

Holonomy spin foam models: boundary Hilbert spaces and time evolution operators

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Holonomy spin foam models: boundary Hilbert spaces and time evolution operators

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Abstract

In this and the companion paper, a novel holonomy formulation of the so-called spin foam models of lattice gauge gravity is explored. After giving a natural basis for the space of simplicity constraints, we define a universal boundary Hilbert space on which the imposition of different forms of the simplicity constraints can be studied. We detail under which conditions this Hilbert space can be mapped to a Hilbert space of projected spin networks or an ordinary spin network space. These considerations allow us to derive the general form of the transfer operators which generates discrete time evolution. We will describe the transfer operators for some current models on the different boundary Hilbert spaces and highlight the role of the simplicity constraints determining the concrete form of the time evolution operators.

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1. Introduction

Together with the companion paper [15], this paper introduces and studies a class of lattice gauge theories that occur as spin foam models of quantum gravity [89, 90, 10, 23, 11, 24, 79, 78, 54, 53, 55, 57, 18, 84, 32].

As opposed to previous formulations that stressed the relationship to the loop quantum gravity Hilbert space [93, 6, 8, 7, 9, 94, 71], and thus was given in terms of spin networks and their geometric interpretation, we will here focus on a presentation that is as close as possible to lattice gauge theory. There is some overlap with the formulations explored in the context of auxiliary field theories on the group manifold [40, 80, 90, 91, 42, 88, 92, 82, 31, 63, 21], though our perspective is markedly different. Furthermore, there are similarities to the formulations of spin foam models in connection variables [83, 85–87, 81, 74, 30] and as more ordinary discretized theories [56, 78, 57, 35, 36], and non-commutative first-order formulations [20, 21]. In a forthcoming work [65], we will show how to extract geometric meaning directly from the formulation given here.

The formulation we explore here takes as its starting point the heuristic formulation of lattice BF theory on a 2-complex. We then insert simplicity constraints in the holonomy picture of this formulation, assuming the most general form common to most simplicity constraints. This allows us to define a very natural space of simplicity constraints covering almost all models in the literature, the exception being the model due to KKL [66, 67]. Combined with the results on the structure of the simplicity constraints of spin foam models in [65], this enables us to give natural extensions of the EPRL/FK and BC models to arbitrary, including finite, groups that we explored in [15].

The key aim of this paper is to explore the consequences of this generic formalism. We will give an explicit basis for the space of simplicity constraints in section 2.3 and discuss which models lead to real partition functions. In section 3, we then explore different ways of introducing boundaries and composing 2-complexes. These will lead to different notions of boundary Hilbert spaces. Next to the known spin network [94] and projected spin network [3, 72] spaces, we introduce a new universal boundary space common to all spin foam models built on the same group. We will also show how the assumptions we make on the structure of the simplicity constraints translate to the operator spin foam formalism, show how the usual models fall into this category and give the basis coefficients of the established models in the space of simplicity constraints in section 5.

In section 6, we use the results obtained so far to derive the general form of the transfer operator for any holonomy spin foam model in the different boundary Hilbert spaces. Transfer operators arise from a spacetime decomposition of the partition function and indeed generate the (discrete) time evolution on the given Hilbert space. Thus, transfer operators can be seen as the discrete time equivalent of Hamiltonian operators. Indeed for standard lattice systems, Hamiltonians can be derived from transfer operators by a limiting procedure. We will shortly discuss the issue of how to take the limit in section 6. The general form of the transfer operator allows us to illuminate the dynamics of spin foams, to highlight the role of the simplicity constraints and to clarify the connections between the different boundary Hilbert spaces.

We end with a discussion and outlook in section 7.

2. Holonomy formulation of spin foam models

We will begin by recalling the formulation of spin foam models based on two arbitrary finite or Lie groups G and $H \subset G$ in terms of holonomies on an arbitrary 2-complex.

2.1. The data

We start with an arbitrary, finite, combinatorial 2-complex \mathcal{C} . The 2-complex \mathcal{C} consists of faces, edges and vertices and we write $f \in \mathcal{C}_f$, $e \in \mathcal{C}_e$ and $v \in \mathcal{C}_v$, respectively.

Each face of the 2-complex comes with a fiducial orientation given by the order of edges and vertices around it, as well as a fiducial base vertex. This orientation is unrelated to any orientation on a manifold from which the 2-complex might be constructed. The combinatorial information, together with the fiducial orientation and base vertex, can be conveniently encoded by thinking of a face as an ordered set of the vertices and edges bordering it, $f = (v, e, v', e', \dots v)$. The notation $(a, b) \subset f$ will always mean that the ordered set (a, b) exists as an uninterrupted subset in the ordered set f , i.e. $f = (v, \dots, a, b, \dots v)$. We will similarly write $v \in e$ and $e \in f$ to denote adjacency relationships.

As per the data on these 2-complexes, we introduce one G element $g_{ev} = g_{ve}^{-1}$ per half-edge, and one G element g_{ef} per neighbouring edge and face, $e \in f$.⁴ In order to have a natural composition of group elements, this should be read from right to left, that is, the group elements g_{ab} and g_{bc} associated with the ordered sets (a, b) and (b, c) , respectively, compose naturally to $g_{ab}g_{bc}$ if we read g_{ab} as going from b to a and g_{bc} as going from c to b .

From these data, we define two types of face holonomies as

$$g_f = \prod_{\substack{(a,b,c) \subset f \\ b \in \mathcal{C}_e}} g_{ab}g_{bf}g_{bc} \tag{1}$$

$$\tilde{g}_f = \prod_{(a,b) \subset f} g_{ab},$$

or, more elliptically, $g_f = g_{ve}g_{ef}g_{ev'}g_{v'e'}g_{e'f} \dots$ and $\tilde{g}_f = g_{ve}g_{ev'}g_{v'e'} \dots$

2.2. The partition function

We will first consider the partition function for 2-complexes without boundary, that is, we treat all edges and faces as internal. We will see later that most spin foam models considered in the literature fall under the following definition.

⁴ Note that these g_{ef} have nothing to do with the holonomy from the midpoint of the edge to the midpoint of the face that is introduced in the wedge formalism.

Definition 2.1 (Holonomy spin foam model (no boundary)). *Let \mathcal{C} be a 2-complex with orientations and base points on the faces, G a unimodular Lie or discrete group with Haar measure dg and $H \subset G$ a subgroup of G . Then, given two complex-valued distributions E and ω on G satisfying*

- $E(hg(h)^{-1}) = E(g)$ for all $h \in H$,
- $E(g) = E(g^{-1})$ and $\omega(g) = \omega(g^{-1})$,
- $\omega(g) = \omega(\tilde{g}\tilde{g}^{-1})$ for all $\tilde{g} \in G$,

we define the holonomy spin foam partition function defined by E and ω to be

$$\mathcal{Z}(\mathcal{C}) = \int \left(\prod_{e \subset f} dg_{ef} \right) \left(\prod_{v \subset e} dg_{ev} \right) \left(\prod_{e \subset f} E(g_{ef}) \right) \left(\prod_f \omega(g_f) \right). \quad (2)$$

If we further have $\omega(g) = \overline{\omega(g)}$ and $E(g) = \overline{E(g)}$, we say that we have a real holonomy spin foam model.

An immediate consequence of the properties of E and ω is that $\mathcal{Z}(\mathcal{C})$ is independent of the orientations and base vertices chosen.

Note that the partition function as given will usually diverge, even for compact groups. For noncompact groups, there is also an ambiguity in the normalization of the Haar measure and additional divergences due to gauge orbits. However, if ω is a regular function and the group is compact, the model will be well defined. For finite groups, this is always the case.

In the rest of this section, we will study some general consequences of this definition, arising mostly from the structure of the integrand. Thus, we will ignore issues of convergence from here on.

2.3. A basis for the space of theories

Note that as the conditions on E and ω are linear, the space of partition functions given two groups $H \subset G$ carries a linear structure. Thus, we can parametrize it by giving an explicit basis for the space of distributions E and ω .

The distribution ω is a class function; thus, the characters form a basis, and for compact groups, it can be expanded as

$$\omega(g) = \sum_{\rho} \dim(\rho) \tilde{\omega}^{\rho} \text{tr}_{\rho}(D_{\rho}(g)), \quad (3)$$

with $D_{\rho}(g)$ being the representation matrix of g in the unitary irrep ρ .

We will often set $\omega = \delta_G$, or $\tilde{\omega}^{\rho} = 1$ in which case the space of theories is simply parametrized by the functions E satisfying the conditions of definition 2.1.

E encodes the analogue of the simplicity constraints for the spin foam model at hand. The presence of the delta function on the face ensures that the product of group elements around the face g_f is flat. This is however not the usual holonomy around the face \tilde{g}_f , but \tilde{g}_f interwoven with g_{ef} . If we force $g_{ef} = 1$, we have $g_f = \tilde{g}_f$ and obtain a theory of flat connections. The presence of g_{ef} and functions E that allow them to differ from the identity thus relaxes the constraints on flatness.

Thus, they exactly play the role of the simplicity constraints in ordinary spin foam models. We will see the precise relationship between the simplicity constraints in the usual operator spin foam models and the functions E in the next section. We call this space of simplicity functions $\mathcal{E}(G, H)$.

We can give an explicit basis for this space by expanding the functions in terms of the irreducible unitary representations of G and H , which we denote by ρ and k , respectively. We

have chosen for every irreducible representation some specific realization. We then chose an orthonormal basis in $\text{Hom}_H(\rho, k)$

$$I(\rho, k)_d \tag{4}$$

satisfying

$$I(\rho, k)_{d'}^\dagger I(\rho, k)_d = \delta_{dd'} \mathbf{1}_k. \tag{5}$$

Every E function can be written as

$$E(g) = \sum_{\rho, k} \dim(\rho) e_{k, dd'}^\rho \text{tr}_\rho(D_\rho(g) I(\rho, k)_d I^\dagger(\rho, k)_{d'}). \tag{6}$$

$\mathcal{E}(G, H)$ can then be parametrized through $e_{k, dd'}^\rho$. The index d can be seen as a degeneracy index for the case where the H reducible representation ρ contains more than one copy of the irreducible k . We have a real holonomy spin foam model if

$$e_k^\rho = e_k^{\rho^\dagger}. \tag{7}$$

e_k^ρ satisfy certain (linear) relations due to the fact that $E(G) = E(g^{-1})$. The precise form of these depends on the groups in question and is given in appendix A.

3. Boundaries and Hilbert spaces

We can now introduce a notion of boundary, which will lead us to a new and more general notion of boundary Hilbert space.

3.1. Boundaries

We can now introduce and study boundaries into the formalism. To do so, we chose an arbitrary graph Γ in \mathcal{C} , with edges $\Gamma_e \subset \mathcal{C}_e$ and vertices $\Gamma_v \subset \mathcal{C}_v$ subsets of the edges and vertices of \mathcal{C} , respectively, as the boundary graph of \mathcal{C} . We then write Γ_{ev} for the set of pairs $v \in e$ in $\Gamma_v \times \Gamma_e$.

The partition function is then an element of the space $L^2(G^{|\Gamma_{ev}|})$ by dropping the integration over the group elements associated with these pairs:

$$\mathcal{Z}^\Gamma(\mathcal{C})[g_{ev}] = \int \left(\prod_{e \subset f} dg_{ef} \right) \left(\prod_{\substack{v \subset e \\ ev \notin \Gamma_{ev}}} dg_{ev} \right) \left(\prod_{e \subset f} E(g_{ef}) \right) \left(\prod_f \omega(g_f) \right). \tag{8}$$

This definition has the advantage that the inner product of the partition functions corresponds to the gluing along the graph. That is, for two complexes $\mathcal{C}, \mathcal{C}'$ with isomorphic boundary graphs, we have

$$\overline{\langle \mathcal{Z}^\Gamma(\mathcal{C}), \mathcal{Z}^\Gamma(\mathcal{C}') \rangle}^\Gamma = \mathcal{Z}(\mathcal{C} \cup_\Gamma \mathcal{C}'), \tag{9}$$

where $\mathcal{C} \cup_\Gamma \mathcal{C}'$ indicates the 2-complex with Γ in both complexes identified and now internal. This follows immediately from definitions. For a real spin foam model, we, of course, can drop the complex conjugation. Note that the edges and vertices of Γ become edges and vertices of $\mathcal{C} \cup_\Gamma \mathcal{C}'$. In particular, these edges can be bivalent.

The integrand in (2), given by

$$\left(\prod_{e \subset f} E(g_{ef}) \right) \left(\prod_f \omega(g_f) \right), \tag{10}$$

has the following symmetries:

$$\begin{aligned} g_{ev} &\rightarrow h_e^{-1} g_{ev} g_v \\ g_{ef} &\rightarrow h_e^{-1} g_{ef} h_e \end{aligned} \quad (11)$$

for $h_e \in H \subset G$ and $g_v \in G$.

Due to the symmetries of the integrand, the partition function actually can be considered to live in a smaller subspace of $L^2(G^{|\Gamma_{ev}|})$, i.e.

$$\mathcal{Z} \in \mathcal{H}^\Gamma = L^2(G^{|\Gamma_{ev}|} / (G^{|\Gamma_v|} \times H^{|\Gamma_{e^l}|})), \quad (12)$$

with the action of $(g_v, h_e) \in (G^{|\Gamma_v|} \times H^{|\Gamma_{e^l}|})$ on g_{ve} by left and right multiplication: $(g_v, h_e) \triangleright g_{ve} = g_v g_{ve} h_e$.

We call this space the universal boundary space for the class of models $\mathcal{E}(G, H)$ and write $\mathcal{H}_{\text{UBS}}^\Gamma$.

This should be contrasted with the usual projected spin networks space:

$$\mathcal{Z} \in \mathcal{H}_{\text{PSN}}^\Gamma = L^2(G^{|\Gamma_{e^l}|} / (H^{|\Gamma_v|})). \quad (13)$$

The spin network basis for this space is discussed in appendix B. In the case of no degeneracies, basis elements are labelled by a ρ per half-edge, an intertwiner between the ρ per vertex and a k per edge, and we denote them as

$$|\rho_{ev}, \eta_v, k_e\rangle.$$

3.2. Trimmed complexes and projected spin networks

If the neighbourhood of the boundary of \mathcal{C} is of the form $\Gamma \times [0, 1]$, we can make contact to the projected spin network space $L^2(G^{|\Gamma_{e^l}|} / H^{|\Gamma_v|})$. In projected spin networks, the subgroup invariance is on the vertices of the boundary graph, rather than on the edges. If the neighbourhood of Γ is $\Gamma \times [0, 1]$, every boundary vertex has an associated internal edge $v \times [0, 1]$; by ‘splitting’ this associated internal edge, we can move the subgroup invariance to the boundary vertices. To do so, we need a square root of the E function, with the same subgroup covariance

$$E(g) = \int dg' F(g') F(g'^{-1} g). \quad (14)$$

In terms of the basis coefficients, this gives $f_k^\rho f_k^\rho = e_k^\rho$. This means that whenever we have a term of the form

$$\int dg dg' f(gg') F(g) F(g')$$

with $g, g' \in G$, we can reparametrize with $\tilde{g} = gg'$ and obtain

$$\int dg d\tilde{g} f(\tilde{g}) F(g) F(g^{-1} \tilde{g}) = \int dg f(g) E(g);$$

thus, if the E function defines a projector, we obtain $E = F$.

While F inherits the symmetries of E , i.e. $F(hgh^{-1}) = F(g)$, for all $h \in H$, we generally have that $F(g^{-1}) \neq F(g)$. Therefore, the amplitudes constructed from F will depend on the orientations of the faces, and we will need to keep an explicit track of the orientation of the group elements associated with the edges in question. We do this by writing $g_{vev'} = g_{v'ev}^{-1}$ for $(vev') \subset f$ instead of g_{ef} , and we will split these as $g_{ve}^f g_{ev'}^f$.

Consider now the partition function associated with a ‘cylindrical’ 2-complex $\mathcal{C} = \Gamma \times [0, 1]$ with boundary equal to two copies of Γ , Γ^1 and Γ^2 . For simplicity for this section,

we will specialize to the case $\omega = \delta$. It is straightforward but notationally cumbersome to extend the discussion to the general case by introducing a square root of the face weight.

This partition function can then be factorized into the operators defined by

$$\begin{aligned} \mu^\Gamma[g_{ve}, \tilde{g}_{v'ev}] &= \int \prod_{\substack{v \in \Gamma_v, e' \notin \Gamma_e \\ v \in e'}} dg_{ve'} \prod_{\substack{v \in \Gamma_v, e' \notin \Gamma_e \\ v \in e', e' \in f}} dg_{ve'}^f \prod_{\substack{e \in \Gamma_e \\ (vev') \subset f}} dg_{vev'} \\ &\times \prod_{e \in \Gamma_e} \delta(g_{ve} g_{vev'} g_{ev'} g_{v'e'} g_{v'e'}^f \tilde{g}_{v'ev} g_{e'v}^f g_{e'v}), \end{aligned} \quad (15)$$

where $(e''v)$ (ve) , (ev') , $(v'e')$ and $(vev') \subset f$, and $e \in \Gamma_e$. e'' and e' are the edges $v \times [0, 1]$, and $v' \times [0, 1]$. The group elements in the delta function are those corresponding to the half of the face f near the boundary edge e . Note that the group elements $\tilde{g}_{v'ev}$ are reversely oriented with respect to the boundary; they are on the ‘opposite side’ of the half-face in the delta.

μ defines a map from the projected spin network space

$$\mathcal{H}_{\text{PSN}}^\Gamma = L^2(G^{|\Gamma_e|}/H^{|\Gamma_v|})$$

to the universal boundary space

$$\mathcal{H}_{\text{UBS}}^\Gamma = L^2(G^{|\Gamma_{ev}|}/(G^{|\Gamma_v|} \times H^{|\Gamma_e|}))$$

via

$$\psi_{\text{UBS}}(\{g_{ve}\}) = \int \prod_{e \in \Gamma} d\tilde{g}_{v'ev} \mu^\Gamma[g_{ve}, \tilde{g}_{v'ev}] \psi_{\text{PSN}}(\{\tilde{g}_{v'ev}\}). \quad (16)$$

This is constructed such that we have

$$\mathcal{Z}^{\Gamma+\Gamma^2}(\Gamma \times [0, 1])[g_{ve}^1, g_{ve}^2] = \int dg_{vev'}^1 dg_{vev'}^2 \mu^{\Gamma^1}[g_{ve}^1, g_{vev'}^1] \mu^{\Gamma^2}[g_{ve}^2, g_{vev'}^2] \prod_{e \in \Gamma} \delta(g_{vev'}^1 g_{vev'}^2). \quad (17)$$

This can be seen by an explicit calculation; however, these calculations are greatly facilitated by the graphical notation we will introduce in the next section, and we will illustrate them using examples there.

In the spin network basis for the case without degeneracies, the μ map can be expressed as

$$\langle \rho_{ev}, \eta_v, k_e | \mu | k_{ev}, \iota_v, \rho_e \rangle = \prod_{ev} \delta_{\rho_{ev}, \rho_e} \prod_v \langle \eta_v | \bigotimes_{e \ni v} I(\rho_e, k_{ev}) f_{k_{ev}}^{\rho_e} | \iota_v \rangle \prod_e \sqrt{\frac{\dim(k_e)}{\dim(\rho_e)}} e_{k_e}^{\rho_e}. \quad (18)$$

In general, if the boundary of the 2-complex is of the form $\Gamma \times [0, 1]$, we can factorize its spin foam amplitude into the amplitude on the ‘trimmed complex’ \mathcal{C}_t , and μ for the boundary graph. The trimmed complex is the complex with ‘half of the boundary faces taken off’, or, more technically, where we consider the boundary edges and vertices not to be part of the edge set and vertex set of the 2-complex but to be in a separate set of boundary edges. We thus have $\mathcal{C}_v, \mathcal{C}_e, \mathcal{C}_f, \Gamma_v, \Gamma_e$ as separate spaces, however still with adjacency relations and orientations amongst each other as before.

$$\mathcal{Z}^\Gamma(\mathcal{C}) = \mu^\Gamma \tilde{\mathcal{Z}}^\Gamma(\mathcal{C}_t) \quad (19)$$

with

$$\tilde{\mathcal{Z}}^\Gamma(\mathcal{C}_t)[g_{ef}] = \int \left(\prod_{e \subset f} dg_{ef} \right) \left(\prod_{\substack{v \subset e \\ e \in \mathcal{C}_e}} dg_{ev} \right) \left(\prod_f \delta(g_f) \right) \left(\prod_{\substack{e \in \mathcal{C}_e, \subset f \\ \nexists v \in e, \in \Gamma_v}} E(g_{ef}) \right) \left(\prod_{\substack{e \in \mathcal{C}_e, \subset f \\ \exists v \in e, \in \Gamma_v}} F(g_{ef}) \right). \quad (20)$$

Note that this crucially depends on the orientations of the faces touching the boundary, and the composition of amplitudes only has the natural interpretation in terms of combining complexes if the orientations match up.

We see that μ embeds the projected spin network state space into the, in some sense larger, universal boundary space defined above, and clearly $\mathcal{Z}^\Gamma(\mathcal{C})$ lives in the image of μ^Γ ; thus, we can equally well see the partition function as an element of the projected spin network space.

3.2.1. Subgroup spin networks. μ will generically have a non-zero kernel depending on E or F , respectively. Thus, we can actually see $\tilde{\mathcal{Z}}$ as living in the coimage of μ , which will allow us to go to subspaces of projected spin networks, for example, H spin networks, as discussed in [52]. This is the case, for example, in the EPRL spin foam model.

We can realize this restriction to subgroup spin networks explicitly if there is an ω' with the properties of ω such that the E function satisfies

$$E(g)\omega(gg'g''g''')E(g'') = \int_H dh\omega'(g'gh)E(g)\omega'(g''g''h^{-1})E(g''). \quad (21)$$

We can then glue via subgroup integrations on the boundary edges. Again, it is easy to see that this leads to the correct gluing using the graphical notation in the next section. We assume that every face has at most one boundary edge and replace ω on those faces with ω' , as well as reducing the boundary group element to live in the subgroup. The partition function then becomes

$$\tilde{\mathcal{Z}}^\Gamma(\mathcal{C}_t)[h_e] = \int \left(\prod_{e \subset f} dg_{ef} \right) \left(\prod_{\substack{v \subset e \\ e \in \mathcal{C}_e}} dg_{ev} \right) \left(\prod_f \omega^*(g_f) \right) \left(\prod_{\substack{e \subset f \\ e \notin \Gamma_e}} E(g_{ef}) \right) \left(\prod_{\substack{e \subset f \\ e \in \Gamma_e}} E(h_e g_{ef}) \right), \quad (22)$$

where ω^* is ω if the face does not contain a boundary edge, and ω' if it does.

3.3. A graphical notation

It is illuminating to illustrate the structure of the convolutions in the partition function using a graphical notation. This will allow us to explicitly keep track of the way group elements in different faces are identified. The graphical notation will have three ingredients, corresponding to the face amplitude ω , the insertion of E around a face and the insertion and integration of g_{ev} .

We will indicate ω by a solid line. White circles crossed by the line indicate group elements that are multiplied together to form the argument of ω . The fins of the line indicate the order in which the inserted group elements should be multiplied. Two circles joined by a dotted line indicate the same group element. If one of them is crossed, the group elements should be inverse of each other. The E function is indicated by a grey circle; the cross indicates that the argument in the E function should be the inverse of that it is linked to. We will represent its convolution square root F with a grey half-circle. A white box indicates an insertion of oriented group elements into the lines that pass through it. This is given in figure 1. A set of relations that the weights typically satisfy can be succinctly expressed in this language, see for example figure 2.

Using these ingredients, we can represent the structure of a face containing four edges as in figure 3. The structure of two half-faces, and their composition in the projected spin network space, is given in figure 4. It is now a straightforward application of the relations illustrated in figure 2 to see that the two half-faces do indeed compose to a full face.

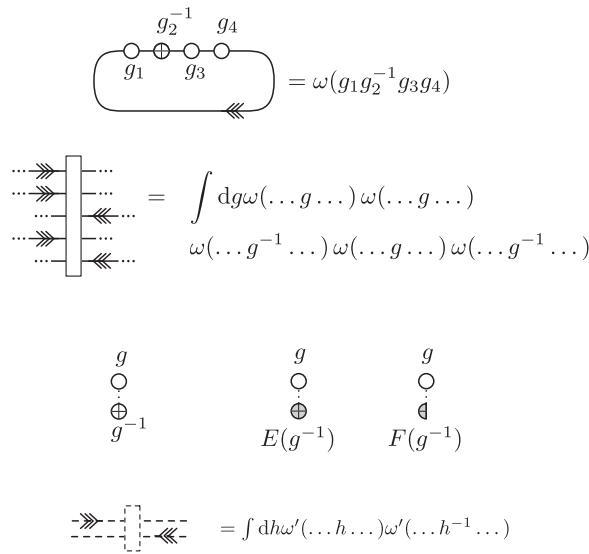


Figure 1. Ingredients of the graphical notation.

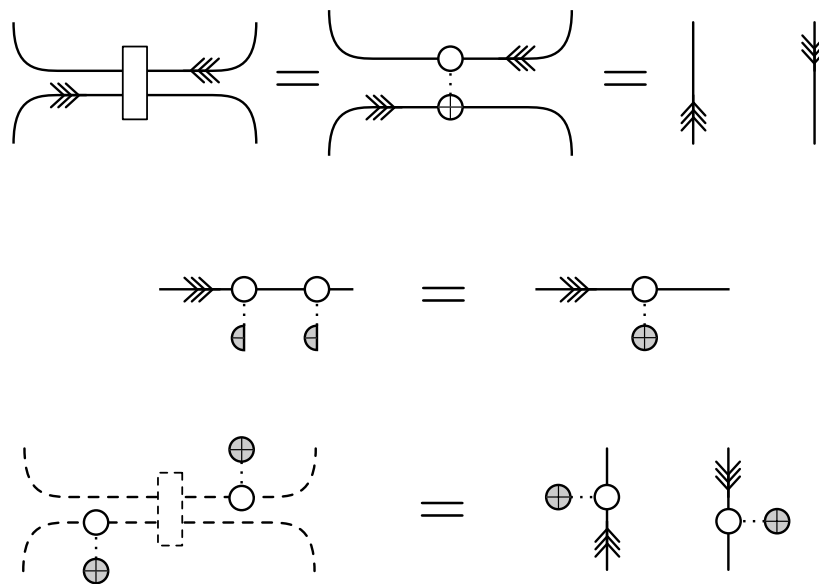


Figure 2. Some possible relations among the ingredients. The upper identity holds for $\omega = \delta$, the second is a consequence of the definition of F and the third is the subgroup property.

We can write out the composition illustrated in figure 4. Let that face be $f = (012345670)$, with $0, 2, 4, 6 \in \mathcal{C}_v$ and $1, 3, 5, 7 \in \mathcal{C}_e$. Taking care of the orientations in the delta function and F , the integrand for a face on the right-hand side reads

$$\delta(g_{01}g_{012}g_{12}g_{23}g_{23}^f\tilde{g}_{210}g_{70}^f g_{70})\delta(g_{45}g_{456}g_{56}g_{67}g_{67}^f\tilde{g}_{654}g_{34}^f g_{34}) \times E(g_{012})F(g_{23}^f)F(g_{70}^f)E(g_{456})F(g_{67}^f)F(g_{34}^f)\delta(\tilde{g}_{210}\tilde{g}_{674}). \tag{23}$$

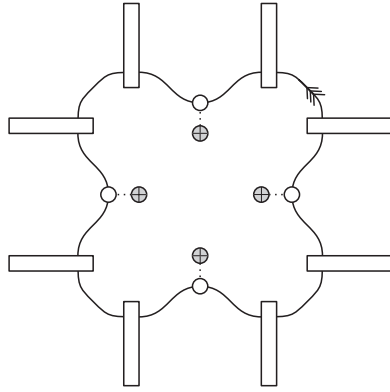


Figure 3. The structure of a face.

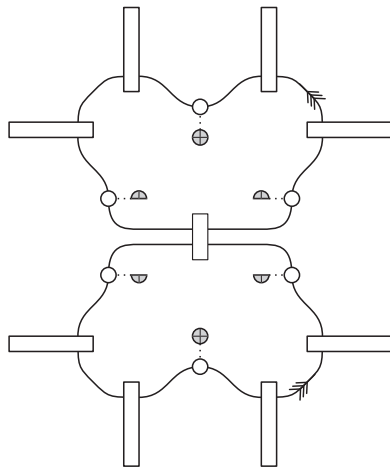


Figure 4. Two half-faces convoluted.

After combining the delta functions, we obtain

$$\delta(g_{01}g_{012}g_{12}g_{23}g_{23}^f g_{34}^f g_{34}g_{45}g_{456}g_{56}g_{67}g_{67}^f g_{70}^f g_{70}) \times E(g_{012})F(g_{23}^f)F(g_{70}^f)E(g_{456})F(g_{67}^f)F(g_{34}^f), \tag{24}$$

which, with equation (14), gives the amplitude for a face in the normal partition function, pictorial represented in figure 3,

$$\delta(g_{01}g_{012}g_{12}g_{23}g_{234}g_{34}g_{45}g_{456}g_{56}g_{67}g_{670}g_{70})E(g_{012})E(g_{234})E(g_{456})E(g_{670}). \tag{25}$$

If we further have the subgroup property on E , we can replace the convolution in G with a convolution in H , thus reducing to the subgroup spin networks. This is illustrated in figure 5.

4. Gluing of 2-complexes

We will now show how the different Hilbert spaces discussed above correspond to different gluing operations on the 2-complexes. To simplify illustrations, we will focus on the case of the 2-complex dual to a triangulated surface, and various related 2-complexes. In that

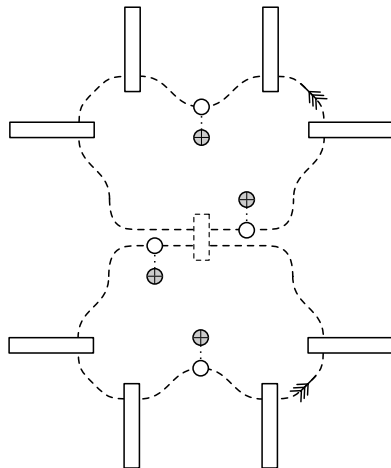


Figure 5. Two half-faces convoluted using the subgroup property.

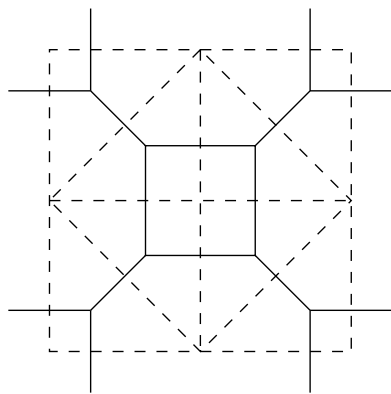


Figure 6. A part of a triangulation (dashed lines) and its dual (solid lines).

case, the dual 2-complex has trivalent vertices and bivalent edges; see, for example, figure 6. The 2-complex contains a central face with four edges, which has the structure illustrated in figure 3.

The partition function for this part of the amplitude is given in figure 7. This amplitude can be obtained by various gluings from different building blocks, depending on how we take it apart.

4.1. Face to face: the universal boundary space

The first gluing simply uses composition in the universal boundary space. As noted above, arbitrary 2-complexes can be glued along arbitrary edges. Thus, we can in particular simply take the faces of the 2-complex as individual partition functions. These partition functions for single faces were introduced in the companion paper as effective face weights ω_f :

$$\omega_f = \mathcal{Z}(f). \tag{26}$$

We obtain one such ω_f per type of face. In particular, we have only one such effective face weight for a regular complex. These can then be composed simply by equation (9) to yield arbitrary 2-complexes. This is sketched for the complex above in figure 8.

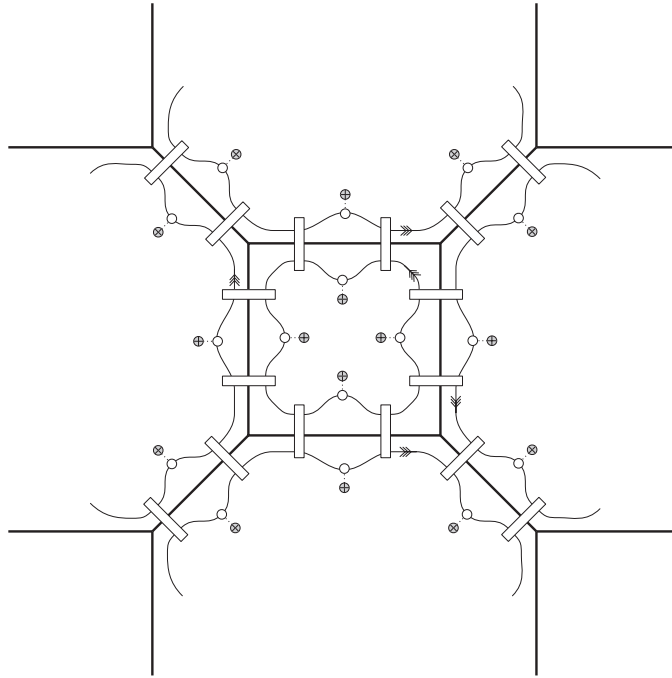


Figure 7. The part of the partition function corresponding to the part of the triangulation shown in figure 6.

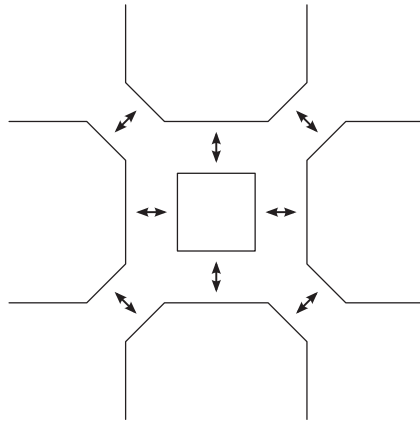


Figure 8. The composition of the 2-complex from faces. Each small double arrow indicates a composition in the universal boundary space associated with one edge.

Note that the composition in $\mathcal{H}_{\text{UBS}}^\Gamma$ as given by (9) can be localized to a subset of the boundary graph. That is, given $\Gamma \in \mathcal{C}$ and $\Gamma' \in \mathcal{C}'$ and a graph Γ^b that is a subgraph of Γ and Γ' , we can treat only the common subgraph as boundary, and glue amongst it, yielding,

$$\text{tr}^{\Gamma^b} \mathcal{Z}^\Gamma(\mathcal{C}) \mathcal{Z}^{\Gamma'}(\mathcal{C}') = \mathcal{Z}^{\Gamma \cup_{\Gamma^b} \Gamma'}(\mathcal{C} \cup_{\Gamma^b} \mathcal{C}'), \quad (27)$$

where $\Gamma \cup_{\Gamma^b} \Gamma'$ is the graph obtained by identifying $\Gamma^b \in \Gamma, \Gamma'$ and then deleting the identified graph, and tr^{Γ^b} indicates integrating the elements of $\mathcal{H}_{\text{UBS}}^{\Gamma^b} \otimes \mathcal{H}_{\text{UBS}}^{\Gamma'^b}$ against the element $\prod_{ev} \delta(g_{ev} g'_{ev}{}^{-1})$.

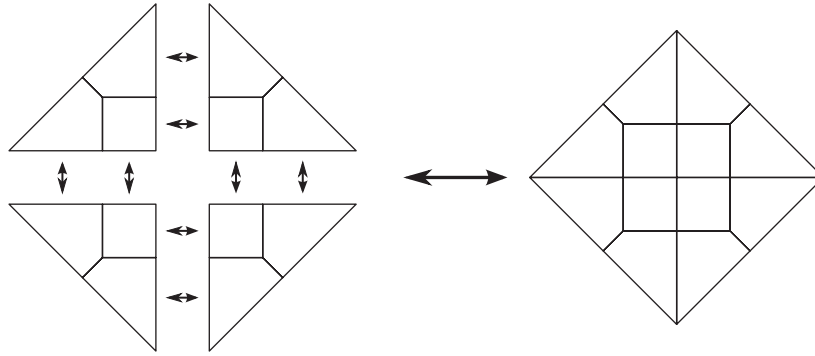


Figure 9. The composition of the 2-complex from the content of the simplex. Each small double arrow indicates a composition in the universal boundary space associated with one edge.

Note that due to gauge invariance at the two-valent vertices, ω_f depends only on as many variables as f has edges. It will be convenient to also introduce ω'_f such that

$$\omega_f(g_{ve}, g_{ev'}, g_{v'e'}, g_{e'v''}, \dots) = \omega'_f(g_{ve}g_{ev'}, g_{v'e'}g_{e'v''}, \dots).$$

4.1.1. A special case: wedges to wedges. In the case of the complex dual to a triangulation, each type of face can occur; thus, effective face weights are not a convenient choice for constructing theories. However, we can construct a second kind of dual 2-complex \mathcal{C}' constructed from the so-called wedges. That is, we take as faces the intersections of the faces of the dual 2-complex and the simplices. In our two-dimensional example, this means that each triangle now contains three such wedges. These can then be composed in the universal boundary space again. The advantage is that now we only need one type of amplitude that we are gluing: the content of a simplex. This is illustrated in figure 9; the new complex \mathcal{C}' is on the right.

Calling the complex of wedges σ^* , and using the trace from equation (27), we can thus write the entire partition function for a dual complex made from wedges:

$$\mathcal{Z}(\mathcal{C}') = \text{tr}^\Gamma \bigotimes_v \mathcal{Z}(\sigma^*). \tag{28}$$

4.2. Half-face to half-face: the spin network spaces

As noted above, the gluing of two trimmed partition functions in $\mathcal{H}_{\text{PSN}}^\Gamma$ also generates a natural composition:

$$\overline{\langle \tilde{\mathcal{Z}}^\Gamma(\mathcal{C}), \mathcal{Z}^\Gamma(\mathcal{C}') \rangle_{\text{PSN}}} = \tilde{\mathcal{Z}}(\mathcal{C} \tilde{\cup}_\Gamma \mathcal{C}'). \tag{29}$$

Due to the topological restrictions near the boundary required for trimming the partition function, all edges of Γ in $\mathcal{C} \cup_\Gamma \mathcal{C}'$ are bivalent, and $\mathcal{C} \tilde{\cup}_\Gamma \mathcal{C}'$ is the 2-complex obtained by identifying the two copies of Γ , erasing the bivalent edges in Γ_e and then the bivalent vertices in Γ_v . This is the composition usually done in spin foam models.

For the case of a 2-complex dual to a triangulation, this is a natural type of gluing; an example is illustrated in figure 10. The half-faces and composition on the left-hand side are exactly those illustrated in figure 4.

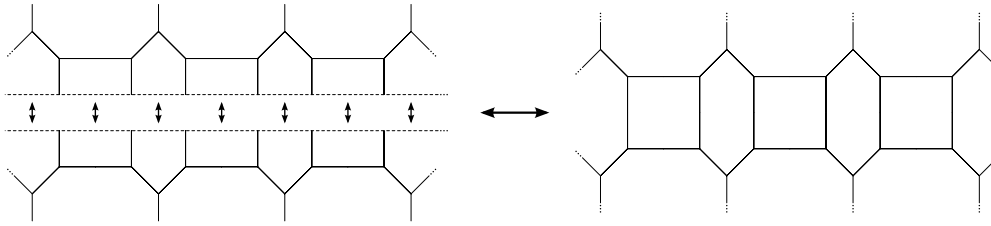


Figure 10. The composition of the 2-complex from half-faces. Each small double arrow indicates a composition in the projected spin network space associated with one edge.

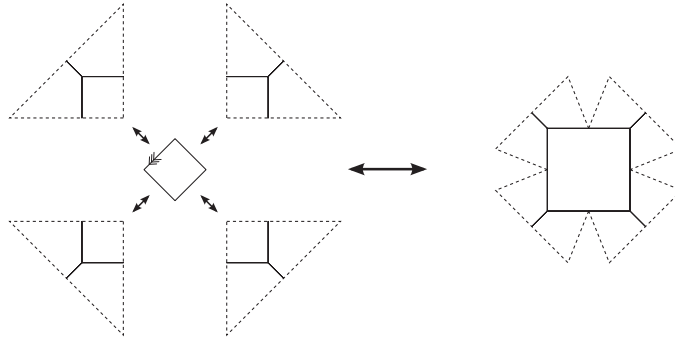


Figure 11. The composition of half-wedges around a face.

4.3. Half-faces around a face

We can generalize the above gluing by using the partition function with δ functions on the half-faces, and gluing around a face amplitude ω in the following sense.

For a set of boundary edges $e_a, a = 1 \dots n$ in a set of trimmed complexes \tilde{C}_a , we can form the new partition function on a trimmed complex with an interior face $f = (v, e_1, v', e_2, \dots, e_n, v)$:

$$\tilde{Z} \left(\bigcup_f \tilde{C}_a \right) = \int dg_{e_a} \omega \left(\prod_a g_{e_a} \right) \prod_a \tilde{Z}_a(\tilde{C}_a). \quad (30)$$

This is illustrated in figure 11.

In this way, we can parametrize the partition function of the dual complex of a triangulation by the complex of trimmed wedges in a simplex and the face weight.

5. Standard spin foam models in the holonomy language

In this section, we show that the BC, EPRL and FK models can be expressed in the language above. To do so, we will relate the holonomy model defined above to the operator models of [18]. We can then give natural generalizations of the BC, EPRL and FK models to arbitrary and, in particular, finite groups.

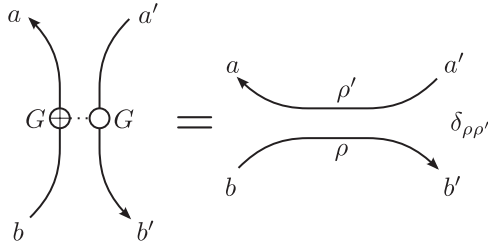


Figure 12. Orthogonality in the graphical notation.

5.1. From operators to holonomies

We can relate the OSFM to the ones by the insertion of the group integrations g_{ef} at each pair of edge and face by using the orthogonality of group elements:

$$\dim(\rho) \int dg D_{\rho_a}^b(g^{-1}) D_{\rho_{a'}}^{b'}(g) = \delta_a^{a'} \delta_b^{b'} \delta_{\rho\rho'}. \quad (31)$$

The structure of the manipulation is the easiest to see using an extension of the graphical notation. We will decorate the lines with irreducible representations. We can then break them into segments with indices living in the representation with arrows indicating the ingoing and outgoing indices. In figure 12, we use this calculus to express equation (31). A line joining two objects indicates the composition of tensors. A line coming into a tensor indicates an index in a representation space (downstairs) and a line outgoing indicates an index in the dual space (upstairs).

We will start by deriving the holonomy formulation given in the previous section, from the edge operator formalism of [18]. There, we have, for a colouring of the faces by irreducible representations ρ , an operator P on each edge. These operators are then contracted according to the structure of the 2-complex. For simplicity, we will focus on the case where the edge orientation and the face orientation agree. Then, we have $P_e \in \text{Hom}(\rho_f \otimes \rho_{f'} \dots \rho_{f''})$, where $f, f', f'' \ni e$. In all models studied so far except the KKL model [67, 66], P has an additional factorization property which implies that it can be expressed in the following way:

$$P_{e(a')}^{(a)} = P_{G(b)}^{(a)} \tilde{E}_{c_f}^{b_f} \otimes \tilde{E}_{c_{f'}}^{b_{f'}} \otimes \dots \otimes \tilde{E}_{c_{f''}}^{b_{f''}} P_{G(a')}^{(c)}, \quad (32)$$

where (a) is a multi-index ranging over a_f , with $f \ni e$, and P_G is the projector on the gauge-invariant subspace. $\tilde{E} \in L(\rho_f)$ are a set of one linear operator per representation space which satisfy relations ensuring that the fiducial orientations do not enter in contracting the edge operators. These are not group-covariant. We will usually suppress the dependence of \tilde{E} on the representation space as it should be clear from the context on which space it acts. If we write this projector explicitly as a gauge averaging, we obtain the following:

$$P_{e(a')}^{(a)} = \int dg_{ve} dg_{ev} \bigotimes_{f \ni e} D_{\rho_f b_f}^{a_f}(g_{ve}) \tilde{E}_{c_f}^{b_f} D_{\rho_f a_f}^{c_f}(g_{ev}). \quad (33)$$

The graphical representation of the edge operator is given in figure 13.

Inserting the resolution of the identity (31) and introducing a sum over representations, we obtain

$$\begin{aligned} P_{e(a')}^{(a)} &= \int dg_{ve} dg_{ev'} \prod_{f \ni e} dg_{ef} \bigotimes_{f \ni e} D_{\rho_f b_f}^{a_f}(g_{ve}) D_{\rho_f c_f}^{b_f}(g_{ef}) D_{\rho_f a_f}^{c_f}(g_{ev'}) \\ &\quad \times \sum_{\rho_{ef}} \dim(\rho_{ef}) D_{\rho_{ef} b'_f}^{c'_f}(g_{ef}^{-1}) \tilde{E}_{c'_f}^{b'_f}, \end{aligned} \quad (34)$$

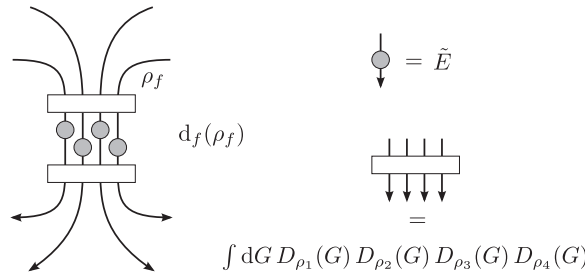


Figure 13. Factorization of the edge operator in the graphical notation.

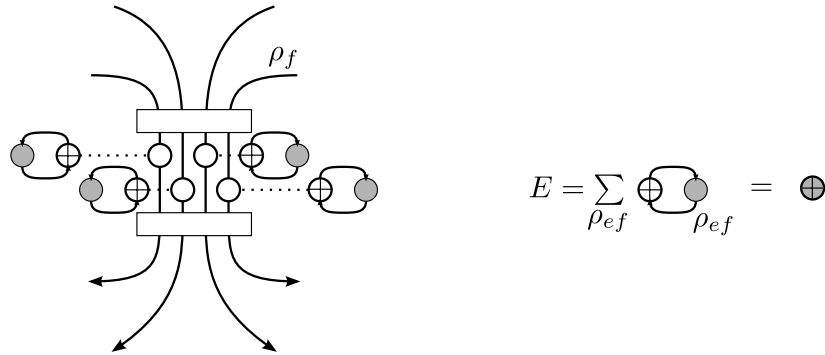


Figure 14. The E functions.

with $\sum_{\rho_{ef}} \dim(\rho_{ef}) D_{\rho_{ef} b_f}^{c_f}(g_{ef}^{-1}) \tilde{E}_{c_f}^{b_f}$ being the function E . The representation matrices around a face can be contracted to a character, and as we decoupled the representation label on the operators \tilde{E} , the sum over ρ_f can be performed exactly to arrive at the distribution (3) and we arrive at (2). Note that the conditions on \tilde{E} that imply independence of the fiducial orientations now imply $E(g) = E(g^{-1})$ and the end result is indeed invariant under reversing the orientations.

The graphical representation of this insertion of identities is given in figure 14.

In the companion paper [15] as well as in [12], more general edge operators that do not have this factorization property are also considered. Then, there still is a holonomy formulation, given that the face amplitude is simply the dimension of ρ_f . We briefly recall this construction and illustrate it in the graphical calculus in figure 15.

The dimension as a face amplitude can be given by the trace of the identity operator, or a closed circle in the graphical notation. The crucial ingredient is then a function on n copies of the group, given simply by

$$C(g_{ef}, \dots, g_{ef'}) = \sum_{\substack{\rho_{ef} \\ e \in f}} \left(\prod_{f \ni e} \dim(\rho_{ef}) D_{\rho_{ef} b_f}^{a_f}(g_{ef}) \right) P_{(a)}^{(b)}. \tag{35}$$

The reason for this formula is clear by figure 15. From this diagram, we can also see that the C functions are glued with delta functions. In particular, the integrand is schematically given by

$$\prod_e C(g_{ef}^v g_{ef'}^{v-1}, \dots) \prod_v \prod_{(ev'e') \subset f} \delta(g_{ef}^v g_{e'f}^{v-1}). \tag{36}$$

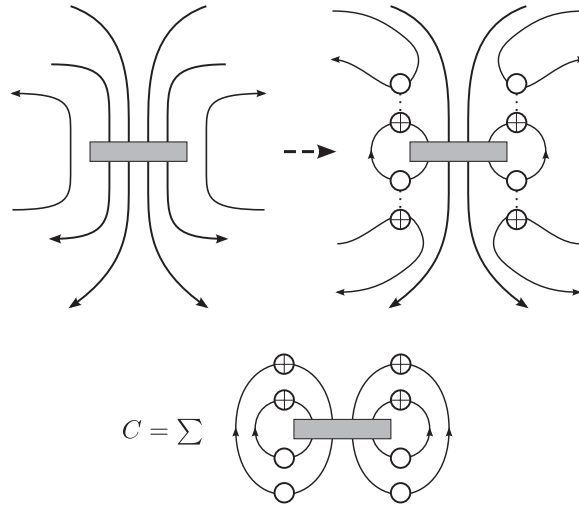


Figure 15. The C functions.

In the GFT language, C are simply the propagators and $\prod_{(eve') \subset f} \delta(g_{ef}^v g_{e'f}^{v^{-1}})$ is the vertex function, or interaction, that glues them.

5.2. The specific models for $Spin(4)$

Having in hand the relationship to the operator formalism, we can now easily give the formulation of the various established spin foam models in the holonomy language. We begin with the BF, BC [23] and EPRL models [53] which are quite straightforward and then discuss the FK [57] and BO [21, 20] in the subsequent subsections.

5.2.1. *BF, Barrett–Crane and Engle–Pereira–Rovelli–Livine.* We specify now to $G = Spin(4)$ and $H = SU(2) = Spin(4)_{diag}$ the diagonal $SU(2)$ subgroup, with irreps labelled by ρ and k , respectively. The operators \tilde{E} in $L(\rho)$ for the first set of models are given by the following:

$$\tilde{E}_{BF} = \mathbf{1}, \tag{37}$$

for BF theory,

$$\tilde{E}_{BC} = d_e(\rho) \sum_k \delta(\rho, (k, k)) I(\rho, 0) I(\rho, 0)^\dagger, \tag{38}$$

for the Barrett–Crane model, where $d_e(\rho)$ is an arbitrary edge measure factor, and for the EPRL model [54, 55, 53], we have

$$\tilde{E}_{EPRL} = d_e(\rho) \sum_k \delta(\rho, \rho_\gamma(k)) I(\rho, k) I(\rho, k)^\dagger, \tag{39}$$

where we write $\rho_\gamma(k) = (\frac{1+\gamma}{2}k, \frac{1-\gamma}{2}k)$.

We can then obtain $E(g)$ by the formulas of the previous section. For the EPRL model, the functions $E_{\text{EPRL}}^\gamma(g)$ is given by

$$E_{\text{EPRL}}^\gamma(g) = \sum_{\rho, k} \dim(\rho) d_e(\rho) \delta(\rho, \rho_\gamma(k)) \text{tr}_\rho(D_\rho(g) I(\rho, k) I(\rho, k)^\dagger), \quad (40)$$

and for Barrett–Crane, we obtain

$$E_{\text{BC}}(g) = \sum_{\rho, k, k'} \dim(\rho) d_e(\rho) \delta(k) \delta(\rho, (k', k')) \text{tr}_\rho(D_\rho(g) I(\rho, k) I(\rho, k)^\dagger). \quad (41)$$

The SU(2) irrep k' plays a very different role than in the EPRL model, namely it restricts the form of the irrep ρ but does not appear in the injection maps $I(\rho, k)$. We can further simplify this by noting that

$$\begin{aligned} \text{tr}_{(k, k)}(D_{(k, k)}(g) I((k, k), 0) I((k, k), 0)^\dagger) &= I((k, k), 0)^\dagger D_{(k, k)}(g) I((k, k), 0) \\ &= \text{tr}_k(g^+ g^{-1}) \dim(k)^{-1}, \end{aligned} \quad (42)$$

with the Spin(4) element g decomposing into the left and right SU(2) as $g = (g^+, g^-)$. Thus for choice $d_e(\rho) = 1$, we have simply

$$E_{\text{BC}}(g) = \delta(g^+ g^{-1}) = \delta_{\text{Spin}(4)_{\text{diag}}}(g), \quad (43)$$

where, for a general subgroup $H \subset G$, we write

$$\delta_H(g) = \int_H dh \delta(gh^{-1}) \quad (44)$$

for the delta function that forces a group element to lie in the subgroup.

Thus, we arrive at a particularly simple form for the Barrett–Crane model as an integral over a product of SU(2) delta functions:

$$\mathcal{Z}_{\text{BC}}(\mathcal{C}) = \int \left(\prod_{e \subset f} dh_{ef}^\pm \right) \left(\prod_{v \subset e} dg_{ev}^\pm \right) \left(\prod_{e \subset f} \delta(g^+(g^-)^{-1}) \right) \left(\prod_f \delta(g_f^+) \delta(g_f^-) \right). \quad (45)$$

For the FK model, the E function is most easily expressed in terms of coherent states, that is, the eigenstates of the Lie algebra generators. For an SU(2) representation labelled by the half-integer k , these are the states $\alpha(\mathbf{n})$ that satisfy $(\mathbf{n} \cdot L)\alpha_k(\mathbf{n}) = ik\alpha_k(\mathbf{n})$, where \mathbf{n} is a unit vector in \mathbb{R}^3 . All $\alpha_k(\mathbf{n})$ for the same \mathbf{n} differ at most by a phase. The E function for the FK model is given by

$$E_{\text{FK}}(g^+, g^-) = \sum_k \dim(k) \int d\mathbf{n} (\alpha_k^\dagger(\mathbf{n}) D_k(g^+) \alpha_k(\mathbf{n})) \overline{(\alpha_k^\dagger(\mathbf{n}) D_k(g^-) \alpha_k(\mathbf{n}))}, \quad (46)$$

which is well defined as the phases of $\alpha_k(\mathbf{n})$ and $\alpha_k^\dagger(\mathbf{n})$ cancel.

The BF theory, of course, is simply given by setting

$$E_{\text{BF}}(g) = \delta(g). \quad (47)$$

The coefficients in the basis of section 2.3 for the various models then are

- BF: $e_k^\rho = 1$
- BC: $e_k^\rho = d_e(\rho) \delta_{k,0} \sum_{k'} \delta(\rho, (k', k'))$
- EPRL: $e_k^\rho = d_e(\rho) \delta\left(\left(\frac{1+\gamma}{2}k, \frac{1-\gamma}{2}k\right), \rho\right)$.

We see from the conditions derived in section 2.3 that for real $d_e(\rho)$, all these spin foam models are indeed real holonomy spin foams.

5.2.2. *Freidel–Krasnov for $\gamma > 1$.* The FK model is defined in terms of coherent states. For $\gamma < 1$, it is equivalent to the EPRL model. To obtain its basis coefficients for $\gamma > 1$, we need to work some more. In this section, we use the shorthand $|m\rangle = \alpha(m)$.

The operator \tilde{E}_ρ for the FK model is given by

$$\tilde{E}_\rho = d_e(\rho) \int \frac{dm}{4\pi} |m\rangle^{2j^+} \langle m|^{2j^+} \otimes |\bar{m}\rangle^{2j^-} \langle \bar{m}|^{2j^-}, \quad (48)$$

with the representations $\rho = (j^+, j^-)$ subject to constraints similar to those in the EPRL model. Let

$$R_\gamma = \left\{ \left(\frac{\gamma+1}{2} \tilde{k}, \frac{\gamma-1}{2} \tilde{k} \right) : \tilde{k} = 0, \frac{1}{2}, \dots \right\}. \quad (49)$$

We can write

$$E(g) = \sum_{\rho} \dim(\rho) d_e(\rho) \delta_{R_\gamma}(\rho) \int \frac{dm}{4\pi} \langle m|g^+|m\rangle^{2j^+} \langle \bar{m}|g^+|\bar{m}\rangle^{2j^-}. \quad (50)$$

Now we want to compute

$$e_k^\rho = \frac{1}{2k+1} \text{Tr} I^\dagger(\rho, k) \tilde{E}_\rho I(\rho, k). \quad (51)$$

Note that $I(\rho, k)$ is given by its matrix elements in the coherent state basis; thus, if $\rho = (j^+, j^-)$ with $j^\pm = \frac{\gamma \pm 1}{2} \tilde{k}$ and j^+, j^-, k admissible, we have that

$$\begin{aligned} \langle I(\rho, k) | n \rangle^{2k}, | n^+ \rangle^{2j^+} \otimes | n^- \rangle^{2j^-} \rangle \\ = \frac{\sqrt{2k+1}}{C} \langle n, n^+ \rangle^{2(k+j^+-j^-)} \langle n, n^- \rangle^{2(k+j^- - j^+)} \epsilon(n^+, n^-)^{2(j^- + j^+ - k)}, \end{aligned} \quad (52)$$

with $\epsilon(\cdot, \cdot)$ being the invariant bilinear form and

$$C = \frac{(j^+ + j^- + k)! (j^- + j^+ - k)! (k + j^+ - j^-)! (k + j^- - j^+)!}{(2k)! (2j^+)! (2j^-)!} \quad (53)$$

is derived in the book by Kauffman and Lins [68].

Since $I^\dagger(\rho, k) \tilde{E}_\rho I(\rho, k)$ is proportional to identity, we only need to find

$$\langle \langle 1/2 |^{2k}, I^\dagger(\rho, k) \tilde{E}_\rho I(\rho, k) | 1/2 \rangle^{2k} \rangle, \quad (54)$$

for the case $\rho \in R_\gamma$. Then, this is equal to

$$\begin{aligned} d_e \frac{2k+1}{C^2} \int \frac{dm}{4\pi} \langle 1/2, m \rangle^{2(k+j^+-j^-)} \langle 1/2, \bar{m} \rangle^{2(k+j^- - j^+)} \epsilon(m, \bar{m})^{2(j^- + j^+ - k)} \\ \times \langle m, 1/2 \rangle^{2(k+j^+ - j^-)} \langle \bar{m}, 1/2 \rangle^{2(k+j^- - j^+)} \epsilon(m, \bar{m})^{2(j^- + j^+ - k)}. \end{aligned} \quad (55)$$

But we know

$$|\langle 1/2, m \rangle|^2 = \left(\cos \frac{\theta}{2} \right)^2, \quad |\langle 1/2, \bar{m} \rangle|^2 = \left(\sin \frac{\theta}{2} \right)^2, \quad (56)$$

where θ is the angle between the direction of m and the north pole. We can introduce polar coordinates

$$\begin{aligned} e_k^\rho &= \langle \langle 1/2 |^{2k}, I^\dagger(\rho, k) \tilde{E}_\rho I(\rho, k) | 1/2 \rangle^{2k} \rangle \\ &= d_e \frac{2k+1}{4\pi C^2} \int_0^{2\pi} d\phi \int_0^\pi d\theta \sin \theta \left(\cos \frac{\theta}{2} \right)^{2(k+j^+-j^-)} \left(\sin \frac{\theta}{2} \right)^{2(k+j^- - j^+)} \\ &= d_e \frac{2k+1}{4\pi C^2} 2\pi \int_0^{\pi/2} d\eta 2 \sin \eta \cos \eta (\cos \eta)^{2(k+j^+ - j^-)} (\sin \eta)^{2(k+j^- - j^+)} \\ &= d_e \frac{2k+1}{C^2} \frac{(k+j^+ - j^-)! (k+j^- - j^+)!}{(2k+1)!}. \end{aligned} \quad (57)$$

We thus obtained

$$e_k^\rho = d_e \frac{(2j^+)!(2j^-)!}{(j^+ + j^- + k)!(j^- + j^+ - k)!}. \quad (58)$$

Substituting $j^\pm = \frac{\gamma \pm 1}{2} \tilde{k}$, we have that for $\rho = (j^+, j^-)$, the basis coefficient is given by

$$e_k^\rho = d_e \frac{((\gamma + 1)\tilde{k})!((\gamma - 1)\tilde{k})!}{(\gamma\tilde{k} + k)!(\gamma\tilde{k} - k)!}. \quad (59)$$

For $\gamma > 1$, \tilde{k} is the minimal representation in the decomposition and

$$e_k^\rho = d_e. \quad (60)$$

Finally, making the conditions on ρ explicit again, we can write

- $\text{FK}_{\gamma > 1}: e_k^\rho = d_e(\rho) \sum_{\tilde{k}} \delta\left(\left(\frac{\gamma+1}{2}\tilde{k}, \frac{\gamma-1}{2}\tilde{k}\right), \rho\right) \frac{((\gamma+1)\tilde{k})!((\gamma-1)\tilde{k})!}{(\gamma\tilde{k}+k)!(\gamma\tilde{k}-k)!}$.

Thus, the FK model also falls into the class of real holonomy spin foam models.

5.2.3. Baratin–Oriti. Let $\beta = \frac{\gamma-1}{\gamma+1}$. We will use the following γ -dependent transformation from $\text{SU}(2)$ to $\text{SU}(2)$:

$$\text{SU}(2) \ni \cos \theta + i\vec{n}\vec{\sigma} \sin \theta = u \rightarrow u^\beta = \cos \theta_\beta + i\vec{n}_\beta\vec{\sigma} \sin \theta_\beta, \quad (61)$$

where the class angle θ_β and the unit vector \vec{n}_β are determined by conditions [21]

$$\sin \theta_\beta = |\beta| \sin \theta, \quad \text{sign}(\sin \cos \theta_\beta) = \text{sign}(\cos \theta), \quad \vec{n}_\beta = \text{sign}(\beta)\vec{n}. \quad (62)$$

Let us note that

$$(u^\beta)^{-1} = (u^{-1})^\beta, \quad (gu^\beta g^{-1})^\beta = gu^\beta g^{-1}. \quad (63)$$

The fusion coefficients for the BO model are given in [21] (equations (37) and (58)). From them, one can derive

$$\tilde{E}_{(m^+m^-)(\tilde{m}^+\tilde{m}^-)}^{j^+j^-} = \int du D_{m^-n^-}^{j^-}(u^{-1}) D_{m^+n^+}^{j^+}((u^\beta)^{-1}) \int d\tilde{u} D_{n^-\tilde{m}^-}^{j^-}(\tilde{u}) D_{n^+\tilde{m}^+}^{j^+}(\tilde{u}^\beta). \quad (64)$$

Thus, the E function

$$E(g^+, g^-) = \sum_{j^+ j^-} \dim(j^+) \dim(j^-) \tilde{E}_{(m^+m^-)(\tilde{m}^+\tilde{m}^-)}^{j^+j^-} D_{\tilde{m}^+m^+}^{j^+}(g^{+-1}) D_{\tilde{m}^-m^-}^{j^-}(g^{-1}) \quad (65)$$

is equal to

$$E(g^+, g^-) = \sum_{j^+ j^-} \dim(j^+) \dim(j^-) \int du d\tilde{u} \chi_{j^-}(u^{-1}\tilde{u}g^{-1}) \chi_{j^+}((u^\beta)^{-1}\tilde{u}^\beta g^{+-1}) \quad (66)$$

$$= \int du d\tilde{u} \delta(u^{-1}\tilde{u}g^{-1}) \delta((u^\beta)^{-1}\tilde{u}^\beta g^{+-1}). \quad (67)$$

We can simplify the equation by solving the first delta function for \tilde{u} so that $\tilde{u} = ug_-$

$$E(g^+, g^-) = \int du \delta((u^\beta)^{-1}(ug^-)^\beta g^{+-1}). \quad (68)$$

In the case of $\beta = 1$, this E function reduces to one of the BC model. The BO model is also a real holonomy spin foam model.

6. The transfer operator for holonomy spin foams

So far, we have discussed at length the structural aspects of holonomy spin foam models, their boundary Hilbert spaces and their gluings. This raises the question if, having the boundary Hilbert spaces at hand, we can define a Hamiltonian dynamics reflecting the one defined by the spin foam models.

The first step in deriving such a Hamiltonian dynamics from a given partition function is to obtain transfer operators. In standard lattice theories, for example, lattice gauge Yang–Mills theory, such transfer operators correspond to finite time steps. To obtain the Hamiltonians as infinitesimal time evolution generators for such systems, one would have to take the limit of infinitesimal time by scaling the coupling constants in time—and space directions in a certain way defined by the dynamics of the system [69, 70]

The issue of obtaining the Hamiltonians is more involved in gravitational systems, as the lattice constants and the time separation are rather encoded in the boundary states of the system. Furthermore, the question of exact and broken diffeomorphism symmetry comes in [13, 44, 46]. Only in the case that exact diffeomorphism symmetry is preserved in the discretization, we can expect the appearance of Hamiltonian constraints in the canonical formulation [44, 13]. If this holds also for the partition function, the transfer operator is a product of projection operators from which the Hamiltonian and diffeomorphism constraints can be read off [77, 76, 17, 16]. In this case, no limiting procedure is necessary. If diffeomorphism constraints are broken, one can either attempt a limiting procedure involving the boundary states or alternatively attempt to obtain an improved model by coarse graining which then carries a notion of diffeomorphism symmetry [13, 14, 47, 46, 45, 15, 43].

A third possibility is to adopt the viewpoint that the dynamics is inherently discrete, a viewpoint which is, for instance, emphasized in the framework of consistent discretizations [58, 59]. In this case, the transfer operator can only be defined for finite time steps, and a limit cannot be taken (in general).

Here, we will consider the finite time transfer operator and comment more on the issues of taking the limit to obtain the time evolution generators afterwards. We will consider a spacetime lattice with a regular slicing in the time direction, i.e. each (thick) time slice is of the form $\Gamma_s \times [0, 1]$. The discussion can be generalized to some extent to an irregular lattice and a notion of local time evolution; see, for instance, [48, 49] for a discussion in classical Regge calculus.

The definition of the transfer operator requires a choice of slicing of the underlying lattice and initially we choose one which will make the transfer operator as similar as possible to the one encountered in lattice gauge theory [69, 70, 95, 16]. As we will see, such a slicing fits well to the universal boundary Hilbert space introduced in section 3.1. In this formulation, the effective face weights introduced in [15] and recalled in section 4.1 will play a prominent role. From a simplicial geometry viewpoint, the faces are dual to the bones of the triangulation, which carry the curvature. Thus, this slicing offers a new perspective on the Hamiltonian dynamics and the semiclassical limit: it does not concentrate on the vertex (i.e. simplex) amplitude but on the gluing of simplices around the bones, where the curvature and hence the essential dynamical information reside.

On the other hand, this slicing is somewhat unusual in discrete gravity, where one often builds a transfer operator by gluing simplices to the hypersurface [1, 48, 49]. In this case, equal time hypersurfaces can be understood as (dual to) $(D - 1)$ -dimensional triangulations. As we will see in the course of the discussion, we can switch to a slicing more adapted to a simplicial viewpoint by using the μ map between the projected spin network and the universal Hilbert space introduced in section 3.2. The μ map can then be understood to project onto the

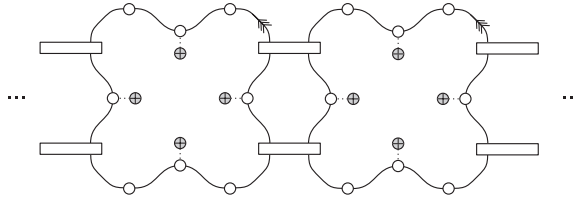


Figure 16. The composition of two effective face weights as it occurs in the definition of K .

solutions of the simplicity constraints—the (stripped) transfer operator will be sandwiched between such projectors.

6.1. Transfer operator for general models

Transfer operators can be defined if we have a discrete ‘time’ direction in our 2-complex in the sense that the complex \mathcal{C} is the 2-skeleton of the complex $(\mathcal{C}_s \times [0, 1])^n$, with $\mathcal{C}_s^n \times [1] = \mathcal{C}_s^{n+1} \times [0]$. In that case, we call the edges and faces in the various \mathcal{C}_s^n spatial, and the other edges and faces temporal.

We call the graph of horizontal edges Γ^s . Then, the partition functions $\mathcal{Z}(\mathcal{C}_s)$ and $\mathcal{Z}(\Gamma^s \times [0, 1])$ act naturally on the space $\mathcal{H}_{\text{UBS}}^{\Gamma^s}$, $\mathcal{Z}(\mathcal{C}_s)$ simply by multiplication.

It follows directly from the gluing in $\mathcal{H}_{\text{UBS}}^{\Gamma^s}$ according to (9) that the partition function of \mathcal{C} can then be written as

$$\mathcal{Z}(\mathcal{C}) = (\mathcal{Z}(\mathcal{C}_s)\mathcal{Z}(\Gamma^s \times [0, 1]))^n \mathcal{Z}(\mathcal{C}_s). \tag{69}$$

This has the structure of a partition function written in terms of transfer operators. For the rest of this section, we call $\mathcal{Z}(\mathcal{C}_s) = W$, $\mathcal{Z}(\Gamma^s \times [0, 1]) = K$ and $\mathcal{Z}(\mathcal{C}_s)\mathcal{Z}(\Gamma^s \times [0, 1]) = T$, i.e.

$$T = WK, \tag{70}$$

and

$$\mathcal{Z}(\mathcal{C}) = T^n W. \tag{71}$$

The operators W and K can be written very efficiently in terms of the effective face weights ω_f which we recalled in section 4.1 in (26). Recall that these are simply given by the amplitude of a face, as illustrated in figure 3.

As the entire 1-skeleton of \mathcal{C}_s is in the boundary space, the only integrations in W are those involving g_{ef} . These are exactly the integrations one performs to obtain the effective face weights ω_f . The operator W acts as a multiplication operator in the holonomy basis of the universal Hilbert space and is just given as a product of the effective face weights

$$W[g_{ev}] = \prod_{f \in \mathcal{C}_s} \omega_f(\{g\}^f), \tag{72}$$

where the notation $\{g\}^f$ indicates the set of all group elements belonging to the face f .

For K , we have a similar simplification. Its structure is illustrated in figure 16. In addition to the integrations over g_{ef} , we have also to take into account the integrations over g_{ev} associated with the time-like edges, which we will denote by g_{ev}^t . By g_{ev}^s and g_{ev}^s , we denote holonomy variables associated with edges in the spatial hypersurfaces at two consecutive time steps. The integration kernel of the operator K in the holonomy representation is given by

$$K[g_{ev}^s, g_{ev}^s] = \int \left(\prod dg_{ev}^t \right) \prod_{f \in \Gamma^s \times [0, 1]} \omega_f(\{g\}^f), \tag{73}$$

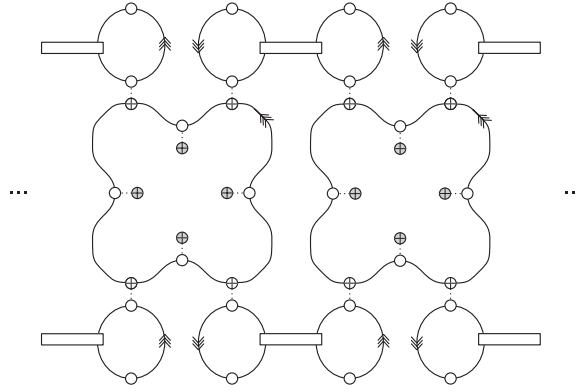


Figure 17. The decomposition of the effective face weights into reduced face weights and gauge projectors.

where each effective face weight $\omega'_f(\{g\}^f)$ depends on four variables g_{ev}^t associated with the two time-like edges of the time-like face, a set of variables g_{ev}^s associated with the space-like edges of this face shared with the graph $\Gamma_s \times [0]$ and a corresponding set g_{ev}^s shared with $\Gamma_s \times [1]$.

Each time-like edge has one boundary vertex in $\Gamma_s \times [0]$ and one in $\Gamma_s \times [1]$, and each vertex in $\Gamma_s \times [0]$ or $\Gamma_s \times [1]$ has only one temporal edge going out. Thus, we can drop e in g_{ve}^t and simply write g_v^t .

Recall that the effective face weights ω_f naturally live in the universal boundary space of a segmented line and thus have a gauge freedom that acts as $g_{ve} \rightarrow g_v g_{ve}$. Hence, we can apply gauge transformations to the vertices in $\Gamma_s \times [0]$ and $\Gamma_s \times [1]$ such that all the group variables associated with the time-like edges are equal to the identity. This allows us to write the integration over these group elements associated with the time-like edges as gauge projectors on the vertices of the graph Γ_s :

$$K = P_G K_0 P_G, \tag{74}$$

where

$$P_G[g_{ev}^s, g_{ev}^s] = \int_G \left(\prod_{v \in \Gamma_s} dg_v^t \right) \left(\prod_{e \in v} \delta(g_{ev}^s g_v^t g_{ve}^s) \right) \tag{75}$$

is simply the projection on gauge-invariant functions on the vertices of the graph.

K_0 is obtained by setting the group variables in ω_f or equivalently ω'_f associated with the time-like edges equal to the identity element:

$$K_0[g_{ev}^s, g_{ev}^s] = \prod_e \omega_{f(e)}(g_{v_1 e}, g_{ev_2}, g'_{v_2 e}, g'_{ev_1}), \tag{76}$$

where v_1, v_2 are the source and target vertexes of the edge e and $f(e) = e \times [0, 1]$ is the time-like face associated with the edge $e \in \Gamma_s$.

This structure is illustrated in figure 17. A simple application of the top relation in figure 2 will return us to figure 16.

The operator K_0 does not map from the universal boundary space to the universal boundary space as it does not produce a state with G invariance at the vertices but acts in $L^2(G^{|\Gamma_{ev}|})$ instead. The projector P_G then brings us back to the gauge-invariant universal boundary space.

It will be convenient to write $K_0 = \prod_e K_e$ as an operator in the usual sense, in terms of left and right shift operators $(L(g) \triangleright f)(\cdot) = f(g^{-1} \cdot)$ and $(R(g) \triangleright f)(\cdot) = f(\cdot g)$, respectively. To

this end, we introduce a bra–ket notation such that we write an element $f(g_1, g_2) \in L^2(G \times G)$ as $f(g_1, g_2) = \langle g_1, g_2 | f \rangle$.

The operator K_e can then be written as

$$K_e = \int dg_A dg_B \prod_{i=1}^4 (dg_i E(g_i)) \omega_f(g_B g_A^{-1}) L_1(g_A) R_1(g_1 g_A) L_2(g_B g_3) R_2(g_2), \quad (77)$$

where the left and right shift operators act as

$$L_1(g_a) R_1(g_b) L_2(g_c) R_2(g_d) |g_1, g_2\rangle = |g_a^{-1} g_1 g_b, g_c^{-1} g_2 g_d\rangle. \quad (78)$$

In summary, the transfer operator is given by

$$T = W \cdot K = \left(\prod_{f \in \mathcal{C}_s} \omega_f \right) \left(\prod_{v \in \mathcal{C}_s} P_{G,v} \right) \left(\prod_{e \in \mathcal{C}_s} K_e \right) \left(\prod_{v \in \mathcal{C}_s} P_{G,v} \right), \quad (79)$$

where W acts as a multiplication operator in the holonomy basis and factorizes over the faces, $P_{G,v}$ is the projector on the G gauge-invariant subspace at the vertex v and we defined the action of K_e in equation (77).

6.2. Transfer operator in the spin network basis and simplicity constraints

The operator K_e is the simplest in the spin basis. As K_e just acts on one edge, it is sufficient to consider the one-edge Hilbert space $L^2(G \times G)$. A basis for this space would be given by $|\rho_1, i_1, j_1; \rho_2, i_2, j_2\rangle$, where i_1, j_1, i_2, j_2 are magnetic indices in the representations ρ_1, ρ_2 , respectively.

However, to compactify the notation, it is useful to introduce a basis adapted to the H group in a given ρ representation. The basis we introduce is of Gelfand–Tsetlin type [62]. It is labelled by

$$j \leftrightarrow \{k, d, m\}, \quad |\{k, d, m\}\rangle = I(\rho, k)^d |m\rangle, \quad (80)$$

where k is the label of the H representation, d is the multiplicity index (in the multiplicity-free case, it will be omitted) and $|m\rangle$ is a basis in the k representation. Thus, we replace the (four) magnetic indices of the ρ representations in $L^2(G \times G)$ by four indices $j_i \leftrightarrow \{k_i, d_i, m_i\}$.

The basis in the space of H -invariant functions is thus spanned by

$$|\rho_1, \{k_1, m_1\}; k; \rho_2, \{k_2, m_2\}\rangle = \frac{1}{\sqrt{\dim(k)}} \sum_m |\rho_1, \{k_1, m_1\}, \{k, m\}; \rho_2, \{k_2, m_2\}, \{k, m\}\rangle, \quad (81)$$

the same functions as given in (B.8).

As follows from (77) and figure 17, the operator K_e can be decomposed into the following more elementary components:

$$K_e = K_1^s K^s K_3^s K_2^t K_4^t, \quad (82)$$

where

$$K^s = \int dg_A dg_B \omega_f(g_B g_A^{-1}) R_1(g_A) L_2(g_B) = \int dg \omega_f(g) R_1(g) \int dg' R_1(g') L_2(g') \quad (83)$$

and

$$K_1^s = \int dg_1 E(g_1) R_1(g_1) \quad K_2^t = \int dg_2 E(g_2) R_2(g_2) \quad (84)$$

$$K_3^s = \int dg_3 E(g_3) L_2(g_3) \quad K_4^t = \int dg_4 E(g_4) L_1(g_4). \quad (85)$$

Let us note that K_2^t and K_4^t commute with the rest of the operators and $K_1^s K^s K_3^s$ commutes with P_G .

We also have

$$K^s = \sum_{\rho} \tilde{\omega}^{\rho} P_{\rho}, \quad (86)$$

where P_{ρ} is the projection onto the subspace in $L^2(G \times G)$ spanned by the following orthonormal basis labelled by j_1, j_2

$$\frac{1}{\sqrt{\dim(\rho)}} \sum_j |\rho, j_1, j; \rho, j, j_2\rangle. \quad (87)$$

The operators $K_n^{s/t}$ act on the basis $|\rho_1, i_1, j_1; \rho_2, i_2, j_2\rangle$ as follows. The operator K_1^s changes only the index $j_1 = \{k, d, m\}$ by the multiplication of the matrix

$$e_{\rho_1}^k \tilde{d}_d \delta_{\tilde{m}m} \delta_{\tilde{k}k}. \quad (88)$$

Similarly, the operators K_2^t, K_3^s, K_4^t act only on the indices j_2, i_2, i_1 , respectively.

In the multiplicity-free case, we have a straightforward eigenfunction expansion of both $K_2^t K_4^t$ and $K_1^s K^s K_3^s$. The eigenvectors with non-vanishing eigenvalues for $K_1^s K^s K_3^s$ are given by

$$|\rho, \{k_1, m_1\}; \{k_2, m_2\}\rangle = \frac{1}{\sqrt{\sum_k \dim(k) (e_{\rho}^k)^2}} \sum_{k,m} e_{\rho}^k |\rho, \{k_1, m_1\}, \{k, m\}; \rho, \{k, m\}, \{k_2, m_2\}\rangle, \quad (89)$$

with corresponding eigenvalues

$$K_1^s K^s K_3^s |\rho, \{k_1, m_1\}; \{k_2, m_2\}\rangle = \frac{\tilde{\omega}^{\rho}}{\dim(\rho)} \left(\sum_k \dim(k) (e_{\rho}^k)^2 \right) |\rho, \{k_1, m_1\}; \{k_2, m_2\}\rangle. \quad (90)$$

These are also eigenvectors for $K_2^t K_4^t$ with eigenvalues

$$e_{\rho}^{k_1} e_{\rho}^{k_2}. \quad (91)$$

In summary, the eigenvectors with *a priori* non-vanishing eigenvalues for K_e are given by (89) with eigenvalues

$$e_{\rho}^{k_1} e_{\rho}^{k_2} \frac{\tilde{\omega}^{\rho}}{\dim(\rho)} \left(\sum_k \dim(k) (e_{\rho}^k)^2 \right). \quad (92)$$

Thus, the eigenvalues are independent of the labels m_1, m_2 and come with a multiplicity $\dim(k_1) \dim(k_2)$. Note that K_e vanishes on states $|\rho_1, i_1, j_1; i_2, j_2\rangle$ with $\rho_1 \neq \rho_2$ as well as on states with $\rho = \rho_1 = \rho_2$ but orthogonal to (89).

In the gravitational models, the E functions and therefore e_k^{ρ} impose the simplicity constraints. Hence, we can say the same of K_e —it maps onto a subspace of the universal boundary Hilbert space on which the (primary) simplicity constraints hold in some form⁵. As the K_e map to a subspace which can be interpreted as solutions to the primary simplicity constraints, let us also consider the question whether W , or some suitable subset of holonomy operators, leaves this subspace of the universal boundary Hilbert space-invariant.

W is a multiplication operator that factorizes over the spatial plaquettes. The contribution from a given plaquette is of the form

$$w'_f = \sum_{\rho} \dim(\rho) \tilde{\omega}^{\rho} \prod_{e \in f_s} D_{\rho}^{\{k_{ev}, d_{ev}, m_{ev}\}}^{\{k'_{ev}, d'_{ev}, m'_{ev}\}} (g_{ev}^{-1}) e_{\rho, d_{ev}}^{k_{ev}} \delta_{k'_{ev} k_{ev}} \delta_{m'_{ev} m_{ev}} D_{\rho}^{\{k'_{ev}, d'_{ev}, m'_{ev}\}}^{\{k_{ev}, d_{ev}, m_{ev}\}} (g_{ev'}), \quad (93)$$

⁵ As the discrete form of the primary simplicity constraints does not even commute weakly, one has the choice to impose them strongly, i.e. as operator equations, as in the BC model [24] or in a certain weak form as in the EPRL model [55].

where the edge e joins v with v' . From now on, we will consider only the multiplicity-free case, thus omit the index d .

These are contractions between $\delta_{\rho_1 \rho} \delta_{\rho_2 \rho} e_k^\rho$ and basic holonomy operators

$$\psi_{\rho_1, k_1, m_1; k; \rho_2, k_2, m_2} = \sum_m D_{\rho \{k, m\}}^{\{k_1, m_1\}}(g_{ev_1}^{-1}) D_{\rho \{k_2, m_2\}}^{\{k, m\}}(g_{ev_2}). \quad (94)$$

Note that $\psi_{\rho_1, k_1, m_1; k; \rho_2, k_2, m_2}$ acting on the constant function creates the states

$$|\rho, \{k_1, m_1\}; k; \rho_2, \{k_2, m_2\}\rangle.$$

Thus, the action of W involves the multiplication of holonomy operators of the form (94). We will therefore consider the product of two such holonomy operators. This is a straightforward calculation, in which one first rewrites

$$D_{\rho j}^i(g) D_{\rho' j'}^{i'}(g) = \sum_{\rho'', i'', j''} \bar{C}^{\rho \rho' \rho''}{}_{i i'}{}^{i''} C^{\rho \rho' \rho''}{}_{j j'}{}^{j''} D_{\rho'' j''}^{i''}(g), \quad (95)$$

with $C_{j j'}^{\rho \rho' \rho''}$ being the Clebsch–Gordan coefficients of G . Here, we assume that G is multiplicity free, i.e. there is maximally one copy of a given irrep in the tensor product of two irreps. We will assume the same property to hold for the subgroup H ; furthermore, we already assumed that there is maximally one copy of a given H representation k in a given G representation.

In this case, the Clebsch–Gordan coefficient for G contracted with the maps $I(\rho, k)$ reduces to the Clebsch–Gordan coefficients of H which we can write in the Gelfand–Tsetlin-like basis as

$$C^{\rho \rho' \rho''}{}_{\{k, m\}, \{k', m'\}}^{\{k'', m''\}} = C^{kk'k''}{}_{mm' m''}. \quad (96)$$

Finally, the summation over the index m in the holonomy operators (94) leads to the following contraction of Clebsch–Gordan coefficients

$$\sum_{m, m'} C_{mm' m''}^{kk'k''} \bar{C}_{mm' m''}^{k k' k''} = \theta(k, k', k'') \delta_{m'', \bar{m}''}, \quad (97)$$

where $\theta(k, k', k'') = 1$ if k, k', k'' couple to the trivial representation and is vanishing otherwise.

The product of two holonomy operators of the form (94) is therefore given by

$$\begin{aligned} & \psi_{\rho_1, k_1, m_1; k; \rho_2, k_2, m_2} \times \psi_{\rho'_1, k'_1, m'_1; k'; \rho'_2, k'_2, m'_2} \\ &= \sum_{\rho''_1, \rho''_2, k'', k'_1, k'_2} \bar{C}_{m_1 m'_1 m''_1}^{k_1 k'_1 k''_1} \theta(k, k', k'') C_{m_2 m'_2 m''_2}^{k_2 k'_2 k''_2} \psi_{\rho''_1, k''_1, m''_1; k''; \rho''_2, k''_2, m''_2}. \end{aligned} \quad (98)$$

Here, we sum over all repeated magnetic indices. The result is again a linear combination of basis states (94).

From expression (98), we note the following: (a) Even if initially the representations satisfy $\rho_1 = \rho_2$ and $\rho'_1 = \rho'_2$, this will in general not hold for the basis states appearing on the right-hand side of (98). (b) Consider the case that we multiply two basic holonomies (94) satisfying $\rho = \rho_1 = \rho_2$ and $\rho' = \rho'_1 = \rho'_2$ which have been contracted with e_k^ρ and $e_{k'}^{\rho'}$ in the k and k' indexes, respectively. The basis states in the product holonomy are then contracted with

$$\sum_{k, k'} e_k^\rho e_{k'}^{\rho'} \theta(k, k', k'') \quad (99)$$

in the k'' index. In general, the product of holonomy operators of the form (94) will not generate a proper subspace. An exception is the Barrett–Crane model in which the E function has an enhanced symmetry that allows a restriction to spin network states with $k = 0$. In this case, $\theta(0, 0, k'') \neq 0$ indeed leads to the condition $k'' = 0$.

The fact that the holonomy operators (94) do not lead in general to a proper subspace might not be a surprise to the expert as the secondary simplicity constraints, which are conditions on the holonomies [100, 5, 51, 50], are usually not imposed in spin foam models. Indeed, the hope is that the imposition of the primary simplicity constraints on two consecutive time slices leads to the automatic imposition of the secondary constraints, whose function in a canonical formulation is to ensure that the primary simplicity constraints are preserved under time evolution. Later on, we will redefine the transfer operator to make this notion more explicit. Here, we just note that one possibility is to consider $T' = K_0^{\frac{1}{2}} P_G W P_G K_0^{\frac{1}{2}}$ as we can interpret K_0 to impose the simplicity constraints.

The operator W naturally factorizes over plaquettes and we can expect that it leads to the curvature term F in the gravitational Hamiltonian constraints, which are of the form FEE with E representing flux (infinitesimal shift) operators. On the other hand, we can also seek an expression which factorizes over the vertices of Γ_s . Such a form brings us back to the usual vertex amplitude representation of spin foams and shows the consistency of the procedure.

For this calculation, the gauge invariance at the vertices of Γ_s is essential; hence, we choose the basic states $|\rho_{ev}, \eta_v, k_e\rangle$ given in (B.12) which, using the GT basis, are defined by

$$|\rho_{ev}, \eta_v, k_e\rangle = \frac{\prod_{(ev)} \sqrt{\dim(\rho)_{ev}}}{\prod_e \sqrt{\dim(k_e)}} \prod_v \eta_{v,j_{ev}} \dots \prod_e D_{\rho_{ve}}(g_{ve})_{j_{ve}\{k_e, m_e\}} D_{\rho_{e'v'}}(g_{e'v'})_{\{k_e, m_e\} j_{e'v'}}. \quad (100)$$

Let us note that W preserves the gauge-invariant \mathcal{H}_{UBS} (with respect to the gauge action at the vertices of the underlying graph).

We want to compute the matrix elements

$$\langle \rho'_{ev}, \eta'_v, k'_e | W | \rho_{ev}, \eta_v, k_e \rangle = \langle \rho'_{ev}, \eta'_v, k'_e | P_G W P_G | \rho_{ev}, \eta_v, k_e \rangle. \quad (101)$$

W is a multiplication operator in the holonomy basis; hence, to compute the matrix elements, we introduce two resolutions of unity into the matrix elements (101). These resolutions of unity lead to an integration over the group elements g_{v_1e} and g_{ev_2} . The holonomy operator associated with a given edge e is then given by

$$(W_e)_{\{i_{fe} i_{f(e+1)}\}(\{\rho_f\})} = \prod_{f \ni e} \sum_{k_{fe}, m_{fe}} D_{\rho_f}(g_{v_1e})_{i_{ve}, \{k_{fe}, m_{fe}\}} e_{k_{fe}}^{\rho_f} D_{\rho_f}(g_{ev_2})_{\{k_{fe}, m_{fe}\}, i_{f(e+1)}}, \quad (102)$$

where the contribution (W_e) comes with magnetic indices $\{i_{fe} i_{f(e+1)}\}$ that are contracted between the different edges of a face. (Relative orientation of the edges and the face is unimportant if the model is real; otherwise, we assume here that these orientations agree.) W_e are contracted and then summed over ρ_f (multiplied with $\tilde{\omega}^{\rho_f} \dim(\rho_f)$) to obtain the full operator W .

The computation then proceeds in the following steps which are completely analogous to the construction of vertex amplitudes in spin foam models; see, for instance, [16, 84].

- (a) The integration over the group elements g_{v_1e} and g_{ev_2} leads to the Haar projector

$$P_{i_1 j_1 \dots i_n j_n}(\rho_1, \dots, \rho_n) = \int dg D_{\rho_1}(g)_{i_1 j_1} \dots D_{\rho_n}(g)_{i_n j_n} \quad (103)$$

on each half-edge. For instance, for the first half-edge, the projector is on the invariant subspace in the tensor product

$$V_{\rho_{v_1e}} \otimes V_{\tilde{\rho}_{v_1e}} \otimes \bigotimes_{f \ni e} V_{\rho_f}. \quad (104)$$

- (b) The Haar projectors on each half-edge can be split into a sum over a basis of orthonormal-invariant vectors or intertwiners η of the corresponding representation space.

$$P_{i_1 j_1 \dots i_n j_n}(\rho_1, \dots, \rho_n) = \sum_{\eta} |\eta\rangle_{i_1 \dots i_n} \langle \eta |_{j_1 \dots j_n}. \quad (105)$$

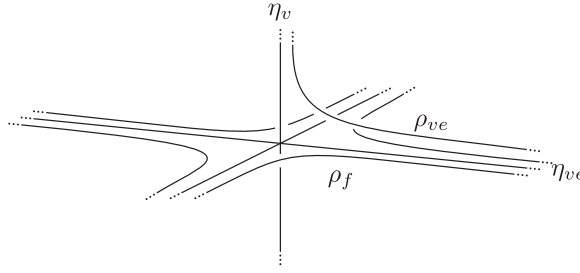


Figure 18. The labelling of the faces and edges around a spatial vertex.

- (c) The magnetic indices of the invariant vectors on the left in (105) associated with half-edges v_1e and of the invariant vectors on the right in (105) associated with half-edges ev_2 contract now at the vertices of the graph Γ_s . Thus, all the magnetic indices of $(W_e)_{\{i_f e i_{f(e+1)}\}}$ contract among each other. The magnetic indices $i_{v_1e}, j_{ev_2}, i'_{v_1e}, j'_{ev_2}$ associated with the representation matrices of our spin networks are contracted with the intertwiners η_v, η'_v of these spin networks.
- (d) The resulting amplitude is the vertex amplitude A_v^{BF} for G -BF theory. Here, a vertex of Γ_s is to be understood as a vertex in the following 2-complex: The edges e of Γ_s are ‘horizontal’ edges in this 2-complex and labelled with intertwiners η_{ve} , which appeared in the expansion of the Haar projectors. Additionally, we have the spatial faces, on which W is defined and which are labelled by ρ_f . There are additional ‘vertical’ edges and faces, which carry the labels of the spin networks between which we compute the matrix elements. For each $v \in \Gamma_s$, we have an edge pointing down and labelled with η'_v and an edge pointing up labelled by η_v . There are also two vertical (half-)faces attached to each (half-)edge $(ve) \in \Gamma_s$ which are labelled by ρ_{ve} and ρ'_{ve} . The orientation of these faces is such that these agree with the orientation of e for the ‘up’ faces and are opposite with respect to the orientation of e for the ‘down’ faces. This 2-complex around a vertex is depicted in figure 18.

Thus, A_v^{BF} depends on all the algebraic data attached to (half-)edges and faces adjacent to the vertex v in this 2-complex. This includes the intertwiners $\eta_{ve}, \eta_v, \eta'_v$ which involve the representation spaces ρ_f for faces sharing f as well as ρ_{ve}, ρ'_{ve} for edges in Γ_s sharing v . The vertex BF amplitude is then defined by the contraction of the invariant vectors in the corresponding representation spaces:

$$A_v^{\text{BF}}(\eta_{ev}, \eta_v, \eta'_v) = \text{tr}_{\rho_{ev}} \text{tr}_{\rho'_{ev}} \text{tr}_{\rho_f \otimes \rho_{f'}} \dots \eta_v \eta'_v \prod_{ev} \eta_{ev}. \quad (106)$$

- (e) We are left with half of the invariant vectors arising from the Haar projectors associated with the edges of Γ_s . These are contracted with the $I(\rho_f, k_{fe})_{j_{fe} m_{fe}} e_{k_{fe}}^{\rho_f} I^\dagger(\rho_f, k_{fe})_{m_{fe} i'_{fe}}$ part of the operators W_e as well as with the $\sum_{m_e} I(\rho_{v_1e}, k_e)_{i_{v_1e} m_e} I^\dagger(\rho_{ev_2}, k_e)_{m_e i_{ev_2}}$ part of the spin network state $|\rho_{ev}, \eta_v, k_e\rangle$ and the corresponding primed counterpart. Thus to each edge in Γ_s , we associate the amplitude

$$(\mathcal{P}_e)_{\{k_{ef}\}_{f \ni e}, k_e, k'_e}^{\{\rho_f\}_{f \ni e}, \rho_{v_1e}, \rho'_{v_1e}, \rho_{ev_2}, \rho'_{ev_2}}(\eta_{v_1e}, \eta_{ev_2}) = \langle \eta_{v_1e} | \left(I(\rho_{ev_2}, k_e) I(\rho_{v_1e}, k_e)^\dagger \right. \\ \left. \otimes I(\rho'_{v_1e}, k'_e) I(\rho'_{ev_2}, k'_e)^\dagger \right) \bigotimes_{f \ni e} e_{k_{ef}}^{\rho_f} I(\rho_f, k_{ef}) I(\rho_f, k_{ef})^\dagger | \eta_{ev_2} \rangle. \quad (107)$$

(f) Finally, the matrix elements of W are given by

$$\begin{aligned} \langle \rho'_{ev}, \eta'_v, k'_e | W | \rho_{ev}, \eta_v, k_e \rangle &= \sum_{\eta_{ev}, \rho_f, k_{ef}} \mathcal{N} \prod_v A_v^{\text{BF}}(\eta_{ev}, \eta_v, \eta'_v) \\ &\times \prod_e (P_e)_{\{k_{ef}\}_{f \ni e}, k_e, k'_e}^{\{\rho_f\}_{f \ni e}, \rho_{v_1 e}, \rho'_{v_1 e}, \rho_{v_2 e}, \rho'_{v_2 e}}(\eta_{v_1 e}, \eta_{v_2 e}), \end{aligned} \quad (108)$$

where \mathcal{N} collects all the dimension factors and face amplitudes

$$\mathcal{N} = \prod_f \dim(\rho_f) \tilde{\omega}^{\rho_f} \prod_{ev} \sqrt{\frac{\dim(\rho_{ev})}{\dim(k_e)}} \sqrt{\frac{\dim(\rho'_{ev})}{\dim(k'_e)}}. \quad (109)$$

Thus, the matrix elements of W provide us almost with the full spin foam amplitude for an equal time slice. What is missing to obtain the full amplitude are the insertions of e_k^o for the edge face pairs consisting of time-like faces and space-like edges as well as the edge amplitudes P_e for the time-like edges. These are indeed provided by the operators K_e .

6.3. The transfer operator on the projected spin network Hilbert space

So far, we discussed the transfer operator in the unsymmetric form $T = W \cdot K$. In lattice gauge theory, one often chooses rather a symmetric form $T_{\text{LG}} = W^{1/2} K W^{1/2}$. In the case of spin foams, W might not be a positive operator. Indeed, for the gravitational spin foams, it is rather easy to construct a square root of K_e so that we can consider $T' = \prod_e K_e^{1/2} P_G W P_G \prod_e K_e^{1/2}$. We have seen that K_e is almost a projection operator and have given the eigenvectors and eigenvalues (89), (92). Indeed for the EPRL model, the non-null eigenvectors (89) reduce to

$$v_{k, m_1, m_2}^{\text{EPRL}} := \left| \rho = \left(\frac{1+\gamma}{2} k, \frac{|1-\gamma|}{2} k \right), \{k, m_1\}; \{k, m_2\} \right\rangle, \quad (110)$$

so that the index structure is the same as for (non-gauge-invariant) H spin network functions, i.e. the spin network basis of the standard loop quantum gravity Hilbert space. By formally identifying the eigenvectors with this spin network basis, we can define the transfer operator on the LQG Hilbert space. In principle, this applies also to the Barrett–Crane model; the non-null eigenvectors are however just labelled by one H representation label

$$v_k^{\text{BC}} := |\rho = (k, k), \{0, 0\}, \{0, 0\}\rangle, \quad (111)$$

whose functionality is in addition quite different from the EPRL model.

A more elegant and geometric method is to use the μ_{Γ_s} map introduced in section 3.2, which maps from the Hilbert space of projected spin network functions to the universal boundary Hilbert space. (As in this section, we will therefore assume that the face weights are given by delta functions $\omega_f = \delta_G$; however, as mentioned there, this can be generalized.) The μ map was defined such that the effect of the time-like plaquettes, i.e. K , can be written as

$$K = \mu_{\Gamma_s} \mu_{\Gamma_s}^\dagger. \quad (112)$$

In the following, we will suppress the index Γ_s .

Thus, we can define the transfer operator in the projected spin network space by

$$\tilde{\mathcal{Z}}(\mathcal{C}_t) = (\mu^\dagger W \mu)^n \quad \text{i.e.} \quad T_{\text{PSN}} = \mu^\dagger W \mu. \quad (113)$$

Remember that the integration kernel of μ is given by

$$\begin{aligned} \langle g_{v_1 e}, g_{ev_2} | \mu | \tilde{g}_{v_1 e v_2} \rangle &= \int \prod_{e \in \Gamma_s} dg_{v_1 e v_2} \prod_{(ve) \in \Gamma_s} dg_{vt}^e \prod_{v \in \Gamma_s} dg_{vt} \\ &\times \prod_{e \in \Gamma_s} \delta(g_{v_1 e} g_{v_1 e v_2} g_{ev_2} g_{v_2 t} g_{v_2 t}^e \tilde{g}_{v_2 e v_1} g_{tv_1}^e g_{tv_1}) E(g_{v_1 e v_2}) F(g_{v_2 t}^e) F(g_{tv_1}^e), \end{aligned} \quad (114)$$

where F satisfies $\int E(g)f(g) dg = \int F(g)F(g^{-1}g')f(g') dg dg'$. F can be expanded in the same way as E and for the coefficients, we have $e_k^\rho = f_k^\rho f_k^\rho$. In the following, we will assume that E is a projector, i.e. $f_k^\rho = e_k^\rho$. Here, t indicates a time-like edge, i.e. g_{vt} is a group element associated with a time-like edge starting at the vertex v , g_{tv} a group element along a time-like edge ending in v' .

In the same way as for the operator K , we can extract a gauge-invariant projector from μ , i.e.

$$\mu = P_G \prod_e \mu_e, \quad (115)$$

where μ_e is given by

$$\begin{aligned} \langle g_{v_1e}, g_{ev_2} | \mu_e | \tilde{g}_{v_1ev_2} \rangle &= \int dg_{v_1ev_2} dg_{v_1t} dg_{v_2t} \\ &\times \delta(g_{v_1e}g_{v_1ev_2}g_{ev_2}g_{v_2t}^e \tilde{g}_{v_2ev_1}g_{tv_1}^e) E(g_{v_1ev_2}) F(g_{v_2t}^e) F(g_{tv_1}^e). \end{aligned} \quad (116)$$

In a spin network basis of the universal boundary Hilbert space and the projected spin network space, respectively, we obtain

$$\begin{aligned} \langle \rho_1, \{k_1, m_1\}; k; \rho_2, \{k_2, m_2\} | \mu_e | \rho', \{k'_1, m'_1\}, \{k'_2, m'_2\} \rangle \\ = \sqrt{\frac{\dim(k)}{\dim(\rho')}} \delta_{\rho_1\rho'} \delta_{\rho_2\rho'} e_k^{\rho'} f_{k_1}^{\rho'} \delta_{k_1k'_1} f_{k_2}^{\rho'} \delta_{k_2k'_2}, \end{aligned} \quad (117)$$

where we also used a Gelfand–Tsetlin basis for the non-gauge-invariant projected spin networks.

Thus, the image of μ_e is spanned by the non-null eigenvectors for K_e discussed previously and given in (110) and (111) for the EPRL and BC models, respectively. The co-kernel of μ_e or image of μ_e^\dagger in the projected spin network space is labelled by the same indices, i.e.

$$\begin{aligned} \tilde{v}_k^{\text{BC}} &= |(k, k), \{0, 0\}, \{0, 0\}\rangle \\ \tilde{v}_{k, m_1, m_2}^{\text{EPRL}} &= \left| \left(\frac{1+\gamma}{2}k, \frac{|1-\gamma|}{2}k \right), \{k, m_1\}, \{k, m_2\} \right\rangle. \end{aligned} \quad (118)$$

Thus, we can, for instance, for the EPRL model formally understand the transfer operator as an operator on the LQG Hilbert space. This might however not be very useful, if one wants to understand the structure of the transfer operator in terms of holonomy and flux operators. In particular, the holonomy operators appearing in the transfer operator are G holonomies and thus act on either the universal boundary Hilbert space or the projected spin network space, both of which are Hilbert spaces over copies of the group G .

6.4. Example: the BF model

Let us first consider the BF model; see also [16] for a discussion of the corresponding transfer operator in the context of standard lattice gauge theory. The BF theory will be the only case where the transfer operator will be actually a projector.

In the case of the BF model, we have $e_k^\rho = \theta(\rho, k)$, where $\theta(\rho, k) = 1$ if k appears in the reduction of ρ over H and $\theta(\rho, k) = 0$ otherwise. Also, as the face weights are given by delta functions on the group, we have $\tilde{\omega}^\rho = 1$.

K_e projects onto states

$$\begin{aligned} v_{\rho, k_1, m_1; k_2, m_2} &= \sum_k \sqrt{\dim(k)} \theta(\rho, k) |\rho, \{k_1, m_1\}; k; \rho, \{k_2, m_2\}\rangle \\ &= \sum_{i_1, j_1, i_2, j_2, m} I^\dagger(\rho, k_1)_{m_1 i_1} \delta_{j_1 i_2} I(\rho, k_2)_{j_2 m_2} |\rho, i_1, j_1; \rho, i_2, j_2\rangle, \end{aligned} \quad (119)$$

which are just linear combinations of spin network states that have been subdivided with a trivial two-valent edge. The eigenvalues of K_e are given by

$$\lambda_{\rho, k_1, m_1; k_2, m_2} = \frac{1}{\dim(\rho)} \theta(\rho, k_1) \theta(\rho, k_2) \sum_k \dim(k) \theta(\rho, k)^2 = \theta(\rho, k_1) \theta(\rho, k_2), \quad (120)$$

i.e. are equal to 1 as long as the indices in $v_{\rho, k_1, m_1; k_2, m_2}$ define a non-vanishing vector. Thus, K_e is a proper projection operator and acts as identity on states that can be embedded into the standard lattice gauge theory Hilbert space for G .

The operator W is a multiplication operator in the holonomy basis given by the effective face weights associated with the spatial plaquettes. As $E(g) = \delta(g)$, the effective face weights are also given by delta functions evaluated on the holonomy around the plaquette

$$\omega'_f(g_{v_1 e_1} g_{e_1 v_2} g_{v_2 e_2} \dots) = \delta(g_{v_1 e_1} g_{e_1 v_2} g_{v_2 e_2} \dots). \quad (121)$$

Hence, W is a projection⁶ onto the states satisfying the flatness conditions.

Hence, the transfer operator T is a projector implementing the Gauss constraints, which impose gauge invariance, as well as the flatness constraints on the plaquettes.

6.5. Example: the BC model

Let us also discuss the Barrett–Crane model as here an enhanced symmetry of the E function allows us to reduce the boundary Hilbert spaces.

For the Barrett–Crane model, the e_k^ρ coefficients factorize $e_k^\rho = \delta_{k,0} \sum_{k'} \delta_{\rho, (k', k')}$. The operator K_e projects onto states

$$v_{k'} = |(k', k'), \{0, 0\}; 0; (k', k'), \{0, 0\}\rangle \quad (122)$$

and the corresponding eigenvalue is given by $\lambda_{k'} = \frac{1}{(\dim(k'))^2}$.

More in general, we can see that the holonomy operators (94) with $k = 0$, i.e. $\psi_{\rho_1, k_1, m_1; 0; \rho_2, k_2, m_2}$, generate a closed subspace. Hence, the dynamics only involves this subspace of the universal boundary Hilbert space. Indeed, the integrand of the partition function, that is, the effective face weights, does show an enhanced symmetry in the case of the Barrett–Crane model: the E function is given as $E(g^L, g^R) = \delta_{\text{SU}(2)}(g^L (g^R)^{-1})$, where $(g^L, g^R) \in \text{SU}(2) \times \text{SU}(2)$. It is invariant under $\text{SU}(2)$ multiplication (of the diagonal subgroup in $\text{SU}(2) \times \text{SU}(2)$) from the left and from the right. This is a stronger symmetry than in the general case where E is just required to be invariant under the adjoint action of the subgroup H . Because of this enhanced symmetry, the effective face weights just depend on variables $G \times G/H$, associated with every half-edge. The subspace of the Hilbert space spanned by basis states with $k = 0$ is exactly the subspace-invariant under this H group action.

Let us turn to the W operator and see in which sense it is a constraint on the curvature (as the case in BF theory). The effective face weights can be computed to

$$\omega'_f(\kappa_{e_1 v_1} \kappa_{v_1 e_2}, \kappa_{e_2 v_2} \kappa_{v_2 e_3}, \dots) = \int \left(\prod_{I=1}^N d\gamma_I \right) \delta_H \left(\prod_{J=1}^N \gamma_J \kappa_{e_J v_J} \kappa_{v_J e_{J+1}} \gamma_J^{-1} \right). \quad (123)$$

Here, γ_I are group variables in $H = \text{SU}(2)$ and κ_{ve} abbreviates $g_{ve}^L (g_{ve}^R)^{-1}$, the product of left and (inverse) right copy of the group. N denotes the number of edges in the face f .

The requirement for a non-zero face weight is that there exists a set of group elements κ'_I , which have to be in the conjugacy class of $\kappa_I := \kappa_{e_I v_I} \kappa_{v_I e_{I+1}}$, such that the product of κ'_I is

⁶ As it involves the delta function, it is not a proper projection operator in the case of Lie groups. A mathematical clean description can be obtained by interpreting W as a rigging map which maps to the dual of some dense subspace of the Hilbert space; see, for instance, [77].

equal to the identity. For faces with more than two edges, this condition is generically satisfied and the effective face weights only vanish on measure-zero submanifolds in the configuration space. Thus, we cannot read off a curvature constraint for the general case.

6.6. Discussion of transfer operator and its limit

A question of intense research is the relation between the dynamics as defined by the spin foam models on the one hand and the dynamics as defined by the Hamiltonian constraints in loop quantum gravity [98] on the other hand [77, 1, 2, 39, 37]. The model example is 3D gravity, which is equivalent to BF theory [77]. In this case, the transfer operator is a projector and its image can be described by quantum constraints which reflect the diffeomorphism symmetry of the model. These constraints can also be encoded into recursion relations (in the spin representation) which can be interpreted as the Wheeler–DeWitt equations of 3D quantum gravity [22, 38].

In 4D gravitational models, the situation is complicated by two main issues. One is that diffeomorphism symmetry in discrete 4D gravity models is generically broken even on the classical level, so that one cannot expect the transfer operators to be a pure projector and to lead to constraints [44, 13, 48]. There is an exception to this general picture in cases where the dynamics allows only for flat geometries. This is the case for special triangulations described in [51, 49] and more specifically for the so-called tent moves at four-valent vertices [48], on which we will comment more below.

The second main issue is the appearance of simplicity constraints. The natural boundary Hilbert spaces for the current spin foam models are based on $G = SO(4)$ (or $G = SO(3, 1)$) holonomies, whereas the LQG Hilbert space is based on $H = SU(2)$ holonomies. Although we can formally define the transfer operator (i.e. for the EPRL model) as an operator on the LQG Hilbert space, it rather involves the multiplication operator W with G holonomies.

Let us comment more on the two issues and point out avenues for further research. In the case of broken diffeomorphism symmetry, the transfer operator is rather an evolution operator corresponding to some finite time step instead of a projection operator. Thus, one could ask for the limit in which this finite time step is taken to be small in order to extract the time generator, that is, the Hamiltonian. Indeed, this is the standard procedure for lattice gauge theory in which the time step is encoded in the lattice constant (in the time direction). In gravity, however, we do not have such an explicit lattice constant (or another coupling constant). Rather, the time distance is boundary state dependent and might only emerge semiclassically. That is, to take the limit of infinitesimal time steps, we need to entangle this procedure with some semiclassical limit. We discussed different boundary Hilbert spaces; hence, different types of semiclassical states are possible, including Hall-like semiclassical states [99, 19] adapted to the universal boundary Hilbert space and coherent states used in the discussion of the semiclassical limit of spin foam models [26, 28, 29, 25, 27, 64, 41].

An alternative to considering the limit of infinitesimal small time-like distances is to ‘improve’ the transfer operator by basically coarse graining [14, 17, 43], i.e. considering an effective transfer operator $T' = T^N$. This again is similar to standard statistical lattice theories, where the limit T^N for $N \rightarrow \infty$ leads to a projector on the eigenspace corresponding to the highest eigenvalue of T . However, in general, some scaling is required in order to obtain an interesting projector, i.e. the highest eigenvalue should be rather highly degenerate. To this end, it might be necessary to refine also the spatial discreteness, as the limit of continuous time but discrete space might rather not lead to a restoration of diffeomorphism symmetry [75].

Here, we discussed a global transfer operator which acts on the entire hypersurface. An alternative, more adapted to the ‘multi-fingered’ time evolutions, is the so-called tent moves,

discussed in [96, 44, 13, 48], which rather evolve single vertices in the hypersurface. To define the corresponding transfer operator, one needs to provide the possibility of gluing wedges, which is discussed in section 4. Tent moves are especially interesting as these would lead to localized, vertex-based (Hamiltonian) constraints (in case the symmetries are realized). Indeed for classical Regge calculus, tent moves at four-valent edges lead to constraints as the simplicial geometry remains flat in this case [48, 49] and diffeomorphism symmetry is preserved. Here, it would be interesting to know if the spin foam transfer operator corresponding to a tent move at a four-valent vertex leads to a projector and therefore constraints or not. More generally, the question is whether we can obtain a dynamics describing flat geometries for triangulations that in Regge calculus only allows for flat metrics due to topological reasons [51].

The second more technical main point concerns the simplicity constraints. As we have seen, the transfer operator naturally involves W as a G -group holonomy and boundary Hilbert spaces based on G . Hence, even if we can formally define matrix elements of the transfer operator on the LQG Hilbert space (based on H holonomies), a comparison to a (Hamiltonian) operator expressed in terms of H holonomies and fluxes is rather difficult. An alternative is provided by the recent work [33, 34] which provides a canonical connection formulation based on G holonomies. In this case, the Hamiltonian constraints have to be augmented by a term that makes them gauge-invariant with respect to the (primary) simplicity constraints. One could argue that this term is taken care off in spin foams, as K_e (or $K_e^{1/2}$ or μ) projects back onto the solutions of the (primary) simplicity constraints. However, heuristically the transfer operator is the exponential of the Hamiltonians. The additional term in the continuum Hamiltonian constraints takes care of staying on the simplicity constraint hypersurface at all times. In contrast, with the spin foam transfer operator, we rather project onto the simplicity constraint hypersurface in between discrete time steps. The discrete time steps itself involve G -group holonomies which in general map out of the subspace defined by the simplicity constraints. This is reminiscent of discussions in [4], which argues that simplicity projectors should be inserted at each point of the G holonomies. More generally, this problem is connected with the issue of how secondary simplicity constraints [5, 100, 51, 50] are implemented into spin foams. The work here offers the possibility to check in which sense the imposition of primary simplicity constraints at consecutive time steps leads to an imposition of secondary simplicity constraints, as, for instance, argued in [73]. These insights might allow a relation between G holonomies and holonomies involving the Ashtekar–Barbero connection [97], which in turn will ease the comparison of the transfer operator with the Hamiltonian constraints.

7. Discussion

We have shown that the new holonomy representation presented in this and the companion paper [15] provides several advantages. There is a clear parametrization of the space of models. From the parameters, one can easily read off the reality of the amplitudes, and it turns out that the BC, EPRL, FK and BO models have real amplitudes in this representation. The holonomy models lead to a natural boundary Hilbert space, which is introduced in this work. This universal boundary Hilbert space is the same Hilbert space for different choices of simplicity functions E . Thus, different possibilities to impose the simplicity constraints can be compared in one and the same Hilbert space. Furthermore, as the models can be naturally defined on arbitrary 2-complexes, we also can obtain arbitrary (and not only four-valent) graphs on which the boundary Hilbert spaces are based. A (dynamical) notion of cylindrical consistency, related to coarse graining, can also be introduced on these Hilbert spaces [12, 45].

The universal boundary Hilbert space results from a definition of ‘equal time’ slices which is most natural for lattice gauge theory. Alternatively, we can adopt slices and gluings more

custom to spin foams and obtain as boundary Hilbert space the Hilbert space of projected spin networks. We detail the conditions under which such a map between the different slicings is possible and construct the corresponding μ -map. These considerations allowed also a discussion on the different kind of basic building blocks, i.e. faces, half-faces and wedges, and the possible gluings which allow a combination of these building blocks to the full partition function.

We explicitly constructed the representation of different current spin foam models in our new representation—the main difference between the models is the simplicity functions E that imposes the simplicity constraints.

Finally, we derived a general form of the transfer operator on the different boundary Hilbert spaces. On the universal Hilbert space, this form is given by $T = K_0^{1/2} P_G W P_G K_0^{1/2}$ with W representing a product over holonomy operators over faces, P_G is the projector on the G -invariant subspace and $K_0^{1/2}$ can be seen as imposing the simplicity constraints on the Hilbert space. Similarly on the Hilbert space of projected spin network functions, we have $T' = \mu^\dagger W \mu$ with μ, μ^\dagger taking over the role of K to impose the simplicity constraints. This form might shed some light on the discussion of how to best impose simplicity constraints into spin foams [5, 50, 34, 61]. In the current models, the secondary simplicity constraints are not imposed. It is argued that the imposition of primary simplicity constraints on each time step should also lead to the imposition of secondary constraints. (Note however that even the primary constraints are imposed weakly in the EPRL-type models.) Indeed, here this is made obvious in the form of the transfer operator, who is projected by either K_0 or μ . Of course, the projections are only inserted at discrete time steps and not continuously in time. Related to this issue is that fundamentally, the transfer operator still includes a $G = SO(4)$ -holonomy operator and not a holonomy operator based on the Ashtekar–Barbero connection. A question for future research is how this form relates to a discretion which starts from the beginning with the $SU(2)$ Ashtekar–Barbero connection, which arises by classically solving for the simplicity constraints [60]. This question can be also studied in a semiclassical limit using semiclassical states in either of the boundary Hilbert spaces or the techniques presented in [65].

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Appendix A. The basis of E functions

In this appendix, we discuss in some detail the precise derivation of the basis of E functions (6), and the conditions satisfied by the coefficients e .

In the space $\text{Hom}_H(\rho, k)$, we can introduce a scalar product by

$$\text{Hom}_H(\rho, k) \times \text{Hom}_H(\rho, k) \ni (I, I') \rightarrow \langle I, I' \rangle_{\text{Hom}_H(\rho, k)} 1_k = I^\dagger I'. \quad (\text{A.1})$$

For every ρ , we have an antiunitary, group-covariant map

$$J_\rho : \rho \rightarrow \bar{\rho}. \quad (\text{A.2})$$

If ρ and $\bar{\rho}$ are distinct, we assume that $J_{\bar{\rho}} = J_\rho^{-1}$. If $\rho = \bar{\rho}$, then we may assume in addition that $J_\rho^2 = s_\rho 1_\rho$ (where $s_\rho = \pm 1$).

Similarly, for every k , we have an antiunitary map

$$J_k : k \rightarrow \bar{k}. \quad (\text{A.3})$$

If k and \bar{k} are distinct, we assume that $J_{\bar{k}} = J_k^{-1}$. If $k = \bar{k}$, then we may assume in addition that $J_k^2 = s_k 1_k$ (where $s_k = \pm 1$).

We can thus define the antiunitary (in the scalar product (A.1)) map

$$M_{\rho,k} : \text{Hom}_H(\rho, k), \rightarrow \text{Hom}_H(\bar{\rho}, \bar{k}), \quad M_{\rho,k}(I) = J_{\rho} I J_k^{\dagger}. \quad (\text{A.4})$$

Here, J^{\dagger} is defined by

$$\langle \cdot, J^{\dagger} \cdot \rangle = \overline{\langle J \cdot, \cdot \rangle}. \quad (\text{A.5})$$

In the case when $\rho = \bar{\rho}$ and $k = \bar{k}$,

$$M_{\rho,k}^2 = \underbrace{s_k s_{\rho}}_{\pm 1} 1_{\text{Hom}_H(\rho,k)}. \quad (\text{A.6})$$

Let us define

$$\tilde{D}(g)_{\rho,k} : \text{Hom}_H(\rho, k) \rightarrow \text{Hom}_H(\rho, k) \quad (\text{A.7})$$

by the matrix elements

$$\langle I, \tilde{D}(g)_{\rho,k} I' \rangle = \text{tr} I^{\dagger} D_{\rho}(g) I'. \quad (\text{A.8})$$

We can then expand E as such

$$E(g) = \sum_{\rho,k} \dim(\rho) \text{tr}_{\text{Hom}_H(\rho,k)} e_k^{\rho} \tilde{D}_{\rho,k}(g^{-1}), \quad (\text{A.9})$$

where $e_k^{\rho} : \text{Hom}_H(\rho, k) \rightarrow \text{Hom}_H(\rho, k)$. $\mathcal{E}(G, H)$ can then be parametrized through e_k^{ρ} .

We will need the following set of useful relations satisfied by \tilde{D} . We see that

$$\begin{aligned} \langle I, \tilde{D}_{\rho,k}(g^{-1}) I' \rangle &= \text{tr} I^{\dagger} \tilde{D}_{\rho}^{\dagger}(g) I' = \overline{\text{tr} (I')^{\dagger} D_{\rho}(g) I} \\ &= \overline{\langle I', \tilde{D}_{\rho,k}(g) I \rangle} = \langle \tilde{D}_{\rho,k}(g) I, I' \rangle; \end{aligned} \quad (\text{A.10})$$

thus, $\tilde{D}_{\rho,k}(g^{-1}) = \tilde{D}_{\rho,k}(g)^{\dagger}$.

Similarly

$$\begin{aligned} \langle I, M_{\rho,k}^{\dagger} \tilde{D}_{\bar{\rho},\bar{k}}(g) M_{\rho,k}(I') \rangle &= \overline{\langle M_{\rho,k}(I), \tilde{D}_{\bar{\rho},\bar{k}}(g) M_{\rho,k}(I') \rangle} \\ &= \text{tr} J_k I^{\dagger} J_{\rho} J_{\rho} D_{\rho}(g) J_{\rho} J_{\rho} I' J_k^{\dagger} = \langle I, \tilde{D}_{\rho,k} I' \rangle, \end{aligned} \quad (\text{A.11})$$

where we used the identity valid for any antiunitary J and linear A

$$\text{tr} J A J^{\dagger} = \overline{\text{tr} A}. \quad (\text{A.12})$$

Thus, $\tilde{D}_{\rho,k}(g) = M_{\rho,k}^{\dagger} \tilde{D}_{\bar{\rho},\bar{k}}(g) M_{\rho,k}$.

e_k^{ρ} are not completely free, but are restricted by the condition that $E(g) = E(g^{-1})$. This implies that they have to satisfy a set of equations relating the coefficients for complex conjugate representations. By definition, we have that

$$\text{tr} e_k^{\rho} \tilde{D}_{\rho,k}(g^{-1}) = \overline{\text{tr} e_k^{\rho \dagger} \tilde{D}_{\rho,k}(g)} = \overline{\text{tr} e_k^{\rho \dagger} M_{\rho,k}^{\dagger} \tilde{D}_{\bar{\rho},\bar{k}}(g) M_{\rho,k}} = \text{tr} M_{\rho,k} e_k^{\rho \dagger} M_{\rho,k}^{\dagger} \tilde{D}_{\bar{\rho},\bar{k}}(g). \quad (\text{A.13})$$

The condition for the E function reads⁷

$$e_k^{\bar{\rho}} = M_{\rho,k} e_k^{\rho \dagger} M_{\rho,k}^{\dagger}. \quad (\text{A.15})$$

⁷ Matrix elements of $\tilde{D}_{\rho,k}$ satisfy

$$d_{\rho} \int dg \langle I_1, \tilde{D}_{\rho,k}(g^{-1}) I_2 \rangle \langle I_3, \tilde{D}_{\rho,k}(g) I_4 \rangle = \delta_{\rho,\rho'} \delta_{k,k'} d_k \langle I_1, I_4 \rangle \langle I_3, I_2 \rangle; \quad (\text{A.14})$$

thus, the basis forms a set of independent functions of g .

Let us note that if $\rho \neq \bar{\rho}$ or $k \neq \bar{k}$, we can choose bases in such a way that $M_{\rho,k}$ acts by complex conjugation, then

$$e_k^{\bar{\rho}} = e_k^{\rho T}. \quad (\text{A.16})$$

The same is possible if $s_k s_\rho = 1$ but not in the case when $s_k s_\rho = -1$.⁸ From now on, we will assume that $s_\rho s_k = 1$ whenever $\rho = \bar{\rho}$ and $k = \bar{k}$. In many models we consider, ρ and k are indeed isomorphic to $\bar{\rho}$ and \bar{k} , in which case the condition simply says that e_k^ρ has to be symmetric.

Furthermore as

$$\text{tr } e_k^\rho \tilde{D}_{\rho,k}(g^{-1}) = \text{tr } e_k^\rho \tilde{D}_{\rho,k}(g)^\dagger = \overline{\text{tr } e_k^{\rho^\dagger} \tilde{D}_{\rho,k}(g)} \quad (\text{A.17})$$

as mentioned in the main body of the text, we have a real holonomy spin foam model if

$$e_k^\rho = e_k^{\rho^\dagger}. \quad (\text{A.18})$$

In the cases where we have (A.16), this reads

$$\overline{e_k^\rho} = e_k^{\bar{\rho}}. \quad (\text{A.19})$$

Using the formula, $E(g) = \sum_\rho \dim(\rho) \text{tr } e_k^{\bar{\rho}} D_{\rho,k}(g)$ and (A.16).

Appendix B. The spin network basis

In this appendix, we give some more detail on the spin network basis for the space $\mathcal{H}_{\text{UBS}}^\Gamma$.

It is convenient to start with the basis for the larger space $L^2(G^{|\Gamma_{\text{evl}}|})$. By the Peter–Weyl theorem, this is given by the matrix elements of representations, i.e.

$$|\rho, i, j\rangle = \sqrt{\dim(\rho)} D_\rho(g)_{ij}, \quad (\text{B.1})$$

with the dimension factor providing the correct normalization,

$$\begin{aligned} \langle \rho', i', j' | \rho, i, j \rangle &= \sqrt{\dim(\rho) \dim(\rho')} \int_G dg \overline{D_{\rho'}(g)}_{i'j'} D_\rho(g)_{ij} \\ &= \sqrt{\dim(\rho) \dim(\rho')} \int_G dg D_{\rho'}(g^{-1})_{j'i'} D_\rho(g)_{ij} \\ &= \delta_{i'i'} \delta_{j'j} \delta_{\rho\rho'}. \end{aligned} \quad (\text{B.2})$$

A basis is thus simply given by the tensor product of basis elements

$$\bigotimes_{ev} |\rho_{ev}, i_{ev}, j_{ev}\rangle = \prod_{ev} \sqrt{\dim(\rho_{ev})} D_{\rho_{ev}}(g_{ev})_{i_{ev}j_{ev}} \in L^2(G^{|\Gamma_{\text{evl}}|}). \quad (\text{B.3})$$

$\mathcal{H}_{\text{UBS}}^\Gamma$ is the subspace of states in $L^2(G^{|\Gamma_{\text{evl}}|})$ that are invariant under the action of the symmetries. In order to give a basis of this subspace, it will actually be more convenient to use the orientation on the edges to introduce the oriented basis. For this, we choose an arbitrary orientation for each edge. We will encode this by writing $(v, e, v') \in \Gamma$ for the oriented edge e running from vertex v' to v .

$$\bigotimes_{(v', e, v) \in \Gamma} \langle \rho_{ev'}, i_{ev'}, j_{ev'} | \otimes |\rho_{ev}, i_{ev}, j_{ev}\rangle. \quad (\text{B.4})$$

We now want to go to the G -invariant subspace of ρ_{ev} at the vertices, and the H -invariant subspace at the edges. We begin by implementing the invariance under H . To do so, we contract the inner indices at each oriented edge with an H -covariant operator. These are parametrized

⁸ If the degeneracy is 1, then always $s_k s_\rho = 1$ since the restriction of J_ρ to k is equal to J_k .

similarly to the E functions, by the matrices $\Xi_{k,dd'}$ which can be contracted with a basis of H -invariant maps between ρ and ρ' giving

$$\Xi_{i\bar{i}'} = \sum_{k,d,d',m} \Xi_{k,dd'} I(\rho, k)_{d,im} I(\rho', k)_{d',m\bar{i}'}^\dagger.$$

Contracting these on the indices i_{ev} in the middle of the edge gives

$$|\rho_{ev}, j_{ev}, \Xi_e\rangle = \prod_{(v',e,v) \subset \Gamma} \sqrt{\dim(\rho_{ev'}) \dim(\rho_{ev})} D_{\rho_{ev'}}(g_{ev'}^{-1})_{j_{ev'} i_{ev'}} \Xi_{e,i_{ev'}, i_{ev}} D_{\rho_{ev}}(g_{ev})_{i_{ev} j_{ev}}. \quad (\text{B.5})$$

This is normalized as

$$\langle \rho'_{ev}, j'_{ev}, \Xi'_e | \rho_{ev}, j_{ev}, \Xi_e \rangle = \prod_{ev} \delta_{\rho_{ev} \rho'_{ev}} \delta_{j_{ev} j'_{ev}} \prod_e \text{tr} \Xi'_e{}^\dagger \Xi_e. \quad (\text{B.6})$$

Note that j_{ev} is in the dual to ρ_{ev} if $(e, v) \subset \Gamma$, and in ρ_{ev} directly if $(v, e) \subset \Gamma$.

For the case where there are no degeneracies d, d' , the coefficients $\Xi_{k,dd'}$ simplify to Ξ_k . Thus, we can directly work with the basis of H -invariant operators

$$\Xi'_{k,i\bar{i}'} = \frac{1}{\sqrt{\dim(k)}} I(\rho', k)_{im} I(\rho, k)_{m\bar{i}'}^\dagger$$

labelled by k with the property

$$\text{tr} \Xi_k{}^\dagger \Xi_{k'} = \delta_{kk'}. \quad (\text{B.7})$$

We then obtain the states

$$\begin{aligned} |\rho_{ev}, j_{ev}, k_e\rangle &= \prod_{(v',e,v) \subset \Gamma} \frac{\sqrt{\dim(\rho_{ev'}) \dim(\rho_{ev})}}{\sqrt{\dim(k_e)}} \\ &\times D_{\rho_{ev'}}(g_{ev'}^{-1})_{j_{ev'} i_{ev'}} I(\rho_{ev'}, k_e)_{i_{ev'} m_e} I(\rho_{ev}, k_e)_{m_e i_{ev}}^\dagger D_{\rho_{ev}}(g_{ev})_{i_{ev} j_{ev}}, \end{aligned} \quad (\text{B.8})$$

which are normalized as

$$\langle \rho'_{ev}, j'_{ev}, k'_e | \rho_{ev}, j_{ev}, k_e \rangle = \prod_{ev} \delta_{\rho_{ev} \rho'_{ev}} \delta_{j_{ev} j'_{ev}} \prod_e \delta_{k_e k'_e}. \quad (\text{B.9})$$

To implement the G invariance at the vertices, we can simply contract with intertwiners $\eta_v \in \text{Inv}(\otimes \rho_{ev}^*)$, where ρ_{ev}^* is the dual representation if $(v, e) \subset \Gamma$ and the usual representation if $(e, v) \subset \Gamma$. Contracting all these, we obtain the state

$$\begin{aligned} |\rho_{ev}, \eta_v, \Xi_e\rangle &= \prod_v \eta_{v,j_{ev}, \dots} \prod_{(v',e,v) \subset \Gamma} \sqrt{\dim(\rho_{ev'}) \dim(\rho_{ev})} \\ &\times D_{\rho_{ev'}}(g_{ev'}^{-1})_{j_{ev'} i_{ev'}} \Xi_{e,i_{ev'}, i_{ev}} D_{\rho_{ev}}(g_{ev})_{i_{ev} j_{ev}}. \end{aligned} \quad (\text{B.10})$$

This is normalized as

$$\langle \rho'_{ev}, \eta'_v, \Xi'_e | \rho_{ev}, \eta_v, \Xi_e \rangle = \prod_{ev} \delta_{\rho_{ev} \rho'_{ev}} \prod_v \langle \eta'_v | \eta_v \rangle \prod_e \text{tr} \Xi'_e{}^\dagger \Xi_e. \quad (\text{B.11})$$

For the case without degeneracies, this again simplifies to the states

$$\begin{aligned} |\rho_{ev}, \eta_v, k_e\rangle &= \prod_v \eta_{v,j_{ev}, \dots} \prod_{(v',e,v) \subset \Gamma} \frac{\sqrt{\dim(\rho_{ev'}) \dim(\rho_{ev})}}{\sqrt{\dim(k_e)}} \\ &\times D_{\rho_{ev'}}(g_{ev'}^{-1})_{j_{ev'} i_{ev'}} I(\rho_{ev'}, k_e)_{i_{ev'} m_e} I(\rho_{ev}, k_e)_{m_e i_{ev}}^\dagger D_{\rho_{ev}}(g_{ev})_{i_{ev} j_{ev}}, \end{aligned} \quad (\text{B.12})$$

which are normalized as

$$\langle \rho'_{ev}, \eta'_v, k'_e | \rho_{ev}, \eta_v, k_e \rangle = \prod_{ev} \delta_{\rho_{ev} \rho'_{ev}} \prod_v \langle \eta'_v | \eta_v \rangle \prod_e \delta_{k_e k'_e}. \quad (\text{B.13})$$

This should be contrasted to the basis of projected spin networks which is given in terms of H intertwiners ι as

$$|k_{ev}, \iota_v, \rho_e\rangle = \prod_v \iota_{v, m_{ev}, \dots} \prod_e \sqrt{\dim(\rho_e)} I(\rho_e, k_{ev})_{m_{ev}, j_{ev}}^\dagger D_{\rho_e}(g_{vev'})_{j_{ev}, j_{ev'}} I(\rho_e, k_{ev'})_{j_{ev'}, m_{ev'}}, \quad (\text{B.14})$$

and is normalized as

$$\langle k'_{ev}, \iota'_v, \rho'_e | k_{ev}, \iota_v, \rho_e \rangle = \prod_{ev} \delta_{k_{ev} k'_{ev}} \prod_v \langle \iota'_v | \iota_v \rangle \prod_e \delta_{\rho_e \rho'_e}. \quad (\text{B.15})$$

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