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Abstract

We derive the propagators for HS master fields in the anti-de Sitter space of arbitrary dimension. A method is developed to construct the propagators directly without solving any differential equations. The use of the ambient space, where AdS is represented as a hyperboloid and its conformal boundary as a projective light cone, simplifies the approach and makes a direct contact between boundary-to-bulk propagators and two-point functions of conserved currents.

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1. Introduction

A canonical playground for the AdS/CFT correspondence [1–3] $\mathcal{N} = 4$ SYM versus superstring theory on $AdS_5 \times S^5$ is quite complicated. One wishes to have a simpler model that captures all essential features of the AdS/CFT paradigm. At the same time, it seems natural and technically appealing to look for the AdS duals of the simplest CFTs, the free ones, rather than strongly coupled. In the OPE of two free conformal fields, one finds an infinite set of conserved currents of increasing tensor rank, suggesting the AdS dual be a theory of gauge, and therefore massless fields of all spins. Such theories, called higher spin (HS) theories, do exist in arbitrary dimension [4–9], see [10–12] for reviews. The conjectures that relate them to various CFTs, which are not always free, have been proposed in [13–16], see also [17–23].

In addition to being dual to 'almost' free CFTs, the simplest HS theories have a nondegenerate spectrum of states and they are believed to be dual to vector models, which do not have long trace operators. Therefore HS theories provide a promising model for AdS/CFT, in which one can expect to prove everything. However, it was not until the main breakthroughs [24, 25] in AdS_4/CFT^3 and [26, 27] in AdS_3/CFT^2 that the topic attracted considerable attention.

One of the goals of the present research is to pave the way for a search for the AdS/CFT dual to d-dimensional bosonic HS field theory [9]. The AdS/CFT analysis in terms of the

correlation functions requires among other things explicit form of the bulk-to-boundary propagators. We develop a method that allows us to find boundary-to-bulk propagators for all HS fields without solving any differential equations at all.

HS gauge theory has its dynamical content totally encoded in terms of some *master* fields. These are W gauge connection 1-form and B HS curvature zero-form. Physical information can be equivalently extracted from either of the two fields, pretty much as in the case of gravity where the degrees of freedom reside in either the metric (gauge field) or the Riemann tensor (curvature field). At the nonlinear level, however, the perturbative sector of HS connections is way more involved as compared with the curvature B-sector. Practically, it makes the HS-curvature analysis sometimes more preferable to the HS connections. Particularly, the AdS/CFT correspondence test carried out in [24, 25] heavily rests upon the B boundary-to-bulk propagator calculation.

The aim of this paper is to explicitly derive W- and B-propagators for all spins in arbitrary dimension. The straightforward approach based on solving the equations of motion (e.o.m.) encounters formidable technical problems though. It calls for some more refined methods to push the matter through. One of the results of our paper is the method that effectively allows us to build B-propagators by purely algebraic means. The method that we call self-similarity virtually represents some motley combination of three ingredients. These are the ambient, the unfolding and the star-product. In its essence, it makes it possible to construct a generating function out of the spin-s Weyl tensor which uniquely defines the full B-propagator via a simple integral map. We have applied this machinery to explicitly find all the boundary-to-bulk B-propagators.

The dynamics of HS fields is described in terms of differential equations and algebraic constraints that set the fields on-shell. The algebraic constraints are quite complicated to work with in perturbation theory. Fortunately, in lower dimensions the most complicated part of the algebraic constraints can be easily resolved by introducing twistor-like variables instead of vector-like. It is because of these simplifications that the computations of three-point functions in 4D HS theory are quite simple, [25]. Unfortunately, no analogue of this twistor resolution is known in arbitrary dimension.

The paper is organized as follows. In section 2, we present a simple route from the free conformal scalar on the boundary to the HS theory in the bulk, the goal being to show that it is the HS master connection W that couples naturally to the currents built of scalar fields. In section 3 we briefly review the ambient approach, while a more specific discussion is presented in section 4. The linearized Vasiliev HS equations [9] are reviewed in section 5. Boundary-to-bulk propagators for master connection and field strength are derived in sections 6 and 7, respectively, where two-point functions are also discussed. Conclusions are given in section 8.

2. From boundary to higher spin fields in the bulk

Here, we would like to present a short path to the gauge fields introduced by Vasiliev in [28] and then used in [9] to construct a classical theory of interacting HS fields in AdS_{d+1}. One starts with a $u(N)$, $so(N)$,...multiplet $\phi^I(x)$ of free scalar fields in the flat space of dimension d. As is well known, having two $\phi^I(x)$ and s derivatives, one can construct a conserved current that is a traceless tensor of any rank $s = 1, 2, 3, \ldots,$[3] the trace over the vector indices

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[3] This is true for $u(N)$. In the case of $so(N)$ there exist currents of even ranks only. Indices $\mu, \nu, \ldots$ run over d values. A group of s (anti)symmetric or to be (anti)symmetrized indices $\mu_1, \ldots, \mu_s$ is denoted by $(\mu[s])$ $\mu(s)$.

being implicit,

$$j_{\mu(s)} = \phi(x)(\overleftarrow{\partial_\mu} - \overrightarrow{\partial_\mu})^s \phi(x) - \text{traces}, \qquad \partial^\nu j_{\nu\mu(s-1)} = 0. \tag{1}$$

For $s = 2$ one recognizes the usual stress–energy tensor. The conformal dimension of a spin-s current is $2\Delta + s$, where $\Delta = (d-2)/2$ is the dimension of a free scalar. These currents form a representation of the conformal algebra $so(d, 2)$, which means that the charges are labeled by certain modules of $so(d, 2)$. Indeed, the conservation condition

$$\partial^\mu j_\mu^s = 0, \qquad j_\mu^s = j_\mu{}^{\nu(s-1)} K_{\nu(s-1)}(x), \tag{2}$$

for yet unknown traceless $K_{\nu(s-1)}(x)$ implies that

$$\text{the traceless part of } [\partial^\nu K^{\nu(s-1)}] = 0, \tag{3}$$

the latter equation being (i) conformally covariant, (ii) overdetermined, as such admitting a finite number of solutions. Its solutions are called the conformal Killing tensors (CKTs), which generalize conformal Killing vectors. Any given CKT allows one to define a conserved charge in a standard way as an integral of a $(d-1)$-form

$$Q = \int \Omega^s, \qquad \Omega^s = j_\mu^s \epsilon^\mu{}_{\sigma[d-1]} \mathrm{d}x^\sigma \wedge \cdots \wedge \mathrm{d}x^\sigma. \tag{4}$$

(i)+(ii) implies that CKTs are just $so(d, 2)$-tensors, although this is not easy to see, [29]. Namely, various components in the Taylor expansion of $K_{\nu(s-1)}(x)$ can be organized into an irreducible $so(d, 2)$-tensor[4] $K^{A(s-1),B(s-1)}$ that has the symmetry of a two-row rectangular Young diagram of length $(s-1)$

$$\boxed{\begin{array}{|c|c|c|} \hline & s-1 & \\ \hline & s-1 & \\ \hline \end{array}}. \tag{5}$$

The map from the explicitly conformal $K^{A(s-1),B(s-1)}$ to the hidden conformal CKT $K_{\nu(s-1)}(x)$ reads [29]

$$K_{\nu(s-1)}(x) = M_\nu^{AB} ... M_\nu^{AB} K_{A(s-1),B(s-1)}, \tag{6}$$

$$M_\nu^{AB} = X^A \partial_\nu X^B - X^B \partial_\nu X^A, \qquad X^A = \{1, x^{\mathsf{a}}, -x^{\mathsf{a}} x_{\mathsf{a}}/2\}, \tag{7}$$

where $X^A(x)$ is a Poincaré slice of the zero cone $X^A X_A = 0$ (see section 3 below). The generators M^{AB} decompose into dilatation $D = M^{+-}$, translations $P^{\mathsf{a}} = M^{+\mathsf{a}}$, Lorentz rotations $L^{\mathsf{ab}} = M^{\mathsf{ab}}$ and conformal boosts $K^{\mathsf{a}} = M^{-\mathsf{a}}$. In particular, it is evident that a Killing tensor gets decomposed into a product of Killing vectors. The Killing vectors associated with P^{a}, D, L^{ab} and K^{a} read

$$P_\nu^{\mathsf{a}} = \delta_\nu^{\mathsf{a}}, \qquad D_\nu = x_\nu, \tag{8}$$

$$L_\nu^{\mathsf{ab}} = \delta_\nu^{\mathsf{a}} x^{\mathsf{b}} - \delta_\nu^{\mathsf{b}} x^{\mathsf{a}}, \qquad K_\nu^{\mathsf{a}} = 2x_\nu x^{\mathsf{a}} - \delta_\nu^{\mathsf{a}} x^{\mathsf{m}} x_{\mathsf{m}}. \tag{9}$$

Therefore, the on-shell closed form Ω, defining the whole conformal multiplet of the charges associated with $j_{\mu(s)}$, is naturally a carrier of the label of a Killing tensor,

$$\Omega = \Omega^{A(s-1),B(s-1)} K_{A(s-1),B(s-1)}. \tag{10}$$

The last but one step is to couple currents $j_{\mu(s)}$ to some external fields, $\phi^{\mu(s)}$, the Fradkin–Tseytlin fields [30], which are gauge fields as the currents are conserved,

$$\Delta S = \int \phi^{\mu(s)} j_{\mu(s)}, \qquad \delta\phi^{\mu(s)} = \partial^\mu \xi^{\mu(s-1)} - \text{traces}. \tag{11}$$

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[4] Indices $A, \ldots$ run over $d + 2$ values $\mathsf{a}, +, -$, where $\mathsf{a}, \mathsf{b}, \ldots$ are fiber $so(d - 1, 1)$ indices. $so(d, 2)$-irreducibility means that (1) the tensor indices have a symmetry of some Young diagram, (2) the tensor is traceless.

Such a coupling, however, involves only one component of the whole $so(d, 2)$-multiplet, the one associated with the $P^a...P^a$ part of the conformal Killing tensor $K_{A(s-1),B(s-1)}$.

It is now natural to introduce a gauge field that *incorporates* all the components of the multiplet; such conformal fields were considered in [31]:

$$\Delta S = \int \Omega_{A(s-1),B(s-1)} \wedge W^{A(s-1),B(s-1)}, \qquad \delta W^{A(s-1),B(s-1)} = D\xi^{A(s-1),B(s-1)}. \tag{12}$$

It has to be a 1-form since Ω is a $(d - 1)$-form and the gauge parameter ξ is a zero-form

$$W^{A(s-1),B(s-1)} \equiv W_\mu^{A(s-1),B(s-1)} \, dx^\mu \tag{13}$$

that take values in the same irreducible $so(d, 2)$-module as Ω does. D is a flat covariant derivative of $so(d, 2)$. Then, $\phi^{\mu(s)}$ is associated with one particular component of W,

$$\phi_\mu{}^{a(s-1)} = \text{totally symmetric and traceless part of } W_\mu{}^{a(s-1),++\cdots+}. \tag{14}$$

Let us note that the current associated with $K^{a(s-1),++\cdots+}$ plays a distinguished role, of course, as it is the highest weight current in the multiplet and all other currents can be thought of as its descendants. However, the importance of having some symmetry manifest rather than playing with a small part of it should not be underestimated.

For the case $s = 2$, i.e. for the energy–momentum tensor $j_{\mu\mu}$, this tells us that $W = W_\mu^{A,B} \, dx^\mu$ is a usual Yang–Mills connection of the conformal algebra and it couples naturally to all currents that can be built out of $j_{\mu\mu}$ with Killing vectors associated with the generators D, P^a, L^{ab}, K^a of the conformal group.

The last step is to interpret W in the spirit of AdS/CFT correspondence as a boundary value of a bulk field, which might be called the HS gauge connection [28]. Taking into account that there are currents of all spins (at least of even ranks) in the theory of free scalars, the dual theory is expected to be a gauge theory of all connections of type (13) and it does exist [9].

3. Bulk and boundary in ambient space

A relation between theories in anti-de Sitter space and their conformal partners should become more transparent and less technically involved if both types of theories are put into the same space where the conformal symmetries/anti-de Sitter global symmetries act geometrically. This is the ambient space, a flat pseudo-Euclidean space $\mathbb{R}^{d,2}$; see [32–45] for original works and developments. Below we introduce the notations we need rather than reviewing the ambient approach.

Bulk. The anti-de Sitter space AdS_{d+1} is understood by definition as the hyperboloid

$$X^A X_A \equiv X \cdot X = -R^2 \tag{15}$$

and we put $R = 1$ for simplicity. It is obvious that any $SO(d, 2)$ rotation $\Lambda^A{}_B$ preserves the hyperboloid, so that the representation of $SO(d, 2)$ is simply $\rho(\Lambda)X = \Lambda^A{}_B X^B$.

A symmetric tensor field of the Lorentz algebra $so(d, 1)$ is defined on AdS as a tensor field $\phi^{A(s)}(X)$ satisfying $X_B \phi^{BA(s-2)} = 0$ with the latter condition properly reducing the number of independent components. So the local Lorentz algebra at X is defined as the stability algebra of X. One can assume that the fields live on $X^2 = -1$ or extend them in the radial direction by imposing certain conditions, e.g. homogeneity, see [45] for recent progress.

The Lorentz covariant derivative is defined as

$$D_M \phi^{A(s)} = G^N{}_M G^A{}_B...G^A{}_B \partial_N \phi^{B(s)}, \qquad G_{AB} = \eta^{AB} + X^A X^B, \tag{16}$$

where G is a projector onto X-transverse subspace in the tangent space and it brings the indices back to the Lorentz subspace. G also serves as the ambient realization of the Lorentz metric

and as a vielbein. Despite the unusual from of D_M, one can verify that it amounts to the usual Lorentz covariant derivative in any local coordinates. A convenient parameterization of AdS is given by Poincaré coordinates

$$X^A = \frac{1}{x^0}(1, x^{\mathsf{a}}, -x^{\mathsf{m}}x_{\mathsf{m}}/2 - (x^0)^2/2). \tag{17}$$

Boundary. The conformal boundary of AdS is understood as a projective light cone:

$$Z^A Z_A = 0, \qquad Z^A \sim \lambda Z^A, \qquad \lambda \neq 0. \tag{18}$$

The fact that Z is null allows one to impose an additional factorization condition. In order to work with the equivalence relation effectively, one may choose a gauge, which can be imposed with the help of an auxiliary vector V. A *convenient* gauge $V \cdot Z = 1$ with $V^A = \delta^A_-$ leads to the Poincaré slice of the cone

$$Z^A = (1, z^{\mathsf{a}}, -z^2/2). \tag{19}$$

Then $SO(d, 2)$ becomes acting as $\rho(\Lambda)Z = \Lambda^A{}_B Z^B/(V\Lambda Z)$.

A symmetric tensor field $T^{A(s)}(Z)$ is a conformal quasi-primary if (i) it is homogeneous of some degree δ, $T(\lambda Z) = \lambda^{-\delta} T(Z)$, which is the conformal weight, (ii) if it is irreducible, i.e. it is traceless, (iii) if it is transverse to the cone, i.e. $Z_B T^{BA(s-1)} = 0$; (iv) it is defined modulo gauge transformations $\delta T^{A(s)} = Z^A \xi^{A(s-1)}$. (ii)+(iii)+(iv) reduce the number of independent components to that of an irreducible $so(d-1, 1)$ tensor. Again one might wish to extend the fields off the cone by imposing further restrictions on Z-dependence. (ii)+(iii)+(iv) can be encoded by contracting the indices with a polarization vector η^A, such that $\eta \cdot \eta = 0, Z \cdot \eta = 0$, $\eta \sim \eta + Z$. Various conformal structures, e.g. the ones appearing in the correlators, can be effectively written in the ambient space [43].

Let us note that up to Fourier transform and the scale condition, the tensor fields on AdS are just massive fields in $\mathbb{R}^{d,2}$, for which AdS is a mass-shell, while fields on the boundary are just massless fields in $\mathbb{R}^{d,2}$, for which the cone is a mass-shell and the extra equivalence relations are just usual gauge symmetries of massless fields.

4. Geometry of the boundary-to-bulk problem in ambient space

It is useful to list, see table 1 and figure 1 below, all the variables that are relevant for the boundary-to-bulk problem where a source field on the boundary is a totally symmetric traceless tensor, which as mentioned above can be encoded with the help of a null polarization vector η, so we will consider polynomials in η instead.

To perform computations, it is important to have explicitly all the derivatives of the quantities given in table 1; fortunately, these are closed on themselves and are given below (we omit the reference point superscript, e.g. just P_A instead of ${}^X P_A$):

$$D_A(XZ) = (XZ)P_A, \quad D_A P_B = G_{AB} - P_A P_B, \quad D_A \xi_B = -\xi_A P_B,$$
$$D_M G_{AB} = 0, \quad D_M X^A = 0. \tag{20}$$

Simplest boundary-to-bulk propagators

To get the feeling that the ambient approach makes things simpler, let us consider (1) propagators for scalars and (2) totally symmetric fields.
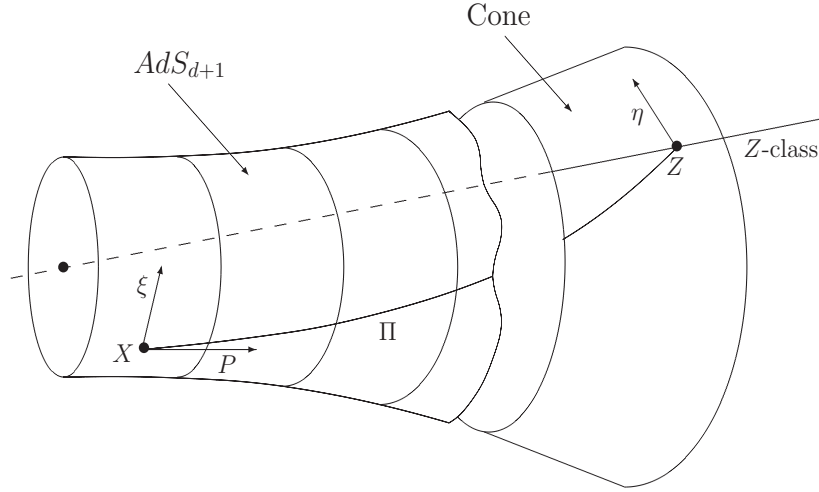
Figure 1. Geometry of boundary-to-bulk problem in the ambient space.

Table 1. Relevant geometric quantities for the boundary-to-bulk problem.

Quantity	Description
X^A	A point in the bulk, i.e. on the hyperboloid, $X^2 = -1$.
Z^A	A point on the boundary, i.e. on the cone, $Z^2 = 0$.
(XZ)	The 'geodesic distance' between X and Z.
η^A	Polarization vector on the boundary at point Z, i.e. $\eta \cdot \eta = 0$, $\eta \cdot Z = 0$, $\eta \sim \eta + Z$.
$^X G_{AA} = \eta_{AA} + X_A X_A$	Induced Lorentz metric on AdS, $G_{AB} X^B = 0$. It also plays the role of the frame field $E^A_{\underline{M}} = G^A_{\underline{M}}$ in the ambient space.
$^X P_A = X_A + Z_A/(XZ)$	A vector that is tangent to the 'geodesic' connecting X with a boundary point Z. It is a 'wave-vector' of a plane-wave toward Z. It respects the equivalence class of Z.
$\Pi(X,Z)^A{}_B = \delta^A{}_B - \dfrac{Z^A X_B}{(XZ)}$	A parallel transport tensor, that propagates tensor indices from the boundary point Z to the bulk point X. It respects the equivalence classes of Z and η.
$^X \xi^A = \Pi(X,Z)^A{}_B \eta^B$	The polarization vector η that is parallel transported to the bulk point X. In addition to being tangent $\xi \cdot X = 0$ and null $\xi \cdot \xi = 0$, it is also orthogonal to the 'wave-vector' P, $\xi \cdot P = 0$.

1. *Scalar b-to-b*. From the pioneering AdS/CFT works [2, 3, 46], the boundary-to-bulk propagator for a weight δ scalar is up to normalization simply

$$K_\delta(X|Z) = \frac{1}{(X \cdot Z)^\delta}, \qquad (D^2 - \delta(\delta - d))K_\delta(X|Z) = 0,$$

$$-\frac{1}{2}(XZ)^{-1} = \frac{x^0}{(x^0)^2 + (x-z)^2} \quad \text{(in Poincaré coordinates)}; \qquad (21)$$

this is so because $\partial_C \partial^C K_\delta \sim Z^2 = 0$.

2. *Spin-s b-to-b*. Starting from spin-1 the boundary-to-bulk propagators become more and more complicated in intrinsic coordinates [47–49]. A totally symmetric spin-*s* field, whose boundary value is the Fradkin–Tseytlin field (11) that couples to the current $j_{\mu(s)}$, can be

described [50] by the Fronsdal field $\phi^{A(s)}(X)$ that obeys

$$(D^2 - m^2)\phi^{A(s)} - sD^A D_M \phi^{A(s-1)M} + \frac{s(s-1)}{2}\left(D^A D^A - 2G^{AA}\right)\phi^{A(s-2)C}{}_C = 0,$$

$$m^2 = E(E-d) - s \qquad G_{BB}G_{BB}\phi^{A(s-4)B(4)} \equiv 0,$$

$$\delta\phi^{A(s)}(X) = D^A\xi^{A(s-1)}, \qquad G_{BB}\xi^{A(s-3)BB} \equiv 0, \tag{22}$$

where $E = 2\Delta + s$ is the lowest energy of the field [37]. It coincides with the dimension of the spin-s current $j_{\mu(s)}$ (1). Such a gauge field with somewhat strange double-trace constraints comes naturally as a part of the HS connection [28]. A propagator for a spin-s field was proposed in [51]; when slightly refined, it reads

$$K_\delta(X|Z,\eta)^{A(s)} = \frac{1}{(XZ)^\delta}\xi^{A(s)}, \qquad G_{BB}K_\delta^{A(s-2)BB} = 0,$$

$$D_M K_\delta^{A(s-1)M} = 0, \qquad (D^2 - \delta(\delta-d)+s)K_\delta^{A(s)} = 0, \tag{23}$$

where the above conditions are satisfied for any δ, but only for $\delta = 2\Delta + s$ K_δ does it become a propagator for the Fronsdal field we need. Equation (23) is the lowest part of the HS master field W that is a part of the Vasiliev formulation [9].

Boundary limit prescription

Within the context of AdS/CFT, it is important to know the limit of all geometric quantities introduced above. There are two natural prescriptions. The first is to recover X^2 factors, which were dropped as $X^2 = -1$, and then try to take $X^2 \to 0$. This is more complicated, however. Second is to use the Poincaré coordinates' experience. Naively, the fact that X grows near the boundary as φ^{-1}, where $\varphi \sim z$ is a defining function of the conformal boundary, suggests the leading order

$$P^A \to X^A, \qquad G^{AA} \to X^A X^A, \qquad \Pi(X,Z) \to \Pi(\overline{X}, Z), \tag{24}$$

where $\Pi(X,Z)$ is formally unchanged, but X gets replaced by the Poincaré slice $\overline{X}$, (19), of the cone, $\overline{X} = \lim zX$.

As is well known [52], given a boundary-to-bulk propagator K_δ, which tends to $(x^0)^{d-\delta}\delta(x-z)$, the coefficient of the second asymptotic $(x^0)^\delta$ is directly proportional to the two-point function. From

$$K_\delta = \frac{1}{(XZ)^\delta} = \frac{(x^0)^\delta}{(\overline{X}\cdot Z)^\delta}\sum_k \frac{\Gamma[\delta+k]}{\Gamma[\delta]}\frac{(-)^k (x^0)^{2k}}{(-2\overline{X}\cdot Z)^k} \tag{25}$$

one observes that this is the coefficient of $1/(\overline{X}\cdot Z)^\delta$, i.e. the two-point functions up to some numerical, but still important factors are obtained just by changing the meaning of X to be that of a point on the cone.

With the above prescription one sees that (21) immediately gives $\langle\phi(X)\phi(Z)\rangle = (XZ)^{-\delta}$. Introducing an auxiliary polarization ζ^A at point X to contract the indices of the boundary-to-bulk propagator (23) for the Fronsdal field, one finds

$$\langle j_{s_1}(X) j_{s_2}(Z)\rangle = \delta_{s_1,s_2}\frac{1}{(XZ)^{2\Delta}}\left(\frac{\zeta^A \Pi(X,Z)_A{}^B \eta_B}{(XZ)}\right)^s \tag{26}$$

which is a correct expression for the two-point function of conserved currents [43].

7

Twisted-adjoint action of the conformal group

Below it will be also important to have a somewhat unusual action of $so(d, 2)$ on ξ_A and P_A. For reasons that become clear in section 7, we call this the twisted-adjoint action. Let the polarization $^V\xi$ and 'wave-vector' ^VP be given at some bulk point V^A. Then an AdS rotation $\Lambda^A{}_B$ is performed that takes V to X. The boundary point Z is kept fixed by hand, i.e. we would like to have a transformation that takes $^VP(V, Z)$ to $^XP(X, Z)$ (*idem.* for ξ) without acting on Z, which means that the hyperboloid is rotated while the cone is not. The corresponding transformations read

$$X^A = \Lambda^A{}_B V^B,$$

$$^XP_A = \frac{1}{\sigma}(^VP_A - V_A) + X_A, \qquad \sigma = (^VP - V) \cdot X,$$

$$^X\xi_A = {}^V\xi_A - \frac{1}{\sigma}(^VP_A - V_A)(^V\xi \cdot X). \tag{27}$$

5. Higher spin fields

In this section, we present the equations that describe free HS fields in anti-de Sitter space of any dimension $d + 1$ in terms of certain *master* fields. The basic material is of course well known, e.g. see [9, 12, 28, 29, 53], but the exposition is somewhat new. Firstly, the master fields that take values in the HS algebra are introduced. Secondly, the background geometry, i.e. the anti-de Sitter space, is given in a form analogous to higher spins themselves. Thirdly, the equations are presented and few properties thereof are discussed. At the end an effective oscillator realization is reviewed.

Algebra

We start with the generators T_{AB} of the anti-de Sitter or conformal algebra $\mathfrak{h} = so(d, 2)$:

$$[T_{AB}, T_{CD}]_\star = T_{AD}\eta_{BC} - T_{BD}\eta_{AC} - T_{AC}\eta_{BD} + T_{BC}\eta_{AD}, \tag{28}$$

where $\star$ is the product in the universal enveloping algebra $U(\mathfrak{h})$ of $\mathfrak{h} = so(d, 2)$. What we would like to review is that there exists an algebra $\mathfrak{g}$, called HS algebra, whose connection $W(T|X)$ gets decomposed under $\mathfrak{h}$ in terms of connections (13) whose boundary values couple to all currents build of free scalar fields, i.e.

$$W(T|X) = \sum_s W^{A(s-1),B(s-1)} T_{AB} \star \cdots \star T_{AB} \equiv \sum_s W^{A(s-1),B(s-1)} T_{AB(s-1)}. \tag{29}$$

$U(\mathfrak{h})$ is a good starting point as it is a quite large extension of $\mathfrak{h}$, which should have enough room. $U(\mathfrak{h})$ is an $\mathfrak{h}$ module itself, whose decomposition in terms of irreducible $\mathfrak{h}$-modules can be worked out using the Poincaré–Birkhoff–Witt theorem, the first levels being given by

$$\tag{30}$$

where the first singlet $\bullet$ is just the unit of $U(\mathfrak{h})$ and $\square$ represents T^{AB}. At level 2, the singlet is the quadratic Casimir operator $C_2 = -\frac{1}{2} T_{AB} \star T^{AB}$ and two more elements

$$\begin{array}{c}\square\\\square\end{array} = T_{[AB} \star T_{CD]}, \qquad \square\square = T_A{}^C \star T_{AC} - \frac{2}{(d+1)} \eta_{AA} C_2 \tag{31}$$

are the first ones that do not fit into the pattern of (13), (29) as they do not have the symmetry of a rectangular two-row Young diagram (5). It is necessary to quotient them , defining a two-sided ideal I :

$$I \cong U(\mathfrak{h}) \star \left(\begin{array}{c}\square\\\square\end{array} \oplus \square\square \right) \star U(\mathfrak{h}). \tag{32}$$

Despite not being immediately obvious, the procedure is consistent and there is nothing else to care about (see [29, 9, 12, 53, 54] for an extended elaboration); the HS algebra defined by $\mathfrak{g} = U(\mathfrak{h})/I$ has the desired decomposition in terms of $\mathfrak{h}$-modules:[5]

$$\mathfrak{g} \cong \bullet \oplus \square \oplus \square\square \oplus \square\square\square \oplus \ldots \tag{33}$$

In addition to the master 1-form connection $W(T|X)$, one should introduce the field strengths that are packed into the master zero-form field $B(T|X)$.

From the Fronsdal $\phi^{A(s)}$ field (22) vantage point, the $W(T|X)$ field will encode non-gauge invariant derivatives of $\phi^{A(s)}$, while $B(T|X)$ will encode gauge-invariant ones. Introduction of auxiliary fields to encode the derivatives of the $\phi^{A(s)}$ field is a matter of convenience as the HS interactions contain higher order derivatives. One then may split a problem of interactions into (i) writing constraints that encode all derivatives of $\phi^{A(s)}$ in terms of master fields, (ii) looking for purely algebraic couplings of master fields that preserve the constraints [9].

Geometry

The anti-de Sitter space can be defined via a flat connection $\Omega = \frac{1}{2} \Omega^{A,B} T_{AB}$, $\Omega^{A,B} \equiv \Omega_\mu^{A,B} \, \mathrm{d}x^\mu$, which is the vacuum value of $W(T|X)$,

$$\mathrm{d}\Omega + \Omega \star \Omega = 0. \tag{34}$$

In order to define the notion of a Lorentz tensor at any point of the anti-de Sitter space, one should introduce [56, 28] an external field $V^A(X)$, $V \cdot V = -1$, called compensator, that defines the splitting of the local $so(d,2)$ into the Lorentz subalgebra $so(d,1)$, which is a stability subalgebra of V, and translations, cf section 3. Roughly speaking, this amounts to splitting $\Omega^{A,B}$ into vielbein h_μ^a and spin-connection $\varpi_\mu^{a,b}$, which as a consequence of (34) will satisfy[6]

$$\mathrm{d}\varpi^{a,b} + \varpi^{a,}{}_c \wedge \varpi^{c,b} = -\Lambda h^a \wedge h^b, \qquad \mathrm{d}h^a + \varpi^{a,}{}_c \wedge h^c = 0. \tag{35}$$

This can be done in a fully $so(d,2)$-covariant way as in [28, 56], and the expressions for the vielbein E^A and Lorentz-covariant derivative D read

$$E^A = \mathrm{d}V^A + \Omega^A{}_B V^B, \qquad E^A V_A = 0, \tag{36}$$

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[5] All the other unwanted diagrams in the spectrum turn out to be removed by I. Actually, I is the annulator of a free conformal scalar [29, 53, 55].

[6] Indices $a, b, \ldots$ are fiber indices of the AdS Lorentz algebra $so(d,1)$. Λ is the cosmological constant.

$$D = d + \tfrac{1}{2}T_{AB}(\Omega^{A,B} + \Sigma^{A,B}), \qquad \Sigma^{A,B} = E^A V^B - E^B V^A. \tag{37}$$

The Lorentz-covariant derivative D is determined by $DV^A = 0$, $DE^A = 0$. It is worth mentioning that $E^A_M \, \mathrm{d}X^M$ must have the maximal rank, i.e. $(d+1)$, which is a standard requirement for the vielbein; otherwise, one cannot interpret the theory given below in terms of Lorentz tensors.

A standard choice for the compensator is $V^A = \mathrm{const} = \delta^A_{d+1}$. Let us note that within the ambient approach, it is natural to choose V to be just an ambient coordinate X; then the vielbein is $E^A_M \, \mathrm{d}X^M = G^A_M \, \mathrm{d}X^M = \partial_M X^A \, \mathrm{d}X^M$. The local Lorentz generators are the V-orthogonal components of T^{AB}. The translation generators are simply $P^A = T^{AB}V_B$.

Equations

The equations that describe free HS fields read

$$DW - \frac{1}{2}\Sigma_{A,B}[T^{AB}, W]_\star = E^A \wedge E^B \frac{\partial}{\partial T^{AB}} B \bigg|_{P^A = 0}, \tag{38a}$$

$$DB - \frac{1}{2}\Sigma_{A,B}\{T^{AB}, B\}_\star = 0. \tag{38b}$$

Let us mention briefly several important properties of these equations.

(a) The equations are consistent and complete (integrable) in a sense that applying d and using the equations again gives zero and does not produce any new constraints on the fields. Equations have the unfolded form [57, 58], i.e. are of first order, written by making use of the exterior products of differential forms and de Rham differential d, which now is hidden inside the Lorentz covariant derivative D. The unfolded equations enjoy a number of nice properties [11, 59, 60]. In particular, all 1-forms, i.e. W in our case, take values in some Lie algebra, which follows from the integrability requirement. Then, all the structures appearing in the unfolded equations have an interpretation in terms of this Lie algebra.

(b) The full nonlinear equations for HS fields [9] are given in the unfolded form and are certain nonlinear deformations of (38a) and (38b).

(c) If it were not for the rhs (38a) would be a covariant constancy condition in the adjoint representation of the HS algebra. Its lhs is simply

$$D_\Omega W = \mathrm{d}W + [\Omega, W]_\star = \cdots \tag{39}$$

which from the point of view of nonlinear theory is to be understood as a linearization of $\mathrm{d}W + W \star W$ over the Ω background. At the linearized level, the HS algebra is just a highly reducible $so(d, 2)$-module, see (33), as Ω does not have any components beyond $so(d, 2)$.

(d) Equations (38a) and (38b) decompose under $so(d, 2)$ into independent sets of equations: one set for each spin $s = 0, 1, 2, \ldots$. A scalar field has all its derivatives in the B field, while any $0 < s$-field has its derivatives split between W and B.

(e) Importantly, the consistency of the equations is not spoiled by the rhs of (38a), which has an interpretation as a Chevalley–Eilenberg cocycle of $so(d, 2)$.

(f) If it were not for the rhs of (38a), then the equation $\mathrm{d}W + W \star W = 0$ would have pure gauge solutions only, $g^{-1} \star \mathrm{d}g$, describing no propagating degrees of freedom in the bulk. It is the gluing term that makes W propagating and it is the difficulty of finding a nonlinear completion of the gluing term that makes the HS problem so complicated. The fact that $P^A = 0$ on the rhs of (38a) tells us that not all components of W are sourced by B, but only

those that are transverse to the compensator; these are called Weyl tensors. Schematically, (38*a*) reads

$$dW^{A(s-1),B(s-1)} + \cdots = E_M \wedge E_N C^{A(s-1)M,B(s-1)N}, \qquad V_M C^{A(s-1)M,B(s)} \equiv 0. \qquad (40)$$

(g) Equation (38*b*) is a covariant constancy equation, but given with respect to the twisted-adjoint action of $so(d, 2)$. Given any automorphism π, one can define a twisted-adjoint action $T(B) = T \star B - B \star \pi(T)$, which is still a representation of the algebra. In the HS case, π reflects the local translation generators $P^A = T^{AC}V_C$ while not affecting the Lorentz ones, explicitly,

$$\pi(T^{AB}) = T^{AB} + 2P^A V^B - 2P^B V^A, \qquad \pi^2 = \mathrm{id}, \qquad \pi(P^A) = -P^A. \qquad (41)$$

One may rewrite (38*b*) in a more Ω-covariant form, emphasizing its representation theory origin as

$$dB + \Omega \star B - B \star \pi(\Omega) - B \star T^{AB} dV_A V_B = 0. \qquad (42)$$

The last term accounts properly for the x-dependence of V^A as the frame field E^A is $(d + \Omega)V$ and π knows nothing about dV. It also restores the integrability since (42) is consistent up to $d\pi$, which is compensated by the last term.

(h) A standard example to demystify (38*a*)–(38*b*) is provided by the spin-2, where $W^{A,B}$ component of $W(T)$ can be decomposed into the vielbein W^a and spin-connection $W^{a,b}$, and then (38*a*) amounts to the linearized zero-torsion constraint and the condition that the only nonzero components of the linearized over AdS Riemann 2-form are given by the Weyl tensor $C^{ab,cd}$, cf (35):

$$DW^a = 0, \quad DW^{a,b} + \Lambda h^a \wedge W^b - \Lambda h^b \wedge W^a = h_c \wedge h_d C^{ac,bd}. \qquad (43)$$

Equation (38*b*) just encodes all derivatives of the Weyl tensor that are compatible with differential Bianchi identities.

(i) Equation (38*a*) contains Fronsdal equation (22), which by virtue of (38*a*) is imposed on the maximally V-parallel component of the spin-s part of the $W(T|X)$ field,

$$\phi^{A(s)} = E^{AM} e_M{}^{A(s-1)}, \qquad e^{A(s-1)} = W^{A(s-1),B(s-1)} V_{B(s-1)}, \qquad (44)$$

where $E^A{}_M$ is the ambient vielbein and $e_M{}^{A(s-1)}$ is a HS vielbein or the frame field.

Oscillator realization

One can develop a quite effective technique for dealing with (38*a*)–(38*b*) directly [53]. Unfortunately, it is not known how to extend this technique beyond the linearized level. Fortunately, everything can be given by means of oscillator realization, which does extend to the interaction level [9]. One introduces an $sp(2)$ pair Y_α^A, $\alpha = 1, 2$, of oscillators, satisfying [9]

$$\left[Y_\alpha^A, Y_\beta^B \right]_\star = 2\mathrm{i}\eta^{AB}\epsilon_{\alpha\beta}, \qquad \epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}, \qquad \epsilon_{12} = 1, \qquad (45)$$

where $\star$ is the Moyal–Weyl $\star$-product

$$f(Y) \star g(Y) = \frac{1}{(2\pi)^{2(d+2)}} \int dU dV f(Y + U) g(Y + V) \exp\left(\mathrm{i}U_\alpha^A V_\beta^B \epsilon^{\alpha\beta} \eta_{AB}\right). \qquad (46)$$

It is easy to see that

$$T^{AB} = \frac{\mathrm{i}}{4}\left\{ Y_\alpha^A, Y_\beta^B \right\}_\star \epsilon^{\alpha\beta} \qquad (47)$$

satisfy the *so(d, 2)* commutation relations (28). The ⋆-product realization makes computations easier than those with the universal enveloping algebra. The fact that there are only two species $Y_{1,2}^A$ of oscillators quotients automatically out the first generator of (31). The absence of the second one, which corresponds to various traces, is not granted for free and it must be factorized by hand via imposing conditions of type

$$\eta^{AB} \frac{\partial^2}{\partial Y_\alpha^A \partial Y_\beta^B} \ldots = 0 \tag{48}$$

that removes proliferation due to traces. Equation (38*b*) can be left unchanged, as it does not depend on the details of realization, while (38*a*) now reads

$$\mathrm{d}W + [\Omega, W]_\star = -\mathrm{i}E^A \wedge E^B \epsilon_{\alpha\beta} \frac{\partial^2}{\partial Y_\alpha^A \partial Y_\beta^B} B(Y)\bigg|_{Y_\alpha^A V_A = 0}. \tag{49}$$

6. W-propagator, two-point functions

Once the boundary-to-bulk propagator for the Fronsdal field is given (23), it is straightforward to determine what the frame field part (44) of $W(T|X)$ is

$$\mathrm{e}^{A(s-1)} = \frac{1}{(XZ)^\alpha} \xi^{A(s-1)} \xi_N \, \mathrm{d}X^N, \tag{50}$$

where we keep for the moment the weight α free. One has to take the derivatives of (23) up to order $s-1$ and to take the traces into account appropriately. The most general ansatz reads (the ambient X now serves also as the compensator)

$$W^s = \frac{1}{(XZ)^\alpha} \sum_{0 \leqslant k+2i \leqslant s-1} A_{k,i}^s \left[\frac{\xi^{s-1}}{X^{s-1-k-2i} P^k G^i} \right] \xi_N \, \mathrm{d}X^N, \tag{51}$$

$$\left[\frac{\xi^{s-1}}{X^{s-1-k-2i} P^k G^i} \right] \equiv T_{AB(s-1)} \xi^{A(s-1)} X^{B(s-1-k-2i)} P^{B(k)} G^{BB(i)}. \tag{52}$$

Equation (49) implies that $D_\Omega W$ equals zero almost everywhere in the parameter space, which with the help of (20) gives

$$A_{k,i}^s = A^s \frac{(-1)^i \Gamma[\alpha+k-1]\Gamma\left[\frac{1}{2}(s+\alpha)\right]}{i!k!\Gamma[\alpha-1]\Gamma\left[\frac{1}{2}(s+\alpha)-i\right]}, \tag{53}$$

where A^s reflects the freedom in normalizing any W^s separately. If one now checks whether the trace constraint (48) is satisfied, then he/she finds up to some nonvanishing function

$$\partial_\mu^C \partial_{Cv} W \sim (2\Delta + s - \alpha), \qquad s > 2, \tag{54}$$

which implies that the trace constraint (48) singles out the weight of a massless spin-*s* field; a similar phenomenon was observed in [45]. One may expect that the connection W with relaxed trace constraints is still a good starting point for the description of massive fields too, as $D_\Omega W \approx 0$ for any weight α. For $\alpha = 2\Delta + s$ one finds

$$A_{k,i}^s = A^s \frac{(-1)^i \Gamma[s+\Delta]\Gamma[k+s+2\Delta-1]}{i!k!\Gamma[s+\Delta-i]\Gamma[s+2\Delta-1]}. \tag{55}$$

Weyl tensor. The terms that do not cancel inside $D_\Omega W$ with (55) are given by the Weyl tensor part $C^{A(s),B(s)}$ of $B(T|X)$. Actually, the form of the Weyl tensor,

$$C^s = \frac{1}{(XZ)^{2\Delta+s}} \sum_{i=0}^{i=[\frac{s}{2}]} H_i^s \left[\frac{\xi^s}{P^{s-2i} G^i} \right], \tag{56}$$

$$\left[\frac{\xi^s}{P^{s-2i}G^i}\right] \equiv T_{AB(s)}\xi^{A(s)}P^{B(s-2i)}G^{BB(i)}, \tag{57}$$

$$H_i^s = H^s H_i^s, \qquad H_i^s = \frac{(-)^i s!\,\Gamma\left[s+\Delta-\frac{1}{2}-i\right]}{4^i i!(s-2i)!\,\Gamma\left[s+\Delta-\frac{1}{2}\right]}, \tag{58}$$

is completely fixed up to the overall factor H^s by the requirement for it to be traceless. It is constructed in terms of variables that are all tangent to the AdS-hyperboloid. Therefore, $so(d, 2)$-tracelessness implies $so(d, 1)$-tracelessness, as it should. The only thing to do is to determine the relative normalization of H^s that does cancel $D_\Omega W$, which gives

$$H^s = A^s \frac{\Gamma[2s+2\Delta-1]}{(s+1)!\,\Gamma[s+2\Delta-1]}. \tag{59}$$

Let us note that for (49) to hold for the traceless Weyl tensor, the massless fall-off $(2\Delta + s)$ is now mandatory.

Two-point functions. Using the prescriptions given in section 4, it is easy to see that on approaching the boundary, all terms in W^s tend to

$$\frac{1}{(XZ)^{2\Delta+s}}T_{AB(s-1)}\xi^{A(s-1)}X^{B(s-1)}\xi_N\,dX^N; \tag{60}$$

the latter expression suggests that the role of a polarization vector in the HS theory is played by $\zeta^A = T^{AB}X_B$, which is by definition orthogonal to the would-be soon boundary point X^A. Unfortunately, it is not null, which manifests the fact that the HS theory is formulated in an extended space of variables, where the conditions like tracelessness do not hold automatically and require a separate and rather cumbersome treatment [9, 12]. On the boundary (60) reduces to (26), which is a desired correlation function of two spin-s conserved currents.

More easily the two-point functions can be extracted out of the Weyl tensor, which directly tends to

$$\frac{1}{(XZ)^{2\Delta+s}}T_{AB(s)}\xi^{A(s)}X^{B(s)} = \frac{1}{(XZ)^{2\Delta}}\left(\frac{\zeta_A \Pi(X, Z)^A{}_B \eta^B}{(XZ)}\right)^s \tag{61}$$

which is (26) again. The phenomenon that the Weyl tensor is relevant for extracting correlation functions of currents was observed in [24, 25].

7. *B*-propagator

The most complicated part is to construct the boundary-to-bulk propagator for the B field as it contains arbitrarily high derivatives of the fields. A straightforward approach based on solving (38b) seems to be too tedious for spins greater than zero, calling for more refined methods. Nevertheless, it is first useful to work up the scalar case directly.

7.1. s=0

The case of the scalar field with dimension $2\Delta = d - 2$, which is the lowest component of the bulk HS multiplet, can be approached rather directly, the most general ansatz being

$$B^0 = \frac{1}{(XZ)^\alpha}F(v, u), \quad v = \left[\frac{P}{X}\right] \equiv T^{AB}P_A X_B, \quad u = \left[\frac{G}{XX}\right] \equiv T^{AB(2)}G_{AA}X_{B(2)}, \tag{62}$$

where again we keep the fall-off α free. The equations of motion (38b) lead to

$$\left(-\alpha - N_v + \frac{1}{2}(N_v + 2N_u + 2)\partial_v\right)F(v, u) = 0,$$
$$(-2 + \partial_v + (N_v + 2N_u + 3)\partial_u)F(v, u) = 0, \tag{63}$$

where $N_\nu = \nu\partial_\nu$ and $N_u = u\partial_u$ are the Euler operators. There are two solutions to these equations: the first one, which is simple and corresponds to the shadow partner of a dimension 2Δ scalar, which has dimension $2 = d - 2\Delta$, reads

$$B^0 = \frac{1}{(XZ)^2} \exp 2\nu. \tag{64}$$

The second solution is the one we need. Let us mention that the shadow solution is simple as it does not depend on variable u, which is of the fourth order in Y_α^A. The latter property is due to its dimension, which does not involve d, so there is no need of u.

To find the second solution, it is easier to convert equations (63) into recurrent relations and solve for $F^{k,m}$:

$$F(\nu, u) = \sum_{k,m} F^{k,m} \nu^k u^m, \tag{65}$$

the solution being

$$F(\nu, u) = \sum_{k,m} \frac{2^{k+2m}(-)^m \Gamma[\Delta]\Gamma[k+2\Delta]}{k!m!(1+k+2m)!\Gamma[\Delta-m]\Gamma[2\Delta]} \nu^k u^m. \tag{66}$$

Let us make several comments on the solutions obtained.

 (i) If one discards $(1 + k + 2m)!$ factor in $F^{k,m}$, (66), which, as will become evident soon, appears naturally from the $\star$-product integration, then the solution has a very simple generating function

$$\widetilde{F}(\nu, u) = (1 - 2\nu)^{-2\Delta}(1 + 4u)^{-1+\Delta}. \tag{67}$$

The additional interfering factorial can be treated with the help of the Hankel representation for the Γ-function, which in our case of integer argument reduces to

$$\frac{1}{\Gamma(n)} = \frac{1}{2\pi i} \oint_{\mathcal{C}} z^{-n} e^z \, dz, \tag{68}$$

with the closed contour around the origin. The solution is the transform

$$F(\nu, u) = \frac{1}{2\pi i} \oint_{\mathcal{C}} dz \widetilde{F}(\nu/z, u/z^2) z^{-2} e^z. \tag{69}$$

 (ii) The form of the solution (66) depends on the spacetime dimension modulo 2, which is a general phenomenon. Indeed, for d even, i.e. Δ integer, the Taylor expansion in u stops at $u^{\Delta-1}$ as is seen from (67), while for d odd, i.e. Δ half-integer, the solution contains all powers of u.

(iii) Using a prescription of section 4 for extracting two-point functions, one observes that both ν and u tend to zero on approaching the boundary; therefore

$$B^0\Big|_{\text{boundary}} = \frac{1}{(XZ)^{2\Delta}} \tag{70}$$

as was expected for $\langle j_0(X) j_0(Z) \rangle$, where $j_0 =: \phi(X)\phi(X)$.

(iv) As one can readily check, both solutions satisfy a twisted analogue of the trace constraint (48), which has the form

$$\left(G^{AB} \frac{\partial^2}{\partial Y_\alpha^A \partial Y_\beta^B} + Y_A^\alpha V^A Y_B^\beta V^B \right) B^0(Y) = 0. \tag{71}$$

It is what should have been expected once the scalar is associated with $B(Y = 0)$. Let us note that there is no freedom in choosing trace factorization condition once the equations of motion are satisfied, and it is stated which component of B is a scalar field (we have assumed a canonical choice $B(Y = 0)$ is a scalar field).

(v) The exponent $\exp 2\nu$ that appears in the shadow solution is a distinguished one as it is a $\star$-algebra projector analogous to the one used in [61], $\exp 2\nu \star \exp 2\nu = \exp 2\nu$, which is important for going beyond the linearized approximation. Among other things, it guarantees that the potentially divergent self-interaction terms for HS fields cancel. One might think of extracting $\exp 2\nu$ out of the second solution, the result being

$$B^0 = \frac{\exp 2\nu}{(XZ)^{2\Delta}} \sum_{k,m} \frac{(-1)^m 2^k \Gamma\left[\Delta - m - \frac{1}{2}\right] \Gamma[2\Delta - 1]}{k! m! (1 + k + 2m)! \Gamma\left[\Delta - \frac{1}{2}\right] \Gamma[2\Delta - k - 2m - 1]} \nu^k u^m. \tag{72}$$

There exists a simpler route to (69) to appreciate which we need to go into the details of explicit solutions to (38*b*).

7.2. Twisted-adjoint transformation and the self-similarity method

Solving the e.o.m. to derive B-propagator for a spin greater than zero turns out to be a highly nontrivial problem. To avoid this problem, we propose another approach which is based on the following propositions.

(i) The lowest component of the unfolded B-propagator which is the Weyl tensor is known and given by (56).
(ii) Vectors ξ^A and P^A the propagator depends upon are related at different points of AdS space according to (27).
(iii) The solution to the twisted-adjoint equation (38*b*) is pure gauge

$$B = g^{-1} \star B_0 \star \pi(g). \tag{73}$$

The idea is as follows. Suppose we know the solution at some point $X_0^A = V^A$: $B = B(Y_\alpha^A | {}^V\xi, {}^VP, (VZ))$; then at an arbitrary point X, it amounts to $B(\overline{Y}_\alpha^A | {}^X\xi, {}^XP, (XZ))$, where $\overline{Y}_\alpha^A = \overline{Y}_\alpha^A(X)$ receive X-dependence. Both solutions are related by twisted-similarity transformation (73):

$$g^{-1} \star B\left(Y_\alpha^A | {}^V\xi, {}^VP, (VZ)\right) \star \pi(g) = B\left(\overline{Y}_\alpha^A | {}^X\xi, {}^XP, (XZ)\right). \tag{74}$$

Note that the same function B enters both sides of (74). Here we assume that $g^{-1} \star \mathrm{d}g$ has all its components in the $so(d, 2)$ subalgebra of the HS algebra, i.e. $g = g(T|X)$ defines some global rotation of AdS, which takes compensator V^A to X^A, $X^A = \Lambda^A{}_B V^B$; the simplest such $g(T|x)$ reads

$$g(T|X) = \exp[-2T^{AB}X_A V_B (1 - X \cdot V)^{-1}]. \tag{75}$$

$\overline{Y}_\alpha^A$-oscillators should preserve their commutation relations and, therefore, transform in the adjoint of $so(d, 2)$,

$$\overline{Y}_\alpha^A = g^{-1} \star Y_\alpha^A \star g = \Lambda^A{}_B Y_\alpha^B. \tag{76}$$

We are going to refer to equation (74) as the *self-similarity* condition. Restricting (74) to the Lorentz sector, i.e. setting $V^A Y_{A\alpha} \equiv X^A \overline{Y}_{A\alpha} = 0$, one arrives at some integral equation with the only Weyl tensor of (56) remaining on the rhs. This will eventually allow us to determine the B-function completely. Before going into the details, let us consider now lhs of (74). For an arbitrary function $F(Y)$,

$$\widehat{F}(Y) = g^{-1}(Y) \star F(Y) \star \pi(g(Y)), \tag{77}$$

simple Gaussian integration in terms of $\overline{Y}_\alpha^A$ and $\tau = (XV)$ yields

$$\widehat{F}(\overline{Y}_\alpha^A) = \int \mathrm{d}s\, \mathrm{d}t \exp \mathrm{i}\left(\overline{Y}_\alpha^A(s^\alpha V_A + t^\alpha X_A) + \tau s^\alpha t_\alpha\right) F\left(\overline{Y}_\alpha^A + V^A s_\alpha + X^A t_\alpha\right), \tag{78}$$

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where it is implied that the 'initial data' $F(Y)$ are given at the point V, at which the ambient coordinate V^A coincides with the compensator field V^A. Setting further $X^A \overline{Y}_{A\alpha} = 0$, as we are interested to end up with the Weyl tensor on the rhs of (74), one obtains

$$\widehat{F}\big|_{X^A \overline{Y}_{A\alpha}=0} = \frac{1}{\tau^2} \int \mathrm{d}s\, \mathrm{d}t\, F\left(\hat{Y}^A_\alpha + V^A s_\alpha + \frac{1}{\tau} X^A t_\alpha\right) \mathrm{e}^{\mathrm{i} t_\alpha s^\alpha}, \tag{79}$$

$$\hat{Y}^A_\alpha = \Pi(V,X)^A{}_B \overline{Y}^B_\alpha, \qquad \Pi_A{}^C \Pi_{CB} = \Pi_{AB}, \tag{80}$$

where $\Pi(V,X)^A{}_B$ (see table 1) now comes as the projector to Lorentz directions. This means, in particular, that $\hat{Y}^A_\alpha V_A = 0$. Another useful in what follows observation is

$$^X\xi_A \Pi(V,X)^A{}_B = {}^X\xi_B, \qquad {}^X P_A \Pi(V,X)^A{}_B = {}^X P_B. \tag{81}$$

Having a simple propagator for the shadow scalar field (64) at hand, let us illustrate how the self-similarity equation (74) indeed performs the desired transformation:

$$Y \to \overline{Y} \qquad V \to X \qquad {}^V P \to {}^X P \qquad {}^V\xi \to {}^X\xi, \tag{82}$$

where $^{X,V}P$ and $^{X,V}\xi$ are the 'wave-vector' and polarization at points X and V, while the boundary point Z and polarization vector η are kept fixed; these were defined in (27). To do so we take the propagator for the shadow field (64), replace X by V and apply (78); one then finds

$$g^{-1} \star B^0(Y|^V P, (VZ)) \star \pi(g) = B^0\left(\overline{Y}|^X P, (XZ)\right), \tag{83}$$

$$g^{-1} \star \frac{1}{(VZ)^2} \exp 2[{}^V P_A T^{AB} V_B] \star \pi(g) = \frac{1}{(XZ)^2} \exp 2[{}^X P_A \overline{T}^{AB} X_B]. \tag{84}$$

The latter means that the twisted-adjoint rotation transforms the boundary-to-bulk propagator from Z to V to the one from Z to X. In particular, the integration produces a prefactor that changes $(VZ)^{-2}$ to $(XZ)^{-2}$.

For $s > 0$ analysis, we need to elaborate (79) a bit further. Since all of the functions depend only on T^{AB}, it is useful to look at what these transform into. Let us expand $F(Y)$ in terms of T^{AB},

$$F(Y) = F(T) = \sum_N T^{AB(N)} C_{A(N)|B(N)}, \tag{85}$$

where $C_{A(N)|B(N)}$ are symmetric in each group of indices. Using (78) and the orthogonality condition

$$\int \mathrm{d}s\, \mathrm{d}t\, (s^\alpha t_\alpha)^m (s^\alpha \xi_\alpha)^k (t^\beta \eta_\beta)^l \exp \mathrm{i}\,(s^\alpha t_\alpha) = \delta_{k,l} \frac{\mathrm{i}^{k+m}(k+m+1)!}{k+1} (\xi^\alpha \eta_\alpha)^k, \tag{86}$$

where ξ and η are auxiliary spinors, the term-wise result is

$$\widehat{F}\big|_{X^A \overline{Y}_{A\alpha}=0} = \frac{1}{\tau^2} \sum_{N=0}^{\infty} \sum_{s=0}^{N} D_N^s C_N^s \hat{T}^{AB(s)} \Theta^{AB(N-s)} C_{A(N)|B(N)},$$

$$D_N^s = \frac{(-)^{N+s}(N+1)!}{2^{N-s}(s+1)!}, \qquad C_N^s = \frac{N!}{(N-s)!s!} \tag{87}$$

where we introduced the notation

$$\Theta^{A,B} = \frac{1}{\tau}(X^A V^B - X^B V^A). \tag{88}$$

The idea of obtaining the spin-s propagator is to reconstruct the unknown function B from the Weyl tensor (56) using the self-similarity equation (74) at $V^A Y_{A\alpha} \equiv X^A \overline{Y}_{A\alpha} = 0$. It is instructive to first consider the spin-zero case separately before we proceed with an arbitrary spin propagator.

7.3. Self-similarity for spin zero

Let us apply the elaborated method to the simplest case of spin zero. Following the logic of the previous section, suppose that we already know the boundary-to-bulk propagator at a point V, where by a coincidence the ambient coordinate V^A equals the compensator field V^A. It is given by an expression similar to (62):

$$^VB = \frac{1}{(VZ)^{2\Delta}} F(v, u), \qquad v = {}^VP_A T^{AB} V_B, \qquad u = {}^VG_{AC} T^{AB} T^{CD} V_B V_D. \tag{89}$$

When rotated by $g(Y)$ that takes V to X, the propagator must coincide with (62). In particular, ^XB at $\overline{Y}_\alpha^A = 0$ must be[7] the 'two-point function' $(XZ)^{-2\Delta}$. With (79), one finds, $\tau = (XV)$,

$$^XB\big|_{\overline{Y}=0} = \frac{1}{\tau^2} \int \mathrm{d}s \, \mathrm{d}t \, \exp \mathrm{i} \left(s^\alpha t_\alpha\right) {}^VB \left(T^{AB} s^\alpha t_\alpha\right) \Big|_{T^{AB}=\Theta^{AB}} \tag{90}$$

which leads us to a transform

$$\int \mathrm{d}s \, \mathrm{d}t \, \exp \mathrm{i} \left(s^\alpha t_\alpha\right) f(s^\alpha t_\alpha x) = \sum_k f_k x^k \mathrm{i}^k (k+1)! \qquad f(x) = \sum_k f_k x^k \tag{91}$$

that brings an additional factorial due to $\int \mathrm{d}s \, \mathrm{d}t \, (s^\alpha t_\alpha)^k \exp \mathrm{i} \left(s^\alpha t_\alpha\right) = \mathrm{i}^k (k+1)!$. The factorial just counts the number of T^{AB}. Let us note that the transformation (91) is to some extent reminiscent to the one used in [7] in a different context. For a function of two variables (89), the transformation reads

$$^XB^0\big|_{\overline{Y}=0} = \frac{1}{\tau^2} \frac{1}{(VZ)^{2\Delta}} \widetilde{F}(v, u)\big|_{T^{AB}=\Theta^{AB}}, \tag{92}$$

$$\widetilde{F}(x, y) = \sum_{k,m} F_{k,m} x^k y^m \frac{(k+2m+1)!(-)^k}{2^{k+2m}}, \tag{93}$$

with the inverse map given by

$$F(x, y) = \frac{1}{2\pi \mathrm{i}} \oint \mathrm{d}z \frac{\mathrm{e}^z}{z^2} \widetilde{F}\left(-\frac{2x}{z}, \frac{4y}{z^2}\right). \tag{94}$$

Using convenient variables $\sigma = (XZ)/(VZ)$ and $\tau = (XV)$, we have

$$v\Big|_{T^{AB}=\Theta^{AB}} = -1 - \frac{(XZ)}{(VZ)(XV)} = -1 - \sigma\tau^{-1} \qquad u\Big|_{T^{AB}=\Theta^{AB}} = 1 - \tau^{-2}. \tag{95}$$

Amazingly, the condition for $^XB^0(\overline{Y} = 0)$ to have the correct behavior is sufficient to fix the function of two variables, thus determining the propagator completely without solving any differential equations at all! Indeed,

$$\frac{1}{(XZ)^{2\Delta}} = {}^XB^0(\overline{Y} = 0) = g^{-1} \star {}^VB^0(Y) \star \pi(g)\big|_{\overline{Y}=0} = \frac{\widetilde{F}(-1 - \sigma\tau^{-1}, 1 - \tau^{-2})}{\tau^2 (VZ)^{2\Delta}} \tag{96}$$

which immediately gives $\widetilde{F}(-1 - \sigma\tau^{-1}, 1 - \tau^{-2}) = \tau^2 \sigma^{-2\Delta}$. The straightforward inverse transform $\widetilde{F}(v, u) \to F(v, u)$ gives (66), which has been obtained by directly solving the field equations. The simple form of the generating function (67) comes now without surprise as well as the dependence of the solution on d mod 2.

A form of the scalar propagator which explicitly contains the star-product projector might be of use for application. To obtain it one should take

$$^VB = \frac{1}{(VZ)^{2\Delta}} \mathrm{e}^{2v} F(v, \omega), \qquad \omega = u - v^2. \tag{97}$$

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[7] For $s = 0$ setting $X^A \overline{Y}_{A\alpha} = 0$ is equivalent to $\overline{Y}_{A\alpha} = 0$.

Repeating the above procedure, one arrives at

$$B = \frac{\exp 2\nu}{(XZ)^{2\Delta}} \frac{1}{2\pi i} \oint dz \frac{e^z}{z^2} \left(1 + \frac{4\nu}{z} - \frac{4\omega}{z^2}\right)^{\Delta-1}. \tag{98}$$

The residue in (98) is some polynomial of degree $\Delta - 1$ for integer Δ and infinite series for Δ half-integer. Particularly, the residue equals just 1 for $\Delta = 1$. As we see, this method is simple yet effective and will be applied to a general case of spin-s field below.

7.4. Any s

Again, instead of solving (49) directly we will use the self-similarity of the propagator (74), where g rotates V to X. This means that within the ambient approach the propagator looks the same at any point. Bearing in mind the spin-zero case, we may have a look only at the Weyl tensor at point X as viewed from point V via (74), i.e. try to solve for

$$(56) = B(^X P, ^X \xi, ^X G, (XZ))\big|_{\bar{Y}^A_\alpha X_A=0} = g^{-1} \star B(^V P, ^V \xi, ^V G, (VZ)) \star \pi(g)\big|_{\bar{Y}^A_\alpha X_A=0} \tag{99}$$

with the hope that it again determines the dependence on all variables, and it does.

First of all, one faces the problem of parameterizing various structures that can appear in the B-field. If factorized, it amounts to ten variables, some of them satisfying quadratic relations, which makes a direct solving somewhat complicated. All the descendants of order $N - s$ of the spin-s Weyl tensor can be parameterized as

$$\left[\frac{\xi^s P^{k+q} G^{m-q}}{P^{s-2n-q} G^n V^{k+2m}} G^q\right] \equiv T^{AB(N)} \xi_{A(s)} P_{A(k+q)} G_{AA(m-q)} G_{AB(q)} P_{B(s-2n-q)} G_{BB(n)} V_{B(N-s)}$$
$$0 \leqslant 2n + q \leqslant s, \qquad k + 2m = N - s, \qquad q \leqslant m,$$

which gives for the B-field in the spin-s sector, B^s,

$$B^s = \sum_N \sum_{k+2m=N} \sum_{0\leqslant 2n+q\leqslant s} F^{k,m}_{n,q} \left[\frac{\xi^s P^{k+q} G^{m-q}}{P^{s-2n-q} G^n V^{k+2m}} G^q\right]. \tag{100}$$

The fact that it is a complete basis can be seen either by evaluating the tensor product $\xi^s \otimes V^N \otimes G^{m+n} \otimes P^{s+k-2n}$ or by noting that given any arrangement of ξ, P, G, Vs inside a tensor having the symmetry properties of a rectangular two-row Young diagram, one can always push all ξs to the first group of symmetrized indices and all Vs to the second. After that there is no freedom left in rearranging the indices while preserving ξs and Vs, so the rest of the Ps and Gs can appear in any combination, as they do above.

At the new point X, the role of the compensator field is played by X itself. Thus, one sets $\bar{Y}^A_\alpha X_A = 0$ and notes that $\hat{T}^{AB}$ becomes orthogonal to the old compensator V, $\hat{T}^{AB} V_B = 0$, while $\Theta^{A,B}$ has no components in the Lorentz subspace, i.e. contraction of $\Theta^{A,B}$ with any Lorentz tensor vanishes identically. Therefore (100) can be contracted with $\hat{T}^{AB(s)} \Theta^{AB(N-s)}$ only and (87) simplifies to

$$\widehat{F}(\hat{T}^{AB}) = \frac{1}{\tau^2} \sum_{N=0}^\infty D^s_N C^s_N \hat{T}^{AB(s)} \Theta^{AB(N-s)} C_{A(N)|B(N)}. \tag{101}$$

Now one just needs (1) to express the new Weyl tensor at point X, (56), in terms of the old variables $^V P, ^V \xi, ^V G$, the relevant transformations having been already given in (27); (2) substitute (100) into (101) and expand. Matching various structures on both sides of (99), one finds all the $F^{k,m}_{n,q}$.

(1) There are two structures that contribute to the Weyl tensor (56); with the help of (27) and (81), one derives

$$C^s = \frac{1}{(XZ)^{2\Delta+s}} \sum_i B_i^s \left[\frac{X\xi}{XP}\right]^{s-2i} \left[\frac{X\xi X\xi}{XG}\right]^i \tag{102}$$

$$\left[\frac{X\xi}{XP}\right] = \frac{1}{\sigma}\left[\frac{\xi}{P}\right] - \Upsilon \qquad \Upsilon = \tau\left[\frac{\xi}{V}G\right] + \frac{\tau^2}{\sigma}\left[\frac{P}{V}G\right]\left[\frac{\xi}{V}\right] \tag{103}$$

$$\left[\frac{X\xi X\xi}{XG}\right] = \left[\frac{\xi\xi}{G}\right] + \frac{2\tau}{\sigma}\left[\frac{\xi P}{G}\right]\left[\frac{\xi}{V}\right] + \frac{\tau^2}{\sigma^2}\left[\frac{PP}{G}\right]\left[\frac{\xi}{V}\right]^2 + \Upsilon^2, \tag{104}$$

where all variables on the rhs refer to the point V, so the superscript V is dropped. In order to determine all the $F_{n,q}^{k,m}$, one does not need to expand the Weyl tensor in full, matching some signature terms is sufficient. We would like to look at

$$\mathbf{S}_{n,q} = \left[\frac{\xi\xi}{G}\right]^n \left[\frac{\xi}{P}\right]^{s-2n-q} \left[\frac{\xi}{V}G\right]^q, \tag{105}$$

for which one finds

$$C^s = \frac{1}{(XZ)^{2\Delta+s}} \sum_{0\leqslant 2n+q\leqslant s} H^s L_{n,q}^s \mathbf{S}_{n,q} \qquad L_{n,q}^s = \sum_{i=n}^{i=n+[\frac{q}{2}]} H_i^s C_{s-2i}^{2n+q-2i} C_i^n \tag{106}$$

$$L_{n,q}^s = \frac{s!(-)^n \Gamma\left[\Delta+s-n-\left[\frac{q}{2}\right]-\frac{1}{2}\right]\Gamma\left[\Delta+s-n-\left[\frac{q+1}{2}\right]\right]}{4^n(s-2n-q)!n!q!\Gamma\left[\Delta+s-\frac{1}{2}\right]\Gamma\left[\Delta+s-n-q\right]}. \tag{107}$$

(2) Simple combinatorics with (100) results in

$$\left[\frac{\xi^s P^{k+q} G^{m-q}}{P^{s-2n-q} G^n V^{k+2m}} G^q\right] = \mathbf{S}_{n,q}(C_N^s)^{-1}\left[\frac{P}{V}\right]^{k+q}\left[\frac{G}{VV}\right]^{m-q} + \cdots,$$

where $\cdots$ denotes the terms with other rearrangements of ξ. It is useful do define $\widetilde{F}_{n,q}$ as

$$\widetilde{F}_{n,q}(x,y) = \sum_N \sum_{k+2m=N} D_N^s F^{k,m} x^{k+q} y^{m-q}, \tag{108}$$

where x and y are the same as for the spin-zero case, (95), given by

$$x = \left[\frac{P}{V}\right]\bigg|_{T=\Theta} = -1 - \sigma\tau^{-1}, \qquad y = \left[\frac{G}{VV}\right]\bigg|_{T=\Theta} = 1 - \tau^{-2} \tag{109}$$

and introduce $\widehat{F}_{n,q}(\tau,\sigma) = \widetilde{F}_{n,q}(-1-\sigma\tau^{-1}, 1-\tau^{-2})$. Matching $\mathbf{S}_{n,q}$ terms on both sides, one directly finds, $\vartheta = \Delta + s - n$,

$$\widehat{F}_{n,q} = H^s L_{n,q}^s (-)^q \tau^{q+2} \sigma^{-2\vartheta+q} \tag{110}$$

and performing an inverse transform results in

$$F_{n,q}^{k,m} = H^s L_{n,q}^s \frac{(-)^m 2^{k+2m}(s+1)!}{(s+k+2m+1)!(m-q)!(k+q)!} \frac{\Gamma[\vartheta-m]\Gamma[2\vartheta+k]}{\Gamma[\vartheta-q]\Gamma[2\vartheta-q]}. \tag{111}$$

One might worry about the subleading terms that appear on both sides, these should match automatically, with the explicit computations being more involved though. To have additional control over the computations, we have explicitly checked that the first subleading terms in which one of the ξs is rearranged in a different way do match. This check is equivalent to certain nontrivial differential equations that involve functions $F_{n,q}$ (a derivative of $F_{n,q}$ brings a factor that is related to the conformal weight $2\Delta + s$) as well as the coefficients B_i^s that were determined independently from the trace conditions on the Weyl tensor. Of course, for $s = 0$, for which also $n = q = 0$, one finds complete agreement with the earlier computations.

8. Conclusions

The boundary-to-bulk propagators for HS master 1-form W (51), (55) and zero-form B (100), (111) have been constructed in arbitrary dimension for arbitrary integer spin. The pursuit has forced us to elaborate an appropriate formalism to tackle this problem as the direct approach based on solving the e.o.m. appeared to be too involved. The developed self-similarity method is essentially based on the mixture of the ambient and the unfolded machineries along with the star-product integration. In fact, the unfolding approach reduces the problem to purely algebraic. Having the spin-s Weyl tensor propagator, one defines the generating function that reproduces its all on-shell derivatives through a simple integral transformation. The very existence of such an approach was possible for a number of reasons. First, within the ambient formalism, the propagator is fixed once it is known at a single point. Second, its pure gauge form within the unfolded formalism allows one to relate the solution at different points using similarity transformation, realized as large twisted-adjoint rotations via the $\star$-product. Finally, the latter being presented in terms of the star-product transformation can be explicitly evaluated resulting in the above-mentioned generating function.

The results obtained in the paper may have at least two applications. First, one may compute three-point functions in d-dimensional HS theory [9], demonstrating how three-point functions of currents built of a free scalar emerge if the arguments of [22] can be extended to higher dimensions. As an alternative approach, one may think of putting the HS theory to the boundary directly using [61] or developing the AdS/CFT technique with the recently proposed action principle [62]. Second, according to the general recipe of AdS/CFT three-point functions of conserved currents, $\langle j_{s_1} j_{s_2} j_{s_3} \rangle$ are in one-to-one correspondence with various cubic vertices one can construct for massless fields with spins s_1, s_2, s_3, [43]. There are a number of such vertices, [63, 64], which result in a number of independent structures that can contribute to $\langle j_{s_1} j_{s_2} j_{s_3} \rangle$, found recently in [65]. One may plug the propagators into the cubic vertices, considered in [66], to obtain the AdS part of the cubic vertices/three-point functions dictionary.

Acknowledgments

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