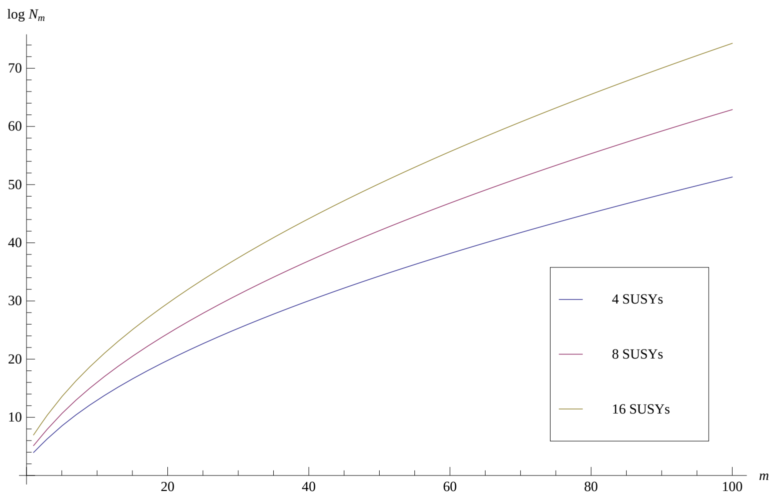

Figure 2: The logarithmic plot of the number of states N_m against the mass level m for the case of 4 and 16 supercharges. The values of N_m are taken from the asymptotic formulae (4.17), (5.12) and (6.10), which work well for large m.

such numbers are written in red. Since this set of numbers stabilises in the large spin limit, we refer to it as a *stable pattern*. In fact, such a pattern appears not only in superstring spectra we are considering, it also does so in spectra of the bosonic and various other types of string theories as pointed out in [4]; there, the stable pattern is referred to as the *leading Regge trajectory*. We shall henceforth use these two terms interchangably.

Let us explore the stable pattern in more details. For a fixed sufficiently large mass level M, the stable pattern for a certain supermultiplet family starts appearing when the spin j increases and reaches a certain value $j_{\text{min}}(M)$. It then extends up to some maximum value $j_{\text{max}}(M)$ where the multiplicity becomes zero for spins $j > j_{\text{max}}(M)$. As an empirical speculation, we observe that for a sufficiently large M, the stable pattern appearing in the spin range $j_{\text{min}}(M) \leq j \leq j_{\text{max}}(M)$ occupies approximately *half* of the spin range $0 \leq j \leq j_{\text{max}}(M)$ of all non-zero multiplicities. As an example, such a phenomenon is highlighted in red in each row of table 3.

Stated differently, for a given super-Poincaré representation, the highest spin with nonzero multiplicity approximately scales linearly $j_{\text{max}}(M) \approx (M - M_0)$ for large M where M_0 is the mass level at which the first non-zero red number appears. The onset of the stable pattern, on the other hand, roughly follows a linear scaling, $j_{\min}(M) \approx \frac{1}{2}(M - M_0)$. The region of validity for the stable pattern is therefore bounded by two straight lines whose slopes have the ratio $\frac{1}{2}$. In this sense, the stable pattern gives control over the essential part of the spectrum.

In addition to the stable pattern or the leading trajectory, there is also a notion of subleading trajectories bounded by linear spin-mass relations with approximate slopes $\frac{1}{3}, \frac{1}{4}, \ldots$. We shall not go over any detail here and postpone the quantitative discussions to subsequent sections.

Outlines and key results

This article can be roughly divided into two parts. The first part develops the SCFT foundations for refined superstring partition functions, using conventions from appendix A. Section 2 introduces $SO(d-1)$ covariant characters for the degrees of freedom due to the superstring oscillators from the spacetime SCFT. In order to describe compactification scenarios, the spacetime sector has to be supplemented by SCFTs describing the internal dimensions. The SCFTs discussed in section 3 capture the universal states present in *any* compactification that preserves four and eight supercharges, respectively.

Starting from section 4, we proceed to the second part of this work where spacetime- and internal characters are combined to super Poincaré covariant partition functions. Universal states of four dimensional $\mathcal{N}_{4d} = 1$ supersymmetric string compactifications are thoroughly investigated in section 4: We analytically derive the stable pattern for supermultiplet multiplicities, in manifest agreement with the tabulated particle content up to mass level 25. Similarly, section 5 is devoted to scenarios with eight supercharges – in both six and four spacetime dimensions. Finally, spectra of maximally supersymmetric open superstring theories are discussed in section 6, a chain of dimensional reductions encompasses $d = 10, 8, 6$ and $d = 4$ compactifications.

The analysis of universal $\mathcal{N}_{4d} = 1$ supermultiplets in section 4 enjoys the highest phenomenological relevance and provides the most compact results. Hence, the reader might want to skip subsections 2.4, 2.5 and 3.2 on higher dimensional generalizations upon the first reading.

Let us summarise the key results in this paper below.

- The exact unrefined partition functions and asymptotic expressions for the number of states at each large mass level are given in (4.8)–(4.17), (5.5)–(5.12) and (6.6)–(6.10) for theories with four, eight and sixteen supercharges respectively. The graphs of these numbers versus the mass level are depicted in Figure 2.

- The exact multiplicity generating functions for theories of four, eight and sixteen supercharges are respectively given in (4.61)–(4.62), (5.41) and (6.33).

- The asymptotic expressions for the multiplicity generating functions for the theory with four supercharges are presented in (4.63) and (4.64)

Even though the tools for expanding the refined partition function to any mass level are presented for all the scenarios, exact formulae for multiplicities of particular multiplets generically involve nested infinite sums. In particular, the bookkeeping of $SO(d-1)$ quantum numbers becomes increasingly difficult in $d > 4$ spacetime dimensions. That is why we elaborate the phenomenologically relevant and mathematically most accessible $\mathcal{N}_{4d} = 1$ case in particular depth.

2 The spacetime CFT in various dimensions

The aim of this section is to derive a refined partition function for the oscillator modes of the worldsheet fields $\partial X^\mu, \psi^\mu$ associated with the d directions of Minkowski spacetime. They carry an $SO(1,d-1)$ vector index $\mu = 0,1,\ldots,d-1$. In the framework of lightcone quantization the physical spectrum is obtained from transverse oscillators $\partial X^{i=2,3,\ldots,d-1}, \psi^{i=2,3,\ldots,d-1}$ which carry charges with respect to the $\frac{1}{2}(d-2)$ Cartan generators of $SO(1,d-1)$ outside the lightcone directions. We assign a separate fugacity y_k to each pair of $\partial X^i, \psi^i$ components (say $(\partial X^{2k}, \partial X^{2k+1})$) such that the fugacity subscript lies in the range $1 \leq k \leq \frac{1}{2}(d-2)$. Since massive particles with d dimensional timelike momentum form representations of the little group $SO(d-1)$, the dependence on Lorentz fugacities y_k necessarily arranges into characters of the massive little group.

It is instructive to first of all study the simplest non-trivial example $d=4$ with one spacetime fugacity. The first three subsections are devoted to the $SO(3)$ covariant partition function of the four dimensional spacetime SCFT. As we will explain in later subsections, higher dimensional cases follow by combining several copies of $d=4$ building blocks.

2.1 Bosonic partition function in $d = 4$

The contribution of the lightcone bosons to the refined partition function is

$$\begin{aligned} \chi_B^{SO(3)}(q,y) &= \mathrm{PE}\left[([2]_y - 1)\left(q+q^2+q^3+q^4+\ldots\right)\right] \;=\; \mathrm{PE}\left[([2]_y-1)\frac{q}{1-q}\right] \\ &= \prod_{n=1}^{\infty} \frac{1}{(1-y^2q^n)(1-y^{-2}q^n)} = \frac{1}{(qy^2;q)_\infty (qy^{-2};q)_\infty} \qquad (2.1) \\ &= -iq^{\frac{1}{12}}(y-y^{-1})\frac{\eta(q)}{\vartheta_1(y^2,q)} \ . \qquad (2.2) \end{aligned}$$

The representation $[2]_y - 1 = y^2 + y^{-2}$ in the plethystic exponential corresponds to the two components $\partial X^+, \partial X^-$ perpendicular to the lightcone. The geometric series $\frac{q}{1-q} = q + q^2 + q^3 + \ldots$, on the other hand, represents the infinite tower of positive frequency modes of $\partial X^\pm$ which act as creation operators.

Explicitly, the first few terms in the power series of $\chi_B^{SO(3)}(q,y)$ can be written in terms of $SO(3)$ characters $[k]_y$ as

$$\begin{aligned} \chi_B^{SO(3)}(q,y) = 1 &+ q([2]_y - 1) + q^2[4]_y + q^3([2]_y + [6]_y) + q^4([0]_y + 2[4]_y + [8]_y) \\ &+ q^5(2[2]_y + [4]_y + 2[6]_y + [10]_y) + q^6(2[0]_y + [2]_y + 3[4]_y + 2[6]_y + 2[8]_y + [12]_y) \\ &+ q^7(4[2]_y + 3[4]_y + 4[6]_y + 2[8]_y + 2[10]_y + [14]_y) + \ldots \ . \end{aligned} \qquad (2.3)$$

From such a power series, we are motivated to rewrite (2.1) as an infinite sum of the form

$$\chi_B^{SO(3)}(q,y) = \sum_{k=0}^{\infty} [k]_y f_k(q) \ , \qquad (2.4)$$

for some function $f_k(q)$ which depends only on q and not on y. The use of this form of the partition function will become clear later.

In order to do so, we rewrite (2.1) using the q-binomial theorem[5] as

$$\chi_B^{SO(3)}(q,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{y^{2(m-n)}}{(q;q)_m (q;q)_n} q^{m+n} =: \sum_{k=0}^{\infty} [k]_y f_k(q) \ . \qquad (2.5)$$

Before proceeding further, let us state an identity that we are going to use many times later. From (A.5) and the residue theorem, we find that

$$\int \mathrm{d}\mu_{SO(3)}(y)\; y^m [n]_y = \begin{cases} \delta_{0,n} & \text{for } m = 0 \ , \\ \frac{1}{2}(\delta_{|m|,n} - \delta_{|m|,n+2}) & \text{for } m \neq 0 \ , \end{cases} \qquad (2.6)$$

[5]The version we use states that $\frac{1}{(z;q)_\infty} = \sum_{n=0}^{\infty} \frac{z^n}{(q;q)_\infty}$.

where the Haar measure of $SO(3)$ is given by (A.9). It is clear from the absence of odd y powers in (2.5) that only integer spin representations of $SO(3)$ occur. We therefore have $f_{2k+1}(q) = 0$ for all k, and the nontrivial coefficients to compute are[6]

$$
\begin{aligned}
f_{2k}(q) &= \int \mathrm{d}\mu_{SO(3)}(y)\ \chi_B^{SO(3)}(q,y)[2k]_y \\
&= \sum_{m=0}^{\infty}\left[\frac{q^{2m}}{(q;q)_m^2}\delta_{k,0} + \frac{1}{2}\sum_{n=0}^{m-1}\frac{\delta_{m-n,k}-\delta_{m-n,k+1}}{(q;q)_m(q;q)_n}q^{m+n} + \frac{1}{2}\sum_{n=m+1}^{\infty}\frac{\delta_{n-m,k}-\delta_{n-m,k+1}}{(q;q)_m(q;q)_n}q^{m+n}\right] \\
&= \sum_{n=0}^{\infty}\frac{q^{2n+k}}{(q;q)_n(q;q)_{n+k+1}}\left(1-q-q^{n+k+1}\right) \\
&= (q;q)_\infty^{-2}\sum_{p=0}^{\infty}(-1)^p q^{(p+1)k+\frac{1}{2}p(p+1)}(1-q^{p+1})(q;q)_p^{-1}\sum_{m=0}^{\infty}(-1)^m\frac{q^{\frac{1}{2}m(m+1)}}{1-q^{2+m+p}}(q;q)_m^{-1} \\
&= (q;q)_\infty^{-2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)^2 q^{nk+\frac{1}{2}n(n-1)}\ .
\end{aligned} \tag{2.7}
$$

We obtain an $SO(3)$ character expansion of the bosonic partition function:

$$
\chi_B^{SO(3)}(q,y) = (q;q)_\infty^{-2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)^2\sum_{k=0}^{\infty}q^{nk+\frac{1}{2}n(n-1)}[2k]_y\ . \tag{2.8}
$$

Note that the pattern $\sum_{n=1}^{\infty}(-1)^{n-1}q^{nk}[2k]_y\ldots$ (where the $\ldots$ ellipsis does not depend on y and k) is described in section 6 of [4] as an alternating sequence of additive and subtractive Regge trajectories of slope $\frac{1}{n}$. This is the source of stable patterns as described in the introduction in bosonic string theory. We will rediscover these patterns in the counting of SUSY multiplets later on.

2.1.1 Multiplicities of representations $[2m]$ and their asymptotics

Let us determine the multiplicity of irreducible $SO(3)$ representations $[2m]$ at each mass level. Recall the orthogonality of characters with respect to the Haar measure:

$$
\int \mathrm{d}\mu_{SO(3)}(y)[m]_y[n]_y = \delta_{mn}\ . \tag{2.9}
$$

From (2.8), we find that the generating function of the multiplicity of $[2m]$ is

$$
\begin{aligned}
M(\chi_B^{SO(3)},[2m];q) &= \int \mathrm{d}\mu_{SO(3)}(y)\ [2m]_y\chi_B^{SO(3)}(q,y) \\
&= (q;q)_\infty^{-2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)^2 q^{\frac{1}{2}n(n-1)}q^{nm}\ .
\end{aligned} \tag{2.10}
$$

Asymptotics as $m \to \infty$. The expression (2.10) found for multiplicity generating functions greatly simplifies in the limit $m \to \infty$ of large spin and mass level. In order to compute an asymptotic formula in this regime, we apply Laplace's method (see *e.g.* section 6.7 of [24]) to our question. Since $0 < q < 1$, the terms in the series peak sharply near the $n = 1$ term as $m \to \infty$. Therefore, it is expected that for any $\epsilon > 0$

$$
M(\chi_B^{SO(3)},[2m];q) \sim (q;q)_\infty^{-2}\sum_{n=1}^{1+\lfloor\epsilon\rfloor}(-1)^{n-1}(1-q^n)^2 q^{\frac{1}{2}n(n-1)}q^{nm}, \qquad m\to\infty\ . \tag{2.11}
$$

[6] In intermediate steps, we are making use of identities like $\sum_{r=0}^{\infty}q^{r(1+p)}(q^{1+r};q)_\infty = (q;q)_\infty(q^{1+p};q)_\infty$.

Now let us write $n = 1 + t$, where t is small compared with 1. Note that

$$q^{\frac{1}{2}n(n-1)} = 1 + \frac{1}{2}(\log q)t + O(t^2) \ , \tag{2.12}$$

Substituting the leading term of this power series into the right hand side of (2.11) and extending the region of summation to ∞, we find that the leading behaviour of $M(\chi_B^{SO(3)}, [2m]; q)$ is given by

$$\begin{aligned} M(\chi_B^{SO(3)}, [2m]; q) &\sim (q;q)_\infty^{-2} \sum_{t=0}^{\infty} (-1)^t (1-q^{t+1})^2 q^{m(t+1)} \\ &= (q;q)_\infty^{-2} \frac{q^m (1-q)^2 \left(1-q^{1+m}\right)}{(1+q^m)\left(1+q^{1+m}\right)\left(1+q^{2+m}\right)} \\ &= (q^2;q)_\infty^{-2} \ \frac{q^m \left(1-q^m\right)}{\left(1+q^m\right)^3} \ , \qquad m \to \infty \ . \end{aligned} \tag{2.13}$$

The higher order corrections can be computed by taking into account the subleading terms of (2.12). Note that the next to leading term of (2.13) is of order $O(q^{2m}\log q)$. Thus, asymptotic formula (2.13) reproduces the exact result up to $O(q^{2m-1})$.

Interpretation and stable pattern. We can extract some information about bosonic string states from (2.13).

- The representation $[2m]$ appears first time in the bosonic partition function $\chi_B^{SO(3)}(q, y)$ at mass level q^m.
- The multiplicities of $[2m]$ at levels $q^{m+\ell}$, for $0 \le \ell \le m-1$, are independent of m. We refer to a set of these numbers as a *stable pattern* for bosonic string theory. The generating function for such a stable pattern can be determined by taking a formal limit $m \to \infty$ in (2.13):

$$\begin{aligned} &\lim_{m\to\infty} q^{-m} M(\chi_B^{SO(3)}, [2m]; q) \\ &= (q^2;q)_\infty^{-2} = \prod_{k=2}^{\infty} (1-q^k)^{-2} \qquad (2.14) \\ &= 1 + 2q^2 + 2q^3 + 5q^4 + 6q^5 + 13q^6 + 16q^7 + 30q^8 + 40q^9 + 66q^{10} + 90q^{11} \\ &\quad + 142q^{12} + 192q^{13} + 290q^{14} + 396q^{15} + 575q^{16} + 782q^{17} + 1112q^{18} \\ &\quad + 1500q^{19} + 2092q^{20} + 2808q^{21} + 3848q^{22} + 5132q^{23} + 6945q^{24} \\ &\quad + 9192q^{25} + O(q^{26}) \ . \qquad (2.15) \end{aligned}$$

  Note that terms with low orders in this power series are in agreement with the data presented in Table 6b of [4].

2.2 The NS sector in $d = 4$

Under NS boundary conditions, the worldsheet superpartners ψ^i of the lightcone bosons contribute

$$\begin{aligned} f_{\rm NS}(q;y) &= \mathrm{PE}_F \left[([2]_y - 1) \frac{q^{\frac{1}{2}}}{1-q} \right] \\ &= \prod_{n=1}^{\infty} (1 + y^2 q^{n-1/2})(1 + y^{-2} q^{n-1/2}) \qquad (2.16) \\ &= q^{\frac{1}{24}} \frac{\vartheta_3(y^2, q)}{\eta(q)} \ . \qquad (2.17) \end{aligned}$$

to the spacetime partition functions. We shall rewrite this function as an infinite sum by means of Jacobi's triple product identity (see, *e.g.*, subsection 19.8 of [25]) :

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n-1}z)(1+x^{2n-1}z^{-1}) = \sum_{m=-\infty}^{\infty} x^{m^2} z^m \ . \tag{2.18}$$

Applying identity (2.18) with $x = q^{1/2}$ and $z = y^2$ to (2.16), we obtain

$$f_{\rm NS}(q,y) = (q;q)_\infty^{-1} \sum_{m=-\infty}^{+\infty} y^{2m} q^{m^2/2} \tag{2.19}$$

$$= (q;q)_\infty^{-1} \sum_{m=0}^{\infty} q^{\frac{1}{2}m^2}(1-q^{m+\frac{1}{2}})[2m]_y \ , \tag{2.20}$$

where (2.20) can be obtained by applying (2.6) and the orthogonality of the characters to (2.19) as follows:

$$\begin{aligned}\int \mathrm{d}\mu_{SO(3)}(y)\ f_{\rm NS}(q,y)[2k]_y &= (q;q)_\infty^{-1}\left[\sum_{m=0}^{\infty} q^{m^2/2}\delta_{m,k} - \sum_{m=-\infty}^{-1} q^{m^2/2}\delta_{-m,k+1}\right] \\ &= (q;q)_\infty^{-1}\left(q^{\frac{1}{2}k^2} - q^{\frac{1}{2}(k+1)^2}\right) \\ &= (q;q)_\infty^{-1} q^{\frac{1}{2}k^2}(1-q^{k+\frac{1}{2}}) \ . \end{aligned} \tag{2.21}$$

Let us combine the bosonic partition function with the NS-sector contribution. Using (2.1), (2.20) and the multiplication rule $[2m]\cdot[2k] = \sum_{l=|k-m|}^{k+m}[2l]$, we find that

$$\chi_{\rm NS}^{SO(3)}(q,y) := \chi_B^{SO(3)}(q,y) f_{\rm NS}(q,y) = -iq^{1/8}(y-y^{-1})\frac{\vartheta_3(y^2,q)}{\vartheta_1(y^2,q)} \tag{2.22}$$

$$= \frac{-1}{(q;q)_\infty^3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^n(1-q^{m+\frac{1}{2}})(1-q^n)^2 q^{\frac{1}{2}n(n-1)+\frac{1}{2}m^2}\sum_{k=0}^{\infty} q^{nk}\sum_{\ell=|k-m|}^{k+m}[2\ell] \ . \tag{2.23}$$

The expression in the curly brackets $\{\cdots\}$ can be rewritten as $\sum_{k=0}^{\infty} f_{kmn}(q)[2k]$, for some function $f_{kmn}(q)$. In order to determine this function, we use the orthogonality of characters:

$$\begin{aligned} f_{kmn}(q) &= \int \mathrm{d}\mu_{SO(3)}(y)\ \sum_{k'=0}^{\infty} q^{nk'} \sum_{\ell=|k'-m|}^{k'+m} [2\ell]_y\ [2k]_y \\ &= \sum_{k'=0}^{\infty} q^{nk'} \sum_{\ell=|k'-m|}^{k'+m} \delta_{\ell,k} \\ &= \sum_{\ell'=|k-m|}^{k+m} q^{n\ell'} = \frac{q^{n|k-m|}-q^{n(k+m+1)}}{1-q^n} \ . \end{aligned} \tag{2.24}$$

Therefore, we obtain

$$\begin{aligned}\chi_{\rm NS}^{SO(3)}(q,y) = (q;q)_\infty^{-3}\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^{m+\frac{1}{2}})(1-q^n)q^{\frac{1}{2}[n(n-1)+m^2]}\times \\ \sum_{k=0}^{\infty}\left(q^{n|k-m|}-q^{n(k+m+1)}\right)[2k] \ . \end{aligned} \tag{2.25}$$

We emphasise that the $SO(3)$ irreducible representations with odd Dynkin labels do not appear in the partition function $\chi_{\rm NS}^{SO(3)}(q,y)$.

In terms of a power series in q, this can be written as

$$\begin{aligned}\chi_{\rm NS}^{SO(3)}(q,y) = 1 &+ q^{1/2}([2]-1) + q[2] + q^{3/2}([0]+[4]) + q^2([0]+2[4]) + q^{5/2}(2[2]+[4]+[6]) \\ &+ q^3(3[2]+[4]+2[6]) + q^{7/2}(2[0]+2[2]+4[4]+[6]+[8]) \\ &+ q^4(3[0]+3[2]+5[4]+2[6]+2[8]) \\ &+ q^{9/2}([0]+7[2]+4[4]+6[6]+[8]+[10]) \\ &+ q^5([0]+9[2]+7[4]+7[6]+2[8]+2[10]) + \dots \end{aligned} \tag{2.26}$$

Setting $y=1$, we obtain the unrefined partition function

$$\begin{aligned}\chi_{\rm NS}^{SO(3)}(q,y=1) &= \chi_B^{SO(3)}(q,y) f_{\rm NS}(q,y) \\ &= \prod_{n=1}^{\infty}\left(\frac{1+q^{n-1/2}}{1-q^n}\right)^2 \\ &= (q;q)_\infty^{-3}\vartheta_3(1,q) = q^{-1/8}\frac{\vartheta_3(1,q)}{\eta(q)^3} \ . \end{aligned} \tag{2.27}$$

2.2.1 Multiplicities of representations $[2j]$ and their asymptotics

Similarly to the bosonic partition function, we can read off the generating function for the multiplicities of the representations $[2j]$ at different mass levels of the NS superstring

$$\begin{aligned} M(\chi_{\rm NS}^{SO(3)},[2j],q) = (q;q)_\infty^{-3}\sum_{m=0}^{\infty}(1-q^{m+\frac{1}{2}})q^{\frac{1}{2}m^2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)q^{\frac{1}{2}n(n-1)}\times \\ (q^{n|j-m|}-q^{n(j+m+1)}) \ . \end{aligned} \tag{2.28}$$

Asymptotics as $j\to\infty$. In this limit, we have $|j-m|\sim j-m$ for a finite m. Futhermore, the summand as a function of n is sharply peaked near $n=1$, and so we can determine the leading behaviour of the sum over n using Laplace's method as follows:

$$\begin{aligned} &\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)q^{\frac{1}{2}n(n-1)}(q^{n(j-m)}-q^{n(j+m+1)}) \\ &\sim (1-q)\sum_{n=1}^{1+\lfloor\epsilon\rfloor}\left[q^{n(j-m)}-q^{n(j+m+1)}\right] \qquad \text{for } \epsilon>0 \\ &\sim (1-q)\sum_{t=0}^{\infty}\left[q^{(t+1)(j-m)}-q^{(t+1)(j+m+1)}\right] \\ &= (1-q)\left[\frac{q^{j-m}}{1-q^{j-m}}-\frac{q^{j+m+1}}{1-q^{j+m+1}}\right] \\ &= q^{j-m}(1-q)\frac{1-q^{2m+1}}{(1-q^{1+j+m})(1-q^{j-m})} \ . \end{aligned} \tag{2.29}$$

Therefore, we find that

$$\begin{aligned} &M(\chi_{\rm NS}^{SO(3)},[2j],q) \\ &\quad\sim (q;q)_\infty^{-3}q^j(1-q)\left[\sum_{m=0}^{\infty}q^{-m+\frac{m^2}{2}}\frac{\left(1-q^{2m+1}\right)\left(1-q^{\frac{1}{2}+m}\right)}{\left(1+q^{1+j-m}\right)\left(1+q^{j-m}\right)}\right] \end{aligned}$$

$$
\begin{aligned}
&\sim (q;q)_\infty^{-3} q^j \frac{1-q}{(1-q^j)^2}\left[\sum_{m=0}^{\infty} q^{\frac{1}{2}(m-1)^2-\frac{1}{2}}\left(1-q^{2m+1}\right)\left(1-q^{\frac{1}{2}+m}\right)\right] \\
&= (q;q)_\infty^{-3} q^{j-\frac{1}{2}}\left(\frac{1-q}{1+q^j}\right)^2 \vartheta_3(1,q)\ . \qquad (2.30)
\end{aligned}
$$

Note that asymptotic formula (2.30) reproduces the exact result up to the order $q^{2j-\frac{3}{2}}$. We emphasise that the representation $[2j]$ appears first time at mass level $q^{j-\frac{1}{2}}$.

In [4], the individual n summands of (2.28) are interpreted as an alternating sequence of additive and subtractive Regge trajectories of slope $\frac{1}{n}$. In the notation of equation (6.2) of that reference, the $M(\chi_{\rm NS}^{SO(3)},[2j],q)$ are expanded as

$$
\begin{aligned}
M(\chi_{\rm NS}^{SO(3)},[2j],q) &= q^j\,\tau_1^{\rm NS}(q) - q^{2j}\,\tau_2^{\rm NS}(q) + q^{3j}\,\tau_3^{\rm NS}(q) - \dots \\
&= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,q^{\ell j}\,\tau_\ell^{\rm NS}(q)\ . \qquad (2.31)
\end{aligned}
$$

Setting $|j-m| = j-m$ in (2.28) leads to the following asymptotic expressions for the $\tau_\ell^{\rm NS}$:

$$
\tau_\ell^{\rm NS}(q) = (q;q)_\infty^{-3}\,q^{-\frac{1}{2}\ell}\,(1-q^\ell)\sum_{m=0}^{\infty} q^{\frac{1}{2}(m-\ell)^2}\,(1-q^{m+\frac{1}{2}})\,(1-q^{2m\ell+\ell}) \qquad (2.32)
$$

We will later on rediscover this trajectory structure in the counting of SUSY multiplets.

The stable pattern. The generating function of the stable pattern can be determined by projecting the sum in (2.31) to the first term (or, equivalently, by taking the limit $j\to\infty$):

$$
\begin{aligned}
\lim_{j\to\infty} q^{-j} M(\chi_{\rm NS}^{SO(3)},[2j],q) &= \tau_1^{\rm NS}(q) \\
&= (q;q)_\infty^{-3} q^{-1/2}(1-q)^2\vartheta_3(1,q) \qquad (2.33) \\
&= \Big(2+2q+8q^2+14q^3+34q^4+58q^5+120q^6+204q^7+378q^8+632q^9 \\
&\quad +1096q^{10}+1786q^{11}+2968q^{12}+4722q^{13}+7578q^{14}+11818q^{15}+\dots\Big)+ \\
&\quad +\Big(\frac{1}{\sqrt{q}}+\sqrt{q}+6q^{3/2}+9q^{5/2}+24q^{7/2}+42q^{9/2}+88q^{11/2}+151q^{13/2} \\
&\quad +287q^{15/2}+480q^{17/2}+846q^{19/2}+1388q^{21/2}+2326q^{23/2}+3724q^{25/2} \\
&\quad +6025q^{27/2}+9438q^{29/2}+\dots\Big)\ . \qquad (2.34)
\end{aligned}
$$

Note that terms with low orders in the power series (2.34) are in agreement with the data presented in Table 6c of [4].

2.3 The R sector in $d = 4$

The R sector of the worldsheet superpartners ψ^i of the lightcone bosons contributes

$$
\begin{aligned}
f_{\rm R}(q,y) &= (y+y^{-1}){\rm PE}_F\left[([2]_y-1)\frac{q}{1-q}\right] \\
&= (y+y^{-1})\prod_{n=1}^{\infty}(1+y^2q^n)(1+y^{-2}q^n) \qquad (2.35) \\
&= q^{-\frac{1}{12}}\frac{\vartheta_2(y^2,q)}{\eta(q)} \qquad (2.36)
\end{aligned}
$$

to the spacetime partition function. Again, it will turn out to be beneficial to rewrite this function as an infinite sum. We proceed as follows. Replacing z by xz in (2.18), we obtain

$$\prod_{n=1}^{\infty}(1-x^{2n})(1+x^{2n}z)(1+x^{2n-2}z^{-1}) = \sum_{m=-\infty}^{+\infty} x^{m^2+m} z^m \ . \tag{2.37}$$

Using the identity

$$\prod_{n=1}^{\infty}(1+x^{2n-2}z^{-1}) = (1+z^{-1}) \prod_{n=1}^{\infty}(1+x^{2n}z^{-1}) \ , \tag{2.38}$$

we arrive at

$$(z^{\frac{1}{2}}+z^{-\frac{1}{2}}) \prod_{n=1}^{\infty}(1+x^{2n}z)(1+x^{2n}z^{-1}) = \frac{\sum_{m=-\infty}^{+\infty} x^{m^2+m} z^{m+1/2}}{\prod_{n=1}^{\infty}(1-x^{2n})} \ . \tag{2.39}$$

Applying identity (2.39) to (2.35) with $x=q^{1/2}$ and $z=y^2$, we have

$$\begin{aligned} f_{\mathrm{R}}(q,y) &= (q;q)_{\infty}^{-1} \sum_{m=-\infty}^{+\infty} y^{2m+1} q^{m(m+1)/2} \\ &= (q;q)_{\infty}^{-1} \sum_{m=0}^{\infty} q^{\frac{1}{2}m(m+1)}(1-q^{m+1})[2m+1]_y \\ &= q^{-1/8}(q;q)_{\infty}^{-1} \sum_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} q^{\frac{1}{2}m^2}(1-q^{m+\frac{1}{2}})[2m]_y \ , \end{aligned} \tag{2.40}$$

where the second equality follows from (2.6) and the orthogonality of the characters.

Let us combine the contribution from the R-sector with the bosonic part. Using (2.1) and (2.35), we find that

$$\chi_{\mathrm{R}}^{SO(3)}(q,y) := \chi_B^{SO(3)}(q,y) f_{\mathrm{R}}(q,y) = -i(y-y^{-1})\frac{\vartheta_2(y^2,q)}{\vartheta_1(y^2,q)} \tag{2.41}$$

$$\begin{aligned} &= q^{-\frac{1}{8}}(q;q)_{\infty}^{-3} \sum_{m\in\mathbb{Z}_{\geq 0}+\frac{1}{2}} \sum_{n=1}^{\infty}(-1)^{n-1}(1-q^{m+\frac{1}{2}})(1-q^n)q^{\frac{1}{2}[n(n-1)+m^2]} \\ &\qquad \times \sum_{k=0}^{\infty}\left(q^{n|k-m|}-q^{n(k+m+2)}\right)[2k+1] \ . \end{aligned} \tag{2.42}$$

This resembles (2.25) up to a shift in the summations over m, k by $\pm\frac{1}{2}$. We emphasise that $SO(3)$ irreducible representation with even Dynkin labels do not appear in the R sector partition function $\chi_{\mathrm{R}}^{SO(3)}(q,y)$.

In terms of a power series, this partition function can be written as

$$\begin{aligned} \chi_{\mathrm{R}}^{SO(3)}(q,y) = [1] &+ 2[3]q + 2([1]+[3]+[5])q^2 + (4[1]+4[3]+4[5]+2[7])q^3 \\ &+ (6[1]+10[3]+8[5]+4[7]+2[9])q^4 \\ &+ (12[1]+18[3]+16[5]+10[7]+4[9]+2[11])q^5 \\ &+ (22[1]+32[3]+30[5]+22[7]+10[9]+4[11]+2[13])q^6 \\ &+ (36[1]+58[3]+56[5]+40[7]+24[9]+10[11]+4[13]+2[15])q^7+\ldots \end{aligned} \tag{2.43}$$

Setting $y=1$, we obtain the unrefined partition function

$$\chi_{\mathrm{R}}^{SO(3)}(q,y=1) = 2\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^2 = q^{-\frac{1}{8}}(q;q)_{\infty}^{-3}\vartheta_2(1,q) = \frac{\vartheta_2(1,q)}{\eta(q)^3} \ . \tag{2.44}$$

2.3.1 Multiplicities of representations $[2j+1]$ and their asymptotics

The generating function for the multiplicities of the representations $[2j+1]$ at different mass levels are

$$M(\chi_{\rm R}^{SO(3)},[2j+1],q) = q^{-\frac{1}{8}}(q;q)_\infty^{-3}\sum_{m=0}^{\infty}(1-q^{m+1})q^{\frac{1}{2}(m+\frac{1}{2})^2}\sum_{n=1}^{\infty}(-1)^{n-1}(1-q^n)q^{\frac{1}{2}n(n-1)}\times (q^{n|j-m|}-q^{n(j+m+2)}) \tag{2.45}$$

in close analogy to (2.28). In fact, one can obtain the above formula by shifting $m\to m+\frac{1}{2}$ and $j\to j+\frac{1}{2}$ in (2.28) and multiply by an overall factor $q^{-\frac{1}{8}}$.

Asymptotics as $j\to\infty$. Similarly to the NS-sector, we find that the leading behaviour of $M(\chi_{\rm R}^{SO(3)},[2j+1],q)$ is

$$\begin{aligned} &M(\chi_{\rm R}^{SO(3)},[2j+1],q)\\ &\quad\sim q^{-\frac{1}{8}}(q;q)_\infty^{-3}q^{j+\frac{1}{2}}\frac{1-q}{(1-q^j)^2}\left[\sum_{m=0}^{\infty}q^{\frac{1}{2}(m-\frac{1}{2})^2-\frac{1}{2}}\left(1-q^{2m+2}\right)\left(1-q^{m+1}\right)\right]\\ &\quad=(q;q)_\infty^{-3}q^{j-\frac{1}{8}}\left(\frac{1-q}{1+q^j}\right)^2\vartheta_2(1,q)\ . \end{aligned} \tag{2.46}$$

Note that the representation $[2j+1]$ appears first time at mass level q^j and the asymptotic formula reproduces the exact result up to the order q^{2j-1}.

Also the multiplicity generating functions of the Ramond sector are suitable for an expansion in terms of Regge trajectories:

$$\begin{aligned} M(\chi_{\rm R}^{SO(3)},[2j+1],q) &= q^j\,\tau_1^{\rm R}(q) - q^{2j}\,\tau_2^{\rm R}(q) + q^{3j}\,\tau_3^{\rm R}(q) - \ldots\\ &= \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\,q^{\ell j}\,\tau_\ell^{\rm R}(q)\ . \end{aligned} \tag{2.47}$$

The $|j-m|=j-m$ asymptotics yield the following expressions for the ℓ'th Ramond trajectory $\tau_\ell^{\rm R}$:

$$\tau_\ell^{\rm R}(q) \;=\; (q;q)_\infty^{-3}\,q^{-\frac{1}{8}}\,(1-q^\ell)\sum_{m=\frac{1}{2}}^{\infty}q^{\frac{1}{2}(m-\ell)^2}\,(1-q^{m+\frac{1}{2}})\,(1-q^{2m\ell+\ell}) \tag{2.48}$$

The stable pattern. The generating function of the stable pattern can be determined by taking the limit $j\to\infty$:

$$\begin{aligned} \lim_{j\to\infty}q^{-j}M(\chi_{\rm R}^{SO(3)},[2j+1],q) \;&=\; \tau_1^{\rm R}(q)\\ &=(q;q)_\infty^{-3}q^{-1/8}(1-q)^2\vartheta_2(1,q) &(2.49)\\ &=2+4q+10q^2+24q^3+48q^4+96q^5+184q^6+336q^7+600q^8+1048q^9+1784q^{10}\\ &\quad+2984q^{11}+4912q^{12}+7952q^{13}+12704q^{14}+20048q^{15}+31256q^{16}+48224q^{17}\\ &\quad+73680q^{18}+111520q^{19}+167368q^{20}+O(q^{21})\ . &(2.50) \end{aligned}$$

Note that terms with low orders in the power series (2.50) are in agreement with the data presented in Table 6d of [4].

2.4 Bosonic partition function in $d > 4$

The bosonic partition function in $d = 2n + 2$ space-time dimensions can be written as

$$\chi_B^{SO(2n+1)}(q, \boldsymbol{y}) = \mathrm{PE}\left[([1, 0, \ldots, 0]_y^{SO(2n+1)} - 1)\frac{q}{1-q}\right] \ , \tag{2.51}$$

where $\boldsymbol{y} = (y_1, \ldots, y_n)$ and the character of the vector representation $[1, 0, \ldots, 0]$ of $SO(2n + 1)$ is given by (A.6). The $2n = d-2$ summands in $[1, 0, \ldots, 0]_y^{SO(2n+1)} - 1 = \sum_{k=1}^{n}(y_k^2 + y_k^{-2})$ reflect the ∂X^i components outside the lightcone. Using (A.8), we see that this choice of character allows us to write

$$\begin{aligned}\chi_B^{SO(2n+1)}(q, \boldsymbol{y}) &= \mathrm{PE}\left[\frac{q}{1-q}\sum_{k=1}^{n}([2]_{y_k} - 1)\right] \\ &= \prod_{A=1}^{n} \chi_B^{SO(3)}(y_A) \ .\end{aligned} \tag{2.52}$$

Observe that the $(2n+2)$-dimensional partition function is simply a product of n copies of the four dimensional partition function. From (2.8), we have

$$\chi_B^{SO(2n+1)}(q, \boldsymbol{y}) = (q; q)_\infty^{-2n} \sum_{\boldsymbol{n}\in\mathbb{Z}_+^n} \sum_{\boldsymbol{k}\in\mathbb{Z}_{\geq 0}^n} \prod_{A=1}^{n} (-1)^{n_A-1}(1 - q^{n_A})^2 q^{n_A k_A + \frac{1}{2}n_A(n_A-1)} [2k_A]_{y_A} \tag{2.53}$$

with $\mathbb{Z}_+$ denoting the set of positive integers and $\mathbb{Z}_{\geq 0} = \mathbb{Z}_+ \cup \{0\}$. For our purpose of resolving the $SO(2n + 1)$ content of the partition function, the aim is to rewrite (2.53) in the form

$$\chi_B^{SO(2n+1)}(q, \boldsymbol{y}) = \sum_{\lambda_1\geq\cdots\geq\lambda_n\geq 0} (\lambda_1, \ldots, \lambda_n)_{\boldsymbol{y}} \ G_{\lambda_1,\ldots,\lambda_n}^{B,SO(2n+1)}(q) \ , \tag{2.54}$$

where the summations run over highest weight vectors $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n) \in \mathbb{Z}^n$ subject to inequalities $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, see Appendix A for the conversion rule to Dynkin label notation $[a_1, \ldots, a_n]$. Since (2.51) involves only the vector representation and the plethystic exponential generates symmetrisations of the representation, there is no spinor representation of $SO(2n + 1)$ appearing in $\chi_B^{SO(2n+1)}(q, \boldsymbol{y})$; therefore,

$$G^{B,SO(2n+1)}_{\lambda_1+\frac{1}{2},\ldots,\lambda_n+\frac{1}{2}}(q) = 0 \ , \qquad \lambda_k \in \mathbb{Z} \ . \tag{2.55}$$

In general, $G_{\lambda_1,\ldots,\lambda_n}^{B,SO(2n+1)}(q)$ can be interpreted as a *generating function for the multiplicities* of the $SO(2n + 1)$ representation $(\lambda_1, \ldots, \lambda_n)$ in the bosonic string partition function.

Some useful relations between $SO(2n + 1)$ and $SO(3)$ representations

In order to obtain compact formulae for the multiplicity generating functions $G_{\lambda_1,\ldots,\lambda_n}^{B,SO(2n+1)}(q)$, we have to convert the $SO(3)$ character products in (2.53) into a basis of $(\lambda_1, \ldots, \lambda_n)_{\boldsymbol{y}}$, i.e. we have to find the Δ coefficients in the basis transformation

$$\prod_{A=1}^{n} [2k_A]_{y_A} = \sum_{\lambda_1\geq\ldots\geq\lambda_n\geq 0} \Delta\left(\lambda_1, \ldots, \lambda_n; 2k_1, \ldots, 2k_n\right)(\lambda_1, \ldots, \lambda_n)_{\boldsymbol{y}} \ . \tag{2.56}$$

In general, according to (5.10) of [4], it can be shown that the coefficients in this basis transformation are given by

$$\Delta(\lambda_1, \ldots, \lambda_n; 2k_1, \ldots, 2k_n) := \int \mathrm{d}\mu_{SO(2n+1)}(\boldsymbol{y}) \ (\lambda_1, \ldots, \lambda_n)_{\boldsymbol{y}} \prod_{A=1}^{n} [2k_A]_{y_A}$$

$$= \frac{1}{n!} \sum_{\sigma_1,\sigma_2 \in S_n} \mathrm{sgn}(\sigma_2) \prod_{A=1}^{n} \theta^{2n+\lambda_A-A-\sigma_2(A)}_{|\lambda_A-A+\sigma_2(A)|} \left(k_{\sigma_1(A)}\right)$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \det \left(\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|} \left(k_{\sigma(A)}\right)\right)^{n}_{A,B=1} \tag{2.57}$$

where the function $\theta^n_m(k)$ is defined as

$$\theta^n_m(k) = \begin{cases} 1 & \text{if } m \le k \le n \ , \\ 0 & \text{otherwise} \ . \end{cases} \tag{2.58}$$

Note that for spinorial representations $(\lambda_1 + \frac{1}{2}, \ldots, \lambda_n + \frac{1}{2})$, (2.57) vanishes identically:

$$\Delta\left(\lambda_1 + \frac{1}{2}, \ldots, \lambda_n + \frac{1}{2}; 2k_1, \ldots, 2k_n\right) = 0 \ , \quad \forall\, \boldsymbol{\lambda} \in \mathbb{Z}^n \text{ and } \lambda_1 \ge \cdots \ge \lambda_n \ge 0 \ . \tag{2.59}$$

Thus, Eq. (2.57) implies the following expansion rule for $SO(3)$ character products in terms of $SO(2n+1)$ characters in Dynkin label notation:

$$\prod_{A=1}^{n} [2k_A]_{y_A} = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n_{\ge 0}} [\ell_1, \ldots, \ell_{n-1}, 2\ell_n]_{\boldsymbol{y}}\, \Delta\left(2k_1, \ldots, 2k_n; \ell_1 + \ell_2 + \ldots + \ell_n, \ell_2 + \ldots + \ell_n, \ldots, \ell_n\right) \tag{2.60}$$

The inverse decomposition formula for an integer spin representation follows from the $SO(2n+1)$ Haar measure (A.10):

$$[\ell_1, \ldots, \ell_{n-1}, 2\ell_n]_{\boldsymbol{y}} = \frac{1}{\rho(\boldsymbol{y})} \sum_{\boldsymbol{k} \in \mathbb{Z}^n_{\ge 0}} \prod_{A=1}^{n} [2k_A]_{y_A}\, \Delta\left(\ell_1 + \ell_2 + \ldots + \ell_n, \ell_2 + \ldots + \ell_n, \ldots, \ell_n; 2k_1, \ldots, 2k_n\right) \ , \tag{2.61}$$

where $\rho(\boldsymbol{y})$ is defined as in (A.11) and $\boldsymbol{\ell} = (\ell_1, \ldots, \ell_n) \in \mathbb{Z}^n_{\ge 0}$.

Similarly, one can convert spinorial $SO(3)$ character products to $SO(2n+1)$ characters via

$$\Delta(\lambda_1, \ldots, \lambda_n; 2k_1 + 1, \ldots, 2k_n + 1) := \int \mathrm{d}\mu_{SO(2n+1)}(\boldsymbol{y})\ (\lambda_1, \ldots, \lambda_n)_{\boldsymbol{y}} \prod_{A=1}^{n} [2k_A + 1]_{y_A}$$

$$= \frac{1}{n!} \sum_{\sigma \in S_n} \det \left(\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|} \left(k_{\sigma(A)} + \frac{1}{2}\right)\right)^{n}_{A,B=1} \ . \tag{2.62}$$

For integer spin representations of $SO(2n+1)$, (2.62) vanishes identically:

$$\Delta\left(\lambda_1 \ldots, \lambda_n; 2k_1 + 1, \ldots, 2k_n + 1\right) = 0 \ , \quad \forall\, \boldsymbol{\lambda} \in \mathbb{Z}^n \text{ and } \lambda_1 \ge \cdots \ge \lambda_n \ge 0 \ . \tag{2.63}$$

We thus have the following decomposition for products of spinorial $SO(3)$ characters

$$\prod_{A=1}^{n} [2k_A + 1]_{y_A} = \sum_{\boldsymbol{\ell} \in \mathbb{Z}^n_{\ge 0}} [\ell_1, \ldots, \ell_{n-1}, 2\ell_n + 1]_{\boldsymbol{y}}$$
$$\times\, \Delta\left(2k_1 + 1, \ldots, 2k_n + 1; \ell_1 + \ell_2 + \ldots + \frac{1}{2}\ell_n, \ell_2 + \ldots + \frac{1}{2}\ell_n, \ldots, \frac{1}{2}\ell_n\right) \ , \tag{2.64}$$

with inverse

$$[\ell_1, \ldots, \ell_{n-1}, 2\ell_n + 1]_{\boldsymbol{y}} = \frac{1}{\rho(\boldsymbol{y})} \sum_{\boldsymbol{k} \in \mathbb{Z}^n_{\ge 0}} \prod_{A=1}^{n} [2k_A + 1]_{y_A}$$
$$\times\, \Delta\left(\ell_1 + \ell_2 + \ldots + \ell_n + \frac{1}{2}, \ell_2 + \ldots + \ell_n + \frac{1}{2}, \ldots, \ell_n + \frac{1}{2}; 2k_1 + 1, \ldots, 2k_n + 1\right) \ . \tag{2.65}$$

Generating function for the multiplicities

According to (2.53), the bosonic spacetime partition function in $2n+2$ dimensions depends on Lorentz fugacities through the factor

$$\begin{aligned}&\sum_{k_1,\ldots,k_n\geq 0}\Delta(\lambda_1,\ldots,\lambda_n;2k_1,\ldots,2k_n)q^{n_1k_1+\ldots+n_nk_n}\\&=\sum_{k_1,\ldots,k_n\geq 0}\det(\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|}(k_A))^n_{A,B=1}q^{n_1k_1+\ldots+n_nk_n}\\&=\det\left(\sum_{k_A\geq 0}\theta^{2n+\lambda_A-A-B}_{|\lambda_A-A+B|}(k_A)q^{n_Ak_A}\right)^n_{A,B=1}\,.\end{aligned}\tag{2.66}$$

Let us apply this to (2.53) to compute $G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)$. For $\lambda_1\geq\cdots\geq\lambda_n\geq n-1$, the argument in the absolute value is non-negative and so

$$\begin{aligned}&\sum_{k_1,\ldots,k_n\geq 0}\Delta(\lambda_1,\ldots,\lambda_n;2k_1,\ldots,2k_n)q^{n_1k_1+\ldots n_nk_n}\\&=\prod_{A=1}^n q^{n_A(\lambda_A-A+1)}\prod_{1\leq B<C\leq n}(q^{n_C}-q^{n_B})(1-q^{n_C+n_B})\quad\text{for }\lambda_1\geq\cdots\geq\lambda_n\geq n-1\,.\end{aligned}\tag{2.67}$$

It is pointed out by [4] and can be checked directly that the contribution from $\lambda_n<n-1$ to the bosonic string partition function is zero. Therefore, we have

$$\begin{aligned}G^{B,SO(2n+1)}_{\lambda_1,\ldots,\lambda_n}(q)=(q;q)^{-2n}_\infty&\sum_{\boldsymbol{n}\in\mathbb{Z}^n_+}\prod_{A=1}^n(-1)^{n_A-1}(1-q^{n_A})^2q^{n_A(\lambda_A-A+1)+\frac{1}{2}n_A(n_A-1)}\\&\times\prod_{1\leq B<C\leq n}(q^{n_C}-q^{n_B})(1-q^{n_C+n_B})\,,\end{aligned}\tag{2.68}$$

for all $\lambda_1,\ldots,\lambda_n\in\mathbb{Z}$ and $\lambda_1\geq\ldots\geq\lambda_n\geq 0$.

2.5 The contributions from the NS and R sectors in $d>4$

The contribution from the NS sector can be obtained by taking a product of n copies of (2.25):

$$\begin{aligned}\chi^{SO(2n+1)}_{\text{NS}}(q,\boldsymbol{y})&=\prod_{A=1}^n\chi^{SO(3)}_{\text{NS}}(q;y_A)\\&=(q;q)^{-3n}_\infty\sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}}\sum_{\boldsymbol{n}\in\mathbb{Z}^n_+}\prod_{A=1}^n(-1)^{n_A+1}\left(1-q^{m_A+\frac{1}{2}}\right)\left(1-q^{n_A}\right)q^{\frac{1}{2}[n_A(n_A-1)+m_A^2]}\times\\&\qquad\sum_{\boldsymbol{k}\in\mathbb{Z}^n_{\geq 0}}^\infty\prod_{A=1}^n\left(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)}\right)[2k_A]_{y_A}\,.\end{aligned}\tag{2.69}$$

Similarly for the contribution from the R sector, the product of n copies of (2.42):

$$\begin{aligned}\chi^{SO(2n+1)}_{\text{R}}(q,\boldsymbol{y})&=\prod_{A=1}^n\chi^{SO(3)}_{\text{R}}(q;y_A)\\&=q^{-\frac{n}{8}}(q;q)^{-3n}_\infty\sum_{\boldsymbol{m}\in\mathbb{Z}^n_{\geq 0}}\sum_{\boldsymbol{n}\in\mathbb{Z}^n_+}\prod_{A=1}^n(-1)^{n_A+1}(1-q^{m_A+1})(1-q^{n_A})q^{\frac{1}{2}[n_A(n_A-1)+\left(m_A+\frac{1}{2}\right)^2]}\times\end{aligned}$$

$$\sum_{\boldsymbol{k}\in\mathbb{Z}^n_{\geq 0}} \prod_{A=1}^{n} (q^{n_A|k_A-m_A|} - q^{n_A(k_A+m_A+2)})\ [2k_A+1]_{y_A}\ . \tag{2.70}$$

The unrefined partition functions can be written as

$$\chi_{\rm NS}^{SO(2n+1)}(q,\{y_i=1\}) = q^{-n/8}\frac{\vartheta_3(1,q)^n}{\eta(q)^{3n}}\ , \tag{2.71}$$

$$\chi_{\rm R}^{SO(2n+1)}(q,\{y_i=1\}) = \frac{\vartheta_2(1,q)^n}{\eta(q)^{3n}}\ . \tag{2.72}$$

3 Internal SCFTs

The SCFT description of four- and six dimensional string compactifications with $\mathcal{N}_{4d}=1, \mathcal{N}_{4d}=2$ or $\mathcal{N}_{6d}=(1,0)$ spacetime SUSY comprises universal sectors with enhanced $\mathcal{N}_{2d}=2,4$ worldsheet SUSY [26–29]. The purpose of this section is to collect the associated charged characters, starting from the expressions given in [30, 31] but adapting the dependence on fugacities s, x and z of the internal symmetries to the R symmetries of the spectrum.

3.1 $\mathcal{N}_{2d}=2$ worldsheet superconformal algebra at $c=9$

The internal SCFT universal to any four dimensional string compactification with $\mathcal{N}_{4d}=1$ spacetime SUSY enjoys $\mathcal{N}_{2d}=2$ worldsheet SUSY. The resulting model independent partition function receives contributions from characters of the $\mathcal{N}_{2d}=2$ superconformal algebra with central charge $c=9$. Its representations are characterized by the conformal weight h and the $U(1)$ charge ℓ of their highest weight state. The representations needed to describe $\mathcal{N}_{4d}=1$ compactifications have $(h,\ell)=(0,0)$ in the NS sector and $(h,\ell)=\left(\frac{3}{8},\frac{3}{2}\right)$ in the R sector.

The $\mathcal{N}_{2d}=2$ SCFT at $c=9$ can be split into two decoupled sectors, each of which enjoys a $U(1)$ symmetry. The first one carries central charge $c=1$ and can be completely bosonized; let us denote the $U(1)$ occurring in this sector by $U(1)_1$ and its $h=1$ current by $\mathcal{J}_1$. In addition, there exists a second decoupled sector with $c=8$ which involves conformal primiaries $g^\pm$ of weight $\frac{4}{3}$, see e.g. [29]. It enjoys an independent $U(1)_2$ under which the $g^\pm$ have opposite charges. The $c=9$ supercurrent can be split into two components $G^\pm_{\rm int}$ that carry opposite charges under both $U(1)_1$ and $U(1)_2$ and factorize into conformal primaries of both sectors. The following figure 3 summarizes the decoupling SCFT ingredients.

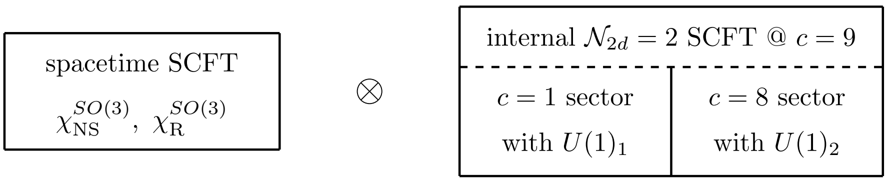

Figure 3: Universal SCFT ingredients of $\mathcal{N}_{4d}=1$ scenarios.

Spacetime symmetries are generated by BRST invariant $h=1$ SCFT operators, and it turns out that only the current $\mathcal{J}_1+\mathcal{J}_2$ associated with the diagonal subgroup $S(U(1)_1\times U(1)_2)$ is BRST closed. Hence, only $S(U(1)_1\times U(1)_2)$ can take the role of the $U(1)_R$ symmetry of the spectrum. Accordingly, we have to define the charged internal character with respect to the diagonal current $\mathcal{J}_1+\mathcal{J}_2$ to see the $U(1)_R$ at the level of the partition function[7].

[7]We cannot give a local representation of $\mathcal{J}_2$ in terms of the $g^\pm$ fields from the $c=8$ sector, but we can make its existence plausible through an analogy: The currents of the $SO(d)$ Lorentz symmetry schematically read

We denote the fugacity for charge under the $S(U(1)_1 \times U(1)_2) \cong U(1)_R$ subgroup by s. On the level of the charged characters, this leads to a different fugacity dependence compared to (3.15)[8] of [30] where the internal charge is defined through the $\mathcal{J}_1$ eigenvalue rather than the $\mathcal{J}_1 + \mathcal{J}_2$ eigenvalue[9]. For instance, the supercurrent components are products of operators from both sectors, so $G^{\pm}_{\text{int}}$ are charged under both $U(1)_1$ and $U(1)_2$ but neutral under the diagonal subgroup $S(U(1)_1 \times U(1)_2)$. The $\chi^{SO(3)}_{\text{NS,R}}$ factors in the following character formulae are due to the oscillator modes of the stress energy tensor, the internal current and the supercurrents. $U(1)_R$ neutrality of the latter forbids an s dependence at this point and sets the second argument of the $\chi^{SO(3)}_{\text{NS,R}}$ characters to unity.

The NS-sector

The internal character in this sector is given by

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=2,c=9}_{\text{NS},h=0,\ell=0}(q;s) &= (1-q)\chi^{SO(3)}_{\text{NS}}(q,1)\sum_{p\in\mathbb{Z}} \frac{q^{p^2+p-\frac{1}{2}}\, s^{2p}}{(1\,+\,q^{p-\frac{1}{2}})\,(1\,+\,q^{p+\frac{1}{2}})} \\
&= (q;q)^{-3}_{\infty}(1-q)\vartheta_3(1,q)\sum_{p\in\mathbb{Z}} \frac{q^{p^2+p-\frac{1}{2}}\, s^{2p}}{(1\,+\,q^{p-\frac{1}{2}})\,(1\,+\,q^{p+\frac{1}{2}})} \\
&= 1 + q + (2+s_2)q^{3/2} + (3+s_2)q^2 + (4+s_2)q^{5/2} + (6+2s_2)q^3 \\
&\quad + (10+4s_2)q^{7/2} + (15+6s_2)q^4 + (20+8s_2)q^{9/2} + (28+12s_2)q^5 \\
&\quad + (42+19s_2+s_4)q^{11/2} + (59+27s_2+2s_4)q^6 \\
&\quad + (78+36s_2+2s_4)q^{13/2} + (107+51s_2+3s_4)q^7 + O(q^{15/2})\ ,
\end{aligned} \tag{3.1}$$

where we have introduced the notation

$$s_n = \begin{cases} s^n + s^{-n} & : \ n > 0 \\ 1 & : \ n = 0 \end{cases} \tag{3.2}$$

to compactly represent the fugacity dependence.

The *unrefined* internal character (*i.e.* setting s to unity) can be rewritten in terms of modular functions as follows:

$$\chi^{\mathcal{N}_{2d}=2,c=9}_{\text{NS},h=0,\ell=0}(q;s=1) = q^{1/8}\frac{\vartheta_3(1,q)}{\eta(q)^3}\left[\vartheta_3(1,q^2) - q^{1/4}\vartheta_2(1,q^2)\right]\ . \tag{3.3}$$

The R-sector

The internal character in this sector is given by

$$\chi^{\mathcal{N}_{2d}=2,c=9}_{\text{R},h=3/8,\ell=3/2}(q;s) = (1-q)\chi^{SO(3)}_{\text{R}}(q,1)\sum_{p\in\mathbb{Z}} \frac{q^{p^2-1}\, s^{2p-1}}{(1\,+\,q^{p})\,(1\,+\,q^{p-1})}$$

$\psi^\mu\psi^\nu + X^{[\mu}\partial X^{\nu]}$. Even though X^μ itself is not a conformal field involved in the construction of the spectrum, the product $X^{[\mu}\partial X^{\nu]}$ is inevitable to form a BRST invariant completion of the $h=1$ primary $\psi^\mu\psi^\nu$. The addition of $X^{[\mu}\partial X^{\nu]}$ for the sake of BRST closure is the spacetime SCFT analogue of the $\mathcal{J}_2$ current.

[8] The R sector analogue of the NS character (3.15) is not explicitly displayed in [30] but must be inferred through spectral flow.

[9] The author of [30] denotes by z the fugacity of charge under $U(1)_1$. For us, it makes sense to rescale the units of internal charge by 3/2 which amounts to the correspondence $s \leftrightarrow z^{3/2}$ (in addition to the aforementioned inclusion of $\mathcal{J}_2$). Moreover, the character in (3.15) of [30] is defined as the trace over $q^{L_0-c/24}$, with $c=9$, instead of q^{L_0}. The reason we consider the latter is because we are dealing with critical string theories, and so the total central charge of all matter and (super) ghost sectors taken together vanishes; this explains the presence of $q^{-9/24}$ factor in (3.15) of [30] but not in (3.1).

$$= (q;q)_\infty^{-3}(1-q)\vartheta_2(1,q)\sum_{p\in\mathbb{Z}}\frac{q^{p^2-\frac{9}{8}}\,s^{2p-1}}{(1\,+\,q^p)\,(1\,+\,q^{p-1})} \tag{3.4}$$
$$= s_1 + 2s_1 q + 6s_1 q^2 + (2s_3+14s_1)q^3 + (4s_3+30s_1)q^4$$
$$+ (10s_3+62s_1)q^5 + (24s_3+122s_1)q^6 + (50s_3+230s_1)q^7 + O(q^8)\ .$$

The unrefined internal character can be rewritten in terms of modular functions as

$$\chi^{\mathcal{N}_{2d}=2,c=9}_{\mathrm{R},h=3/8,\ell=3/2}(q;s=1) = q^{-1/4}\frac{\vartheta_2(1,q)}{\eta(q)^3}\left[\vartheta_2(1,q^2) - q^{1/4}\vartheta_3(1,q^2)\right]\ . \tag{3.5}$$

Some features

Let us discuss some properties of the above internal characters.

- The units of $U(1)_R$ charge are normalized such that all integer powers of s occur. According to the infinite sums within (3.1) and (3.4), even powers s_{2p} firstly occur along with $q^{p^2+p-1/2}$, i.e. in the NS sector at mass level p^2+p-1. Odd powers s_{2p-1} of the $U(1)_R$ fugacity, on the other hand, firstly show up at power $q^{p^2-5/8}$, i.e. in the R sector at mass level p^2-1 [10].

- The unrefined internal R character (3.5) can be derived from the NS counterpart (3.3) by exchanging ϑ_2 and ϑ_3 and multiplying by an overall factor $q^{-3/8}$.

- In contrast to their cousins in [30], the charged characters (3.1) and (3.4) of the NS- and R sector are not related by spectral flow because the internal fugacity s is defined through the $U(1)_R$ symmetry current $\mathcal{J}_1+\mathcal{J}_2$ and not through the bosonizable $U(1)_1$ current $\mathcal{J}_1$.

- Both of the unrefined internal characters (3.3) and (3.5) are *not* modular invariant. This can be seen from the modular transformation $q\mapsto\tilde{q}=e^{-2\pi i/\tau}$,

$$\vartheta_2(1,\tilde{q}) = \vartheta_4(1,q)\sqrt{-i\tau}\ ,\qquad \eta(\tilde{q}) = \eta(q)\sqrt{-i\tau}\ ,\qquad \vartheta_3(1,\tilde{q}) = \vartheta_3(1,q)\sqrt{-i\tau}\ . \tag{3.6}$$

3.2 $\mathcal{N}_{2d}=4$ worldsheet superconformal algebra at $c=6$

The existence of eight supercharges in four or six dimensional spacetime implies that the universal part of the internal SCFT contains a sector with central charge $c=6$, enhanced $\mathcal{N}_{2d}=4$ worldsheet SUSY and $SU(2)$ Kac Moody symmetry at level 1. The $c=6$ representations contributing to the NS sector and R sector of $\mathcal{N}_{4d}=2$ and $\mathcal{N}_{6d}=(1,0)$ spectra are characterized by values $(h,\ell)=(0,0)$ and $(h,\ell)=\left(\frac{1}{4},\frac{1}{2}\right)$, respectively, of the conformal weight h and the spin ℓ with respect to the $SU(2)$ Kac Moody symmetry.

The $c=6$ SCFT is governed by $\mathcal{N}_{2d}=4$ worldsheet SUSY and $SU(2)$ Kac Moody symmetry at level $k=1$. In the notation of [29], the supercurrent components are built from two spin fields $\lambda^{1,2}$ of conformal weight $\frac{1}{4}$ which form a doublet under the $SU(2)_1$ Kac Moody currents $\mathcal{J}^{A=1,2,3}$ and additional weight $\frac{5}{4}$ fields $g_{1,2}$ which decouple from the $\mathcal{J}^A$. The $g_{1,2}$ form a doublet under another $SU(2)_2$ which is embedded into the SCFT sector decoupling from $\mathcal{J}^A$. Figure 4 summarizes the mutually decoupling SCFT sectors involved in $\mathcal{N}_{6d}=(1,0)$ compactifications:

[10] The onset of the s_m at q power $q^{\frac{1}{4}m^2+\frac{1}{2}m+const}$ might seem counterintuitive in view of the bosonized operators $e^{\pm\frac{i}{2}\sqrt{3}mH}$ (with H a free boson) which contribute $s_m q^{\frac{3}{8}m^2}$ to the character. The mismatch between the q exponents $\frac{1}{4}m^2+\frac{1}{2}m$ and $\frac{3}{8}m^2$ is caused by the fact that generic contributions to s_m at fixed $U(1)_R\cong S(U(1)_1\times U(1)_2)$ charge stem from composite fields with both $U(1)_1$ and $U(1)_2$ charges. The operator of lowest conformal weight along with some $s_{m\geq 3}$ is charged under both $U(1)_1$ and $U(1)_2$. Since the internal fugacities in [30] only count $U(1)_1$ charges and are insensitive to $U(1)_2$, the leading q power associated with some s_m in the characters of the reference can be directly traced back to the aforementioned operators $e^{\pm\frac{i}{2}\sqrt{3}mH}$.

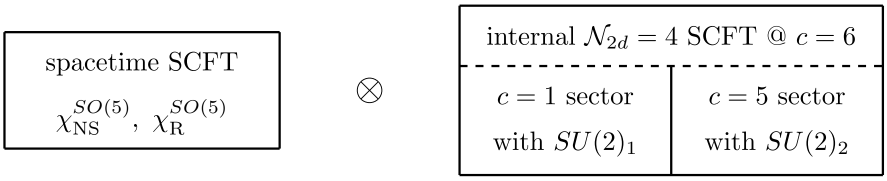

Figure 4: Universal SCFT ingredients of $\mathcal{N}_{6d}=(1,0)$ scenarios.

We shall use charged characters in the following where the fugacity r is defined with respect to the diagonal subgroup within the two decoupling $SU(2)$'s acting on the $\lambda^{1,2}$ and $g_{1,2}$ doublets. In other words, the insertion into the character trace is the BRST invariant sum of the two $SU(2)_{1,2}$ Cartan generators associated with the $SU(2)_R$ symmetry of the spectrum. This makes sure that the diagonal component $\lambda^1 g_1 + \lambda^2 g_2$ of the supercurrent is a singlet of the diagonal $SU(2)$, as required by the BRST invariance. The character formulae (21) and (22) in [31][11] are therefore slightly modified in their r dependence.

The NS-sector

The internal character in this sector is given by

$$\begin{aligned}
\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(q;r) &= \chi^{SO(3)}_{\mathrm{NS}}(q,1)\sum_{m\in\mathbb{Z}} q^{\frac{1}{2}m^2+\frac{1}{4}}\, r^{2m}\, \frac{q^{m-\frac{1}{2}}\;-\;r^{-2}}{1\;+\;q^{m-\frac{1}{2}}} \\
&= (q;q)^{-3}_{\infty}\vartheta_3(1,q)\sum_{k=0}^{\infty}[2k]_r \frac{(1-q)(1-q^{k+\frac{1}{2}})}{(1+q^{k-\frac{1}{2}})(1+q^{k+\frac{3}{2}})} q^{\frac{1}{2}k^2+k-\frac{1}{2}} \\
&= [0]_r + [2]_r q + ([2]_r+[0]_r)q^{3/2} + ([2]_r+2[0]_r)q^2 + (2[2]_r+2[0]_r)q^{5/2} \\
&\quad + (4[2]_r+2[0]_r)q^3 + ([4]_r+5[2]_r+4[0]_r)q^{7/2} + (2[4]_r+6[2]_r+7[0]_r)q^4 \\
&\quad + (2[4]_r+10[2]_r+8[0]_r)q^{9/2} + (3[4]_r+16[2]_r+9[0]_r)q^5 \\
&\quad + (6[4]_r+21[2]_r+15[0]_r)q^{11/2} + (9[4]_r+27[2]_r+23[0]_r)q^6 \\
&\quad + (12[4]_r+39[2]_r+27[0]_r)q^{13/2} + ([6]_r+17[4]_r+56[2]_r+33[0]_r)q^7 \\
&\quad + O(q^{15/2})\ .
\end{aligned} \tag{3.7}$$

The unrefined internal character for the NS-sector can be written as

$$\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{NS},h=0,\ell=0}(q;r=1) = q^{1/8}\frac{\vartheta_3(1,q)^2}{\eta(q)^3}\left[1-2iq^{1/8}\mu\left(\frac{1+\tau}{2},\tau\right)\right]\ , \tag{3.8}$$

where $\mu(u,\tau)$ is an Appell-Lerch sum defined in (A.20); for our purpose, we have[12]

$$\mu\left(\frac{1+\tau}{2},\tau\right) = -\frac{i}{\vartheta_3(1,q)}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{1}{2}m^2-\frac{1}{8}}}{1+q^{m-\frac{1}{2}}}\ , \tag{3.9}$$

where we have used the fact that $\vartheta_1\left(e^{2\pi i(1+\tau)/2},q\right) = q^{-1/8}\vartheta_3(1,q)$.

The R-sector

The internal character in this sector is given by

$$\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{R},h=\frac{1}{4},\ell=\frac{1}{2}}(q;r) = \chi^{SO(3)}_{\mathrm{R}}(q,1)\sum_{m\in\mathbb{Z}} r^{2m+1}\,\frac{q^m\;-\;r^{-2}}{1\;+\;q^m} q^{\frac{1}{2}m^2+\frac{1}{2}m}$$

[11] Note that the sign in the second pair of brackets in the numerator of Eq. (24) of [31] should be +.
[12] This function is also closely related to the function $h_3(q)$ introduced in [31–33].

$$= q^{-\frac{1}{8}}(q;q)_\infty^{-3}\vartheta_2(1,q)\sum_{k=0}^{\infty}[2k+1]_r\frac{(1-q)(1-q^{k+1})}{(1+q^k)(1+q^{k+2})}q^{\frac{1}{2}k^2+\frac{3}{2}k} \tag{3.10}$$
$$= [1]_r + 2[1]_r q + (2[3]_r + 4[1]_r)q^2 + (4[3]_r + 10[1]_r)q^3 + (10[3]_r + 20[1]_r)q^4 + (2[5]_r + 22[3]_r + 38[1]_r)q^5 + (6[5]_r + 44[3]_r + 72[1]_r)q^6 + (14[5]_r + 86[3]_r + 130[1]_r)q^7 + O(q^8)\ .$$

The unrefined internal character for the R-sector can be written as

$$\begin{aligned}\chi^{\mathcal{N}_{2d}=4,c=6}_{\mathrm{R},h=\frac{1}{4},\ell=\frac{1}{2}}(q;r=1) &= \frac{\vartheta_2(1,q)}{\eta(q)^3}\sum_{m\in\mathbb{Z}}\left(\frac{q^m-1}{1+q^m}\right)q^{\frac{1}{2}m(m+1)}\\ &= \frac{\vartheta_2(1,q)}{\eta(q)^3}\sum_{m\in\mathbb{Z}}\left[\left(1-\frac{2}{1+q^m}\right)q^{\frac{1}{2}m(m+1)}\right]\\ &= q^{-1/8}\frac{\vartheta_2(1,q)^2}{\eta(q)^3}\left[1-2iq^{1/8}\mu\left(1/2,\tau\right)\right]\ ,\end{aligned} \tag{3.11}$$

where we have[13]

$$\mu\left(1/2,\tau\right) = -\frac{i}{\vartheta_2(1,q)}\sum_{m\in\mathbb{Z}}\frac{q^{\frac{1}{2}m(m+1)}}{1+q^m}\ , \tag{3.12}$$

where we have used the fact that $\vartheta_1(-1,q)=\vartheta_2(1,q)$.

Some features

- According to appendix A, characters $[n]_r$ of $SU(2)_R$ follow the same highest weight notation as for $SO(3)$, i.e. we have $[1]_r = r + r^{-1}$ for the fundamental representation and $[n]_r = \sum_{k=-n/2}^{+n/2} r^{2k}$ in the general spin $n/2$ case. Again, the infinite sum representations allow to read off the lowest level where individual $SU(2)_R$ representations contribute: Integer spin representations $[2k]_r$ firstly occur at power $q^{k^2/2+k-1/2}$, i.e. at mass level[14] $\lfloor k^2/2+k-1/2\rfloor$. Spinorial representations $[2k+1]_r$, on the other hand, firstly show up at $q^{k^2/2+3k/2-1/4}$, i.e. at mass level $k(k+3)/2$.[15]
- Observe that the unrefined internal characters in both NS and R sectors involve Appell-Lerch sums, which are mock modular forms. Since the characters and are holomorphic in q, it is immediate that they are *not* modular invariant. Also, as before in the $\mathcal{N}_{2d}=2$ SCFT, the relation between NS and R characters through spectral flow is absent due to the adaption of the internal fugacity to the $SU(2)_R$ symmetry.

4 Spectrum in $\mathcal{N}_{4d}=1$ supersymmetric compactifications

This section opens up the main body of this work where the SCFT ingredients introduced so far are applied to counting universal super Poincaré multiplets in the perturbative string spectrum[16].

[13] This function is also closely related to the function $h_2(q)$ introduced in [31–33].

[14] The floor function $\lfloor\cdot\rfloor$ picks out the nearest integer smaller than or equal to its argument.

[15] The lowest q power along with some $SU(2)_R$ representation $[n]_r$ is generically caused by an operator charged under both $SU(2)_1$ and $SU(2)_2$. That is why one cannot identify these leading q exponents with the conformal dimension of a simple CFT operator such as an exponential $e^{\pm iqH}$, see the footnote at the end of subsection 3.1.

[16] The methods within this work are adapted to the representatives of physical states in the canonical superghost pictures: After stripping off the superghost contributions $e^{q\phi}$ from the $h=1$ vertex operators (with $q=-1$ and $h[e^{-\phi}]=\frac{1}{2}$ in the NS sector as well as $q=-\frac{1}{2}$ and $h[e^{-\phi/2}]=\frac{3}{8}$ in the R sector), this amounts to counting operators in the matter part of the SCFTs with weight $h=\frac{1}{2}$ in the NS sector and $h=\frac{3}{8}$ in the R sector.

We start with the phenomenologocially relevant and mathematically most tractable $\mathcal{N}_{4d}=1$ supersymmetric scenario. Its SCFT description requires the internal sector with enhanced $\mathcal{N}_{2d}=2$ worldsheet SUSY introduced in subsection 3.1, independently on the compactification details. The BRST invariant completion of the internal current takes the role of the $U(1)_R$ symmetry generator. Lorentz quantum numbers enter through the partition functions (2.25) and (2.42) of the spacetime SCFT for the ∂X^μ and ψ^μ oscillators, expressed in terms of characters of the massive little group $SO(3)$ in four dimension.

The universal part of the $\mathcal{N}_{4d}=1$ spectrum is built from both spacetime oscillators and internal operators. On the level of its partition function $\chi^{\mathcal{N}_{4d}=1}(q;y,s)$, this amounts to forming a GSO projected product of NS- and R characters from the spacetime- and internal SCFT, see (3.1) and (3.4) for the latter. In a power series expansion in q, the coefficient of the n'th power q^n comprises characters for the $\mathcal{N}_{4d}=1$ super Poincaré multiplets occurring at the n'th mass level with $m^2=n/\alpha'$. The aforementioned massive supercharacters are functions of $SO(3)$ fugacity y and $U(1)_R$ fugacity s.

The fundamental $\mathcal{N}_{4d}=1$ multiplet[17] consists of 2 real bosonic degrees of freedom and a Majorana fermion with 2 real fermonic on-shell degrees of freedom after taking the Dirac equation into account, see e.g. [34]. The two real bosonic degrees of freedom can be complexified to yield a complex scalar and its complex conjugate; they transform as a singlet under the little group $SO(3)$ and each of them carries opposite R-charges $+1$ and -1. On the other hand, the two real fermonic degrees of freedom transform as a doublet under the little group $SO(3)$ and each of them carries zero R-charge. Thus, these 2+2 states yield the character

$$Z(\mathcal{N}_{4d}=1) \;\;=\;\; [1]_y+(s+s^{-1})\;. \tag{4.1}$$

Any other massive representation of $\mathcal{N}_{4d}=1$ super Poincaré is specified by the little group $SO(3)$ quantum number n and the $U(1)_R$ charge Q of its highest weight state or Clifford vacuum. Its $SO(3)\times U(1)_R$ constituents follow from a tensor product:

$$\begin{aligned}[\![n,Q]\!] \;&:=\; Z(\mathcal{N}_{4d}=1)\cdot s^Q[n]_y \;\;=\;\; s^Q[n]_y\,\big([1]_y+(s+s^{-1})\big) \\ &=\; \begin{cases} s^Q\,\big([n+1]\;+\;(s+s^{-1})\,[n]\;+\;[n-1]\big) & \text{for } n\geq 1 \\ s^Q\,\big([1]\;+\;(s+s^{-1})\,[0]\big) & \text{for } n=0\end{cases}\end{aligned} \tag{4.2}$$

The super-Poincaré character $[\![n,Q]\!]$ corresponds to $4(n+1)$ states of spin $\frac{n+1}{2}$, $\frac{n}{2}$ and (if $n\neq 0$) $\frac{n-1}{2}$ that can be generated from a Clifford vacuum with spin $n/2$ and $U(1)_R$ charge $Q+1$[18]. Note that Q is even whenever the maximum spin quantum number $n+1$ is.

In this setting, we find the (GSO projected) $\mathcal{N}_{4d}=1$ partition function

$$\chi^{\mathcal{N}_{4d}=1}(q;y,s) \;\;:=\;\; \chi^{\mathcal{N}_{4d}=1}_{\text{NS}}\,|_{\text{GSO}}\,(q;y,s)+\chi^{\mathcal{N}_{4d}=1}_{\text{R}}\,|_{\text{GSO}}\,(q;y,s)\;, \tag{4.3}$$

where GSO projection removes half odd integer mass level $\alpha' m^2\in\mathbb{Z}-\frac{1}{2}$ from the NS sector and interlocks spacetime chirality with $U(1)_R$ charges in the R sector. We can capture this projection

[17]As we shall see below, the fundamental multiplet does not appear on its own in both massless and massive spectra. Representations appearing in the massive spectrum arise from certain non-trivial products with the fundamental multiplet.

[18]In this terminology, the first label of $[\![n,Q]\!]$ refers to the average spin of the $SO(3)$ irreducibles. We deviate from the common practice that supermultiplets are referred to through the highest spin therein. The supercharacter $[\![3,0]\!]=[4]+[2]+(s+s^{-1})[3]$, for instance, describes $U(1)_R$ neutral bosons of spin two and one, and two massive gravitinos of opposite $U(1)_R$ charges.

through[19]:

$$\chi_{\rm NS}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q) \;=\; \frac{1}{2}\,q^{-\frac{1}{2}}\left[\chi_{\rm NS}^{SO(3)}(q;y)\,\chi^{\mathcal{N}_{2d}=2,c=9}_{{\rm NS},h=0,\ell=0}(q;s)\;-\;\chi_{\rm NS}^{SO(3)}(e^{2\pi i}q;y)\,\chi^{\mathcal{N}_{2d}=2,c=9}_{{\rm NS},h=0,\ell=0}(e^{2\pi i}q;s)\right],$$

$$\chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q) \;=\; \frac{1}{2}\,\chi_{\rm R}^{SO(3)}(q;y)\,\chi^{\mathcal{N}_{2d}=2,c=9}_{{\rm R},h=3/8,\ell=3/2}(q;s)\;. \qquad (4.4)$$

In order to compactly represent the leading terms in a power series expansion of the partition function $\chi^{\mathcal{N}_{4d}=1}$, let us introduce the shorthand

$$[\![n,\pm Q]\!] \;:=\; \begin{cases} [\![n,+Q]\!] \;+\; [\![n,-Q]\!] & :\; Q\neq 0 \\ [\![n,0]\!] & :\; Q=0 \end{cases} \qquad (4.5)$$

which exploits that $U(1)_R$ charges always appear on symmetric footing $Q \leftrightarrow -Q$. The pairing of supermultiplets with opposite (nonzero) $U(1)_R$ charges combines Majorana fermions as they appear in the fundamental multiplet (4.1) to Dirac fermions.

$$\begin{aligned}\chi^{\mathcal{N}_{4d}=1}(q;y,s) \;=\;& \underbrace{\left(y^2+y^{-2}\;+\;\frac{1}{2}\,(y+y^{-1})\,(s+s^{-1})\right)q^0}_{\text{4 massless states}} \;+\; \underbrace{\big(\,[\![3,0]\!]\;+\;[\![0,\pm1]\!]\,\big)\,q}_{\text{24 states at level 1}} \\ &+\;\underbrace{\big(\,[\![5,0]\!]\;+\;[\![3,0]\!]\;+\;2\,[\![2,\pm1]\!]\;+\;2\,[\![1,0]\!]\,\big)\,q^2}_{\text{104 states at level 2}} \\ &+\;\big(\,[\![7,0]\!]\;+\;[\![5,0]\!]\;+\;3\,[\![4,\pm1]\!]\;+\;5\,[\![3,0]\!]\;+\;2\,[\![2,\pm1]\!] \\ &\quad+\;[\![1,\pm2]\!]\;+\;5\,[\![1,0]\!]\;+\;3\,[\![0,\pm1]\!]\,\big)\,q^3\;+\;\mathcal{O}(q^4)\end{aligned} \qquad (4.6)$$

The content of the first eight $\mathcal{N}_{4d}=1$ levels is summarized in Table 2. The explicit form of the vertex operators at mass level one[20] can be found in section 5 of [29] (equations (5.3) to (5.6) for bosons and equations (5.14) to (5.18) for fermions) in the RNS framework, and references [35, 36] provide their superspace description.

Character multiplicities up to mass level $\alpha' m^2 = 25$ are gathered in table 3 and in the tables of appendix B.1.

4.1 The total number of states at a given mass level

In this subsection, we focus on the total number of states present at a given mass level. These numbers can indeed be obtained by adding up the dimensions of representations presented in table 2. Our aim here is to compute such numbers analytically and asymptotically for large mass levels.

[19]The formula for the GSO projected R sector is reliable for positive powers $q^{\geq 1}$ only and inaccurate at the massless level: The coefficient of q^0 in $\chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}$ is $\frac{1}{2}(y+y^{-1})(s+s^{-1})$ instead of the desired value $ys+(ys)^{-1}$. One can just add to the former $\frac{1}{2}(y-y^{-1})(s-s^{-1})$ to compensate this mismatch. This artifact of the mismatch between massive and massless little groups does not affect the main focus our analysis – the massive particle content. Indeed, the character ys corresponds to the left-handed gaugino and the character $(ys)^{-1}$ corresponds to the right-handed gaugino; they carry opposite R-charge $+1$ and -1 and opposite helicities $+1/2$ and $-1/2$.

[20]Let us discuss about the states at the first mass level. The 24 total states consist of the following multiplets:

(1) **the massive spin 3/2 multiplet** $[\![3,0]\!]$: it contains a massive spin 2 field with 5 on-shell degrees of freedoms (OSDOFs), a massive spin 1 field with 3 OSDOFs, a massive spin 3/2 field with 4 OSDOFs, and a Dirac fermion with 4 OSDOFs; so we have 8+8 real OSDOFs in total

(2) **the massive spin 0 multiplet** $[\![0,\pm1]\!]$: the two constituents $[\![0,1]\!]$ and $[\![0,-1]\!]$ of the massive scalar multiplet correspond to two massless chiral fields, Φ and $\widetilde{\Phi}$ (not complex conjugate to each other) at $Q=\pm1$. The opposite Q-charges are necessary to form an invariant mass term $\Phi\widetilde{\Phi}$ in the superpotential. This multiplet contains 4 + 4 real OSDOFs coming from two complex scalars plus two Majorana fermions; the latter are equivalent to one massive Dirac fermion. Note that the spin 0 multiplet is also referred to as two spin 1/2 multiplets in [29].

$\alpha' m^2$	Representations of $\mathcal{N}_{4d}=1$ super Poincaré
1	$[\![3,0]\!] + [\![0,\pm1]\!]$
2	$[\![5,0]\!] + [\![3,0]\!] + 2\,[\![2,\pm1]\!] + 2\,[\![1,0]\!]$
3	$[\![7,0]\!] + [\![5,0]\!] + 3\,[\![4,\pm1]\!] + 5\,[\![3,0]\!] + 2\,[\![2,\pm1]\!] + [\![1,\pm2]\!] + 5\,[\![1,0]\!] + 3\,[\![0,\pm1]\!]$
4	$[\![9,0]\!] + [\![7,0]\!] + 3\,[\![6,\pm1]\!] + 7\,[\![5,0]\!] + 4\,[\![4,\pm1]\!] + 2\,[\![3,\pm2]\!] + 12\,[\![3,0]\!]$ $+11\,[\![2,\pm1]\!] + 2\,[\![1,\pm2]\!] + 12\,[\![1,0]\!] + 3\,[\![0,\pm1]\!]$
5	$[\![11,0]\!] + [\![9,0]\!] + 3\,[\![8,\pm1]\!] + 7\,[\![7,0]\!] + 5\,[\![6,\pm1]\!] + 2\,[\![5,\pm2]\!] + 17\,[\![5,0]\!] + 18\,[\![4,\pm1]\!]$ $+6\,[\![3,\pm2]\!] + 31\,[\![3,0]\!] + 20\,[\![2,\pm1]\!] + 6\,[\![1,\pm2]\!] + 28\,[\![1,0]\!] + [\![0,\pm3]\!] + 15\,[\![0,\pm1]\!]$
6	$[\![13,0]\!] + [\![11,0]\!] + 3\,[\![10,\pm1]\!] + 7\,[\![9,0]\!] + 5\,[\![8,\pm1]\!] + 2\,[\![7,\pm2]\!] + 19\,[\![7,0]\!]$ $+21\,[\![6,\pm1]\!] + 8\,[\![5,\pm2]\!] + 45\,[\![5,0]\!] + 39\,[\![4,\pm1]\!] + 15\,[\![3,\pm2]\!] + 72\,[\![3,0]\!]$ $+3\,[\![2,\pm3]\!] + 58\,[\![2,\pm1]\!] + 17\,[\![1,\pm2]\!] + 64\,[\![1,0]\!] + 21\,[\![0,\pm1]\!]$
7	$[\![15,0]\!] + [\![13,0]\!] + 3\,[\![12,1]\!] + 7\,[\![11,0]\!] + 5\,[\![10,1]\!] + 2\,[\![9,2]\!] + 19\,[\![9,0]\!] + 22\,[\![8,1]\!]$ $+8\,[\![7,2]\!] + 51\,[\![7,0]\!] + 49\,[\![6,1]\!] + 22\,[\![5,2]\!] + 108\,[\![5,0]\!] + 4\,[\![4,3]\!] + 105\,[\![4,1]\!]$ $+43\,[\![3,2]\!] + 166\,[\![3,0]\!] + 5\,[\![2,3]\!] + 115\,[\![2,1]\!] + 38\,[\![1,2]\!] + 136\,[\![1,0]\!] + 6\,[\![0,3]\!]$ $+66\,[\![0,1]\!]$
8	$[\![17,0]\!] + [\![15,0]\!] + 3\,[\![14,1]\!] + 7\,[\![13,0]\!] + 5\,[\![12,1]\!] + 2\,[\![11,2]\!] + 19\,[\![11,0]\!]$ $+22\,[\![10,1]\!] + 8\,[\![9,2]\!] + 53\,[\![9,0]\!] + 52\,[\![8,1]\!] + 24\,[\![7,2]\!] + 125\,[\![7,0]\!] + 4\,[\![6,3]\!]$ $+135\,[\![6,1]\!] + 62\,[\![5,2]\!] + 254\,[\![5,0]\!] + 10\,[\![4,3]\!] + 223\,[\![4,1]\!] + 101\,[\![3,2]\!] + 357\,[\![3,0]\!]$ $+21\,[\![2,3]\!] + 274\,[\![2,1]\!] + [\![1,4]\!] + 89\,[\![1,2]\!] + 289\,[\![1,0]\!] + 7\,[\![0,3]\!] + 112\,[\![0,1]\!]$

Table 2. The content of the first eight $\mathcal{N}_{4d}=1$ levels.

The starting point is the unrefined partition function obtained by setting the fugacities y and s in (4.3) to unity. The total number of states N_m at the mass level m can be read off from the coefficient of q^m in the power series of $\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)$.

Supersymmetry implies that

$$\chi_{\rm NS}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y=1, s=1) = \chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y=1, s=1)\ . \tag{4.7}$$

which can, of course, be checked directly using (4.4), (2.27), (2.44), (3.3) and (3.5). Since the formula for the R sector is simpler, we proceed from there.

$$\begin{aligned}\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1) &= 2\chi_{\rm R}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q; y=1, s=1)\\ &= \chi_{\rm R}^{SO(3)}(q, y=1)\chi_{{\rm R}, h=3/8, \ell=3/2}^{\mathcal{N}_{2d}=2, c=9}(q; s=1)\\ &= q^{-1/4}\frac{\vartheta_2(1,q)^2}{\eta(q)^6}\left[\vartheta_2(1,q^2) - q^{1/4}\vartheta_3(1,q^2)\right]\ .\end{aligned} \tag{4.8}$$

Indeed, the power series of $\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)$ in q reproduces the numbers presented in the first column of Table 1. We mention in passing that $\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)$ is *not* a modular form.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can be computed from

$$N_m = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{{\rm d}q}{q^{m+1}}\ \chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)\ , \tag{4.9}$$

where $\mathcal{C}$ is a contour around the origin.

Let us compute the number of states N_m in the limit $m\to\infty$. Since the integrand of (4.9) is sharply peaked near $q=1$, we need to examine the behaviour of $\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1)$ as $q\to 1^-$.

The $q \to 1^-$ regime in question is related to the easily accessible $q \to 0$ limit

$$\eta(q) \sim q^{1/24} \ , \qquad \vartheta_3(1,q) \sim 1 \ , \qquad \vartheta_4(1,q) \sim 1 \ , \qquad q \to 0 \tag{4.10}$$

through modular transformation $q = e^{2\pi i\tau} \mapsto \widetilde{q} = e^{-2\pi i/\tau}$:

$$\begin{aligned}
&\vartheta_2\textbf{-function}: && \vartheta_4\left(1,\tilde{q}\right) = \vartheta_2(1,q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\vartheta_2(1,q) \\
&\Rightarrow && \vartheta_2(1,q) \sim \sqrt{2\pi}(1-q)^{-1/2}, \qquad q \to 1^- \ , && (4.11) \\
&\vartheta_3\textbf{-function}: && \vartheta_3\left(1,\tilde{q}\right) = \vartheta_3(1,q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\vartheta_3(1,q) \\
&\Rightarrow && \vartheta_3(1,q) \sim \sqrt{2\pi}(1-q)^{-1/2}, \qquad q \to 1^- \ , && (4.12) \\
&\eta\textbf{-function}: && \eta\left(\tilde{q}\right) = \eta(q)\sqrt{-i\tau} \sim \frac{1}{\sqrt{2\pi}}(1-q)^{1/2}\eta(q) \\
&\Rightarrow && \eta(q) \sim \sqrt{2\pi}(1-q)^{-1/2}\exp\left(\frac{\pi^2}{6\log q}\right), \qquad q \to 1^- \ , && (4.13)
\end{aligned}$$

Hence, we have

$$\vartheta_2(1,q^2) \sim \vartheta_3(1,q^2) \sim \sqrt{2\pi}(1-q^2)^{-1/2} \ , \quad q \to 1^- \ , \tag{4.14}$$

and so as $q \to 1^-$,

$$\chi^{\mathcal{N}_{4d}=1}(q; y=1, s=1) \sim (2\pi)^{-3/2}(1-q)^2(1-q^{1/4})(1-q^2)^{-1/2}\exp\left(-\frac{\pi^2}{\log q}\right) \ . \tag{4.15}$$

Hence, as $m \to \infty$,

$$N_m \sim (2\pi)^{-3/2}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}q}{q}\ (1-q)^2(1-q^{1/4})(1-q^2)^{-1/2}\exp\left(-\frac{\pi^2}{\log q} - m\log q\right) \ . \tag{4.16}$$

Observe that the argument of the exponential function has a critical value at $q_0 = \exp(-\pi/\sqrt{m})$; this is the saddle point. The direction of steepest descent at this point is the imaginary direction in q. We deform the contour $\mathcal{C}$ such that it passes through $q = q_0$ and tangent to this direction. The leading contribution comes from expansions around $q = q_0$ in the steepest descent direction. Writing $q = q_0 e^{i\theta}$, we have

$$\begin{aligned}
N_m &\sim (2\pi)^{-3/2}(1-q_0)^2(1-q_0^{1/4})(1-q_0^2)^{-1/2}\times \\
&\qquad\qquad \frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\mathrm{d}\theta\exp\left(-\frac{\pi^2}{i\theta+\log q_0} - m(i\theta + \log q_0)\right), \quad \epsilon > 0 \\
&\sim (2\pi)^{-3/2}(1-q_0)^2(1-q_0^{1/4})(1-q_0^2)^{-1/2}\times \\
&\qquad\qquad e^{2\pi\sqrt{m}}\frac{1}{2\pi}\int_{-\epsilon}^{\epsilon}\mathrm{d}\theta\exp\left(-\frac{m^{3/2}}{\pi}\theta^2 + O(\theta^3)\right), \qquad \epsilon > 0 \\
&\sim (2\pi)^{-3/2}(1-q_0)^2(1-q_0^{1/4})(1-q_0^2)^{-1/2}e^{2\pi\sqrt{m}}\frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\theta\exp\left(-\frac{m^{3/2}}{\pi}\theta^2\right) \\
&\sim \frac{\pi}{32}m^{-2}\exp\left(2\pi\sqrt{m}\right) \ , \qquad m \to \infty \ .
\end{aligned} \tag{4.17}$$

The plot of the exact and asymptotic values for N_m against m is depicted in Figure 5.

4.2 The GSO projected NS- and R sectors

In what follows, we compute analytic expressions of the refined partition function $\chi^{\mathcal{N}_{4d}=1}(q; y, s)$ and discuss its asymptotic behaviour.

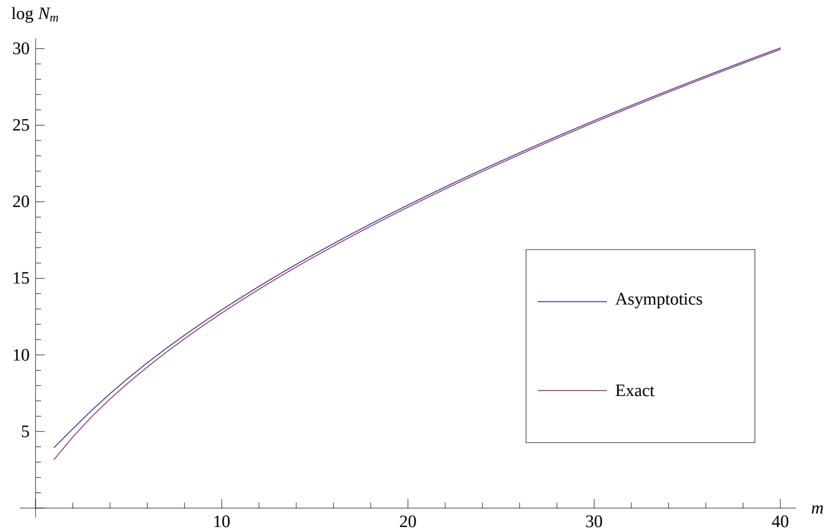

Figure 5: The plot of the exact and asymptotic values of $\log N_m$ against the mass level m for the case of 4 supercharges.

The NS sector

Let us write the partition function $\chi_{\rm NS}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q;y,s)$, defined in (4.4), as

$$\chi_{\rm NS}^{\mathcal{N}_{4d}=1}\,|_{\rm GSO}\,(q;y,s) = \sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}[2k]_y s^{2p}\; F^{\rm NS}_{k,p}(q)\ , \tag{4.18}$$

where the function $F^{\rm NS}_{k,p}(q)$ follows from (2.25), (3.1) and (4.4):

$$\begin{aligned} F^{\rm NS}_{k,p}(q) &= (q;q)_\infty^{-6}(1-q)q^{p^2+p-1}\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}\sum_{m=0}^{\infty}\left(q^{n|k-m|}-q^{n(k+m+1)}\right) \\ &\times \frac{1}{2}q^{\frac{1}{2}m^2}\left[\frac{(1-q^{m+\frac{1}{2}})\vartheta_3(1,q)}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}+(-1)^{m^2}\frac{(1+q^{m+\frac{1}{2}})\vartheta_4(1,q)}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right]\ . \end{aligned} \tag{4.19}$$

This expression can be simplified further in the asymptotic limit $k\to\infty$. In this limit, $q^{n|k-m|}\sim q^{n(k-m)}$ and the dominant contribution in the summation over n comes from $n=1$. The summation over n can be asymptotically evaluated as follows (assume that m is finite):

$$\begin{aligned} \sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}(q^{n|k-m|}-q^{n(k+m+1)}) &\sim \sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{n(k-m)}(1-q^{n(2m+1)}) \\ &\sim \frac{q^k(1-q)\left(1-q^{2k}\right)}{(1+q^k)^4}\left\{q^{-m}\left(1-q^{2m+1}\right)\right\}\ . \end{aligned} \tag{4.20}$$

The summation over m can be evaluated by considering

$$\sum_{m=0}^{\infty}q^{\frac{1}{2}m^2-m}\left(1-q^{m+\frac{1}{2}}\right)\left(1-q^{2m+1}\right) = q^{-\frac{1}{2}}(1-q)\vartheta_3(1,q)\ , \tag{4.21}$$

$$\sum_{m=0}^{\infty}(-1)^{m^2}q^{\frac{1}{2}m^2-m}\left(1+q^{m+\frac{1}{2}}\right)\left(1-q^{2m+1}\right) = -q^{-\frac{1}{2}}(1-q)\vartheta_4(1,q)\ . \tag{4.22}$$

In such a limit, the function $F^{\mathrm{NS}}_{k,p}(q)$ becomes

$$
\begin{aligned}
F^{\mathrm{NS}}_{k,p}(q) &\sim \frac{1}{2}(q;q)_\infty^{-6}(1-q)^3 q^{p^2+p+k-\frac{3}{2}}\,\frac{1-q^{2k}}{(1+q^k)^4}\left[\frac{\vartheta_3(1,q)^2}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}-\frac{\vartheta_4(1,q)^2}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right]\\
&\sim \frac{1}{2}(q;q)_\infty^{-6}(1-q)^3 q^{p^2+p+k-\frac{3}{2}}\left[\frac{\vartheta_3(1,q)^2}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{1}{2}})}-\frac{\vartheta_4(1,q)^2}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{1}{2}})}\right] \ , \qquad k\to\infty \ .
\end{aligned}
\tag{4.23}
$$

The R sector

Similarly the partition function $\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q;y,s)$, defined in (4.4), can be written as

$$
\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\,(q;y,s)=\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}[2k+1]_y s^{2p-1}\;F^{\mathrm{R}}_{k,p}(q) \ , \tag{4.24}
$$

where the function $F^{\mathrm{R}}_{k,p}(q)$ follows from (2.42), (3.4) and (4.4):

$$
\begin{aligned}
F^{\mathrm{R}}_{k,p}(q) = \frac{1}{2}(q;q)_\infty^{-6}(1-q)\frac{q^{p^2-\frac{5}{4}}}{(1+q^p)(1+q^{p-1})}\vartheta_2(1,q)\times\\
\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}\sum_{m=0}^{\infty}q^{\frac{1}{2}(m+\frac{1}{2})^2}(1-q^{m+1})(q^{n|k-m|}-q^{n(k+m+2)}) \ .
\end{aligned}
\tag{4.25}
$$

In the limit $k\to\infty$, this function can be simplified further. The summation over n can be asymptotically evaluated as follows (assume that m is finite):

$$
\begin{aligned}
\sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{\binom{n}{2}}(q^{n|k-m|}-q^{n(k+m+2)}) &\sim \sum_{n=1}^{\infty}(-1)^{n+1}(1-q^n)q^{n(k-m)}(1-q^{n(2m+2)})\\
&\sim \frac{q^k(1-q)\left(1-q^{2k}\right)}{\left(1+q^k\right)^4}\left\{q^{-m}\left(1-q^{2m+2}\right)\right\} \ ,
\end{aligned}
\tag{4.26}
$$

and the summation over m can be computed as follows:

$$
\sum_{m=0}^{\infty}q^{\frac{1}{2}(m+\frac{1}{2})^2-m}\left(1-q^{m+1}\right)\left(1-q^{2m+2}\right)=(1-q)\vartheta_2(1,q) \ . \tag{4.27}
$$

Therefore, we have the following asymptotic formula:

$$
\begin{aligned}
F^{\mathrm{R}}_{k,p}(q) &\sim \frac{1}{2}(q;q)_\infty^{-6}\frac{q^{p^2+k-\frac{5}{4}}(1-q)^3\left(1-q^{2k}\right)}{(1+q^p)\left(1+q^{p-1}\right)\left(1+q^k\right)^4}\vartheta_2(1,q)^2\\
&\sim \frac{1}{2}(q;q)_\infty^{-6}\frac{q^{p^2+k-\frac{5}{4}}(1-q)^3}{(1+q^p)\left(1+q^{p-1}\right)}\vartheta_2(1,q)^2 \ , \qquad k\to\infty \ .
\end{aligned}
\tag{4.28}
$$

Combining both sectors

Combining the NS- and R contributions from the previous subsections gives rise to the following $SO(3)\times U(1)_R$ covariant partition function

$$
\begin{aligned}
\chi^{\mathcal{N}_{4d}=1}(q;y,s) &= \sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}\left([2k]_y s^{2p}\;F^{\mathrm{NS}}_{k,p}(q)+[2k+1]_y s^{2p-1}\;F^{\mathrm{R}}_{k,p}(q)\right)\\
&= \sum_{k=0}^{\infty}\left\{[2k]\left(F^{\mathrm{NS}}_{k,0}(q)+\sum_{p=1}^{\infty}s_{2p}F^{\mathrm{NS}}_{k,p}(q)\right)+[2k+1]\sum_{p=1}^{\infty}s_{2p-1}F^{\mathrm{R}}_{k,p}(q)\right\} \ ,
\end{aligned}
\tag{4.29}
$$

where s_m is defined by (3.2). Even though the $F^{\rm NS}_{k,p}$ and $F^{\rm R}_{k,p}$ functions are known, the representation (4.29) of the overall partition function does not make $\mathcal{N}_{4d}=1$ SUSY manifest to all mass levels. In order to do so, we have to combine $SO(3)\times U(1)_R$ representations to supermultiplets (4.2) and rewrite (4.29) as[21]

$$\chi^{\mathcal{N}_{4d}=1}(q;y,s)=\sum_{n=0}^{\infty}\sum_{Q=0}^{\infty}[\![n,\pm Q]\!]\,M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)\ . \tag{4.30}$$

This introduces a *multiplicity generating function* $M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)$ for the supermultiplet $[\![n,Q]\!]$ appearing in the partition function $\chi^{\mathcal{N}_{4d}=1}$. To lighten our notation in the subsequent steps, we shall use the shorthand

$$G_{n,Q}(q):=M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)\ . \tag{4.31}$$

By comparing (4.29) with (4.30), it is immediate that

$$G_{2n,2Q}(q)=G_{2n+1,2Q+1}(q)=0\ ,\qquad \text{for all } n\geq 0 \text{ and } Q\geq 0\ . \tag{4.32}$$

Recurrence relations

In order to relate the supersymmetric multiplicity generating functions $G_{n,Q}$ to their $SO(3)\times U(1)_R$ relatives $F^{\rm NS}_{k,p}$ and $F^{\rm R}_{k,p}$, we use (4.2) to rewrite (4.30) in terms of characters of irreducible $SO(3)$ characters and the fugacity s as

$$\begin{aligned}\chi^{\mathcal{N}_{4d}=1}(q;y,s)=[0]\Big[(G_{1,0}+2G_{0,1})+\sum_{Q=1}^{\infty}s_{2Q}\left(G_{0,2Q-1}+G_{1,2Q}+G_{0,2Q+1}\right)\Big]\\ +\sum_{k=1}^{\infty}[2k]\Big[\left(G_{2k-1,0}+2G_{2k,1}+G_{2k+1,0}\right)+\sum_{Q=1}^{\infty}s_{2Q}\left(G_{2k-1,2Q}+G_{2k,2Q-1}+G_{2k,2Q+1}+G_{2k+1,2Q}\right)\Big]\\ +\sum_{k=0}^{\infty}[2k+1]\sum_{Q=1}^{\infty}s_{2Q-1}\left(G_{2k,2Q-1}+G_{2k+1,2Q-2}+G_{2k+1,2Q}+G_{2k+2,2Q-1}\right)\ ,\end{aligned} \tag{4.33}$$

where $G_{n,Q}$ is a shorthand notation for $G_{n,Q}(q)$.

Comparing (4.29) with (4.33), we have the following relations:

$$2G_{0,1}(q)+G_{1,0}(q)=F^{\rm NS}_{0,0}(q)\ , \tag{4.34}$$

$$G_{2k-1,0}(q)+2G_{2k,1}(q)+G_{2k+1,0}(q)=F^{\rm NS}_{k,0}(q)\ ,\quad k\geq 1 \tag{4.35}$$

$$G_{0,2Q-1}(q)+G_{0,2Q+1}(q)+G_{1,2Q}(q)=F^{\rm NS}_{0,Q}(q)\ ,\quad Q\geq 1 \tag{4.36}$$

$$G_{2k-1,2Q}(q)+G_{2k,2Q-1}(q)+G_{2k,2Q+1}(q)+G_{2k+1,2Q}(q)=F^{\rm NS}_{k,Q}(q)\ ,\quad k,Q\geq 1 \tag{4.37}$$

$$G_{2k,2Q-1}(q)+G_{2k+1,2Q-2}(q)+G_{2k+1,2Q}(q)+G_{2k+2,2Q-1}(q)=F^{\rm R}_{k,Q}(q)\ ,\quad k\geq 0, Q\geq 1\ . \tag{4.38}$$

These relations are useful for computing a multiplicity generating function for a representation $[\![\text{odd},\text{even}]\!]$ (or $[\![\text{even},\text{odd}]\!]$) when the one for opposite parity is known. However, the recursion is not powerful enough to directly determine all the $G_{n,Q}$ in terms of $F^{\rm NS}_{k,p}$ and $F^{\rm R}_{k,p}$. The following subsection follows an alternative approach to determine the $G_{n,Q}$.

[21]The symmetry of (4.29) under $s\to s^{-1}$ guarantees that $M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)=M(\chi^{\mathcal{N}_{4d}=1},[\![n,-Q]\!],q)$, so we shall henceforth assume that $Q\geq 0$.

4.3 Multiplicities of representations in the $\mathcal{N}_{4d}=1$ partition function

Our aim in this subsection is to factor out the fundamental $\mathcal{N}_{4d}=1$ super Poincaré character $Z(\mathcal{N}_{4d}=1)=[1]_y+s+s^{-1}$ and to compute explicitly the multiplicity generating functions $G_{n,Q}(q)$ for $[\![n,Q]\!]$ in

$$\chi^{\mathcal{N}_{4d}=1}(q;y,s)=\sum_{n=0}^{\infty}\sum_{Q=-\infty}^{\infty}[\![n,Q]\!]G_{n,Q}(q)\ . \tag{4.39}$$

Using the second equality of (4.2) and orthogonality of $SO(3)\times U(1)_R$ representations, we have

$$G_{n,Q}(q)=M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)=\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}s}{s}\int \mathrm{d}\mu_{SO(3)}(y)\ [n]_y s^{-Q}\,\frac{\chi^{\mathcal{N}_{4d}=1}(q;y,s)}{[1]_y+(s+s^{-1})}\ , \tag{4.40}$$

where $\mathcal{C}$ is a contour in the complex s-plane enclosing the origin. In order to proceed, we use the geometric series expansion of the inverse $Z(\mathcal{N}_{4d}=1)$,[22]

$$\frac{1}{[1]_y+(s+s^{-1})}=\frac{1}{s+s^{-1}}\frac{1}{1+\frac{[1]_y}{s+s^{-1}}}=\sum_{m=0}^{\infty}(-1)^m\frac{[1]_y^m}{(s+s^{-1})^{m+1}}\ . \tag{4.42}$$

In what follows, we consider the contributions from $\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\ (q;y,s)$ and $\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\ (q;y,s)$ separately and then add up these results to yield the overall multiplicity generating function defined by (4.39),

$$M(\chi^{\mathcal{N}_{4d}=1},[\![n,Q]\!],q)=M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}},[\![n,Q]\!],q)+M(\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}},[\![n,Q]\!],q)\ , \tag{4.43}$$

where $\chi_{\mathrm{NS,R}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}$ are given by (4.18) and (4.24).

Multiplicities in the NS-sector

The series expansion of $(Z(\mathcal{N}_{4d}=1))^{-1}$ leads to the following NS sector contribution to the multiplicity generating function of the supermultiplet $[\![n,Q]\!]$

$$\begin{aligned} M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}},[\![n,Q]\!],q) &:= \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}s}{s}\int \mathrm{d}\mu_{SO(3)}(y)\ \frac{[n]_y}{s^Q}\times\frac{\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\,|_{\mathrm{GSO}}\ (q;y,s)}{[1]_y+(s+s^{-1})} \\ &= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty}(-1)^m F_{k,p}^{\mathrm{NS}}(q)\ \frac{1}{2\pi i}\oint_{|s|=1-\epsilon}\frac{\mathrm{d}s}{s}\frac{s^{2p}}{s^Q(s+s^{-1})^{m+1}}\ \int \mathrm{d}\mu_{SO(3)}(y)\ [n]_y[1]_y^m[2k]_y\ , \end{aligned} \tag{4.44}$$

We shall henceforth take $\mathcal{C}$ to be a circle centred at the origin with the radius $1-\epsilon$, with $0<\epsilon<1$. The quantities in the curly brackets can be computed as follows:

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{|s|=1-\epsilon}\frac{\mathrm{d}s}{s}\frac{s^{2p}}{s^Q(s+s^{-1})^{m+1}} \\ &=\begin{cases}(-1)^{\frac{1}{2}(Q-m-2p-1)}\binom{\frac{1}{2}(Q+m-2p-1)}{m} & \text{for } Q-m \text{ odd and } Q+m\geq 2p+1 \\ 0 & \text{otherwise}\ ,\end{cases} \end{aligned} \tag{4.45}$$

[22] Note that $\frac{1}{[1]_y+(s+s^{-1})}$ can also be written in another way as follows:

$$\frac{1}{[1]_y+(s+s^{-1})}=\sum_{m=0}^{\infty}(-1)^m s^{m+1}[m]_y\ . \tag{4.41}$$

However, we shall not take this approach, since otherwise this would lead to tensor products in (4.46) and (4.47) which are harder to evaluate in comparison with our current approach.

and

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n]_y[1]_y^m[2k]_y = \begin{cases} T_{2n+1}\left(m, \frac{1}{2}m+n-k\right) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd}\ , \end{cases} \tag{4.46}$$

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n+1]_y[1]_y^m[2k]_y = \begin{cases} T_{2n+2}\left(m, \frac{1}{2}m+n+\frac{1}{2}-k\right) & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}\ , \end{cases} \tag{4.47}$$

where

$$T_p(m,k) = \binom{m}{k} - \binom{m}{k-p}\ . \tag{4.48}$$

Note that (4.45), (4.46) and (4.47) are in perfect agreement with the selection rule

$$M(\chi^{\mathcal{N}_{4d}=1}, [\![2n, 2Q]\!], q) = M(\chi^{\mathcal{N}_{4d}=1}, [\![2n+1, 2Q+1]\!], q) = 0\ . \tag{4.49}$$

The nonzero multiplicities of $[\![2n, 2Q+1]\!]$ and $[\![2n+1, 2Q]\!]$ receive the following NS sector contributions:

$$\begin{aligned} &M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\ |_{\mathrm{GSO}}, [\![2n, 2Q+1]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p} F_{k,p}^{\mathrm{NS}}(q) \binom{Q+m-p}{2m} T_{2n+1}(2m, m+n-k)\ , \end{aligned} \tag{4.50}$$

$$\begin{aligned} &M(\chi_{\mathrm{NS}}^{\mathcal{N}_{4d}=1}\ |_{\mathrm{GSO}}, [\![2n+1, 2Q]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p} F_{k,p}^{\mathrm{NS}}(q) \binom{Q+m-p}{2m+1} T_{2n+2}(2m+1, m+n+1-k)\ . \end{aligned} \tag{4.51}$$

Multiplicities in the R-sector

Similarly to the NS-sector, the generating function for the multiplicity of the representation $[\![n, Q]\!]$ in the function $\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\ |_{\mathrm{GSO}}\ (q; y, s)$ is given by

$$\begin{aligned} M(\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\ |_{\mathrm{GSO}}, [\![n, Q]\!], q) &:= \frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s} \int \mathrm{d}\mu_{SO(3)}(y)\ \frac{[n]_y}{s^Q} \times \frac{\chi_{\mathrm{R}}^{\mathcal{N}_{4d}=1}\ |_{\mathrm{GSO}}\ (q; y, s)}{[1]_y + (s+s^{-1})} \\ = \sum_{m=0}^{\infty}\sum_{k=0}^{\infty}\sum_{p=-\infty}^{\infty} (-1)^m F_{k,p}^{\mathrm{R}}(q)\ &\frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s}\frac{s^{2p-1}}{s^Q(s+s^{-1})^{m+1}} \int \mathrm{d}\mu_{SO(3)}(y)\ [n]_y[1]_y^m[2k+1]_y\ , \end{aligned} \tag{4.52}$$

with $0 < \epsilon < 1$,

$$\begin{aligned} &\frac{1}{2\pi i}\oint_{|s|=1-\epsilon} \frac{\mathrm{d}s}{s}\frac{s^{2p-1}}{s^Q(s+s^{-1})^{m+1}} \\ &= \begin{cases} (-1)^{\frac{1}{2}(Q-m-2p)} \binom{\frac{1}{2}(Q+m-2p)}{m} & \text{for } Q-m \text{ even and } Q+m \geq 2p \\ 0 & \text{otherwise}\ , \end{cases} \end{aligned} \tag{4.53}$$

and

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n]_y[1]_y^m[2k+1]_y = \begin{cases} T_{2n+1}\left(m, \frac{1}{2}m+n-k-\frac{1}{2}\right) & \text{if } m \text{ is odd} \\ 0 & \text{if } m \text{ is even}\ , \end{cases} \tag{4.54}$$

$$\int \mathrm{d}\mu_{SO(3)}(y)\ [2n+1]_y[1]_y^m[2k+1]_y = \begin{cases} T_{2n+2}\left(m, \frac{1}{2}m+n-k\right) & \text{if } m \text{ is even} \\ 0 & \text{if } m \text{ is odd}\ , \end{cases} \tag{4.55}$$

where $T_p(m,k)$ is defined as above and the zeros once again confirm the selection rule (4.49).

The multiplicities of $[\![2n, 2Q+1]\!]$ are given by

$$\begin{aligned} &M(\chi_{\rm R}^{\mathcal{N}_{4d}=1}|_{\rm GSO}, [\![2n, 2Q+1]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p+1} F^{\rm R}_{k,p}(q) \binom{Q+m-p+1}{2m+1} T_{2n+1}(2m+1, m+n-k) \ . \end{aligned} \tag{4.56}$$

The multiplicities of $[\![2n+1, 2Q]\!]$ are given by

$$\begin{aligned} &M(\chi_{\rm R}^{\mathcal{N}_{4d}=1}|_{\rm GSO}, [\![2n+1, 2Q]\!], q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} (-1)^{Q-m-p} F^{\rm R}_{k,p}(q) \binom{Q+m-p}{2m} T_{2n+2}(2m, m+n-k) \ . \end{aligned} \tag{4.57}$$

Combining the NS and R sectors

Now we can assemble the NS- and R sector results to obtain the full multiplicities of the representation $[\![n, Q]\!]$ in $\chi^{\mathcal{N}_{4d}=1}(q; y, s)$. First, it is clear that

$$G_{2n,2Q}(q) = G_{2n+1,2Q+1}(q) = 0 \ . \tag{4.58}$$

The nonzero multiplicities of $[\![2n, 2Q+1]\!]$ and $[\![2n+1, 2Q]\!]$ are most conveniently presented in terms of the shorthands

$$\begin{aligned} \mathfrak{M}_{[\![2n,2Q+1]\!]}(m,p,k;q) \ &:= \ (-1)^{Q-m-p}\Bigg[F^{\rm NS}_{k,p}(q)\binom{Q+m-p}{2m} T_{2n+1}(2m, m+n-k) \\ &\qquad - F^{\rm R}_{k,p}(q)\binom{Q+m-p+1}{2m+1} T_{2n+1}(2m+1, m+n-k)\Bigg] \end{aligned} \tag{4.59}$$

$$\begin{aligned} \mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) \ &:= \ (-1)^{Q-m-p}\Bigg[F^{\rm NS}_{k,p}(q)\binom{Q+m-p}{2m+1} T_{2n+2}(2m+1, m+n+1-k) \\ &\qquad + F^{\rm R}_{k,p}(q)\binom{Q+m-p}{2m} T_{2n+2}(2m, m+n-k)\Bigg] \end{aligned} \tag{4.60}$$

for the contributions $\mathfrak{M}_{[\![\cdot,\cdot]\!]}(m,p,k;q)$ of individual terms in the m,p,k triple sum to the multiplicity generating function. The result for $[\![2n, 2Q+1]\!]$ supermultiplets is

$$\begin{aligned} G_{2n,2Q+1}(q) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} \mathfrak{M}_{[\![2n,2Q+1]\!]}(m,p,k;q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Bigg[\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n,2Q+1]\!]}(m+p,p,k;q)\Big\} \\ &\qquad + \sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n,2Q+1]\!]}(m,m+p+1,k;q)\Bigg] \ . \end{aligned} \tag{4.61}$$

whereas the multiplicities of $[\![2n+1, 2Q]\!]$ are given by

$$\begin{aligned} G_{2n+1,2Q}(q) &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=-\infty}^{Q+m} \mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) \\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Bigg[\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big\} \end{aligned}$$

$$+\sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\Bigg] \,. \tag{4.62}$$

4.4 Asymptotic analysis for the multiplicities

This subsection is devoted to the multiplicity generating function $G_{n,Q}(q)$ in the limit $n \to \infty$. We shall present analytic expressions for their $n \to \infty$ asymptotics whose derivation is defered to appendix D. The method essentially relies on identifying the dominant contribution to the triple sums in (4.61) and (4.62). The end result for multiplicity generating functions $G_{n,Q}(q)$ reads

$$G_{2n+1,2Q}(q) \sim \frac{(1-q)^2 q^{n-\frac{3}{2}}}{2(q;q)_\infty^6}\mathcal{F}(q,Q) \,, \quad n\to\infty \,, \tag{4.63}$$

$$G_{2n,2Q+1}(q) \sim \frac{(1-q)^2 q^{n-\frac{3}{2}}}{2(q;q)_\infty^6(1+q)} \times \left[\frac{q^{(Q+1)^2+\frac{1}{4}}(1-q)}{(1+q^Q)\,(1+q^{Q+1})}\vartheta_2(1,q)^2 - \mathcal{F}(q,Q)-\mathcal{F}(q,Q+1)\right] \tag{4.64}$$

with the function $\mathcal{F}(q,Q)$ given by

$$\begin{aligned}\mathcal{F}(q,Q) = \vartheta_2(1,q)^2\Big[&q^{1-Q}u_1(\sqrt{q},Q)+(-1)^Q(1-q)(v_1(\sqrt{q},Q)+q^{-1/4}w_1(\sqrt{q},Q))\Big]\\ +\vartheta_3(1,q)^2\Big[&-q^{1-Q}u_2(\sqrt{q},Q)+(-1)^Q(1-q)(v_2(\sqrt{q},Q)+q^2w_2(\sqrt{q},Q))\Big]\\ +\vartheta_4(1,q)^2\Big[&q^{1-Q}u_2(-\sqrt{q},Q)-(-1)^Q(1-q)(v_2(-\sqrt{q},Q)+q^2w_2(-\sqrt{q},Q))\Big] \,. \end{aligned}\tag{4.65}$$

The three pairs of functions u_i, v_i and w_i correspond to the three summations in (4.61) and (4.62):

$$\begin{aligned} u_1(q,Q) &= \sum_{p=0}^{\infty} q^{2\left(p+\frac{3}{2}\right)^2}\frac{1-q^{4p+4Q+6}}{(1+q^{2p+2})(1+q^{2p+4})} \,,\\ u_2(q,Q) &= \sum_{p=0}^{\infty} q^{2(p+1)^2}\frac{1-q^{4p+4Q+4}}{(1+q^{2p+1})(1+q^{2p+3})} \,. \end{aligned}\tag{4.66}$$

$$\begin{aligned} v_1(q,Q) &= \sum_{p=0}^{\lfloor Q/2\rfloor}\frac{q^{2(p-\frac{1}{2})^2}(1+q^2)^{2p}}{(1+q^{2p-2})(1+q^{2p})}\binom{Q}{2p}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p-Q\\ p+1/2,\ p+1\end{matrix};\frac{(1+q)^2}{4q}\right] \,,\\ v_2(q,Q) &= \sum_{p=0}^{\lfloor Q/2\rfloor}\frac{(1+q)q^{2p^2}(1+q^2)^{2p}}{(1+q^{2p-1})(1+q^{2p+1})}\binom{Q}{2p+1}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p+1-Q\\ p+1,\ p+3/2\end{matrix};\frac{(1+q)^2}{4q}\right] \,, \end{aligned}\tag{4.67}$$

$$\begin{aligned} w_1(q,Q) &= \sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{1+2(1+m+p)^2-2m}\left(1+q^2\right)^{2m}\binom{Q-1-p}{2m}}{\left(1+q^{2(m+p)}\right)\left(1+q^{2(1+m+p)}\right)} \,,\\ w_2(q,Q) &= q^{-\frac{9}{2}}\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{2\left(m+p+\frac{3}{2}\right)^2-2m}\left(1+q^2\right)^{2m+1}\binom{Q-1-p}{1+2m}}{\left(1+q^{1+2m+2p}\right)\left(1+q^{3+2m+2p}\right)} \,. \end{aligned}\tag{4.68}$$

Note that the leading orders in the power series are

$$G_{2n+1,2Q}(q) \ \sim\ q^{n+Q(Q+2)} \,, \qquad G_{2n,2Q+1}(q) \ \sim\ q^{n+Q^2+3Q+1} \,, \qquad q\to 0 \,, \tag{4.69}$$

i.e. the supermultiplet $[\![2n+1,2Q]\!]$ firstly occurs at mass level $n+Q(Q+2)$ whereas the $[\![2n,2Q+1]\!]$ multiplet firstly occurs at mass level $n+Q^2+3Q+1$.

For reference, we list the leading q powers for the $G_{n\to\infty,Q}$ regime for some small values of the $U(1)_R$ charge, obtained by expansion of (4.63) and (4.64): firstly for even values $Q\in 2\mathbb{N}_0$

$$G_{2n+1,0}(q) \sim q^n(1+q+7q^2+19q^3+53q^4+133q^5+328q^6+752q^7+1689q^8+3635q^9+O(q^{10})) \,,$$

$$G_{2n+1,2}(q) \sim q^{n+3}(2+8q+24q^2+73q^3+187q^4+467q^5+1090q^6+2457q^7+5314q^8+O(q^9)) \ ,$$
$$G_{2n+1,4}(q) \sim q^{n+8}(2+10q+36q^2+110q^3+306q^4+773q^5+1861q^6+4245q^7+9327q^8+O(q^9)) \ ,$$
$$G_{2n+1,6}(q) \sim q^{n+15}(2+10q+38q^2+124q^3+352q^4+928q^5+2282q^6+5335q^7+O(q^8)) \ , \quad (4.70)$$

and secondly for odd values $Q \in 2\mathbb{N}-1$

$$G_{2n,1}(q) \sim q^{n+1}(3+5q+22q^2+53q^3+150q^4+345q^5+836q^6+1824q^7+4011q^8+O(q9)) \ ,$$
$$G_{2n,3}(q) \sim q^{n+5}(4+11q+46q^2+117q^3+331q^4+784q^5+1876q^6+4133q^7+O(q^8)) \ ,$$
$$G_{2n,5}(q) \sim q^{n+11}(4+12q+55q^2+150q^3+437q^4+1078q^5+2640q^6+5951q^7+O(q^8)) \ ,$$
$$G_{2n,7}(q) \sim q^{n+19}(4+12q+56q^2+159q^3+474q^4+1197q^5+2994q^6+6882q^7+O(q^8)) \ . \quad (4.71)$$

Note that the general formula greatly simplifies at $U(1)_R$ charges $Q=0$ and $Q=1$,

$$G_{2n+1,0}(q) \sim \frac{q^n}{(q;q)_\infty^6}\Bigg\{\frac{1}{2}(1-q)^2 q^{-\frac{1}{2}}\Big(u_1(\sqrt{q})\vartheta_2(1,q)^2 - \big[u_2(\sqrt{q})\vartheta_3(1,q)^2 - u_2(-\sqrt{q})\vartheta_4(1,q)^2\big]\Big)$$
$$+\frac{1}{4}\frac{(1-q)^3}{1+q}q^{-\frac{1}{4}}\vartheta_2(1,q)^2\Bigg\} \ , \qquad n\to\infty \quad (4.72)$$

$$G_{2n,1}(q) \sim \frac{(1-q)^3 q^{n+1}}{4(q;q)_\infty^6}\Bigg[q^{-\frac{5}{2}}\Bigg\{\frac{\vartheta_3(1,q)^2}{(1+q^{-\frac{1}{2}})(1+q^{\frac{1}{2}})} - \frac{\vartheta_4(1,q)^2}{(1-q^{-\frac{1}{2}})(1-q^{\frac{1}{2}})}\Bigg\} - \frac{1}{2}q^{-\frac{9}{4}}\vartheta_2(1,q)^2$$
$$- q^{-\frac{5}{2}}\frac{1+q}{1-q}\Big(u_1(\sqrt{q})\vartheta_2(1,q)^2 - \big[u_2(\sqrt{q})\vartheta_3(1,q)^2 - u_2(-\sqrt{q})\vartheta_4(1,q)^2\big]\Big)\Bigg] \quad (4.73)$$

where $u_i(q) \equiv u_i(q;0)$, see the first subsection of appendix D.

4.5 Empirical approach to $\mathcal{N}_{4d}=1$ asymptotic patterns

In the previous subsection, we have derived the large spin asymptotics for multiplicity generating functions $G_{k,Q}(q)$ of individual $\mathcal{N}_{4d}=1$ multiplets (at finite Q while $k\to\infty$), the main results being (4.63) and (4.64). The asymptotic formulae can be viewed as the supersymmetric generalization of truncating the infinite sum expression (2.10) for the $SO(3)$ multiplicity generating function in the $d=4$ bosonic partition function to its $n=1$ term. In [4], this $n=1$ term is interpreted as the leading (additive) Regge trajectory of unit slope, followed by an infinite tower of sister trajectories of fractional slope and alternating sign. Let us borrow the τ notation from equation (6.2) of [4] and expand the $G_{k,Q}(q)$ in an infinite series of trajectories:

$$G_{2n+1,2Q}(q) = q^n\,\tau_1^{2Q}(q) - q^{2n}\,\tau_2^{2Q}(q) + q^{3n}\,\tau_3^{2Q}(q) - \ldots = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}q^{\ell n}\,\tau_\ell^{2Q}(q) \quad (4.74)$$
$$G_{2n,2Q+1}(q) = q^n\,\tau_1^{2Q+1}(q) - q^{2n}\,\tau_2^{2Q+1}(q) + q^{3n}\,\tau_3^{2Q+1}(q) - \ldots = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}q^{\ell n}\,\tau_\ell^{2Q+1}(q)$$

All our $\mathcal{N}_{4d}=1$ data suggests that both of $\tau_\ell^{2Q}(q)$ and $\tau_\ell^{2Q+1}(q)$ are power series in q with non-negative coefficients. Our analytic results (4.63) and (4.64) identify the first coefficient functions $\tau_1(q)$ in (4.74):

$$\tau_1^{2Q}(q) = \frac{(1-q)^2 q^{-\frac{3}{2}}}{2(q;q)_\infty^6}\mathcal{F}(q,Q) \quad (4.75)$$
$$\tau_1^{2Q+1}(q) = \frac{(1-q)^2 q^{-\frac{3}{2}}}{2(q;q)_\infty^6(1+q)} \times \left[\frac{q^{(Q+1)^2+\frac{1}{4}}(1-q)}{(1+q^Q)\,(1+q^{Q+1})}\vartheta_2(1,q)^2 - \mathcal{F}(q,Q) - \mathcal{F}(q,Q+1)\right] \quad (4.76)$$

$\alpha' m^2$	$\#[\![1,0]\!]$	$\#[\![3,0]\!]$	$\#[\![5,0]\!]$	$\#[\![7,0]\!]$	$\#[\![9,0]\!]$	$\#[\![11,0]\!]$	$\#[\![13,0]\!]$	$\#[\![15,0]\!]$	$\#[\![17,0]\!]$	$\#[\![19,0]\!]$	$\#[\![21,0]\!]$
1	0	1	0								
2	2	1	1	0							
3	5	5	1	1	0						
4	12	12	7	1	1	0					
5	28	31	17	7	1	1	0				
6	64	72	45	19	7	1	1	0			
7	136	166	108	51	19	7	1	1	0		
8	289	357	254	125	53	19	7	1	1	0	
9	588	757	557	302	131	53	19	7	1	1	0
10	1175	1548	1200	675	320	133	53	19	7	1	1
11	2293	3100	2482	1479	726	326	133	53	19	7	1
12	4399	6053	5028	3106	1611	744	328	133	53	19	7
13	8267	11620	9910	6373	3422	1663	750	328	133	53	19
14	15325	21855	19173	12713	7098	3557	1681	752	328	133	53
15	27949	40496	36322	24856	14297	7428	3609	1687	752	328	133
16	50306	73846	67720	47539	28216	15061	7564	3627	1689	752	328
17	89367	132860	124161	89401	54430	29909	15394	7616	3633	1689	752
18	156930	235871	224479	165210	103182	58054	30687	15530	7634	3635	1689
19	272424	413879	400257	300837	192109	110702	59786	31021	15582	7640	3635
20	468130	717909	705032	539962	352279	207282	114437	60567	31157	15600	7642
21	796410	1232463	1227214	956883	636445	382179	215074	116183	60901	31209	15606
22	1342531	2094716	2113394	1674933	1134836	694090	398007	218848	116965	61037	31227
23	2243232	3527456	3602086	2899342	1997955	1243836	725457	405910	220597	117299	61089
24	3717405	5887668	6081317	4965411	3477396	2200438	1304682	741559	409698	221379	117435
25	6111615	9745995	10173766	8420331	5986079	3847540	2316123	1336712	749501	411448	221713

Table 3. $\mathcal{N}_{4d} = 1$ multiplets at $U(1)_R$ charge $Q = 0$

The methods presented in appendix D and applied in the previous subsection are not suitable to extract subleading Regge trajectories $\tau_{\ell\geq 2}(q)$, i.e. $\mathcal{N}_{4d} = 1$ analogues of $n \geq 2$ terms in the sum (2.10). Instead, we shall rely on an empirical approach, more specifically on explicit results obtained from a supercharacter expansion of the partition function (4.4) up to the 25th mass level.

As an illustrative example, let us first of all investigate the family of $Q = 0$ supermultiplets: The following table 3 gathers $[\![2n+1, 0]\!]$ multiplicities in the first 25 levels. Numbers marked in red directly correspond to the leading trajectory $\tau_1^0(q)$ whereas those in blue are additionally affected by the subleading trajectory $\tau_2^0(q)$. Given the leading trajectories (4.75), our data from table 3 can be used to determine the following subleading behaviour for $Q = 0$ multiplets:

$$\begin{aligned}
G_{2n+1,0}(q) \quad \sim \quad & q^n\,(1 + q + 7q^2 + 19q^3 + 53q^4 + 133q^5 + 328q^6 + 752q^7 + 1689q^8 + 3635q^9 \\
& \quad + 7642q^{10} + 15608q^{11} + 31235q^{12} + 61115q^{13} + 117513q^{14} + 221927q^{15} + 412778q^{16} \\
& \quad + 756372q^{17} + 1367753q^{18} + 2441849q^{19} + 4309132q^{20} + 7520092q^{21} \\
& \quad + 12989357q^{22} + 22216885q^{23} + 37651970q^{24} + 63252874q^{25} + \dots) \\
- \ & q^{2n+1}\,(2 + 8q + 26q^2 + 78q^3 + 214q^4 + 548q^5 + 1330q^6 + 3080q^7 + 6872q^8 + 14832q^9 \\
& \quad + 31102q^{10} + 63574q^{11} + 127020q^{12} + 248590q^{13} + 477504q^{14} + \dots) \\
+ \ & q^{3n+1}\,(1 + 4q + 19q^2 + 61q^3 + 187q^4 + 503q^5 + 1294q^6 + 3113q^7 \\
& \quad + 7217q^8 + 16036q^9 + 34584q^{10} + \dots) \\
- \ & q^{4n+2}\,(2 + 10q + 38q^2 + 124q^3 + 364q^4 + 978q^5 + 2476q^6 + \dots) \\
+ \ & q^{5n+2}\,(1 + 4q + 21q^2 + 72q^3 + \dots) \ + \ \dots \,, \qquad n \to \infty
\end{aligned} \tag{4.77}$$

The first term linear in q^n simply reproduces (4.72) for $\tau_1^{Q=0}(q)$ whereas higher powers of q^n allow to read off subleading $\tau^{Q=0}_{\ell\geq 2}(q)$ to certain order in q:

$$\begin{aligned}
\tau_2^{Q=0}(q) &= q\,(2+8q+26q^2+78q^3+214q^4+548q^5+1330q^6+3080q^7+6872q^8+14832q^9 \\
&\qquad +31102q^{10}+63574q^{11}+127020q^{12}+248590q^{13}+477504q^{14}+\ldots) \qquad (4.78)\\
\tau_3^{Q=0}(q) &= q\,(1+4q+19q^2+61q^3+187q^4+503q^5+1294q^6+3113q^7 \\
&\qquad +7217q^8+16036q^9+34584q^{10}+\ldots) \qquad (4.79)\\
\tau_4^{Q=0}(q) &= q^2\,(2+10q+38q^2+124q^3+364q^4+978q^5+2476q^6+\ldots) \qquad (4.80)\\
\tau_5^{Q=0}(q) &= q^2\,(1+4q+21q^2+72q^3+\ldots) \qquad (4.81)
\end{aligned}$$

Determining higher order terms in the $\tau^{Q=0}_{\ell\geq 2}(q)$ would require $\mathcal{O}(q^{26})$ parts of (4.4), this is where we stopped the explicit evaluation.

Similarly, the $[\![2n+1,2Q]\!]$ multiplicities up to level q^{25} as tabulated in appendix B.1 determine the associated $\tau^{2Q}_{\ell}(q)$ coefficients for low charges Q to the following orders:

- $U(1)_R$ charge $Q=2$:

$$\begin{aligned}
\tau_2^{Q=2}(q) &= q^3\,(2+11q+37q^2+114q^3+319q^4+822q^5+2000q^6+4645q^7+10354q^8 \\
&\qquad +22317q^9+46702q^{10}+95210q^{11}+189656q^{12}+\ldots)\\
\tau_3^{Q=2}(q) &= q^3\,(2+8q+33q^2+104q^3+310q^4+826q^5+2093q^6+4991q^7+11454q^8+\ldots)\\
\tau_4^{Q=2}(q) &= q^3\,(1+5q+22q^2+77q^3+237q^4+664q^5+\ldots)\\
\tau_5^{Q=2}(q) &= q^4\,(3+12q+49q^2+\ldots) \qquad (4.82)
\end{aligned}$$

- $U(1)_R$ charge $Q=4$:

$$\begin{aligned}
\tau_2^{Q=4}(q) &= q^8\,(2+14q+57q^2+187q^3+542q^4+1438q^5+3563q^6 \\
&\qquad +8376q^7+18846q^8+40866q^9+\ldots)\\
\tau_3^{Q=4}(q) &= q^8\,(2+14q+58q^2+200q^3+591q^4+1612q^5+\ldots)\\
\tau_4^{Q=4}(q) &= q^8\,(2+13q+53q^2+\ldots) \qquad (4.83)
\end{aligned}$$

- $U(1)_R$ charge $Q=6$:

$$\begin{aligned}
\tau_2^{Q=6}(q) &= q^{15}\,(2+14q+60q^2+209q^3+633q^4+\ldots)\\
\tau_3^{Q=6}(q) &= q^{15}\,(2+14q+64q^2+\ldots) \qquad (4.84)
\end{aligned}$$

Note that the analytic result (4.63) for $\tau^{2Q}_1(q)$ was used as an extra input, in addition to the explicit results for the first 25 mass level, to make a few more orders of the subleading $\tau^{2Q}_{\ell\geq 2}(q)$ accessible.

Also in the $[\![2n,2Q+1]\!]$ sector, we can use the data from appendix B.1 to expand the subleading trajectories $\tau^{2Q+1}_{\geq 2}(q)$:

- $U(1)_R$ charge $Q=1$:

$$\begin{aligned}
\tau_2^{Q=1}(q) &= 1+4q+15q^2+50q^3+143q^4+379q^5+947q^6+2244q^7+5103q^8+11196q^9 \\
&\qquad +23804q^{10}+49252q^{11}+99465q^{12}+196522q^{13}+380719q^{14}+\ldots\\
\tau_3^{Q=1}(q) &= 1+5q+22q^2+70q^3+212q^4+568q^5+1458q^6+3496q^7 \\
&\qquad +8093q^8+17936q^9+\ldots\\
\tau_4^{Q=1}(q) &= 1+6q+24q^2+83q^3+252q^4+698q^5+\ldots\\
\tau_5^{Q=1}(q) &= 1+6q+25q^2+\ldots \qquad (4.85)
\end{aligned}$$

- $U(1)_R$ charge $Q=3$:

$$\begin{aligned}\tau_2^{Q=3}(q) &= q^4\,(1+9q+37q^2+120q^3+347q^4+922q^5+2287q^6+5385q^7+12142q^8\\ &\quad +26395q^9+55605q^{10}+113973q^{11}+\ldots)\\ \tau_3^{Q=3}(q) &= q^4\,(4+17q+68q^2+208q^3+603q^4+1573q^5+3919q^6+9195q^7+\ldots)\\ \tau_4^{Q=3}(q) &= q^3\,(1+7q+28q^2+99q^3+304q^4+851q^5+\ldots)\\ \tau_5^{Q=3}(q) &= q^3\,(2+9q+38q^2+\ldots)\end{aligned} \tag{4.86}$$

- $U(1)_R$ charge $Q=5$:

$$\begin{aligned}\tau_2^{Q=5}(q) &= q^{10}\,(1+9q+43q^2+151q^3+462q^4+1277q^5+3264q^6+7865q^7+\ldots)\\ \tau_3^{Q=5}(q) &= q^{10}\,(4+20q+89q^2+292q^3+\ldots)\\ \tau_4^{Q=5}(q) &= q^9\,(1+9q+\ldots)\end{aligned} \tag{4.87}$$

- $U(1)_R$ charge $Q=7$:

$$\tau_2^{Q=7}(q) = q^{18}\,(1+9q+\ldots) \tag{4.88}$$

Again, the set of accessible coefficients in $\tau_{\ell\geq 2}^{2Q+1}(q)$ could be slightly improved by making use of the $\tau_1^{2Q+1}(q)$ expression (4.64) to order q^{25}.

Note that for all values of the $U(1)_R$ charge Q considered here, the leading q powers of the $\tau_\ell^Q(q)$ at fixed Q hardly vary with ℓ (at $Q=2$, for instance, we can read off $\tau_1^2,\tau_2^2,\tau_3^2,\tau_4^2 \sim \mathcal{O}(q^3)$ and $\tau_5^2 \sim \mathcal{O}(q^4)$ from (4.82)). In particular, the approximate agreement of the leading q powers of $\tau_1(q)$ and $\tau_2(q)$ supports our claim in the introduction that half of the nonzero multiplicities exactly match with the stable patterns.

5 Spectra in compactifications with 8 supercharges

In six dimensional Minkowski space, the minimal realization of SUSY involves eight supercharges. They form two left- handed Weyl spinors of $SO(6)$ which are related through an $SU(2)_R$ R symmetry. Our notation for such minimally supersymmetric theories in $d=6$ is $\mathcal{N}_{6d}=(1,0)$. Superstring compactification subject to $\mathcal{N}_{6d}=(1,0)$ SUSY are described by a universal SCFT sector with $c=6$ and $\mathcal{N}_{2d}=4$ SUSY on the worldsheet, see subsection 3.2 for details. In addition, the SCFT introduces $SO(5)$ quantum numbers for the massive string states through a six dimensional spacetime sector for which the methods of subsections 2.4 and 2.5 are applicable.

The fundamental multiplet of $\mathcal{N}_{6d}=(1,0)$ theories consists of 8+8 states

$$Z(\mathcal{N}_{6d}=(1,0)) \;:=\; [1,0] \;+\; [2]_R \;+\; [1]_R\,[0,1]\;. \tag{5.1}$$

where $[p]_R$ is the character of the $p+1$ dimensional representation of $SU(2)_R$. Generic multiplets follow through the tensor product with some $SO(5)\times SU(2)_R$ representation with little group quantum numbers $[n_1,n_2]$ and R symmetry content $[k]_R$. This leads to the general supercharacter

$$[\![n_1,n_2;p]\!] \;:=\; Z(\mathcal{N}_{6d}=(1,0))\cdot[p]_R\,[n_1,n_2]\;. \tag{5.2}$$

The partition function capturing the universal spectrum of six dimensional $\mathcal{N}_{6d}=(1,0)$ compactifications is obtained thorugh a GSO projected product of internal $\chi_{\ldots}^{\mathcal{N}_{2d}=4,c=6}(q;r)$ characters (with $SU(2)_R$ fugacity r) defined by (3.7) as well as (3.10) and $SO(5)$ spacetime characters (2.69) and

(2.70). The GSO projection removes half odd integer mass leves from the NS sector and enforces the R spin field to be a left handed $SO(6)$ spinor, therefore:

$$\begin{aligned}
\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r) &= \chi^{\mathcal{N}_{6d}=(1,0)}_{\text{NS}}\,|_{\text{GSO}}\,(q;\boldsymbol{y},r) + \chi^{\mathcal{N}_{6d}=(1,0)}_{\text{R}}\,|_{\text{GSO}}\,(q;\boldsymbol{y},r) \\
\chi^{\mathcal{N}_{6d}=(1,0)}_{\text{NS}}\,|_{\text{GSO}}\,(q;\boldsymbol{y},r) &= \frac{1}{2}\,q^{-\frac{1}{2}}\,\big[\,\chi^{SO(5)}_{\text{NS}}(q;\boldsymbol{y})\,\chi^{\mathcal{N}_{2d}=4,c=6}_{\text{NS},h=0,\ell=0}(q;r) \\
&\qquad - \chi^{SO(5)}_{\text{NS}}(e^{2\pi i}q;\boldsymbol{y})\,\chi^{\mathcal{N}_{2d}=4,c=6}_{\text{NS},h=0,\ell=0}(e^{2\pi i}q;r)\,\big] \\
\chi^{\mathcal{N}_{6d}=(1,0)}_{\text{R}}\,|_{\text{GSO}}\,(q;\boldsymbol{y},r) &= \frac{1}{2}\,\chi^{SO(5)}_{\text{R}}(q;\boldsymbol{y})\,\chi^{\mathcal{N}_{2d}=4,c=6}_{\text{R},h=1/4,\ell=1/2}(q;r)\;.
\end{aligned}\tag{5.3}$$

The power series expansion of (5.3) starts as[23]

$$\begin{aligned}
\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r) &= \underbrace{\left(y_1^2+y_1^{-2}+y_2^2+y_2^{-2} + \frac{1}{2}[1]_r\prod_{i=1}^{2}(y_i+y_i^{-1})\right)q^0}_{\text{8 massless states}} + \underbrace{[\![1,0;0]\!]\,q}_{\text{80 states at level 1}} \\
&+ \underbrace{\big([\![2,0;0]\!] + [\![0,2;0]\!] + [\![0,1;1]\!]\big)\,q^2}_{\text{512 states at level 2}} + \big([\![3,0;0]\!] + 2\,[\![1,0;0]\!] + [\![0,0;0]\!] \\
&\quad + [\![1,2;0]\!] + [\![0,2;0]\!] + [\![0,0;2]\!] + 2\,[\![1,1;1]\!] + [\![0,1;1]\!]\big)\,q^3 + \mathcal{O}(q^4)\;.
\end{aligned}\tag{5.4}$$

The $q^{\leq 6}$ coefficients are listed in table 4, further information on the particle content up to level 25 is tabulated in appendix B.2.

$\alpha' m^2$	representations of $\mathcal{N}_{6d}=(1,0)$ super Poincaré
1	$[\![1,0;0]\!]$
2	$[\![2,0;0]\!] + [\![0,2;0]\!] + [\![0,1;1]\!]$
3	$[\![3,0;0]\!] + 2\,[\![1,0;0]\!] + [\![0,0;0]\!] + [\![1,2;0]\!] + [\![0,2;0]\!] + [\![0,0;2]\!] + 2\,[\![1,1;1]\!] + [\![0,1;1]\!]$
4	$[\![4,0;0]\!] + 3\,[\![2,0;0]\!] + 2\,[\![1,0;0]\!] + 2\,[\![0,0;0]\!] + [\![2,2;0]\!] + 2\,[\![1,2;0]\!] + 4\,[\![0,2;0]\!]$ $+\,2\,[\![1,0;2]\!] + [\![0,2;2]\!] + 3\,[\![1,1;1]\!] + 4\,[\![0,1;1]\!] + 2\,[\![2,1;1]\!]$
5	$[\![5,0;0]\!] + 3\,[\![3,0;0]\!] + 4\,[\![2,0;0]\!] + 9\,[\![1,0;0]\!] + 3\,[\![0,0;0]\!] + [\![3,2;0]\!] + 2\,[\![2,2;0]\!]$ $+\,7\,[\![1,2;0]\!] + 6\,[\![0,2;0]\!] + [\![0,4;0]\!] + 3\,[\![2,0;2]\!] + 3\,[\![1,0;2]\!] + 3\,[\![0,0;2]\!] + [\![1,2;2]\!]$ $+\,3\,[\![0,2;2]\!] + 2\,[\![3,1;1]\!] + 4\,[\![2,1;1]\!] + 9\,[\![1,1;1]\!] + 8\,[\![0,1;1]\!] + [\![1,3;1]\!] + 4\,[\![0,3;1]\!]$ $+\,[\![0,1;3]\!]$
6	$[\![6,0;0]\!] + [\![4,2;0]\!] + 2\,[\![4,1;1]\!] + 3\,[\![4,0;0]\!] + 2\,[\![3,2;0]\!] + 4\,[\![3,1;1]\!] + 3\,[\![3,0;2]\!]$ $+\,5\,[\![3,0;0]\!] + [\![2,3;1]\!] + [\![2,2;2]\!] + 8\,[\![2,2;0]\!] + 12\,[\![2,1;1]\!] + 4\,[\![2,0;2]\!] + 14\,[\![2,0;0]\!]$ $+\,[\![1,4;0]\!] + 5\,[\![1,3;1]\!] + 6\,[\![1,2;2]\!] + 13\,[\![1,2;0]\!] + 2\,[\![1,1;3]\!] + 23\,[\![1,1;1]\!] + 9\,[\![1,0;2]\!]$ $+\,12\,[\![1,0;0]\!] + 4\,[\![0,4;0]\!] + 9\,[\![0,3;1]\!] + 9\,[\![0,2;2]\!] + 19\,[\![0,2;0]\!] + 3\,[\![0,1;3]\!]$ $+\,18\,[\![0,1;1]\!] + 4\,[\![0,0;2]\!] + 8\,[\![0,0;0]\!]$

Table 4. $\mathcal{N}_{6d}=(1,0)$ multiplets occurring up to mass level 6

5.1 The total number of states at a given mass level

In this subsection, we compute the total number of states present at a given mass level through the unrefined partition function, *i.e.* by setting the fugacities y_1, y_2 and r in (5.4) to unity. The total

[23] Again, there is a subtlety in applying (5.3) to the massless R sector, see the footnote before (4.4). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y_1-y_1^{-1})(y_2-y_2^{-1})(r-r^{-1})$ to get the correct massless character in R sector.

number of states N_m at the mass level m can be read off from the coefficient of q^m in the power series of $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$.

We follow the analysis presented in subsection 4.1. The unrefined partition function is given by

$$\begin{aligned}
\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\}) &= 2\chi_{\mathrm{R}}^{\mathcal{N}_{6d}=(1,0)}\ |_{\mathrm{GSO}}\ (q;y=1,s=1) \\
&= \chi_{\mathrm{R}}^{SO(5)}(q;\{y_i=1\})\,\chi_{\mathrm{R},h=1/4,\ell=1/2}^{\mathcal{N}_{2d}=4,c=6}(q;r=1) \\
&= q^{-1/8}\frac{\vartheta_2(1,q)^4}{\eta(q)^9}\left[1-2iq^{1/8}\mu\left(1/2,\tau\right)\right]\ .
\end{aligned} \tag{5.5}$$

Indeed, the power series of $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$ in q reproduces the numbers presented in the second column of Table 1. Note that $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})$ is *not* a modular form, since the Appell-Lerch sum is a mock modular form and it is not added by a suitable non-holomorphic component to be modular.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can also be computed from

$$N_m=\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{\mathrm{d}q}{q^{m+1}}\ \chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\})\ , \tag{5.6}$$

where $\mathcal{C}$ is a contour around the origin.

Now let us compute an asymptotic formula for the number of states N_m at a mass level m when $m\to\infty$. We focus on the limit $q\to 1^-$ and proceed in a similar way to subsection 4.1.

Let us first examine the leading behaviour of $\mu\left(1/2,\tau\right)$ as $q\to1^-$ or $\tau\to0$. Using the second point of Proposition 1.5 of [37], we find that

$$\frac{1}{\sqrt{-i\tau}}\mu\left(\frac{1}{2\tau},-\frac{1}{\tau}\right)+\mu\left(\frac{1}{2},\tau\right)=\frac{1}{2i}\ . \tag{5.7}$$

Let us consider $\mu\left(\frac{1}{2\tau},-\frac{1}{\tau}\right)$ as $q\to1^-$ or equivalently $\tau=i\epsilon$ as $\epsilon\to0^+$. It follows from the definition of Appell-Lerch sum that

$$\begin{aligned}
\mu\left(\frac{1}{2\tau},-\frac{1}{\tau}\right) &= -\frac{e^{i\pi/(2\tau)}}{\vartheta_1(e^{2\pi i/(2\tau)},e^{-2\pi i/\tau})}\sum_{m\in\mathbb{Z}}(-1)^m\frac{e^{-i\pi m^2/\tau}}{1-e^{-2\pi i m/\tau+\pi i/\tau}} \\
&\sim -\frac{e^{\pi/(2\epsilon)}}{-ie^{\pi/(4\epsilon)}}\times\left(-2e^{-\pi/\epsilon}\right)\ , \qquad \tau=i\epsilon,\ \epsilon\to0^+ \\
&= 2i\exp\left(-\frac{3\pi}{4\epsilon}\right)\ ,
\end{aligned} \tag{5.8}$$

where in the second 'equality' only $m=0,1$ in the infinite sum contribute to the leading behaviour and we have used the fact that $\vartheta_1(e^{2\pi i/(2\tau)},e^{-2\pi i/\tau})=-ie^{\pi/(4\epsilon)}$, as $\tau=i\epsilon,\ \epsilon\to0^+$. Hence, to the leading order, one can neglect the first term in (5.7) in comparison with $1/(2i)$ on the right hand side and so

$$\mu\left(\frac{1}{2},\tau\right)\sim\frac{1}{2i}\ ,\qquad q\to1^-\ . \tag{5.9}$$

Therefore it follows from (5.5) that, as $q\to1^-$,

$$\begin{aligned}
\chi^{\mathcal{N}_{6d}=(1,0)}(q;\{y_i=1,r=1\}) &\sim q^{-1/8}\frac{\vartheta_2(1,q)^4}{\eta(q)^9}\left(1-q^{1/8}\right) \\
&\sim (2\pi)^{-5/2}(1-q^{1/8})(1-q)^{5/2}\exp\left(-\frac{3\pi^2}{2\log q}\right)\ ,
\end{aligned} \tag{5.10}$$

where we have used (4.13) and (4.11). Hence, as $m \to \infty$,

$$N_m \sim (2\pi)^{-5/2} \frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\mathrm{d}q}{q} \, (1-q^{1/8})(1-q)^{5/2} \exp\left(-\frac{3\pi^2}{2\log q} - m \log q\right) . \tag{5.11}$$

The saddle point is at $q_0 = \exp\left(-\pi\sqrt{3}/\sqrt{2m}\right)$ and the steepest descent direction is the imaginary direction in q. We proceed in a similar way to (4.17) by writing $q = q_0 e^{i\theta}$ and using Laplace's method to obtain

$$\begin{aligned} N_m &\sim (2\pi)^{-5/2}(1-q_0^{1/8})(1-q_0)^{5/2} e^{\pi\sqrt{6m}} \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\theta \exp\left(-\frac{1}{\pi}\sqrt{\frac{2}{3}} m^{3/2}\theta^2\right) \\ &\sim \frac{9\pi}{2^{17/2}} m^{-5/2} \exp\left(\pi\sqrt{6m}\right) , \qquad m \to \infty . \end{aligned} \tag{5.12}$$

The plot of the exact and asymptotic values for N_m against m is depicted in Figure 6.

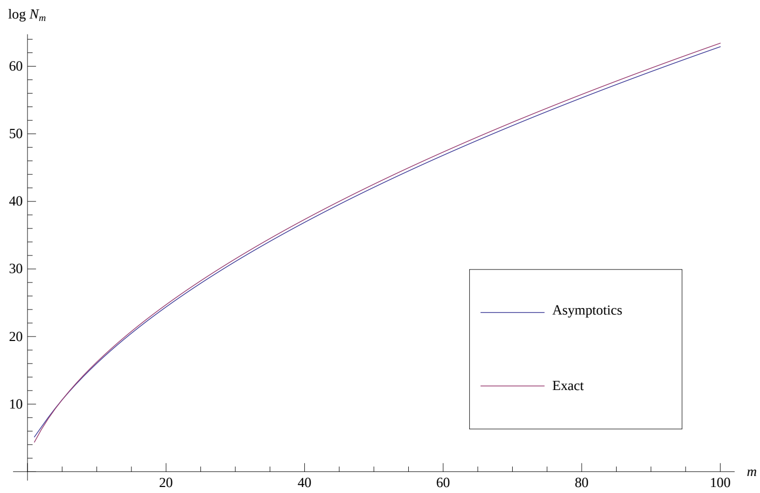

Figure 6: The plot of the exact and asymptotic values of $\log N_m$ against the mass level m for the case of 8 supercharges.

5.2 The GSO projected NS and R sectors

The NS sector

From (6.53), the partition function of the GSO projected NS sector is

$$\chi_{\mathrm{NS}}^{\mathcal{N}_{6d}=(1,0)} |_{\mathrm{GSO}} \, (q; y, s) = \sum_{k_1,k_2,p=0}^{\infty} [2k_1]_{y_1} [2k_2]_{y_2} [2p]_r \, F^{\mathrm{NS}}_{k_1,k_2,p}(q) , \tag{5.13}$$

where the function $F^{\mathrm{NS}}_{k,p}(q)$ is given by

$$\begin{aligned} F^{\mathrm{NS}}_{k_1,k_2,p}(q) &= (q;q)_\infty^{-9}(1-q) q^{\frac{1}{2}p^2+p-1} \\ &\times \sum_{\boldsymbol{n}\in\mathbb{Z}_+^2} \sum_{\boldsymbol{m}\in\mathbb{Z}_{\geq 0}^2} \prod_{A=1}^{2} (-1)^{n_A+1}(1-q^{n_A}) q^{\frac{1}{2}m_A^2+\binom{n_A}{2}} (q^{n_A|k_A-m_A|} - q^{n_A(k_A+m_A+1)}) \\ &\times \frac{1}{2}\left[\frac{(1-q^{p+\frac{1}{2}})\vartheta_3(1,q)}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{3}{2}})} \prod_{A=1}^{2} (1-q^{m_A+\frac{1}{2}}) \right. \\ &\left. \qquad + (-1)^{m_1^2+m_2^2+p^2} \frac{(1+q^{p+\frac{1}{2}})\vartheta_4(1,q)}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{3}{2}})} \prod_{A=1}^{2} (1+q^{m_A+\frac{1}{2}}) \right] . \end{aligned} \tag{5.14}$$

Asymptotics. This expression can be simplified further in the asymptotic limit $k_1, k_2 \to \infty$. Using (4.20), we have

$$\begin{aligned}\sum_{\boldsymbol{n}\in\mathbb{Z}_+^2}&\prod_{A=1}^{2}(-1)^{n_A+1}(1-q^{n_A})q^{\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)})\\&\sim(1-q)^2\prod_{A=1}^{2}\frac{q^{k_A}\left(1-q^{2k_A+2}\right)}{\left(1+q^{k_A}\right)^4}\left\{q^{-m_A}\left(1-q^{2m_A+1}\right)\right\}\ ,\end{aligned}\tag{5.15}$$

and using (4.22) we have

$$\sum_{\boldsymbol{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}q^{\frac{1}{2}m_A^2-m_A}\left(1-q^{m_A+\frac{1}{2}}\right)\left(1-q^{2m_A+1}\right)=q^{-1}(1-q)^2\vartheta_3(1,q)^2\ ,\tag{5.16}$$

$$\sum_{\boldsymbol{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}(-1)^{m_A^2}q^{\frac{1}{2}m_A^2-m_A}\left(1+q^{m_A+\frac{1}{2}}\right)\left(1-q^{2m_A+1}\right)=q^{-1}(1-q)^2\vartheta_4(1,q)^2\ ,\tag{5.17}$$

Therefore we arrive at an asymptotic formula for $F^{\rm NS}_{k_1,k_2,p}(q)$ when $k_1, k_2 \to \infty$:

$$\begin{aligned}F^{\rm NS}_{k_1,k_2,p}(q)\sim\frac{1}{2}(q;q)_\infty^{-9}(1-q)^5q^{\frac{1}{2}p^2+p+k_1+k_2-2}&\left[\frac{(1-q^{p+\frac{1}{2}})}{(1+q^{p-\frac{1}{2}})(1+q^{p+\frac{3}{2}})}\vartheta_3(1,q)^3\right.\\&\left.+(-1)^{p^2}\frac{(1+q^{p+\frac{1}{2}})}{(1-q^{p-\frac{1}{2}})(1-q^{p+\frac{3}{2}})}\vartheta_4(1,q)^3\right]\ ,\qquad k_1,k_2\to\infty\ .\end{aligned}\tag{5.18}$$

The R sector

The partition function of the GSO projected R sector is

$$\chi_{\rm R}^{\mathcal{N}_{6d}=(1,0)}\,|_{\rm GSO}\,(q;y,s)=\sum_{k_1,k_2,p=0}^{\infty}[2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p+1]_r\ F^{\rm R}_{k_1,k_2,p}(q)\ ,\tag{5.19}$$

where $F^{\rm R}_{k_1,k_2,p}(q)$ is given by

$$\begin{aligned}F^{\rm R}_{k_1,k_2,p}(q)&=(q;q)_\infty^{-9}(1-q)q^{\frac{1}{2}p^2+\frac{3}{2}p-\frac{3}{8}}\times\frac{1}{2}\frac{(1-q^{p+1})\vartheta_2(1,q)}{(1+q^p)(1+q^{p+2})}\\&\times\sum_{\boldsymbol{n}\in\mathbb{Z}_+^2}\sum_{\boldsymbol{m}\in\mathbb{Z}_{\geq 0}^2}\prod_{A=1}^{2}(-1)^{n_A}(1-q^{n_A})q^{\frac{1}{2}(m_A+\frac{1}{2})^2+\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+2)})(1-q^{m_A+1})\end{aligned}\tag{5.20}$$

Similarly to the NS sector, an asymptotic formula for $F^{\rm NS}_{k_1,k_2,p}(q)$ when $k_1, k_2 \to \infty$ is given by

$$F^{\rm R}_{k_1,k_2,p}(q)\sim\frac{1}{2}(q;q)_\infty^{-9}(1-q)^5q^{\frac{1}{2}p^2+\frac{3}{2}p+k_1+k_2-\frac{3}{8}}\frac{(1-q^{p+1})}{(1+q^p)(1+q^{p+2})}\vartheta_2(1,q)^3\ .\tag{5.21}$$

5.3 Multiplicities of representations in the $\mathcal{N}_{6d} = (1,0)$ partition function

Combining the contributions from the NS and R sectors, we have

$$\begin{aligned}\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r)&=\chi_{\rm NS}^{\mathcal{N}_{6d}=(1,0)}\,|_{\rm GSO}\,(q;\boldsymbol{y},r)\ +\ \chi_{\rm R}^{\mathcal{N}_{6d}=(1,0)}\,|_{\rm GSO}\,(q;\boldsymbol{y},r)\\&=\sum_{k_1,k_2,p=0}^{\infty}\Big([2k_1]_{y_1}[2k_2]_{y_2}[2p]_r\ F^{\rm NS}_{k_1,k_2,p}(q)\end{aligned}$$

$$+ [2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p+1]_r \; F^{\rm R}_{k_1,k_2,p}(q)\Big) \; . \tag{5.22}$$

Making SUSY manifest amounts to rewriting the partition function as

$$\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r) = \sum_{n_1,n_2\geq 0}\sum_{p=0}^{\infty} [\![n_1,n_2;p]\!] \; G_{n_1,n_2,p}(q) \; , \tag{5.23}$$

and the aim is to compute explicitly a *multiplicity generating function* $G_{n_1,n_2,p}(q)$.

Before proceeding further, we observe the selection rule

$$G_{n_1,2n_2,2p+1}(q) = 0 \; , \qquad G_{n_1,2n_2+1,2p}(q) = 0 \; . \tag{5.24}$$

It follows from (5.22) that $[k_1]_{y_1}[k_2]_{y_2}[p]_r$ with odd (respectively even) values of p only enter with a product of two representations with both odd (resp. even) k_1 and k_2. According to (2.60) and (2.64), the product $[k_1]_{y_1}[k_2]_{y_2}$ with both odd (resp. even) k_1 and k_2 decomposes into only spin (resp. non-spin) representations of $SO(5)$. In other words, a spin (resp. non-spin) representation only comes with an odd (resp. even) value of p, and hence (5.24) follows.

The multiplicity of $[\![n_1,n_2;p]\!]$ appearing in $\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r)$ can be determined as follows:

$$\begin{aligned} G_{n_1,n_2,p}(q) &= \int \mathrm{d}\mu_{SU(2)}(r)[p]_r \int \mathrm{d}\mu_{SO(5)}(\boldsymbol{y})[n_1,n_2]_{\boldsymbol{y}} \; \frac{\chi^{\mathcal{N}_{6d}=(1,0)}(q;\boldsymbol{y},r)}{Z(\mathcal{N}_{6d}=(1,0))(\boldsymbol{y},r)} \; , \\ &= G^{\rm NS}_{n_1,n_2,p}(q) + G^{\rm R}_{n_1,n_2,p}(q) \; , \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} G^{\rm NS}_{n_1,n_2,p}(q) &= \int \mathrm{d}\mu_{SU(2)}(r)[p]_r \; \int \mathrm{d}\mu_{SO(5)}(\boldsymbol{y})[n_1,n_2]_{\boldsymbol{y}} \times \\ &\qquad \sum_{k_1,k_2,p'\geq 0} \frac{[2k_1]_{y_1}[2k_2]_{y_2}[2p']_r}{Z(\mathcal{N}_{6d}=(1,0))(\boldsymbol{y},r)} \; F^{\rm NS}_{k_1,k_2,p'}(q) \; , \end{aligned} \tag{5.26}$$

$$\begin{aligned} G^{\rm R}_{n_1,n_2,p}(q) &= \int \mathrm{d}\mu_{SU(2)}(r)[p]_r \; \int \mathrm{d}\mu_{SO(5)}(\boldsymbol{y})[n_1,n_2]_{\boldsymbol{y}} \times \\ &\qquad \sum_{k_1,k_2,p'\geq 0} \frac{[2k_1+1]_{y_1}[2k_2+1]_{y_2}[2p'+1]_r}{Z(\mathcal{N}_{6d}=(1,0))(\boldsymbol{y},r)} \; F^{\rm R}_{k_1,k_2,p'}(q) \; , \end{aligned} \tag{5.27}$$

and the inverse of the character of the fundamental multiplet in (5.1) can be written as a geometric series[24] similar to (4.42)

$$\begin{aligned} [Z(\mathcal{N}_{6d}=(1,0))(\boldsymbol{y},r)]^{-1} &= \frac{r^2}{\left(1+\frac{r}{y_1y_2}\right)\left(1+\frac{ry_1}{y_2}\right)\left(1+\frac{ry_2}{y_1}\right)(1+ry_1y_2)} \\ &= \sum_{m_1,\ldots,m_4\geq 0} (-1)^{m_1+m_2+m_3+m_4} r^{2+m_1+m_2+m_3+m_4} \times \\ &\qquad y_1^{-m_1+m_2-m_3+m_4} y_2^{-m_1-m_2+m_3+m_4} \; . \end{aligned} \tag{5.29}$$

[24]Note that this can also be rewritten as

$$\begin{aligned} [Z(\mathcal{N}_{6d}=(1,0))(\boldsymbol{y},r)]^{-1} &= r^2 \, \mathrm{PE}\left[s[0,1]_{\boldsymbol{y}}\right] \quad \text{with } s=-r \\ &= \sum_{m=0}^{\infty} (-1)^m r^{m+2} [0,m]_{\boldsymbol{y}} \; . \end{aligned} \tag{5.28}$$

5.3.1 Some useful identities

Before we proceed further, let us derive some useful identities for the elementary building blocks of $G_{n_1,n_2,p}$. The first one follows from (2.6):

$$\begin{aligned}\mathcal{I}_0(w;p_1,p_2) &:= \int \mathrm{d}\mu_{SO(3)}(r)\ r^w [p_1]_r [p_2]_r \\ &= \begin{cases} \delta_{p_1,p_2} & \text{for } w=0 \\ \frac{1}{2}\sum_{p=0}^{\frac{1}{2}(p_1+p_2-|p_1-p_2|)} \left(\delta_{|w|,2p+|p_1-p_2|} - \delta_{|w|,2p+2+|p_1-p_2|}\right) & \text{for } w\neq 0 \end{cases}\end{aligned} \tag{5.30}$$

Next, we are interested in the following integral:

$$\mathcal{I}(\boldsymbol{w};\boldsymbol{k};\boldsymbol{n}) := \int \mathrm{d}\mu_{SO(5)}(\boldsymbol{y})\ y_1^{w_1} y_2^{w_2} [k_1]_{y_1} [k_2]_{y_2} [n_1,n_2]_{\boldsymbol{y}}\ . \tag{5.31}$$

There are four cases to be considered. Each of them can be computed using the decomposition formula (2.61) or (2.65), together with (5.30). In what follows, we assume that $\boldsymbol{k},\boldsymbol{n}\in\mathbb{Z}^2_{\geq 0}$ and $\boldsymbol{w}\in\mathbb{Z}^2$.

$$\begin{aligned}&\mathcal{I}(\boldsymbol{w};2k_1,2k_2;n_1,2n_2) \\ &\quad = \sum_{\boldsymbol{k}'\in\mathbb{Z}^2_{\geq 0}} \Delta(n_1+n_2,n_2;2k_1',2k_2') \prod_{A=1}^{2} \mathcal{I}_0(w_A;2k_A,2k_A')\ ,\end{aligned} \tag{5.32}$$

$$\begin{aligned}&\mathcal{I}(\boldsymbol{w};2k_1+1,2k_2+1;n_1,2n_2) \\ &\quad = \sum_{\boldsymbol{k}'\in\mathbb{Z}^2_{\geq 0}} \Delta(n_1+n_2,n_2;2k_1',2k_2') \prod_{A=1}^{2} \mathcal{I}_0(w_A;2k_A+1,2k_A')\ ,\end{aligned} \tag{5.33}$$

$$\begin{aligned}&\mathcal{I}(\boldsymbol{w};2k_1,2k_2;n_1,2n_2+1) \\ &\quad = \sum_{\boldsymbol{k}'\in\mathbb{Z}^2_{\geq 0}} \Delta(n_1+n_2+\frac{1}{2},n_2+\frac{1}{2};2k_1'+1,2k_2'+1) \prod_{A=1}^{2} \mathcal{I}_0(w_A;2k_A,2k_A'+1)\ ,\end{aligned} \tag{5.34}$$

$$\begin{aligned}&\mathcal{I}(\boldsymbol{w};2k_1+1,2k_2+1;n_1,2n_2+1) \\ &\quad = \sum_{\boldsymbol{k}'\in\mathbb{Z}^2_{\geq 0}} \Delta(n_1+n_2+\frac{1}{2},n_2+\frac{1}{2};2k_1'+1,2k_2'+1) \prod_{A=1}^{2} \mathcal{I}_0(w_A;2k_A+1,2k_A'+1)\ ,\end{aligned} \tag{5.35}$$

where from (2.57) and (2.62)

$$\Delta(\lambda_1,\lambda_2;2k_1,2k_2) = \frac{1}{2}\sum_{\sigma\in S_2} \det\left(\theta^{4+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}\right)\right)^2_{A,B=1}\ , \tag{5.36}$$

$$\Delta(\lambda_1,\lambda_2;2k_1+1,2k_2+1) = \frac{1}{2}\sum_{\sigma\in S_2} \det\left(\theta^{4+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}+\frac{1}{2}\right)\right)^2_{A,B=1}\ . \tag{5.37}$$

5.3.2 Multiplicity generating function

The NS- and R sector contributions to the multiplicity generating function for the representation $[\![n_1,n_2;p]\!]$ can be rewritten as

$$\begin{aligned}G^{\mathrm{NS}}_{n_1,n_2,p}(q) = \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \mathcal{I}_0(W_1(\boldsymbol{m}),p,2p')\times \\ \sum_{k_1,k_2\geq 0} \mathcal{I}(\boldsymbol{W}_2(\boldsymbol{m});2k_1,2k_2;n_1,n_2)\ F^{\mathrm{NS}}_{k_1,k_2,p'}(q)\end{aligned} \tag{5.38}$$

$$G^{\mathrm{R}}_{n_1,n_2,p}(q) = \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \mathcal{I}_0(W_1(\boldsymbol{m}), p, 2p'+1)\times$$
$$\sum_{k_1,k_2\geq 0} \mathcal{I}(\boldsymbol{W}_2(\boldsymbol{m}); 2k_1+1, 2k_2+1; n_1, n_2)\ F^{\mathrm{R}}_{k_1,k_2,p'}(q)\ , \qquad (5.39)$$

where we define

$$W_1(\boldsymbol{m}) = 2 + m_1 + m_2 + m_3 + m_4\ ,$$
$$\boldsymbol{W}_2(\boldsymbol{m}) = (-m_1 + m_2 - m_3 + m_4, -m_1 - m_2 + m_3 + m_4)\ . \qquad (5.40)$$

As stated in (5.25), the multiplicity of the representation $[\![n_1, n_2; p]\!]$ in the $\mathcal{N}_{6d} = (1,0)$ partition function is given by

$$\begin{aligned} G_{n_1,n_2,p}(q) &= G^{\mathrm{NS}}_{n_1,n_2,p}(q) + G^{\mathrm{R}}_{n_1,n_2,p}(q) \\ &= \sum_{m_1,\ldots,m_4\geq 0} (-1)^{\sum_{j=1}^4 m_j} \sum_{p'\geq 0} \Big[\mathcal{I}_0(W_1(\boldsymbol{m}); p, 2p') \sum_{k_1,k_2\geq 0} \mathcal{I}(\boldsymbol{W}_2(\boldsymbol{m}); 2k_1, 2k_2; n_1, n_2)\ F^{\mathrm{NS}}_{k_1,k_2,p'}(q) \\ &\quad + \mathcal{I}_0(W_1(\boldsymbol{m}), p, 2p'+1) \sum_{k_1,k_2\geq 0} \mathcal{I}(\boldsymbol{W}_2(\boldsymbol{m}); 2k_1+1, 2k_2+1; n_1, n_2)\ F^{\mathrm{R}}_{k_1,k_2,p'}(q)\Big]\ . \end{aligned} \qquad (5.41)$$

5.4 Empirical approach to $\mathcal{N}_{6d} = (1,0)$ asymptotic patterns

In this subsection, we follow the lines of subsection 4.5 and investigate the large spin asymptotics of multiplicity generating functions $G_{n,k,p}(q)$ for universal $\mathcal{N}_{6d} = (1,0)$ supermultiplets $[\![n,k;p]\!]$. Similar to the $\mathcal{N}_{4d} = 1$ strategy, the $G_{n,k,p}(q)$ are expanded in powers of q^n where n denotes the first Dynkin label that we loosely identify with the spin. The coefficients $\tau^{k,p}_\ell(q)$ of $(q^n)^\ell$ turn out to be power series with non-negative coefficients which enter with alternating sign $(-1)^{\ell-1}$:

$$\begin{aligned} G_{n,k,p}(q) &= q^n\,\tau^{k,p}_1(q) - q^{2n}\,\tau^{k,p}_2(q) + q^{3n}\,\tau^{k,p}_3(q) - \ldots \\ &= \sum_{\ell=1}^{\infty} (-1)^{\ell-1}\, q^{\ell n}\,\tau^{k,p}_\ell(q) \end{aligned} \qquad (5.42)$$

In spacetime dimensions higher than four, the analytic methods of subsection 4.4 are no longer efficiently applicable. We could not find an asymptotic formula for (5.41) resembling (4.63) and (4.64) for the large spin regime of the $\mathcal{N}_{4d} = 1$ multiplicity generating functions. Hence, we determine the $\tau^{k,p}_\ell(q)$ including the leading trajectory $\tau^{k,p}_1(q)$ from our data found by expanding the partition function (5.3) up to mass level 25. The multiplicities of $[\![n,0;0]\!]$ multiplets are shown in the following table 5, data for nonzero values of k or p can be found in appendix B.2. Table entries marked in red are only affected by the stable pattern $\tau^{k,p}_{\ell=1}(q)$ whereas the blue numbers arise from $q^n\tau^{k,p}_1(q) - q^{2n}\tau^{k,p}_2(q)$, i.e. by including the (subtractive) subleading trajectory.

5.4.1 Levels of first appearance

Let us firstly determine the level of first appearance for various families $\{[\![n,k;p]\!],\ n = 0, 1, \ldots\}$ of $\mathcal{N}_{6d} = (1,0)$ supermultiplets with second $SO(5)$ Dynkin label k and R symmetry quantum number p fixed. It is identical to the leading q power of the multiplicity generaing function $G_{0,k,p}(q)$ or its expansion coefficients $\tau^{k,p}_\ell(q)$ defined by (5.42). The following table 6 gathers the mass levels $\alpha' m^2 \leq 25$ where the first instance of a $\{[\![n,k;p]\!],\ n = 0, 1, \ldots\}$ member can be found:

$\alpha' m^2$	$\#[\![0,0;0]\!]$	$\#[\![1,0;0]\!]$	$\#[\![2,0;0]\!]$	$\#[\![3,0;0]\!]$	$\#[\![4,0;0]\!]$	$\#[\![5,0;0]\!]$	$\#[\![6,0;0]\!]$	$\#[\![7,0;0]\!]$	$\#[\![8,0;0]\!]$	$\#[\![9,0;0]\!]$	$\#[\![10,0;0]\!]$	$\#[\![11,0;0]\!]$
1	0	1	0									
2	0	0	1	0								
3	1	2	0	1	0							
4	2	2	3	0	1	0						
5	3	9	4	3	0	1	0					
6	8	12	14	5	3	0	1	0				
7	13	35	24	17	5	3	0	1	0			
8	30	58	63	29	18	5	3	0	1	0		
9	53	135	116	82	32	18	5	3	0	1	0	
10	107	243	265	153	88	33	18	5	3	0	1	0
11	193	505	503	358	172	91	33	18	5	3	0	1
12	376	918	1044	696	403	178	92	33	18	5	3	0
13	670	1803	1975	1474	801	423	181	92	33	18	5	3
14	1246	3269	3887	2839	1711	846	429	182	92	33	18	5
15	2220	6136	7235	5687	3355	1824	866	432	182	92	33	18
16	4005	11015	13691	10754	6784	3605	1870	872	433	182	92	33
17	7025	20052	25041	20649	13021	7348	3718	1890	875	433	182	92
18	12407	35469	45971	38304	25243	14213	7606	3764	1896	876	433	182
19	21469	63030	82532	71226	47411	27774	14790	7720	3784	1899	876	433
20	37182	109838	147906	129443	89013	52547	29015	15048	7766	3790	1900	876
21	63492	191293	260818	234646	163536	99387	55177	29600	15162	7786	3793	1900
22	108142	328527	457957	418298	299140	183903	104797	56431	29859	15208	7792	3794
23	182254	562391	794256	741961	538495	338749	194850	107476	57016	29973	15228	7795
24	306007	952431	1369976	1299438	963344	613928	360467	200360	108738	57275	30019	15234
25	509309	1605996	2339762	2261945	1702039	1105604	656324	371692	203052	109324	57389	30039

Table 5. $\mathcal{N}_{6d} = (1,0)$ multiplets with $SO(5)$ quantum numbers $[n,0]$ and $SU(2)_R$ spin 0

$\downarrow p,\ \overrightarrow{k}$	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
0	1		2		5		8		11		14		17		20		23	
1		2		4		7		10		13		16		19		22		25
2	3		4		7		10		13		16		19		22		25	
3		5		7		10		13		16		19		22		25		
4	7		8		10		13		16		19		22		25			
5		9		11		14		17		20		23						
6	11		12		15		18		21		24							
7		14		16		19		22		25								
8	17		18		20		23											
9		20		22		25												
10	23		24															

Table 6. Mass level where the $[\![0,k;p]\!]$ multiplet of $\mathcal{N}_{6d} = (1,0)$ firstly occurs. Empty spaces indicate that the representations in question do not occur at levels ≤ 25.

We observe that, roughly speaking, the level of first appearance for supermultiplets $[\![n,k;p]\!]$ depends linearly[25] on the $SO(5)$ Dynkin label k (with slope $\frac{3}{2}$) but quadratically on the R symmetry

[25]The linear k dependence can be partially understood from the $\lambda_{1,2}$ dependence in (2.68). However, the bosonic string suggests that an $SO(5)$ representation $[n,k]$ is delayed by *two* levels under $k \mapsto k+1$ whereas the observations from table 6 clearly show a delay of *three* levels per $k \mapsto k+1$. Even though we cannot give a detailed explanation on analytical grounds, it is clear that this extra delay in mass level must be due to the worldsheet fermions, see e.g.

spin $p/2$, in agreement with the final remark in subsection 3.2.

5.4.2 Explicit formulae for the $\tau_\ell^{k,p}(q)$

Let us now list the leading terms in various $\tau_\ell^{k,p}(q)$, obtained through the entries of table 5 and its $(k,p) \neq (0,0)$ relatives displayed in appendix B.2. This allows to reconstruct the large spin asymptotics of the multiplicity generating functions $G_{n,k,p}(q)$ via (5.42).

At zero $SU(2)_R$ charge, we have

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[0]$

$$\begin{aligned}
\tau_1^{0,0}(q) &= 1 + 0q + 3q^2 + 5q^3 + 18q^4 + 33q^5 + 92q^6 + 182q^7 + 433q^8 + 876q^9 + 1900q^{10} \\
&\quad + 3794q^{11} + 7796q^{12} + 15238q^{13} + 30049q^{14} + 57465q^{15} + 109773q^{16} \\
&\quad + 205349q^{17} + 382249q^{18} + 700520q^{19} + \ldots \\
\tau_2^{0,0}(q) &= q\,(1 + 4q^1 + 10q^2 + 30q^3 + 76q^4 + 190q^5 + 449q^6 + 1035q^7 + 2298q^8 + 4999q^9 \\
&\quad + 10580q^{10} + 21976q^{11} + 44727q^{12} + 89543q^{13} + \ldots) \\
\tau_3^{0,0}(q) &= q\,(1 + q + 10q^2 + 23q^3 + 81q^4 + 194q^5 + 531q^6 + 1232q^7 + 2967q^8 + 6586q^9 + \ldots) \\
\tau_4^{0,0}(q) &= q^2\,(1 + 5q + 16q^2 + 53q^3 + 153q^4 + 417q^5 + \ldots) \\
\tau_5^{0,0}(q) &= q^2\,(1 + q + 11q^2 + \ldots)
\end{aligned} \tag{5.43}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[0]$

$$\begin{aligned}
\tau_1^{2,0}(q) &= q^2\,(1 + 2q + 8q^2 + 17q^3 + 52q^4 + 117q^5 + 293q^6 + 645q^7 + 1468q^8 \\
&\quad + 3119q^9 + 6667q^{10} + 13674q^{11} + 27913q^{12} + 55446q^{13} + 109165q^{14} \\
&\quad + 210717q^{15} + 402714q^{16} + 757889q^{17} + 1412208q^{18} + \ldots) \\
\tau_2^{2,0}(q) &= q^3\,(1 + 4q + 14q^2 + 41q^3 + 118q^4 + 306q^5 + 764q^6 + 1818q^7 + 4191q^8 \\
&\quad + 9344q^9 + 20318q^{10} + 43083q^{11} + 89493q^{12} + 182239q^{13} + \ldots) \\
\tau_3^{2,0}(q) &= q^5\,(3 + 9q + 40q^2 + 114q^3 + 345q^4 + 890q^5 + 2297q^6 + 5481q^7 + 12871q^8 + \ldots) \\
\tau_4^{2,0}(q) &= q^6\,(1 + 5q + 23q^2 + 79q^3 + 251q^4 + 717q^5 + \ldots) \\
\tau_5^{2,0}(q) &= q^8\,(3 + 10q + 48q^2 + \ldots)
\end{aligned} \tag{5.44}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 4]$ and $SU(2)_R$ representation $[0]$

$$\begin{aligned}
\tau_1^{4,0}(q) &= q^5\,(1 + 5q + 14q^2 + 43q^3 + 113q^4 + 294q^5 + 698q^6 + 1648q^7 + 3677q^8 \\
&\quad + 8090q^9 + 17182q^{10} + 35919q^{11} + 73211q^{12} + 147036q^{13} + 289598q^{14} \\
&\quad + 562694q^{15} + 1076373q^{16} + \ldots) \\
\tau_2^{4,0}(q) &= q^6\,(1 + 5q + 18q^2 + 56q^3 + 166q^4 + 446q^5 + 1143q^6 + 2787q^7 + 6549q^8 \\
&\quad + 14864q^9 + 32811q^{10} + 70532q^{11} + 148268q^{12} + \ldots) \\
\tau_3^{4,0}(q) &= q^9\,(4 + 14q + 61q^2 + 184q^3 + 561q^4 + 1495q^5 + 3896q^6 + 9478q^7 + \ldots) \\
\tau_4^{4,0}(q) &= q^{11}\,(1 + 8q + 36q^2 + 131q^3 + \ldots)
\end{aligned} \tag{5.45}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 6]$ and $SU(2)_R$ representation $[0]$

$$\tau_1^{6,0}(q) = q^8\,(1 + 5q + 18q^2 + 53q^3 + 158q^4 + 407q^5 + 1033q^6 + 2452q^7 + 5686q^8$$

(2.69) and (2.70).

$$
\begin{aligned}
& \quad + 12640q^9 + 27521q^{10} + 58151q^{11} + 120616q^{12} + 244647q^{13} + \ldots) \\
\tau_2^{6,0}(q) &= q^9 \,(1 + 5q + 18q^2 + 57q^3 + 173q^4 + 473q^5 + 1234q^6 + 3060q^7 \\
& \quad + 7308q^8 + 16835q^9 + \ldots) \\
\tau_3^{6,0}(q) &= q^{13} \,(4 + 15q + 67q^2 + 209q^3 + \ldots)
\end{aligned} \tag{5.46}
$$

Observe that the leading q powers of $\tau_1^{k,p}, \tau_2^{k,p}, \tau_3^{k,p}, \tau_4^{k,p}, \ldots$ increase more rapidly at higher values of k. In other words, the series (5.42) converges more quickly as the second Dynkin label k increases.

The simplest examples with vanishing second $SO(5)$ Dynkin label and nonzero R symmetry charge are the following:

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[2]$

$$
\begin{aligned}
\tau_1^{0,2}(q) &= q^3 \,(3 + 5q + 20q^2 + 46q^3 + 128q^4 + 288q^5 + 696q^6 + 1513q^7 + 3354q^8 \\
& \quad + 7025q^9 + 14707q^{10} + 29736q^{11} + 59679q^{12} + 116933q^{13} \\
& \quad + 226900q^{14} + 432515q^{15} + 816089q^{16} + \ldots) \\
\tau_2^{0,2}(q) &= q^2 \,(1 + 3q^1 + 13q^2 + 37q^3 + 109q^4 + 285q^5 + 727q^6 + 1737q^7 + 4050q^8 + 9075q^9 \\
& \quad + 19868q^{10} + 42302q^{11} + 88278q^{12} + \ldots) \\
\tau_3^{0,2}(q) &= q^2 \,(1 + 2q + 13q^2 + 37q^3 + 124q^4 + 331q^5 + 906q^6 + 2233q^7 + 5456q^8 + \ldots) \\
\tau_4^{0,2}(q) &= q^3 \,(2 + 7q + 29q^2 + 92q^3 + 282q^4 + \ldots) \\
\tau_5^{0,2}(q) &= q^3 \,(1 + 3q + 18q^2 + \ldots)
\end{aligned} \tag{5.47}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[4]$

$$
\begin{aligned}
\tau_1^{0,4}(q) &= q^6 \,(1 + 4q + 18q^2 + 47q^3 + 142q^4 + 353q^5 + 887q^6 + 2049q^7 + 4692q^8 \\
& \quad + 10215q^9 + 21942q^{10} + 45608q^{11} + 93377q^{12} + 186790q^{13} + 368341q^{14} + \ldots) \\
\tau_2^{0,4}(q) &= q^6 \,(3 + 10q + 41q^2 + 124q^3 + 362q^4 + 952q^5 + 2424q^6 + 5811q^7 \\
& \quad + 13526q^8 + 30317q^9 + \ldots) \\
\tau_3^{0,4}(q) &= q^5 \,(1 + 3q + 17q^2 + 53q^3 + 179q^4 + 501q^5 + 1392q^6 + \ldots) \\
\tau_4^{0,4}(q) &= q^5 \,(1 + 3q + 16q^2 + 53q^3 + \ldots)
\end{aligned} \tag{5.48}
$$

- $SO(5)$ Dynkin labels $[n \to \infty, 0]$ and $SU(2)_R$ representation $[6]$

$$
\begin{aligned}
\tau_1^{0,6}(q) &= q^{11} \,(3 + 8q + 35q^2 + 98q^3 + 291q^4 + 733q^5 + 1856q^6 + 4339q^7 \\
& \quad + 9987q^8 + 21954q^9 + \ldots) \\
\tau_2^{0,6}(q) &= q^{10} \,(1 + 5q^1 + 27q^2 + 88q^3 + 286q^4 + 804q^5 + 2171q^6 + \ldots) \\
\tau_3^{0,6}(q) &= q^{10} \,(3 + 10q + 46q^2 + 148q^3 + \ldots) \\
\tau_4^{0,6}(q) &= q^9 \,(1 + 3q + \ldots)
\end{aligned} \tag{5.49}
$$

Observe that the leading q powers of $\tau_1^{k,p}, \tau_2^{k,p}, \tau_3^{k,p}, \tau_4^{k,p}, \ldots$ tend to decrease at higher values of p. In other words, the series (5.42) converges more slowly as the $SU(2)_R$ spin p increases.

Finally, we shall display asymptotic data for the simplest family of fermionic multiplets with $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation $[1]$

$$
\begin{aligned}
\tau_1^{1,1}(q) &= q^2 \,(2 + 4q + 13q^2 + 35q^3 + 89q^4 + 216q^5 + 508q^6 + 1145q^7 + 2521q^8 \\
& \quad + 5402q^9 + 11320q^{10} + 23238q^{11} + 46856q^{12} + 92850q^{13}
\end{aligned}
$$

$$
\begin{aligned}
& \quad + 181217q^{14} + 348612q^{15} + 661792q^{16} + 1240786q^{17} + \ldots) \\
\tau_2^{1,1}(q) &= q^2\,(1 + 4q + 13q^2 + 43q^3 + 122q^4 + 323q^5 + 814q^6 + 1962q^7 + 4550q^8 \\
& \quad + 10233q^9 + 22370q^{10} + 47718q^{11} + 99574q^{12} + \ldots) \\
\tau_3^{1,1}(q) &= q^3\,(1 + 5q + 21q^2 + 70q^3 + 211q^4 + 584q^5 + 1529q^6 + 3798q^7 + \ldots) \\
\tau_4^{1,1}(q) &= q^4\,(1 + 6q + 24q^2 + 85q^3 + \ldots) \\
\tau_5^{1,1}(q) &= q^5\,(1 + \ldots)
\end{aligned} \tag{5.50}
$$

Further $\tau_\ell^{k,p}(q)$ for $[\![n,k;p]\!]$ multiplets at $(k,p) = (2,2),(4,2),(2,4),(3,1),(1,3),(5,1),(3,3),(1,5)$ are listed in appendix C.1. They confirm our observations that the $\tau_\ell^{k,p}(q)$ expansion (5.42) coverges more quickly with larger values of k and smaller values of p.

5.5 Four dimensional $\mathcal{N}_{4d} = 2$ spectra

In order to determine universal string spectra with $\mathcal{N}_{4d} = 2$ SUSY, we shall now compactify two dimensions of minimally supersymmetric $\mathcal{N}_{6d} = (1,0)$ theories on a T^2. This preserves all the eight supercharges and the internal rotation symmetry becomes an R symmetry factor of $SO(2)_R \cong U(1)_R$. Hence, the dimensionally reduced theory in $d = 4$ spacetime dimensions enjoys $\mathcal{N}_{4d} = 2$ SUSY and R symmetry $SU(2)_R \times U(1)_R$. The fundamental $\mathcal{N}_{4d} = 2$ super Poincaré multiplet encompasses 8+8 states,

$$
Z(\mathcal{N}_{4d} = 2) \;=\; [2]_y \;+\; [2]_r\,[0]_y \;+\; (z^2 + z^{-2})\,[0]_y \;+\; (z + z^{-1})\,[1]_r\,[1]_y \tag{5.51}
$$

where z denotes the $U(1)_R$ fugacity. The tensor product of (5.51) with a Clifford vacuum in some $SO(3) \times SU(2)_R \times U(1)_R$ representation yields a family of supermultiplets characterized by three quantum numbers – n for $SO(3)$ spin, m for $SU(2)_R$ spin and p for $U(1)_R$ charge. The resulting $16(n+1)(m+1)$ states are described by the supercharacter [26]

$$
[\![n;m,p]\!] \;:=\; Z(\mathcal{N}_{4d} = 2) \cdot z^p\,[m]_r\,[n]_y\,. \tag{5.57}
$$

The position of the semicolon in the arguments of the supercharacter allows to distinguish $\mathcal{N}_{4d} = 2$ multiplets $[\![\cdot;\cdot,\cdot]\!]$ from $\mathcal{N}_{6d} = (1,0)$ multiplets $[\![\cdot,\cdot;\cdot]\!]$.

The universal partition function of $\mathcal{N}_{4d} = 2$ scenarios is obtained through GSO projection of the following character products:

$$
\chi^{\mathcal{N}_{4d}=2}(q;y,r,z) \;=\; \chi_{\rm NS}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z) \;+\; \chi_{\rm R}^{\mathcal{N}_{4d}=2}\,|_{\rm GSO}\,(q;y,r,z)
$$

---

[26] The simplicity of the $SO(3)$ tensor product $[2m]\cdot[2k] = \sum_{l=|k-m|}^{k+m}[2l]$ allows for compact closed formulae for the $SO(3)\times SU(2)_R \times U(1)_R$ decomposition of a general $\mathcal{N}_{4d} = 2$ supercharacter:

$$
\begin{aligned}
[\![n;m,p]\!] &= z^p\,\big\{\,[m]_r\,[n+2] \;+\; [m]_r\,[n-2] \;+\; [m+2]_r\,[n] \;+\; [m-2]_r\,[n] \;+\; 2\,[m]_r\,[n] \\
& \quad +\; (z^2+z^{-2})\,[m]_r\,[n] \;+\; (z+z^{-1})\,\big([m+1]_r \;+\; [m-1]_r\big)\,\big([n+1] \;+\; [n-1]\big)\,\big\}
\end{aligned} \tag{5.52}
$$

This generic character formula (5.52) holds for values $n,m \geq 2$ of the Clifford vacuum's $SO(3)\times SU(2)_R$ spin quantum numbers and specializes otherwise:

$$
\begin{aligned}
[\![n;0,p]\!] &= z^p\,\big\{\,[n+2] \;+\; [n-2] \;+\; [2]_r\,[n] \;+\; (1+z^2+z^{-2})\,[n] \\
& \quad +\; (z+z^{-1})\,[1]_r\,\big([n+1] \;+\; [n-1]\big)\,\big\}\,, \qquad n \geq 2
\end{aligned} \tag{5.53}
$$

$$
\begin{aligned}
[\![0;m,p]\!] &= z^p\,\big\{\,[m]_r\,[2] \;+\; [m]_r\,[0] \;+\; [m+2]_r\,[0] \;+\; [m-2]_r\,[0] \;+\; (z^2+z^{-2})\,[m]_r\,[0] \\
& \quad +\; (z+z^{-1})\,\big([m+1]_r \;+\; [m-1]_r\big)\,[1]\,\big\}\,, \qquad m \geq 2
\end{aligned} \tag{5.54}
$$

$$
[\![0;0,p]\!] = z^p\,\big\{\,[2] \;+\; [2]_r\,[0] \;+\; (z^2+z^{-2})\,[0] \;+\; (z+z^{-1})\,[1]_r\,[1]\,\big\} \tag{5.55}
$$

$$
\begin{aligned}
[\![1;1,p]\!] &= z^p\,\big\{\,[1]_r\,[3] \;+\; [3]_r\,[1] \;+\; (2+z^2+z^{-2})\,[1]_r\,[1] \\
& \quad (z+z^{-1})\,\big([2]_r\,[2] \;+\; [2]_r\,[0] \;+\; [2] \;+\; [0]\big)\,\big\}
\end{aligned} \tag{5.56}
$$

We observe the general selection rule that either none or all of n,m,p are odd, hence, there is no need to consider $[\![1;0,p]\!]$ or $[\![0;1,p]\!]$.

$$\begin{aligned}\chi^{\mathcal{N}_{4d}=2}_{\rm NS}\,|_{\rm GSO}\,(q;y,r,z) \;&=\; \frac{1}{2}\,q^{-\frac{1}{2}}\,\big[\,\chi^{SO(3)}_{\rm NS}(q;y)\,\chi^{\mathcal{N}_{2d}=4,c=6}_{{\rm NS},h=0,\ell=0}(q;r)\,\chi^{SO(3)}_{\rm NS}(q;z)\\ &\qquad -\;\chi^{SO(3)}_{\rm NS}(e^{2\pi i}q;y)\,\chi^{\mathcal{N}_{2d}=4,c=6}_{{\rm NS},h=0,\ell=0}(e^{2\pi i}q;r)\,\chi^{SO(3)}_{\rm NS}(e^{2\pi i}q;z)\,\big]\\ \chi^{\mathcal{N}_{4d}=2}_{\rm R}\,|_{\rm GSO}\,(q;y,r,z) \;&=\; \frac{1}{2}\,\chi^{SO(3)}_{\rm R}(q;y)\,\chi^{\mathcal{N}_{2d}=4,c=6}_{{\rm R},h=1/4,\ell=1/2}(q;r)\,\chi^{SO(3)}_{\rm R}(q;z)\end{aligned} \tag{5.58}$$

Its symmetry under reversal $p\mapsto -p$ of $U(1)_R$ charges motivates the definition

$$[\![n;m,\pm p]\!] \;:=\; \begin{cases} [\![n;m,p]\!] \;+\; [\![n;m,-p]\!] & : \; p\neq 0 \\ \qquad [\![n;m,0]\!] & : \; p=0 \end{cases}\,, \tag{5.59}$$

then the power series expansion of (5.58) starts like[27]

$$\begin{aligned}\chi^{\mathcal{N}_{4d}=2}(q;y,r,z) \;&=\; \underbrace{\Big(y^2+y^{-2}+z^2+z^{-2}\;+\;\frac{1}{2}(y+y^{-1})[1]_z[1]_r\Big)\,q^0}_{\text{8 massless states}} \;+\; \underbrace{\big(\,[\![2;0,0]\!]\;+\;[\![0;0,\pm 2]\!]\,\big)\,q}_{\text{80 states at level 1}}\\ &+\;\underbrace{\big([\![4;0,0]\!]\;+\;2\,[\![2;0,\pm2]\!]\;+\;[\![2;0,0]\!]\;+\;[\![1;1,\pm1]\!]\;+\;[\![0;0,\pm4]\!]\;+\;2\,[\![0;0,0]\!]\,\big)\,q^2}_{\text{512 states at level 2}}\\ &+\;\big([\![6;0,0]\!]\;+\;2\,[\![4;0,\pm2]\!]\;+\;[\![4;0,0]\!]\;+\;2\,[\![3;1,\pm1]\!]\;+\;2\,[\![2;0,\pm4]\!]\;+\;2\,[\![2;0,\pm2]\!]\;+\;6\,[\![2;0,0]\!]\\ &\quad+\;2\,[\![1;1,\pm3]\!]\;+\;3\,[\![1;1,\pm1]\!]\;+\;[\![0;2,0]\!]\;+\;[\![0;0,\pm6]\!]\;+\;4\,[\![0;0,\pm2]\!]\;+\;2\,[\![0;0,0]\!]\,\big)\,q^3\;+\;\mathcal{O}(q^4)\end{aligned} \tag{5.60}$$

The vertex operators occurring in the three multiplets of the first mass level have been constructed in [29], see equations (6.3) to (6.11) of that reference for bosons and equations (6.22) to (6.30) for fermions. The content of the first five levels is summarized in table 7:

$\alpha' m^2$	representations of $\mathcal{N}_{4d}=2$ super Poincaré
1	$[\![2;0,0]\!] + [\![0;0,\pm2]\!]$
2	$[\![4;0,0]\!] + 2\,[\![2;0,\pm2]\!] + [\![2;0,0]\!] + [\![1;1,\pm1]\!] + [\![0;0,\pm4]\!] + 2\,[\![0;0,0]\!]$
3	$[\![6;0,0]\!] + 2\,[\![4;0,\pm2]\!] + [\![4;0,0]\!] + 2\,[\![3;1,\pm1]\!] + 2\,[\![2;0,\pm4]\!] + 2\,[\![2;0,\pm2]\!] + 6\,[\![2;0,0]\!]$ $+\,2\,[\![1;1,\pm3]\!] + 3\,[\![1;1,\pm1]\!] + [\![0;2,0]\!] + [\![0;0,\pm6]\!] + 4\,[\![0;0,\pm2]\!] + 2\,[\![0;0,0]\!]$
4	$[\![8;0,0]\!] + 2\,[\![6;0,\pm2]\!] + [\![6;0,0]\!] + 2\,[\![5;1,\pm1]\!] + 2\,[\![4;0,\pm4]\!] + 3\,[\![4;0,\pm2]\!] + 8\,[\![4;0,0]\!]$ $+\,3\,[\![3;1,\pm3]\!] + 6\,[\![3;1,\pm1]\!] + [\![2;2,\pm2]\!] + 3\,[\![2;2,0]\!] + 2\,[\![2;0,\pm6]\!] + 3\,[\![2;0,\pm4]\!]$ $+\,12\,[\![2;0,\pm2]\!] + 11\,[\![2;0,0]\!] + 2\,[\![1;1,\pm5]\!] + 5\,[\![1;1,\pm3]\!] + 10\,[\![1;1,\pm1]\!] + 2\,[\![0;2,\pm2]\!]$ $+\,[\![0;2,0]\!] + [\![0;0,\pm8]\!] + 5\,[\![0;0,\pm4]\!] + 4\,[\![0;0,2]\!] + 11\,[\![0;0,0]\!]$
5	$[\![10;0,0]\!] + 2\,[\![8;0,\pm2]\!] + [\![8;0,0]\!] + 2\,[\![7;1,\pm1]\!] + 2\,[\![6;0,\pm4]\!] + 3\,[\![6;0,\pm2]\!] + 8\,[\![6;0,0]\!]$ $+\,3\,[\![5;1,\pm3]\!] + 7\,[\![5;1,\pm1]\!] + [\![4;2,\pm2]\!] + 4\,[\![4;2,0]\!] + 2\,[\![4;0,\pm6]\!] + 4\,[\![4;0,\pm4]\!]$ $+\,16\,[\![4;0,\pm2]\!] + 17\,[\![4;0,0]\!] + 3\,[\![3;1,\pm5]\!] + 11\,[\![3;1,\pm3]\!] + 21\,[\![3;1,\pm1]\!] + [\![2;2,\pm4]\!]$ $+\,7\,[\![2;2,\pm2]\!] + 8\,[\![2;2,0]\!] + 2\,[\![2;0,\pm8]\!] + 3\,[\![2;0,\pm6]\!] + 15\,[\![2;0,\pm4]\!] + 23\,[\![2;0,\pm2]\!]$ $+\,38\,[\![2;0,0]\!] + [\![1;3,\pm1]\!] + 2\,[\![1;1,\pm7]\!] + 6\,[\![1;1,\pm5]\!] + 16\,[\![1;1,\pm3]\!] + 28\,[\![1;1,\pm1]\!]$ $+\,3\,[\![0;2,\pm4]\!] + 4\,[\![0;2,\pm2]\!] + 9\,[\![0;2,0]\!] + [\![0;0,\pm10]\!] + 5\,[\![0;0,\pm6]\!] + 6\,[\![0;0,\pm4]\!]$ $+\,21\,[\![0;0,\pm2]\!] + 16\,[\![0;0,0]\!]$

Table 7. $\mathcal{N}_{4d}=2$ multiplets occurring up to mass level 5

[27] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.4). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y-y^{-1})(z-z^{-1})(r-r^{-1})$ to get the correct massless character in R sector.

Comparison with the partition function (5.4) of the $\mathcal{N}_{6d} = (1,0)$ ancestor theory (and table 4) clearly demonstrates that the six dimensional viewpoint gives a more streamlined handle on the spectrum in terms of fewer supermultiplets. This is why we do not provide an asymptotic analysis and data tables for the universal $\mathcal{N}_{4d} = 2$ spectrum like we did for the $d = 6$ ancestor in subsection 5.4 and appendix B.2.

6 Spectra in compactifications with 16 supercharges

This section is devoted to maximally supersymmetric type I superstring compactifications on even dimensional tori where all the sixteen supercharges are preserved [1]. The methods introduced in subsections 2.4 and 2.5 are applied to decompose the partition function of the $(\partial X^i, \psi^i)$ CFT describing $d = 10, 8, 6, 4$ spacetime dimensions into characters of the little group $SO(d-1)$. According to figure 1, the $d = 10$ case takes the role of the ancestor theory for 16 supercharges, so its spectrum will be analyzed in particular detail. In the remaining cases $d = 8, 6, 4$, dimensional reduction converts part of the higher dimensional Lorentz symmetry into an internal R symmetry, i.e. we branch the ten dimensional little group into $SO(9) \rightarrow SO(d-1) \times SO(10-d)_R$. In this process, individual Lorentz fugacities y_k with $k > \frac{1}{2}(d-2)$ are reinterpreted as R symmetry fugacities r_k.

Before looking at individual dimensionalities in detail, let us fix the notation for describing supersymmetric spectra with R symmetries: Characters of the spacetime little group $SO(d-1)$ are denoted by $[a_1, \ldots, a_n]$ with fugacities $y_1, \ldots, y_n$ and $n = \frac{1}{2}(d-2)$ whereas those of the R symmetry $SO(10-d)_R$ receive an extra subscript $[b_1, \ldots, b_\ell]_R$ with fugacities $r_1, \ldots, r_\ell$ and $\ell = 5 - \frac{d}{2}$. Our notation for supercharacters makes use of double brackets $[\![a_1, \ldots, a_n; b_1, \ldots, b_\ell]\!]$ enclosing the $SO(d-1) \times SO(10-d)_R$ quantum numbers of the highest weight state. The semicolon between a_n and b_1 separates spacetime from R symmetry Dynkin labels and eliminates any ambiguity about the spacetime dimension under consideration.

6.1 Ten dimensional $\mathcal{N}_{10d} = 1$ spectra

In this subsection, we want to revisit the results of [5] on $SO(9)$ covariant partition functions for ten dimensional open string excitations and examine further symmetry patterns. The minimal massive $\mathcal{N}_{10d} = 1$ SUSY multiplet encompasses $SO(9)$ representations of a spin two tensor, a three-form and a massive gravitino[28]

$$Z(\mathcal{N}_{10d} = 1) \;\; := \;\; [2,0,0,0] \; + \; [0,0,1,0] \; + \; [1,0,0,1] \; . \tag{6.1}$$

This is precisely the particle content of the first mass level, its vertex operators can for instance be found in equations (2.8), (2.9) and (2.22) of [29].

The generic multiplet is obtained as a tensor product of $Z(\mathcal{N}_{10d} = 1)$ with some $SO(9)$ representation and therefore described by the following $\mathcal{N}_{10d} = 1$ supercharacter:

$$[\![a_1, a_2, a_3, a_4]\!] \;\; := \;\; Z(\mathcal{N}_{10d} = 1) \cdot [a_1, a_2, a_3, a_4] \tag{6.2}$$

This is the basic building blocks of the refined ten dimensional partition function. The latter can be obtained through standard GSO projection of the spacetime CFT

$$\begin{aligned} \chi^{\mathcal{N}_{10d}=1}(q; \boldsymbol{y}) \;\; &= \;\; \chi^{\mathcal{N}_{10d}=1}_{\rm NS} \,|_{\rm GSO}\, (q; \boldsymbol{y}) \; + \; \chi^{\mathcal{N}_{10d}=1}_{\rm R} \,|_{\rm GSO}\, (q; \boldsymbol{y}) \\ \chi^{\mathcal{N}_{10d}=1}_{\rm NS} \,|_{\rm GSO}\, (q; \boldsymbol{y}, r) \;\; &= \;\; \frac{1}{2}\, q^{-\frac{1}{2}} \left[\chi^{SO(9)}_{\rm NS}(q; \boldsymbol{y}) \; - \; \chi^{SO(9)}_{\rm NS}(e^{2\pi i} q; \boldsymbol{y}) \right] \\ \chi^{\mathcal{N}_{10d}=1}_{\rm R} \,|_{\rm GSO}\, (q; \boldsymbol{y}, r) \;\; &= \;\; \frac{1}{2}\, \chi^{SO(9)}_{\rm R}(q; \boldsymbol{y}) \; , \end{aligned} \tag{6.3}$$

[28] Note that $Z(\mathcal{N}_{10d} = 1)$ is denoted by Z_Q in [5].

where $\chi_{\rm NS}^{SO(9)}(q;\boldsymbol{y})$ and $\chi_{\rm R}^{SO(9)}(q;\boldsymbol{y})$ are given by (2.69) and (2.70).

In a power series expansion in q, the coefficient of the n'th power q^n comprises the super Poincaré characters of the n'th mass level $m^2 = n/\alpha'$:[29]

$$\begin{aligned}\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y}) \;&=\; \underbrace{\left(\sum_{j=1}^{4}(y_j^2+y_j^{-2}) \;+\; \frac{1}{2}\prod_{j=1}^{4}(y_j+y_j^{-1})\right)q^0}_{\text{16 massless states}} \;+\; \underbrace{[\![0,0,0,0]\!]\,q}_{\text{256 states at level 1}}\\ &+\; \underbrace{[\![1,0,0,0]\!]\,q^2}_{\text{2304 states at level 2}} \;+\; \underbrace{\big([\![2,0,0,0]\!] \;+\; [\![0,0,0,1]\!]\big)\,q^3}_{\text{15360 states at level 3}}\\ &+\; \big([\![3,0,0,0]\!] \;+\; [\![1,0,0,1]\!] \;+\; [\![1,0,0,0]\!] \;+\; [\![0,1,0,0]\!]\big)\,q^4 \;+\; \mathcal{O}(q^5)\,. \end{aligned} \tag{6.4}$$

The supermultiplets up to level eight are listed in table 8 and the complete first 25 mass levels can be found in table 9 and appendix B.3.

$\alpha' m^2$	representations of $\mathcal{N}_{10d}=1$ super Poincaré
1	$[\![0,0,0,0]\!]$
2	$[\![1,0,0,0]\!]$
3	$[\![2,0,0,0]\!] + [\![0,0,0,1]\!]$
4	$[\![3,0,0,0]\!] + [\![1,0,0,1]\!] + [\![1,0,0,0]\!] + [\![0,1,0,0]\!]$
5	$[\![4,0,0,0]\!] + [\![2,0,0,1]\!] + [\![2,0,0,0]\!] + [\![1,1,0,0]\!] + [\![1,0,0,1]\!] + [\![0,1,0,0]\!]$ $+ [\![0,0,1,0]\!] + [\![0,0,0,1]\!] + [\![0,0,0,0]\!]$
6	$[\![5,0,0,0]\!] + [\![3,0,0,1]\!] + [\![3,0,0,0]\!] + [\![2,1,0,0]\!] + [\![2,0,0,1]\!] + [\![2,0,0,0]\!] + 2\,[\![1,1,0,0]\!]$ $+ [\![1,0,1,0]\!] + 2\,[\![1,0,0,1]\!] + 2\,[\![1,0,0,0]\!] + [\![0,1,0,1]\!] + [\![0,1,0,0]\!] + [\![0,0,0,2]\!] + 2\,[\![0,0,0,1]\!]$
7	$[\![6,0,0,0]\!] + [\![4,0,0,1]\!] + [\![4,0,0,0]\!] + [\![3,1,0,0]\!] + [\![3,0,0,1]\!] + [\![3,0,0,0]\!] + 2\,[\![2,1,0,0]\!]$ $+ [\![2,0,1,0]\!] + 3\,[\![2,0,0,1]\!] + 3\,[\![2,0,0,0]\!] + [\![1,1,0,1]\!] + 2\,[\![1,1,0,0]\!] + [\![1,0,1,0]\!] + [\![1,0,0,2]\!]$ $+ 4\,[\![1,0,0,1]\!] + 2\,[\![1,0,0,0]\!] + [\![0,2,0,0]\!] + 2\,[\![0,1,0,1]\!] + 2\,[\![0,1,0,0]\!] + 3\,[\![0,0,1,0]\!]$ $+ [\![0,0,0,2]\!] + 2\,[\![0,0,0,1]\!] + 2\,[\![0,0,0,0]\!]$
8	$[\![7,0,0,0]\!] + [\![5,0,0,1]\!] + [\![5,0,0,0]\!] + [\![4,1,0,0]\!] + [\![4,0,0,1]\!] + [\![4,0,0,0]\!] + 2\,[\![3,1,0,0]\!]$ $+ [\![3,0,1,0]\!] + 3\,[\![3,0,0,1]\!] + 4\,[\![3,0,0,0]\!] + [\![2,1,0,1]\!] + 3\,[\![2,1,0,0]\!] + [\![2,0,1,0]\!] + [\![2,0,0,2]\!]$ $+ 5\,[\![2,0,0,1]\!] + 3\,[\![2,0,0,0]\!] + [\![1,2,0,0]\!] + 3\,[\![1,1,0,1]\!] + 5\,[\![1,1,0,0]\!] + 4\,[\![1,0,1,0]\!]$ $+ 2\,[\![1,0,0,2]\!] + 7\,[\![1,0,0,1]\!] + 5\,[\![1,0,0,0]\!] + [\![0,2,0,0]\!] + [\![0,1,1,0]\!] + 4\,[\![0,1,0,1]\!]$ $+ 5\,[\![0,1,0,0]\!] + [\![0,0,1,1]\!] + 2\,[\![0,0,1,0]\!] + 3\,[\![0,0,0,2]\!] + 4\,[\![0,0,0,1]\!] + [\![0,0,0,0]\!]$

Table 8. $\mathcal{N}_{10d}=1$ multiplets occurring up to mass level eight

6.1.1 The total number of states at a given mass level

The total number of states at a given mass level m can be read off from the coefficient of q^m in the partition function $\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y})$ when the $SO(9)$ fugacities $y_1,\ldots,y_4$ are set to unity. The function $\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\})$ is referred to as the *unrefined partition function*. From (2.71), (6.3)

[29]Note the usual subtlety about the massless R sector which was explained in the footnote before (4.4). One can simply fix this by adding $\frac{1}{2}([0,0,0,1]_{SO(8)} - [0,0,1,0]_{SO(8)}) = \frac{1}{2}\prod_{i=1}^4(y_i - y_i^{-1})$ to the present result and obtain the correct answer; see also (3.16) of [5]. The $\frac{1}{2}[1,0,0,0]_9$ factor in the massive sector of the aforementioned (3.16) exactly matches our formula at any positive q power.

and SUSY[30], we have

$$\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\}) = 2\chi_{\rm R}^{\mathcal{N}_{10d}=1}|_{\rm GSO}\,(q;\{y_i=1\}) = \frac{\vartheta_2(1,q)^4}{\eta(q)^{12}} = 16\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8 \ . \tag{6.6}$$

The coefficients in the power series of this formula reproduces the third column of Table 1. It also agrees with (5.3.37) of [21]. Note that $\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\})$ is *not* a modular form.

The number of states at each mass level and its asymptotics

The number of states at the mass level m can be determined by

$$N_m = \frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{{\rm d}q}{q^{m+1}}\ \chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\})\ , \tag{6.7}$$

where $\mathcal{C}$ is a contour around the origin.

Now let us compute an asymptotic formula for the number of states N_m at mass level m when $m\to\infty$. Note that a similar discussion can be found in subsections 4.3.3 and 5.3.1 of [21]. For completeness, let us go over some details here. We focus on the limit $q\to 1^-$ and proceed in a similar way to subsection 4.1. The asymptotic behaviour (4.11) and (4.13) of $\vartheta_2(1,q)$ and $\eta(q)$, respectively, leads to

$$\chi^{\mathcal{N}_{10d}=1}(q;\{y_i=1\}) \sim \frac{1}{(2\pi)^4}(1-q)^4\exp\left(-\frac{2\pi^2}{\log q}\right)\ , \quad q\to 1^-\ . \tag{6.8}$$

Let us now combine (6.7) with (6.8). As $m\to\infty$,

$$N_m \sim \frac{1}{(2\pi)^4}\frac{1}{2\pi i}\oint_{\mathcal{C}}\frac{{\rm d}q}{q}\ (1-q)^4\exp\left(-\frac{2\pi^2}{\log q}-m\log q\right)\ . \tag{6.9}$$

The saddle point is at $q_0=\exp\left(-\pi\sqrt{\frac{2}{m}}\right)$ and the steepest descent direction is the imaginary direction in q. We proceed in a similar way to (4.17) by writing $q=q_0e^{i\theta}$ and using Laplace's method to obtain

$$\begin{aligned} N_m &\sim \frac{1}{4}m^{-2}\exp\left(2\pi\sqrt{2m}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty}{\rm d}\theta\exp\left(-\frac{m^{3/2}}{\pi\sqrt{2}}\theta^2\right) \\ &\sim \frac{1}{2^{11/4}}m^{-11/4}e^{2\pi\sqrt{2m}}\ , \qquad m\to\infty\ . \end{aligned} \tag{6.10}$$

The plot of the exact and asymptotic values for N_m against m is depicted in Figure 7.

6.1.2 The GSO projected NS and R sectors

In this section we compute the contributions from the NS and R sectors to the partition function given in (6.3). Here we consider the refined partition function, *i.e.* the fugacities y's are kept explicit.

[30] The agreement of GSO projected partition functions for NS and R sectors follows from Jacobi's abstruse identity:

$$\vartheta_3(1,q)^4-\vartheta_4(1,q)^4-\vartheta_2(1,q)^4=0\ . \tag{6.5}$$

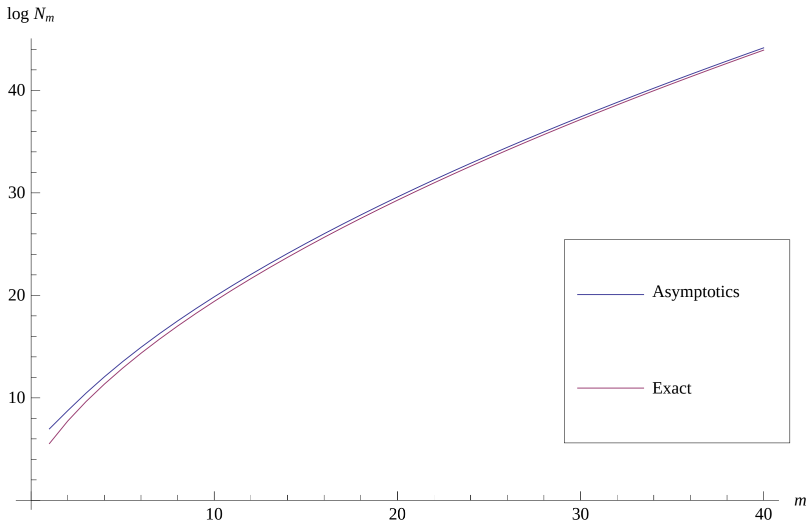

Figure 7: The plot of the exact and asymptotic values of $\log N_m$ against the mass level m for the case of 16 supercharges.

The NS sector. From (6.3) and (2.69), the partition function of the GSO projected NS sector has the structure

$$\chi_{\rm NS}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;y) = \sum_{\boldsymbol{k}\in\mathbb{Z}^4_{\geq 0}} F^{\rm NS}_{k_1,\ldots,k_4}(q)\prod_{A=1}^{4}[2k_A]_{y_A}\ , \tag{6.11}$$

where the functions $F^{\rm NS}_{k_1,\ldots,k_4}(q)$ are given by

$$\begin{aligned} F^{\rm NS}_{k_1,\ldots,k_4}(q) &= (q;q)_\infty^{-12}\sum_{\boldsymbol{n}\in\mathbb{Z}^4_+}\sum_{\boldsymbol{m}\in\mathbb{Z}^4_{\geq 0}}\prod_{A=1}^{4}(-1)^{n_A+1}(1-q^{n_A})q^{\frac{1}{2}m_A^2+\binom{n_A}{2}}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+1)}) \\ &\times\frac{1}{2}\left[\prod_{A=1}^{4}(1-q^{m_A+\frac{1}{2}})+(-1)^{m_1^2+m_2^2+m_3^2+m_4^2}\prod_{A=1}^{4}(1+q^{m_A+\frac{1}{2}})\right]. \end{aligned} \tag{6.12}$$

The R sector. From (6.3) and (2.70), the partition function of the GSO projected R sector is

$$\chi_{\rm R}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;y,s) = \sum_{\boldsymbol{k}\in\mathbb{Z}^4_{\geq 0}} F^{\rm R}_{k_1,\ldots,k_4}(q)\prod_{A=1}^{4}[2k_A+1]_{y_A}\ , \tag{6.13}$$

where the function $F^{\rm R}_{k_1,\ldots,k_4}(q)$ is given by

$$\begin{aligned} F^{\rm R}_{k_1,\ldots,k_4}(q) &= \frac{1}{2}q^{-\frac{1}{2}}(q;q)_\infty^{-12}\sum_{\boldsymbol{m}\in\mathbb{Z}^4_{\geq 0}}\sum_{\boldsymbol{n}\in\mathbb{Z}^4_+}\prod_{A=1}^{4}(-1)^{n_A+1}(1-q^{m_A+1})(1-q^{n_A})q^{\frac{1}{2}(m_A+\frac{1}{2})^2+\binom{n_A}{2}} \\ &\times\prod_{A=1}^{4}(q^{n_A|k_A-m_A|}-q^{n_A(k_A+m_A+2)})\ . \end{aligned} \tag{6.14}$$

6.1.3 Multiplicities of representations in the $\mathcal{N}_{10d}=1$ partition function

Combining the contributions from the NS and R sectors, we have

$$\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y}) = \chi_{\rm NS}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;\boldsymbol{y})\ +\ \chi_{\rm R}^{\mathcal{N}_{10d}=1}\,|_{\rm GSO}\,(q;\boldsymbol{y})$$

$$= \sum_{\boldsymbol{k}\in\mathbb{Z}^4_{\geq 0}} \left(F^{\rm NS}_{\boldsymbol{k}}(q) \prod_{A=1}^{4} [2k_A]_{y_A} + F^{\rm R}_{\boldsymbol{k}} \prod_{A=1}^{4} [2k_A+1]_{y_A} \right) . \tag{6.15}$$

Supersymmetry implies that this partition function can be rewritten as

$$\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^4_{\geq 0}} [\![n_1,n_2,n_3,n_4]\!] \; G_{n_1,n_2,n_3,n_4}(q) \ , \tag{6.16}$$

and the aim is to compute explicitly a *multiplicity generating function* $G_{n_1,n_2,n_3,n_4}(q)$.

The multiplicity of $[\![n_1,n_2,n_3,n_4]\!]$ appearing in $\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y})$ can be determined as follows:

$$\begin{aligned} G_{n_1,n_2,n_3,n_4}(q) &= \int \mathrm{d}\mu_{SO(9)}(\boldsymbol{y})[n_1,n_2,n_3,n_4]_{\boldsymbol{y}} \; \frac{\chi^{\mathcal{N}_{10d}=1}(q;\boldsymbol{y})}{Z(\mathcal{N}_{10d}=1)(\boldsymbol{y})} \ , \\ &= G^{\rm NS}_{n_1,n_2,n_3,n_4}(q) + G^{\rm R}_{n_1,n_2,n_3,n_4}(q) \ , \end{aligned} \tag{6.17}$$

where

$$G^{\rm NS}_{n_1,n_2,n_3,n_4}(q) = \int \mathrm{d}\mu_{SO(9)}(\boldsymbol{y})[n_1,n_2,n_3,n_4]_{\boldsymbol{y}} \sum_{\boldsymbol{k}\in\mathbb{Z}^4_{\geq 0}} \frac{\prod_{A=1}^4 [2k_A]_{y_A}}{Z(\mathcal{N}_{10d}=1)(\boldsymbol{y})} F^{\rm NS}_{k_1,\ldots,k_4}(q) \ , \tag{6.18}$$

$$G^{\rm R}_{n_1,n_2,n_3,n_4}(q) = \int \mathrm{d}\mu_{SO(9)}(\boldsymbol{y})[n_1,n_2,n_3,n_4]_{\boldsymbol{y}} \sum_{\boldsymbol{k}\in\mathbb{Z}^4_{\geq 0}} \frac{\prod_{A=1}^4 [2k_A+1]_{y_A}}{Z(\mathcal{N}_{10d}=1)(\boldsymbol{y})} F^{\rm R}_{k_1,\ldots,k_4}(q) \ . \tag{6.19}$$

The inverse of the character of the fundamental multiplet in (6.1) can be written as a geometric series[31] similar to (4.42) and (5.29)

$$\begin{aligned} &[Z(\mathcal{N}_{10d}=1)(\boldsymbol{y},r)]^{-1} \\ &= \frac{y_4^4}{\left(1+\frac{y_4}{y_1y_2y_3}\right)\left(1+\frac{y_1y_4}{y_2y_3}\right)\left(1+\frac{y_2y_4}{y_1y_3}\right)\left(1+\frac{y_1y_2y_4}{y_3}\right)\left(1+\frac{y_3y_4}{y_1y_2}\right)\left(1+\frac{y_1y_3y_4}{y_2}\right)\left(1+\frac{y_2y_3y_4}{y_1}\right)(1+y_1y_2y_3y_4)} \\ &= \sum_{\boldsymbol{m}\in\mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} y_1^{\sum_{j=1}^8 (-1)^j m_j} y_2^{\sum_{j=1}^8 (-1)^{\lfloor (j+1)/2\rfloor} m_j} y_3^{\sum_{j=1}^8 (-1)^{\lfloor (j+3)/4\rfloor} m_j} y_4^{4+\sum_{j=1}^8 m_j} \ . \end{aligned} \tag{6.21}$$

6.1.4 Some useful identities

In this section, we derive some useful identities that will be put into use later. Once we plug the series expansion (6.21) of the inverse $Z(\mathcal{N}_{10d}=1)$ into the integrand of (6.17), the elementary contributions to multiplicity generating functions G_{n_1,n_2,n_3,n_4} are integrals of type

$$\mathcal{J}_0(w;\boldsymbol{p}) := \int \mathrm{d}\mu_{SO(3)}(r)\; r^w \prod_{A=1}^{4} [p_A]_r \ . \tag{6.22}$$

as well as

$$\mathcal{J}(\boldsymbol{w};\boldsymbol{k};\boldsymbol{n}) := \int \mathrm{d}\mu_{SO(9)}(\boldsymbol{y})[n_1,n_2,n_3,n_4]_{\boldsymbol{y}} \prod_{A=1}^{4} y_A^{w_A} [k_A]_{y_A} \ . \tag{6.23}$$

[31] Note that this can also be rewritten as

$$[Z(\mathcal{N}_{10d}=1)(\boldsymbol{y})]^{-1} = \lim_{s\to -1} (\mathrm{PE}\,[s[0,0,0,1]_{\boldsymbol{y}}])^{1/2} = \left[\sum_{m=0}^{\infty} (-1)^m \mathrm{Sym}^m [0,0,0,1]_{\boldsymbol{y}}\right]^{1/2} . \tag{6.20}$$

There are four cases to be considered, namely spin/non-spin representations of $SO(9)$ and for each of these cases $k_1, \ldots, k_4$ can be all even or all odd. In what follows, we assume that $\boldsymbol{k}, \boldsymbol{n} \in \mathbb{Z}^4_{\geq 0}$ and $\boldsymbol{w} \in \mathbb{Z}^4$. For non-spin representations,

$$\mathcal{J}(\boldsymbol{w};\ 2k_1, \ldots, 2k_4;\ n_1, \ldots, 2n_4) = \sum_{\boldsymbol{k'} \in \mathbb{Z}^4_{\geq 0}} \Delta(\boldsymbol{\lambda}_{ns}; 2k'_1, \ldots, 2k'_4) \prod_{A=1}^{4} \mathcal{J}_0(w_A; 2k_A, 2k'_A) \ , \tag{6.24}$$

$$\mathcal{J}(\boldsymbol{w};\ 2k_1+1, \ldots, 2k_4+1;\ n_1, \ldots, 2n_4) = \sum_{\boldsymbol{k'} \in \mathbb{Z}^4_{\geq 0}} \Delta(\boldsymbol{\lambda}_{ns}; 2k'_1, \ldots, 2k'_4) \prod_{A=1}^{4} \mathcal{J}_0(w_A; 2k_A+1, 2k'_A) \ , \tag{6.25}$$

where $\boldsymbol{\lambda}_{ns} = (n_1+n_2+n_3+n_4, n_2+n_3+n_4, n_3+n_4, n_4)$. For spin representations,

$$\mathcal{J}(\boldsymbol{w};\ 2k_1, \ldots, 2k_4;\ n_1, \ldots, 2n_4+1) = \sum_{\boldsymbol{k'} \in \mathbb{Z}^4_{\geq 0}} \Delta(\boldsymbol{\lambda}_{s}; 2k'_1+1, \ldots, 2k'_4+1) \prod_{A=1}^{4} \mathcal{J}_0(w_A; 2k_A, 2k'_A+1) \ , \tag{6.26}$$

$$\mathcal{J}(\boldsymbol{w};\ 2k_1+1, \ldots, 2k_4+1; n_1, \ldots, 2n_4+1) = \sum_{\boldsymbol{k'} \in \mathbb{Z}^4_{\geq 0}} \Delta(\boldsymbol{\lambda}_{s}; 2k'_1+1, \ldots, 2k'_4+1) \prod_{A=1}^{4} \mathcal{J}_0(w_A; 2k_A+1, 2k'_A+1) \ , \tag{6.27}$$

where $\boldsymbol{\lambda}_s = \left(n_1+n_2+n_3+n_4+\frac{1}{2}, n_2+n_3+n_4+\frac{1}{2}, n_3+n_4+\frac{1}{2}, n_4+\frac{1}{2}\right)$. Recall from (2.57) and (2.62) that

$$\Delta(\boldsymbol{\lambda}; 2k_1, \ldots, 2k_4) = \frac{1}{4!} \sum_{\sigma \in S_4} \det\left(\theta^{8+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}\right)\right)^4_{A,B=1} \ , \tag{6.28}$$

$$\Delta(\boldsymbol{\lambda}; 2k_1+1, \ldots, 2k_4+1) = \frac{1}{4!} \sum_{\sigma \in S_4} \det\left(\theta^{8+\lambda_A-A-B}_{|\lambda_A-A+B|}\left(k_{\sigma(A)}+\frac{1}{2}\right)\right)^4_{A,B=1} \ . \tag{6.29}$$

6.1.5 Multiplicity generating function

The NS- and R sector contributions to the multiplicity generating function for the representation $[\![n_1, n_2, n_3, n_4]\!]$ can be rewritten as

$$G^{\mathrm{NS}}_{n_1,\ldots,n_4}(q) = \sum_{\boldsymbol{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\boldsymbol{k} \in \mathbb{Z}^4_{\geq 0}} \mathcal{J}(\boldsymbol{W}(\boldsymbol{m}); 2k_1, \ldots, 2k_4; \boldsymbol{n})\ F^{\mathrm{NS}}_{k_1,\ldots,k_4}(q) \ , \tag{6.30}$$

$$G^{\mathrm{R}}_{n_1,\ldots,n_4}(q) = \sum_{\boldsymbol{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\boldsymbol{k} \in \mathbb{Z}^4_{\geq 0}} \mathcal{J}(\boldsymbol{W}(\boldsymbol{m}); 2k_1+1, \ldots, 2k_4+1; \boldsymbol{n})\ F^{\mathrm{R}}_{k_1,\ldots,k_4}(q) \ , \tag{6.31}$$

where

$$\boldsymbol{W}(\boldsymbol{m}) = \left(\sum_{j=1}^{8}(-1)^j m_j,\ \sum_{j=1}^{8}(-1)^{\lfloor (j+1)/2 \rfloor} m_j,\ \sum_{j=1}^{8}(-1)^{\lfloor (j+3)/4 \rfloor} m_j,\ 4+\sum_{j=1}^{8} m_j\right) \ . \tag{6.32}$$

As stated in (6.3), the multiplicity of the representation $[\![n_1, n_2, n_3, n_4]\!]$ in the $\mathcal{N}_{10d} = 1$ partition function is given by

$$G_{n_1,n_2,n_3,n_4}(q) = \sum_{\boldsymbol{m} \in \mathbb{Z}^8_{\geq 0}} (-1)^{\sum_{j=1}^8 m_j} \sum_{\boldsymbol{k} \in \mathbb{Z}^4_{\geq 0}} \Big[\mathcal{J}(\boldsymbol{W}(\boldsymbol{m}); 2k_1, \ldots, 2k_4; \boldsymbol{n})\ F^{\mathrm{NS}}_{k_1,\ldots,k_4}(q) + \mathcal{J}(\boldsymbol{W}(\boldsymbol{m}); 2k_1+1, \ldots, 2k_4+1; \boldsymbol{n})\ F^{\mathrm{R}}_{k_1,\ldots,k_4}(q)\Big] \ . \tag{6.33}$$

$\alpha' m^2$	# $[\![0,0,0,0]\!]$	# $[\![1,0,0,0]\!]$	# $[\![2,0,0,0]\!]$	# $[\![3,0,0,0]\!]$	# $[\![4,0,0,0]\!]$	# $[\![5,0,0,0]\!]$	# $[\![6,0,0,0]\!]$	# $[\![7,0,0,0]\!]$	# $[\![8,0,0,0]\!]$	# $[\![9,0,0,0]\!]$	# $[\![10,0,0,0]\!]$	# $[\![11,0,0,0]\!]$	# $[\![12,0,0,0]\!]$	# $[\![13,0,0,0]\!]$	# $[\![14,0,0,0]\!]$
1	1	0													
2	0	1	0												
3	0	0	1	0											
4	0	1	0	1	0										
5	1	0	1	0	1	0									
6	0	2	1	1	0	1	0								
7	2	2	3	1	1	0	1	0							
8	1	5	3	4	1	1	0	1	0						
9	3	5	9	4	4	1	1	0	1	0					
10	3	12	10	11	5	4	1	1	0	1	0				
11	8	15	23	14	12	5	4	1	1	0	1	0			
12	8	30	31	31	16	13	5	4	1	1	0	1	0		
13	19	41	61	45	36	17	13	5	4	1	1	0	1	0	
14	22	77	89	87	53	38	18	13	5	4	1	1	0	1	0
15	41	109	164	132	104	58	39	18	13	5	4	1	1	0	1
16	57	190	245	244	162	113	60	40	18	13	5	4	1	1	0
17	100	282	426	378	299	179	118	61	40	18	13	5	4	1	1
18	138	471	656	657	473	332	188	120	62	40	18	13	5	4	1
19	235	710	1097	1040	830	532	350	193	121	62	40	18	13	5	4
20	336	1153	1699	1751	1333	938	565	359	195	122	62	40	18	13	5
21	544	1750	2778	2769	2263	1523	1000	583	364	196	122	62	40	18	13
22	799	2785	4309	4561	3630	2600	1635	1034	592	366	197	122	62	40	18
23	1261	4237	6907	7201	6025	4212	2803	1697	1052	597	367	197	122	62	40
24	1860	6634	10700	11637	9629	7034	4567	2918	1731	1061	599	368	197	122	62
25	2895	10082	16893	18301	15694	11337	7662	4774	2981	1749	1066	600	368	197	122

Table 9. $\mathcal{N}_{10d} = 1$ multiplets with $SO(9)$ quantum numbers $[n, 0, 0, 0]$

6.2 Empirical approach to $\mathcal{N}_{10d} = 1$ asymptotic patterns

In this subsection, we proceed like in subsections 4.5 and 5.4 to obtain large spin asymptotics of multiplicity generating functions $G_{n,x,y,z}(q)$ for $\mathcal{N}_{10d} = 1$ supermultiplet $[\![n, x, y, z]\!]$. The supermultiplet content of the first 25 mass levels is used to determine the q expansion of the leading coefficients $\tau_\ell^{x,y,z}(q)$ defined by:

$$\begin{aligned} G_{n,x,y,z}(q) &= q^n \, \tau_1^{x,y,z}(q) \; - \; q^{2n} \, \tau_2^{x,y,z}(q) \; + \; q^{3n} \, \tau_3^{x,y,z}(q) \; - \; \dots \\ &= \sum_{\ell=1}^{\infty} (-1)^{\ell-1} \, q^{\ell n} \, \tau_\ell^{x,y,z}(q) \end{aligned} \tag{6.34}$$

Again, the $\tau_\ell^{x,y,z}(q)$ are found to be power series in q with non-negative coefficients.

Having $d > 4$ spacetime dimensions makes the analytic methods of subsection 4.4 inefficient, i.e. we did not find a manageable asymptotic formula for (6.33). Hence, we compute the $\tau_\ell^{x,y,z}(q)$ at $\ell \leq 5$ on the basis of an $\mathcal{O}(q^{25})$ expansion of the partition function (6.3). The multiplicities of $[\![n, 0, 0, 0]\!]$ multiplets are shown in the following table 9, and analogous data tables for $[\![n, x, y, z]\!]$ at nonzero values of x, y, z can be found in appendix B.3. The numbers marked in red match with the leading trajectory contribution $q^n \tau_1^{x,y,z}(q)$ whereas blue numbers correspond to $q^n \tau_1^{x,y,z}(q) - q^{2n} \tau_2^{x,y,z}(q)$ including one subleading trajectory.

6.2.1 Levels of first appearance

The mass level where some $[\![0, x, y, z]\!]$ multiplet firstly occurs can be studied by inspecting the leading power of the multiplicity generating function $G_{0,x,y,z}(q)$ and therefore $\tau_\ell^{x,y,z}(q)$. The following tables 10 and 11 give an overview of this mass level threshold for various values of x, y, z.

x	y	z	level
0	0	0	1
1	0	0	4
2	0	0	7
3	0	0	10
4	0	0	13
5	0	0	16
6	0	0	19
7	0	0	22
8	0	0	25
9	0	0	
0	1	0	5
1	1	0	8
2	1	0	11
3	1	0	14
4	1	0	17
5	1	0	20
6	1	0	23

x	y	z	level
7	1	0	
0	2	0	11
1	2	0	14
2	2	0	17
3	2	0	20
4	2	0	23
5	2	0	
0	3	0	17
1	3	0	20
2	3	0	23
3	3	0	
0	4	0	23
1	4	0	
0	5	0	
0	0	2	6
1	0	2	9

x	y	z	level
2	0	2	12
3	0	2	15
4	0	2	18
5	0	2	21
6	0	2	24
7	0	2	
0	1	2	12
1	1	2	15
2	1	2	18
3	1	2	21
4	1	2	24
5	1	2	
0	2	2	18
1	2	2	21
2	2	2	24
3	2	2	

x	y	z	level
0	3	2	24
1	3	2	
0	4	2	
0	0	4	14
1	0	4	17
2	0	4	20
3	0	4	23
4	0	4	
0	1	4	20
1	1	4	23
2	1	4	
0	2	4	
0	0	6	24
1	0	6	
0	1	6	
0	0	8	

Table 10. First mass level where bosonic supermultiplets $[\![0,x,y,z]\!]$ of $\mathcal{N}_{10d}=1$ firstly occur. Empty spaces indicate that the representations in question do not occur at levels ≤ 25.

x	y	z	level
0	0	1	3
1	0	1	6
2	0	1	9
3	0	1	12
4	0	1	15
5	0	1	18
6	0	1	21
7	0	1	24
8	0	1	
0	1	1	8
1	1	1	11
2	1	1	14
3	1	1	17

x	y	z	level
4	1	1	20
5	1	1	23
6	1	1	
0	2	1	14
1	2	1	17
2	2	1	20
3	2	1	23
4	2	1	
0	3	1	20
1	3	1	23
2	3	1	
0	4	1	

x	y	z	level
0	0	3	10
1	0	3	13
2	0	3	16
3	0	3	19
4	0	3	22
5	0	3	25
6	0	3	
0	1	3	16
1	1	3	19
2	1	3	22
3	1	3	25
4	1	3	

x	y	z	level
0	2	3	22
1	2	3	25
2	2	3	
0	3	3	
0	0	5	19
1	0	5	22
2	0	5	25
3	0	5	
0	1	5	25
1	1	5	
0	2	5	
0	0	7	

Table 11. First mass level where fermionic supermultiplets $[\![0,x,y,z]\!]$ of $\mathcal{N}_{10d}=1$ firstly occur. Empty spaces indicate that the representations in question do not occur at levels ≤ 25.

For all supermultiplets $[\![0,x,y,z]\!]$ considered in tables 10 and 11, the level of first appearance is delayed by three whenever the second Dynkin label is incremented as $x \mapsto x+1$. This suggests to look for a similar linear effect of $y \mapsto y+1$ and $z \mapsto z+1$. Up to the two exceptions $[\![0,0,0,0]\!]$ and $[\![0,0,0,1]\!]$, the data in the tables shows that the value y of the third Dynkin label increases the level of first appearance by $6y$.

The influence of the last Dynkin label z is much more difficult to probe without any explicit multiplicities beyond level 25 at hand. If an asymptotically linear relation between z and the level of first appearance of $[\![0,x,y,z]\!]$ exists, then it certainly admits even more exceptions than in the $y \mapsto y+1$ case. The onset of $[\![n,0,0,4]\!]$, $[\![n,0,0,5]\!]$ and $[\![n,0,0,6]\!]$ multiplets at levels 14, 19 and 24, respectively, suggests that an increment $z \mapsto z+1$ delays the $[\![0,x,y,z]\!]$ multiplet by five levels – at least in the regime of sufficiently large values of x, y, z.

On the basis of this reasonning, we conjecture that sufficiently high mass levels of first occurrence for general supermultiplets $[\![n,x,y,z]\!]$ are determined by the following overall prefactor in their multiplicity generating function:

$$G_{n,x,y,z}(q) \quad \sim \quad q^{n+3x+6y+5z-6} \times \mathcal{O}(1) \ , \qquad x,y,z \text{ large} \tag{6.35}$$

Note that also the six dimensional $\mathcal{N}_{6d} = (1,0)$ spectrum exhibis an asymptotic linear relation between the second $SO(5)$ label k and the level of first appearance: Table 6 shows that sufficiently high levels of first appearance for $[\![n,k;p]\!]$ are delayed by three under $k \mapsto k+2$.

6.2.2 Explicit formulae for the $\tau_\ell^{x,y,z}(q)$

We shall now give the explicit results for a large class of $\tau_\ell^{x,y,z}(q)$, obtained through the entries of table 9 and its generalizations to $(x,y,z) \neq (0,0,0)$ gathered in appendix B.3. This reflects large spin information on the multiplicity generating functions $G_{n,x,y,z}(q)$ via (6.34).

We firstly consider supermultiplet families of vanishing third and fourth Dynkin labels $y = z = 0$ and vary the second Dynkin label through values $x = 0, 1, 2, 3$:

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 0]$

$$\begin{aligned}
\tau_1^{0,0,0}(q) &= q^1\,(1 + 0q + 1q^2 + 1q^3 + 4q^4 + 5q^5 + 13q^6 + 18q^7 + 40q^8 + 62q^9 + 122q^{10} \\
&\quad + 197q^{11} + 368q^{12} + 601q^{13} + 1070q^{14} + 1767q^{15} + 3051q^{16} + 5022q^{17} \\
&\quad + 8489q^{18} + 13897q^{19} + \ldots) \\
\tau_2^{0,0,0}(q) &= q^1\,(1 + 2q + 4q^2 + 9q^3 + 18q^4 + 36q^5 + 70q^6 + 133q^7 + 249q^8 + 460q^9 \\
&\quad + 836q^{10} + 1503q^{11} + 2672q^{12} + 4699q^{13} + \ldots) \\
\tau_3^{0,0,0}(q) &= q^1\,(1 + 1q + 5q^2 + 9q^3 + 26q^4 + 48q^5 + 112q^6 + 211q^7 + 439q^8 + 818q^9 + \ldots) \\
\tau_4^{0,0,0}(q) &= q^1\,(1 + 3q + 8q^2 + 20q^3 + 48q^4 + 106q^5 + \ldots) \\
\tau_5^{0,0,0}(q) &= q^1\,(1 + 1q + 6q^2 + \ldots)
\end{aligned} \tag{6.36}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 0, 0]$

$$\begin{aligned}
\tau_1^{1,0,0}(q) &= q^4\,(1 + 2q + 3q^2 + 7q^3 + 14q^4 + 28q^5 + 53q^6 + 103q^7 + 189q^8 + 352q^9 + 634q^{10} \\
&\quad + 1146q^{11} + 2026q^{12} + 3578q^{13} + 6209q^{14} + 10752q^{15} + 18378q^{16} + 31279q^{17} + \ldots) \\
\tau_2^{1,0,0}(q) &= q^5\,(1 + 2q + 5q^2 + 11q^3 + 26q^4 + 54q^5 + 114q^6 + 227q^7 + 449q^8 + 863q^9 \\
&\quad + 1639q^{10} + 3050q^{11} + 5618q^{12} + 10187q^{13} + \ldots) \\
\tau_3^{1,0,0}(q) &= q^8\,(2 + 5q + 15q^2 + 35q^3 + 86q^4 + 185q^5 + 403q^6 + 825q^7 + \ldots) \\
\tau_4^{1,0,0}(q) &= q^{10}\,(1 + 3q + 11q^2 + 30q^3 + \ldots)
\end{aligned} \tag{6.37}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 2, 0, 0]$

$$\tau_1^{2,0,0}(q) = q^7\,(1 + 2q + 6q^2 + 11q^3 + 27q^4 + 52q^5 + 112q^6 + 212q^7 + 423q^8$$

$$\begin{aligned}
&\quad + 787q^9 + 1496q^{10} + 2724q^{11} + 5001q^{12} + 8927q^{13} + 15950q^{14} + \ldots) \\
\tau_2^{2,0,0}(q) &= q^8\,(1 + 2q^1 + 6q^2 + 14q^3 + 34q^4 + 74q^5 + 161q^6 + 333q^7 + 680q^8 \\
&\quad + 1346q^9 + 2627q^{10} + \ldots) \\
\tau_3^{2,0,0}(q) &= q^{12}\,(3 + 7q + 23q^2 + 54q^3 + 138q^4 + \ldots) \\
\tau_4^{2,0,0}(q) &= q^{15}\,(1 + \ldots) \qquad (6.38)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 3, 0, 0]$

$$\begin{aligned}
\tau_1^{3,0,0}(q) &= q^{10}\,(1 + 2q + 6q^2 + 14q^3 + 32q^4 + 69q^5 + 147q^6 + 299q^7 + 597q^8 + 1168q^9 \\
&\quad + 2239q^{10} + 4226q^{11} + 7854q^{12} + \ldots) \\
\tau_2^{3,0,0}(q) &= q^{11}\,(1 + 2q + 6q^2 + 14q^3 + 35q^4 + 77q^5 + 172q^6 + 361q^7 + 752q^8 + 1513q^9 + \ldots) \\
\tau_3^{3,0,0}(q) &= q^{16}\,(3 + 8q + 25q^2 + 63q^3 + \ldots) \qquad (6.39)
\end{aligned}$$

Let us next freeze the second and fourth Dynkin label to $x = z = 0$ and vary the third Dynkin label to $y = 1$ and $y = 2$:

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 1, 0]$

$$\begin{aligned}
\tau_1^{0,1,0}(q) &= q^5\,(1 + 1q + 5q^2 + 8q^3 + 22q^4 + 40q^5 + 90q^6 + 165q^7 + 338q^8 + 619q^9 + 1190q^{10} \\
&\quad + 2149q^{11} + 3969q^{12} + 7048q^{13} + 12630q^{14} + 22060q^{15} + 38603q^{16} + \ldots) \\
\tau_2^{0,1,0}(q) &= q^6\,(1 + 2q + 7q^2 + 17q^3 + 41q^4 + 91q^5 + 199q^6 + 412q^7 + 841q^8 + 1665q^9 \\
&\quad + 3241q^{10} + 6178q^{11} + 11611q^{12} + \ldots) \\
\tau_3^{0,1,0}(q) &= q^8\,(1 + 2q + 11q^2 + 25q^3 + 71q^4 + 160q^5 + 381q^6 + 809q^7 + \ldots) \\
\tau_4^{0,1,0}(q) &= q^{11}\,(2 + 7q + 23q^2 + \ldots) \qquad (6.40)
\end{aligned}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 2, 0]$

$$\begin{aligned}
\tau_1^{0,2,0}(q) &= q^{11}\,(2 + 4q + 17q^2 + 36q^3 + 97q^4 + 207q^5 + 473q^6 + 963q^7 + 2016q^8 \\
&\quad + 3957q^9 + 7809q^{10} + 14838q^{11} + \ldots) \\
\tau_2^{0,2,0}(q) &= q^{12}\,(2 + 6q + 22q^2 + 59q^3 + 153q^4 + 365q^5 + 842q^6 + 1842q^7 + \ldots) \\
\tau_3^{0,2,0}(q) &= q^{14}\,(2 + 5q + 24q^2 + 62q^3 + \ldots) \qquad (6.41)
\end{aligned}$$

Finally, at $x = y = 0$ and nonzero fourth Dynkin label, we have the bosonic supermultiplets

$$\begin{aligned}
\tau_1^{0,0,2}(q) &= q^6\,(1 + 2q + 7q^2 + 13q^3 + 33q^4 + 66q^5 + 143q^6 + 277q^7 + 559q^8 + 1053q^9 \\
&\quad + 2019q^{10} + 3715q^{11} + 6859q^{12} + 12338q^{13} + 22156q^{14} + 39043q^{15} + \ldots) \\
\tau_2^{0,0,2}(q) &= q^7\,(1 + 4q^1 + 11q^2 + 28q^3 + 68q^4 + 155q^5 + 339q^6 + 716q^7 + 1469q^8 \\
&\quad + 2938q^9 + 5755q^{10} + 11054q^{11} + \ldots) \\
\tau_3^{0,0,2}(q) &= q^9\,(2 + 5q + 19q^2 + 48q^3 + 130q^4 + 301q^5 + 703q^6 + 1518q^7 + \ldots) \\
\tau_4^{0,0,2}(q) &= q^{11}\,(1 + 4q + 16q^2 + 49q^3 + \ldots) \qquad (6.42)
\end{aligned}$$

and the simplest fermionic supermultiplets $[\![n, 0, 0, 1]\!]$:

$$\begin{aligned}
\tau_1^{0,0,1}(q) &= q^3\,(1 + 1q + 3q^2 + 6q^3 + 12q^4 + 24q^5 + 48q^6 + 90q^7 + 171q^8 + 317q^9 + 579q^{10} \\
&\quad + 1045q^{11} + 1870q^{12} + 3299q^{13} + 5777q^{14} + 10017q^{15} + 17222q^{16} + 29370q^{17} + \ldots)
\end{aligned}$$

$$
\begin{aligned}
\tau_2^{0,0,1}(q) &= q^4\,(1+2q^1+5q^2+13q^3+29q^4+62q^5+130q^6+263q^7+520q^8+1008q^9 \\
&\quad +1916q^{10}+3583q^{11}+6609q^{12}+\ldots) \\
\tau_3^{0,0,1}(q) &= q^6\,(1+3q^1+10q^2+26q^3+63q^4+143q^5+315q^6+664q^7+\ldots) \\
\tau_4^{0,0,1}(q) &= q^8\,(1+4q+12q^2+35q^3+\ldots) \\
\tau_5^{0,0,1}(q) &= q^{10}\,(1+\ldots)
\end{aligned} \tag{6.43}
$$

The leading q powers of various $\tau_\ell^{x,y,z}(q)$ suggest that the $\tau_\ell^{x,y,z}(q)$ expansion (6.34) converges more quickly at higher value of x,y,z. The rapid increase of leading q powers of $\tau_1^{x,y,z},\tau_2^{x,y,z},\tau_3^{x,y,z},\ldots$ with growing x becomes particular obvious from the $\tau_\ell^{x,y,z}(q)$ shown. These trends are supported by further $\tau_\ell^{x,y,z}(q)$ at $(x,y,z)=(1,1,0),(1,0,2),(2,1,0),(1,0,1),(0,1,1),(2,0,1),(0,0,3),(1,1,1)$, listed in appendix C.2.

6.3 Eight dimensional $\mathcal{N}_{8d}=1$ spectra

Starting from this subsection, we consider even dimensional type I superstring compactifications on T^2 tori preserving all the sixteen supercharges. The highest dimensional example is $\mathcal{N}_{8d}=1$ SUSY in eight spacetime dimensions. As explained in [38, 39], dimensional reduction of the open superstring from $d=10$ to $d=8$ paves the way towards powerful on-shell SUSY techniques to manifest hidden simplicity of scattering amplitudes among massive string modes (further examples following in [40]): One technical advantage of the eight dimensional setting is the possibility to covariantly single out a Clifford vacuum which is annihilated by half of the supercharges, say the right handed $SO(8)$ spinor of SUSY generators [38, 39]. This is a particular motivation to focus on the covariant particle content of the maximally supersymmetric open superstring in $d=8$.

Let r denote the fugacity with respect to the R symmetry $SO(2)_R\cong U(1)_R$ and y_i the fugacities of the massive little group $SO(7)$, then the fundamental $\mathcal{N}_{8d}=1$ super Poincaré multiplet is described by the supercharacter

$$
\begin{aligned}
Z(\mathcal{N}_{8d}=1) \;&:=\; (r^4+r^{-4})\,[0,0,0] \;+\; (r^3+r^{-3})\,[0,0,1] \;+\; (r^2+r^{-2})\,\big([0,1,0]\;+\;[1,0,0]\big) \\
&+\; (r+r^{-1})\,\big([1,0,1]\;+\;[0,0,1]\big) \;+\; [2,0,0]\;+\;[0,0,2]\;+\;[1,0,0]\;+\;[0,0,0]
\end{aligned} \tag{6.44}
$$

which is obtained by branching the $SO(9)$ representations contributing to the $\mathcal{N}_{10d}=1$ analogue (6.1) to $SO(7)\times U(1)_R$. The minimal multiplet (6.44) can be generated from a scalar Clifford vacuum of $U(1)_R$ charge $+4$, and the generic $\mathcal{N}_{8d}=1$ multiplet follows from a Clifford vacuum with nontrivial $SO(7)\times U(1)_R$ quantum numbers[32]. This gives rise to the supercharacter

$$
[\![a_1,a_2,a_3;Q]\!] \;:=\; Z(\mathcal{N}_{8d}=1)\cdot r^Q\,[a_1,a_2,a_3]\,. \tag{6.45}
$$

The eight dimensional partition function is obtained from its ten dimensional ancestor (6.3) by singling out an internal factor $\chi^{SO(3)}_{\rm NS,R}$ within $\chi^{SO(9)}_{\rm NS,R}(\boldsymbol{y})=\prod_{k=1}^4\chi^{SO(3)}_{\rm NS,R}(y_k)$ and reinterpreting its argument as an R-symmetry fugacity:

$$
\begin{aligned}
\chi^{\mathcal{N}_{8d}=1}(q;\boldsymbol{y},r) &= \chi^{\mathcal{N}_{8d}=1}_{\rm NS}\,|_{\rm GSO}\,(q;\boldsymbol{y},r) \;+\; \chi^{\mathcal{N}_{8d}=1}_{\rm R}\,|_{\rm GSO}\,(q;\boldsymbol{y},r) \\
\chi^{\mathcal{N}_{8d}=1}_{\rm NS}\,|_{\rm GSO}\,(q;\boldsymbol{y},r) &= \frac{1}{2}\,q^{-\frac{1}{2}}\,\big[\chi^{SO(7)}_{\rm NS}(q;\boldsymbol{y})\,\chi^{SO(3)}_{\rm NS}(q;r) \;-\; \chi^{SO(7)}_{\rm NS}(e^{2\pi i}q;\boldsymbol{y})\,\chi^{SO(3)}_{\rm NS}(e^{2\pi i}q;r)\big] \\
\chi^{\mathcal{N}_{8d}=1}_{\rm R}\,|_{\rm GSO}\,(q;\boldsymbol{y},r) &= \frac{1}{2}\,\chi^{SO(7)}_{\rm R}(q;\boldsymbol{y})\,\chi^{SO(3)}_{\rm R}(q;r)
\end{aligned} \tag{6.46}
$$

[32] Recall that the semicolon in $[\![a_1,a_2,a_3;b]\!]$ separating the $U(1)_R$ quantum number b from the $SO(7)$ Dynkin labels a_1,a_2,a_3 eliminates potential confusion with $\mathcal{N}_{10d}=1$ supercharacters (6.2).

Let us display the first four coefficients of the power series expansion in q:[33]

$$\chi^{\mathcal{N}_{8d}=1}(q;\boldsymbol{y},r) \;=\; \underbrace{\left(\sum_{j=1}^{3}(y_j^2+y_j^{-2}) \;+\; r^2 \;+\; r^{-2} \;+\; \frac{1}{2}\prod_{j=1}^{3}(y_j+y_j^{-1})\,(r+r^{-1})\right) q^0}_{16 \text{ massless states}}$$
$$+ \quad \underbrace{[\![0,0,0;0]\!]\,q}_{256 \text{ states at level } 1} \quad + \underbrace{\big(\,[\![0,0,0;\pm 2]\!] \,+\, [\![1,0,0;0]\!]\,\big)\,q^2}_{2304 \text{ states at level } 2}$$
$$+ \;\big(\,[\![0,0,0;\pm 4]\!] \,+\, [\![1,0,0;\pm 2]\!] \,+\, [\![0,0,1;\pm 1]\!]$$
$$+ \,[\![2,0,0;0]\!] \,+\, [\![0,0,0;0]\!]\,\big)\,q^3 \;+\; \mathcal{O}(q^4)\;. \tag{6.47}$$

The pairing of opposite $U(1)_R$ charges $\pm Q$ motivates the following shorthand:

$$[\![a_1,a_2,a_3;\pm Q]\!] \;\;:=\;\; \begin{cases} [\![a_1,a_2,a_3;Q]\!] + [\![a_1,a_2,a_3;-Q]\!] & \text{for } Q\neq 0\;, \\ [\![a_1,a_2,a_3;0]\!] & \text{for } Q=0\;. \end{cases} \tag{6.48}$$

The supermultiplets up to level six are listed in table 12, some of their scattering amplitudes are discussed in [39, 40]. The branching process obviously increases the number and diversity of multiplets compared to the ten dimensional analogue, cf. table 8. This is why we do not repeat the higher level analysis carried out for the $d=10$ ancestor in dimensionally reduced settings.

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d}=1$ partition function (6.4) into $SO(7)\times U(1)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(7)\times U(1)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = y_1, \qquad z_2=y_2, \qquad z_3=y_3, \qquad z_4 = s\;, \tag{6.49}$$

where $z_1,\ldots,z_4$ are fugacities of $SO(9)$, y_1,y_2,y_3 are fugacities of $SO(7)$ and s is a fugacity of $U(1)_R$. For example,

$$\begin{aligned}
[1,0,0,0]_{\boldsymbol{z}} &= 1+\frac{1}{z_1^2}+z_1^2+\frac{1}{z_2^2}+z_2^2+\frac{1}{z_3^2}+z_3^2+\frac{1}{z_4^2}+z_4^2 \\
&= 1+\frac{1}{y_1^2}+y_1^2+\frac{1}{y_2^2}+y_2^2+\frac{1}{y_3^2}+y_3^2+\frac{1}{s^2}+s^2 \\
&= [1,0,0;0]_{\boldsymbol{y};s}+[0,0,0;+2]_{\boldsymbol{y};s}+[0,0,0;-2]_{\boldsymbol{y};s}\;,
\end{aligned} \tag{6.50}$$

where the notation $[b_1,b_2,b_3;Q]$ denotes the $SO(7)\times U(1)_R$ representation.

6.4 Six dimensional $\mathcal{N}_{6d}=(1,1)$ spectra

Six dimensional type I compactifications with sixteen supercharges are said to possess $\mathcal{N}_{6d}=(1,1)$ SUSY. The spacetime symmetry branches to $SO(9)\to SO(5)\times SO(4)_R$, i.e. two Cartan generators of ten dimensional Lorentz group take the role of R symmetry generators probing fugacities r_1,r_2 of $SO(4)_R\cong SU(2)_R\times SU(2)_R$. The fundamental supermultiplet of the $\mathcal{N}_{6d}=(1,1)$ super Poincaré group has the following $SO(5)\times SU(2)_R\times SU(2)_R$ particle content:

$$Z(\mathcal{N}_{6d}=(1,1)) \;\;:=\;\; [2,0]\cdot[0,0]_R \;+\; [0,2]\cdot[0,0]_R \;+\; [0,2]\cdot[1,1]_R \;+\; [1,0]\cdot[1,1]_R$$

[33] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.4). However, this can be fixed easily: one can simply add to it $\frac{1}{2}\prod_{j=1}^{3}(y_j-y_j^{-1})\,(r-r^{-1})$ to get the correct massless character in R sector.

$\alpha' m^2$	representations of $\mathcal{N}_{8d}=1$ super Poincaré
1	$[\![0,0,0;0]\!]$
2	$[\![0,0,0;\pm 2]\!] + [\![1,0,0;0]\!]$
3	$[\![0,0,0;\pm 4]\!] + [\![1,0,0;\pm 2]\!] + [\![0,0,1;\pm 1]\!] + [\![2,0,0;0]\!] + [\![0,0,0;0]\!]$
4	$[\![0,0,0;\pm 6]\!] + [\![1,0,0;\pm 4]\!] + [\![0,0,1;\pm 3]\!] + [\![2,0,0;\pm 2]\!] + [\![1,0,0;\pm 2]\!] + 2\,[\![0,0,0;\pm 2]\!]$ $+ [\![1,0,1;\pm 1]\!] + [\![0,0,1;\pm 1]\!] + [\![3,0,0;0]\!] + 2\,[\![1,0,0;0]\!] + [\![0,1,0;0]\!] + [\![0,0,0;0]\!]$
5	$[\![0,0,0;\pm 8]\!] + [\![1,0,0;\pm 6]\!] + [\![0,0,1;\pm 5]\!] + [\![2,0,0;\pm 4]\!] + [\![1,0,0;\pm 4]\!] + 2\,[\![0,0,0;\pm 4]\!]$ $+ [\![1,0,1;\pm 3]\!] + 2\,[\![0,0,1;\pm 3]\!] + [\![3,0,0;\pm 2]\!] + [\![2,0,0;\pm 2]\!] + 3\,[\![1,0,0;\pm 2]\!] + 2\,[\![0,1,0;\pm 2]\!]$ $+ [\![0,0,0;\pm 2]\!] + [\![2,0,1;\pm 1]\!] + 2\,[\![1,0,1;\pm 1]\!] + 3\,[\![0,0,1;\pm 1]\!] + [\![4,0,0;0]\!] + 2\,[\![2,0,0;0]\!]$ $+ [\![1,1,0;0]\!] + 3\,[\![1,0,0;0]\!] + [\![0,1,0;0]\!] + [\![0,0,2;0]\!] + 4\,[\![0,0,0;0]\!]$
6	$[\![0,0,0;\pm 10]\!] + [\![1,0,0;\pm 8]\!] + [\![0,0,1;\pm 7]\!] + [\![2,0,0;\pm 6]\!] + [\![1,0,0;\pm 6]\!] + 2\,[\![0,0,0;\pm 6]\!]$ $+ [\![1,0,1;\pm 5]\!] + 2\,[\![0,0,1;\pm 5]\!] + [\![3,0,0;\pm 4]\!] + [\![2,0,0;\pm 4]\!] + 3\,[\![1,0,0;\pm 4]\!] + 2\,[\![0,1,0;\pm 4]\!]$ $+ 2\,[\![0,0,0;\pm 4]\!] + [\![2,0,1;\pm 3]\!] + 3\,[\![1,0,1;\pm 3]\!] + 3\,[\![0,0,1;\pm 3]\!] + [\![4,0,0;\pm 2]\!] + [\![3,0,0;\pm 2]\!]$ $+ 3\,[\![2,0,0;\pm 2]\!] + 2\,[\![1,1,0;\pm 2]\!] + 5\,[\![1,0,0;\pm 2]\!] + [\![0,1,0;\pm 2]\!] + 2\,[\![0,0,2;\pm 2]\!]$ $+ 4\,[\![0,0,0;\pm 2]\!] + [\![3,0,1;\pm 1]\!] + 2\,[\![2,0,1;\pm 1]\!] + 4\,[\![1,0,1;\pm 1]\!] + [\![0,1,1;\pm 1]\!]$ $+ 5\,[\![0,0,1;\pm 1]\!] + [\![5,0,0;0]\!] + 2\,[\![3,0,0;0]\!] + [\![2,1,0;0]\!] + 4\,[\![2,0,0;0]\!] + [\![1,1,0;0]\!]$ $+ [\![1,0,2;0]\!] + 5\,[\![1,0,0;0]\!] + 5\,[\![0,1,0;0]\!] + [\![0,0,2;0]\!] + 3\,[\![0,0,0;0]\!]$

Table 12. $\mathcal{N}_{8d}=1$ multiplets occurring up to mass level six

$$\begin{aligned} &+ [1,0]\cdot\big([2,0]_R + [0,2]_R\big) + [0,0]\cdot[2,2]_R + [0,0]\cdot[1,1]_R + [0,0]\cdot[0,0]_R \\ &+ [1,1]\cdot\big([1,0]_R + [0,1]_R\big) + [0,1]\cdot\big([2,1]_R + [1,2]_R + [1,0]_R + [0,1]_R\big) \end{aligned} \tag{6.51}$$

Note that the R-symmetry characters $[\ldots]_R$ carry a subscript to avoid confusion with the Lorentz symmetry of identical rank.

The most general multiplet follows from (6.51) by taking tensor products with $SO(5)\times SU(2)_R\times SU(2)_R$ representations, this leads to the supercharacter

$$[\![a_1,a_2;b_1,b_2]\!] \;:=\; Z(\mathcal{N}_{6d}=(1,1))\cdot[a_1,a_2]\cdot[b_1,b_2]_R \tag{6.52}$$

The six dimensional partition function is obtained from its ten dimensional ancestor (6.3) by singling out two internal factor $\chi^{SO(3)}_{\rm NS,R}$ within $\chi^{SO(9)}_{\rm NS,R}(\boldsymbol{y}) = \prod_{k=1}^4 \chi^{SO(3)}_{\rm NS,R}(y_k)$ and reinterpreting their second argument as an R-symmetry fugacity:

$$\begin{aligned} \chi^{\mathcal{N}_{6d}=(1,1)}(q;\boldsymbol{y},\boldsymbol{r}) &= \chi^{\mathcal{N}_{6d}=(1,1)}_{\rm NS}\,|_{\rm GSO}\,(q;\boldsymbol{y},\boldsymbol{r}) + \chi^{\mathcal{N}_{6d}=(1,1)}_{\rm R}\,|_{\rm GSO}\,(q;\boldsymbol{y},\boldsymbol{r}) \\ \chi^{\mathcal{N}_{6d}=(1,1)}_{\rm NS}\,|_{\rm GSO}\,(q;\boldsymbol{y},\boldsymbol{r}) &= \frac{1}{2}\,q^{-\frac{1}{2}}\,\big[\,\chi^{SO(5)}_{\rm NS}(q;\boldsymbol{y})\,\chi^{SO(5)}_{\rm NS}(q;\boldsymbol{r}) - \chi^{SO(5)}_{\rm NS}(e^{2\pi i}q;\boldsymbol{y})\,\chi^{SO(5)}_{\rm NS}(e^{2\pi i}q;\boldsymbol{r})\big] \\ \chi^{\mathcal{N}_{6d}=(1,1)}_{\rm R}\,|_{\rm GSO}\,(q;\boldsymbol{y},\boldsymbol{r}) &= \frac{1}{2}\,\chi^{SO(5)}_{\rm R}(q;\boldsymbol{y})\,\chi^{SO(5)}_{\rm R}(q;\boldsymbol{r}) \end{aligned} \tag{6.53}$$

Its q expansion starts like[34]

$$\begin{aligned} \chi^{\mathcal{N}_{6d}=(1,1)}(q;\boldsymbol{y},\boldsymbol{r}) &= \underbrace{\left(\sum_{j=1}^{2}(y_j^2+y_j^{-2}) + \sum_{j=1}^{2}(r_j^2+r_j^{-2}) + \frac{1}{2}\prod_{j=1}^{2}(y_j+y_j^{-1})\prod_{j=1}^{2}(r_j+r_j^{-1})\right)q^0}_{16\ \text{massless states}} \\ &\quad + \underbrace{[\![0,0;0,0]\!]\,q}_{256\ \text{states at level 1}} + \underbrace{\big([\![0,0;1,1]\!] + [\![1,0;0,0]\!]\big)\,q^2}_{2304\ \text{states at level 2}} \end{aligned}$$

[34]Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.4). However, this can be fixed easily: one can simply add to it $\frac{1}{2}\prod_{j=1}^2(y_j-y_j^{-1})\prod_{j=1}^2(r_j-r_j^{-1})$ to get the correct massless character in R sector.

$$+ \ \big([\![0,0;2,2]\!] + [\![1,0;1,1]\!] + [\![0,1;1,0]\!] \\ + [\![0,1;0,1]\!] + [\![2,0;0,0]\!] + [\![0,0;0,0]\!] \big) q^3 + \mathcal{O}(q^4) \ , \tag{6.54}$$

and supermultiplets at higher levels ≤ 5 are listed in table 13.

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d} = 1$ partition function (6.4) into $SO(5) \times SU(2)_R \times SU(2)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(5) \times SU(2)_R \times SU(2)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = y_1, \qquad z_2 = y_2, \qquad z_3 = r_1 r_2, \qquad z_4 = r_1 r_2^{-1} \ , \tag{6.55}$$

where $z_1, \ldots, z_4$ are fugacities of $SO(9)$, y_1, y_2 are fugacities of $SO(5)$, and r_1, r_2 are fugacities for the two $SU(2)_R$ factors. For example,

$$\begin{aligned} [1,0,0,0]_{\boldsymbol{z}} &= 1 + \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + z_2^2 + \frac{1}{z_3^2} + z_3^2 + \frac{1}{z_4^2} + z_4^2 \\ &= 1 + \frac{1}{y_1^2} + y_1^2 + \frac{1}{y_2^2} + y_2^2 + (r_1 + r_1^{-1})(r_2 + r_2^{-1}) \\ &= [1,0;0,0]_{\boldsymbol{y};\boldsymbol{r}} + [0,0;1,1]_{\boldsymbol{y};\boldsymbol{r}} \ , \end{aligned} \tag{6.56}$$

where the notation $[a_1, a_2; b_1, b_2]$ denotes the $SO(5) \times SU(2)_R \times SU(2)_R$ representation.

$\alpha' m^2$	representations of $\mathcal{N}_{6d} = (1,1)$ super Poincaré
1	$[\![0,0;0,0]\!]$
2	$[\![0,0;1,1]\!] + [\![1,0;0,0]\!]$
3	$[\![0,0;2,2]\!] + [\![1,0;1,1]\!] + [\![0,1;1,0]\!] + [\![0,1;0,1]\!] + [\![2,0;0,0]\!] + [\![0,0;0,0]\!]$
4	$[\![0,0;3,3]\!] + [\![1,0;2,2]\!] + [\![0,1;2,1]\!] + [\![0,0;2,0]\!] + [\![0,1;1,2]\!] + [\![2,0;1,1]\!] + [\![1,0;1,1]\!]$ $+ 2\,[\![0,0;1,1]\!] + [\![1,1;1,0]\!] + [\![0,1;1,0]\!] + [\![0,0;0,2]\!] + [\![1,1;0,1]\!] + [\![0,1;0,1]\!]$ $+ [\![3,0;0,0]\!] + 2\,[\![1,0;0,0]\!] + [\![0,2;0,0]\!]$
5	$[\![0,0;4,4]\!] + [\![1,0;3,3]\!] + [\![0,1;3,2]\!] + [\![0,0;3,1]\!] + [\![0,1;2,3]\!] + [\![2,0;2,2]\!] + [\![1,0;2,2]\!]$ $+ 2\,[\![0,0;2,2]\!] + [\![1,1;2,1]\!] + 2\,[\![0,1;2,1]\!] + 2\,[\![1,0;2,0]\!] + [\![0,0;2,0]\!] + [\![0,0;1,3]\!]$ $+ [\![1,1;1,2]\!] + 2\,[\![0,1;1,2]\!] + [\![3,0;1,1]\!] + [\![2,0;1,1]\!] + 3\,[\![1,0;1,1]\!] + 2\,[\![0,2;1,1]\!]$ $+ 2\,[\![0,0;1,1]\!] + [\![2,1;1,0]\!] + 2\,[\![1,1;1,0]\!] + 3\,[\![0,1;1,0]\!] + 2\,[\![1,0;0,2]\!] + [\![0,0;0,2]\!]$ $+ [\![2,1;0,1]\!] + 2\,[\![1,1;0,1]\!] + 3\,[\![0,1;0,1]\!] + [\![4,0;0,0]\!] + 2\,[\![2,0;0,0]\!] + [\![1,2;0,0]\!]$ $+ [\![1,0;0,0]\!] + 2\,[\![0,2;0,0]\!] + 3\,[\![0,0;0,0]\!]$

Table 13. $\mathcal{N}_{6d} = (1,1)$ multiplets occurring up to mass level five

6.5 Four dimensional $\mathcal{N}_{4d} = 4$ spectra

Finally, four dimensional theories with maximal $\mathcal{N}_{4d} = 4$ SUSY follow from the ten dimensional ancestor through compactification on T^6. The internal rotation group is identified with the R symmetry $SO(6)_R$, its characters are denoted by $[b_1, b_2, b_3]_R$. The universal partition function decomposes into characters of the $\mathcal{N}_{4d} = 4$ super Poincaré algebra, the fundamental one being

$$Z(\mathcal{N}_{4d} = 4) \ = \ [0] \big([0,0,2]_R + [0,2,0]_R + [2,0,0]_R + 2 \big) + [2]\,[0,1,1]_R + 2\,[2]\,[1,0,0]_R + [4] \\ + [1] \big([0,0,1]_R + [0,1,0]_R + [1,0,1]_R + [1,1,0]_R \big) + [3] \big([0,0,1]_R + [0,1,0]_R \big) . \tag{6.57}$$

Any other supermultiplet follows by taking a tensor product of (6.57) with the $SO(3) \times SO(6)_R$ representation $[n]\,[b_1, b_2, b_3]_R$ of the the Clifford vacuum,

$$[\![n; b_1, b_2, b_3]\!] \quad := \quad Z(\mathcal{N}_{4d} = 4) \cdot [n]\,[b_1, b_2, b_3]_R \ . \tag{6.58}$$

The four dimensional partition function is obtained through the usual procedure from the ten dimensional ancestor (6.3), this time we have to interpret three factors of $\chi^{SO(3)}_{\rm NS,R}$ as carrying R-symmetry fugacities r_j:

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=4}(q; y, \boldsymbol{r}) \quad &= \quad \chi^{\mathcal{N}_{4d}=4}_{\rm NS}\,|_{\rm GSO}\,(q; y, \boldsymbol{r}) \ + \ \chi^{\mathcal{N}_{4d}=4}_{\rm R}\,|_{\rm GSO}\,(q; y, \boldsymbol{r}) \\
\chi^{\mathcal{N}_{4d}=4}_{\rm NS}\,|_{\rm GSO}\,(q; y, \boldsymbol{r}) \quad &= \quad \frac{1}{2}\, q^{-\frac{1}{2}} \left[\chi^{SO(3)}_{\rm NS}(q; y)\, \chi^{SO(7)}_{\rm NS}(q; \boldsymbol{r}) \ - \ \chi^{SO(3)}_{\rm NS}(e^{2\pi i} q; y)\, \chi^{SO(7)}_{\rm NS}(e^{2\pi i} q; \boldsymbol{r}) \right] \\
\chi^{\mathcal{N}_{4d}=4}_{\rm R}\,|_{\rm GSO}\,(q; y, \boldsymbol{r}) \quad &= \quad \frac{1}{2}\, \chi^{SO(3)}_{\rm R}(q; y)\, \chi^{SO(7)}_{\rm R}(q; \boldsymbol{r})
\end{aligned} \tag{6.59}$$

The power series in q starts with[35]

$$\begin{aligned}
\chi^{\mathcal{N}_{4d}=4}(q; y, r_j) \quad = \quad & \underbrace{\left(y^2 + y^{-2} \ + \ \sum_{j=1}^{3} (r_j^2 + r_j^{-2}) \ + \ \frac{1}{2}\,[1]_y \prod_{j=1}^{3} (r_j + r_j^{-1}) \right) q^0}_{\text{16 massless states}} \ + \ \underbrace{[\![0; 0, 0, 0]\!]\, q}_{\text{256 states at level 1}} \\
& + \ \underbrace{\big([\![0; 1, 0, 0]\!] + [\![2; 0, 0, 0]\!] \big)\, q^2}_{\text{2304 states at level 2}} \ + \ \big([\![0; 0, 0, 0]\!] + [\![0; 2, 0, 0]\!] + [\![1; 0, 0, 1]\!] \\
& + [\![1; 0, 1, 0]\!] + [\![2; 1, 0, 0]\!] + [\![4; 0, 0, 0]\!] \big)\, q^3 \ + \ \mathcal{O}(q^4) \ ,
\end{aligned} \tag{6.60}$$

the coefficients of q^4 and q^5 can be found in table 14. The explicit vertex operators from the first level are listed in section 4 of [29].

Note that this partition function can also be obtained by branching the $SO(9)$ representations appearing in the the $\mathcal{N}_{10d} = 1$ partition function (6.4) into $SO(3) \times SO(6)_R$ representations. In terms of characters, one simply maps $SO(9)$ fugacities into $SO(3) \times SO(6)_R$ fugacities; a possible fugacity map is as follows:

$$z_1 = r_1, \qquad z_2 = r_2, \qquad z_3 = r_3, \qquad z_4 = y \ , \tag{6.61}$$

where $z_1, \ldots, z_4$ are fugacities of $SO(9)$, r_1, r_2, r_3 are fugacities of $SO(6)_R$ and y is a fugacity of $SO(3)$. For example,

$$\begin{aligned}
[1, 0, 0, 0]_{\boldsymbol{z}} &= 1 + \frac{1}{z_1^2} + z_1^2 + \frac{1}{z_2^2} + r_2^2 + \frac{1}{z_3^2} + z_3^2 + \frac{1}{z_4^2} + z_4^2 \\
&= \frac{1}{r_1^2} + r_1^2 + \frac{1}{r_2^2} + r_2^2 + \frac{1}{r_3^2} + r_3^2 + \left(1 + \frac{1}{y^2} + y^2 \right) \\
&= [0; 1, 0, 0]_{\boldsymbol{r};y} + [2; 0, 0, 0]_{\boldsymbol{r};y} \ ,
\end{aligned} \tag{6.62}$$

where the notation $[a; b_1, b_2, b_3]$ denotes the $SO(3) \times SO(6)_R$ representation for which the $SO(3)$ representation is $[a]$ and $SO(6)_R$ representation is $[b_1, b_2, b_3]_R$.

[35] Again, there is a subtlety in applying the above formula to the massless R sector; see the footnote before (4.4). However, this can be fixed easily: one can simply add to it $\frac{1}{2}(y - y^{-1}) \prod_{j=1}^{3} (r_j - r_j^{-1})$ to get the correct massless character in R sector.

$\alpha' m^2$	representations of $\mathcal{N}_{4d} = 4$ super Poincaré
1	$[\![0;0,0,0]\!]$
2	$[\![0;1,0,0]\!] + [\![2;0,0,0]\!]$
3	$[\![0;0,0,0]\!] + [\![0;2,0,0]\!] + [\![1;0,0,1]\!] + [\![1;0,1,0]\!] + [\![2;1,0,0]\!] + [\![4;0,0,0]\!]$
4	$[\![0;0,1,1]\!] + 2[\![0;1,0,0]\!] + [\![0;3,0,0]\!] + [\![1;0,0,1]\!] + [\![1;0,1,0]\!] + [\![1;1,0,1]\!]$ $+ [\![1;1,1,0]\!] + 3[\![2;0,0,0]\!] + [\![2;1,0,0]\!] + [\![2;2,0,0]\!] + [\![3;0,0,1]\!] + [\![3;0,1,0]\!]$ $+ [\![4;1,0,0]\!] + [\![6;0,0,0]\!]$
5	$4[\![0;0,0,0]\!] + [\![0;0,0,2]\!] + [\![0;0,1,1]\!] + [\![0;0,2,0]\!] + [\![0;1,0,0]\!] + [\![0;1,1,1]\!]$ $+ 2[\![0;2,0,0]\!] + [\![0;4,0,0]\!] + 3[\![1;0,0,1]\!] + 3[\![1;0,1,0]\!] + 2[\![1;1,0,1]\!]$ $+ 2[\![1;1,1,0]\!] + [\![1;2,0,1]\!] + [\![1;2,1,0]\!] + 2[\![2;0,0,0]\!] + 2[\![2;0,1,1]\!] + 5[\![2;1,0,0]\!]$ $+ [\![2;2,0,0]\!] + [\![2;3,0,0]\!] + 2[\![3;0,0,1]\!] + 2[\![3;0,1,0]\!] + [\![3;1,0,1]\!] + [\![3;1,1,0]\!]$ $+ 3[\![4;0,0,0]\!] + [\![4;1,0,0]\!] + [\![4;2,0,0]\!] + [\![5;0,0,1]\!] + [\![5;0,1,0]\!] + [\![6;1,0,0]\!]$ $+ [\![8;0,0,0]\!]$

Table 14. $\mathcal{N}_{4d} = 4$ multiplets occurring up to mass level 5

7 Conclusion

We have investigated model independent superstring states common to all type I compactifications that preserve $\mathcal{N}_{4d} = 1$ and $\mathcal{N}_{6d} = (1,0)$ SUSY, respectively, and identified the underlying super Poincaré multiplets at individual mass levels. Part of our results are the associated unrefined partition functions together with their asymptotics for large mass levels, see (4.8)–(4.17) and (5.5)–(5.12). The refined versions of the universal partition functions are given by (4.4) and (5.3) and rewritten in terms of super Poincaré characters in (4.39), (4.61), (4.62), (5.23) and (5.41). Moreover, we have presented dimensional reductions of the universal $\mathcal{N}_{6d} = (1,0)$ and $\mathcal{N}_{10d} = 1$ spectra to even dimensions $d \geq 4$ in subsections 5.5, 6.3, 6.4 and 6.5.

Multiplicity generating functions for individual supermultiplets tend to stabilize in the regime where the spin j (or more generally the first $SO(d-1)$ Dynkin label) is comparable to the mass level $M = \alpha' m^2$. More specifically, the validity for the stable pattern roughly ranges between $\frac{1}{2}(M - M_0) \lessapprox j \lessapprox M - M_0$ where the offset M_0 depends on the remaining super Poincaré quantum numbers of the multiplets beyond the spin. In the mathematically most tractable $\mathcal{N}_{4d} = 1$ case, we have derived closed formulae (4.63) and (4.64) for the leading Regge trajectory. In the highest dimensional scenarios with given number of supercharges – $\mathcal{N}_{4d} = 1$, $\mathcal{N}_{6d} = (1,0)$ and $\mathcal{N}_{10d} = 1$ – we extracted both leading and subleading Regge trajectories from explicitly computed multiplicities up to level $\alpha' m^2 = 25$, see subsections 4.5, 5.4 and 6.2.

Identifying the super Poincaré covariant spectrum in scenarios with different numbers of supercharges provides a significant step towards a better understanding of the string S-matrix. As pointed out in [39], cubic tree level vertices among all the massive states are the seeds for superstring amplitudes of higher multiplicity and genus. The results of this work appear inspiring to push this programme further, using on-shell superspace techniques in various dimenions [34, 38]. Refined partition functions as computed here serve as generating functions for helicitiy supertraces [41] which allow to disentangle contribution of individual supermultiplets to loop amplitudes.

Extending flat space results as presented in this work to curved spacetime provides an exciting direction of further research. Anti-de-Sitter backgrounds are of particular interest in view of their conjectured correspondence to conformal field theories [42, 43]. For instance, the model independent higher spin string spectrum at the first massive level in $AdS_3 \times S^3$ compactifications with pure NSNS background has been pioneered in [44]. This is a motivating starting point towards generalizations to nonzero RR flux and superstrings in $AdS_5 \times S^5$, see [45] for a review. Also, we would like to mention reference [46] which extracts information on the $AdS_5 \times S^5$ Kaluza Klein excitations from the $\mathcal{N}_{10d} = 1$ flat space spectrum, in particular from its large spin regime investigated in detail here. Finally it would be also very interesting to explore the extended symmetry structure of the universal higher spin states in supersymmetric string compactifications, in analogy to the $\mathcal{W}_N$-symmetries in three-dimensional higher spin theories on AdS_3 [47–49].

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Appendices

A Notation and conventions

Unless stated otherwise, the following notation and conventions will be used throughout the paper.

Group and representation theoretic objects

- The plethystic exponential of a multivariate function $f(t_1,\ldots,t_n)$ that vanishes at the origin, $f(0,\ldots,0)=0$, is defined as

$$\mathrm{PE}\left[f(t_1,\ldots,t_n)\right] = \exp\left(\sum_{k=1}^{\infty}\frac{1}{k}f(t_1^k,\ldots,t_n^k)\right) \ . \tag{A.1}$$

  The ferminonic plethystic exponential is defined by

$$\mathrm{PE}_F\left[f(t_1,\ldots,t_n)\right] = \exp\left(\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}f(t_1^k,\ldots,t_n^k)\right) \ . \tag{A.2}$$

- An irreducible representation of a simple group G can be denoted by its highest weight vector.
  - With respect to a basis consisting of the fundamental weights (the ω-basis), we write the highest vector as $[a_1,\ldots,a_r]$ with $r = \mathrm{rank}\ G$. This is also known as the *Dynkin label*.
  - With respect to a basis of the dual Cartan subalgebra (the e-basis), we write the the highest vector as $(\lambda_1,\ldots,\lambda_r)$.
  - Note that we use the round brackets to distinguish the latter from the former for which the square brackets are used.

- For $SO(2n+1)$, the label $[a_1,a_2,\ldots,a_n]$ is related to the label $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ by the formula

$$\begin{aligned}\lambda_i &= a_i + a_{i+1} + \ldots + a_{n-1} + \frac{1}{2}a_n \ , \quad 1 \le i \le n-1 \ ,\\ \lambda_n &= \frac{1}{2}a_n \ ,\end{aligned} \tag{A.3}$$

  or equivalently

$$\begin{aligned}a_i &= \lambda_i - \lambda_{i+1} \ , \quad 1 \le i \le n-1 \ ,\\ a_n &= 2\lambda_n \ ,\end{aligned} \tag{A.4}$$

- Note that a representation is uniquely specified by its character. We use the notation $[a_1,a_2,\ldots,a_r]_{\boldsymbol{y}}$ (resp. $(\lambda_1,\ldots,\lambda_r)_{\boldsymbol{y}}$) to denote the character of the representation $[a_1,a_2,\ldots,a_r]$ (resp. $(\lambda_1,\ldots,\lambda_r)$) written in terms of the variables $\boldsymbol{y}=(y_1,\ldots,y_r)$. Whenever there is no potential confusion, we drop the subscript $\boldsymbol{y}$ to avoid cluttered notation.

- A representation of a product group $G_1 \times G_2$ is denoted by $[a_1,\ldots,a_{r_1};b_1,b_2,\ldots,b_{r_2}]$ where $[a_1,\ldots,a_{r_1}]$ is an irreducible representation of G_1 and $[b_1,\ldots,b_{r_2}]$ is that of G_2. We use a semi-colon (;) to separate each representation.

- We use the notation $[n]$ to denote the $(n+1)$-dimensional irreducible representation of $SU(2)$ and $SO(3)$. Its character is given by

$$[n]_y = \sum_{k=-n/2}^{+n/2} y^{2k} \ . \tag{A.5}$$

- The character of the vector representation of $SO(2n+1)$, with $n>1$, is taken to be

$$(1,0,\ldots,0)_y = [1,0,\ldots,0]_y = 1 + \sum_{k=1}^{n} \left(y_k^2 + y_k^{-2}\right) \ . \tag{A.6}$$

In general, the character of the irreducible representation $(\lambda_1,\ldots,\lambda_n)$ of $SO(2n+1)$ is given by the Weyl character formula:

$$(\lambda_1,\ldots,\lambda_n)_y = \frac{\det\left(y_j^{2(\lambda_i+n-i+\frac{1}{2})} - y_j^{-2(\lambda_i+n-i+\frac{1}{2})}\right)_{i,j=1}^{n}}{\det\left(y_j^{2(n-i+\frac{1}{2})} - y_j^{-2(n-i+\frac{1}{2})}\right)_{i,j=1}^{n}} \ . \tag{A.7}$$

- The choice of the character in (A.6) has a great advantage: One can relate the character of the vector representation of $SO(2n+1)$ to that of the vector representation of $SO(3)$ in a simple way:

$$[1,0,\ldots,0]_y^{SO(2n+1)} = \sum_{k=1}^{n} [2]_{y_k}^{SO(3)} - (n-1) \ . \tag{A.8}$$

As we shall see in subsequent sections, this helps simplify a number of computations.

- The Haar measures of $SO(3)$ and $SU(2)$ are taken to be

$$\int \mathrm{d}\mu_{SO(3)}(y) = \int \mathrm{d}\mu_{SU(2)}(y) = \frac{1}{2}\frac{1}{2\pi i}\oint_{|y|=1} \frac{\mathrm{d}y}{y}(1-y^2)(1-y^{-2}) \ . \tag{A.9}$$

In general, the Haar measure for $SO(2n+1)$ can be written as

$$\int \mathrm{d}\mu_{SO(2n+1)}(\boldsymbol{y}) = \int \mathrm{d}\mu_{SO(3)}(y_1)\cdots\int \mathrm{d}\mu_{SO(3)}(y_n)\ \rho(\boldsymbol{y}) \ , \tag{A.10}$$

where

$$\rho(\boldsymbol{y}) = \frac{1}{n!}\prod_{1\le i<j\le n} (1-y_i^2 y_j^2)(1-y_i^{-2}y_j^{-2})\left(1-y_i^2 y_j^{-2}\right)\left(1-y_i^{-2}y_j^2\right) \ . \tag{A.11}$$

Special functions

- The q-Pochhammer symbols are defined as

$$(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k) \ , \qquad (a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) \ . \tag{A.12}$$

- Our conventions for the Dedekind eta and the Jacobi theta functions are [36]

$$\eta(q) = q^{\frac{1}{24}}\prod_{n=1}^{\infty}(1-q^n) = q^{\frac{1}{24}}(q;q)_\infty \ , \tag{A.13}$$

$$\vartheta_1(y,q) = -iq^{\frac{1}{8}}(y^{\frac{1}{2}} - y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1-yq^n)(1-y^{-1}q^n) \ , \tag{A.14}$$

$$\vartheta_2(y,q) = q^{\frac{1}{8}}(y^{\frac{1}{2}} + y^{-\frac{1}{2}})\prod_{n=1}^{\infty}(1-q^n)(1+yq^n)(1+y^{-1}q^n) \ , \tag{A.15}$$

[36] These conventions are related to, for example, those adopted in Appendix A of [50] by $y = \exp(2\pi i v)$, $q = \exp(2\pi i\tau)$. We refer the reader to this reference for further properties of such functions.

$$\vartheta_3(y,q) = \prod_{n=1}^{\infty}(1-q^n)\,(1+yq^{n-1/2})(1+y^{-1}q^{n-1/2})\ , \tag{A.16}$$

$$\vartheta_4(y,q) = \prod_{n=1}^{\infty}(1-q^n)\,(1-yq^{n-1/2})(1-y^{-1}q^{n-1/2})\ , \tag{A.17}$$

In terms of an infinite sum, the Jacobi theta functions can be written as

$$\vartheta[{}^a_b](y,q) = \sum_{m\in\mathbb{Z}} q^{\frac{1}{2}(m-a/2)^2}(e^{-i\pi b}y)^{(m-a/2)}\ , \tag{A.18}$$

where

$$\vartheta_1 = \vartheta[{}^1_1]\ , \qquad \vartheta_2 = \vartheta[{}^1_0]\ , \qquad \vartheta_3 = \vartheta[{}^0_0]\ , \qquad \vartheta_4 = \vartheta[{}^0_1]\ . \tag{A.19}$$

- The Appell-Lerch sum is defined as follows [37]:[37]

$$\mu(u,\tau) = -\frac{e^{i\pi u}}{\vartheta_1(y,q)} \sum_{m\in\mathbb{Z}} (-1)^m \frac{e^{\pi i m(m+1)\tau + 2\pi i m u}}{1-e^{2\pi i m\tau + 2\pi i u}}\ , \tag{A.20}$$

where

$$y = \exp(2\pi i u)\ , \qquad q = \exp(2\pi i\tau)\ . \tag{A.21}$$

[37] The notation in this paper and that in Proposition 1.4 of [37] can be related as follows. Our notation is on the left hand sides of the following equalities: $\mu(u,q) = \mu(u,u,q)$, and $\vartheta_1(u,\tau) = -\vartheta(u,\tau)$.

B Data tables for super Poincaré multiplicities

This appendix contains data tables for multiplicities of super Poincaré representations up to mass level $\alpha' m^2 = 25$. We only display tables for the ancestor theories with 4, 8 and 16 supercharges, respectively, since these highest dimensional theories organize the states in the most economic number of supermultiplets. Particular attention is paid to stable patterns, i.e. to the asymptotics of multiplicity generating functions for large spins and mass levels.

Each of the following tables is devoted to family of supermultiplets whose quantum numbers differ in the first $SO(d-1)$ Dynkin label and match in the remaining $SO(d-1)$ and R symmetry quantum numbers. Rows are associated with mass levels, and columns are associated with the value of the first $SO(d-1)$ Dynkin label to which we loosely refer to as the spin. Independently of spacetime dimensions and supercharges, the multiplicity generating functions $G_{\ldots}(q)$ tend to stabilize for large values of the spin and the mass level in the limit where both of them are uniformly increased. This leading Regge trajectory (corresponding to the $\tau_1^{\cdots}(q)$ contribution in (4.74), (5.42) and (6.34)) is exact when numbers occur repeatedly along diagonal lines in the tables, these entries are marked in red.

Moreover, once the asymptotic numbers in red are subtracted from the data outside the first stable region, further subleading trajectory emerge. The leftover after this subtraction tends to stabilize along lines where the mass level grows twice as fast as the spin. This can be understood as the second Regge trajectory (corresponding to the $\tau_2^{\cdots}(q)$ contribution in (4.74), (5.42) and (6.34)) with slope $\frac{1}{2}$ and subtractive sign. Its region of exact validity is highlighted in blue.

B.1 4 supercharges in four dimensions

The tables in this subsection are based on the $\mathcal{N}_{4d} = 1$ partition function (4.4), organized in terms of multiplicity generating functions $G_{n,Q}(q)$, see (4.39).

$\alpha' m^2$	$[\![1;2]\!]$	$[\![3;2]\!]$	$[\![5;2]\!]$	$[\![7;2]\!]$	$[\![9;2]\!]$	$[\![11;2]\!]$	$[\![13;2]\!]$	$[\![15;2]\!]$	$[\![17;2]\!]$	$[\![19;2]\!]$	$[\![21;2]\!]$	$[\![23;2]\!]$
1	0											
2	0											
3	1	0										
4	2	2	0									
5	6	6	2	0								
6	17	15	8	2	0							
7	38	43	22	8	2	0						
8	89	101	62	24	8	2	0					
9	195	233	152	71	24	8	2	0				
10	411	512	361	176	73	24	8	2	0			
11	843	1089	803	430	185	73	24	8	2	0		
12	1694	2231	1734	978	456	187	73	24	8	2	0	
13	3302	4483	3602	2146	1053	465	187	73	24	8	2	0
14	6336	8758	7304	4525	2343	1079	467	187	73	24	8	2
15	11919	16795	14402	9300	4997	2420	1088	467	187	73	24	8
16	22053	31582	27835	18548	10383	5200	2446	1090	467	187	73	24
17	40173	58428	52685	36227	20921	10878	5277	2455	1090	467	187	73
18	72204	106359	98044	69217	41236	22068	11083	5303	2457	1090	467	187
19	128014	191004	179419	129896	79473	43785	22569	11160	5312	2457	1090	467
20	224337	338384	323661	239545	150345	84906	44955	22774	11186	5314	2457	1090
21	388651	592391	575773	435174	279322	161591	87520	45458	22851	11195	5314	2457
22	666314	1025226	1011672	779119	510970	301946	167204	88696	45663	22877	11197	5314
23	1131024	1755809	1756589	1377070	920804	555389	313632	169841	89199	45740	22886	11197
24	1902209	2976969	3017219	2404087	1637411	1006121	579053	319310	171019	89404	45766	22888
25	3170935	5000934	5129359	4150179	2874993	1798156	1052851	590920	321953	171522	89481	45775

$\alpha' m^2$	[[1; 4]]	[[3; 4]]	[[5; 4]]	[[7; 4]]	[[9; 4]]	[[11; 4]]	[[13; 4]]	[[15; 4]]	[[17; 4]]	[[19; 4]]	[[21; 4]]	[[23; 4]]	[[25; 4]]	[[27; 4]]
7	0													
8	1	0												
9	3	2	0											
10	9	8	2	0										
11	25	24	10	2	0									
12	63	65	34	10	2	0								
13	145	166	96	36	10	2	0							
14	327	387	251	108	36	10	2	0						
15	701	870	600	292	110	36	10	2	0					
16	1455	1868	1375	716	304	110	36	10	2	0				
17	2935	3884	2994	1676	759	306	110	36	10	2	0			
18	5784	7830	6304	3717	1804	771	306	110	36	10	2	0		
19	11124	15422	12839	7947	4058	1847	773	306	110	36	10	2	0	
20	21013	29656	25499	16409	8787	4188	1859	773	306	110	36	10	2	0
21	38962	55955	49404	32977	18350	9140	4231	1861	773	306	110	36	10	2
22	71109	103656	93817	64563	37270	19232	9270	4243	1861	773	306	110	36	10
23	127858	188982	174756	123758	73674	39339	19587	9313	4245	1861	773	306	110	36
24	226848	339385	320180	232485	142472	78301	40233	19717	9325	4245	1861	773	306	110
25	397364	601382	577497	429191	269832	152411	80412	40588	19760	9327	4245	1861	773	306

$\alpha' m^2$	[[1; 6]]	[[3; 6]]	[[5; 6]]	[[7; 6]]	[[9; 6]]	[[11; 6]]	[[13; 6]]	[[15; 6]]	[[17; 6]]	[[19; 6]]	[[21; 6]]	[[23; 6]]	[[25; 6]]
14	0												
15	1	0											
16	3	2	0										
17	10	8	2	0									
18	29	26	10	2	0								
19	73	76	36	10	2	0							
20	178	195	110	38	10	2	0						
21	406	474	294	122	38	10	2	0					
22	888	1086	733	338	124	38	10	2	0				
23	1876	2382	1711	868	350	124	38	10	2	0			
24	3845	5028	3815	2075	914	352	124	38	10	2	0		
25	7657	10304	8160	4716	2222	926	352	124	38	10	2	0	

$\alpha' m^2$	[[0; 1]]	[[2; 1]]	[[4; 1]]	[[6; 1]]	[[8; 1]]	[[10; 1]]	[[12; 1]]	[[14; 1]]	[[16; 1]]	[[18; 1]]	[[20; 1]]
1	1	0									
2	0	2	0								
3	3	2	3	0							
4	3	11	4	3	0						
5	15	20	18	5	3	0					
6	21	58	39	21	5	3	0				
7	66	115	105	49	22	5	3	0			
8	112	274	223	135	52	22	5	3	0		
9	267	543	521	296	146	53	22	5	3	0	
10	487	1159	1066	698	330	149	53	22	5	3	0
11	1027	2248	2258	1467	786	341	150	53	22	5	3
12	1872	4483	4465	3133	1682	821	344	150	53	22	5
13	3684	8456	8874	6300	3637	1774	832	345	150	53	22
14	6654	16077	16929	12629	7413	3868	1809	835	345	150	53
15	12430	29505	32174	24376	15014	7960	3961	1820	836	345	150
16	22104	54085	59444	46663	29304	16246	8195	3996	1823	836	345
17	39831	96778	109017	86997	56583	31974	16809	8288	4007	1824	836
18	69495	172263	195931	160521	106459	62184	33250	17045	8323	4010	1824
19	121751	301246	348996	290518	197927	117845	64978	33817	17138	8334	4011
20	208588	523209	612069	520208	360936	220529	123748	66270	34053	17173	8337
21	356951	896281	1063839	917434	650566	404759	232640	126586	66838	34146	17184
22	601090	1524153	1825894	1601735	1154779	733851	428967	238668	127882	67074	34181
23	1008432	2562971	3106955	2761714	2027692	1310137	781160	441385	241522	128450	67167
24	1670909	4278549	5231334	4717314	3515675	2312784	1400641	806110	447457	242819	128686
25	2755277	7075262	8737282	7973033	6035514	4030732	2482787	1449609	818653	450315	243387

$\alpha' m^2$	$[\![0;3]\!]$	$[\![2;3]\!]$	$[\![4;3]\!]$	$[\![6;3]\!]$	$[\![8;3]\!]$	$[\![10;3]\!]$	$[\![12;3]\!]$	$[\![14;3]\!]$	$[\![16;3]\!]$	$[\![18;3]\!]$	$[\![20;3]\!]$	$[\![22;3]\!]$
4	0											
5	1	0										
6	0	3	0									
7	6	5	4	0								
8	7	21	10	4	0							
9	29	44	37	11	4	0						
10	50	122	84	45	11	4	0					
11	135	254	227	108	46	11	4	0				
12	249	588	498	294	116	46	11	4	0			
13	569	1191	1136	668	322	117	46	11	4	0		
14	1061	2504	2359	1546	747	330	117	46	11	4	0	
15	2184	4885	4938	3278	1756	775	331	117	46	11	4	0
16	4044	9638	9770	6932	3790	1839	783	331	117	46	11	4
17	7804	18183	19255	13918	8113	4013	1867	784	331	117	46	11
18	14160	34268	36625	27663	16509	8671	4096	1875	784	331	117	46
19	26159	62704	69034	53180	33151	17810	8898	4124	1876	784	331	117
20	46461	114071	126973	100951	64405	36059	18381	8981	4132	1876	784	331
21	82968	203202	231136	187165	123324	70634	37407	18608	9009	4133	1876	784
22	144356	359209	413075	342732	230632	136240	73668	37982	18691	9017	4133	1876
23	250925	624938	730729	616388	425446	256624	142806	75029	38209	18719	9018	4133
24	428144	1078397	1274031	1095794	770702	476487	270343	145887	75604	38292	18727	9018
25	727755	1837377	2199827	1920245	1378855	868644	504339	277036	147252	75831	38320	18728

$\alpha' m^2$	$[\![0;5]\!]$	$[\![2;5]\!]$	$[\![4;5]\!]$	$[\![6;5]\!]$	$[\![8;5]\!]$	$[\![10;5]\!]$	$[\![12;5]\!]$	$[\![14;5]\!]$	$[\![16;5]\!]$	$[\![18;5]\!]$	$[\![20;5]\!]$	$[\![22;5]\!]$	$[\![24;5]\!]$	$[\![26;5]\!]$
10	0													
11	1	0												
12	0	3	0											
13	7	6	4	0										
14	10	26	11	4	0									
15	37	58	46	12	4	0								
16	70	163	111	54	12	4	0							
17	188	355	305	141	55	12	4	0						
18	359	832	696	394	149	55	12	4	0					
19	821	1726	1616	931	428	150	55	12	4	0				
20	1574	3664	3429	2198	1035	436	150	55	12	4	0			
21	3240	7267	7266	4762	2489	1069	437	150	55	12	4	0		
22	6100	14444	14582	10210	5493	2597	1077	437	150	55	12	4	0	
23	11809	27539	28985	20800	11934	5800	2631	1078	437	150	55	12	4	0
24	21646	52203	55668	41719	24651	12729	5908	2639	1078	437	150	55	12	4
25	40108	96213	105581	80976	49997	26553	13040	5942	2640	1078	437	150	55	12

$\alpha' m^2$	$[\![0;7]\!]$	$[\![2;7]\!]$	$[\![4;7]\!]$	$[\![6;7]\!]$	$[\![8;7]\!]$	$[\![10;7]\!]$	$[\![12;7]\!]$	$[\![14;7]\!]$
18	0							
19	1	0						
20	0	3	0					
21	7	6	4	0				
22	11	27	11	4	0			
23	41	63	47	12	4	0		
24	78	180	120	55	12	4	0	
25	214	402	336	150	56	12	4	0

B.2 8 supercharges in six dimensions

The tables in this subsection are based on the $\mathcal{N}_{6d} = (1,0)$ partition function (5.3), organized in terms of multiplicity generating functions $G_{n_1,n_2,p}(q)$, see (5.23).

$\alpha' m^2$	[[0, 2; 0]]	[[1, 2; 0]]	[[2, 2; 0]]	[[3, 2; 0]]	[[4, 2; 0]]	[[5, 2; 0]]	[[6, 2; 0]]	[[7, 2; 0]]	[[8, 2; 0]]	[[9, 2; 0]]	[[10, 2; 0]]	[[11, 2; 0]]
1	0											
2	1	0										
3	1	1	0									
4	4	2	1	0								
5	6	7	2	1	0							
6	19	13	8	2	1	0						
7	34	38	16	8	2	1	0					
8	81	79	48	17	8	2	1	0				
9	156	184	103	51	17	8	2	1	0			
10	332	378	252	113	52	17	8	2	1	0		
11	636	813	530	279	116	52	17	8	2	1	0	
12	1276	1623	1171	604	289	117	52	17	8	2	1	0
13	2404	3290	2395	1350	631	292	117	52	17	8	2	1
14	4614	6386	4962	2816	1427	641	293	117	52	17	8	2
15	8537	12406	9823	5912	3001	1454	644	293	117	52	17	8
16	15853	23445	19436	11896	6361	3078	1464	645	293	117	52	17
17	28748	44075	37346	23836	12913	6549	3105	1467	645	293	117	52
18	52034	81247	71315	46446	26104	13368	6626	3115	1468	645	293	117
19	92579	148705	133388	89732	51295	27149	13556	6653	3118	1468	645	293
20	163950	268145	247448	169908	99935	53631	27607	13633	6663	3119	1468	645
21	286638	479693	451900	318623	190744	104983	54682	27795	13660	6666	3119	1468
22	498178	848018	818105	588270	360520	201413	107347	55140	27872	13670	6667	3119
23	856969	1487396	1462590	1075628	670688	382510	206529	108401	55328	27899	13673	6667
24	1465054	2583018	2592572	1942043	1235427	715151	393379	208899	108859	55405	27909	13674
25	2483037	4452127	4547623	3474093	2246578	1323605	737611	398523	209953	109047	55432	27912

$\alpha' m^2$	[[0, 4; 0]]	[[1, 4; 0]]	[[2, 4; 0]]	[[3, 4; 0]]	[[4, 4; 0]]	[[5, 4; 0]]	[[6, 4; 0]]	[[7, 4; 0]]	[[8, 4; 0]]	[[9, 4; 0]]	[[10, 4; 0]]	[[11, 4; 0]]	[[12, 4; 0]]
4	0												
5	1	0											
6	4	1	0										
7	9	5	1	0									
8	25	13	5	1	0								
9	61	38	14	5	1	0							
10	142	95	42	14	5	1	0						
11	312	238	108	43	14	5	1	0					
12	681	536	276	112	43	14	5	1	0				
13	1415	1216	642	289	113	43	14	5	1	0			
14	2909	2595	1482	680	293	113	43	14	5	1	0		
15	5804	5486	3235	1592	693	294	113	43	14	5	1	0	
16	11416	11186	6961	3511	1630	697	294	113	43	14	5	1	0
17	21988	22514	14456	7644	3621	1643	698	294	113	43	14	5	1
18	41816	44165	29554	16043	7924	3659	1647	698	294	113	43	14	5
19	78176	85560	58907	33146	16736	8034	3672	1648	698	294	113	43	14
20	144486	162571	115712	66723	34776	17016	8072	3676	1648	698	294	113	43
21	263440	305182	222926	132356	70428	35473	17126	8085	3677	1648	698	294	113
22	475248	564283	423773	257348	140501	72068	35753	17164	8089	3677	1648	698	294
23	847638	1031812	793186	493656	274795	144249	72765	35863	17177	8090	3677	1648	698
24	1497518	1863142	1466875	931993	530067	283053	145893	73045	35901	17181	8090	3677	1648
25	2619670	3330628	2677934	1738092	1006402	547844	286811	146590	73155	35914	17182	8090	3677

$\alpha' m^2$	[[0, 6; 0]]	[[1, 6; 0]]	[[2, 6; 0]]	[[3, 6; 0]]	[[4, 6; 0]]	[[5, 6; 0]]	[[6, 6; 0]]	[[7, 6; 0]]	[[8, 6; 0]]	[[9, 6; 0]]	[[10, 6; 0]]	[[11, 6; 0]]	[[12, 6; 0]]	[[13, 6; 0]]
7	0													
8	1	0												
9	4	1	0											
10	13	5	1	0										
11	35	17	5	1	0									
12	101	48	18	5	1	0								
13	238	140	52	18	5	1	0							
14	575	350	153	53	18	5	1	0						
15	1285	860	389	157	53	18	5	1	0					
16	2834	1983	976	402	158	53	18	5	1	0				
17	5972	4467	2279	1015	406	158	53	18	5	1	0			
18	12413	9647	5213	2395	1028	407	158	53	18	5	1	0		
19	24997	20422	11410	5513	2434	1032	407	158	53	18	5	1	0	
20	49629	41963	24476	12167	5629	2447	1033	407	158	53	18	5	1	0
21	96355	84692	50910	26287	12467	5668	2451	1033	407	158	53	18	5	1
22	184497	167219	103990	55095	27048	12583	5681	2452	1033	407	158	53	18	5
23	347237	324945	207612	113323	56917	27348	12622	5685	2452	1033	407	158	53	18
24	645476	620525	407840	227879	117556	57678	27464	12635	5686	2452	1033	407	158	53
25	1183084	1168737	786848	450666	237343	119382	57978	27503	12639	5686	2452	1033	407	158

$\alpha' m^2$	⟦0, 0; 2⟧	⟦1, 0; 2⟧	⟦2, 0; 2⟧	⟦3, 0; 2⟧	⟦4, 0; 2⟧	⟦5, 0; 2⟧	⟦6, 0; 2⟧	⟦7, 0; 2⟧	⟦8, 0; 2⟧	⟦9, 0; 2⟧	⟦10, 0; 2⟧	⟦11, 0; 2⟧
2	0											
3	1	0										
4	0	2	0									
5	3	3	3	0								
6	4	9	4	3	0							
7	13	20	17	5	3	0						
8	20	50	34	19	5	3	0					
9	53	101	93	43	20	5	3	0				
10	93	224	192	115	45	20	5	3	0			
11	203	449	446	252	125	46	20	5	3	0		
12	369	924	903	589	275	127	46	20	5	3	0	
13	743	1798	1920	1241	659	285	128	46	20	5	3	0
14	1355	3523	3792	2664	1405	683	287	128	46	20	5	3
15	2585	6673	7601	5410	3071	1476	693	288	128	46	20	5
16	4662	12617	14601	10981	6311	3245	1500	695	288	128	46	20
17	8585	23303	28083	21538	13007	6741	3317	1510	696	288	128	46
18	15272	42800	52540	41953	25810	13982	6916	3341	1512	696	288	128
19	27351	77315	97864	79808	50933	28012	14422	6988	3351	1513	696	288
20	47902	138661	178789	150444	97964	55666	29010	14598	7012	3353	1513	696
21	83950	245476	324415	278690	186802	107982	57944	29451	14670	7022	3354	1513
22	144814	431357	580136	511315	349601	207363	112896	58952	29627	14694	7024	3354
23	249137	750026	1029661	925300	648055	391117	217862	115197	59394	29699	14704	7025
24	423589	1294613	1806340	1658994	1183895	730037	412771	222852	116206	59570	29723	14706
25	717200	2214733	3145140	2940833	2142556	1343353	774118	423453	225163	116648	59642	29733

$\alpha' m^2$	⟦0, 2; 2⟧	⟦1, 2; 2⟧	⟦2, 2; 2⟧	⟦3, 2; 2⟧	⟦4, 2; 2⟧	⟦5, 2; 2⟧	⟦6, 2; 2⟧	⟦7, 2; 2⟧	⟦8, 2; 2⟧	⟦9, 2; 2⟧	⟦10, 2; 2⟧	⟦11, 2; 2⟧
3	0											
4	1	0										
5	3	1	0									
6	9	6	1	0								
7	22	16	6	1	0							
8	54	47	19	6	1	0						
9	122	114	57	19	6	1	0					
10	269	282	147	60	19	6	1	0				
11	570	628	372	157	60	19	6	1	0			
12	1182	1397	867	408	160	60	19	6	1	0		
13	2384	2944	1973	966	418	160	60	19	6	1	0	
14	4720	6137	4285	2249	1002	421	160	60	19	6	1	0
15	9164	12349	9114	4962	2351	1012	421	160	60	19	6	1
16	17509	24540	18781	10746	5247	2387	1015	421	160	60	19	6
17	32937	47598	37992	22468	11461	5349	2397	1015	421	160	60	19
18	61121	91162	75102	46159	24208	11749	5385	2400	1015	421	160	60
19	111963	171440	146106	92470	50163	24932	11851	5395	2400	1015	421	160
20	202707	318632	279173	182328	101434	51941	25220	11887	5398	2400	1015	421
21	362956	583695	526058	352627	201679	105547	52668	25322	11897	5398	2400	1015
22	643253	1057824	976881	672443	393429	210967	107334	52956	25358	11900	5398	2400
23	1129052	1894240	1792109	1262534	756265	413603	215118	108061	53058	25368	11900	5398
24	1963846	3359194	3247454	2341077	1431348	799141	423000	216908	108349	53094	25371	11900
25	3386710	5896540	5821871	4284997	2674272	1520012	819640	427160	217635	108451	53104	25371

$\alpha' m^2$	⟦0, 4; 2⟧	⟦1, 4; 2⟧	⟦2, 4; 2⟧	⟦3, 4; 2⟧	⟦4, 4; 2⟧	⟦5, 4; 2⟧	⟦6, 4; 2⟧	⟦7, 4; 2⟧	⟦8, 4; 2⟧	⟦9, 4; 2⟧	⟦10, 4; 2⟧	⟦11, 4; 2⟧	⟦12, 4; 2⟧
6	0												
7	3	0											
8	9	3	0										
9	33	12	3	0									
10	81	45	12	3	0								
11	218	126	48	12	3	0							
12	504	345	138	48	12	3	0						
13	1169	849	393	141	48	12	3	0					
14	2525	2025	989	405	141	48	12	3	0				
15	5415	4556	2426	1037	408	141	48	12	3	0			
16	11115	9997	5574	2569	1049	408	141	48	12	3	0		
17	22527	21139	12502	5988	2617	1052	408	141	48	12	3	0	
18	44383	43734	26921	13577	6131	2629	1052	408	141	48	12	3	0
19	86277	88152	56723	29598	13994	6179	2632	1052	408	141	48	12	3
20	164309	174452	116181	63019	30686	14137	6191	2632	1052	408	141	48	12
21	308983	338438	233542	130513	65753	31103	14185	6194	2632	1052	408	141	48
22	571846	646421	459542	264959	136982	66844	31246	14197	6194	2632	1052	408	141
23	1046250	1215097	889787	526615	279815	139729	67261	31294	14200	6194	2632	1052	408
24	1889540	2253670	1693826	1029156	559415	286341	140820	67404	31306	14200	6194	2632	1052
25	3377343	4124779	3179821	1977217	1099765	574444	289091	141237	67452	31309	14200	6194	2632

$\alpha' m^2$	$[\![0,0;4]\!]$	$[\![1,0;4]\!]$	$[\![2,0;4]\!]$	$[\![3,0;4]\!]$	$[\![4,0;4]\!]$	$[\![5,0;4]\!]$	$[\![6,0;4]\!]$	$[\![7,0;4]\!]$	$[\![8,0;4]\!]$	$[\![9,0;4]\!]$	$[\![10,0;4]\!]$	$[\![11,0;4]\!]$	$[\![12,0;4]\!]$	$[\![13,0;4]\!]$
6	0	0												
7	1	1	0											
8	2	2	1	0										
9	5	10	4	1	0									
10	12	20	15	4	1	0								
11	30	58	38	18	4	1	0							
12	61	125	104	44	18	4	1	0						
13	135	296	245	132	47	18	4	1	0					
14	273	613	575	313	139	47	18	4	1	0				
15	555	1320	1260	766	343	142	47	18	4	1	0			
16	1087	2639	2719	1704	846	350	142	47	18	4	1	0		
17	2115	5333	5628	3792	1926	877	353	142	47	18	4	1	0	
18	3999	10325	11477	7967	4333	2008	884	353	142	47	18	4	1	0
19	7521	19947	22744	16616	9280	4568	2039	887	353	142	47	18	4	1
20	13858	37496	44413	33421	19571	9854	4651	2046	887	353	142	47	18	4
21	25303	70043	84963	66421	39975	20993	10091	4682	2049	887	353	142	47	18
22	45553	128294	160356	128808	80349	43201	21580	10174	4689	2049	887	353	142	47
23	81270	233155	297815	246711	157849	87619	44657	21818	10205	4692	2049	887	353	142
24	143279	417523	546529	463836	305575	173443	90956	45246	21901	10212	4692	2049	887	353
25	250518	741533	989832	861982	581093	338524	180996	92425	45484	21932	10215	4692	2049	887

$\alpha' m^2$	$[\![0,2;4]\!]$	$[\![1,2;4]\!]$	$[\![2,2;4]\!]$	$[\![3,2;4]\!]$	$[\![4,2;4]\!]$	$[\![5,2;4]\!]$	$[\![6,2;4]\!]$	$[\![7,2;4]\!]$	$[\![8,2;4]\!]$	$[\![9,2;4]\!]$	$[\![10,2;4]\!]$	$[\![11,2;4]\!]$	$[\![12,2;4]\!]$
7	0												
8	3	0											
9	6	4	0										
10	25	13	4	0									
11	57	47	14	4	0								
12	152	128	57	14	4	0							
13	338	338	159	58	14	4	0						
14	782	808	447	169	58	14	4	0					
15	1644	1886	1098	481	170	58	14	4	0				
16	3493	4153	2657	1219	491	170	58	14	4	0			
17	7041	8937	5997	2996	1253	492	170	58	14	4	0		
18	14124	18564	13258	6912	3120	1263	492	170	58	14	4	0	
19	27439	37778	28108	15522	7263	3154	1264	492	170	58	14	4	0
20	52817	74981	58430	33506	16489	7387	3164	1264	492	170	58	14	4
21	99411	146275	118038	70651	35926	16843	7421	3165	1264	492	170	58	14
22	185238	279950	234313	144914	76519	36905	16967	7431	3165	1264	492	170	58
23	339430	527948	455350	291435	158361	78991	37259	17001	7432	3165	1264	492	170
24	615770	980532	871500	573877	321433	164388	79973	37383	17011	7432	3165	1264	492
25	1102442	1798020	1640298	1111406	638384	335362	166872	80327	37417	17012	7432	3165	1264

$\alpha' m^2$	$[\![0,0;6]\!]$	$[\![1,0;6]\!]$	$[\![2,0;6]\!]$	$[\![3,0;6]\!]$	$[\![4,0;6]\!]$	$[\![5,0;6]\!]$	$[\![6,0;6]\!]$	$[\![7,0;6]\!]$	$[\![8,0;6]\!]$	$[\![9,0;6]\!]$	$[\![10,0;6]\!]$	$[\![11,0;6]\!]$	$[\![12,0;6]\!]$	$[\![13,0;6]\!]$
10	0													
11	1	0												
12	0	2	0											
13	5	5	3	0										
14	8	16	7	3	0									
15	27	42	30	8	3	0								
16	50	110	74	34	8	3	0							
17	129	253	212	93	35	8	3	0						
18	255	581	490	264	97	35	8	3	0					
19	565	1258	1184	648	286	98	35	8	3	0				
20	1101	2674	2587	1580	706	290	98	35	8	3	0			
21	2258	5480	5674	3580	1768	728	291	98	35	8	3	0		
22	4314	11042	11782	7961	4056	1829	732	291	98	35	8	3	0	
23	8389	21690	24263	16956	9193	4251	1851	733	291	98	35	8	3	0
24	15646	41956	48269	35421	19829	9701	4312	1855	733	291	98	35	8	3
25	29297	79620	94929	71854	42078	21153	9899	4334	1856	733	291	98	35	8

$\alpha' m^2$	$[\![0,1;1]\!]$	$[\![1,1;1]\!]$	$[\![2,1;1]\!]$	$[\![3,1;1]\!]$	$[\![4,1;1]\!]$	$[\![5,1;1]\!]$	$[\![6,1;1]\!]$	$[\![7,1;1]\!]$	$[\![8,1;1]\!]$	$[\![9,1;1]\!]$	$[\![10,1;1]\!]$	$[\![11,1;1]\!]$
1	0											
2	1	0										
3	1	2	0									
4	4	3	2	0								
5	8	9	4	2	0							
6	18	23	12	4	2	0						
7	39	51	31	13	4	2	0					
8	82	114	76	34	13	4	2	0				
9	165	249	174	85	35	13	4	2	0			
10	333	519	391	203	88	35	13	4	2	0		
11	652	1064	843	465	212	89	35	13	4	2	0	
12	1260	2137	1776	1024	495	215	89	35	13	4	2	0
13	2396	4202	3645	2203	1102	504	216	89	35	13	4	2
14	4499	8128	7330	4609	2399	1132	507	216	89	35	13	4
15	8321	15488	14450	9428	5080	2478	1141	508	216	89	35	13
16	15236	29063	28022	18898	10511	5280	2508	1144	508	216	89	35
17	27556	53844	53451	37201	21297	10997	5359	2517	1145	508	216	89
18	49336	98540	100527	71985	42376	22425	11198	5389	2520	1145	508	216
19	87449	178260	186521	137212	82828	44899	22915	11277	5398	2521	1145	508
20	153595	319063	341843	257835	159430	88321	46042	23116	11307	5401	2521	1145
21	267352	565412	619252	478197	302417	171054	90889	46533	23195	11316	5402	2521
22	461595	992485	1109824	876142	565992	326453	176672	92036	46734	23225	11319	5402
23	790578	1726764	1968850	1587104	1046065	614658	338400	179255	92527	46813	23234	11320
24	1343972	2979088	3459778	2844391	1910959	1142740	639492	344063	180403	92728	46843	23237
25	2268336	5098709	6025145	5046950	3452679	2099666	1193279	651564	346650	180894	92807	46852

$\alpha' m^2$	$[\![0,3;1]\!]$	$[\![1,3;1]\!]$	$[\![2,3;1]\!]$	$[\![3,3;1]\!]$	$[\![4,3;1]\!]$	$[\![5,3;1]\!]$	$[\![6,3;1]\!]$	$[\![7,3;1]\!]$	$[\![8,3;1]\!]$	$[\![9,3;1]\!]$	$[\![10,3;1]\!]$	$[\![11,3;1]\!]$
3	0											
4	1	0										
5	4	1	0									
6	9	5	1	0								
7	26	15	5	1	0							
8	61	42	16	5	1	0						
9	140	109	48	16	5	1	0					
10	311	261	127	49	16	5	1	0				
11	669	604	318	133	49	16	5	1	0			
12	1387	1343	756	336	134	49	16	5	1	0		
13	2833	2883	1726	815	342	134	49	16	5	1	0	
14	5638	6031	3797	1887	833	343	134	49	16	5	1	0
15	11026	12313	8123	4213	1946	839	343	134	49	16	5	1
16	21191	24598	16912	9138	4376	1964	840	343	134	49	16	5
17	40119	48224	34431	19284	9563	4435	1970	840	343	134	49	16
18	74828	92924	68660	39746	20332	9726	4453	1971	840	343	134	49
19	137838	176248	134437	80231	42221	20759	9785	4459	1971	840	343	134
20	250749	329537	258807	158890	85837	43278	20922	9803	4460	1971	840	343
21	451108	608030	490719	309257	171219	88345	43705	20981	9809	4460	1971	840
22	802990	1108150	917317	592528	335580	176928	89404	43868	20999	9810	4460	1971
23	1415399	1996715	1692631	1118817	647375	348202	179445	89831	43927	21005	9810	4460
24	2471579	3559576	3085506	2084291	1230561	674467	353944	180504	89994	43945	21006	9810
25	4278524	6282467	5561480	3834679	2307511	1287320	687192	356463	180931	90053	43951	21006

$\alpha' m^2$	$[\![0,5;1]\!]$	$[\![1,5;1]\!]$	$[\![2,5;1]\!]$	$[\![3,5;1]\!]$	$[\![4,5;1]\!]$	$[\![5,5;1]\!]$	$[\![6,5;1]\!]$	$[\![7,5;1]\!]$	$[\![8,5;1]\!]$	$[\![9,5;1]\!]$	$[\![10,5;1]\!]$	$[\![11,5;1]\!]$	$[\![12,5;1]\!]$
6	0												
7	1	0											
8	6	1	0										
9	17	7	1	0									
10	54	23	7	1	0								
11	138	73	24	7	1	0							
12	341	202	79	24	7	1	0						
13	797	518	221	80	24	7	1	0					
14	1795	1254	584	227	80	24	7	1	0				
15	3879	2912	1441	603	228	80	24	7	1	0			
16	8183	6485	3410	1507	609	228	80	24	7	1	0		
17	16780	14008	7731	3599	1526	610	228	80	24	7	1	0	
18	33692	29414	16985	8239	3665	1532	610	228	80	24	7	1	0
19	66268	60280	36213	18272	8428	3684	1533	610	228	80	24	7	1
20	128089	120877	75329	39321	18782	8494	3690	1533	610	228	80	24	7
21	243471	237770	153142	82512	40618	18971	8513	3691	1533	610	228	80	24
22	456134	459491	305209	169218	85661	41128	19037	8519	3691	1533	610	228	80
23	842758	873960	597152	340066	176532	86960	41317	19056	8520	3691	1533	610	228
24	1537763	1638041	1149250	670793	356528	179691	87470	41383	19062	8520	3691	1533	610
25	2773038	3028963	2178141	1301158	706690	363883	180990	87659	41402	19063	8520	3691	1533

$\alpha' m^2$	$[\![0,1;3]\!]$	$[\![1,1;3]\!]$	$[\![2,1;3]\!]$	$[\![3,1;3]\!]$	$[\![4,1;3]\!]$	$[\![5,1;3]\!]$	$[\![6,1;3]\!]$	$[\![7,1;3]\!]$	$[\![8,1;3]\!]$	$[\![9,1;3]\!]$	$[\![10,1;3]\!]$	$[\![11,1;3]\!]$
4	0											
5	1	0										
6	3	2	0									
7	7	7	2	0								
8	19	20	9	2	0							
9	44	53	27	9	2	0						
10	100	130	76	29	9	2	0					
11	215	303	195	84	29	9	2	0				
12	454	675	472	223	86	29	9	2	0			
13	925	1453	1084	552	231	86	29	9	2	0		
14	1854	3036	2403	1302	581	233	86	29	9	2	0	
15	3630	6184	5144	2948	1387	589	233	86	29	9	2	0
16	6990	12327	10721	6442	3183	1416	591	233	86	29	9	2
17	13233	24088	21797	13674	7043	3269	1424	591	233	86	29	9
18	24712	46250	43391	28292	15133	7283	3298	1426	591	233	86	29
19	45490	87411	84717	57218	31670	15751	7369	3306	1426	591	233	86
20	82763	162815	162618	113413	64772	33187	15992	7398	3308	1426	591	233
21	148802	299261	307244	220754	129748	68318	33810	16078	7406	3308	1426	591
22	264749	543354	572296	422630	255152	137754	69852	34051	16107	7408	3308	1426
23	466300	975347	1051966	797014	493286	272632	141358	70476	34137	16115	7408	3308
24	813740	1732302	1910295	1482317	939075	530438	280808	142897	70717	34166	16117	7408
25	1407443	3046334	3429687	2721679	1762389	1016082	548377	284429	143521	70803	34174	16117

$\alpha' m^2$	$[\![0,3;3]\!]$	$[\![1,3;3]\!]$	$[\![2,3;3]\!]$	$[\![3,3;3]\!]$	$[\![4,3;3]\!]$	$[\![5,3;3]\!]$	$[\![6,3;3]\!]$	$[\![7,3;3]\!]$	$[\![8,3;3]\!]$	$[\![9,3;3]\!]$	$[\![10,3;3]\!]$	$[\![11,3;3]\!]$
6	0											
7	2	0										
8	7	2	0									
9	24	10	2	0								
10	63	38	10	2	0							
11	163	109	41	10	2	0						
12	385	295	124	41	10	2	0					
13	879	736	351	127	41	10	2	0				
14	1915	1740	902	366	127	41	10	2	0			
15	4066	3931	2202	959	369	127	41	10	2	0		
16	8365	8576	5105	2378	974	369	127	41	10	2	0	
17	16851	18124	11412	5604	2435	977	369	127	41	10	2	0
18	33194	37328	24640	12713	5781	2450	977	369	127	41	10	2
19	64238	75100	51777	27847	13222	5838	2453	977	369	127	41	10
20	122171	148039	106067	59296	29185	13399	5853	2453	977	369	127	41
21	228951	286468	212660	123042	62633	29695	13456	5856	2453	977	369	127
22	422965	545251	417987	249674	130948	63981	29872	13471	5856	2453	977	369
23	771624	1022124	807305	496442	267714	134322	64491	29929	13474	5856	2453	977
24	1390866	1889717	1534140	969373	536185	275750	135671	64668	29944	13474	5856	2453
25	2479819	3449211	2873001	1861540	1054472	554615	279134	136181	64725	29947	13474	5856

$\alpha' m^2$	$[\![0,1;5]\!]$	$[\![1,1;5]\!]$	$[\![2,1;5]\!]$	$[\![3,1;5]\!]$	$[\![4,1;5]\!]$	$[\![5,1;5]\!]$	$[\![6,1;5]\!]$	$[\![7,1;5]\!]$	$[\![8,1;5]\!]$	$[\![9,1;5]\!]$	$[\![10,1;5]\!]$	$[\![11,1;5]\!]$	$[\![12,1;5]\!]$	$[\![13,1;5]\!]$
8	0													
9	1	0												
10	3	2	0											
11	9	8	2	0										
12	26	25	10	2	0									
13	62	73	34	10	2	0								
14	148	188	105	36	10	2	0							
15	332	457	283	116	36	10	2	0						
16	721	1056	717	322	118	36	10	2	0					
17	1511	2343	1708	839	333	118	36	10	2	0				
18	3097	5020	3902	2053	880	335	118	36	10	2	0			
19	6181	10457	8566	4793	2183	891	335	118	36	10	2	0		
20	12114	21231	18249	10747	5170	2224	893	335	118	36	10	2	0	
21	23284	42177	37794	23329	11740	5302	2235	893	335	118	36	10	2	0
22	44053	82157	76466	49173	25807	12125	5343	2237	893	335	118	36	10	2
23	82070	157249	151421	101106	55044	26833	12257	5354	2237	893	335	118	36	10
24	150888	296196	294293	203277	114478	57629	27220	12298	5356	2237	893	335	118	36
25	273843	549904	562169	400661	232669	120665	58663	27352	12309	5356	2237	893	335	118

B.3 16 supercharges in ten dimensions

The tables in this subsection are based on the $\mathcal{N}_{10d} = 1$ partition function (6.3), organized in terms of multiplicity generating functions $G_{n_1,n_2,n_3,n_4}(q)$, see (6.16).

$\alpha' m^2$	[[0, 1, 0, 0]]	[[1, 1, 0, 0]]	[[2, 1, 0, 0]]	[[3, 1, 0, 0]]	[[4, 1, 0, 0]]	[[5, 1, 0, 0]]	[[6, 1, 0, 0]]	[[7, 1, 0, 0]]	[[8, 1, 0, 0]]	[[9, 1, 0, 0]]	[[10, 1, 0, 0]]	[[11, 1, 0, 0]]	[[12, 1, 0, 0]]	[[13, 1, 0, 0]]	[[14, 1, 0, 0]]
3	0														
4	1	0													
5	1	1	0												
6	1	2	1	0											
7	2	2	2	1	0										
8	5	5	3	2	1	0									
9	7	9	6	3	2	1	0								
10	13	17	12	7	3	2	1	0							
11	21	29	23	13	7	3	2	1	0						
12	37	54	42	26	14	7	3	2	1	0					
13	60	90	77	48	27	14	7	3	2	1	0				
14	101	159	137	92	51	28	14	7	3	2	1	0			
15	165	268	243	163	98	52	28	14	7	3	2	1	0		
16	274	457	422	298	178	101	53	28	14	7	3	2	1	0	
17	441	760	732	522	326	184	102	53	28	14	7	3	2	1	0
18	717	1276	1248	924	580	341	187	103	53	28	14	7	3	2	1
19	1149	2088	2121	1592	1032	608	347	188	103	53	28	14	7	3	2
20	1847	3443	3551	2750	1801	1092	623	350	189	103	53	28	14	7	3
21	2928	5585	5929	4656	3134	1912	1120	629	351	189	103	53	28	14	7
22	4647	9060	9790	7886	5361	3351	1972	1135	632	352	189	103	53	28	14
23	7310	14538	16095	13160	9148	5762	3464	2000	1141	633	352	189	103	53	28
24	11482	23301	26221	21906	15414	9894	5982	3524	2015	1144	634	352	189	103	53
25	17908	36995	42535	36063	25846	16754	10303	6095	3552	2021	1145	634	352	189	103

$\alpha' m^2$	[[0, 2, 0, 0]]	[[1, 2, 0, 0]]	[[2, 2, 0, 0]]	[[3, 2, 0, 0]]	[[4, 2, 0, 0]]	[[5, 2, 0, 0]]	[[6, 2, 0, 0]]	[[7, 2, 0, 0]]	[[8, 2, 0, 0]]	[[9, 2, 0, 0]]	[[10, 2, 0, 0]]	[[11, 2, 0, 0]]	[[12, 2, 0, 0]]	[[13, 2, 0, 0]]	[[14, 2, 0, 0]]
6	0														
7	1	0													
8	1	1	0												
9	4	2	1	0											
10	5	5	2	1	0										
11	13	9	6	2	1	0									
12	21	21	10	6	2	1	0								
13	45	38	25	11	6	2	1	0							
14	74	78	46	26	11	6	2	1	0						
15	143	141	98	50	27	11	6	2	1	0					
16	240	269	178	106	51	27	11	6	2	1	0				
17	437	477	349	198	110	52	27	11	6	2	1	0			
18	731	870	629	389	206	111	52	27	11	6	2	1	0		
19	1280	1515	1170	713	409	210	112	52	27	11	6	2	1	0	
20	2126	2673	2067	1335	753	417	211	112	52	27	11	6	2	1	0
21	3619	4576	3709	2394	1422	773	421	212	112	52	27	11	6	2	1
22	5952	7867	6438	4328	2563	1462	781	422	212	112	52	27	11	6	2
23	9908	13251	11235	7604	4668	2650	1482	785	423	212	112	52	27	11	6
24	16128	22320	19168	13377	8250	4840	2690	1490	786	423	212	112	52	27	11
25	26386	37038	32718	23070	14611	8594	4927	2710	1494	787	423	212	112	52	27

$\alpha' m^2$	[[0, 3, 0, 0]]	[[1, 3, 0, 0]]	[[2, 3, 0, 0]]	[[3, 3, 0, 0]]	[[4, 3, 0, 0]]	[[5, 3, 0, 0]]	[[6, 3, 0, 0]]	[[7, 3, 0, 0]]	[[8, 3, 0, 0]]	[[9, 3, 0, 0]]	[[10, 3, 0, 0]]	[[11, 3, 0, 0]]	[[12, 3, 0, 0]]	[[13, 3, 0, 0]]	[[14, 3, 0, 0]]
9	0														
10	1	0													
11	1	1	0												
12	4	2	1	0											
13	8	5	2	1	0										
14	18	12	6	2	1	0									
15	34	26	13	6	2	1	0								
16	73	55	30	14	6	2	1	0							
17	135	112	63	31	14	6	2	1	0						
18	261	222	133	67	32	14	6	2	1	0					
19	479	428	264	141	68	32	14	6	2	1	0				
20	885	815	520	285	145	69	32	14	6	2	1	0			
21	1577	1512	996	562	293	146	69	32	14	6	2	1	0		
22	2822	2776	1881	1091	583	297	147	69	32	14	6	2	1	0	
23	4922	5005	3482	2067	1133	591	298	147	69	32	14	6	2	1	0
24	8567	8930	6366	3865	2162	1154	595	299	147	69	32	14	6	2	1
25	14672	15706	11460	7105	4054	2204	1162	596	299	147	69	32	14	6	2

$\alpha' m^2$	[[0,0,1,0]]	[[1,0,1,0]]	[[2,0,1,0]]	[[3,0,1,0]]	[[4,0,1,0]]	[[5,0,1,0]]	[[6,0,1,0]]	[[7,0,1,0]]	[[8,0,1,0]]	[[9,0,1,0]]	[[10,0,1,0]]	[[11,0,1,0]]	[[12,0,1,0]]	[[13,0,1,0]]	[[14,0,1,0]]
4	0														
5	1	0													
6	0	1	0												
7	3	1	1	0											
8	2	4	1	1	0										
9	7	6	5	1	1	0									
10	10	15	7	5	1	1	0								
11	22	24	20	8	5	1	1	0							
12	30	51	33	21	8	5	1	1	0						
13	64	85	73	38	22	8	5	1	1	0					
14	97	164	125	83	39	22	8	5	1	1	0				
15	179	276	249	148	88	40	22	8	5	1	1	0			
16	282	502	431	297	158	89	40	22	8	5	1	1	0		
17	496	842	803	529	321	163	90	40	22	8	5	1	1	0	
18	784	1473	1379	993	578	331	164	90	40	22	8	5	1	1	0
19	1335	2449	2462	1748	1099	602	336	165	90	40	22	8	5	1	1
20	2117	4164	4181	3153	1951	1149	612	337	165	90	40	22	8	5	1
21	3497	6853	7238	5454	3559	2058	1173	617	338	165	90	40	22	8	5
22	5546	11401	12131	9549	6218	3770	2108	1183	618	338	165	90	40	22	8
23	8981	18557	20509	16261	10990	6637	3878	2132	1188	619	338	165	90	40	22
24	14141	30342	33931	27794	18890	11791	6849	3928	2142	1189	619	338	165	90	40
25	22570	48846	56288	46628	32585	20406	12218	6957	3952	2147	1190	619	338	165	90

$\alpha' m^2$	[[0,1,1,0]]	[[1,1,1,0]]	[[2,1,1,0]]	[[3,1,1,0]]	[[4,1,1,0]]	[[5,1,1,0]]	[[6,1,1,0]]	[[7,1,1,0]]	[[8,1,1,0]]	[[9,1,1,0]]	[[10,1,1,0]]	[[11,1,1,0]]	[[12,1,1,0]]	[[13,1,1,0]]	[[14,1,1,0]]
7	0														
8	1	0													
9	1	1	0												
10	6	2	1	0											
11	10	7	2	1	0										
12	23	17	8	2	1	0									
13	43	36	18	8	2	1	0								
14	90	77	43	19	8	2	1	0							
15	162	157	91	44	19	8	2	1	0						
16	312	307	194	98	45	19	8	2	1	0					
17	554	591	385	208	99	45	19	8	2	1	0				
18	1010	1110	763	423	215	100	45	19	8	2	1	0			
19	1764	2041	1453	844	437	216	100	45	19	8	2	1	0		
20	3105	3701	2741	1636	882	444	217	100	45	19	8	2	1	0	
21	5310	6608	5043	3111	1718	896	445	217	100	45	19	8	2	1	0
22	9113	11636	9178	5810	3297	1756	903	446	217	100	45	19	8	2	1
23	15325	20254	16405	10673	6191	3379	1770	904	446	217	100	45	19	8	2
24	25728	34873	29035	19314	11467	6378	3417	1777	905	446	217	100	45	19	8
25	42607	59411	50676	34509	20876	11851	6460	3431	1778	905	446	217	100	45	19

$\alpha' m^2$	[[0,2,1,0]]	[[1,2,1,0]]	[[2,2,1,0]]	[[3,2,1,0]]	[[4,2,1,0]]	[[5,2,1,0]]	[[6,2,1,0]]	[[7,2,1,0]]	[[8,2,1,0]]	[[9,2,1,0]]	[[10,2,1,0]]	[[11,2,1,0]]	[[12,2,1,0]]	[[13,2,1,0]]	[[14,2,1,0]]
10	0														
11	1	0													
12	1	1	0												
13	7	2	1	0											
14	13	8	2	1	0										
15	36	20	9	2	1	0									
16	70	50	21	9	2	1	0								
17	160	109	57	22	9	2	1	0							
18	307	243	123	58	22	9	2	1	0						
19	629	497	283	130	59	22	9	2	1	0					
20	1176	1016	583	297	131	59	22	9	2	1	0				
21	2259	1983	1219	623	304	132	59	22	9	2	1	0			
22	4119	3837	2400	1306	637	305	132	59	22	9	2	1	0		
23	7570	7206	4727	2606	1346	644	306	132	59	22	9	2	1	0	
24	13461	13400	8972	5157	2693	1360	645	306	132	59	22	9	2	1	0
25	23950	24383	16923	9892	5364	2733	1367	646	306	132	59	22	9	2	1

$\alpha' m^2$	[[0,0,0,2]]	[[1,0,0,2]]	[[2,0,0,2]]	[[3,0,0,2]]	[[4,0,0,2]]	[[5,0,0,2]]	[[6,0,0,2]]	[[7,0,0,2]]	[[8,0,0,2]]	[[9,0,0,2]]	[[10,0,0,2]]	[[11,0,0,2]]	[[12,0,0,2]]	[[13,0,0,2]]	[[14,0,0,2]]
5	0														
6	1	0													
7	1	1	0												
8	3	2	1	0											
9	4	6	2	1	0										
10	10	9	7	2	1	0									
11	16	22	12	7	2	1	0								
12	32	40	29	13	7	2	1	0							
13	52	80	55	32	13	7	2	1	0						
14	98	141	115	62	33	13	7	2	1	0					
15	160	267	211	132	65	33	13	7	2	1	0				
16	286	463	409	249	139	66	33	13	7	2	1	0			
17	469	835	733	491	266	142	66	33	13	7	2	1	0		
18	805	1431	1351	900	531	273	143	66	33	13	7	2	1	0	
19	1314	2489	2375	1685	985	548	276	143	66	33	13	7	2	1	0
20	2199	4199	4218	3018	1864	1025	555	277	143	66	33	13	7	2	1
21	3558	7131	7270	5438	3378	1951	1042	558	277	143	66	33	13	7	2
22	5837	11842	12571	9530	6148	3560	1991	1049	559	277	143	66	33	13	7
23	9361	19709	21279	16701	10888	6520	3647	2008	1052	559	277	143	66	33	13
24	15106	32300	35990	28688	19266	11624	6704	3687	2015	1053	559	277	143	66	33
25	23999	52855	59966	49138	33418	20692	11999	6791	3704	2018	1053	559	277	143	66

$\alpha' m^2$	[[0,1,0,2]]	[[1,1,0,2]]	[[2,1,0,2]]	[[3,1,0,2]]	[[4,1,0,2]]	[[5,1,0,2]]	[[6,1,0,2]]	[[7,1,0,2]]	[[8,1,0,2]]	[[9,1,0,2]]	[[10,1,0,2]]	[[11,1,0,2]]	[[12,1,0,2]]	[[13,1,0,2]]	[[14,1,0,2]]
8	0														
9	1	0													
10	3	1	0												
11	8	4	1	0											
12	17	11	4	1	0										
13	38	27	12	4	1	0									
14	76	61	30	12	4	1	0								
15	153	133	71	31	12	4	1	0							
16	290	273	158	74	31	12	4	1	0						
17	548	547	336	168	75	31	12	4	1	0					
18	1003	1058	687	361	171	75	31	12	4	1	0				
19	1819	2012	1365	752	371	172	75	31	12	4	1	0			
20	3227	3732	2646	1511	777	374	172	75	31	12	4	1	0		
21	5674	6825	5017	2973	1576	787	375	172	75	31	12	4	1	0	
22	9821	12252	9337	5702	3121	1601	790	375	172	75	31	12	4	1	0
23	16851	21737	17080	10752	6035	3186	1611	791	375	172	75	31	12	4	1
24	28565	38015	30794	19888	11457	6183	3211	1614	791	375	172	75	31	12	4
25	48036	65800	54747	36281	21354	11792	6248	3221	1615	791	375	172	75	31	12

$\alpha' m^2$	[[0,0,2,0]]	[[1,0,2,0]]	[[2,0,2,0]]	[[3,0,2,0]]	[[4,0,2,0]]	[[5,0,2,0]]	[[6,0,2,0]]	[[7,0,2,0]]	[[8,0,2,0]]	[[9,0,2,0]]	[[10,0,2,0]]	[[11,0,2,0]]	[[12,0,2,0]]	[[13,0,2,0]]	[[14,0,2,0]]
10	0														
11	2	0													
12	2	2	0												
13	11	4	2	0											
14	16	15	4	2	0										
15	43	30	17	4	2	0									
16	78	75	34	17	4	2	0								
17	169	150	91	36	17	4	2	0							
18	297	325	185	95	36	17	4	2	0						
19	593	622	414	201	97	36	17	4	2	0					
20	1043	1236	812	451	205	97	36	17	4	2	0				
21	1935	2296	1656	904	467	207	97	36	17	4	2	0			
22	3369	4316	3139	1863	941	471	207	97	36	17	4	2	0		
23	6003	7793	6029	3594	1957	957	473	207	97	36	17	4	2	0	
24	10261	14093	11090	6972	3804	1994	961	473	207	97	36	17	4	2	0
25	17753	24813	20426	13020	7444	3898	2010	963	473	207	97	36	17	4	2

$\alpha' m^2$	⟦0, 0, 0, 1⟧	⟦1, 0, 0, 1⟧	⟦2, 0, 0, 1⟧	⟦3, 0, 0, 1⟧	⟦4, 0, 0, 1⟧	⟦5, 0, 0, 1⟧	⟦6, 0, 0, 1⟧	⟦7, 0, 0, 1⟧	⟦8, 0, 0, 1⟧	⟦9, 0, 0, 1⟧	⟦10, 0, 0, 1⟧	⟦11, 0, 0, 1⟧	⟦12, 0, 0, 1⟧	⟦13, 0, 0, 1⟧	⟦14, 0, 0, 1⟧
2	0														
3	1	0													
4	0	1	0												
5	1	1	1	0											
6	2	2	1	1	0										
7	2	4	3	1	1	0									
8	4	7	5	3	1	1	0								
9	8	12	10	6	3	1	1	0							
10	12	22	19	11	6	3	1	1	0						
11	20	38	35	22	12	6	3	1	1	0					
12	34	66	62	43	23	12	6	3	1	1	0				
13	54	113	112	77	46	24	12	6	3	1	1	0			
14	89	190	197	142	85	47	24	12	6	3	1	1	0		
15	147	318	342	256	158	88	48	24	12	6	3	1	1	0	
16	233	532	587	452	288	166	89	48	24	12	6	3	1	1	0
17	376	877	1001	792	517	304	169	90	48	24	12	6	3	1	1
18	603	1438	1686	1376	916	550	312	170	90	48	24	12	6	3	1
19	954	2345	2823	2354	1610	983	566	315	171	90	48	24	12	6	3
20	1511	3795	4684	4003	2789	1740	1016	574	316	171	90	48	24	12	6
21	2383	6105	7716	6745	4795	3037	1808	1032	577	317	171	90	48	24	12
22	3727	9775	12620	11265	8164	5260	3169	1841	1040	578	317	171	90	48	24
23	5821	15552	20513	18678	13782	9019	5514	3237	1857	1043	579	317	171	90	48
24	9050	24624	33121	30757	23075	15332	9498	5647	3270	1865	1044	579	317	171	90
25	13998	38797	53183	50273	38366	25850	16217	9754	5715	3286	1868	1045	579	317	171

$\alpha' m^2$	⟦0, 1, 0, 1⟧	⟦1, 1, 0, 1⟧	⟦2, 1, 0, 1⟧	⟦3, 1, 0, 1⟧	⟦4, 1, 0, 1⟧	⟦5, 1, 0, 1⟧	⟦6, 1, 0, 1⟧	⟦7, 1, 0, 1⟧	⟦8, 1, 0, 1⟧	⟦9, 1, 0, 1⟧	⟦10, 1, 0, 1⟧	⟦11, 1, 0, 1⟧	⟦12, 1, 0, 1⟧	⟦13, 1, 0, 1⟧	⟦14, 1, 0, 1⟧
5	0														
6	1	0													
7	2	1	0												
8	4	3	1	0											
9	8	6	3	1	0										
10	15	14	7	3	1	0									
11	29	28	16	7	3	1	0								
12	53	55	34	17	7	3	1	0							
13	96	107	70	36	17	7	3	1	0						
14	171	201	138	76	37	17	7	3	1	0					
15	300	369	268	153	78	37	17	7	3	1	0				
16	520	671	506	301	159	79	37	17	7	3	1	0			
17	891	1195	939	578	316	161	79	37	17	7	3	1	0		
18	1512	2101	1710	1089	611	322	162	79	37	17	7	3	1	0	
19	2541	3654	3071	2012	1163	626	324	162	79	37	17	7	3	1	0
20	4233	6280	5439	3663	2167	1196	632	325	162	79	37	17	7	3	1
21	6999	10680	9518	6573	3978	2241	1211	634	325	162	79	37	17	7	3
22	11481	18008	16466	11648	7199	4135	2274	1217	635	325	162	79	37	17	7
23	18704	30086	28203	20395	12861	7519	4209	2289	1219	635	325	162	79	37	17
24	30270	49864	47842	35340	22696	13500	7676	4242	2295	1220	635	325	162	79	37
25	48683	82031	80451	60618	39634	23943	13822	7750	4257	2297	1220	635	325	162	79

$\alpha' m^2$	⟦0, 2, 0, 1⟧	⟦1, 2, 0, 1⟧	⟦2, 2, 0, 1⟧	⟦3, 2, 0, 1⟧	⟦4, 2, 0, 1⟧	⟦5, 2, 0, 1⟧	⟦6, 2, 0, 1⟧	⟦7, 2, 0, 1⟧	⟦8, 2, 0, 1⟧	⟦9, 2, 0, 1⟧	⟦10, 2, 0, 1⟧	⟦11, 2, 0, 1⟧	⟦12, 2, 0, 1⟧	⟦13, 2, 0, 1⟧
8	0													
9	1	0												
10	2	1	0											
11	6	3	1	0										
12	14	8	3	1	0									
13	30	20	9	3	1	0								
14	62	46	22	9	3	1	0							
15	125	98	52	23	9	3	1	0						
16	241	204	114	54	23	9	3	1	0					
17	460	408	242	120	55	23	9	3	1	0				
18	855	798	493	258	122	55	23	9	3	1	0			
19	1561	1522	982	531	264	123	55	23	9	3	1	0		
20	2806	2848	1904	1069	547	266	123	55	23	9	3	1	0	
21	4977	5233	3621	2094	1107	553	267	123	55	23	9	3	1	0
22	8706	9473	6754	4020	2181	1123	555	267	123	55	23	9	3	1
23	15067	16902	12404	7571	4212	2219	1129	556	267	123	55	23	9	3
24	25791	29782	22437	14033	7976	4299	2235	1131	556	267	123	55	23	9
25	43720	51867	40062	25611	14867	8168	4337	2241	1132	556	267	123	55	23

$\alpha' m^2$	$[\![0,0,1,1]\!]$	$[\![1,0,1,1]\!]$	$[\![2,0,1,1]\!]$	$[\![3,0,1,1]\!]$	$[\![4,0,1,1]\!]$	$[\![5,0,1,1]\!]$	$[\![6,0,1,1]\!]$	$[\![7,0,1,1]\!]$	$[\![8,0,1,1]\!]$	$[\![9,0,1,1]\!]$	$[\![10,0,1,1]\!]$	$[\![11,0,1,1]\!]$	$[\![12,0,1,1]\!]$	$[\![13,0,1,1]\!]$	$[\![14,0,1,1]\!]$
7	0														
8	1	0													
9	3	1	0												
10	5	4	1	0											
11	12	9	4	1	0										
12	25	22	10	4	1	0									
13	47	47	26	10	4	1	0								
14	90	98	58	27	10	4	1	0							
15	169	195	125	62	27	10	4	1	0						
16	304	378	258	136	63	27	10	4	1	0					
17	547	713	516	286	140	63	27	10	4	1	0				
18	966	1322	1001	584	297	141	63	27	10	4	1	0			
19	1677	2402	1903	1151	612	301	141	63	27	10	4	1	0		
20	2887	4299	3540	2226	1220	623	302	141	63	27	10	4	1	0	
21	4916	7584	6475	4207	2381	1248	627	302	141	63	27	10	4	1	0
22	8274	13215	11659	7808	4542	2450	1259	628	302	141	63	27	10	4	1
23	13822	22755	20706	14260	8510	4698	2478	1263	628	302	141	63	27	10	4
24	22889	38785	36301	25672	15681	8850	4767	2489	1264	628	302	141	63	27	10
25	37594	65459	62931	45588	28475	16395	9006	4795	2493	1264	628	302	141	63	27

$\alpha' m^2$	$[\![0,1,1,1]\!]$	$[\![1,1,1,1]\!]$	$[\![2,1,1,1]\!]$	$[\![3,1,1,1]\!]$	$[\![4,1,1,1]\!]$	$[\![5,1,1,1]\!]$	$[\![6,1,1,1]\!]$	$[\![7,1,1,1]\!]$	$[\![8,1,1,1]\!]$	$[\![9,1,1,1]\!]$	$[\![10,1,1,1]\!]$	$[\![11,1,1,1]\!]$	$[\![12,1,1,1]\!]$	$[\![13,1,1,1]\!]$	$[\![14,1,1,1]\!]$
10	0														
11	1	0													
12	4	1	0												
13	11	5	1	0											
14	28	15	5	1	0										
15	65	40	16	5	1	0									
16	141	99	44	16	5	1	0								
17	292	224	111	45	16	5	1	0							
18	587	483	259	115	45	16	5	1	0						
19	1143	1007	572	271	116	45	16	5	1	0					
20	2176	2023	1216	607	275	116	45	16	5	1	0				
21	4056	3959	2495	1306	619	276	116	45	16	5	1	0			
22	7420	7580	4977	2710	1341	623	276	116	45	16	5	1	0		
23	13361	14206	9692	5467	2800	1353	624	276	116	45	16	5	1	0	
24	23720	26160	18474	10762	5683	2835	1357	624	276	116	45	16	5	1	0
25	41558	47429	34562	20726	11258	5773	2847	1358	624	276	116	45	16	5	1

$\alpha' m^2$	$[\![0,0,0,3]\!]$	$[\![1,0,0,3]\!]$	$[\![2,0,0,3]\!]$	$[\![3,0,0,3]\!]$	$[\![4,0,0,3]\!]$	$[\![5,0,0,3]\!]$	$[\![6,0,0,3]\!]$	$[\![7,0,0,3]\!]$	$[\![8,0,0,3]\!]$	$[\![9,0,0,3]\!]$	$[\![10,0,0,3]\!]$	$[\![11,0,0,3]\!]$	$[\![12,0,0,3]\!]$	$[\![13,0,0,3]\!]$	$[\![14,0,0,3]\!]$
9	0														
10	2	0													
11	3	2	0												
12	7	5	2	0											
13	16	13	5	2	0										
14	32	30	15	5	2	0									
15	62	65	36	15	5	2	0								
16	121	135	82	38	15	5	2	0							
17	222	272	176	88	38	15	5	2	0						
18	406	525	368	193	90	38	15	5	2	0					
19	731	997	732	412	199	90	38	15	5	2	0				
20	1291	1848	1431	836	429	201	90	38	15	5	2	0			
21	2247	3367	2722	1662	880	435	201	90	38	15	5	2	0		
22	3879	6033	5078	3218	1769	897	437	201	90	38	15	5	2	0	
23	6601	10664	9300	6100	3457	1813	903	437	201	90	38	15	5	2	0
24	11134	18593	16784	11343	6620	3564	1830	905	437	201	90	38	15	5	2
25	18612	32056	29830	20770	12428	6862	3608	1836	905	437	201	90	38	15	5

C Large spin asymptotics of super Poincaré multiplicities

This appendix contains some more data on the large spin asymptotics of $\mathcal{N}_{6d} = (1,0)$ and $\mathcal{N}_{10d} = 1$ spectra. The leading and subleading Regge trajectories $\tau_\ell^{k,p}(q)$ and $\tau_\ell^{x,y,z}(q)$ are defined through the expansion (5.42) and (6.34) of super Poincaré multiplicity generating functions in terms of q^n powers (with n denoting the first $SO(d-1)$ Dynkin label). They have been computed on the basis of the $\alpha' m^2 \leq 25$ data tabulated in appendix B.2 and B.3, respectively.

C.1 $\mathcal{N}_{6d} = (1,0)$ multiplets at $SO(5)$ Dynkin labels $[n \to \infty, k]$

For the universal $\mathcal{N}_{6d} = (1,0)$ multiplets $[\![n \to \infty, k; p]\!]$ we display some $\tau_{\ell \leq 5}^{k,p}(q)$ associated with super Poincaré quantum numbers $(k,p) = (2,2), (4,2), (2,4), (3,1), (1,3), (5,1), (3,3), (1,5)$ here. The former three multiplets are bosonic and characterized by the following asymptotic behaviour:

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[2]$

$$\begin{aligned}
\tau_1^{2,2}(q) &= q^4\,(1 + 6q + 19q^2 + 60q^3 + 160q^4 + 421q^5 + 1015q^6 + 2400q^7 + 5398q^8 \\
&\quad + 11900q^9 + 25371q^{10} + 53107q^{11} + 108500q^{12} + 218074q^{13} \\
&\quad + 430116q^{14} + 836194q^{15} + 1600889q^{16} + \ldots) \\
\tau_2^{2,2}(q) &= q^5\,(3 + 13q + 49q^2 + 151q^3 + 439q^4 + 1166q^5 + 2956q^6 + 7119q^7 \\
&\quad + 16566q^8 + 37224q^9 + 81414q^{10} + 173493q^{11} + \ldots) \\
\tau_3^{2,2}(q) &= q^6\,(3 + 12q + 53q^2 + 171q^3 + 537q^4 + 1486q^5 + 3960q^6 + 9876q^7 + \ldots) \\
\tau_4^{2,2}(q) &= q^7\,(1 + 8q + 35q^2 + 134q^3 + 434q^4 + \ldots) \\
\tau_5^{2,2}(q) &= q^9\,(4 + \ldots)
\end{aligned} \tag{C.1}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 4]$ and $SU(2)_R$ representation $[2]$

$$\begin{aligned}
\tau_1^{4,2}(q) &= q^7\,(3 + 12q + 48q^2 + 141q^3 + 408q^4 + 1052q^5 + 2632q^6 + 6194q^7 + 14200q^8 \\
&\quad + 31309q^9 + 67467q^{10} + 141443q^{11} + 290805q^{12} + 585447q^{13} + 1159182q^{14} + \ldots) \\
\tau_2^{4,2}(q) &= q^8\,(3 + 15q + 63q^2 + 206q^3 + 623q^4 + 1714q^5 + 4464q^6 + 11006q^7 + 26108q^8 \\
&\quad + 59679q^9 + 132452q^{10} + \ldots) \\
\tau_3^{4,2}(q) &= q^{10}\,(3 + 16q + 76q^2 + 262q^3 + 847q^4 + 2427q^5 + 6599q^6 + \ldots) \\
\tau_4^{4,2}(q) &= q^{12}\,(1 + 11q + 52q^2 + \ldots)
\end{aligned} \tag{C.2}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 2]$ and $SU(2)_R$ representation $[4]$

$$\begin{aligned}
\tau_1^{2,4}(q) &= q^8\,(4 + 14q + 58q^2 + 170q^3 + 492q^4 + 1264q^5 + 3165q^6 + 7432q^7 + 17012q^8 \\
&\quad + 37428q^9 + 80496q^{10} + 168377q^{11} + 345433q^{12} + \ldots) \\
\tau_2^{2,4}(q) &= q^8\,(1 + 11q + 45q^2 + 169q^3 + 523q^4 + 1505q^5 + 3992q^6 + 10086q^7 + 24241q^8 + \ldots) \\
\tau_3^{2,4}(q) &= q^9\,(3 + 15q + 70q^2 + 241q^3 + 781q^4 + \ldots) \\
\tau_4^{2,4}(q) &= q^{10}\,(3 + 15q + \ldots)
\end{aligned} \tag{C.3}$$

In addition, let us display some $\tau_\ell^{k,p}(q)$ associated with fermionic supermultiplets:

- $SO(5)$ Dynkin labels $[n \to \infty, 3]$ and $SU(2)_R$ representation $[1]$

$$\tau_1^{3,1}(q) = q^4\,(1 + 5q + 16q^2 + 49q^3 + 134q^4 + 343q^5 + 840q^6 + 1971q^7 + 4460q^8$$

$$+ 9810q^9 + 21006q^{10} + 43952q^{11} + 90078q^{12} + 181178q^{13} + 358196q^{14} + 697195q^{15} + 1337468q^{16} + \ldots)$$

$$\tau_2^{3,1}(q) = q^5 \left(1 + 7q + 25q^2 + 84q^3 + 247q^4 + 674q^5 + 1733q^6 + 4252q^7 + 10005q^8 + 22774q^9 + 50306q^{10} + 108276q^{11} + \ldots\right)$$

$$\tau_3^{3,1}(q) = q^7 \left(2 + 11q + 46q^2 + 158q^3 + 486q^4 + 1369q^5 + 3622q^6 + \ldots\right)$$

$$\tau_4^{3,1}(q) = q^9 \left(2 + 13q + 57q^2 + \ldots\right) \tag{C.4}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation [3]

$$\tau_1^{1,3}(q) = q^5 \left(2 + 9q + 29q^2 + 86q^3 + 233q^4 + 591q^5 + 1426q^6 + 3308q^7 + 7408q^8 + 16117q^9 + 34176q^{10} + 70842q^{11} + 143887q^{12} + 286959q^{13} + 562767q^{14} + 1086923q^{15} + \ldots\right)$$

$$\tau_2^{1,3}(q) = q^5 \left(2 + 10q + 39q^2 + 125q^3 + 366q^4 + 990q^5 + 2530q^6 + 6157q^7 + 14414q^8 + 32604q^9 + 71640q^{10} + 153380q^{11} + \ldots\right)$$

$$\tau_3^{1,3}(q) = q^5 \left(1 + 6q + 24q^2 + 87q^3 + 275q^4 + 799q^5 + 2168q^6 + 5570q^7 + 13669q^8 + \ldots\right)$$

$$\tau_4^{1,3}(q) = q^6 \left(2 + 9q + 38q^2 + 135q^3 + 428q^4 + \ldots\right)$$

$$\tau_5^{1,3}(q) = q^7 \left(2 + 11q + \ldots\right) \tag{C.5}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 5]$ and $SU(2)_R$ representation [1]

$$\tau_1^{5,1}(q) = q^7 \left(1 + 7q + 24q^2 + 80q^3 + 228q^4 + 610q^5 + 1533q^6 + 3691q^7 + 8520q^8 + 19063q^9 + 41409q^{10} + 87751q^{11} + 181781q^{12} + 369134q^{13} + 735899q^{14} + \ldots\right)$$

$$\tau_2^{5,1}(q) = q^8 \left(1 + 7q + 26q^2 + 92q^3 + 281q^4 + 791q^5 + 2090q^6 + 5251q^7 + 12618q^8 + 29264q^9 + 65731q^{10} + \ldots\right)$$

$$\tau_3^{5,1}(q) = q^{11} \left(2 + 12q + 55q^2 + 196q^3 + 625q^4 + 1808q^5 + \ldots\right)$$

$$\tau_4^{5,1}(q) = q^{14} \left(2 + 15q + \ldots\right) \tag{C.6}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 3]$ and $SU(2)_R$ representation [3]

$$\tau_1^{3,3}(q) = q^7 \left(2 + 10q + 41q^2 + 127q^3 + 369q^4 + 977q^5 + 2453q^6 + 5856q^7 + 13474q^8 + 29947q^9 + 64743q^{10} + 136433q^{11} + 281245q^{12} + 568184q^{13} + 1127435q^{14} + \ldots\right)$$

$$\tau_2^{3,3}(q) = q^8 \left(3 + 18q + 75q^2 + 252q^3 + 762q^4 + 2111q^5 + 5496q^6 + 13580q^7 + 32188q^8 + 73580q^9 + 163122q^{10} + \ldots\right)$$

$$\tau_3^{3,3}(q) = q^9 \left(1 + 11q + 49q^2 + 189q^3 + 617q^4 + 1841q^5 + 5079q^6 + \ldots\right)$$

$$\tau_4^{3,3}(q) = q^{11} \left(3 + 19q + 84q^2 + \ldots\right) \tag{C.7}$$

- $SO(5)$ Dynkin labels $[n \to \infty, 1]$ and $SU(2)_R$ representation [5]

$$\tau_1^{1,5}(q) = q^9 \left(2 + 10q + 36q^2 + 118q^3 + 335q^4 + 893q^5 + 2237q^6 + 5356q^7 + 12311q^8 + 27406q^9 + 59236q^{10} + 124892q^{11} + \ldots\right)$$

$$\tau_2^{1,5}(q) = q^9 \left(2 + 13q + 54q^2 + 186q^3 + 573q^4 + 1609q^5 + 4237q^6 + 10575q^7 + \ldots\right)$$

$$\tau_3^{1,5}(q) = q^9 \left(2 + 10q + 45q^2 + 161q^3 + 518q^4 + \ldots\right)$$

$$\tau_4^{1,5}(q) = q^9 \left(1 + 6q + 26q^2 + \ldots\right) \tag{C.8}$$

The results listed in this appendix confirm the trend observed in subsection 5.4: The $\tau_\ell^{k,p}(q)$ expansion (5.42) of $G_{n,k,p}(q)$ converges more rapidly at large vales of the second Dynkin label k and small values $SU(2)_R$ spin p.

C.2 $\mathcal{N}_{10d} = 1$ multiplets at $SO(9)$ Dynkin labels $[n \to \infty, x, y, z]$

Also for $\mathcal{N}_{10d} = 1$ multiplets $[\![n \to \infty, x, y, z]\!]$ we would like to list some more $\tau^{x,y,z}_{\ell \leq 5}(q)$ beyond those of subsection 6.2, specifically for Dynkin labels $(x, y, z) = (1, 1, 0), (1, 0, 2), (2, 1, 0), (1, 0, 1), (0, 1, 1), (2, 0, 1), (0, 0, 3), (1, 1, 1)$. We focus on three bosonic families

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 1, 0]$

$$\begin{aligned} \tau_1^{1,1,0}(q) &= q^8\,(1 + 2q + 8q^2 + 19q^3 + 45q^4 + 100q^5 + 217q^6 + 446q^7 + 905q^8 + 1779q^9 \\ &\quad + 3440q^{10} + 6521q^{11} + 12181q^{12} + 22396q^{13} + \ldots) \\ \tau_2^{1,1,0}(q) &= q^9\,(1 + 2q + 9q^2 + 23q^3 + 61q^4 + 143q^5 + 330q^6 + 715q^7 + 1524q^8 + 3128q^9 + \ldots) \\ \tau_3^{1,1,0}(q) &= q^{12}\,(1 + 4q^1 + 16q^2 + 46q^3 + 125q^4 + \ldots) \end{aligned} \tag{C.9}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 0, 2]$

$$\begin{aligned} \tau_1^{1,0,2}(q) &= q^9\,(1 + 4q + 12q^2 + 31q^3 + 75q^4 + 172q^5 + 375q^6 + 791q^7 + 1615q^8 \\ &\quad + 3225q^9 + 6287q^{10} + 12044q^{11} + 22652q^{12} + \ldots) \\ \tau_2^{1,0,2}(q) &= q^{10}\,(1 + 4q + 14q^2 + 39q^3 + 104q^4 + 252q^5 + 587q^6 + 1300q^7 + 2794q^8 + \ldots) \\ \tau_3^{1,0,2}(q) &= q^{13}\,(2 + 8q + 30q^2 + 87q^3 + \ldots) \end{aligned} \tag{C.10}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 2, 1, 0]$

$$\begin{aligned} \tau_1^{2,1,0}(q) &= q^{11}\,(1 + 2q + 9q^2 + 22q^3 + 59q^4 + 132q^5 + 306q^6 + 646q^7 + 1369q^8 + 2756q^9 \\ &\quad + 5514q^{10} + 10682q^{11} + \ldots) \\ \tau_2^{2,1,0}(q) &= q^{12}\,(1 + 2q + 9q^2 + 23q^3 + 63q^4 + 150q^5 + 357q^6 + 791q^7 + 1728q^8 + \ldots) \\ \tau_3^{2,1,0}(q) &= q^{16}\,(1 + 4q + 18q^2 + 51q^3 + \ldots) \end{aligned} \tag{C.11}$$

and five fermionic families of supermultiplets:

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 0, 1]$

$$\begin{aligned} \tau_1^{1,0,1}(q) &= q^6\,(1 + 3q + 7q^2 + 17q^3 + 37q^4 + 79q^5 + 162q^6 + 325q^7 + 635q^8 + 1220q^9 \\ &\quad + 2298q^{10} + 4266q^{11} + 7807q^{12} + 14110q^{13} + 25197q^{14} + 44530q^{15} + \ldots) \\ \tau_2^{1,0,1}(q) &= q^7\,(1 + 3q + 9q^2 + 24q^3 + 57q^4 + 131q^5 + 288q^6 + 610q^7 + 1256q^8 + 2523q^9 \\ &\quad + 4957q^{10} + 9557q^{11} + \ldots) \\ \tau_3^{1,0,1}(q) &= q^{10}\,(2 + 7q + 22q^2 + 61q^3 + 155q^4 + 367q^5 + 835q^6 + \ldots) \\ \tau_4^{1,0,1}(q) &= q^{13}\,(2 + 9q + 31q^2 + \ldots) \end{aligned} \tag{C.12}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 1, 1]$

$$\begin{aligned} \tau_1^{0,1,1}(q) &= q^8\,(1 + 4q + 10q^2 + 27q^3 + 63q^4 + 141q^5 + 302q^6 + 628q^7 + 1264q^8 \\ &\quad + 2494q^9 + 4811q^{10} + 9119q^{11} + 17005q^{12} + 31260q^{13} + \ldots) \\ \tau_2^{0,1,1}(q) &= q^9\,(1 + 5q + 16q^2 + 44q^3 + 113q^4 + 269q^5 + 610q^6 + 1330q^7 \\ &\quad + 2804q^8 + 5748q^9 + \ldots) \\ \tau_3^{0,1,1}(q) &= q^{11}\,(1 + 6q + 19q^2 + 59q^3 + 160q^4 + 404q^5 + \ldots) \\ \tau_4^{0,1,1}(q) &= q^{14}\,(2 + 9q + \ldots) \end{aligned} \tag{C.13}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 2, 0, 1]$

$$\begin{aligned} \tau_1^{2,0,1}(q) &= q^9\,(1 + 3q + 9q^2 + 23q^3 + 55q^4 + 123q^5 + 267q^6 + 556q^7 + 1132q^8 \\ &\quad + 2244q^9 + 4362q^{10} + 8318q^{11} + 15616q^{12} + 28873q^{13} + \dots) \\ \tau_2^{2,0,1}(q) &= q^{10}\,(1 + 3q + 9q^2 + 25q^3 + 63q^4 + 150q^5 + 342q^6 + 749q^7 + 1591q^8 \\ &\quad + 3289q^9 + 6640q^{10} + \dots) \\ \tau_3^{2,0,1}(q) &= q^{14}\,(2 + 8q + 27q^2 + 77q^3 + 204q^4 + \dots) \end{aligned} \tag{C.14}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 0, 0, 3]$

$$\begin{aligned} \tau_1^{0,0,3}(q) &= q^{10}\,(2 + 5q + 15q^2 + 38q^3 + 90q^4 + 201q^5 + 437q^6 + 905q^7 + 1838q^8 \\ &\quad + 3633q^9 + 7038q^{10} + 13374q^{11} + \dots) \\ \tau_2^{0,0,3}(q) &= q^{11}\,(2 + 8q + 25q^2 + 69q^3 + 176q^4 + 418q^5 + 949q^6 + 2069q^7 + \dots) \\ \tau_3^{0,0,3}(q) &= q^{13}\,(3 + 11q + 38q^2 + 109q^3 + \dots) \\ \tau_4^{0,0,3}(q) &= q^{15}\,(1 + \dots) \end{aligned} \tag{C.15}$$

- $SO(9)$ Dynkin labels $[n \to \infty, 1, 1, 1]$

$$\begin{aligned} \tau_1^{1,1,1}(q) &= q^{11}\,(1 + 5q + 16q^2 + 45q^3 + 116q^4 + 276q^5 + 624q^6 + 1358q^7 + 2852q^8 \\ &\quad + 5825q^9 + 11616q^{10} + 22669q^{11} + \dots) \\ \tau_2^{1,1,1}(q) &= q^{12}\,(1 + 5q + 17q^2 + 52q^3 + 142q^4 + 358q^5 + 855q^6 + 1950q^7 + 4279q^8 + \dots) \\ \tau_3^{1,1,1}(q) &= q^{15}\,(1 + 7q + 26q^2 + 84q^3 + 243q^4 + \dots) \end{aligned} \tag{C.16}$$

These results confirm the trend observed in subsection 6.2: The $\tau_\ell^{x,y,z}(q)$ expansion (6.34) of multiplicity generating functions $G_{n,x,y,z}(q)$ converges more quickly at higher values of the Dynkin labels x, y, z.

D Deriving the asymptotic formulae for $\mathcal{N}_{4d} = 1$ multiplicity generating functions

In this appendix, we derive the asymptotic results on multiplicity generating function $G_{n,Q}(q)$ in the limit $n \to \infty$ presented in subsection 4.4.

In what follows, we will exploit the $n \to \infty$ behaviour of objects $T_p(m,k) := \binom{m}{k} - \binom{m}{k-p}$,

$$\begin{aligned} T_{2n+2}(2m+1, m+n+1-k) &\sim \binom{2m+1}{m+n+1-k}\,, \\ T_{2n+2}(2m, m+n-k) &\sim \binom{2m}{m+n-k}\,. \end{aligned} \tag{D.1}$$

assuming that $m, k \geq 0$

D.1 Warm-up: Multiplicities of $[\![2n+1, 0]\!]$ and $[\![2n, 1]\!]$ as $n \to \infty$

In order to get familiar with the asymptotic methods in the $\mathcal{N}_{4d} = 1$ context, we shall first of all discuss the large spin regime of supermultiplets with $U(1)_R$ neutral Clifford vacuum.

The multiplicity generating function for the representation $[\![2n+1, 0]\!]$ can be written as

$$G_{2n+1,0}(q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty} \mathfrak{M}_{[\![2n+1,0]\!]}(m, -p-1, k; q) + \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \mathfrak{M}_{[\![2n+1,0]\!]}(p, p, k; q)\,, \tag{D.2}$$

where the function $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ and $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ are defined in (4.59) and (4.60) and, as $n\to\infty$,

$$\mathfrak{M}_{[\![2n+1,0]\!]}(m,p,k;q) \sim (-1)^{-m-p}\left[F^{\rm NS}_{k,p}(q)\binom{m-p}{2m+1}\binom{2m+1}{m+n+1-k}\right.$$
$$\left. + F^{\rm R}_{k,p}(q)\binom{m-p}{2m}\binom{2m}{m+n-k}\right] . \tag{D.3}$$

Note that the binomial coefficient $\binom{\alpha}{\beta}$ increases as β increases from 0 to $\lfloor \alpha/2\rfloor$ and then decreases as β increases from $\lfloor\alpha/2\rfloor+1$ to α.

Observe that $\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q)$ is sharply peaked near $(m,p,k)=(0,0,n)$ for n large. Therefore, the dominant contribution to the first set of summations in (D.2) comes from

$$\begin{aligned}
&\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q)\\
&\sim \sum_{m=0}^{\lceil\epsilon_1\rceil}\sum_{p=0}^{\lceil\epsilon_2\rceil}\sum_{k=\lfloor n(1-\epsilon_3)\rfloor}^{\lceil n(1+\epsilon_3)\rceil}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q) \qquad \text{any } \epsilon_1,\epsilon_2,\epsilon_3>0, \quad n\to\infty\\
&\sim \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,n+\delta;q), \qquad n\to\infty .
\end{aligned} \tag{D.4}$$

In the limit of large k, we can use asymptotic formulae (4.23) and (4.28) for $F^{\rm NS}_{k,p}(q)$ and $F^{\rm R}_{k,p}(q)$. The summation over δ from $-\infty$ to ∞ can be readily computed using the fact that

$$\begin{aligned}
\sum_{\delta=-\infty}^{\infty} q^{\delta}\binom{2m}{m-\delta} &= \sum_{\delta=-m}^{m} q^{\delta}\binom{2m}{m-\delta} = q^{-m}(1+q)^{2m} ,\\
\sum_{\delta=-\infty}^{\infty} q^{\delta}\binom{2m+1}{m-\delta+1} &= \sum_{\delta=-(m+1)}^{m+1} q^{\delta}\binom{2m+1}{m-\delta+1} = q^{-m}(1+q)^{2m+1} .
\end{aligned} \tag{D.5}$$

Next, the summation over m from 0 to ∞ can be computed using the following identities:

$$\begin{aligned}
\sum_{m=0}^{\infty}(-q)^{-m}(1+q)^{2m}\binom{1+m+p}{2m} &= (-q)^{-p-1}\frac{1-q^{2p+3}}{1-q} ,\\
\sum_{m=0}^{\infty}(-q)^{-m}(1+q)^{2m+1}\binom{1+m+p}{1+2m} &= (-q)^{-p}\frac{1-q^{2p+2}}{1-q} .
\end{aligned} \tag{D.6}$$

Thus, from (D.4), we find that

$$\begin{aligned}
\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(m,-p-1,k;q) &= \frac{(1-q)^2 q^{n-\frac{1}{2}}}{2(q,q)_\infty^6}\\
&\times\left\{u_1(\sqrt{q})\vartheta_2(1,q)^2-\left[u_2(\sqrt{q})\vartheta_3(1,q)^2-u_2(-\sqrt{q})\vartheta_4(1,q)^2\right]\right\} ,
\end{aligned} \tag{D.7}$$

where the functions $u_1(q)$ and $u_2(q)$ are defined as follows:

$$\begin{aligned}
u_1(q) &= \sum_{p=0}^{\infty} q^{2(p+\frac{3}{2})^2}\frac{1-q^{4p+6}}{(1+q^{2p+2})(1+q^{2p+4})} ,\\
u_2(q) &= \sum_{p=0}^{\infty} q^{2(p+1)^2}\frac{1-q^{4p+4}}{(1+q^{2p+1})(1+q^{2p+3})} .
\end{aligned} \tag{D.8}$$

It remains unclear whether $u_1(q)$ and $u_2(q)$ can be written in terms of known functions (if this is useful at all). In practice, it is easy to compute the power series $u_1(q)$ and $u_2(q)$ up to a high order in q. Moreover, their asymptotic formulae can be easily derived in the limit $q \to 0$. We shall come back to this point later.

Let us now examine the second set of summations in (D.2). The function $\mathfrak{M}_{[\![2n+1,0]\!]}(p,p,k;q)$ is sharply peaked near $(p,k) = (0,n)$ for large n. Thus,

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\mathfrak{M}_{[\![2n+1,0]\!]}(p,p,k;q) &\sim \mathfrak{M}_{[\![2n+1,0]\!]}(0,0,n;q)\ , \qquad n\to\infty \\ &= \frac{1}{4(q;q)_\infty^6}\frac{(1-q)^3}{1+q}q^{n-\frac{1}{4}}\vartheta_2(1,q)^2\ . \end{aligned} \tag{D.9}$$

From (D.2), we simply add (D.4) and (D.9) together and obtain the expression (4.72) for $Q_{2n+1,0}$, in agreement with the stable pattern in table 3.

From recurrence relation (4.35) for $G_{n,Q}$, the asymptotic behaviour of multiplicity generating functions $U(1)_R$ charge $Q=1$ is given by

$$G_{2n,1}(q) = \frac{1}{2}\left[F^{\rm NS}_{n,0}(q) - G_{2n-1,0}(q) - G_{2n+1,0}(q)\right]\ . \tag{D.10}$$

Using the asymptotics $G_{2n-1,Q} \sim q^{-1}G_{2n+1,Q}$ as well as (4.72) for $G_{2n+1,Q}$ and (4.23) for $F^{\rm NS}_{n,0}$, we arrive at (4.73). This also agrees with the stable pattern tabulated in appendix B.1.

D.2 Multiplicities of $[\![2n+1,2Q]\!]$ and $[\![2n,2Q+1]\!]$ as $n\to\infty$, $Q=\mathcal{O}(1)$

This subsection generalizes the asymptotic results from the $Q=0$ (or $Q=1$) sector to generic $U(1)_R$ charges. The multiplicity generating function for $[\![2n+1,2Q]\!]$ can be written as

$$\begin{aligned}G_{2n+1,2Q}(q) = \sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\Bigg[&\sum_{p=0}^{\infty}\Big\{\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q) + \mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big\} \\ &+\sum_{p=0}^{Q-1}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\Bigg]\ .\end{aligned} \tag{D.11}$$

where the $\mathfrak{M}_{[\![2n+1,2Q]\!]}$ function follows the following $n\to\infty$ behaviour:

$$\begin{aligned}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,p,k;q) = (-1)^{Q-m-p}\Bigg[&F^{\rm NS}_{k,p}(q)\binom{Q+m-p}{2m+1}\binom{2m+1}{m+n+1-k} \\ &+F^{\rm R}_{k,p}(q)\binom{Q+m-p}{2m}\binom{2m}{m+n-k}\Bigg]\end{aligned} \tag{D.12}$$

The dominant contribution to $G_{2n+1,2Q}(q)$ comes from

$$\begin{aligned}G_{2n+1,2Q}(q) &\sim \sum_{m=0}^{\infty}\sum_{p=0}^{\lceil\epsilon_2\rceil}\sum_{k=\lfloor n(1-\epsilon_1)\rfloor}^{\lceil n(1+\epsilon_1)\rceil}\Big[\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,k;q)+\mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,k;q)\Big] \\ &\quad+\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{k=\lfloor n(1-\epsilon_1)\rfloor}^{\lceil n(1+\epsilon_1)\rceil}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,k;q)\ , \qquad \epsilon_1,\epsilon_2>0\ , n\to\infty \\ &\sim\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\Big[\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,n+\delta;q)+\mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,n+\delta;q)\Big]\end{aligned}$$

$$+\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,n+\delta;q)\ , \qquad n\to\infty\ . \tag{D.13}$$

The first set of summations can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,-p-1,n+\delta;q)=\frac{(1-q)^2q^{n-Q-\frac{1}{2}}}{2(q;q)_\infty^6}$$
$$\times\left\{u_1(\sqrt{q},Q)\vartheta_2(1,q)^2-\left[u_2(\sqrt{q},Q)\vartheta_3(1,q)^2-u_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}\ , \tag{D.14}$$

where

$$u_1(q,Q)=\sum_{p=0}^{\infty}q^{2\left(p+\frac{3}{2}\right)^2}\frac{1-q^{4p+4Q+6}}{(1+q^{2p+2})(1+q^{2p+4})}\ ,$$
$$u_2(q,Q)=\sum_{p=0}^{\infty}q^{2(p+1)^2}\frac{1-q^{4p+4Q+4}}{(1+q^{2p+1})(1+q^{2p+3})}\ . \tag{D.15}$$

The next set of summations in (4.62) can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m+p,p,n+\delta;q)=\frac{(-1)^Q(1-q)^3q^{n-\frac{3}{2}}}{2(q;q)_\infty^6}$$
$$\times\ \left\{v_1(\sqrt{q},Q)\vartheta_2(1,q)^2+\left[v_2(\sqrt{q},Q)\vartheta_3(1,q)^2-v_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}\ , \tag{D.16}$$

where[38]

$$v_1(q,Q)=\sum_{p=0}^{\lfloor Q/2\rfloor}\frac{q^{2\left(p-\frac{1}{2}\right)^2}(1+q^2)^{2p}}{(1+q^{2p-2})(1+q^{2p})}\binom{Q}{2p}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p-Q\\ p+1/2,\ p+1\end{matrix};\ \frac{(1+q)^2}{4q}\right]\ ,$$
$$v_2(q,Q)=\sum_{p=0}^{\lfloor Q/2\rfloor}\frac{(1+q)q^{2p^2}(1+q^2)^{2p}}{(1+q^{2p-1})(1+q^{2p+1})}\binom{Q}{2p+1}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p+1-Q\\ p+1,\ p+3/2\end{matrix};\ \frac{(1+q)^2}{4q}\right]\ , \tag{D.18}$$

The last set of summations in (4.62) can be evaluated as follows:

$$\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\sum_{\delta=-\infty}^{\infty}\mathfrak{M}_{[\![2n+1,2Q]\!]}(m,m+p+1,n+\delta;q)=\frac{(-1)^Q(1-q)^3q^{n-\frac{7}{4}}}{2(q;q)_\infty^6}$$
$$\times\ \left\{w_1(\sqrt{q},Q)\vartheta_2(1,q)^2+q^{\frac{9}{4}}\left[w_2(\sqrt{q},Q)\vartheta_3(1,q)^2-w_2(-\sqrt{q},Q)\vartheta_4(1,q)^2\right]\right\}\ , \tag{D.19}$$

where

$$w_1(q,Q)=\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{1+2(1+m+p)^2-2m}\left(1+q^2\right)^{2m}\binom{Q-1-p}{2m}}{\left(1+q^{2(m+p)}\right)\left(1+q^{2(1+m+p)}\right)}\ ,$$
$$w_2(q,Q)=q^{-\frac{9}{2}}\sum_{m=0}^{\infty}\sum_{p=0}^{Q-1}\frac{(-1)^{p+1}q^{2\left(m+p+\frac{3}{2}\right)^2-2m}\left(1+q^2\right)^{2m+1}\binom{Q-1-p}{1+2m}}{\left(1+q^{1+2m+2p}\right)\left(1+q^{3+2m+2p}\right)}\ . \tag{D.20}$$

[38]Upon obtaining the hypergeometric functions, we make use of the following identities for $p\geq 0$:

$$\sum_{m=0}^{Q}(-1)^mq^{-m}(1+q)^{2m}\binom{Q+m}{2p+2m}=\binom{Q}{2p}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p-Q\\ p+1/2,\ p+1\end{matrix};\ \frac{(1+q)^2}{4q}\right]\ ,$$
$$\sum_{m=0}^{Q}(-1)^mq^{-m}(1+q)^{2m+1}\binom{Q+m}{1+2p+2m}=\binom{Q}{2p+1}{}_3F_2\left[\begin{matrix}1,\ Q+1,\ 2p+1-Q\\ p+1,\ p+3/2\end{matrix};\ \frac{(1+q)^2}{4q}\right]\ . \tag{D.17}$$

Combining the three sets of summations into (4.62), we have

$$G_{2n+1,2Q}(q)=\frac{(1-q)^2q^n}{2q^{\frac{3}{2}}(q;q)_\infty^6}\Big\{\vartheta_2(1,q)^2\Big[q^{1-Q}u_1(\sqrt{q},Q)+(-1)^Q(1-q)(v_1(\sqrt{q},Q)+q^{-1/4}w_1(\sqrt{q},Q))\Big]$$
$$+\vartheta_3(1,q)^2\Big[-q^{1-Q}u_2(\sqrt{q},Q)+(-1)^Q(1-q)(v_2(\sqrt{q},Q)+q^2w_2(\sqrt{q},Q))\Big]$$
$$+\vartheta_4(1,q)^2\Big[q^{1-Q}u_2(-\sqrt{q},Q)-(-1)^Q(1-q)(v_2(-\sqrt{q},Q)+q^2w_2(-\sqrt{q},Q))\Big]\Big\} \tag{D.21}$$

which exactly (4.63) with the definition (4.65) for the function $\mathcal{F}(q,Q)$ in the curly brackets. Note that this formula reproduces (4.72) when $Q=0$.

This allows to quickly infer asymptotic $[\![2n,2Q+1]\!]$ multiplicities through the recursion (4.38) and the asymptotic relations $G_{2n+2,2Q+1}(q)\sim qG_{2n,2Q+1}(q)$ as $n\to\infty$:

$$G_{2n,2Q+1}(q)\sim\frac{1}{1+q}\left[F^{\rm R}_{n,Q+1}(q)-G_{2n+1,2Q}(q)-G_{2n+1,2Q+2}(q)\right] \tag{D.22}$$

The asymptotic formula (4.28) for $F^{\rm R}_{n,Q+1}(q)$ and the definition (4.65) for the function $\mathcal{F}(q,Q)$ then leads to (4.64).

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