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Nicht-minimale Varianten des Seesaw-Mechanismus als Erweiterung des Standardmodells

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Non-minimal variants

of the Seesaw Mechanism

in extension of the Standard Model

This Diploma thesis has been carried out by Pascal Humbert

at the

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under the supervision of

Herrn Prof. Dr. Manfred Lindner

Nicht-minimale Varianten des Seesaw-Mechanismus:

Wir untersuchen nicht-minimale Varianten des Seesaw-Mechanismus, insbesondere des Typ I und des doppelten Seesaw-Mechanismus, in Verbindung mit einem singulären Majorana-Massenterm in der Neutrino-Massenmatrix. Wir demonstrieren auch, dass die Neutrino-Massenmatrix im invertierten Seesaw-Mechanism eine "pseudo-singuläre" Struktur annehmen kann. Im Weiteren wird gezeigt, dass in allen betrachteten Szenarien im Prinzip aktive Neutrinomassen an der verbindlichen eV-Skala erhalten werden können. Durch das Analysieren der Eigenwert- und Massenskala-Struktur der Neutrino-Massenmatrix im singulären doppelten Seesaw finden wir Szenarien, die eV, keV und MeV bis GeV sterile Neutrinos beinhalten.

Non-minimal variants of the Seesaw Mechanism:

We study non-minimal variants of the seesaw mechanism, especially of the type I and the double seesaw mechanism, in correlation with a singular Majorana mass term in the neutrino mass matrix. Also we demonstrate that in the inverse seesaw mechanism a "pseudo-singular" structure for the neutrino mass matrix can be realized. It is further shown that in all scenarios under consideration active neutrino masses at the compulsory eV scale can be obtained in principle. By analyzing the eigenvalue and mass scale structure of the neutrino mass matrix in the singular double seesaw we find scenarios that feature eV, keV and MeV to GeV sterile neutrinos.

Contents

Chapter 1

Introduction

The Standard Model (SM) of particle physics is a theory of elementary particles and their interactions. It contains the descriptions of the strong, weak and electromagnetic interactions. Predictions made by the SM have been confirmed by many experiments. With the latest results concerning the search for the Higgs particle from the CMS and ATLAS collaboration at CERN, particle physicists may hope that the last missing piece of the SM has been discovered. There exist, however, strong evidences for physics beyond the SM. The velocity distribution of galaxies is interpreted as evidence for Dark Matter (DM). Neutrino oscillations have been observed that can only occur if neutrinos have non-zero and non-degenerate masses and mix among each other. On the other side, neutrino masses must be very small compared to other SM particle's masses. None of the mentioned phenomenona can be explained by the SM.

Before we begin the discussion of how the SM might be modified to possibly include these observations, let us see which experimental data any extended or new model must reproduce. The standard parameters of neutrino oscillation are the mass-squared differences $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ between the massive neutrino states i and j, and the three mixing angles θ_{ij} and the Dirac phase δ_{CP} of the (Pontecorvo-Maki-Nakagawa-Sakata) leptonic mixing matrix U_{PMNS} [1, 2], whose elements are given by the overlap between mass and flavor eigenstates, $U_{\alpha i} = \langle \nu_i | \nu_\alpha \rangle$.[1] Observations from solar and atmospheric neutrino oscillations allow for two different orderings of the three massive neutrino states corresponding to the sign of $\Delta m_{\mathrm{atm}}^2$, the mass-squared difference of atmospheric neutrinos. The first ordering, called "normal hierarchy" (NH), corresponds to $m_1 < m_2 < m_3$, whereas the second, called "inverted hierarchy" (IH), corresponds to $m_3 < m_1 < m_2$.[2] The values for $\Delta m_{\mathrm{sol}}^2 \equiv \Delta m_{21}^2$, $|\Delta m_{\mathrm{atm}}^2|$, θ_{12} and θ_{23} have been measured with relatively high

[1] For a review of neutrino oscillations, see e.g. [3].

[2] Note that the mass-squared difference of atmospheric neutrinos is given by $\Delta m_{\mathrm{atm}}^2(\mathrm{NH}) \equiv \Delta m_{31}^2$, if the neutrinos are normal ordered, and by $\Delta m_{\mathrm{atm}}^2(\mathrm{IH}) \equiv \Delta m_{32}^2$, if they are inverse ordered.

precision. Moreover, it is remarkable that only last year with the help of the youngest reactor neutrino data from Double Chooz [4], Daya Bay [5] and Reno [6] and better statistics in the long-baseline experiments T2K [7] and MINOS [8], the value of θ_{13} was pinned down with good accuracy. The latest update on constraints on the oscillation parameters can be found for example in [9].[3] In this paper a global fit of neutrino data sets is performed, where the reactor fluxes are taken as open parameters and short-baseline reactor data with $L \leq 100$ m are included. The best-fit-value of the mass-squared difference of solar neutrinos is given by $\Delta m^2_{21} = 7.50^{+0.18}_{-0.19} \times 10^{-5}$ eV^2 and for the atmospheric neutrinos by $\Delta m^2_{31} = +2.473^{+0.070}_{-0.069} \times 10^{-3}$ eV^2 for NH or $\Delta m^2_{32} = -2.427^{+0.042}_{-0.065} \times 10^{-3}$ eV^2 for IH.

There are further constraints on the *absolute* mass of neutrinos. From kinematic mass measurements in the energy spectrum of tritium beta-decay the Mainz Collaboration [12] and the Troitsk Collaboration [13] report a limit of $m_\beta^{\rm Mz} < 2.2$ eV and $m_\beta^{\rm Tk} < 2.05$ eV, respectively, both at 95% CL (= confidence level), where $m_\beta \equiv \sqrt{\sum_{i=1}^3 |U_{ei}|^2 m_i^2}$ denotes the effective mass of the electron neutrino involved in the decay $^3{\rm H} \longrightarrow\ ^3{\rm He} + e^- + \widehat{\nu}_\beta$.[4] Currently the KATRIN experiment [14] is being built, where a sensitivity limit of 0.02 eV for m_β is pursued.

A limit on the sum of neutrino masses is given by cosmological observations. Combined WMAP cosmological data restrict the summed mass of neutrinos to $\Sigma \equiv \sum_{i=1}^3 m_i < 0.44$ eV at 95% CL [15].

If neutrinos have a Majorana mass, neutrinoless double beta-decay ($0\nu\beta\beta$-decay) is possible. From non-observation of the $0\nu\beta\beta$-decay a limit on the effective mass of the $0\nu\beta\beta$-decay $\langle m_{\beta\beta}\rangle \equiv |\sum_{i=1}^3 U_{ei}^2 m_i|$ for light neutrinos can be deduced. The KamLAND-Zen Collaboration [16] reports a lower limit on the half-life of $0\nu\beta\beta$-decay from combined results of KamLAND-Zen and EXO-200 of $T^{0\nu}_{1/2} > 3.4 \times 10^{25}$ yr at 90% CL, corresponding to an effective Majorana mass limit of $\langle m_{\beta\beta}\rangle < (120 - 250)$ meV.[5]

Although neutrino oscillations are usually interpreted as transitions among active neutrino states, there could exist other effects that influence the oscillation parameters. A popular theory that could modify the parameters of $U_{\rm PMNS}$ is the hypothesis of the existence of additional particles not present in the SM, namely of sterile neutrinos [18]. They are commonly introduced as fermions that only interact gravitationally. If sterile neutrinos exist, neutrino oscillations could be the result of transitions between active and light sterile neutrinos.

[3] For alternative analyses, see e.g. [10,11].
[4] By $\widehat{\nu}$ we denote the anti-neutrino. For our nomenclature see also section A.1 of the appendix.
[5] Note that there has been a claim of observation of $0\nu\beta\beta$-decay in ^{76}Ge by a part of the Heidelberg-Moscow Collaboration [17] that has been ruled out by the results of the analysis of [16] at more than 97.5% CL.

A recent reevaluation [19] of reactor neutrino flux data [20] at short baselines found that the ratio of observed event rate to predicted rate at baselines < 100 m is shifted by about 3% (compared to earlier evaluations), leading to a deviation from unity at 98.6% CL. If confirmed, this is another hint (apart from the "LSND anomaly" [21] and MiniBooNE anti-neutrino oscillation results [22]) for transitions between active and sterile neutrino states with masses at the eV scale [23]. The suggestion of sterile neutrinos in this mass range is seized by [24] and [25]. In these papers the compatibility of models with one or two sterile neutrinos ($3+1$ or $3+2$ models) at the eV scale with experimental data is examined. The former analysis, considering a $3+1$ model, reports a value of $\Delta m_{41}^2 = 5.6$ eV^2 as the best-fit-value for the mass-squared difference of the sterile state, which is rather large, and three regions within 1σ at $\Delta m_{41}^2 = 1.6,\ 1.2,\ 0.91$ eV^2 in good agreement with data. The latter finds a value of $\Delta m_{41}^2 = 1.78$ eV^2 as best-fit point in the $3+1$ model and in the $3+2$ model $\Delta m_{41}^2 = 0.46$ eV^2 and $\Delta m_{51}^2 = 0.89$ eV^2. Both analyses, however, remark that there arise strong tensions when trying to fit the models to the data of both appearance and disappearance experiments at the same time. Nevertheless, we record the fact that in the light of these analyses sterile neutrinos in the eV range can be regarded as a desirable feature of a theory beyond the SM. Note that there has been also a recent publication [26], which examines the impact of TeV sterile neutrinos on fits to the data.

Concerning the chase for DM candidates neutrinos have come to some attention. Since active neutrinos are too light to explain DM (they only contribute to hot DM), considerations of sterile neutrinos possibly being warm DM have emerged (see e.g. [18]. See also [27] for a recent review of keV sterile neutrinos). The analysis of the DM phase-space distribution in dwarf spheroidal galaxies gives a lower bound of $M_{\mathrm{DM}} \gtrsim \mathcal{O}(1)$ keV for the mass of the lightest DM particle [28] so that there is existing interest in theories with keV sterile neutrinos.

In this thesis we study the possibilities to generate small neutrino masses in the spirit of the seesaw mechanism. Apart from common seesaw variants we develop methods to predict the scale of the active neutrino mass matrix in singular seesaw scenarios. In our considerations we include the constraints on neutrino oscillation parameters as well as the motivations for the different neutrino mass scales described above.

The organization of this thesis is as follows: In chapter 2 the SM and especially the electroweak theory are presented. Possible extensions of the SM in order to generate neutrino masses are discussed in chapter 3. This is followed by a review of the seesaw mechanism in chapter 4. In chapter 5 we present the results of this thesis in the context of the singular seesaw mechanism and conclude in chapter 6. Quantum field theoretic aspects and mathematical methods used in this thesis can be found in the apendices A and B, respectively.

Chapter 2

The Standard Model

2.1 The Standard Model as a gauge theory

The fundamental objects of the SM are quantized fields in spacetime. Consequently, special relativity (SR) and quantum field theory (QFT) are the cornerstones of the SM. From a field theoretical point of view the interactions of the SM are the result of the transformational behavior of the elementary particles under the local gauge group of the SM,

$$G = SU(3)_{\mathrm{C}} \times SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}, \tag{2.1}$$

where the subscripts C, L and Y denote color, left-handedness and hypercharge, respectively. To each subgroup of G belong gauge bosons, corresponding to the generators of the subgroups and transmitting the forces. The subgroup $SU(3)_{\mathrm{C}}$ has eight generators and is responsible for strong interactions. The gauge bosons corresponding to the generators are called gluons. Only particles that carry the quantum number of $SU(3)_{\mathrm{C}}$, color, participate in strong interactions. The theory of strong interactions is called quantum chromo dynamics(QCD).[1] The subgroup $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$, leading to electroweak interactions, is discussed in section 2.2. The $W^{\pm}$ and Z bosons of weak interactions and the photon γ, responsible for electromagnetic interactions are described as superpositions of the gauge bosons corresponding to the generators of $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$. This mixing is due to the spontaneous break down of the symmetry of G to $SU(3)_{\mathrm{C}} \times U(1)_{\mathrm{Q}}$ of strong and electromagnetic interactions. Since there is no further break down of symmetry, strong interactions and electroweak interactions can be discussed in separate theories.

Apart from the integer-spin bosons, the SM contains elementary spin-1/2-particles,

[1] QCD is not discussed here, since it would go beyond this thesis' scope. A good description of QCD, however, can be found e.g. in [3].

called fermions. They are distinguished in quarks, which participate in all fundamental interactions, and charged and uncharged leptons, where the former participate in all interactions except for the strong and the latter interact weakly and gravitationally. The particle content of the SM together with particles' masses is listed in Table 2.1. Note that the fields in Table 2.1 are mass eigenstates and, hence, correspond to the physical particles of the SM. The fermion fields appearing in the electroweak theory, however, are by definition flavor eigenstates. Whether particles are given in the mass basis or in the flavor basis, they are organized in three generations, where two versions of the "same" particle, but from different generations have the same quantum numbers under the SM gauge group.

Apart from the Higgs boson, all SM particles have been detected already. The present state of affairs in the search for the Higgs boson is reflected by the latest LHC results of the CMS and the ATLAS collaboration at CERN, which report a significant signal of a Higgs-like particle with a measured mass of $125.3 \pm 0.4(\text{stat}) \pm 0.5(\text{sys})$ GeV [29] and $126.0 \pm 0.4(\text{stat}) \pm 0.4(\text{sys})$ GeV [30], respectively, consistent with the limits of the Higgs mass in Table 2.1.

2.2 Electroweak theory

We discuss the electroweak theory in the Weinberg-Salam model [32,33] in a similar way as [34], but only include the leptonic part of the electroweak Lagrangian in our discussion. A crucial point of electroweak theory is the Higgs mechanism [35–39], giving masses to the weak gauge bosons, not to be explained here. In this section, however, we will address some attention to another important aspect of the Higgs mechanism, namely the generation of fermion masses.

2.2.1 Gauge group and fields

In the SM the subgroup $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}} \subset G$ is responsible for electroweak interactions. The group $SU(2)_{\mathrm{L}}$ is the symmetry group of weak isospin and only acts (non-trivially) on the left-handed component of a particle. To its generators correspond three gauge bosons, denoted by $\mathbf{W}_\mu$. To the generator of $U(1)_{\mathrm{Y}}$, the symmetry group of hypercharge, corresponds one gauge boson, denoted by B_μ. Under the symmetry group $SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ the covariant derivative is given by[2]

$$D_\mu = \partial_\mu - igI\boldsymbol{\sigma} \cdot \boldsymbol{W}_\mu - ig'\frac{Y}{2}B_\mu, \tag{2.2}$$

[2] The minus signs in eq. (2.2) come from our convention that e denotes the electric charge of the *positron*. If e is taken to denote the electric charge of the *electron*, the minus signs have to be replaced by plus signs.

	particle	mass
quarks	u	$2.3^{+0.7}_{-0.5}$ MeV
	d	$4.8^{+0.7}_{-0.3}$ MeV
	s	95 ± 5 MeV
	c	1.275 ± 0.025 GeV
	b	4.18 ± 0.03 GeV
	t	$173.5 \pm 0.6(\text{stat}) \pm 0.8(\text{sys})$ GeV
leptons	ν_e	< 2.05 eV @ 95% CL
	e	$0.510998928 \pm 0.000000011$ MeV
	ν_μ	< 0.19 MeV @ 90% CL
	μ	$105.6583715 \pm 0.0000035$ MeV
	ν_τ	< 18.2 MeV @ 95% CL
	τ	1776.82 ± 0.16 MeV
gauge bosons	γ	$< 10^{-18}$ eV
	g	0
	$W^\pm$	80.385 ± 0.015 GeV
	Z	91.1876 ± 0.0021 GeV
Higgs	ϕ^0	> 115.5 and none $127 - 600$ GeV @ 95% CL

Table 2.1: Standard Model particles with masses taken from the PDG [3]. The charged lepton masses are pole masses. Masses for u-, d-, and s-quark are "running" masses at renormalization scale $\mu = 2$ GeV in the minimal subtraction scheme. Masses for c- and b-quark are "running" masses at $\mu = \bar{m}_c$ and $\bar{m}_b$, respectively, as well in the minimal subtraction scheme. The top mass is the PDG's best-fit value of the pole mass from combination of published measurements. Zero gluon mass is a theoretical value. For the photon mass PDG refers to [31]. Note that the photon of the electroweak theory presented in section 2.2 is massless as a theoretical consequence of the Higgs mechanism.

where the spacetime derivative ∂_μ and the Pauli matrices $\boldsymbol{\sigma}$ are defined as in [40], g and g' are the coupling constants of $SU(2)_\text{L}$ and $U(1)_\text{Y}$, respectively, and I and Y denote the weak isospin and hypercharge quantum numbers, respectively, of the particle field D_μ acts on. The electric charge Q of a particle is connected to Y and the third component of isospin I_3 by the Gell-Mann-Nishijima relation [41–43]

$$Q = I_3 + \frac{Y}{2} \ . \tag{2.3}$$

In the electroweak theory the left-handed fermion fields of each generation are arranged in weak isospin doublets, while the right-handed fermion fields are weak isospin singlets. Since we pay particular attention to leptons, we can leave out the quark sector here. The left-handed lepton doublets are denoted by $L_{\alpha\mathrm{L}}$ and the right-handed charged lepton singlet by $e_{\alpha\mathrm{R}}$. The index $\alpha = e,\ \mu,\ \tau$ marks the flavor (or generation) of the corresponding lepton field.[3] The leptonic flavor eigenstates with quantum numbers are given in Table 2.2.

Lepton	Q [e]	I	I_3	Y
$L_{\alpha\mathrm{L}} = \begin{pmatrix} \nu_{\alpha\mathrm{L}} \\ e_{\alpha\mathrm{L}} \end{pmatrix}$	$\begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$1/2$	$\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix}$	-1
$e_{\alpha\mathrm{R}}$	-1	0	0	-2

Table 2.2: Leptonic quantum numbers: Charge Q in units of positron charge e. Weak isospin I and its third component I_3. Hypercharge Y.

From the values of I and Y we recognize that left-handed doublets and right-handed singlets will have distinct covariant derivatives. Inserting the leptons' quantum numbers into eq. (2.2) we have for the left-handed doublets $L_{\alpha\mathrm{L}}$, with $I = 1/2$ and $Y = -1$,

$$D_{\mu\mathrm{L}} = \partial_\mu - \frac{i}{2} g \boldsymbol{\sigma} \cdot \boldsymbol{W}_\mu + \frac{i}{2} g' B_\mu, \tag{2.4}$$

and for the right-handed singlets $e_{\alpha\mathrm{R}}$, with $I = 0$ and $Y = -2$,

$$D_{\mu\mathrm{R}} = \partial_\mu + i g' B_\mu. \tag{2.5}$$

2.2.2 Electroweak Lagrangian

- one generation: Having defined the flavor eigenstates (cf. Table 2.2) and the covariant derivative for the theory eqs. (2.4) and (2.5), we are prepared to turn our attention towards the electroweak Lagrangian. In one generation we write the lepton Lagrangian

$$\mathscr{L}_{\mathrm{lept}} = \mathscr{L}_{\mathrm{kin}} + \mathscr{L}_{\mathrm{Yukawa}}, \tag{2.6}$$

where the different parts are given by

[3] We use Greek letters for indices of the flavor basis. Later on, when we change to the mass basis, we will use Latin letters for mass basis indices. Since flavor and mass basis indices are carried only by fields, there should be no confusion between them and Lorentz and spacial indices.

$$\mathscr{L}_{\text{kin}} = i\overline{L}_{\text{L}}\gamma^\mu D_{\mu\text{L}} L_{\text{L}} + i\overline{e}_{\text{R}}\gamma^\mu D_{\mu\text{R}} e_{\text{R}}, \tag{2.7a}$$

$$-\mathscr{L}_{\text{Yukawa}} = y\overline{L}_{\text{L}}\phi e_{\text{R}} + h.c. \ . \tag{2.7b}$$

In the second equation y denotes the Yukawa coupling strength, and ϕ denotes the SM Higgs doublet, to be defined later (cf. eq. (2.13)). With $\overline{L}_{\text{L}}$ we denote the adjoint lepton doublet (cf. eq. (A.3))

$$\overline{L}_{\text{L}} \equiv \begin{pmatrix} \overline{\nu}_{\text{L}} & \overline{e}_{\text{L}} \end{pmatrix} . \tag{2.8}$$

Introducing the fields[4] W_μ, Z_μ and A_μ as superpositions of the generators $\mathbf{W}_\mu$ and B_μ given by

$$W_\mu \equiv \frac{1}{\sqrt{2}}(W_\mu^1 - iW_\mu^2), \tag{2.9a}$$

$$Z_\mu \equiv \frac{gW_\mu^3 - g' B_\mu}{(g^2 + g'^2)^{1/2}} = \cos\theta_{\text{W}} W_\mu^3 - \sin\theta_{\text{W}} B_\mu, \tag{2.9b}$$

$$A_\mu \equiv \frac{g' W_\mu^3 + gB_\mu}{(g^2 + g'^2)^{1/2}} = \sin\theta_{\text{W}} W_\mu^3 + \cos\theta_{\text{W}} B_\mu, \tag{2.9c}$$

where we have defined the Weinberg angle θ_{W} [32, 44] as

$$\cos\theta_{\text{W}} = \frac{g}{(g^2 + g'^2)^{1/2}} \ , \tag{2.10}$$

the covariant derivatives become

$$D_{\mu\text{L}} = \partial_\mu - \frac{i}{2}\begin{pmatrix} \frac{g}{\cos\theta_{\text{W}}} Z_\mu & \sqrt{2} g W_\mu \\ \sqrt{2} g W_\mu^\dagger & -\frac{g}{\cos\theta_{\text{W}}}\cos 2\theta_{\text{W}} Z_\mu - 2g\sin\theta_{\text{W}} A_\mu \end{pmatrix}, \tag{2.11a}$$

$$D_{\mu\text{R}} = \partial_\mu - ig'(\sin\theta_{\text{W}} Z_\mu - \cos\theta_{\text{W}} A_\mu). \tag{2.11b}$$

Inserting expressions (2.11) in eq. (2.7a) the kinetic part of the electroweak lepton Lagrangian reads

$$\begin{aligned}\mathscr{L}_{\text{kin}} = {} & i\overline{\nu}_{\text{L}}\gamma^\mu\partial_\mu\nu_{\text{L}} + i\overline{e}_{\text{L}}\gamma^\mu\partial_\mu e_{\text{L}} + i\overline{e}_{\text{R}}\gamma^\mu\partial_\mu e_{\text{R}} \\ & - g\sin\theta_{\text{W}}(\overline{e}_{\text{L}}\gamma^\mu e_{\text{L}} + \overline{e}_{\text{R}}\gamma^\mu e_{\text{R}})A_\mu \\ & + \frac{g}{\cos\theta_{\text{W}}}(\frac{1}{2}\overline{\nu}_{\text{L}}\gamma^\mu\nu_{\text{L}} - \frac{1}{2}\cos 2\theta_{\text{W}}\overline{e}_{\text{L}}\gamma^\mu e_{\text{L}} + \sin^2\theta_{\text{W}}\overline{e}_{\text{R}}\gamma^\mu e_{\text{R}})Z_\mu \\ & + \frac{g}{\sqrt{2}}(\overline{e}_{\text{L}}\gamma^\mu\nu_{\text{L}} W_\mu^\dagger + \overline{\nu}_{\text{L}}\gamma^\mu e_{\text{L}} W_\mu),\end{aligned} \tag{2.12}$$

[4] These fields are identified with the weak gauge bosons $W^\pm$ and Z, and the electromagnetic photon field A_μ of the SM.

where we have used eq. (2.10) to replace $g' \cos\theta_{\rm W} = g \sin\theta_{\rm W}$. The first line of eq. (2.12) contains the leptons' kinetic terms. The second line describes the electromagnetic interactions of the leptons with the photon field A_μ, where only the charged lepton participates, as it should be. And finally the third and fourth lines contain the (weak) neutral and charged current interactions, respectively. Note that only left-handed particle states participate in charged current interactions.

Now, let us return to eq. (2.7b), the Yukawa interactions of the leptons with the SM Higgs doublet. The electroweak Higgs (iso-)doublet is defined by

$$\phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}, \tag{2.13}$$

where the superscripts refer to the electric charge of its components. It is assigned quantum numbers $(I,\ Y) = (1/2,\ 1)$, leading to electric charges according to eq. (2.3) consistent with the nomenclature of eq. (2.13).[5] Together with the Higgs doublet comes its charge conjugate

$$\widehat{\phi} \equiv i\sigma^2 \phi^*, \tag{2.14}$$

with quantum numbers $(I,\ Y) = (1/2,\ -1)$, to be used when we introduce right-handed neutrinos. Inserting the expressions for ϕ, $L_{\rm L}$ and $e_{\rm R}$ into eq. (2.7b) we get

$$-\mathscr{L}_{\rm Yukawa} = y(\overline{\nu}_{\rm L} e_{\rm R} \phi^+ + \overline{e}_{\rm L} e_{\rm R} \phi^0) + h.c. \ . \tag{2.15}$$

After the neutral component of the Higgs doublet has developed a real non-zero vacuum expectation value (VEV)

$$\langle \phi \rangle = \begin{pmatrix} 0 \\ v \end{pmatrix}, \tag{2.16}$$

the symmetry of the electroweak gauge group $SU(2)_{\rm L} \times U(1)_{\rm Y}$ is spontaneously broken down to the symmetry group of electric charge, $U(1)_{\rm Q}$. In the Lagrangian in eq. (2.15) as consequence of the non-vanishing VEV a mass term for the charged lepton,

$$-\mathscr{L}_{\rm mass} = yv\overline{e}_{\rm L} e_{\rm R} + h.c. \equiv m\overline{e}_{\rm L} e_{\rm R} + h.c. \ , \tag{2.17}$$

is generated with mass $m \equiv yv$. Due to the fact that in the SM there is no right-handed neutrino field, no such mass term arises for the neutrino.[6]

[5] The quantum numbers of the Higgs doublet are chosen in order to form invariant terms in the Lagrangian involving left- and right-handed fermion states.

[6] Actually, it is the other way around. Historically, the SM by convention does not contain right-

- three generations: Next, we consider three generations of leptons. Evoking the flavor indices of the weak lepton doublet and singlet, eq. (2.7) reads

$$\begin{aligned}\mathscr{L}_{\text{lept}} &= i\overline{L}_{\alpha\text{L}}\gamma^{\mu}D_{\text{L}\mu}L_{\alpha\text{L}} + i\overline{e}_{\alpha\text{R}}\gamma^{\mu}D_{\text{R}\mu}e_{\alpha\text{R}} \\ &\quad - Y_{\alpha\beta}\overline{L}_{\alpha\text{L}}\phi e_{\beta\text{R}} - Y^{*}_{\alpha\beta}\overline{e}_{\alpha\text{R}}\phi^{\dagger}L_{\beta\text{L}},\end{aligned} \tag{2.18}$$

where now the Yukawa couplings $Y_{\alpha\beta}$ are the in general complex-valued coefficients of a 3×3 matrix Y. Inserting the expressions for the covariant derivatives eqs. (2.11), the first line of eq. (2.18), i.e. the kinetic part of the Lagrangian, takes on the same form as eq. (2.12) - one simply has to put a flavor index α on each lepton field, to get its correct version in three generations.

Now, we take a look at the second line in eq. (2.18). After the Higgs develops a VEV as in eq. (2.16), the Yukawa couplings read

$$\begin{aligned}-\mathscr{L}_{\text{Yukawa}} &= Y_{\alpha\beta}\begin{pmatrix}\overline{\nu}_{\alpha\text{L}} & \overline{e}_{\alpha\text{L}}\end{pmatrix}\begin{pmatrix}0 \\ v\end{pmatrix}e_{\beta\text{R}} + Y^{*}_{\alpha\beta}\overline{e}_{\alpha\text{R}}\begin{pmatrix}0 & v\end{pmatrix}\begin{pmatrix}\nu_{\beta\text{L}} \\ e_{\beta\text{L}}\end{pmatrix} \\ &= vY_{\alpha\beta}\overline{e}_{\alpha\text{L}}e_{\beta\text{R}} + h.c.,\end{aligned} \tag{2.19}$$

or in matrix notation

$$-\mathscr{L}_{\text{Yukawa}} = v\overline{e}_{\text{L}}Ye_{\text{R}} + h.c.\ . \tag{2.20}$$

The matrix Y can be diagonalized by a bi-unitary transformation[7]

$$V_{\text{L}}^{\dagger}YV_{\text{R}} = Y', \tag{2.21}$$

where V_{L} and V_{R} are unitary matrices and the diagonalized form of Y is given by[8]

$$Y'_{ab} = y_a\delta_{ab}. \tag{2.22}$$

Note that here we used Latin letters for the indices, since the transformation eq. (2.21) leads us to the mass basis, which is *defined* as the basis where Y is diagonal. Defining the lepton states in the mass basis

$$e_{a\text{L}} \equiv V^{\dagger}_{a\alpha\text{L}}e_{\alpha\text{L}}, \qquad\qquad e_{a\text{R}} \equiv V^{\dagger}_{a\alpha\text{R}}e_{\alpha\text{R}}, \tag{2.23}$$

handed neutrinos, *because* otherwise they would get a mass like all the other SM fermions resulting from the spontaneous symmetry breaking by the Higgs doublet's VEV.

[7]For a proof, see [34], chapter 4.1.

[8]No summation over a.

eq. (2.19) reads

$$
\begin{aligned}
-\mathscr{L}_{\text{Yukawa}} &= v(\overline{e}_{\text{L}} V_{\text{L}}) V_{\text{L}}^{\dagger} Y V_{\text{R}} (V_{\text{R}}^{\dagger} e_{\text{R}}) + h.c. \\
&= v y_a \overline{e}_{a\text{L}} \delta_{ab} e_{b\text{R}} + h.c. \\
&= m_a \overline{e}_{a\text{L}} e_{a\text{R}} + h.c. \ , \qquad (2.24)
\end{aligned}
$$

where in the last line we have defined the charged lepton masses $m_a = v y_a$ with $a = 1,\ 2,\ 3$.[9] Note that for quark masses one can find equal expressions $m_a^q = v y_a^q$, with $a = 1,\ 2,\ 3$, where y_a^q denotes an entry of the quarks' Yukawa coupling matrix Y^q in the mass basis. For up-type quarks we have $Y_{ab}^u = y_a^u \delta_{ab}$, and similarly Y_{ab}^d for down-type quarks.

[9] Writing this, we strictly stick to our notation for the mass basis. The unitary transformation matrices, which diagonalize Y, however, can be absorbed into the field definitions of the charged leptons so that flavor and mass eigenstates coincide. Hence, one is free to identify $m_1 = m_e$, and so on.

Chapter 3

Neutrino related extensions of the Standard Model

In this chapter general extensions of the SM that can generate neutrino masses are presented. The main actor of our discussion will be the neutrino mass matrix M in the Majorana basis introduced in eq. (A.16). Note that the diagonal elements of the mass matrix, M_{aa}, represent Majorana mass terms, while the off-diagonal elements, $M_{ab} = M_{ba}$, are Dirac mass terms. We begin by simply adding right-handed neutrino fields to the particle content of the SM and then we discuss the results. Afterwards, we explain how the Higgs sector may be manipulated to show alternative ways for generating neutrino masses.

3.1 Generation of neutrino masses

An obvious way to generate neutrino masses is to extend the SM by three right-handed neutrino fields $N_{\alpha\mathrm{R}}$ in parallel to the other SM fermion fields. Since they are introduced as right-handed particles they are weak isospin singlets ($I = 0$). From eq. (2.3) we derive that they consequently must have hypercharge $Y = 0$, too, making them singlets of the whole electroweak gauge group. Since the neutrinos N_R only interact gravitationally they are often called *sterile* neutrinos, whereas the SM neutrinos ν_L are referred to as *active* neutrinos. Note, however, that the mass eigenstates of the neutrinos in general can be a combination of active and sterile states (cf. section 3.3).

Let us see how we can form mass terms for the neutrinos invariant under the SM gauge group. First, we can only couple left-handed fields to right-handed fields.[1] To our disposal are the active neutrinos ν_L with quantum numbers $(I_3,\ Y) = (1/2,\ -1)$ and their charge conjugate $\widehat{\nu}_\mathrm{R}$ with opposite quantum numbers and also

[1] A mass term of two fields with the same chirality vanishes because of the properties of the chiral projection operators, $P_\mathrm{L} P_\mathrm{R} = P_\mathrm{R} P_\mathrm{L} = 0$.

the sterile neutrinos N_R and their charge conjugate $\widehat{N}_\mathrm{L}$, both with $(I_3,\ Y) = (0,\ 0)$. The mass terms that can theoretically be formed with this setup together with the respective net quantum numbers are listed in Table 3.1. The term $\overline{\nu}_\mathrm{L} N_\mathrm{R} = \overline{\widehat{N}}_\mathrm{L} \widehat{\nu}_\mathrm{R}$ (cf. eq. (A.7)) responsible for Dirac masses can be rendered invariant by means of the neutral component of the charge conjugate of the SM Higgs doublet. The term $\overline{\widehat{N}}_\mathrm{L} N_\mathrm{R}$ leads to a Majorana mass for the sterile neutrinos. It can either be a bare Majorana mass term with some dimension-1 coupling constant, or theoretically it could be coupled to a singlet scalar field with $(I_3,\ Y) = (0,\ 0)$. The Majorana mass term for the active neutrinos coming from $\overline{\nu}_\mathrm{L} \widehat{\nu}_\mathrm{R}$ can only be made invariant if coupled to a triplet scalar field with $(I_3,\ Y) = (1, -2)$. Both scalar fields, the singlet and the triplet do not exist in the SM and would have to be introduced as new fields (cf. section 3.4).

mass term	$\Delta(I_3,\ Y)$
$\overline{\nu}_\mathrm{L} N_\mathrm{R}$	$(1/2, -1)$
$\overline{\widehat{N}}_\mathrm{L} \widehat{\nu}_\mathrm{R}$	$(1/2, -1)$
$\overline{\widehat{N}}_\mathrm{L} N_\mathrm{R}$	$(0,\ 0)$
$\overline{\nu}_\mathrm{L} \widehat{\nu}_\mathrm{R}$	$(1, -2)$

Table 3.1: Possible mass terms for neutrinos and their net quantum numbers.

In the following, we will elaborate the consequences of including a Dirac mass term for neutrinos in our theory. Then we will additionally consider a bare Majorana mass term for the sterile neutrinos. At the end we will discuss the possible extensions of the Higgs sector, enabling us to write down any neutrino mass term.

3.2 Dirac mass terms for neutrinos

In order to form an invariant term in the Lagrangian, containing the newly introduced fields, we can couple the N_R to the left-handed doublets L_L and the charge conjugate of the Higgs doublet $\widehat{\phi}$, defined in eq. (2.14). This leads to Yukawa couplings that we will denote by the 3×3 matrix Y^ν. Then neutrino masses are the result of symmetry breaking by the Higgs as explained for charged lepton masses in section 2.2.2, coming from the Yukawa Lagrangian

$$\begin{aligned} -\mathscr{L}^\nu_\mathrm{Yukawa} &= \overline{L}_\mathrm{L} Y^\nu \widehat{\phi} N_\mathrm{R} + h.c. \\ &\underset{\phi \to \langle\phi\rangle}{=} v \overline{\nu}_\mathrm{L} Y^\nu N_\mathrm{R} + h.c. \ . \end{aligned} \tag{3.1}$$

Changing from flavor basis to mass basis as in eq. (2.21) and defining the neutrino fields in the mass basis by

$$\nu_{a\mathrm{L}} \equiv V^{\nu\dagger}_{a\alpha\mathrm{L}} \nu_{\alpha\mathrm{L}}, \tag{3.2}$$

$$\nu_{a\mathrm{R}} \equiv V^{N\dagger}_{a\alpha\mathrm{R}} N_{\alpha\mathrm{R}}, \tag{3.3}$$

where V^{ν}_{L} and V^{N}_{R} are unitary matrices, eq. (3.1) becomes

$$\begin{aligned} -\mathscr{L}_{\text{Yukawa}} &= v(\overline{\nu}_{\mathrm{L}} V^{\nu}_{\mathrm{L}}) V^{\nu\dagger}_{\mathrm{L}} Y^{\nu} V^{N}_{\mathrm{R}} (V^{N\dagger}_{\mathrm{R}} N_{\mathrm{R}}) + h.c. \\ &= v y^{\nu}_{a} \overline{\nu}_{a\mathrm{L}} \delta_{ab} \nu_{b\mathrm{R}} + h.c. \\ &= m^{\nu}_{a} \overline{\nu}_{a\mathrm{L}} \nu_{a\mathrm{R}} + h.c. \ , \end{aligned} \tag{3.4}$$

where we defined the neutrino masses $m^{\nu}_{a} \equiv v y^{\nu}_{a}$, with $a = 1,\ 2,\ 3$. Defining the Dirac mass matrix

$$M_{\mathrm{D}} \equiv v Y^{\nu}, \tag{3.5}$$

the Yukawa couplings eq. (3.1) can be rewritten in the notation of eq. (A.16) as

$$\begin{aligned} -\mathscr{L}_{\text{Yukawa}} &= \overline{\nu}_{\alpha\mathrm{L}} M_{\mathrm{D}\alpha\beta} N_{\beta\mathrm{R}} + h.c. \\ &= \frac{1}{2}(\overline{\nu}_{\alpha\mathrm{L}} M_{\mathrm{D}\alpha\beta} N_{\beta\mathrm{R}} + \overline{\widehat{N}}_{\alpha\mathrm{L}} M_{\mathrm{D}\beta\alpha} \widehat{\nu}_{\beta\mathrm{R}}) + h.c. \\ &= \frac{1}{2} \begin{pmatrix} \overline{\nu}_{\mathrm{L}} & \overline{\widehat{N}}_{\mathrm{L}} \end{pmatrix} \begin{pmatrix} 0 & M_{\mathrm{D}} \\ M^{T}_{\mathrm{D}} & 0 \end{pmatrix} \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} \\ N_{\mathrm{R}} \end{pmatrix} + h.c. \ . \end{aligned} \tag{3.6}$$

Thus a neutrino mass matrix

$$M = \begin{pmatrix} 0 & M_{\mathrm{D}} \\ M^{T}_{\mathrm{D}} & 0 \end{pmatrix} \tag{3.7}$$

as in eq. (3.6) results in Dirac neutrinos with mass M_{D}.

Since the mass of a SM fermion f_a is given by $m^{f}_{a} = v y^{f}_{a}$, the ratio of Yukawa couplings of two fermions f and g of the same generation is equal to the ratio of masses

$$R_{fg} = \frac{y^{f}_{a}}{y^{g}_{a}} = \frac{m^{f}_{a}}{m^{g}_{a}} \ , \tag{3.8}$$

with a fixed. Inserting the masses (or in the case of neutrinos the upper bounds for the mass) of Table 2.1 into eq. (3.8), for the generations in the SM we have

$$
\begin{aligned}
&R_{du} \sim 1, && R_{sc} \sim 10^{-1}, && R_{bt} \sim 10^{-2}, \\
&R_{ed} \sim 1, && R_{\mu s} \sim 1, && R_{\tau b} \sim 1, \\
&R_{\nu_e e} < 10^{-5}, && R_{\nu_\mu \mu} < 10^{-3}, && R_{\nu_\tau \tau} < 10^{-2}.
\end{aligned} \tag{3.9}
$$

This shows that, if masses are generated by Dirac mass terms, the neutrinos' Yukawa couplings would have to be very small in comparison to the ones of the other SM fermions. In the rest of this chapter we will present the theoretically possible ways to explain the smallness of neutrino masses in the context of Majorana mass terms.

3.3 Majorana mass terms for neutrinos

Among all SM fermions the neutrinos are the only ones which are uncharged so that they can possibly be Majorana particles. If they really are, Majorana mass terms in the Lagrangian can be written for them. In section A.2 of the appendix we point out that one can get mass eigenvalues suppressed by some scale, if one allows for Majorana mass terms in the theory (cf. eqs. (A.20) and (A.22)). We had there a set of two Majorana spinors. In this section we are going to explain how this is done for three left- and three right-handed neutrinos. The diagonalization of the neutrino mass matrix will result in the suppression of active neutrino masses known as the type I seesaw mechanism (cf. section 4.1).

We begin with the Lagrangian given in eq. (3.6). In addition, we permit a bare Majorana mass term for the sterile neutrinos and their charge conjugate. The mass Lagrangian for the neutrinos then reads

$$
-\mathscr{L}_{\text{mass}} = \overline{\nu}_{\alpha\text{L}} M_{\text{D}\alpha\beta} N_{\beta\text{R}} + \frac{1}{2}\overline{\widehat{N}}_{\alpha\text{L}} M_{\text{R}\alpha\beta} N_{\beta\text{R}} + h.c. \;, \tag{3.10}
$$

where M_{R} denotes the Majorana mass matrix for the sterile neutrinos. In the matrix notation eq. (3.10) reads

$$
-\mathscr{L}_{\text{mass}} = \frac{1}{2}\begin{pmatrix} \overline{\nu}_{\text{L}} & \overline{\widehat{N}}_{\text{L}} \end{pmatrix} \begin{pmatrix} 0 & M_{\text{D}} \\ M_{\text{D}}^T & M_{\text{R}} \end{pmatrix} \begin{pmatrix} \widehat{\nu}_{\text{R}} \\ N_{\text{R}} \end{pmatrix} + h.c. \;. \tag{3.11}
$$

Under the assumption $M_{\text{R}} \gg M_{\text{D}}$ we can diagonalize the mass matrix, as explained in section B.2 of the appendix, to find its mass eigenvalues.[2] Performing

[2]Note the approximate nature of this diagonalization. In this subsection, we will emphasize this with approximately signs, but later on we will mostly give the expressions to leading order with equality signs.

a transformation with

$$V \approx \begin{pmatrix} \mathbb{1} & M_{\mathrm{R}}^{-1} M_{\mathrm{D}} \\ -M_{\mathrm{D}}^{T} M_{\mathrm{R}}^{-1} & \mathbb{1} \end{pmatrix} \tag{3.12}$$

eq. (3.11) becomes

$$\begin{aligned} -\mathscr{L}_{\text{mass}} &= \frac{1}{2} \begin{pmatrix} \overline{\nu}_{\mathrm{L}} & \overline{\widehat{N}}_{\mathrm{L}} \end{pmatrix} V V^{T} \begin{pmatrix} 0 & M_{\mathrm{D}} \\ M_{\mathrm{D}}^{T} & M_{\mathrm{R}} \end{pmatrix} V^{T} V \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} \\ N_{\mathrm{R}} \end{pmatrix} + h.c. \\ &= \overline{n}_{\mathrm{L}} M' n_{\mathrm{R}} + h.c. \ , \end{aligned} \tag{3.13}$$

where we have defined the diagonalized mass matrix

$$M' \equiv V^{T} \begin{pmatrix} 0 & M_{\mathrm{D}} \\ M_{\mathrm{D}}^{T} & M_{\mathrm{R}} \end{pmatrix} V \approx \begin{pmatrix} -M_{\mathrm{D}} M_{\mathrm{R}}^{-1} M_{\mathrm{D}}^{T} & 0 \\ 0 & M_{\mathrm{R}} \end{pmatrix} \tag{3.14}$$

and the chiral neutrino states

$$\overline{n}_{\mathrm{L}} \equiv \begin{pmatrix} \overline{\nu}_{\mathrm{L}} & \overline{\widehat{N}}_{\mathrm{L}} \end{pmatrix} V \approx \begin{pmatrix} \overline{\nu}_{\mathrm{L}} - M_{\mathrm{D}}^{T} M_{\mathrm{R}}^{-1} \overline{\widehat{N}}_{\mathrm{L}}, & \overline{\widehat{N}}_{\mathrm{L}} + M_{\mathrm{R}}^{-1} M_{\mathrm{D}} \overline{\nu}_{\mathrm{L}} \end{pmatrix}, \tag{3.15}$$

$$n_{\mathrm{R}} \equiv V^{T} \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} \\ N_{\mathrm{R}} \end{pmatrix} \approx \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} - M_{\mathrm{D}}^{T} M_{\mathrm{R}}^{-1} N_{\mathrm{R}} \\ N_{\mathrm{R}} + M_{\mathrm{R}}^{-1} M_{\mathrm{D}} \widehat{\nu}_{\mathrm{R}} \end{pmatrix}. \tag{3.16}$$

The states with mass $M_1 \approx -M_{\mathrm{D}} M_{\mathrm{R}}^{-1} M_{\mathrm{D}}^{T}$ and, respectively, $M_2 \approx M_{\mathrm{R}}$, then, are given by

$$n_1 \equiv n_{1\mathrm{L}} + n_{1\mathrm{R}} \approx (\nu_{\mathrm{L}} + \widehat{\nu}_{\mathrm{R}}) - M_{\mathrm{D}}^{T} M_{\mathrm{R}}^{-1} (N_{\mathrm{R}} + \widehat{N}_{\mathrm{L}}) \tag{3.17}$$

$$n_2 \equiv n_{2\mathrm{L}} + n_{2\mathrm{R}} \approx (N_{\mathrm{R}} + \widehat{N}_{\mathrm{L}}) + M_{\mathrm{R}}^{-1} M_{\mathrm{D}} (\nu_{\mathrm{L}} + \widehat{\nu}_{\mathrm{R}}). \tag{3.18}$$

Working in one generation only, where the matrices in eq. (3.17) and (3.18) are ordinary numbers, it is obvious that both equations describe a Majorana particle.

3.4 Extended Higgs sector

In this section we discuss possible extensions of the Higgs sector. We introduce new scalar fields, which have Yukawa couplings with the neutrinos leading to neutrino mass terms that are absent in the SM. The Higgs potential has to be minimized with respect to the VEVs of all scalar fields. This includes the potential of the SM Higgs, which will be altered by the introduction of additional scalars. In this thesis we always assume that the VEVs we choose correspond to the minimum of the Higgs potential. Note that to guarantee electric charge conservation, only neutral components of a Higgs multiplet can develop non-zero VEVs.

Depending on the transformation properties of a scalar field with respect to the gauge group of the theory, a field can affect the gauge sector. We will briefly comment on this when discussing the left-right symmetric model in section 3.4.3.

3.4.1 Scalar singlet

In the previous section, we introduced a bare Majorana mass term $\overline{\widehat{N}}_{\mathrm{L}} M_{\mathrm{R}} N_{\mathrm{R}}$ for the sterile neutrinos. If the theory is extended by a singlet scalar field ϕ_{S} with quantum numbers $(I_3,\ Y) = (0,\ 0)$, this Majorana mass term can be thought of as being the result of a Yukawa coupling of the sterile neutrinos with ϕ_{S}, which then develops a VEV. Obviously, the scalar singlet is electrically uncharged, allowing us to assume that it takes on a non-zero VEV $\langle\phi_{\mathrm{S}}\rangle = v_{\mathrm{S}}$. Denoting the matrix of Yukawa couplings by Y_{S}, the Lagrangian contains a term

$$\begin{aligned} -\mathscr{L}_{\text{Yukawa}}^{\phi_{\mathrm{S}}} &= \frac{1}{2}\overline{\widehat{N}}_{\mathrm{L}} Y_{\mathrm{S}} N_{\mathrm{R}} \phi_{\mathrm{S}} + h.c. \\ &\underset{\phi_{\mathrm{S}}\to\langle\phi_{\mathrm{S}}\rangle}{=} \frac{1}{2} v_{\mathrm{S}} \overline{\widehat{N}}_{\mathrm{L}} Y_{\mathrm{S}} N_{\mathrm{R}} + h.c.\ . \end{aligned} \tag{3.19}$$

Identifying $M_{\mathrm{R}} = v_{\mathrm{S}} Y_{\mathrm{S}}$, we see that the bare Majorana mass term in eq. (3.10) and the Yukawa term in eq. (3.19) are mathematically equivalent. The difference between both terms is that the mass scale of the bare Majorana mass term in eq. (3.10) is inserted artificially, while the mass scale of the Majorana mass term in eq. (3.19) comes from the VEV of the scalar field.

3.4.2 Triplet

From the net quantum numbers in Table 3.1 we have seen that in order to get a Majorana mass term for the active neutrinos, we need a scalar field with $(I_3,\ Y) = (1, -2)$. Such a field can be introduced as the neutral component Δ^0 of a scalar triplet $\boldsymbol{\Delta}$ with $I = 1$.[3] The Yukawa coupling of the neutral component of the triplet with the active neutrinos reads

$$-\mathscr{L}_{\text{Yukawa}}^{\Delta} = \frac{1}{2}\overline{\nu}_{\mathrm{L}} Y_{\Delta} \Delta^0 \widehat{\nu}_{\mathrm{R}} + h.c.\ . \tag{3.20}$$

When Δ^0 develops a non-zero VEV $\langle\Delta^0\rangle = v_\Delta$ eq. (3.20) becomes

$$-\mathscr{L}_{\text{Yukawa}}^{\Delta} = \frac{1}{2}\overline{\nu}_{\mathrm{L}} M_{\mathrm{L}} \widehat{\nu}_{\mathrm{R}} + h.c.\ , \tag{3.21}$$

[3] For a detailed introduction of a Higgs triplet see e.g. [45].

where we have defined the Majorana mass matrix $M_{\mathrm{L}} \equiv v_{\Delta} Y_{\Delta}$ for the active neutrinos. Including eq. (3.21) in eq. (3.10) leads to the mass Lagrangian

$$\begin{aligned}
-\mathscr{L}_{\text{mass}}^{\Delta} &= \overline{\nu}_{\alpha\mathrm{L}} M_{\mathrm{D}\alpha\beta} N_{\beta\mathrm{R}} + \frac{1}{2}\overline{\widehat{N}}_{\alpha\mathrm{L}} M_{\mathrm{R}\alpha\beta} N_{\beta\mathrm{R}} + \frac{1}{2}\overline{\nu}_{\alpha\mathrm{L}} M_{\mathrm{L}\alpha\beta} \widehat{\nu}_{\beta\mathrm{R}} + h.c. \\
&= \frac{1}{2}\begin{pmatrix} \overline{\nu}_{\mathrm{L}} & \overline{\widehat{N}}_{\mathrm{L}} \end{pmatrix} \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} \\ N_{\mathrm{R}} \end{pmatrix} + h.c. \ .
\end{aligned} \tag{3.22}$$

Hence the neutrino mass matrix is given by

$$M = \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix}, \tag{3.23}$$

which corresponds to the form of the neutrino mass matrix in the type II seesaw mechanism (cf. section 4.2).

3.4.3 Left-right symmetric models

Another promising theory for generating neutrino masses is the left-right symmetric model [46–49], where the gauge group of the SM is extended by the gauge group $SU(2)_{\mathrm{R}}$ of right-handed isospin, technically analogous to the symmetry group of weak (left-handed) isospin $SU(2)_{\mathrm{L}}$. The quantum number hypercharge is redefined as

$$Y = B - L, \tag{3.24}$$

where B and L denote baryon and lepton number, respectively. With this extension the Gell-Mann-Nishijima relation, eq. (2.3), is changed to

$$Q = I_{3\mathrm{L}} + I_{3\mathrm{R}} + \frac{B-L}{2}, \tag{3.25}$$

where the subscripts L and R serve to distinguish between the isospin quantum numbers of $SU(2)_{\mathrm{L}}$ and $SU(2)_{\mathrm{R}}$, respectively. Now right-handed leptons can be grouped in right-handed iso-doublets $R_{\mathrm{R}} \equiv (N_{\mathrm{R}},\ e_{\mathrm{R}})^T$ with quantum numbers $(I_{\mathrm{L}},\ I_{\mathrm{R}},\ B-L) = (0,\ 1/2,\ -1)$, while the left-handed lepton doublet is a singlet with respect to $SU(2)_{\mathrm{R}}$ with quantum numbers $(I_{\mathrm{L}},\ I_{\mathrm{R}},\ B-L) = (1/2,\ 0,\ -1)$. The new gauge group has additional gauge bosons $W_{\mathrm{R}}^{\pm}$ and Z_{R} analogous to the weak gauge bosons defined in section 2.2.

The left-right symmetric model brings up new possibilities of forming invariant terms in the Lagrangian. From Table 3.2 we see that Dirac mass terms for neutrinos can be formed using a scalar bi-doublet, denoted by Φ, with quantum numbers

mass term	$\Delta(I_{3\mathrm{L}},\ I_{3\mathrm{R}},\ B-L)$
$\overline{\nu}_{\mathrm{L}} N_{\mathrm{R}}$	$(1/2, -1/2,\ 0)$
$\overline{\widehat{N}}_{\mathrm{L}} \widehat{\nu}_{\mathrm{R}}$	$(1/2, -1/2,\ 0)$
$\overline{\widehat{N}}_{\mathrm{L}} N_{\mathrm{R}}$	$(0, -1,\ 2)$
$\overline{\nu}_{\mathrm{L}} \widehat{\nu}_{\mathrm{R}}$	$(1,\ 0, -2)$

Table 3.2: Possible mass terms for neutrinos and their net quantum numbers in left-right symmetric models.

$(I_{\mathrm{L}},\ I_{\mathrm{R}},\ B-L) = (1/2,\ 1/2,\ 0)$ and with a charge decomposition given by

$$\Phi \equiv \begin{pmatrix} \phi_1^0 & \phi_1^+ \\ \phi_2^- & \phi_2^0 \end{pmatrix} . \tag{3.26}$$

Its Yukawa couplings with the left- and right-handed doublet read

$$-\mathscr{L}_{\text{Yukawa}} = \overline{L}_{\mathrm{L}}(F\Phi + G\widehat{\Phi})R_{\mathrm{R}} + h.c. \ , \tag{3.27}$$

where the charge conjugate bi-doublet is defined as $\widehat{\Phi} \equiv \sigma^2\Phi^*\sigma^2$ and F and G denote the Yukawa couplings of Φ and $\widehat{\Phi}$, respectively. For simplicity we assume that only the component ϕ_1^0 of bi-doublet develops a real non-zero VEV,

$$\langle\Phi\rangle = \begin{pmatrix} v & 0 \\ 0 & 0 \end{pmatrix} , \tag{3.28}$$

so that eq. (3.27) becomes

$$\begin{aligned} -\mathscr{L}_{\text{Yukawa}} &= \overline{\nu}_{\mathrm{L}} F N_{\mathrm{R}} \phi_1^0 + \overline{e}_{\mathrm{L}} G e_{\mathrm{R}} (\phi_1^0)^* + h.c. \\ &\underset{\Phi\to\langle\Phi\rangle}{=} v\overline{\nu}_{\mathrm{L}} F N_{\mathrm{R}} + v\overline{e}_{\mathrm{L}} G e_{\mathrm{R}} + h.c. \ . \end{aligned} \tag{3.29}$$

Comparing with eq. (2.19) and (3.1) and identifying $F = Y^\nu$ and $G = Y$ we see that the component ϕ_1^0 of the bi-doublet corresponds to the component $\widehat{\phi}\,^0$ of the (charge conjugate) SM Higgs doublet. Hence, the VEV of Φ must be the VEV of the SM Higgs introduced in eq. (2.16).[4]

The active neutrinos get a Majorana mass term by coupling to the neutral component Δ_{L}^0 of a left-iso-triplet scalar $\boldsymbol{\Delta}_{\mathrm{L}}$ that corresponds to the triplet introduced in section 3.4. Analogously one can generate a Majorana mass term for the sterile neutrinos using the neutral component Δ_{R}^0 of a right-iso-triplet scalar $\boldsymbol{\Delta}_{\mathrm{R}}$. Note

[4] If one generally assumes that ϕ_2^0 takes on a non-zero VEV, too, the VEV of the neutral component of the SM Higgs will be a superposition of $\langle\phi_1^0\rangle$ and $\langle\phi_2^0\rangle$.

that Δ_{R}^0 may be identified with the singlet ϕ_{S}, if the latter is assigned the quantum numbers $(I_{3\mathrm{L}},\ I_{3\mathrm{R}},\ B-L) = (0, -1,\ 2)$.[5]

Now we want to write a Yukawa Lagrangian including all possible mass terms. Let us assume that the scalar fields develop the VEVs $\langle \phi_1^0 \rangle = v$, $\langle \Delta_{\mathrm{L}}^0 \rangle = v_{\mathrm{L}}$, $\langle \Delta_{\mathrm{R}}^0 \rangle = v_{\mathrm{R}}$ and the other VEVs vanish. Then the general Lagrangian of neutrino Yukawa couplings in the left-right symmetric model can be written as

$$\begin{aligned} -\mathscr{L}_{\mathrm{Yukawa}} &= v\overline{\nu}_{\mathrm{L}} Y^{\nu} N_{\mathrm{R}} + \frac{1}{2} v_{\mathrm{L}} \overline{\nu}_{\mathrm{L}} Y_{\mathrm{L}} \widehat{\nu}_{\mathrm{R}} + \frac{1}{2} v_{\mathrm{R}} \overline{\widehat{N}}_{\mathrm{L}} Y_{\mathrm{R}} N_{\mathrm{R}} + h.c. \\ &= \frac{1}{2} \begin{pmatrix} \overline{\nu}_{\mathrm{L}} & \overline{\widehat{N}}_{\mathrm{L}} \end{pmatrix} \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix} \begin{pmatrix} \widehat{\nu}_{\mathrm{R}} \\ N_{\mathrm{R}} \end{pmatrix} + h.c.\ , \end{aligned} \tag{3.30}$$

where we defined $M_{\mathrm{D}} \equiv vY^{\nu}$, $M_{\mathrm{L}} \equiv v_{\mathrm{L}} Y_{\mathrm{L}}$ and $M_{\mathrm{R}} \equiv v_{\mathrm{R}} Y_{\mathrm{R}}$. Note that the mass matrix in eq. (3.30)

$$M \equiv \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix} \tag{3.31}$$

is mathematically the same as the mass matrix in eq. (3.23), but comes from different physics in the Higgs sector.

Finally, we want to point out that the structure of the neutrino mass matrix can easily be extended e.g. by introducing a different type of neutrinos S. The additional neutrinos can possibly form both Dirac mass terms with the active and sterile neutrinos and Majorana mass terms among themselves. This results in a mass matrix with the general form

$$M = \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} & M_{\mathrm{LS}}^T \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} & M_{\mathrm{RS}}^T \\ M_{\mathrm{LS}} & M_{\mathrm{RS}} & M_{\mathrm{S}} \end{pmatrix} \tag{3.32}$$

in the Majorana basis $(\widehat{\nu}_{\mathrm{R}},\ N_{\mathrm{R}},\ S)^T$. Matrices of this type appear in the context of the extended seesaw mechanisms that will be discussed in section 4.4.

[5] In the context of left-right symmetry, obviously, ϕ_{S} is no longer a singlet.

Chapter 4

Variants of the seesaw mechanism

In the last chapter we have shown how to obtain neutrino mass terms of different types leading to a neutrino mass matrix in the general form of

$$M = \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix}. \tag{4.1}$$

In the Lagrangian and hence in the mass matrix one always has to include all theoretically allowed terms. Now that we know how to form mass terms for different types of neutrinos, we can write the mass matrix according to the model we wish to discuss. As long as we use the matrix notation, we do not need to specify the number of neutrinos we consider. For example in a model with sterile neutrinos and a Higgs triplet coupled to active neutrinos, eq. (4.1) is valid for n_{a} active and n_{s} sterile neutrinos. In a model without an $SU(2)$-triplet Higgs the Majorana mass term M_{L} is forbidden and neither appears in the Lagrangian nor in the mass matrix.

To find the (active) neutrino masses, the matrix M needs to be diagonalized by a unitary transformation matrix V obeying

$$M^{\mathrm{d}} \equiv V^T M V = \begin{pmatrix} M_\nu & 0 \\ 0 & M_{\mathrm{st}} \end{pmatrix}, \tag{4.2}$$

where M_ν and M_{st} denote the active and sterile neutrino mass matrix, respectively. By means of the seesaw mechanism, predictions of neutrino masses (or mass scales) according to a certain model can be made. Implications in correlation with the neutrino mass matrix such as neutrino mixing and the smallness of neutrino masses or the possibility of Majorana and DM neutrinos might find an explanation in the seesaw mechanism.

In this chapter we will present analyses of different models that realize a seesaw scenario together with the respective predictions of neutrino masses.

4.1 Type I seesaw

The canonical or type I seesaw mechanism was already presented in section 3.3. For completeness, we recapitulate the major facts. We extend the SM by n_s sterile neutrinos in order to generate a Dirac mass term and permit a Majorana mass term for the sterile neutrinos. The Lagrangian for this setup can be found in eq. (3.10). This scenario corresponds to a mass matrix of the form

$$M_\mathrm{I} = \begin{pmatrix} 0 & M_\mathrm{D} \\ M_\mathrm{D}^T & M_\mathrm{R} \end{pmatrix} . \tag{4.3}$$

Note that one can always take M_R to be diagonal. Indeed, since $M_\mathrm{R} = M_\mathrm{R}^T$ it can be diagonalized by a unitary transformation. Consider a mass matrix

$$M_\mathrm{I}' = \begin{pmatrix} 0 & M_\mathrm{D}' \\ (M_\mathrm{D}')^T & M_\mathrm{R}' \end{pmatrix} , \tag{4.4}$$

with an arbitrary symmetric $n_\mathrm{s} \times n_\mathrm{s}$ matrix M_R' in place of M_R. Assuming that M_R' can be diagonalized by the unitary matrix V, we transform the total mass matrix M_I' like

$$M_\mathrm{I} \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & V^T \end{pmatrix} \begin{pmatrix} 0 & M_\mathrm{D} \\ M_\mathrm{D}^T & M_\mathrm{R} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} 0 & M_\mathrm{D}' V \\ V^T (M_\mathrm{D}')^T & V^T M_\mathrm{R}' V \end{pmatrix} . \tag{4.5}$$

Relabeling $M_\mathrm{D}' V \to M_\mathrm{D}$ and $V^T M_\mathrm{R}' V \to M_\mathrm{R}$ we are back to eq. (4.3), but now with diagonal M_R.

Let us return to eq. (4.3). If we assume $M_\mathrm{R} \gg M_\mathrm{D} \sim m_\mathrm{ew} \sim 10^2$ GeV, we can perform a seesaw-type transformation (see section B.2.1 of the appendix) leading to the diagonal form of the mass matrix

$$M_\mathrm{I}^\mathrm{d} = \begin{pmatrix} M_\mathrm{D} M_\mathrm{R}^{-1} M_\mathrm{D}^T & 0 \\ 0 & M_\mathrm{R} \end{pmatrix} . \tag{4.6}$$

In order to get neutrino masses $m_\nu \lesssim 1$ eV we need the Majorana mass term to be at least at about 10^{13} GeV.

For completeness we emphasize that the matrix form given in eq. (4.6) is a leading order approximation. In the cases we considered this approximation is well justified.[1] There are, however, scenarios, where next to leading order (NLO) effects get in the percent regime (- a detailed examination of consequences of NLO terms can be found in [50]). In section 4.4.4 we give an example for this.

[1] For example in eq. (4.6) we neglected terms of relative order $m_\mathrm{D}^2/m_\mathrm{R}^2 \lesssim 10^{-22}$ and smaller.

4.2 Type II seesaw

In the type II seesaw mechanism the mass Lagrangian in eq. (3.10) is extended by a triplet Majorana mass term for the active neutrinos that fills the zero-block of the mass matrix in eq. (4.3). The resulting mass matrix

$$M_{\mathrm{II}} = \begin{pmatrix} M_{\mathrm{L}} & M_{\mathrm{D}} \\ M_{\mathrm{D}}^T & M_{\mathrm{R}} \end{pmatrix} \tag{4.7}$$

is exactly in the form of eq. (B.9) in section B.2.1 of the appendix. This means that under the assumption $M_{\mathrm{R}} \gg M_{\mathrm{D}}$ and $M_{\mathrm{R}} \gg M_{\mathrm{L}}$ it can be diagonalized by a seesaw-type transformation leading to the diagonal form

$$M_{\mathrm{II}}^{\mathrm{d}} = \begin{pmatrix} M_{\mathrm{L}} - M_{\mathrm{D}} M_{\mathrm{R}}^{-1} M_{\mathrm{D}}^T & 0 \\ 0 & M_{\mathrm{R}} \end{pmatrix} . \tag{4.8}$$

Obviously, the type II seesaw mechanism can be understood as a generalization of the type I seesaw mechanism. Note that the triplet Majorana mass term M_{L} needs to be sufficiently small, since it contributes to the active neutrino masses without suppression by high scale physics.

For the type III seesaw mechanism, where a $SU(2)_{\mathrm{L}}$-triplet fermion is introduced, we refer to [51].

4.3 Minimal seesaw

In minimal seesaw models (for analyses of $3+1$ and $3+2$ models see e.g. [52, 53]) it is examined, which minimal set of extensions to the SM is needed to make the model compatible with experimental data. Since this thesis concerns itself with non-minimal variants we only mention that according to the "seesaw fair play rule" in the unbalanced seesaw [54] a seesaw scenario with p active and $q < p$ sterile neutrinos, leads to q non-zero active neutrino masses. Hence, to explain the two observed mass differences in active neutrino masses, one needs at least two sterile neutrinos. Note that this does not rule out $3+1$ models. Although within these models one sterile neutrino at the eV scale is considered that does not mean that there cannot be more.

4.4 Extended seesaw

As mentioned at the end of chapter 3, the seesaw mass matrix can be upgraded to an extended seesaw scenario (cf. eq. (3.32)). In addition to the canonical seesaw setup, let us introduce n_b uncharged fermion singlets S that have a Majorana mass term. Furthermore we introduce new scalars σ_LS and σ_RS, with VEVs v_LS and v_RS that couple singlets to active and sterile neutrinos, respectively. Including a possible triplet Majorana mass term for the active neutrinos the total neutrino mass matrix is given by eq. (3.32),

$$M = \begin{pmatrix} M_\mathrm{L} & M_\mathrm{D} & M_\mathrm{LS}^T \\ M_\mathrm{D}^T & M_\mathrm{R} & M_\mathrm{RS}^T \\ M_\mathrm{LS} & M_\mathrm{RS} & M_\mathrm{S} \end{pmatrix} \in \mathscr{M}^2[(n_\mathrm{a}+n_\mathrm{s}+n_\mathrm{b})], \tag{4.9}$$

where we write the Dirac mass terms in an obvious way as

$$M_\mathrm{D} \equiv Y_\mathrm{D}\langle\phi_\mathrm{SM}\rangle = vY_\mathrm{D}, \tag{4.10a}$$
$$M_\mathrm{LS} \equiv Y_\mathrm{LS}\langle\sigma_\mathrm{LS}\rangle = v_\mathrm{LS}Y_\mathrm{LS}, \tag{4.10b}$$
$$M_\mathrm{RS} \equiv Y_\mathrm{RS}\langle\sigma_\mathrm{RS}\rangle = v_\mathrm{RS}Y_\mathrm{RS}. \tag{4.10c}$$

As one can imagine, there are many ways to specify the neutrino mass matrix in the extended seesaw scenario. In this thesis we only considered cases with $n_\mathrm{a} = 3$ active neutrinos. In the following we will single out instructive examples and later on in chapter 5 we will discuss the singular extended seesaw mechanism.

4.4.1 Double seesaw

In the double seesaw mechanism [55, 56], the canonical seesaw is extended by introducing fermion singlets S that couple to sterile but not to active neutrinos. Moreover they shall have a Majorana mass term. Other Majorana mass terms are forbidden. Under these assumptions the neutrino mass matrix in the basis $(\widehat{\nu}_\mathrm{R},\ N_\mathrm{R},\ S)^T$ reads

$$M = \begin{pmatrix} 0 & M_\mathrm{D} & 0 \\ M_\mathrm{D}^T & 0 & M_\mathrm{RS}^T \\ 0 & M_\mathrm{RS} & M_\mathrm{S} \end{pmatrix}. \tag{4.11}$$

If $M_\mathrm{S} \gg M_\mathrm{RS} \gg M_\mathrm{D}$, this matrix can be diagonalized by two successive seesaw-type transformations. Using the formulas developed in section B.2 we find the diagonal form

$$M^\mathrm{d} = \begin{pmatrix} M_\nu & 0 & 0 \\ 0 & M_\mathrm{R}^\mathrm{d} & 0 \\ 0 & 0 & M_\mathrm{S} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M_D M_{RS}^{-1} M_S (M_{RS}^T)^{-1} M_D^T & 0 & 0 \\ 0 & M_{RS}^T M_S^{-1} M_{RS} & 0 \\ 0 & 0 & M_S \end{pmatrix} \tag{4.12}$$

with mass scale structure

$$M^d \sim \begin{pmatrix} \frac{m_D^2}{m_{RS}^2} m_S & 0 & 0 \\ 0 & \frac{m_{RS}^2}{m_S} & 0 \\ 0 & 0 & m_S \end{pmatrix}. \tag{4.13}$$

The double seesaw scenario is interesting, since if one inserts the characteristic values $M_D \sim m_{ew} \sim 100$ GeV, $M_{RS} \sim m_{GUT} \sim 10^{16}$ GeV and $M_{RS} \sim m_{Planck} \sim 10^{19}$ GeV, one gets active neutrino masses around 1 eV.

4.4.2 Screening of Dirac flavor structure

In [57] it is proposed to assume that the Yukawa couplings of M_D and M_{RS} are proportional to each other, i.e. $Y_D = rY_{RS}$.[2] The consequence for the active neutrino mass matrix in eq. (4.12)

$$M_\nu = M_D M_{RS}^{-1} M_S (M_{RS}^T)^{-1} M_D^T = r^2 \frac{\langle \phi_{SM} \rangle^2}{\langle \sigma_{RS} \rangle^2} M_S \tag{4.14}$$

is that its structure now only depends on the structure of the Majorana mass term for the singlets, M_S, and hence the flavor structure of the Dirac mass terms that describe mixing among the different types of neutrinos, is screened.

4.4.3 Linear seesaw

The linear seesaw mechanism (see e.g. [58]) follows an idea similar to the one just presented. Imagine an extended seesaw scenario without Majorana mass terms so that the mass matrix in eq. (4.9) takes on the form

$$M = \begin{pmatrix} 0 & M_D & M_{LS}^T \\ M_D^T & 0 & M_{RS}^T \\ M_{LS} & M_{RS} & 0 \end{pmatrix}. \tag{4.15}$$

[2]This proportionality might be the result of e.g. lepton number and/or gauge symmetry [57].

Assume that the VEVs of $\sigma_{\rm LS}$ and $\sigma_{\rm RS}$ are related by $v_{\rm LS} = r v_{\rm RS}$. Then the active neutrino mass matrix after diagonalization becomes

$$\begin{aligned} M_\nu &= -\begin{pmatrix} M_{\rm D} & M_{\rm LS}^T \end{pmatrix} \begin{pmatrix} 0 & M_{\rm RS}^{-1} \\ (M_{\rm RS}^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} M_{\rm D}^T \\ M_{\rm LS} \end{pmatrix} \\ &= -r(M_{\rm D} Y_{\rm RS}^{-1} Y_{\rm LS} + Y_{\rm LS}^T (Y_{\rm RS}^T)^{-1} M_{\rm D}^T), \end{aligned} \tag{4.16}$$

where we have used the proportionality between $v_{\rm LS}$ and $v_{\rm RS}$. Note that the scale $v_{\rm RS}$ has completely dropped out of the formula for the active neutrino masses. Now assuming that the Yukawa couplings are of order 1 and $M_{\rm D} \sim 10^2$ GeV, we see that the proportionality factor r needs to be rather small at $\mathscr{O}(10^{-11})$ or below.

4.4.4 Inverse seesaw

Consider a mass matrix of the form

$$M = \begin{pmatrix} 0 & M_{\rm D} & 0 \\ M_{\rm D}^T & M_{\rm R} & M_{\rm RS}^T \\ 0 & M_{\rm RS} & M_{\rm S} \end{pmatrix} . \tag{4.17}$$

The inverse seesaw scenario [55, 56] is obtained by $M_{\rm R} \to 0$ and assuming $M_{\rm RS} \gg M_{\rm D} \gg M_{\rm S}$. The inverse of the sub-matrix

$$M_{\rm X} \equiv \begin{pmatrix} 0 & M_{\rm RS}^T \\ M_{\rm RS} & M_{\rm S} \end{pmatrix} \tag{4.18}$$

is given by[3]

$$M_{\rm X}^{-1} = \begin{pmatrix} -M_{\rm RS}^{-1} M_{\rm S} (M_{\rm RS}^T)^{-1} & M_{\rm RS}^{-1} \\ (M_{\rm RS}^T)^{-1} & 0 \end{pmatrix} . \tag{4.19}$$

Since in the limit $M_{\rm S} \to 0$ the eigenvalues of $M_{\rm RS} \gg M_{\rm D}$ dominate the scale of $M_{\rm X}$, one can apply the seesaw formula to eq. (4.17) to get

$$M^{\rm d} = \begin{pmatrix} M_\nu & 0 \\ 0 & M_{\rm X} \end{pmatrix} , \tag{4.20}$$

where the active neutrino mass matrix is given by

$$\begin{aligned} M_\nu &= -\begin{pmatrix} M_{\rm D} & 0 \end{pmatrix} \begin{pmatrix} -M_{\rm RS}^{-1} M_{\rm S} (M_{\rm RS}^T)^{-1} & M_{\rm RS}^{-1} \\ (M_{\rm RS}^T)^{-1} & 0 \end{pmatrix} \begin{pmatrix} M_{\rm D}^T \\ 0 \end{pmatrix} \\ &= M_{\rm D} M_{\rm RS}^{-1} M_{\rm S} (M_{\rm RS}^T)^{-1} M_{\rm D}^T, \end{aligned} \tag{4.21}$$

[3]Here we assume that $M_{\rm X}$ is invertible. For the singular case see section 5.2.2.

which is the same expression for M_ν as in eq. (4.12). Putting in the suggestive values $M_\mathrm{S} \sim 0.1$ keV, $M_\mathrm{D} \sim 100$ GeV and $M_\mathrm{RS} \sim 1$ TeV brings the active neutrino masses to the scale of 1 eV. Interestingly, in this scenario the first order correction term to M_ν is of order $M_\mathrm{S} M_\mathrm{D}^4 / M_\mathrm{RS}^4 \sim 10^{-2}$ eV, which is about few percent of M_ν. These corrections become important for example, if leading order terms vanish [50].

4.4.5 Minimal radiative inverse seesaw

The minimal radiative inverse seesaw model (MRISM) [59, 60] can be regarded as a modification of the inverse seesaw, where in eq. (4.17) the Majorana mass term for the fermionic singlets is set equal to zero and instead a Majorana mass term for the sterile neutrinos is assumed. Let us consider a model with three neutrinos of each species. The 9×9 mass matrix, then, is given by

$$M = \begin{pmatrix} 0 & M_\mathrm{D} & 0 \\ M_\mathrm{D}^T & M_\mathrm{R} & M_\mathrm{RS}^T \\ 0 & M_\mathrm{RS} & 0 \end{pmatrix} . \tag{4.22}$$

Let us assume $M_\mathrm{R} \gg M_\mathrm{RS}$ and $M_\mathrm{R} \gg M_\mathrm{D}$ and realign M in the basis $(\widehat{\nu}_\mathrm{R},\ S,\ N_\mathrm{R})$. Now M has the form

$$M = \left(\begin{array}{cc|c} 0 & 0 & M_\mathrm{D} \\ 0 & 0 & M_\mathrm{RS} \\ \hline M_\mathrm{D}^T & M_\mathrm{RS}^T & M_\mathrm{R} \end{array}\right) , \tag{4.23}$$

from which we see that it has rank 6. Also, it is clear that M can be transformed by a seesaw-type transformation yielding

$$M' = \begin{pmatrix} M_\nu^{6\times 6} & 0 \\ 0 & M_\mathrm{R} \end{pmatrix} , \tag{4.24}$$

where the 6×6 matrix $M_\nu^{6\times 6}$ is given by

$$\begin{aligned} M_\nu^{6\times 6} &\equiv - \begin{pmatrix} M_\mathrm{D} \\ M_\mathrm{RS} \end{pmatrix} M_\mathrm{R}^{-1} \begin{pmatrix} M_\mathrm{D}^T & M_\mathrm{RS}^T \end{pmatrix} \\ &= - \begin{pmatrix} M_\mathrm{D} M_\mathrm{R}^{-1} M_\mathrm{D}^T & M_\mathrm{D} M_\mathrm{R}^{-1} M_\mathrm{RS}^T \\ M_\mathrm{RS} M_\mathrm{R}^{-1} M_\mathrm{D}^T & M_\mathrm{RS} M_\mathrm{R}^{-1} M_\mathrm{RS}^T \end{pmatrix} \equiv - \begin{pmatrix} M_\mathrm{a} & M_\mathrm{b} \\ M_\mathrm{b}^T & M_\mathrm{c} \end{pmatrix} . \end{aligned} \tag{4.25}$$

Under the assumption $M_\mathrm{R} \gg M_\mathrm{D}$ the matrix $M_\nu^{6\times 6}$ can be diagonalized by yet another seesaw-type transformation leading to

$$M_\nu'^{6\times 6} = - \begin{pmatrix} M_\mathrm{a} - M_\mathrm{b} M_\mathrm{c}^{-1} M_\mathrm{b}^T & 0 \\ 0 & M_\mathrm{c} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -M_\mathrm{RS} M_\mathrm{R}^{-1} M_\mathrm{RS}^T \end{pmatrix} , \tag{4.26}$$

where both terms in the first diagonal entry have canceled exactly. The matrix $M_\nu^{\prime 6\times 6}$ is of rank 3 as expected. Note that in this scenario the active neutrino masses vanish at tree level. Their masses only receive radiative corrections at loop level. To quantify the corrections we give the expression for the one-loop contribution to $M_\nu^{\prime 6\times 6}$ [60],

$$
\begin{aligned}
M_\nu^{1-\text{loop}} &\approx \begin{pmatrix} M_\text{D} \\ 0 \end{pmatrix} \frac{\alpha_\text{W}}{16\pi m_\text{W}^2} M_\text{R} \left[\frac{m_\text{H}^2}{M_\text{R}^2 - m_\text{H}^2 \mathbb{1}} \ln\left(\frac{M_\text{R}^2}{m_\text{H}^2} \right) \right. \\
&\quad \left. + \frac{3m_\text{Z}^2}{M_\text{R}^2 - m_\text{Z}^2 \mathbb{1}} \ln\left(\frac{M_\text{R}^2}{m_\text{Z}^2} \right) \right] \begin{pmatrix} M_\text{D}^T & 0 \end{pmatrix} \\
&= \begin{pmatrix} M_\text{D} M_\text{R}^{-1} x_\text{R} f(x_\text{R}) M_\text{D}^T & 0 \\ 0 & 0 \end{pmatrix} \equiv \begin{pmatrix} \Delta M & 0 \\ 0 & 0 \end{pmatrix},
\end{aligned} \tag{4.27}
$$

where $\alpha_\text{W} \equiv g^2/4\pi$ denotes the weak fine-structure constant, and m_W, m_Z and m_H denote respectively, the W, Z and Higgs boson masses. The one-loop function $f(x_\text{R})$ is defined by

$$
f(x_\text{R}) \equiv \frac{\alpha_\text{W}}{16\pi} \left[\frac{x_\text{H}}{x_\text{R} - x_\text{H}} \ln\left(\frac{x_\text{R}}{x_\text{H}} \right) + \frac{3x_\text{Z}}{x_\text{R} - x_\text{Z}} \ln\left(\frac{x_\text{R}}{x_\text{Z}} \right) \right], \tag{4.28}
$$

with $x_\text{R} \equiv m_\text{R}^2/m_\text{W}^2$, $x_\text{H} \equiv m_\text{H}^2/m_\text{W}^2$, $x_\text{Z} \equiv m_\text{Z}^2/m_\text{W}^2$ and for simplicity it was assumed that $M_\text{R} = m_\text{R}\mathbb{1}$. Then, at one-loop level we have the effective mass matrix

$$
\begin{aligned}
M_\text{eff}^{6\times 6} &\equiv M_\nu^{6\times 6} + M_\nu^{1-\text{loop}} \\
&= -\begin{pmatrix} M_\text{D} M_\text{R}^{-1} (\mathbb{1} - x_\text{R} f(x_\text{R})) M_\text{D}^T & M_\text{D} M_\text{R}^{-1} M_\text{RS}^T \\ M_\text{RS} M_\text{R}^{-1} M_\text{D}^T & M_\text{RS} M_\text{R}^{-1} M_\text{RS}^T \end{pmatrix} \\
&= -\begin{pmatrix} M_\text{D} M_\text{R}^{-1} M_\text{D}^T - \Delta M & M_\text{D} M_\text{R}^{-1} M_\text{RS}^T \\ M_\text{RS} M_\text{R}^{-1} M_\text{D}^T & M_\text{RS} M_\text{R}^{-1} M_\text{RS}^T \end{pmatrix}.
\end{aligned} \tag{4.29}
$$

In consequence of the one-loop corrections the active neutrino masses are non-vanishing after diagonalization.

4.4.6 Minimal extended seesaw

The minimal extended seesaw mechanism (MES) presented in [61] is based on the type I seesaw mechanism. As the name indicates it extends the model by a minimal set of particles. Assuming a scenario as in the MRISM, but introducing only one singlet fermion instead of three, we have the same mass matrix structure and rank as the mass matrix in eq. (4.22), only now M is a 7×7 matrix. Its diagonalization can be carried out as before leading to the 4×4 matrix in the

$(\nu_\mathrm{L},\ S)$-sector

$$M_\nu^{4\times4} = -\begin{pmatrix} M_\mathrm{D}M_\mathrm{R}^{-1}M_\mathrm{D}^T & M_\mathrm{D}M_\mathrm{R}^{-1}M_\mathrm{RS}^T \\ M_\mathrm{RS}M_\mathrm{R}^{-1}M_\mathrm{D}^T & M_\mathrm{RS}M_\mathrm{R}^{-1}M_\mathrm{RS}^T \end{pmatrix}, \tag{4.30}$$

after integrating out the heavy states N_R. The four neutrino masses corresponding to $M_\nu^{4\times4}$ are light, since all of them are suppressed by the scale of M_R. Note especially that at least one light neutrino is massless at tree level, since M does not have full rank [61]. By the way, this is in direct agreement with the seesaw fair play rule mentioned earlier. Under the assumption $M_\mathrm{D} < M_\mathrm{RS}$ the matrix $M_\nu^{4\times4}$ can further be diagonalized by a second seesaw-type transformation. This leads to

$$M_\nu^{\mathrm{d}4\times4} = \begin{pmatrix} M_\nu & 0 \\ 0 & m_\mathrm{S} \end{pmatrix}, \tag{4.31}$$

where the diagonal entries are given by

$$\begin{aligned} M_\nu \equiv & M_\mathrm{D}M_\mathrm{R}^{-1}M_\mathrm{RS}^T(M_\mathrm{RS}M_\mathrm{R}^{-1}M_\mathrm{RS}^T)^{-1}M_\mathrm{RS}M_\mathrm{R}^{-1}M_\mathrm{D}^T \\ & -M_\mathrm{D}M_\mathrm{R}^{-1}M_\mathrm{D}^T \in \mathscr{M}[3\times3], \end{aligned} \tag{4.32}$$

$$m_\mathrm{S} \equiv -M_\mathrm{RS}M_\mathrm{R}^{-1}M_\mathrm{RS}^T \in \mathscr{M}[1\times1]. \tag{4.33}$$

Note that the expressions in eq. (4.32) do not cancel (like they did in eq. (4.26)), since $M_\mathrm{RS} \in \mathscr{M}[1\times3]$ is a vector. Note as well that both terms in eq. (4.32) have the same order of magnitude. Inserting the naively chosen values $M_\mathrm{D} \sim 100$ GeV, $M_\mathrm{RS} \sim 500$ GeV and $M_\mathrm{R} \sim 2\times10^{14}$ GeV the matrices in eqs. (4.32) and (4.33) are of order $M_\nu \sim 0.05$ eV and $m_\mathrm{S} \sim 1.3$ eV, respectively. Thus the MES is an example for generating one sterile neutrino mass at the eV scale.

4.4.7 Schizophrenic neutrinos

Another interesting theory is the "schizophrenic neutrino" alternative [62]. In this theory the possibility is pointed out that at tree level some neutrino mass eigenstates could have a Dirac mass, while others have a Majorana type mass. In this case the mass matrix in the flavor basis would have a large admixture of both Dirac and Majorana mass terms. To generate active neutrino masses in the eV range the flavor eigenstates forming the Dirac mass neutrinos would need to have small Yukawa couplings of about $\mathscr{O}(10^{-12})$, while sufficiently small Majorana type masses could be obtained by high mass scale suppression as in the seesaw mechanism. Let us illustrate this consideration by an example. Imagine a scenario with three active neutrinos $\nu_{\alpha\mathrm{L}}$ and three sterile neutrinos $N_{\alpha\mathrm{R}}$, where the states with $\alpha = 1$ form a Dirac neutrino, while the other states have mass terms as in the type I seesaw. The neutrino mass matrix corresponding to this scenario written in the basis $(\widehat{\nu}_{1\mathrm{R}},\ \widehat{\nu}_{2\mathrm{R}},\ \widehat{\nu}_{3\mathrm{R}},\ N_{1\mathrm{R}},\ N_{2\mathrm{R}},\ N_{3\mathrm{R}})$ would be given by

$$M = \begin{pmatrix} 0 & 0 & 0 & m_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & m_{22} & m_{23} \\ 0 & 0 & 0 & 0 & m_{32} & m_{33} \\ m_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & m_{22} & m_{32} & 0 & \mu_2 & 0 \\ 0 & m_{23} & m_{33} & 0 & 0 & \mu_3 \end{pmatrix}, \tag{4.34}$$

where for simplicity we have assumed that the Majorana mass term for the sterile neutrinos is diagonal. The structure of the mass matrix in eq. (4.34) could for instance be the result of a flavor symmetry. Indeed, if in our example we identify $\nu_{1\mathrm{L}}$ with the electron-neutrino, electron lepton number is conserved.

This scheme can be generalized in an obvious way to a case with p Dirac and q Majorana masses. Note that we need $2p$ states to generate p Dirac masses as well as $2q$ states, of which q have large Majorana mass terms, to obtain q suppressed Majorana masses. Now, consider a scenario with respectively $n = p + q$ active and sterile neutrinos. We work in the basis

$$(\widehat{\nu}_{1\mathrm{R}}, \ldots,\ \widehat{\nu}_{p\mathrm{R}},\ \widehat{\nu}_{(p+1)\mathrm{R}}, \ldots,\ \widehat{\nu}_{(p+q)\mathrm{R}},\ N_{1\mathrm{R}}, \ldots,\ N_{p\mathrm{R}},\ N_{(p+1)\mathrm{R}}, \ldots,\ N_{(p+q)\mathrm{R}}),$$

where the states with $\alpha = 1, \ldots,\ p$, will form Dirac neutrinos and the remaining states bring forth the Majorana neutrinos. Then the mass matrix is written as

$$M = \begin{pmatrix} 0 & 0 & M_{\mathrm{D}p} & 0 \\ 0 & 0 & 0 & M_{\mathrm{D}q} \\ M_{\mathrm{D}p}^T & 0 & 0 & 0 \\ 0 & M_{\mathrm{D}q}^T & 0 & M_{Rq} \end{pmatrix} \in \mathscr{M}^2[((p+q)+(p+q))], \tag{4.35}$$

where $M_{\mathrm{D}p}$ is responsible for the Dirac masses, while $M_{\mathrm{D}q}$ and $M_{\mathrm{R}q}$ generate the Majorana masses. For later we record that, if a Dirac mass term can be put in such a "schizophrenic" form

$$M_{\mathrm{D}} = \begin{pmatrix} M_{\mathrm{D}p} & 0 \\ 0 & M_{\mathrm{D}q} \end{pmatrix} \tag{4.36}$$

with sufficiently small $M_{\mathrm{D}p}$, it can give rise to p neutrinos with Dirac masses and q Majorana type neutrinos with seesaw suppressed masses.

Chapter 5

Singular seesaw mechanism

Since till now nothing about the possible Majorana nature of neutrinos is known, the Majorana mass terms can in principle be arbitrary. This includes cases where they are singular. The two examples below in eq. (5.4) and (5.5) demonstrate how easily singular mass terms can be constructed e.g. under the assumption of a certain (flavor) symmetry. The possibility of having zero eigenvalues in the Majarona mass matrix so far has not been studied very much. The scenarios we are going to discuss in this chapter, however, will show that it is worthwhile to consider singular cases, since they can lead to different mass scales as well as to a different partition of the eigenstates among the mass scales compared to the non-singular seesaw.

5.1 Canonical singular seesaw

Consider a type I seesaw model as in section 4.1 with three active and three sterile neutrinos. In this context the mass matrix reads

$$M = \begin{pmatrix} 0 & M_\mathrm{D} \\ M_\mathrm{D}^T & M_\mathrm{R} \end{pmatrix} \in \mathscr{M}^2[(3+3)]. \tag{5.1}$$

We work in the basis, where M_R is diagonal (cf. chapter 4),

$$M_\mathrm{R} = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{5.2}$$

Now let us assume that M_R is singular, i.e. $\det(M_\mathrm{R}) = 0$, with $0 \leq n < 3$ non-zero eigenvalues. If $n = 0$, M_R is the zero-matrix and we have the simple Dirac case explained in section 3.2. In the case $n > 0$ we will parametrize the Majorana mass term as

$$M_{\mathrm{R}} = \begin{pmatrix} 0 & 0 \\ 0 & M_n \end{pmatrix} \in \mathscr{M}^2[((3-n)+n)], \tag{5.3}$$

where M_n contains the non-zero eigenvalues of M_{R} and thus is by construction non-singular.[1] An example for a symmetric matrix with one non-zero eigenvalue is

$$M_1' = m_{\mathrm{R}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \longrightarrow M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3m_{\mathrm{R}} \end{pmatrix}, \tag{5.4}$$

and for two non-zero eigenvalues

$$M_2' = m_{\mathrm{R}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \longrightarrow M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & m_{\mathrm{R}} & 0 \\ 0 & 0 & 2m_{\mathrm{R}} \end{pmatrix}. \tag{5.5}$$

With the parametrization as in eq. (5.3) the whole mass matrix takes on the form

$$M = \left(\begin{array}{cc|c} 0 & M_{\mathrm{D1}} & M_{\mathrm{D2}} \\ M_{\mathrm{D1}}^T & 0 & 0 \\ \hline M_{\mathrm{D2}}^T & 0 & M_n \end{array}\right) \in \mathscr{M}^2[(3+(3-n)+n)]. \tag{5.6}$$

Assuming $M_n \gg M_{\mathrm{D}}$, this matrix can be transformed with a seesaw-type transformation (cf. section B.2.1) with the result

$$M' = \left(\begin{array}{cc|c} M_{\mathrm{X}} & M_{\mathrm{D1}} & 0 \\ M_{\mathrm{D1}}^T & 0 & 0 \\ \hline 0 & 0 & M_n \end{array}\right), \tag{5.7}$$

where we have abbreviated $M_{\mathrm{X}} \equiv -M_{\mathrm{D2}} M_n^{-1} M_{\mathrm{D2}}^T \in \mathscr{M}[3\times 3]$. Note that according to the seesaw fair play rule M_{X} will in general have $3-n$ zero eigenvalues.

In the case that $\mathrm{Rk}(M_{\mathrm{D1}}) = 0 \Rightarrow M_{\mathrm{D1}} = 0$, the mass matrix in the form of eq. (5.7) already has diagonal form

$$M^{\mathrm{d}} = M' = \begin{pmatrix} M_{\mathrm{X}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & M_n \end{pmatrix} \in \mathscr{M}^2[(3+(n-3)+n)] \tag{5.8}$$

[1] Hence, M_n can be inverted.

with mass scale structure

$$M^{\mathrm{d}} \sim \begin{pmatrix} \frac{m_{\mathrm{D}}^2}{m_{\mathrm{R}}} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & m_{\mathrm{R}} \end{pmatrix}. \tag{5.9}$$

The case $M_{\mathrm{D1}} = 0$, however, corresponds to a scenario, where the sterile neutrinos with zero Majorana mass have no couplings, either. These neutrinos, then, would be massless particles without interactions, which is not really meaningful. We will not discuss such possibilities any further.

In the case $n = 1$ the matrix M_{X} has one non-vanishing eigenvalue and $M_{\mathrm{D1}} \in \mathscr{M}[3 \times 2]$. Then in eq. (5.7) the sub-matrix of M',

$$M_{\mathrm{a}} \equiv \begin{pmatrix} M_{\mathrm{X}} & M_{\mathrm{D1}} \\ M_{\mathrm{D1}}^T & 0 \end{pmatrix}, \tag{5.10}$$

can be written in the form

$$M_{\mathrm{a}} = \begin{pmatrix} a & 0 & 0 & b_1 & c_1 \\ 0 & 0 & 0 & b_2 & c_2 \\ 0 & 0 & 0 & b_3 & c_3 \\ b_1 & b_2 & b_3 & 0 & 0 \\ c_1 & c_2 & c_3 & 0 & 0 \end{pmatrix}. \tag{5.11}$$

Among the eigenvalues of this matrix there are in general four eigenvalues proportional to m_{D}, which is not consistent with the smallness of active neutrino masses.

In the case $n = 2$ the matrix M_{X} has two non-vanishing eigenvalues and $M_{\mathrm{D1}} \in \mathscr{M}[3 \times 1]$. In a similar way as in the case $n = 1$ we write the sub-matrix M_{a} as

$$M_{\mathrm{a}} = \begin{pmatrix} a_1 & 0 & 0 & b_1 \\ 0 & a_2 & 0 & b_2 \\ 0 & 0 & 0 & b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}. \tag{5.12}$$

To find its eigenvalues we compute the determinant

$$\begin{aligned} \det(\lambda \cdot \mathbb{1} - M_{\mathrm{a}}) = & \lambda^4 - (a_1 + a_2)\lambda^3 + (a_1 a_2 - b_1^2 - b_2^2 - b_3^2)\lambda^2 \\ & + \left[a_1(b_2^2 + b_3^2) + a_2(b_1^2 + b_3^2)\right] \lambda - a_1 a_2 b_3^2. \end{aligned} \tag{5.13}$$

This clearly gives a vanishing eigenvalue, if one puts $b_3 = 0$. On the other hand, if for example one sets $b_1 = b_2 = 0$, it is obvious that M_{a} has the eigenvalues $a_1,\ a_2,\ b_3, -b_3$, where a_1 and a_2 are proportional to $m_{\mathrm{D}}^2/m_{\mathrm{R}}$ and $b_3 \sim m_{\mathrm{D}}$. Hence, the eigenvalues of M_a strongly depend on the concrete form of M_X and M_D1. Neither of the cases ($b_3 = 0$ or $b_1 = b_2 = 0$), however, contain enough suppressed eigenvalues to be consistent with the light active neutrino masses.

Here we mention that the introduction of more than three sterile neutrinos does not overcome the problem of active neutrino masses being too large. Even with more sterile states the three active neutrino states would still receive a contribution from the Dirac mass terms.

Regarding the singular type I seesaw in the light of the schizophrenic neutrino alternative (cf. section 4.4.7) reveals a different opportunity to us. Imagine that in eq. (5.6) the Dirac mass term $M_\mathrm{D} = \begin{pmatrix} M_\mathrm{D1} & M_\mathrm{D2} \end{pmatrix} \in \mathscr{M}[3 \times ((3-n)+n)]$ is responsible for $p := (3-n)$ Dirac and $q := n$ Majorana neutrinos. Then we can write M_D in a "schizophrenic" form as

$$\left(M_\mathrm{D1} \mid M_\mathrm{D2}\right) \equiv \left(\begin{array}{c|c} M_p & 0 \\ 0 & M_q \end{array}\right) \in \mathscr{M}^2[((3-n)+n)]. \tag{5.14}$$

According to the schizophrenic neutrino alternative we assume M_p to be small. In the case $n = 1$ we would have two Dirac neutrinos with mass of order m_p and one Majorana type mass neutrino with mass proportional to m_q^2/m_R. The case of $n = 2$ corresponds to one Dirac mass neutrino and two neutrinos with seesaw suppressed mass. In both cases, of course, there will additionally be the n large scale Majorana type mass neutrinos that were initially assumed.

5.2 Extended singular seesaw

In the following we will examine, which structures are present in two extended singular variants, namely in the double seesaw and the inverse seesaw (cf. sections 4.4.1 and 4.4.4). We begin our discussion with the case of a vanishing Majorana mass term in the double seesaw and go on to the non-vanishing (but still singular) case. Afterward we will reconsider the inverse seesaw scenario from another point of view to demonstrate its pseudo-singular structure.

5.2.1 Singular double seesaw

Under the assumptions described in section 4.4.1 the total neutrino mass matrix has the form

$$M = \begin{pmatrix} 0 & M_\mathrm{D} & 0 \\ M_\mathrm{D}^T & 0 & M_\mathrm{RS}^T \\ 0 & M_\mathrm{RS} & M_\mathrm{S} \end{pmatrix} \in \mathscr{M}^2[(3 + n_\mathrm{s} + n_\mathrm{b})] \tag{5.15}$$

with $M_\text{S} \gg M_\text{RS}$ and $M_\text{RS} \gg M_\text{D}$. To make this scenario singular, we assume additionally $\det(M_\text{S} = 0)$, which means that M_S has at least one zero eigenvalue.

The case, where $M_\text{S} = 0$ and the neutrino mass matrix takes on the form

$$M = \begin{pmatrix} 0 & M_\text{D} & 0 \\ M_\text{D}^T & 0 & M_\text{RS}^T \\ 0 & M_\text{RS} & 0 \end{pmatrix}, \tag{5.16}$$

leads to Dirac neutrinos, just as the case with vanishing Majorana mass term in the type I seesaw. In addition, as can be seen from eq. (5.16), the mass matrix has vanishing eigenvalues. For instance in a (3+3+3) framework the diagonal form of M contains three vanishing eigenvalues and (in the limit $M_\text{RS} \gg M_\text{D}$) six eigenvalues of order m_RS. Since in scenarios with vanishing Majorana mass term no seesaw suppressed masses arise, we will not follow this possibility anymore. Instead, we consider the case where M_S possesses $n < n_\text{b}$ zero eigenvalues. Parametrizing the Majorana mass term as

$$M_\text{S} \equiv \begin{pmatrix} 0 & 0 \\ 0 & M_n \end{pmatrix} \in \mathscr{M}^2[((n_\text{b} - n) + n)] \tag{5.17}$$

the whole mass matrix reads

$$M = \left(\begin{array}{ccc|c} 0 & M_\text{D} & 0 & 0 \\ M_\text{D}^T & 0 & M_\text{RS1}^T & M_\text{RS2}^T \\ 0 & M_\text{RS1} & 0 & 0 \\ \hline 0 & M_\text{RS2} & 0 & M_n \end{array}\right) \in \mathscr{M}^2[(3 + n_\text{s} + (n_\text{b} - n) + n)], \tag{5.18}$$

where we have split up $M_\text{RS}^T \equiv \begin{pmatrix} M_\text{RS1}^T & M_\text{RS2}^T \end{pmatrix} \in \mathscr{M}[n_\text{s} \times ((n_\text{b} - n) + n)]$ in correspondence with M_S. A seesaw-type transformation of M according to the split-up indicated in eq. (5.18) leads to

$$M' = \left(\begin{array}{ccc|c} 0 & M_\text{D} & 0 & 0 \\ M_\text{D}^T & M_\text{Y} & M_\text{RS1}^T & 0 \\ 0 & M_\text{RS1} & 0 & 0 \\ \hline 0 & 0 & 0 & M_n \end{array}\right) \tag{5.19}$$

where we have abbreviated $M_\text{Y} \equiv -M_\text{RS2}^T M_n^{-1} M_\text{RS2}$. Note that M_Y has $n_\text{s} - n$ vanishing eigenvalues according to the seesaw fair play rule. The sub-matrix

$$M_\text{a} \equiv \begin{pmatrix} 0 & M_\text{D} & 0 \\ M_\text{D}^T & M_\text{Y} & M_\text{RS1}^T \\ 0 & M_\text{RS1} & 0 \end{pmatrix} \in \mathscr{M}^2[(3 + n_\text{s} + (n_\text{b} - n))] \tag{5.20}$$

has the same form as M_3 in section B.2.3 of the appendix and can be diagonalized as explained there. The eigenvalues of M_a, however, are hard to predict. From the form of M_a we can tell that, if $n_\mathrm{s} < 3 + n_\mathrm{b} - n$, this matrix will in general have $3 + n_\mathrm{b} - n - n_\mathrm{s}$ vanishing eigenvalues.[2] At which position they will appear in the block diagonal form we cannot predict. Keeping this in mind, we proceed applying the diagonalization techniques.

If M_RS1 is not quadratic or does not have full rank, the diagonal form of M_a is given by

$$M_\mathrm{a}^\mathrm{d} = \begin{pmatrix} K_{3\times3} & 0 & 0 \\ 0 & J_{k\times k} & 0 \\ 0 & 0 & D_{l\times l} \end{pmatrix} \in \mathscr{M}^2[(3+k+l)], \tag{5.21}$$

where $l = 2\mathrm{Rk}(M_\mathrm{RS1})$ and $k = n_\mathrm{s} + (n_\mathrm{b} - n) - l$. Its mass scale structure reads

$$M_\mathrm{a}^\mathrm{d} \sim \begin{pmatrix} \frac{m_\mathrm{D}^2}{m_\mathrm{RS}^2} m_\mathrm{S} & 0 & 0 \\ 0 & \frac{m_\mathrm{RS}^2}{m_\mathrm{S}} & 0 \\ 0 & 0 & m_\mathrm{RS} \end{pmatrix}, \tag{5.22}$$

where in the first diagonal entry we omitted a term of relative order $m_\mathrm{RS}/m_\mathrm{S}$. The full diagonal mass matrix, hence, is given by

$$M^\mathrm{d} = \begin{pmatrix} K_{3\times3} & 0 & 0 & 0 \\ 0 & J_{k\times k} & 0 & 0 \\ 0 & 0 & D_{l\times l} & 0 \\ 0 & 0 & 0 & M_n \end{pmatrix} \in \mathscr{M}^2[(3+k+l+n)], \tag{5.23}$$

with mass scale structure

$$M^\mathrm{d} \sim \begin{pmatrix} \frac{m_\mathrm{D}^2}{m_\mathrm{RS}^2} m_\mathrm{S} & 0 & 0 & 0 \\ 0 & \frac{m_\mathrm{RS}^2}{m_\mathrm{S}} & 0 & 0 \\ 0 & 0 & m_\mathrm{RS} & 0 \\ 0 & 0 & 0 & m_\mathrm{S} \end{pmatrix}. \tag{5.24}$$

As explained before, somewhere in $K_{3\times3}$, $J_{k\times k}$ and/or $D_{l\times l}$ must be $3+n_\mathrm{b}-n-n_\mathrm{s}$ zero eigenvalues. To interpret this we imagine a scenario with three sterile neutrinos and three singlets ($n_\mathrm{s} = n_\mathrm{b} = 3$) and with $n = 1$ non-vanishing eigenvalues in the Majorana mass term. Clearly the condition $n_\mathrm{s} < 3 + n_\mathrm{b} - n$ holds here. Thus the neutrino mass matrix will have 2 zero eigenvalues. This can have consequences for the practicability of this model, if for example there are vanishing

[2] Note that the condition $n_\mathrm{s} < 3 + n_\mathrm{b} - n$ holds in the scenarios we consider.

eigenvalues in $K_{3\times 3}$.

Now, putting the problem with the vanishing eigenvalues aside for the moment, we will discuss the diagonal form of $M^{\rm d}$ given in eq. (5.23) and (5.24). Comparing the structure of $M^{\rm d}$ with eq. (4.12) and (4.13) of the non-singular double seesaw we see that in the singular context a new mass scale appears in the diagonal form of the mass matrix, namely $m_{\rm RS}$. Additionally, the eigenvalue structure $(3+n_{\rm s}+n_{\rm b})$ of the non-singular case is replaced by the structure $(3+k+l+n)$.

Let us examine, which values we can choose for the three different mass scales. Putting $m_{\rm D}\sim 10^{x}$ GeV, $m_{\rm RS}\sim 10^{y}$ GeV and $m_{\rm S}\sim 10^{z}$ GeV with $x<y<z$ in accordance with our assumption $M_{\rm D}\ll M_{\rm RS}\ll M_{\rm S}$, the condition to generate active neutrino masses at about 1 eV reads

$$10^{2x-2y+z}\ \text{GeV}\approx 10^{-9}\ \text{GeV}. \tag{5.25}$$

First, note that inserting the typical double seesaw scales $m_{\rm ew}\sim 10^{2}$ GeV, $m_{\rm GUT}\sim 10^{16}$ GeV and $m_{\rm Planck}\sim 10^{19}$ GeV for $m_{\rm D}$, $m_{\rm RS}$ and $m_{\rm S}$, respectively, leads to eV active neutrinos. But apart from the additional scale at $m_{\rm R}\sim 10^{16}$ GeV, we have found nothing that is new compared to the non-singular double seesaw mechanism. More interesting cases arise, if one puts $m_{\rm D}$ at the eV scale, i.e. $x=-9$. With this choice of x the condition eq. (5.25) reads

$$-2y+z=9. \tag{5.26}$$

Under these circumstances the mass scale of $J_{k\times k}$, $m_{\rm RS}^{2}/m_{\rm S}=10^{2y-z}$ GeV, is fixed at the eV scale. Now if, for example, we want a scenario with keV sterile neutrinos, we could choose $-6\leq y\leq -4$. Then, according to eq. (5.26), we would have a mass scale structure

$$M\sim\begin{pmatrix}1\ \text{eV} & 0 & 0 & 0\\ 0 & 1\ \text{eV} & 0 & 0\\ 0 & 0 & 1-100\ \text{keV} & 0\\ 0 & 0 & 0 & 1\ \text{MeV}-10\ \text{GeV}\end{pmatrix}, \tag{5.27}$$

with eV active neutrinos and eV, keV and MeV to GeV sterile neutrinos, respectively. There are other valid values for y and z. If we, however, take the Planck scale as a limit on $m_{\rm S}$ ($z=19$), the maximal value for $m_{\rm RS}$ is at about 10^{5} GeV ($y=5$).

To study the eigenvalue structure of eq. (5.23) we choose the common scenario with three sterile neutrinos and three singlets ($n_{\rm s}=n_{\rm b}=3$). Then the number of non-zero eigenvalues of $M_{\rm S}$, n, can take on the values 1 or 2. In both cases the matrix $M_{\rm RS1}\in\mathscr{M}[3\times(3-n)]$ determining $l=2{\rm Rk}(M_{\rm RS1})$ and $k=6-n-l$ is not quadratic and thus may have full rank. In Table 5.1 we have listed the scenarios, which can emerge in this constellation. First note that all scenarios

$n=1$			$n=2$		
k	l	structure	k	l	structure
1	4	$(3+1+4+1)$	2	2	$(3+2+2+2)$
3	2	$(3+3+2+1)$			

Table 5.1: Possible eigenvalue structure of the neutrino mass matrix in the singular double seesaw with non-vanishing Majorana mass term. The notation $(3+k+l+n)$ indicates that the corresponding scenario contains 3 masses of order $(m_{\mathrm{D}}/m_{\mathrm{RS}})^2 m_{\mathrm{S}}$, k masses of order $m_{\mathrm{RS}}^2/m_{\mathrm{S}}$, l masses of order m_{RS} and n masses of order m_{S}.

feature three active neutrinos, since the initial number of eigenvalues in the active neutrino sector is not affected by the diagonalization procedure. If we assume a concrete mass scale structure as in eq. (5.27), the scenarios $(3+1+4+1)$ and $(3+2+2+2)$ could represent the hidden eigenvalue structure of a low-energy effective $3+1$ and $3+2$ model with eV sterile neutrinos, respectively. Moreover all scenarios in Table 5.1 provide us with keV sterile neutrinos. In the light of possibly vanishing eigenvalues (cf. the comments after eq. (5.24)), however, the prediction of the mass scales must be taken with a grain of salt.

5.2.2 Inverse seesaw revisited

On closer inspection the inverse seesaw presented in section 4.4.4 emerges as kind of pseudo singular seesaw mechanism, in the sense that the mass matrix in eq. (4.11), with $M_{\mathrm{RS}} \gg M_{\mathrm{D}} \gg M_{\mathrm{S}}$, under certain assumptions realizes a singular structure. This becomes obvious, when realigning the mass matrix in the basis $(\widehat{\nu}_{\mathrm{R}},\ S,\ N_{\mathrm{R}})$ with the form

$$M = \left(\begin{array}{c|cc} 0 & 0 & M_{\mathrm{D}} \\ 0 & M_{\mathrm{S}} & M_{\mathrm{RS}} \\ M_{\mathrm{D}}^T & M_{\mathrm{RS}}^T & 0 \end{array}\right) \equiv \begin{pmatrix} 0 & M_{\mathrm{D}}' \\ M_{\mathrm{D}}'^T & M_{\mathrm{X}} \end{pmatrix} \in \mathscr{M}^2[(3+(n_{\mathrm{b}}+n_{\mathrm{s}}))]. \qquad (5.28)$$

This mass matrix is a variant of the matrix in eq. (B.34) in section B.2.3, so we can follow the steps there to diagonalize it. We see that the sub-matrix M_{X} defined in eq. (4.18) in the new basis reads

$$M_{\mathrm{X}} = \begin{pmatrix} M_{\mathrm{S}} & M_{\mathrm{RS}} \\ M_{\mathrm{RS}}^T & 0 \end{pmatrix} \in \mathscr{M}^2[(n_{\mathrm{b}}+n_{\mathrm{s}})]. \qquad (5.29)$$

Now the diagonalization performed in section 4.4.4 was based on the assumption that M_{X} has eigenvalues proportional to the scale of M_{RS}. But this is only true, if

$M_{\rm RS}$ is assumed to be quadratic and to have full rank. If not, however, $M_{\rm X}$ will in general have a different eigenvalue structure. So let us examine the diagonalization of the neutrino mass matrix M given in eq. (5.28) under the assumption that $M_{\rm RS}$ is not quadratic or does not have full rank.

Obviously, the matrix $M_{\rm X}$ written as the sum

$$M_{\rm X} = \begin{pmatrix} 0 & M_{\rm RS} \\ M_{\rm RS}^T & 0 \end{pmatrix} + \begin{pmatrix} M_{\rm S} & 0 \\ 0 & 0 \end{pmatrix} \equiv D' + A' \tag{5.30}$$

is a second-type matrix and, hence, can be quasi-diagonalized by a second-type transformation matrix S_2. After the transformation the matrix $M_{\rm X}$ has the form

$$M'_{\rm X} \equiv S_2^T M_{\rm X} S_2 = \begin{pmatrix} S_{k\times k} & S_{k\times l} \\ S_{k\times l}^T & D_{l\times l} \end{pmatrix}, \tag{5.31}$$

where the definitions of the block-matrices can be found in section B.2.2 of the appendix and $S \sim M_{\rm S}$ and $D \sim M_{\rm RS}$. Remember that $l = 2\mathrm{Rk}(M_{\rm RS})$ and $k = n_{\rm b} + n_{\rm s} - l$. Defining $M''_{\rm D} \equiv M'_{\rm D} S_2 = \begin{pmatrix} M''_{\rm D1} & M''_{\rm D2} \end{pmatrix} \in \mathscr{M}[3 \times (k+l)]$ the whole neutrino mass matrix now reads

$$\begin{aligned} M' &\equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S_2^T \end{pmatrix} \begin{pmatrix} 0 & M'_{\rm D} \\ M'^T_{\rm D} & M_{\rm X} \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S_2 \end{pmatrix} \\ &= \left(\begin{array}{cc|c} 0 & M''_{\rm D1} & M''_{\rm D2} \\ M''^T_{\rm D1} & S_{k\times k} & S_{k\times l} \\ \hline M''^T_{\rm D2} & S^T_{k\times l} & D_{l\times l} \end{array}\right). \end{aligned} \tag{5.32}$$

A subsequent seesaw-type transformation in accordance with the indicated split-up leads to

$$M'' = \begin{pmatrix} M_{\rm a} & 0 \\ 0 & D_{l\times l} \end{pmatrix} \tag{5.33}$$

with

$$\begin{aligned} M_{\rm a} &= \begin{pmatrix} 0 & M''_{\rm D1} \\ M''^T_{\rm D1} & S_{k\times k} \end{pmatrix} - \begin{pmatrix} M''_{\rm D2} D^{-1}_{l\times l} M''^T_{\rm D2} & M''_{\rm D2} D^{-1}_{l\times l} S^T_{k\times k} \\ S_{k\times l} D^{-1}_{l\times l} M''^T_{\rm D2} & S_{k\times l} D^{-1}_{l\times l} S^T_{k\times l} \end{pmatrix} \\ &\approx \begin{pmatrix} -M''_{\rm D2} D^{-1}_{l\times l} M''^T_{\rm D2} & M''_{\rm D1} \\ M''^T_{\rm D1} & S_{k\times k} \end{pmatrix}, \end{aligned} \tag{5.34}$$

where in the last step we estimated that the non-zero blocks of the first term in the upper line of eq. (5.34) are much larger, than the blocks of the second term. Since $D_{l\times l} \sim M_{\rm RS}$, $M''_{\rm D} \sim M_{\rm D}$, $S_{k\times k} \sim M_{\rm S}$ and $M_{\rm RS} \gg M_{\rm D} \gg M_{\rm S}$, the matrix $M_{\rm a}$ is a second-type matrix, whose eigenvalue structure depends on the rank of $M''_{\rm D1}$.

Under the assumption that M''_{D1} is not quadratic or does not have full rank the diagonalized form of M_{a} has the structure

$$M_{\mathrm{a}}^{\mathrm{d}} = \begin{pmatrix} K_{p\times p} & 0 \\ 0 & J_{q\times q} \end{pmatrix}, \tag{5.35}$$

where $q = 2\mathrm{Rk}(M''_{\mathrm{D1}})$ and $p = 3 + k - q$.[3] The matrix $J_{q\times q}$ is proportional to m_{D}, while $K_{p\times p}$ is proportional to $(m_{\mathrm{D}}^2/m_{\mathrm{RS}} + m_{\mathrm{S}})^2/m_{\mathrm{D}}$. Finally, putting everything together, the diagonalized form of M reads

$$M^{\mathrm{d}} = \begin{pmatrix} K_{p\times p} & 0 & 0 \\ 0 & J_{q\times q} & 0 \\ 0 & 0 & D_{l\times l} \end{pmatrix} \in \mathscr{M}^2[(p+q+l)] \tag{5.36}$$

and its mass scale structure is given by

$$M^{\mathrm{d}} \sim \begin{pmatrix} \frac{1}{m_{\mathrm{D}}} \cdot [\max(\frac{m_{\mathrm{D}}^2}{m_{\mathrm{RS}}},\ m_{\mathrm{S}})]^2 & 0 & 0 \\ 0 & m_{\mathrm{D}} & 0 \\ 0 & 0 & m_{\mathrm{RS}} \end{pmatrix}, \tag{5.37}$$

where the two possibilities in the first diagonal entry come from the different scales of $K_{p\times p}$.

To discuss the mass scale structure of M^{d} let us play with some numbers. We put $m_{\mathrm{S}} \sim 10^x$ GeV, $m_{\mathrm{D}} \sim 10^y$ GeV and $m_{\mathrm{RS}} \sim 10^z$ GeV and insert these values into eq. (5.37). Thus we are led to the condition

$$\max(10^{3y-2z},\ 10^{2x-y})\ \mathrm{GeV} \lesssim 10^{-9}\ \mathrm{GeV} = 1\ \mathrm{eV} \tag{5.38}$$

to get active neutrino masses of the correct order, where we have to demand $x < y < z$ to keep up the initial assumption $M_{\mathrm{S}} \ll M_{\mathrm{D}} \ll M_{\mathrm{RS}}$. The naive choice $y = 2$ to have the Dirac mass term M_{D} at the electroweak scale of 100 GeV leads to $x \leq -4$, i.e. $m_{\mathrm{S}} \lesssim 100$ keV and $z \geq 8$ so that $M_{\mathrm{RS}} \gtrsim 10^8$ GeV.

Another consideration, however, is to put $y \approx -4$ to bring the sterile neutrino states belonging to $J_{q\times q}$ down to the keV scale. In order to generate active neutrino masses of about 1 eV we need to choose $x \leq -7$ and $z \geq -1$, corresponding to $m_{\mathrm{S}} \lesssim 100$ eV and $m_{\mathrm{RS}} \gtrsim 0,1$ GeV.

If we wish to have eV sterile neutrinos, we can put $y = -9$. Then from the constraints on the scale of m_{S} or m_{RS} to be smaller or, respectively, larger than m_{D}, one can derive that the active neutrino masses would be at 10^{-2} eV or smaller.

[3] The assumption that M''_{D1} is quadratic *and* has full rank leads to a poorer and less interesting mass scale structure and is not followed here.

$(n_b,\ n_s)$	q	l	scenario
$(1,\ 3)$	2	2	$(3+2+2)$
$(3,\ 3)$	2	4	$(3+2+4)$
	4	2	$(3+4+2)$

Table 5.2: Possible eigenvalue structure of the neutrino mass matrix in the inverse seesaw under the assumption that M_{RS} is not quadratic or does not have full rank. The notation $(3+q+l)$ indicates that the corresponding scenario contains 3 masses of order $\max(m_{\mathrm{D}}^2/m_{\mathrm{RS}},\ m_{\mathrm{S}})/m_{\mathrm{D}}$, q masses of order m_{D} and l masses of order m_{RS}.

Finally, we comment on the number of neutrino states present in the different scales. Remember that we assumed to have three active neutrinos, n_{b} singlet neutrinos and n_{s} sterile neutrinos. The eigenvalue structure of M depends on the rank of $M_{\mathrm{RS}} \in \mathscr{M}[n_{\mathrm{b}} \times n_{\mathrm{s}}]$, determining the numbers

$$l = 2\mathrm{Rk}(M_{\mathrm{RS}}) \leq 2\min(n_{\mathrm{b}},\ n_{\mathrm{s}}), \tag{5.39}$$

$$k = n_{\mathrm{b}} + n_{\mathrm{s}} - l, \tag{5.40}$$

and on the rank of $M''_{\mathrm{D1}} \in \mathscr{M}[3 \times k]$, responsible for the numbers

$$q = 2\mathrm{Rk}(M''_{\mathrm{D1}}) \leq 2\min(3,\ k), \tag{5.41}$$

$$p = 3 + k - q. \tag{5.42}$$

Note that the formulae for l and k are not affected by the interchange $(n_{\mathrm{b}} \longleftrightarrow n_{\mathrm{s}})$ so that the eigenvalue structure of M^{d} only depends on the unsorted pair $(n_{\mathrm{b}},\ n_{\mathrm{s}}) = (n_{\mathrm{s}},\ n_{\mathrm{b}})$. To get three active neutrinos, we hold the value $p = 3$ fixed. With this condition it follows directly from eq. (5.42) that $q = k$. Since q and l are even numbers, we see that we can only get three active neutrinos, if n_{b} and n_{s} are both either even or odd numbers. In Table 5.2 we listed the possible scenarios for the common choice $(n_{\mathrm{b}},\ n_{\mathrm{s}}) = (3,\ 3)$ as well as $(n_{\mathrm{b}},\ n_{\mathrm{s}}) = (1,\ 3)$ in the style of the MES. As in the singular double seesaw mechanism we have found scenarios that can provide eV or keV sterile neutrinos.

Chapter 6

Conclusion

In this thesis we analyzed the consequences of a singular Majorana mass term in different seesaw scenarios. We considered the type I seesaw mechanism in a scenario with three active and three sterile neutrinos ($(3+3)$ framework) and two versions of the extended seesaw mechanism in a more general scneario with three active neutrinos, n_s sterile neutrinos and n_b fermionic singlets ($(3+n_\mathrm{s}+n_\mathrm{b})$ framework). We showed that in every scenario under consideration there are cases with three active neutrinos at the compulsory eV scale.

We considered a type I seesaw scenario in the standard $(3+3)$ framework. In this scenario we examined, which eigenvalues of the neutrino mass matrix are obtained in the cases of a vanishing and a non-vanishing singular Majorana mass term. The former represents the simple Dirac case and in this sense is not singular. In the latter we found that the eigenvalues strongly depend on the concrete form of the involved sub-matrices of the total neutrino mass matrix. We showed that the cases of one and two non-vanishing eigenvalues of the Majorana mass term in general do not lead to realistic active neutrino masses. Additionally we mentioned that the introduction of more than three sterile neutrinos does not resolve this problem in the singular type I seesaw mechanism.

We gave, however, an example in the context of the schizophrenic neutrino alternative, which leads to the generation of sufficiently small active neutrino masses. There, under the assumption of a concrete “schizophrenic” form of the Dirac mass term, we obtained two(one) Dirac neutrino(s) and one(two) Majorana neutrino(s) within the correct mass range in the active neutrino sector in the case of one(two) non-vanishing eigenvalue(s) of the Majorana mass term.

By means of the inverse seesaw we demonstrated that the diagonalization techniques developed in the course of this thesis can be applied to non-singular scenarios, too, if they realize a certain “pseudo-singular” structure.

Also we studied a double seesaw scenario with a Dirac mass term of order m_{D} between active and sterile neutrinos and another Dirac mass term of order m_{RS} between sterile neutrinos and fermionic singlets. Additionally we assumed a Majorana mass term for the fermionic singlets, only, of order m_{S}. In this scenario we analyzed the consequences of a singular Majorana mass term especially in a $(3+3+3)$ framework.

We pointed out that in the case of a vanishing Majorana mass term no seesaw suppressed masses are generated, but instead Dirac masses are formed. Since this simply represents a more complicated way to generate Dirac masses than with only three additional sterile neutrinos and no Majorana mass term, we did not follow this possibility any further.

In the case of a non-vanishing singular Majorana mass term we carried out the diagonalization of the neutrino mass matrix under general considerations. The result was a diagonal structure with $(3+k+l+n)$ eigenvalues corresponding to four different mass scales of order $m_{\mathrm{S}}(m_{\mathrm{D}}^2/m_{\mathrm{RS}}^2)$, $m_{\mathrm{RS}}^2/m_{\mathrm{S}}$, m_{RS} and m_{S}, respectively. Compared to the non-singular double seesaw the structure of the resulting neutrino mass matrix in the singular case contains an additional mass scale, namely the one proportional to m_{RS}. Note that the partition of the number of eigenvalues to the different scales was influenced. This, however, applies not to the active neutrino sector so that in any case three active neutrinos are obtained.

In a $(3+3+3)$ framework with one or two non-vanishng eigenvalues in the Majorana mass term we evaluated the possible numbers of eigenvalues $(3+k+l+n)$ corresponding to the four different mass scales. In the case of one non-vanishing eigenvalue we found a $(3+1+4+1)$ as well as a $(3+3+2+1)$ scenario as possible partitions of eigenvalues to the scales $m_{\mathrm{S}}(m_{\mathrm{D}}^2/m_{\mathrm{RS}}^2)$, $m_{\mathrm{RS}}^2/m_{\mathrm{S}}$, m_{RS} and m_{S} respectively. In the case of two non-vanishing eigenvalues we found a $(3+2+2+2)$ scenario corresponding to the same scales as just described.

When analyzing the mass scale structure of the neutrino mass matrix we first noted that choosing the scales to be $m_{\mathrm{D}} \sim 10^2$ GeV, $m_{\mathrm{RS}} \sim 10^{16}$ GeV and $m_{\mathrm{S}} \sim 10^{19}$ GeV, as commonly used in the double seesaw, also in the singular context is consistent with the limits on active neutrino masses. A different choice of $m_{\mathrm{D}} \sim 1$ eV leads to the condition that the orders of magnitude of m_{RS} and m_{S} must be related by $m_{\mathrm{S}} \sim 10^9 \times (m_{\mathrm{RS}}/\mathrm{GeV})^2$ GeV to generate active neutrino masses at $m_{\mathrm{S}}(m_{\mathrm{D}}^2/m_{\mathrm{RS}}^2) \sim 1$ eV. By this condition the three remaining mass scales in the neutrino mass matrix were set to $m_{\mathrm{RS}}^2/m_{\mathrm{S}} \sim 1$ eV, $m_{\mathrm{RS}} \sim 1-100$ keV and $m_{\mathrm{S}} \sim 0.001-10$ GeV, respectively. With this choice for the scales the $(3+1+4+1)$ and the $(3+2+2+2)$ scenarios mentioned above contain one and two sterile neutrinos, respectively, at the eV scale. Hence, these scenarios could represent the hidden eigenvalue structures of an effective low energy $3+1$ and $3+2$ model with eV sterile neutrinos. Moreover all scenarios that we found in the singular double seesaw feature keV sterile neutrinos that could possibly be DM particles. Also, all of the three mass scales $m_{\mathrm{RS}}^2/m_{\mathrm{S}}$, m_{RS} and m_{S} are within the sensitivity reach of near future collider experiments.

Having said this we must emphasize two important shortcomings of our prediction of the eigenvalue and the mass scale structure of the neutrino mass matrix in the singular double seesaw. First, in our derivation of the diagonal structure of the neutrino mass matrix we did not take into account that there could be vanishing eigenvalues in the involved sub-matrices in the process of diagonalizing it (we only mentioned that they would appear). If such vanishing eigenvalues are present they could affect the outcome of the diagonalization, leading to an eigenvalue and mass scale structure different from our predictions. A thorough study of these effects, however, would require an exact knowledge of the involved sub-matrices, which are highly model dependent. We refrain from commenting any further on this.

The second shortcoming is that the (total) neutrino mixing matrix directly linked to the diagonalization of the neutrino mass matrix has not been studied in the course of this thesis. To obtain a neutrino mixing matrix in agreement with data, however, for a theory of neutrinos is as important as to generate active neutrino masses in the correct mass range. The diagonalization presented here may lead to a mixing of active and sterile neutrino states too large to be consistent with known constraints of the mixing parameters. In any realistic model one has to analyze the neutrino mixing that accompanies the diagonalization of the neutrino mass matrix corresponding to the individual scenarios.

In future studies on the singular seesaw mechanism one could consider a concrete form of the Dirac mass terms to improve the estimation of the diagonal form of the neutrino mass matrix. Under these circumstances it would be less complicated to make predictions on the neutrino mixing matrix, too. Like this the mixing between active and sterile neutrinos could be quantified to assess the quality of the considered singular seesaw model.

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Appendix A

Spinor fields

In this thesis the same notations and conventions as in [34] are used, unless noted otherwise. In this part of the appendix important properties of spinor fields, especially in correlation with Majorana particles on the basis of [34, 63] are presented. Some useful formulae are also taken from [40]. At first general properties of 4-component spinors are gathered in section A.1. Afterwards differences on one side and correlations on the other between the physical nature of Dirac and Majorana spinors are elaborated in section A.2.

A.1 Properties of spinor fields

Any 4-component spinor field ψ can be decomposed as

$$\psi = (P_{\mathrm{L}} + P_{\mathrm{R}})\psi = \psi_{\mathrm{L}} + \psi_{\mathrm{R}}, \tag{A.1}$$

with the left- and right-handed projection $\psi_{\mathrm{L}} \equiv P_{\mathrm{L}}\psi$ and $\psi_{\mathrm{R}} \equiv P_{\mathrm{R}}\psi$, respectively.[1] In the chiral basis, the projections can be written as

$$\psi_{\mathrm{L}} \equiv \begin{pmatrix} 0 \\ \chi_{\mathrm{L}} \end{pmatrix} \tag{A.2a}$$

$$\psi_{\mathrm{R}} \equiv \begin{pmatrix} \chi_{\mathrm{R}} \\ 0 \end{pmatrix}, \tag{A.2b}$$

where χ_{L} and χ_{R} denote two-component spinors. The spinor field belonging to the adjoint representation of ψ is defined by

$$\overline{\psi} \equiv \psi^{\dagger}\gamma^{0}. \tag{A.3}$$

[1] Often the projections ψ_{L} and ψ_{R} are called chiral projections or chiral fields as well.

The charge conjugate of a spinor field is defined by[2] [63]

$$\widehat{\psi}(x) \equiv \gamma^0 C \psi^*(x) = -C\overline{\psi}^T(x). \tag{A.4}$$

Note that by this definition it follows that

$$\widehat{\widehat{\psi}} = \psi, \tag{A.5}$$

as it should be. Using the properties of the chiral projection operators and the charge conjugation operator and their commutation relations it is easy to verify that the charge conjugate of a left-handed field is right-handed and vice versa,

$$\widehat{(\psi_\mathrm{L})} = \gamma^0 C \psi_\mathrm{L}^* = P_\mathrm{R}\widehat{\psi} \equiv \widehat{\psi}_\mathrm{R}, \tag{A.6a}$$

$$\widehat{(\psi_\mathrm{R})} = \gamma^0 C \psi_\mathrm{R}^* = P_\mathrm{L}\widehat{\psi} \equiv \widehat{\psi}_\mathrm{L}. \tag{A.6b}$$

Here, it is important to point out that in this notation charge conjugation comes *before* chiral projection, i.e. that for example the notation $\widehat{\psi}_\mathrm{R}$ advises us to *first* charge conjugate the field ψ and only *afterward* act on it with the chiral projection operator (in this case P_R).

Eqs. (A.6) and the anti-commutation property of spinor fields, can be used to prove that

$$\overline{\psi}_\mathrm{L}\chi_\mathrm{R} = \overline{\widehat{\chi}_\mathrm{L}}\widehat{\psi}_\mathrm{R} \ , \tag{A.7}$$

for two spinor fields ψ and χ. Note that eq. (A.7) especially applies if $\chi = \psi$. We keep this in mind for later.

A field is called *Majorana field* if it obeys the Majorana condition

$$\psi(x) = \xi\widehat{\psi}, \tag{A.8}$$

where ξ denotes a phase factor with $|\xi|^2 = 1$ that we commonly set equal to unity, if not otherwise noted. A field whose chiral components are related by

$$\psi_\mathrm{R} = \widehat{\psi}_\mathrm{R} = \gamma^0 C \psi_\mathrm{L}^* \tag{A.9}$$

is a Majorana field by construction, since

$$\psi = \psi_\mathrm{L} + \widehat{\psi}_\mathrm{R} = \widehat{(\widehat{\psi}_\mathrm{R})} + \widehat{(\psi_\mathrm{L})} = \widehat{\psi}, \tag{A.10}$$

clearly satisfies the Majorana condition eq. (A.10). The important distinction

[2]Here we stick to the notation of [63] instead of ψ^C for the charge conjugate field.

between Dirac and Majorana particles is that for a *Majorana* field the left- and right-handed projection are dependent on each other, i.e. they are related descriptions of the *same* field. On the other hand, the chiral projections of a *Dirac* field can be regarded as describing two *different* particles that together form the Dirac field.

A.2 Dirac fields and Majorana fields

The Lagrangian of a 4-component Dirac spinor field $\psi(x)$ is given by[3]

$$\mathscr{L}_{\mathrm{Dirac}} = \overline{\psi}(x)(i\gamma^\mu\partial_\mu - m)\psi(x). \tag{A.11}$$

With the decomposition eq. (A.1), the Dirac Lagrangian eq. (A.11) reads

$$\mathscr{L}_{\mathrm{Dirac}} = i\overline{\psi}_{\mathrm{L}}\gamma^\mu\partial_\mu\psi_{\mathrm{L}} + i\overline{\psi}_{\mathrm{R}}\gamma^\mu\partial_\mu\psi_{\mathrm{R}} - m(\overline{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + \overline{\psi}_{\mathrm{R}}\psi_{\mathrm{L}}), \tag{A.12}$$

where we have used the properties of the chiral projection operators to eliminate vanishing terms like $i\overline{\psi}_{\mathrm{L}}\gamma^\mu\partial_\mu\psi_{\mathrm{R}}$ and $m\overline{\psi}_{\mathrm{L}}\psi_{\mathrm{L}}$. The "coupling" between the left- and right-handed component, $m\overline{\psi}_{\mathrm{L}}\psi_{\mathrm{R}} + h.c.$, represents a Dirac mass term.[4]

Using the Euler-Lagrange-formalism with respect to $\overline{\psi}_{\mathrm{L}}$ and $\overline{\psi}_{\mathrm{R}}$, we find the equations of motion

$$i\gamma^\mu\partial_\mu\psi_{\mathrm{L}} = m\psi_{\mathrm{R}}, \tag{A.13a}$$
$$i\gamma^\mu\partial_\mu\psi_{\mathrm{R}} = m\psi_{\mathrm{L}}, \tag{A.13b}$$

respectively. The chiral components ψ_{L} and ψ_{R} are in general independent of each other. The question, if it is possible to satisfy eqs. (A.13) using only one chiral field, historically led to the hypothesis of the existence of Majorana fields. In [34] it is shown that, if one chooses

$$\psi_{\mathrm{R}} = \gamma^0 C\psi_{\mathrm{L}}^*, \tag{A.14}$$

eqs. (A.13) are satisfied.[5] Note that eq. (A.14) is the same as eq. (A.9). But, according to our earlier argument eq. (A.10), this makes ψ a Majorana field. Hence, to satisfy the equations of motion for the chiral components of a spinor field, one can either take a conventional Dirac spinor with independent chiral components, or a Majorana spinor, where the chiral components depend on each other.

[3] In eq. (A.11) we deviate from the form of $\mathscr{L}_{\mathrm{Dirac}}$ given in [34] and instead use the simplified form found in many books as [40]. The Dirac Lagrangian written as in eq. (A.11) still gives the correct equations of motion for the spinor field [34].

[4] By "*h.c.*" we denote the Hermitian conjugate of all prior expressions.

[5] We can do this in principle, since $\gamma^0 C\psi_{\mathrm{L}}^* = P_{\mathrm{R}}(\gamma^0 C\psi^*)$ is right-handed.

This result naturally raises the question, if a Dirac spinor is needed to describe a massive particle [34]. In the following we will show similarly to [63], how a Dirac spinor can be described using two Majorana spinors. To do so, we take a spinor $\psi_{1\mathrm{L}}$. Together with its charge conjugate $\widehat{\psi}_{1\mathrm{R}}$ it can form a Majorana spinor $\psi_1 = \psi_{1\mathrm{L}} + \widehat{\psi}_{1\mathrm{R}}$. Take another Majorana spinor ψ_2 built of $\psi_{2\mathrm{L}}$ and $\widehat{\psi}_{2\mathrm{R}}$. If we forbid couplings between $\psi_{1\mathrm{L}}$ and $\widehat{\psi}_{1\mathrm{R}}$ as well as couplings between $\psi_{2\mathrm{L}}$ and $\widehat{\psi}_{2\mathrm{R}}$, but on the other hand assume couplings like $m\overline{\psi}_{1\mathrm{L}}\widehat{\psi}_{2\mathrm{R}}$ and $m'\overline{\psi}_{2\mathrm{L}}\widehat{\psi}_{1\mathrm{R}}$, we get mass terms[6]

$$\begin{aligned} -\mathscr{L}_{\text{mass}} &= \frac{m}{2}\overline{\psi}_{1\mathrm{L}}\widehat{\psi}_{2\mathrm{R}} + \frac{m'}{2}\overline{\psi}_{2\mathrm{L}}\widehat{\psi}_{1\mathrm{R}} + h.c. \\ &= \frac{1}{2}\begin{pmatrix}\overline{\psi}_{1\mathrm{L}} & \overline{\psi}_{2\mathrm{L}}\end{pmatrix}\begin{pmatrix}0 & m \\ m' & 0\end{pmatrix}\begin{pmatrix}\widehat{\psi}_{1\mathrm{R}} \\ \widehat{\psi}_{2\mathrm{R}}\end{pmatrix} + h.c.\ . \end{aligned} \tag{A.15}$$

A general mass term constructed from a set of chiral fields $\psi_{\alpha\mathrm{L}}$ and $\widehat{\psi}_{\alpha\mathrm{R}}$ can be written [63] as

$$-\mathscr{L}_{\text{mass}} = \frac{1}{2}\overline{\psi}_{\alpha\mathrm{L}} M_{\alpha\beta}\widehat{\psi}_{\beta\mathrm{R}} + h.c.\ , \tag{A.16}$$

where M denotes the mass matrix in the Majorana basis. In [63] it is shown that M is symmetric. We will call the diagonal elements $M_{\alpha\alpha}$ of the mass matrix *Majorana* mass terms and refer to the off-diagonal elements $M_{\alpha\beta} = M_{\beta\alpha}$ as *Dirac* mass terms.[7] From the symmetry of M it follows that $m = m'$ and the mass term eq. (A.15) becomes

$$-\mathscr{L}_{\text{mass}} = \frac{m}{2}(\overline{\psi}_{1\mathrm{L}}\widehat{\psi}_{2\mathrm{R}} + \overline{\psi}_{2\mathrm{L}}\widehat{\psi}_{1\mathrm{R}}) + h.c.\ . \tag{A.17}$$

Now, identify the Majorana spinors ψ_1 and ψ_2 with the left- and right-handed projection ψ_L and ψ_R of a Dirac spinor ψ in the following way:

$$\psi_{1\mathrm{L}} \to \psi_\mathrm{L}, \qquad \widehat{\psi}_{2\mathrm{R}} \to \psi_\mathrm{R}, \qquad \psi_{2\mathrm{L}} \to \widehat{\psi}_\mathrm{L}, \qquad \widehat{\psi}_{1\mathrm{R}} \to \widehat{\psi}_\mathrm{R}. \tag{A.18}$$

With these replacements the mass term eq. (A.17) reads

$$\begin{aligned} -\mathscr{L}_{\text{mass}} &= \frac{m}{2}(\overline{\psi}_\mathrm{L}\psi_\mathrm{R} + \overline{\widehat{\psi}}_\mathrm{L}\widehat{\psi}_\mathrm{R}) + h.c. \\ &= m\overline{\psi}_\mathrm{L}\psi_\mathrm{R} + h.c.\ , \end{aligned} \tag{A.19}$$

[6] The factor of 1/2 is inserted artificially, in order to obtain a conventional Dirac mass of m, where without this factor one would obtain $2m$.

[7] This nomenclature applies as well, if the elements are actually block-matrices.

where in the last step we used eq. (A.7). Comparing this result with eq. (A.12), we see that we have constructed a Dirac spinor with proper Dirac mass term from two Majorana spinors degenerate in mass. If instead of 2 we consider $2n$ Majorana fields (i.e. respectively n fields ψ_1 and ψ_2) the entry m in the mass matrix in the second line of eq. (A.15) would be an $n \times n$ matrix and m' becomes m^T. Such a mass matrix, after diagonalization, leads to n Dirac fields in general [63].

Now let us return to eq. (A.16). For simplicity we consider, again, a system of two Majorana spinors ψ_1 and ψ_2 with couplings $m\overline{\psi}_{1L}\widehat{\psi}_{2R}$ and $m\overline{\psi}_{2L}\widehat{\psi}_{1R}$. This time, however, we permit a Majorana mass term $b\overline{\psi}_{2L}\widehat{\psi}_{2R}$ for the field ψ_2, where b is assumed to be much larger than m. In the matrix notation eq. (A.16) then becomes

$$\begin{aligned} -\mathscr{L}_{\text{mass}} &= \frac{1}{2}\overline{\psi}_{\alpha L} M_{\alpha\beta} \widehat{\psi}_{\beta R} + h.c. \\ &= \frac{1}{2}\begin{pmatrix}\overline{\psi}_{1L} & \overline{\psi}_{2L}\end{pmatrix}\begin{pmatrix}0 & m \\ m & b\end{pmatrix}\begin{pmatrix}\widehat{\psi}_{1R} \\ \widehat{\psi}_{2R}\end{pmatrix} + h.c.\ . \end{aligned} \tag{A.20}$$

The mass matrix

$$M = \begin{pmatrix}0 & m \\ m & b\end{pmatrix} \tag{A.21}$$

has eigenvalues $\lambda_1,\ \lambda_2 = \frac{b}{2}(1 \pm \sqrt{1 + 4m^2/b^2})$. Remember $b \gg m$. With the approximation $\sqrt{1+x} = 1 + x/2$ for small x, the eigenvalues are

$$\lambda_1 = b, \qquad \lambda_2 = -m^2/b. \tag{A.22}$$

We see that the introduction of the large scale b strongly suppresses one of the masses.

Appendix B

Matrix manipulations

In this part of the appendix the reader is equipped with the mathematical tools needed to execute the remodeling of matrices performed in this thesis. After specifying our notations and conventions in section B.1, we begin in section B.2 with the easiest case and work our way through to more complicated ones. In section B.3 we summarize the results for convenience.

B.1 Notations and conventions

When a matrix A is introduced we will use the notation $A \in \mathscr{M}[\mathbb{K}, m \times n]$ to indicate that A is an $m \times n$ matrix with entries $a_{ij} \in \mathbb{K}$, where $\mathbb{K}$ denotes an arbitrary field and $(i;\ j) = (1, \dots,\ m;\ 1, \dots,\ n)$. Since we will almost always consider $a_{ij} \in \mathbb{C}$, we will just write $A \in \mathscr{M}[m \times n]$, in the case of $\mathbb{K} = \mathbb{C}$. Quadratic $m \times m$ matrices are denoted by $A \in \mathscr{M}^2[m]$. Then, for $A \in \mathscr{M}[\mathbb{K}, m \times n]$ we define the function

$$\dim(A) = (m,\ n), \tag{B.1}$$

giving the (maximal) dimension of the row and column space of A. Note that by the definition of $\dim(\cdot)$, it follows that $\dim(A^T) = (n,\ m)$.

If not noted otherwise, we will parametrize a matrix $A \in \mathscr{M}[\mathbb{K}, m \times n]$ as

$$A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}. \tag{B.2}$$

If $A \in \mathscr{M}^2[m]$ and we choose $m = k + l$, with $k > 0$ and $l > 0$, we can split up the matrix into blocks

$$
\begin{aligned}
A &= \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \\
&\equiv \left(\begin{array}{ccc|ccc}
a_{11} & \dots & a_{1k} & a_{1(k+1)} & \dots & a_{1(k+l)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{k1} & \dots & a_{kk} & a_{k(k+1)} & \dots & a_{k(k+l)} \\
\hline
a_{(k+1)1} & \dots & a_{(k+1)k} & a_{(k+1)(k+1)} & \dots & a_{(k+1)(k+l)} \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
a_{(k+l)1} & \dots & a_{(k+l)k} & a_{(k+l)(k+1)} & \dots & a_{(k+l)(k+l)}
\end{array}\right), \qquad \text{(B.3)}
\end{aligned}
$$

where the lines indicate, how we define A_1, A_2, A_3 and A_4. Note that by this definition of the split-up, we have $\dim(A_2) = (k,\ l) = \dim(A_3^T)$. When the notation "$A \in \mathscr{M}^2[(k+l)]$" is used, we imply a split-up of the matrix A according to eq. (B.3). Sometimes, we will use the alternative notation

$$
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \equiv \begin{pmatrix} A_{k\times k} & A_{k\times l} \\ A_{l\times k} & A_{l\times l} \end{pmatrix}, \qquad \text{(B.4)}
$$

where $\dim(A_{i\times j}) = (i, j)$, to emphasize the size of the blocks of A. The principle of splitting up a matrix into blocks can obviously be generalized in choosing numbers $k_1,\ k_2, \dots,\ k_x$ with $k_1 + k_2 + \dots + k_x = m$ and $k_i > 0\ \forall i \in \{1, \dots,\ x\}$.

When performing a matrix-multiplication or an addition of two block-matrices

$$
A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad B = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \qquad \text{(B.5)}
$$

when writing

$$
AB = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix} \quad \text{or} \qquad \text{(B.6)}
$$

$$
A + B = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \qquad \text{(B.7)}
$$

we always imply that the split-up of A and B is the same, i.e. that $\dim(A_i) = \dim(B_i)\ \forall i \in \{1,\ 2,\ 3,\ 4\}$, without mention. This especially applies, when multiplying matrices to perform a transformation, e.g.

$$
B^T AB = \begin{pmatrix} B_1^T & B_3^T \\ B_2^T & B_4^T \end{pmatrix} \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}. \qquad \text{(B.8)}
$$

When handling different matrices, we will often want to compare the scales of their elements or, respectively, their eigenvalues. So we will in general assume that any matrix X has elements/eigenvalues proportional to the scale m_{X}.[1] To indicate this, we will sometimes just write "$X \sim m_{\mathrm{X}}$". Then, if for two matrices X and Y, for example it holds that $m_{\mathrm{X}} \gg m_{\mathrm{Y}}$, we will imply this by the short-hand notation "$X \gg Y$".

B.2 Block-diagonalization of matrices

In this section of the appendix, we will explain how to block-diagonalize *symmetric* matrices with a certain structure, which we encounter in this thesis. The explanation of the diagonalization technique in section B.2.1 is taken from the detailed discussion in [64]. The following sections are applications of this diagonalization technique in more complicated cases that are considered in this thesis.

B.2.1 First-type transformation

Imagine a matrix with the structure

$$M_1 = \begin{pmatrix} G & H^T \\ H & J \end{pmatrix} \in \mathscr{M}^2[(b+c)]. \tag{B.9}$$

Remember that in section B.1 we introduced the convention $X \sim m_{\mathrm{X}}$ for any matrix X. The matrix J is assumed to be non-singular, i.e. that none of the eigenvalues of J are equal to zero ($\det(J) \neq 0$). Additionally, we assume "$J \gg G$", "$J \gg H$".[2] Matrices of this type can be diagonalized by the transformation with a (by construction) unitary matrix

$$S_1 = \begin{pmatrix} C & B \\ -B^\dagger & D \end{pmatrix}, \tag{B.10}$$

where

$$C \equiv \sqrt{\mathbb{1} - BB^\dagger} = \mathbb{1} - \frac{1}{2}BB^\dagger - \frac{1}{8}BB^\dagger BB^\dagger - \dots \tag{B.11a}$$

$$D \equiv \sqrt{\mathbb{1} - B^\dagger B}, \text{ defined analogously to } C. \tag{B.11b}$$

The matrix B depends on the blocks of M_1. We will call matrices of the type of S_1 first-type or seesaw-type transformation matrices.

[1] The proportionality, of course, only applies to non-zero eigenvalues.

[2] If G is set equal to zero, this structure of M_1 corresponds to the structure of the neutrino mass matrix in the type I seesaw mechanism.

After the transformation of M_1 with S_1, the matrix reads

$$M_1^{\mathrm{d}} \equiv S_1^T M_1 S_1 = \begin{pmatrix} C^T & -B^* \\ B^T & D^T \end{pmatrix} \begin{pmatrix} G & H^T \\ H & J \end{pmatrix} \begin{pmatrix} C & B \\ -B^\dagger & D \end{pmatrix}$$
$$= \begin{pmatrix} M_{\mathrm{a}} & X^T \\ X & M_{\mathrm{b}} \end{pmatrix}, \tag{B.12}$$

where the blocks of M_1^{d} are given by

$$M_{\mathrm{a}} = C^T G C - C^T H^T B^\dagger - B^* H C + B^* J B^\dagger, \tag{B.13a}$$
$$M_{\mathrm{b}} = B^T G B + B^T H^T D + D^T H B + D^T J D, \tag{B.13b}$$
$$X = B^T G C - B^T H^T B^\dagger + D^T H C - D^T J B^\dagger. \tag{B.13c}$$

In order to get M_1^{d} to block-diagonal form, we demand $X \overset{!}{=} 0$ and solve this equation for B. To do so, we expand B as a power series in m_{J}^{-1},

$$B = B_1 + B_2 + B_3 + \ldots, \tag{B.14}$$

where $B_i \sim m_{\mathrm{J}}^{-i}$. Note that by the expansion B the matrix C is expanded as

$$C \equiv \sqrt{\mathbb{1} - B B^\dagger} = \mathbb{1} - \frac{1}{2} B B^\dagger - \frac{1}{8} B B^\dagger B B^\dagger - \ldots \tag{B.15}$$
$$= \mathbb{1} - \frac{1}{2} B_1 B_1^\dagger - \frac{1}{2}(B_1 B_2^\dagger + B_2 B_1^\dagger)$$
$$- \frac{1}{2}(B_2 B_2^\dagger + B_1 B_3^\dagger + B_3 B_1^\dagger) - \frac{1}{8} B_1 B_1^\dagger B_1 B_1^\dagger - \ldots\ , \tag{B.16}$$

and D is expanded analogously. The equation $X = 0$ can then be solved for B to arbitrary order. The first three elements of the expansion that solve $X = 0$ up to third order in m_{J}^{-1} in terms of the blocks of M_1 are given by

$$B_1^\dagger = J^{-1} H, \tag{B.17a}$$
$$B_2^\dagger = J^{-1} (J^{-1})^* H^* G, \tag{B.17b}$$
$$B_3^\dagger = J^{-1} (J^{-1})^* J^{-1} H G^* G$$
$$- J^{-1} (J^{-1})^* H^* H^T J^{-1} H$$
$$- \frac{1}{2} J^{-1} H H^\dagger (J^{-1})^* J^{-1} H. \tag{B.17c}$$

Inserting the expansion of B into eq. (B.13a) and (B.13b), the first four orders of

M_a are given by

$$M_{\mathrm{a}0} = G \tag{B.18a}$$
$$M_{\mathrm{a}1} = -H^T B_1^\dagger - B_1^* H + B_1^* J B_1^\dagger \tag{B.18b}$$
$$M_{\mathrm{a}2} = -\frac{1}{2}(B_1^* B_1^T G + G B_1 B_1^\dagger) - H^T B_2^\dagger - B_2^* H + B_1^* J B_2^\dagger + B_2^* J B_1^\dagger \tag{B.18c}$$
$$\begin{aligned} M_{\mathrm{a}3} = {} & \frac{1}{2}(B_1^* B_1^T H^T B_1^\dagger + B_1^* H B_1 B_1^\dagger) - H^T B_3^\dagger - B_3^* H^T \\ & + B_1^* J B_3^\dagger + B_3^* J B_1^\dagger + B_2^* J B_2^\dagger, \end{aligned} \tag{B.18d}$$

while M_b is approximately

$$M_\mathrm{b} \approx J. \tag{B.19}$$

With the condition $X = 0$ satisfied and inserting the expressions for the B_i's, the matrix M_1^d has the block-diagonal form

$$M_1^\mathrm{d} \equiv S_1^T M_1 S_1 \approx \begin{pmatrix} G - H^T J^{-1} H & 0 \\ 0 & J \end{pmatrix}, \tag{B.20}$$

where the diagonal entries are the corresponding expressions for M_a and M_b to leading order. The off-diagonal zero-blocks, however, are zero to arbitrary order.

B.2.2 Second-type transformation

The second type of matrices we encounter has the form

$$M_2 = \begin{pmatrix} G & H^T \\ H & 0 \end{pmatrix} \in \mathscr{M}^2[(b+c)]. \tag{B.21}$$

We assume "$H \gg G$".[3] Matrices of the second type naturally appear, when a singular seesaw mechanism is considered.

These matrices can be written as the sum

$$M_2 = D' + A' = \begin{pmatrix} 0 & H^T \\ H & 0 \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.22}$$

Now, we will explain how to quasi-diagonalize them. First, assume that we know the unitary transformation matrix that diagonalizes D'. We will denote this trans-

[3] Note that this implies $H \neq 0$. Otherwise $H = 0$ together with $H \gg G$ would imply $G = 0$, and hence $M_2 = 0$.

formation matrix by S_2 and the diagonalized form of D' by

$$D = \mathrm{diag}(d_1, \dots, d_{b+c}) \equiv S_2^T D' S_2, \tag{B.23}$$

which is the defining condition for S_2. Note that the rank of any matrix $X \in \mathscr{M}[m \times n]$ is given by

$$\mathrm{Rk}(X) \leq \min(m,\ n). \tag{B.24}$$

A matrix $X \in \mathscr{M}[m \times n]$ is said to have *full* rank, if $\mathrm{Rk}(X) = \min(m,\ n)$. Since $\mathrm{Rk}(D) = 2\mathrm{Rk}(H)$, we know that D can have

$$l := \mathrm{Rk}(D) = 2\mathrm{Rk}(H) \leq 2\min(b,\ c) \tag{B.25}$$

non-zero eigenvalues. Note that we excluded the case $l = 0$ from our discussion, since $l = 0 \Rightarrow \mathrm{Rk}(H) = 0 \Rightarrow H = 0$. The only possibility for $D \in \mathscr{M}^2[(b+c)]$ to have full rank ($l = b + c$) is, when H is quadratic *and* has full rank. Indeed, if $H \in \mathscr{M}[b \times b] (\Rightarrow b = c)$, and if $\mathrm{Rk}(H) = b$, then $l = \mathrm{Rk}(D) = 2\mathrm{Rk}(H) = 2b$. To make the proof complete we have to show that D does not have full rank, if one of the conditions $H \in \mathscr{M}[b \times b]$ or $\mathrm{Rk}(H) = \min(b,\ c)$ is not satisfied. First assume that H is not quadratic: $H \in \mathscr{M}[b \times c]$ with $b \neq c$ ($b < c$ without loss of generality). Then it follows that $\mathrm{Rk}(D) = 2\mathrm{Rk}(H) \leq 2b < b + c$, and D does not have full rank. And secondly assume that H does not have full rank: $\mathrm{Rk}(H) < \min(b,\ c)$ ($b \leq c$ without loss of generality). Then clearly $l = \mathrm{Rk}(D) = 2\mathrm{Rk}(H) < 2b \leq b+c$. Again, D does not have full rank, which completes the proof. Summarizing, $b = c$ is the necessary and $l = 2b$ the sufficient condition for D to have full rank. We write $D \equiv D_{2b}$ in the case $b = c$ and $l = 2b$, where D has full rank. Otherwise, if the conditions $b = c$ and $l = 2b$ are not satisfied, the matrix D will have $k := b+c-l > 0$ zero eigenvalues. In these cases, denoting the (diagonal) matrix that carries the l non-zero eigenvalues of D by $D_{l \times l}$, we will parametrize D according to

$$D = \begin{pmatrix} 0 & 0 \\ 0 & D_{l \times l} \end{pmatrix} \in \mathscr{M}^2[(k+l)], \tag{B.26}$$

with $k + l = b + c$ in agreement with the definitions of k and l. After the transformation of M_2 with S_2, the matrix reads

$$M_2^{\mathrm{qd}} \equiv S_2^T M_2 S_2 = S_2^T D' S_2 + S_2^T A' S_2 \equiv D + A. \tag{B.27}$$

The elements of A, a_{ij}, are given by

$$a_{ij} = \sum_{p=1}^{b+c}\sum_{q=1}^{b+c}(S_2)_{pi}(A')_{pq}(S_2)_{qj} = \sum_{p=1}^{b}\sum_{q=1}^{b}(S_2)_{pi}(G)_{pq}(S_2)_{qj}, \tag{B.28}$$

with $i \in \{1, \ldots, (b+c)\}$ and $j \in \{1, \ldots, (b+c)\}$. Since the elements of S_2 are of order 1, the elements of A are proportional to m_{G}, the scale of G. On the other hand the eigenvalues of D are of order $m_{\mathrm{H}} \gg m_{\mathrm{G}}$. In the case $D = D_{2b}$, this means that

$$M^{\mathrm{qd}} \approx D_{2b} \tag{B.29}$$

and we are done with the diagonalization. Otherwise, we split up A in blocks with the same size as the blocks of D in eq. (B.26) according to

$$A = \begin{pmatrix} A_{k\times k} & A^T_{l\times k} \\ A_{l\times k} & A_{l\times l} \end{pmatrix} \in \mathscr{M}^2[(k+l)]. \tag{B.30}$$

This parametrization follows the convention in eq. (B.3) of section B.1. Note that, since A is symmetric, there was no need to introduce the matrix $A_{k\times l} = A^T_{l\times k}$ as independent block of A. Inserting the parametrizations of D and A into eq. (B.27) yields

$$\begin{aligned} M_2^{\mathrm{qd}} &= \begin{pmatrix} A_{k\times k} & A^T_{l\times k} \\ A_{l\times k} & A_{l\times l} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & D_{l\times l} \end{pmatrix} \\ &\approx \begin{pmatrix} A_{k\times k} & A^T_{l\times k} \\ A_{l\times k} & D_{l\times l} \end{pmatrix}. \end{aligned} \tag{B.31}$$

Later on, we will consider a case, where $G = 0$ and eq. (B.31) becomes

$$M_2^{\mathrm{qd}} = \begin{pmatrix} 0 & 0 \\ 0 & D_{l\times l} \end{pmatrix}. \tag{B.32}$$

Note that in the case of $D = D_{2b}$, no matter whether $G = 0$ or not, we have $M_2^{\mathrm{qd}} \approx D_{2b}$.

Since the eigenvalues of the matrix $D_{l\times l}$ were assumed to be much larger than the elements of $A_{k\times k}$ and $A_{l\times k}$, the matrix M_2^{qd} is quasi-diagonal. We will call transformation matrices of the type of S_2 that led to eq. (B.31), second-type transformation matrices.

With M_2^{qd} in the form of eq. (B.31), it is an easy task to block-diagonalize the matrix by means of a seesaw-type transformation S_1. Introducing the short-hand notation $J_{k\times k} \equiv A_{k\times k} - A^T_{l\times k} D^{-1}_{l\times l} A_{l\times k}$, we have

$$M_2^{\text{d}} \equiv S_1^T M_2^{\text{qd}} S_1 = (S_2 S_1)^T M_2 (S_2 S_1) \approx \begin{pmatrix} J_{k\times k} & 0 \\ 0 & D_{l\times l} \end{pmatrix}. \tag{B.33}$$

The same result can be obtained, of course, by transforming M_2 with a combined transformation matrix $V_{\text{c}} \equiv S_2 S_1$, as indicated in eq.(B.33). In this thesis we encounter second-type matrices as being part of a bigger matrix that needs to be diagonalized. We will show how this can be done in the next subsection.

B.2.3 Combined transformation

The structure of the matrix discussed in this section appears in scenarios of the singular double seesaw mechanism.

Imagine a matrix with the structure

$$M_3 = \begin{pmatrix} 0 & E & 0 \\ E^T & G & H^T \\ 0 & H & 0 \end{pmatrix} \in \mathscr{M}^2[(a+b+c)]. \tag{B.34}$$

We assume "$H \gg G$" and "$G \gg E$". To diagonalize M_3, first, we choose a split-up of M_3 according to

$$M_3 = \left(\begin{array}{c|cc} 0 & E & 0 \\ \hline E^T & G & H^T \\ 0 & H & 0 \end{array}\right) = \begin{pmatrix} 0 & E' \\ E'^T & M_2 \end{pmatrix} \in \mathscr{M}^2[(a+(b+c))]. \tag{B.35}$$

Then, M_2 is a second-type matrix by construction. Writing

$$M_2 = \begin{pmatrix} 0 & H^T \\ H & 0 \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} = D' + A', \tag{B.36}$$

we recognize the structure of eq. (B.22). Hence M_2 can be quasi-diagonalized by a second-type transformation matrix. Introducing the matrix S_2 that diagonalizes D', as explained in the previous section, we can perform the second-type transformation of the complete matrix M_3 using

$$U \equiv \begin{pmatrix} \mathbb{1} & 0 \\ 0 & S_2 \end{pmatrix}. \tag{B.37}$$

After the transformation of M_3 with U, we have

$$M_3' \equiv U^T M_3 U = \begin{pmatrix} 0 & F \\ F^T & M_2^{\mathrm{qd}} \end{pmatrix}, \tag{B.38}$$

introducing the short-hand notation $F \equiv E' S_2$. The components of F are given by

$$f_{ij} = \sum_{p=1}^{b+c} (E')_{ip} (S_2)_{pj}, = \sum_{p=1}^{b} (E)_{ip} (S_2)_{pj} \tag{B.39}$$

where $i \in \{1, \ldots, a\}$ and $j \in \{1, \ldots, (b+c)\}$. Since the elements of S_2 are of order 1, we have $F \sim E \ll H$.

In the case that H is quadratic and has full rank, it follows that $M_2^{\mathrm{qd}} = D_{2b}$, and hence

$$M_3' = \begin{pmatrix} 0 & F \\ F^T & D_{2b} \end{pmatrix}. \tag{B.40}$$

Then, because of $F \ll H \sim D_{2b}$, we can diagonalize M_3' with a seesaw-type transformation, denoted by V_{2b}. Introducing the short-hand notation $J_{a\times a} \equiv -F D_{2b}^{-1} F^T$ the diagonalized form of M_3 reads

$$M_3^{\mathrm{d}} \equiv V_{2b}^T M_3' V_{2b} \approx \begin{pmatrix} J_{a\times a} & 0 \\ 0 & D_{2b} \end{pmatrix} \in \mathscr{M}^2[(a+2b)]. \tag{B.41}$$

Note that this form of M_3^{d} is unaltered if G is set equal to zero.

If H is not quadratic or if the rank of H is not full, M_2^{qd} is given by eq. (B.31). Remember that, under these circumstances, M_2^{qd} has $l = 2\mathrm{Rk}(H)$ non-zero and $k = b + c - l$ zero eigenvalues (cf. section B.2.2). Splitting up $F \in \mathscr{M}[a \times (k+l)]$ in accordance with our convention in eq. (B.3) as

$$F = \left(\begin{array}{ccc|ccc} f_{11} & \cdots & f_{1k} & f_{1(k+1)} & \cdots & f_{1(k+l)} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_{a1} & \cdots & f_{ak} & f_{a(k+1)} & \cdots & f_{a(k+l)} \end{array}\right) \equiv \begin{pmatrix} F_k & F_l \end{pmatrix}, \tag{B.42}$$

where $F_k \in \mathscr{M}[a \times k]$ and $F_l \in \mathscr{M}[a \times l]$, we can write M_3' from eq. (B.38) as

$$M_3' = \begin{pmatrix} 0 & F_k & F_l \\ F_k^T & A_{k\times k} & A_{l\times k}^T \\ F_l^T & A_{l\times k} & D_{l\times l} \end{pmatrix}, \tag{B.43}$$

where F_k and F_l are of order E, the matrices $A_{k\times k}$ and $A_{l\times k}$ are of order G, and $D_{l\times l}$ is of order H. Remembering that we have assumed "$H \gg G$" and "$G \gg E$", we see that M_3' is a first-type matrix. To make this clearer consider the split-up

$$M_3' = \left(\begin{array}{cc|c} 0 & F_k & F_l \\ F_k^T & A_{k\times k} & A_{l\times k}^T \\ \hline F_l^T & A_{l\times k} & D_{l\times l} \end{array}\right) \equiv \begin{pmatrix} G_1 & H_1^T \\ H_1 & D_{l\times l} \end{pmatrix}. \tag{B.44}$$

This means, performing a seesaw-type transformation of M_3', labeled by V, leads to

$$M_3'' \equiv V^T M_3' V \approx \begin{pmatrix} M_\mathrm{a} & 0 \\ 0 & D_{l\times l} \end{pmatrix}, \tag{B.45}$$

where we have introduced $M_\mathrm{a} \equiv G_1 - H_1^T D_{l\times l}^{-1} H_1$.

Let us compute M_a,

$$\begin{aligned} M_\mathrm{a} \equiv G_1 - H_1^T D_{l\times l}^{-1} H_1 &= \begin{pmatrix} 0 & F_k \\ F_k^T & A_{k\times k} \end{pmatrix} - \begin{pmatrix} F_l \\ A_{l\times k}^T \end{pmatrix} D_{l\times l}^{-1} \begin{pmatrix} F_l^T & A_{l\times k} \end{pmatrix} \\ &= \begin{pmatrix} 0 & F_k \\ F_k^T & A_{k\times k} \end{pmatrix} - \begin{pmatrix} F_l D_{l\times l}^{-1} F_l^T & F_l D_{l\times l}^{-1} A_{l\times k} \\ A_{l\times k}^T D_{l\times l}^{-1} F_l^T & A_{l\times k}^T D_{l\times l}^{-1} A_{l\times k} \end{pmatrix}. \end{aligned} \tag{B.46}$$

To get M_3 to block-diagonal form, we still need to diagonalize eq. (B.46). To do so, we introduce the short-hand notation[4]

$$\begin{aligned} M_\mathrm{a} &= \begin{pmatrix} 0 & F_k \\ F_k^T & A_{k\times k} \end{pmatrix} - \begin{pmatrix} F_l D_{l\times l}^{-1} F_l^T & F_l D_{l\times l}^{-1} A_{l\times k} \\ A_{l\times k}^T D_{l\times l}^{-1} F_l^T & A_{l\times k}^T D_{l\times l}^{-1} A_{l\times k} \end{pmatrix} \\ &\equiv \begin{pmatrix} J_{a\times a} & J_{k\times a}^T \\ J_{k\times a} & J_{k\times k} \end{pmatrix}. \end{aligned} \tag{B.47}$$

Then, defining the transformation matrix

$$W \equiv \begin{pmatrix} W_1 & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \tag{B.48}$$

where W_1 is the seesaw-type transformation matrix that diagonalizes M_a, it is by now an easy task for us to perform the seesaw-type transformation of M_3''. Introducing the short-hand notation $K_{a\times a} \equiv J_{a\times a} - J_{k\times a}^T J_{k\times k}^{-1} J_{k\times a}$, the resultant

[4]We choose this notation to emphasize the dimensions of the blocks of M_a.

matrix reads

$$M_3^{\rm d} \equiv W^T M_3'' W = \begin{pmatrix} K_{a\times a} & 0 & 0 \\ 0 & J_{k\times k} & 0 \\ 0 & 0 & D_{l\times l} \end{pmatrix}. \tag{B.49}$$

Now, computing all the elements of $M_3^{\rm d}$ is quite cumbersome, but technically possible with the equations given in this section. We will not do this here. Instead, we want to emphasize that a matrix as in eq. (B.34), where we started off with an $(a+b+c)$ structure for the different blocks, ends up with a block-diagonal form as in eq. (B.49), where we have an $(a+k+l)$ structure. From our discussion we can derive the scales of the blocks of $M_3^{\rm d}$. They are $D_{l\times l} \sim m_{\rm H}$, $J_{k\times k} \sim m_{\rm G}$ and the two terms of $K_{a\times a}$ are proportional to $m_{\rm E}^2/m_{\rm H}$ and $m_{\rm E}^2/m_{\rm G}$, respectively. Hence $M_3^{\rm d} \in \mathscr{M}^2[(a+k+l)]$ has the mass scale structure

$$M_3^{\rm d} \sim \begin{pmatrix} \frac{m_{\rm E}^2}{m_{\rm H}} + \frac{m_{\rm E}^2}{m_{\rm G}} & 0 & 0 \\ 0 & m_{\rm G} & 0 \\ 0 & 0 & m_{\rm H} \end{pmatrix}. \tag{B.50}$$

Remember that $l = 2{\rm Rk}(H)$ and $k = b + c - l$ so that the rank of the matrix H determines the scale structure of $M_3^{\rm d}$. Note that, if $b = c$ and $l = 2b \Rightarrow k = 0$, the diagonalized form of M_3 was given in eq. (B.41) with a eigenvalues of order $m_{\rm E}^2/m_{\rm H}$ and $2b$ eigenvalues of order $m_{\rm H}$.

Summing up the transformations that led to the diagonal form of M_3, eq.(B.49), we define

$$S_3 = UVW, \tag{B.51}$$

where U, V and W are defined in eq. (B.37), (B.10) and (B.48), respectively. M_3, then, is block-diagonalized by the transformation with S_3,

$$M_3^{\rm d} = S_3^T M_3 S_3. \tag{B.52}$$

And in the case of $M_3^{\rm d}$ as in eq. (B.41), the combined transformation is given by

$$S_3^{(2b)} \equiv U V_{2b}. \tag{B.53}$$

B.3 Summary

In this section we simply sum up the results from our discussion for better overview of the different matrix structures and scales.

In section B.2 we have shown that a first-type matrix

$$M_1 = \begin{pmatrix} G & H^T \\ H & J \end{pmatrix} \in \mathscr{M}^2[(b+c)] \tag{B.54}$$

can be block-diagonalized by a seesaw-type transformation matrix parametrized like

$$S_1 = \begin{pmatrix} C & B \\ -B^\dagger & D \end{pmatrix} \in \mathscr{M}^2[(b+c)]. \tag{B.55}$$

To leading order, the transformation of M_1 with S_1 leads to

$$M_1^{\mathrm{d}} = \begin{pmatrix} G - H^T J^{-1} H & 0 \\ 0 & J \end{pmatrix}, \tag{B.56}$$

where the zero-matrices are exact.

In section B.2.2 we explained that a second-type matrix

$$M_2 = D' + A' = \begin{pmatrix} 0 & H^T \\ H & 0 \end{pmatrix} + \begin{pmatrix} G & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} G & H^T \\ H & 0 \end{pmatrix} \in \mathscr{M}^2[(b+c)] \tag{B.57}$$

can be brought into quasi-diagonal form by performing a second-type transformation. The defining condition for the second-type transformation matrix S_2 was given in eq. (B.23). If H is quadratic and has full rank ($\Rightarrow b = c$), the diagonal form of M_2 is

$$M_2^{\mathrm{d}} = S_2^T M_2 S_2 = D_{2b} \in \mathscr{M}[2b \times 2b]. \tag{B.58}$$

If H is not quadratic or does not have full rank, it is given by

$$M_2^{\mathrm{d}} = S_{\mathrm{c}}^T M_2 S_{\mathrm{c}} = \begin{pmatrix} J_{k\times k} & 0 \\ 0 & D_{l\times l} \end{pmatrix} \in \mathscr{M}^2[(k+l)], \tag{B.59}$$

where S_{c} denotes a combined transformation consisting of S_2 and a suitable seesaw-type transformation matrix. In eq. (B.59), $l = 2\mathrm{Rk}(H)$ is defined as the number of non-zero eigenvalues of D', and $k = b + c - l$. Note that setting G equal to zero does not affect eq. (B.58), but in eq. (B.59) leads to $J_{k\times k} = 0$.

Finally, in section B.2.3, we showed how to diagonalize a third-type matrix

$$M_3 = \begin{pmatrix} 0 & E & 0 \\ E^T & G & H^T \\ 0 & H & 0 \end{pmatrix} \in \mathscr{M}^2[(a+b+c)]. \tag{B.60}$$

If H is quadratic and has full rank ($\Rightarrow b = c$), we use the combined transformation $S_3^{(2b)}$ defined in eq. (B.53) to diagonalize M_3 resulting in the structure

$$M_3^{\mathrm{d}} = \begin{pmatrix} J_{a\times a} & 0 \\ 0 & D_{2b} \end{pmatrix} \in \mathscr{M}^2[(a+2b)] \tag{B.61}$$

with scales $J_{a\times a} \sim \frac{m_{\mathrm{E}}^2}{m_{\mathrm{H}}}$ and $D_{2b} \sim m_{\mathrm{H}}$. This structure is not changed if $G = 0$.

If H is not quadratic or does not have full rank, we can diagonalize M_3 using the combined transformation S_3 defined in eq. (B.51), according to

$$M_3^d = S_3^T M_3 S_3 = \begin{pmatrix} K_{a\times a} & 0 & 0 \\ 0 & J_{k\times k} & 0 \\ 0 & 0 & D_{l\times l} \end{pmatrix}, \tag{B.62}$$

where $l = 2\mathrm{Rk}(H)$ and $k = b + c - l$. The scales of the blocks of M_3^d are

$$M_3^d \sim \begin{pmatrix} \frac{m_E^2}{m_H} + \frac{m_E^2}{m_G} & 0 & 0 \\ 0 & m_G & 0 \\ 0 & 0 & m_H \end{pmatrix}. \tag{B.63}$$

We admit that eq. (B.62) seems to tell us not much about the diagonal entries of M_3. We want to point out, however, that in our discussion, we are only interested in the structure of the diagonalized matrix, which can be seen from these equations. Still, we could calculate the elements of the blocks in these equations, using the definitions given in section B.2.3.

Appendix C

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Erklärung:

Ich versichere, dass ich diese Arbeit selbstständig verfasst habe und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe.

Heidelberg, den 15. Februar 2013 ....