

Testing the non-linear flux ansatz for maximal supergravity

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Abstract

We put to test the recently proposed non-linear flux ansatz for maximal supergravity in eleven dimensions, which gives the seven-dimensional flux in terms of the scalars and pseudoscalars of maximal $N = 8$ supergravity, by considering a number of non-trivial solutions of gauged supergravity for which the higher dimensional solutions are known. These include the G_2 and $SU(4)^-$ invariant stationary points. The examples considered constitute a very non-trivial check of the ansatz, which it passes with remarkable success.

1 Introduction

Recently [1], a simple non-linear flux ansatz giving the seven-dimensional components of the 3-form potential of eleven-dimensional supergravity [2] in terms of the scalars and pseudoscalars of maximal (gauged) $N = 8$ supergravity [3] has been proposed. This result arose from an attempt to understand the embedding of a recently discovered continuous family of inequivalent maximal ($N = 8$) gauged supergravities in four dimensions [4]. The emergence of this new family of theories follows from the electric-magnetic duality of the ungauged $N = 8$ theory [5], and can thus be understood in terms of the freedom to rotate between how one chooses to define electric and magnetic vector fields [6]. The inequivalence of the resulting theories is confined to the gauged theory, because in the ungauged theory electric-magnetic duality renders all such theories equivalent. From an eleven-dimensional perspective, the electric vector fields arise from the off-diagonal elfbein components (graviphoton), while the magnetic vector fields emerge from particular components of the 3-form potential.

A standard method by which new theories are obtained in supergravity is by reducing a higher dimensional theory on some group manifold or coset space. The problem of determining whether a coset space reduction is consistent, in the sense that every solution of the lower dimensional theory can be uplifted to a solution of the higher dimensional theory is a subtle one. In fact, the expectation is that such reductions are, in general, inconsistent [7]. A notable exception to this expectation is the consistency of the seven-sphere reduction of eleven-dimensional supergravity [8, 9]. Central to this result is a local $SU(8)$ invariant reformulation of the eleven-dimensional theory [10], which in the reduction on a seven-torus T^7 immediately reduces to the E_7 invariant theory of Cremmer and Julia [5], without the need to dualise tensors to scalars. This reformulation necessitates the introduction of new $SU(8)$ covariant objects in eleven dimensions. The most significant such object is the generalised vielbein, which arises from the study of the supersymmetry transformation of the graviphoton and replaces the siebenbein in the reformulated theory. The intimate connection between the electric vector fields of the four dimensional theory and the graviphoton leads naturally to the non-linear metric ansatz [11]

$$\Delta^{-1}g^{mn}(x, y) = \frac{1}{8}K^{mIJ}(y)K^{nKL}(y)\left[(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\right](x), \quad (1)$$

whereby the seven-dimensional metric g_{mn} is given terms of the scalar and pseudoscalar fields of $N = 8$ supergravity, via the $E_{7(7)}$ matrix components $u(x)$ and $v(x)$ (see (49) below), and of the Killing vectors $K^m(y)$ on the seven-sphere S^7 (where the eleven-dimensional coordinates are split as $z^M = (x^\mu, y^m)$). We note that the above formula has been subjected to numerous tests and also proven its usefulness in other contexts, such as the AdS/CFT correspondence [11–18]. The proof of consistency in [8] also furnishes a formula for the 4-form field strength, modulo a subtlety that is resolved in [9]. However, this formula appears to be too cumbersome for practical applications.

The remarkable result of [1] is that there exists an object analogous to the generalised vielbein that arises from the supersymmetry transformation of the components of the 3-form potential from which the magnetic vector fields of the four-dimensional theory arise. This new generalised vielbein now replaces the components of the 3-form potential A_{mnp} along the seven directions in the local $SU(8)$ invariant reformulation of the eleven-dimensional theory. Furthermore, it leads to a non-linear flux ansatz, which complements the non-linear metric ansatz above [1]:

$$\begin{aligned} \sqrt{2}K^{pIJ}(y)K^{qKL}(y)\left[(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\right](x)A_{mnp}(x, y) \\ = -iK_{mn}{}^{IJ}(y)K^{qKL}(y)\left[(u^{ij}{}_{IJ} - v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})\right](x). \end{aligned} \quad (2)$$

While this non-linear flux ansatz takes a surprisingly simple form, it is not a formula that can be found by asserting consistency with previous results. This is a major difference with the corresponding result for the $\text{AdS}_7 \times S^4$ compactification of maximal supergravity, where the non-linear ansätze can be directly substituted into the higher-dimensional field equations [19]; such a direct substitution is not possible for the $\text{AdS}_4 \times S^7$ compactification. Let us also mention that there exist partial results and uplift formulae for truncated versions of the maximal theory where the scalar sector is much simpler (see for example [20–27] and references therein). However, the formula above cannot be guessed from these. Its derivation is critically dependent on an analysis of the local $\text{SU}(8)$ manifest formulation of eleven-dimensional supergravity. Last but not least we wish to point out that in comparison with other theories, $N = 8$ supergravity is distinguished by an astonishingly rich variety of stationary points [28]¹ that can now be explored by means of the new formula.

The non-linear flux ansatz given in equation (5.11) of [1] differs from the above expression by the factor of $\sqrt{2}$ on the left hand side. In fact, as already pointed out there, this overall factor can so far not be determined from intrinsically Kaluza-Klein theoretic considerations matching the eleven-dimensional 3-form potential with the four-dimensional gauge field. This is in marked contrast to the graviphoton, for which general Kaluza-Klein theory gives its precise relation to the four dimensional gauge field, by matching the non-abelian interaction with the commutator of two Killing vector fields. However, as we will show here, this factor is universally and unambiguously the same for all solutions, and does follow by explicitly computing the 3-form potential using the above ansatz for solutions of gauged supergravity for which the higher dimensional uplift solution is known (the factor $\sqrt{2}$ is most easily checked for the Englert solution [30]).

The aim of this paper, then, is to test the non-linear flux ansatz for a number of solutions of gauged supergravity. In order to make the paper self-contained we begin, in section 2, by describing the main conventions and definitions that are required, summarising various known results. In addition, in appendix C, we list some important Γ -matrix identities, most of which already appear in the appendices of [5] and [10], while some are new.

In section 3, we begin by describing the eleven-dimensional G_2 invariant solution of [11]. Then, we consider the G_2 invariant stationary point of $N = 8$ supergravity [31, 32] that uplifts to the aforementioned solution, rederiving the E_7 matrix components u and v that are parametrised by the scalars and pseudoscalars. These components are essential inputs in the non-linear ansätze. We calculate the 3-form potential using the non-linear flux ansatz, equation (2), verifying its total antisymmetry as is expected from the general argument in [1]. The field strength of this potential is then derived for the G_2 family of solutions. Substituting the G_2 stationary point values yields the flux of the G_2 invariant solution of the eleven-dimensional theory with precise agreement.

As our next test, we consider the $\text{SU}(4)^-$ invariant solution of [33] in section 4. We rewrite this solution of eleven-dimensional supergravity in terms of geometric quantities defined on the seven-sphere. As in section 3, we derive the 3-form potential using the non-linear flux ansatz and confirm that at the stationary point [31, 32] the associated field strength matches precisely with that of the eleven-dimensional solution.

Furthermore, in appendix A, we give the metric and the flux calculated from the non-linear ansätze with the scalars of the $\text{SO}(7)^\pm$ invariant family of maximal gauged supergravity [31, 32]. These examples are simple enough for the reader to immediately match with the known $\text{SO}(7)^+$ [34] and $\text{SO}(7)^-$ [30] solutions of eleven-dimensional supergravity, and are thus included mainly for the reader's convenience.

¹Whereas, for instance, maximal gauged supergravity in seven dimensions has only one non-trivial stationary point besides the trivial vacuum [29].

2 Preliminaries

In this paper, we follow the conventions of reference [10]. The bosonic field equations of eleven-dimensional supergravity [2] read²

$$R_{MN} = \frac{1}{72}g_{MN}F_{PQRS}^2 - \frac{1}{6}F_{MPQR}F_N{}^{PQR}, \quad (3)$$

$$E^{-1}\partial_M(EF^{MNPQ}) = \frac{\sqrt{2}}{1152}i\eta^{NPQR_1\dots R_4S_1\dots S_4}F_{R_1\dots R_4}F_{S_1\dots S_4}, \quad (4)$$

where E is the determinant of the elfbein $E_M{}^A$. We note that solutions to these combined equations are only determined up to an overall constant scaling

$$g_{MN} \rightarrow \lambda g_{MN}, \quad F_{MNPQ} \rightarrow \lambda^{3/2} F_{MNPQ}. \quad (5)$$

Such a rescaling must be taken into account when comparing the various solutions given in the literature with the ones constructed from the non-linear ansätze (1) and (2). We emphasise that the normalisation of all solutions is thus completely fixed by (1) and (2), once the trivial vacuum solution has been specified.

We are interested in solutions of the above equations that are obtained via a compactification to a four-dimensional maximally symmetric spacetime. The most general ansatz for the elfbein that is consistent with this requirement is of the warped form³

$$E_M{}^A(x, y) = \begin{pmatrix} \Delta^{-1/2}(y)\mathring{e}_\mu{}^\alpha(x) & 0 \\ 0 & e_m{}^a(y) \end{pmatrix}, \quad (6)$$

where x^μ are coordinates on the four-dimensional spacetime and y^m are coordinates on the compact seven-dimensional space; $\mathring{e}_\mu{}^\alpha(x)$ is the vierbein of the maximally symmetric four-dimensional spacetime and $e_m{}^a(y)$ is the siebenbein of the compact space. In particular, we assume the siebenbein to be that of a deformed round seven-sphere with the deformation parametrised by a matrix $S_a{}^b(y)$

$$e_m{}^a(y) = \mathring{e}_m{}^b(y) S_b{}^a(y), \quad (7)$$

where $\mathring{e}_m{}^a(y)$ is the siebenbein on the round seven-sphere with inverse radius m_7 , and where

$$\Delta(y) \equiv \det S_a{}^b(y). \quad (8)$$

The presence of the warp factor in (6) is required by consistency with the supersymmetry transformation rules of the fields that would correspond with those of the maximal theory upon reduction to four dimensions [34].

The eight Killing spinors of S^7 , η^I satisfy

$$(\mathring{D}_m + \frac{1}{2}i m_7 \mathring{e}_m{}^a \Gamma_a)\eta^I = 0, \quad (9)$$

where \mathring{D}_m is the covariant derivative on the round seven-sphere and the Γ^a -matrices are flat, Euclidean, seven-dimensional and purely imaginary. In a Majorana representation of the Clifford algebra in Euclidean seven dimensions, the charge conjugation matrix that is used to define spinor

²Note that for consistency with [11], we use a negative curvature convention, i.e. $[D_M, D_N]V^P = -R^P{}_{QMN}V^Q$. Hence, the scalar curvature of a sphere is negative.

³In general, of course, $e_m{}^a$ can also have x dependence. But here we are considering compactifications.

conjugates, or raise and lower spinor indices, can be chosen to be the identity matrix. Here we make such a choice. Furthermore, it is useful to choose Killing spinors that are orthonormal

$$\bar{\eta}^I \eta^J = \delta^{IJ}, \quad \eta^I \bar{\eta}^I = \mathbf{1}, \quad (10)$$

where on the right hand side of the second equation, $\mathbf{1}$ denotes the identity matrix with spinor indices.

These spinors can be used to define a set of vectors and 2-forms as follows:

$$K_a^{IJ} = i\bar{\eta}^I \Gamma^a \eta^J, \quad K_{ab}^{IJ} = \bar{\eta}^I \Gamma^{ab} \eta^J, \quad (11)$$

$$K_m^{IJ} = \hat{e}_m^a K_a^{IJ}, \quad K_{mn}^{IJ} = \hat{e}_m^a \hat{e}_n^b K_{ab}^{IJ}. \quad (12)$$

In the following we will adopt the rule that the curved indices on Killing vectors and their derivatives are always lowered and raised *with the round seven-sphere metric \hat{g}_{mn} and its inverse*. It is now straightforward to show that

$$K_{ab}^{IJ} K_c^{IJ} = 0, \quad K^{aIJ} K_b^{IJ} = 8 \delta_b^a, \quad K^{abIJ} K_{cd}^{IJ} = 16 \delta_{cd}^{ab}. \quad (13)$$

Assuming the four-dimensional spacetime to be maximally symmetric implies that the only non-zero components of the field strength F_{MNPQ} are $F_{\mu\nu\rho\sigma}$ and F_{mnpq} . Following [35], we parametrise $F_{\mu\nu\rho\sigma}$ as follows

$$F_{\mu\nu\rho\sigma} = i\mathfrak{f}_{\text{FR}} \eta_{\mu\nu\rho\sigma}, \quad (14)$$

where $\eta_{\mu\nu\rho\sigma}$ is the alternating tensor in four dimensions.⁴ The Bianchi identities imply that the Freund-Rubin parameter \mathfrak{f}_{FR} is a constant. Beware that switching to flat indices introduces y -dependence

$$F_{\alpha\beta\gamma\delta} = i\mathfrak{f}_{\text{FR}} \Delta^2 \eta_{\alpha\beta\gamma\delta}. \quad (15)$$

Given an elfbein of the form given in equation (6) and using equation (14), it is fairly straightforward to show that the eleven-dimensional equations (3) and (4) reduce to [11]

$$R_\mu{}^\nu = \left(\frac{2}{3} \mathfrak{f}_{\text{FR}}^2 \Delta^4 + \frac{1}{72} F_{mnpq}^2 \right) \delta_\mu^\nu, \quad (16)$$

$$R_m{}^n = -\frac{1}{6} F_{mnpq} F^{mnpq} + \left(\frac{1}{72} F_{mnpq}^2 - \frac{1}{3} \mathfrak{f}_{\text{FR}}^2 \Delta^4 \right) \delta_m^n, \quad (17)$$

$$\hat{D}_q (\Delta^{-1} F^{mnpq}) = \frac{1}{24} \sqrt{2} \mathfrak{f}_{\text{FR}} \hat{\eta}^{mnpqrst} F_{qrst}, \quad (18)$$

where seven-dimensional indices m, n, p, \dots are raised (lowered) with

$$g^{mn} = e_a{}^m e_b{}^n \delta^{ab} \quad (g_{mn} = e_m{}^a e_n{}^b \delta_{ab}),$$

except in cases where the object is denoted with a circle \circ on top, in which case indices are raised (lowered) with \hat{g}^{mn} (\hat{g}_{mn}) analogously defined. Hence, $\hat{\eta}^{mnpqrst}$ is the alternating tensor corresponding to the round seven-sphere metric \hat{g}_{mn} and its indices are raised with \hat{g}^{mn} .

As is well-known, the four-dimensional spacetime must be AdS_4 . We choose to parametrise its radius such that

$$R_{\mu\nu} = 3m_4^2 g_{\mu\nu}. \quad (19)$$

Furthermore, for an S^7 of inverse radius m_7 ,

$$\hat{R}_{mn} = -6m_7^2 \hat{g}_{mn}. \quad (20)$$

⁴Note that the conventions used in this paper are such that $\eta_{\mu\nu\rho\sigma} \eta^{\mu\nu\rho\sigma} = +4!$.

Thus, in our conventions, the S^7 compactification [36] is given by

$$m_4 = 2m_7, \quad \mathfrak{f}_{FR} = \pm 3\sqrt{2}m_7. \quad (21)$$

We repeat that the normalisation of *all* solutions away from the trivial AdS_4 vacuum is fixed by the non-linear ansätze. Thus, they are all expressed in terms of a single dimensionful parameter m_7 .

3 The G_2 invariant solution

3.1 The G_2 invariant solution of eleven-dimensional supergravity

In order to write out the G_2 invariant solution, we must first define the geometrical quantities, respectively preserving the $\text{SO}(7)^+$ and $\text{SO}(7)^-$ subgroups of $\text{SO}(8)$ whose common subgroup is $G_2 = \text{SO}(7)^+ \cap \text{SO}(7)^-$. These are given in terms of the following self-dual C_+^{IJKL} and antiself-dual C_-^{IJKL} $\text{SO}(8)$ tensors, respectively, which satisfy the identities [34, 37]

$$C_+^{IJMN} C_+^{MNKL} = 12\delta_{KL}^{IJ} + 4C_+^{IJKL}, \quad (22)$$

$$C_-^{IJMN} C_-^{MNKL} = 12\delta_{KL}^{IJ} - 4C_-^{IJKL}. \quad (23)$$

These tensors will also appear below in the parametrisation of the scalar and pseudoscalar expectations in $N = 8$ supergravity.

The self-dual tensor C_+ can be used to define $\text{SO}(7)^+$ invariant quantities [34]

$$\xi_a = \frac{1}{16} C_+^{IJKL} K_{ab}{}^{IJ} K_b{}^{KL}, \quad (24)$$

$$\xi_{ab} = -\frac{1}{16} C_+^{IJKL} K_a{}^{IJ} K_b{}^{KL}, \quad (25)$$

$$\xi = \delta^{ab} \xi_{ab}. \quad (26)$$

These quantities satisfy the non-trivial identities [34]

$$\xi_a \xi_a = (21 + \xi)(3 - \xi), \quad (27)$$

$$\xi_{ab} = \frac{1}{6}(3 + \xi)\delta_{ab} - \frac{1}{6(3 - \xi)}\xi_a \xi_b, \quad (28)$$

$$\overset{\circ}{D}_c \xi_{ab} = \frac{1}{3} m_7 (\delta_{ab} - \xi_{(a} \delta_{b)c}), \quad (29)$$

$$\overset{\circ}{D}_a \xi = 2m_7 \xi_a, \quad (30)$$

$$\overset{\circ}{D}_a \xi_b = m_7(3 - \xi)\delta_{ab} - \frac{m_7}{3 - \xi} \xi_a \xi_b. \quad (31)$$

Hence, the variable ξ lies in the range $-21 < \xi < 3$, with the endpoints corresponding to the north and south poles of the seven sphere. Alternatively, in terms of the unit vector

$$\hat{\xi}_a = \frac{1}{\sqrt{(21 + \xi)(3 - \xi)}} \xi_a, \quad (32)$$

the last two equations become [34]

$$\overset{\circ}{D}_a \xi = 2m_7 \sqrt{(21 + \xi)(3 - \xi)} \hat{\xi}_a, \quad (33)$$

$$\overset{\circ}{D}_a \hat{\xi}_b = m_7 \sqrt{\frac{3 - \xi}{21 + \xi}} (\delta_{ab} - \hat{\xi}_a \hat{\xi}_b). \quad (34)$$

The antiself-dual C_-^{IJKL} can similarly be used to define the $\text{SO}(7)^-$ invariant tensor (*alias* the ‘parallelising torsion’ on S^7)

$$S_{abc} = \frac{1}{16} C_-^{IJKL} K_{[ab}^{IJ} K_c^{KL]}, \quad (35)$$

which satisfies the relations

$$\mathring{D}_a S_{bcd} = \frac{1}{6} m_7 \epsilon_{abcdefg} S^{efg}, \quad (36)$$

$$S^{[abc} S^{d]ef} = \frac{1}{4} \epsilon^{abcd[e} S^{f]gh}, \quad (37)$$

$$S^{a[bc} S^{de]f} = \frac{1}{6} \epsilon^{bcde(a} S^{f)gh}, \quad (38)$$

$$S^{abe} S_{cde} = 2\delta_{cd}^{ab} + \frac{1}{6} \epsilon^{ab}{}_{cdefg} S^{efg}. \quad (39)$$

These relations have been derived in [37, 38]. There is a potential ambiguity in the sign of terms with S on the right hand side of the equations above, which is fixed by requiring that C_- is antiself-dual and satisfies equation (23) (see equations (3.6)–(3.17) of [37]). Equation (36) is derived using the Γ -matrix identity (156) and equation (9).

The relations (24), (25) and (35) can be inverted to give the $\text{SO}(8)$ tensors in terms of the $\text{SO}(7)^+$ and $\text{SO}(7)^-$ geometric quantities [9, 34, 38]

$$C_+^{IJKL} = -\frac{1}{12} (9 + \xi) K_a^{[IJ} K_a^{KL]} + \frac{1}{4} (21 + \xi) \hat{\xi}^a \hat{\xi}^b K_a^{[IJ} K_b^{KL]} + \frac{1}{12} \sqrt{(21 + \xi)(3 - \xi)} \hat{\xi}^a K_{ab}^{[IJ} K_b^{KL]}, \quad (40)$$

$$C_-^{IJKL} = \frac{1}{2} S_{abc} K_{ab}^{[IJ} K_c^{KL]}. \quad (41)$$

In appendix B, we explain that C_+, iC_- together with their symmetrised product iD_+ generate the $\text{SU}(1,1)$ algebra in E_7 [11], which commutes with G_2 . This fact can be used to derive the relations listed in equations (145)–(150).

In terms of the $\text{SO}(7)^\pm$ invariant tensors defined above, the G_2 invariant solution of eleven-dimensional supergravity is given by the following expressions. In the uncompactified dimensions, it is the usual AdS_4 metric, while the metric in the internal seven-dimensional space is given by [11]

$$g_{mn} = 6^{2/3} \gamma^{-1/9} (15 - \xi)^{-1/3} \left\{ \left(\mathring{g}_{mn} - \hat{\xi}_m \hat{\xi}_n \right) + \frac{1}{36} (15 - \xi) \hat{\xi}_m \hat{\xi}_n \right\}, \quad (42)$$

where γ is an arbitrary positive constant and the index on $\hat{\xi}_m$ is raised with metric \mathring{g}^{mn} . The determinant of this metric is

$$\det(g_{mn}) = \Delta^2 \det(\mathring{g}_{mn}), \quad (43)$$

where

$$\Delta = 6^{4/3} \gamma^{-7/18} (15 - \xi)^{-2/3}. \quad (44)$$

The internal flux (4-form field strength) is

$$F_{mnpq} = \frac{4\sqrt{6/5}}{15 - \xi} \gamma^{-1/6} m_7 \left\{ \mathring{\eta}_{mnpqrst} \mathring{S}^{rst} - \frac{(21 + \xi)(\xi - 27 \pm 12\sqrt{3})}{12(15 - \xi)} \hat{\xi}_{[m} \mathring{\eta}_{npq]rst} \hat{\xi}^r \mathring{S}^{stu} + \sqrt{(21 + \xi)(3 - \xi)} \frac{(\xi - 51 \pm 12\sqrt{3})}{2(15 - \xi)} \mathring{S}_{[mnp} \hat{\xi}_{q]} \right\}. \quad (45)$$

The \pm ambiguity in the expression above arises from the arbitrariness in the sign of the Freund-Rubin parameter f_{FR} [11]. As shown there, this solution has $N = 1$ residual supersymmetry.

The solution given above solves the Einstein equations for any value of the constant γ (see equation (5)). However, the non-linear metric ansatz gives the solution with a particular value for γ . In anticipation of this fact, and for ease of comparison later, we choose

$$\gamma^{-1/3} = \frac{5}{6\sqrt{3}}. \quad (46)$$

Hence,

$$g_{mn} = 3^{1/6} 10^{1/3} (15 - \xi)^{-1/3} \left\{ \left(\overset{\circ}{g}_{mn} - \hat{\xi}_m \hat{\xi}_n \right) + \frac{1}{36} (15 - \xi) \hat{\xi}_m \hat{\xi}_n \right\} \quad (47)$$

and

$$F_{mnpq} = \frac{4 \times 3^{-1/4}}{15 - \xi} m_7 \left\{ \overset{\circ}{\eta}_{mnpqrst} \overset{\circ}{S}^{rst} - \frac{(21 + \xi)(\xi - 27 \pm 12\sqrt{3})}{12(15 - \xi)} \hat{\xi}_{[m} \overset{\circ}{\eta}_{npq]rstu} \hat{\xi}^r \overset{\circ}{S}^{stu} \right. \\ \left. + \sqrt{(21 + \xi)(3 - \xi)} \frac{(\xi - 51 \pm 12\sqrt{3})}{2(15 - \xi)} \overset{\circ}{S}_{[mnp} \hat{\xi}_{q]} \right\}. \quad (48)$$

3.2 The G_2 invariant stationary point of gauged supergravity

The 70 scalars and pseudoscalars of the $N = 8$ supergravity theory that parametrise an element of the coset space $E_7/SU(8)$ can be described by an element in the fundamental representation of E_7 as follows [5]:

$$\mathcal{V} = \begin{pmatrix} u_{ij}^{IJ} & v_{ijIJ} \\ v^{ijIJ} & u^{ij}_{IJ} \end{pmatrix}. \quad (49)$$

Note that complex conjugation is represented by a respective lowering/raising of indices.

Using an $SU(8)$ transformation, the E_7 matrix \mathcal{V} can be brought into a symmetric gauge of the form

$$\mathcal{V} = \exp \Phi \equiv \exp \begin{pmatrix} 0 & \phi_{IJKL} \\ \phi^{IJKL} & 0 \end{pmatrix}. \quad (50)$$

Once this gauge is fixed, the distinction between i, j, \dots and I, J, \dots indices may be safely ignored, as we shall do so hereafter. For a G_2 invariant configuration, the most general vacuum expectation value that ϕ_{IJKL} can take may be parametrised as follows [31, 32]:

$$\phi_{IJKL} \equiv \phi_{IJKL}(\lambda, \alpha) = \frac{1}{2} \lambda (C_+^{IJKL} \cos \alpha + i C_-^{IJKL} \sin \alpha), \quad (51)$$

where λ and α take a particular value for each stationary point consistent with this configuration. The self-dual C_+^{IJKL} , antiself-dual C_-^{IJKL} and

$$D_{\pm}^{IJKL} = \frac{1}{2} (C_+^{IJMN} C_-^{MNKL} \pm C_-^{IJMN} C_+^{MNKL}), \quad (52)$$

form a basis of G_2 invariant objects in E_7 . In the remainder we will often keep the index structure implicit for brevity; so

$$A \cdot B \equiv (A \cdot B)^{IJKL} \equiv A^{IJMN} B^{MNKL}. \quad (53)$$

Given ϕ of the form above, the components of the $E_7/SU(8)$ coset elements u^{IJ}_{KL} and v^{IJKL} can be written in terms of the G_2 invariant C_{\pm} and D_{\pm} . Given the structure of the matrix Φ , it is not too difficult to see that

$$\mathcal{V} = \sum_{n=0}^{\infty} \frac{1}{n!} \Phi^n = \begin{pmatrix} \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (\varphi\varphi^*)^n & \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!} \varphi(\varphi^*\varphi)^n \\ \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!} \varphi^*(\varphi\varphi^*)^n & \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (\varphi^*\varphi)^n \end{pmatrix}, \quad (54)$$

where

$$\varphi = \cos \alpha C_+ + i \sin \alpha C_-, \quad \varphi^* = \cos \alpha C_+ - i \sin \alpha C_-. \quad (55)$$

Of course, the order in which the tensors appear is now important and indicative of the index structure of the terms. For example, $C_+C_- = D_+ + D_-$, while $C_-C_+ = D_+ - D_-$. The matrix above is clearly compatible with the structure of the matrix given in the defining equation (49).

Consider

$$\varphi\varphi^* = (\cos \alpha C_+ + i \sin \alpha C_-)(\cos \alpha C_+ - i \sin \alpha C_-). \quad (56)$$

Using equations (22) and (23), we find that

$$\varphi\varphi^* = 12 + 4\tilde{\Theta}, \quad (57)$$

where we have omitted a δ symbol in the first term above for brevity and

$$\tilde{\Theta} = \cos^2 \alpha C_+ - \sin^2 \alpha C_- - \frac{1}{4}i \sin 2\alpha D_-. \quad (58)$$

We notice that $\tilde{\Theta}$ has the rather convenient property that

$$\tilde{\Theta}^2 = 12 + 4\tilde{\Theta}. \quad (59)$$

One can simply verify the above equation using equations (145)–(150). Now, define a new quantity

$$\Theta = \frac{1}{8}(\tilde{\Theta} + 2), \quad (60)$$

which has been chosen so that

$$\Theta^2 = \Theta. \quad (61)$$

From equation (57), we have

$$\varphi\varphi^* = 12 + 4\tilde{\Theta} = 4 + 32\Theta. \quad (62)$$

Comparing the components of equations (49) and (54) gives that

$$u_{IJ}{}^{KL} = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (\varphi\varphi^*)^n \quad (63)$$

and

$$v^{IJKL} = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!} \varphi^*(\varphi\varphi^*)^n, \quad (64)$$

where, as before, indices have been suppressed on the right hand side of the above equations.

First, consider u_{IJ}^{KL} .

$$\begin{aligned}
u_{IJ}^{KL} &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (\varphi\varphi^*)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (4 + 32\Theta)^n \\
&= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} \sum_{p=0}^n \binom{n}{p} 4^{n-p} (32\Theta)^p,
\end{aligned}$$

where we have used equation (62) in the second equality and applied the binomial theorem in the third equality. Using the property satisfied by Θ , equation (61), the previous expression can be rewritten as follows:

$$\begin{aligned}
u_{IJ}^{KL} &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} \left\{ 4^n + \sum_{p=1}^n \binom{n}{p} 4^{n-p} (32)^p \Theta \right\} \\
&= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} \left\{ 4^n + \left[\sum_{p=0}^n \binom{n}{p} 4^{n-p} (32)^p - 4^n \right] \Theta \right\} \\
&= \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!} \left\{ 1 + [3^{2n} - 1] \Theta \right\}.
\end{aligned}$$

Identifying the above expressions as the Taylor expansions of the cosh function simplifies the expression to

$$\begin{aligned}
u_{IJ}^{KL} &= \cosh \lambda + (\cosh 3\lambda - \cosh \lambda)\Theta \\
&= \cosh \lambda + \frac{1}{8}(\cosh 3\lambda - \cosh \lambda)(\tilde{\Theta} + 2) \\
&= \cosh^3 \lambda + \frac{1}{2} \cosh \lambda \sinh^2 \lambda \tilde{\Theta},
\end{aligned}$$

where we have used equation (60) in the second equality and well-known multiple angle identities for hyperbolic functions in the final equality. Defining

$$p = \cosh \lambda, \quad q = \sinh \lambda \quad (65)$$

and substituting for $\tilde{\Theta}$ using equation (58) gives

$$u_{IJ}^{KL}(\lambda, \alpha) = p^3 \delta_{IJ}^{KL} + \frac{1}{2} p q^2 \cos^2 \alpha C_+^{IJKL} - \frac{1}{2} p q^2 \sin^2 \alpha C_-^{IJKL} - \frac{1}{8} i p q^2 \sin 2\alpha D_-^{IJKL} \quad (66)$$

or, equivalently,

$$u^{IJ}_{KL}(\lambda, \alpha) = p^3 \delta_{KL}^{IJ} + \frac{1}{2} p q^2 \cos^2 \alpha C_+^{IJKL} - \frac{1}{2} p q^2 \sin^2 \alpha C_-^{IJKL} + \frac{1}{8} i p q^2 \sin 2\alpha D_-^{IJKL} \quad (67)$$

for the complex conjugate.

The derivation of v^{IJKL} is essentially the same as that of u_{IJ}^{KL} . Starting from equation (64)

$$\begin{aligned}
v^{IJKL} &= \varphi^* \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!} (\varphi\varphi^*)^n \\
&= \frac{1}{2} \varphi^* \sum_{n=0}^{\infty} \frac{\lambda^{2n+1}}{(2n+1)!} \left\{ 1 + [3^{2n} - 1] \Theta \right\} \\
&= \frac{1}{2} \varphi^* \left\{ \sinh \lambda + \left(\frac{1}{3} \sinh 3\lambda - \sinh \lambda \right) \Theta \right\},
\end{aligned}$$

where the second equality is a direct application of the results derived above. Substituting for φ^* and Θ using equations (55) and (60), respectively, the above expression simplifies to

$$v^{IJKL} = \frac{1}{48}(\cos \alpha C_+ - i \sin \alpha C_-) \left\{ 2(\sinh 3\lambda + 9 \sinh \lambda) \right. \\ \left. + (\sinh 3\lambda - 3 \sinh \lambda)(\cos^2 \alpha C_+ - \sin^2 \alpha C_- - \frac{i}{4} \sin 2\alpha D_-) \right\} \quad (68)$$

where we have substituted for $\tilde{\Theta}$ using equation (58). Expanding out the bracket above and using equations (145)–(150), one can simply show that the above expression reduces to

$$v^{IJKL}(\lambda, \alpha) = q^3(\cos^3 \alpha + i \sin^3 \alpha) \delta_{KL}^{IJ} + \frac{1}{2} p^2 q \cos \alpha C_+^{IJKL} - \frac{1}{2} i p^2 q \sin \alpha C_-^{IJKL} \\ - \frac{1}{8} q^3 \sin 2\alpha (\sin \alpha + i \cos \alpha) D_+^{IJKL}, \quad (69)$$

where we have used well-known multiple angle identities for hyperbolic functions and definitions (65).

The G_2 invariant stationary point is given by⁵ [31]

$$c^2 = (p^2 + q^2)^2 = \frac{1}{5}(3 + 2\sqrt{3}), \quad (70)$$

$$s^2 = (2pq)^2 = \frac{2}{5}(\sqrt{3} - 1), \quad (71)$$

$$v^2 = \cos^2 \alpha = \frac{1}{4}(3 - \sqrt{3}), \quad (72)$$

where $c = \cosh(2\lambda)$ and $s = \sinh(2\lambda)$. We also define the following useful combinations

$$b_1(\lambda, \alpha) = c^3 + v^3 s^3, \quad b_2(\lambda, \alpha) = csv(c + vs), \quad (73)$$

$$f_1(\lambda, \alpha) = p^2 q^2 (p^2 + q^2) \sin \alpha \cos \alpha = \frac{1}{4} s^2 cv \sqrt{1 - v^2}, \quad (74)$$

$$f_2(\lambda, \alpha) = p^3 q^3 \sin \alpha \cos^2 \alpha = \frac{1}{8} s^3 v^2 \sqrt{1 - v^2}, \quad (75)$$

$$f_3(\lambda, \alpha) = pq(p^2 + q^2)^2 \sin \alpha = \frac{1}{2} s c^2 \sqrt{1 - v^2}. \quad (76)$$

At the G_2 invariant stationary point we have the following simplifying relations

$$b_1 = 3b_2, \quad f_3 = 4(2f_1 - f_2), \quad (77)$$

which will be useful below.

3.3 Derivation of the 3-form potential

In this section we derive the potential of the eleven-dimensional G_2 invariant solution using the non-linear flux ansatz (2), showing that its field strength coincides with the expression found in [11].

We identify the expression multiplying the potential in equation (2) as

$$8\Delta^{-1} g^{pq} = K^{pIJ} K^{qKL} (u^{ij}{}_{IJ} + v^{ijIJ}) (u_{ij}{}^{KL} + v_{ijKL}) \quad (78)$$

via the non-linear metric ansatz (1). Using this, equation (2) takes the form

$$A_{mnp} = -\frac{1}{8\sqrt{2}} (i\Delta g_{pq}) K_{mn}{}^{IJ} K^{qKL} (u^{ij}{}_{IJ} - v^{ijIJ}) (u_{ij}{}^{KL} + v_{ijKL}). \quad (79)$$

⁵The parametrisation of ϕ_{IJKL} used in this paper coincides with that defined in [31, 32] by taking $\lambda \rightarrow \frac{1}{2\sqrt{2}}\lambda$. Thus, the values of c , s and v coincide precisely with those given in [31].

From the metric ansatz [11]

$$g^{mn} = \frac{\Delta}{6} \left\{ [6b_1 - b_2(\xi + 3)] \mathring{g}^{mn} + b_2(21 + \xi) \hat{\xi}^m \hat{\xi}^n \right\}. \quad (80)$$

Equivalently,

$$\Delta g_{mn} = \left\{ \frac{6}{6b_1 - b_2(\xi + 3)} \left(\mathring{g}_{mn} - \hat{\xi}_m \hat{\xi}_n \right) + \frac{1}{b_1 + 3b_2} \hat{\xi}_m \hat{\xi}_n \right\}. \quad (81)$$

Substituting the G_2 invariant stationary point values given in equations (70)–(72) and using (77) gives

$$g_{mn} = 3^{1/6} 10^{1/3} (15 - \xi)^{-1/3} \left\{ \left(\mathring{g}_{mn} - \hat{\xi}_m \hat{\xi}_n \right) + \frac{1}{36} (15 - \xi) \hat{\xi}_m \hat{\xi}_n \right\}, \quad (82)$$

which coincides with the metric given in equation (47).

In order to simplify the right hand side of equation (79), recall that $u^{ij}{}_{IJ}$ and v^{ijIJ} are components of E_7 matrices. In particular, they satisfy the relations [3]

$$u^{ij}{}_{IJ} u_{ij}{}^{KL} - v_{ijIJ} v^{ijKL} = \delta_{IJ}^{KL}, \quad (83)$$

$$u^{ij}{}_{IJ} v_{ijKL} - v_{ijIJ} u^{ij}{}_{KL} = 0. \quad (84)$$

These relations can be explicitly verified for the components of the $E_7/SU(8)$ coset element given in equations (67) and (69) by using identities (145)–(150). Now, the expression for the E_7 matrix components in equation (79)

$$\begin{aligned} (u^{ij}{}_{IJ} - v^{ijIJ}) (u_{ij}{}^{KL} + v_{ijKL}) &= u^{ij}{}_{IJ} u_{ij}{}^{KL} - v^{ijIJ} v_{ijKL} + u^{ij}{}_{IJ} v_{ijKL} - v^{ijIJ} u_{ij}{}^{KL}, \\ &= u^{ij}{}_{IJ} u_{ij}{}^{KL} - (u^{ij}{}_{KL} u_{ij}{}^{IJ} - \delta_{KL}^{IJ}) + u^{ij}{}_{KL} v_{ijIJ} - v^{ijIJ} u_{ij}{}^{KL}. \end{aligned}$$

Recalling that

$$u_{ij}{}^{IJ} = (u^{ij}{}_{IJ})^*, \quad v_{ijIJ} = (v^{ijIJ})^*, \quad (85)$$

the above expression reduces to

$$(u^{ij}{}_{IJ} - v^{ijIJ}) (u_{ij}{}^{KL} + v_{ijKL}) = \delta_{KL}^{IJ} + 2i \text{Im}(u^{ij}{}_{IJ} u_{ij}{}^{KL} - v^{ijIJ} u_{ij}{}^{KL}), \quad (86)$$

where the last term is the imaginary part of the expression in the bracket.

Using equations (145)–(150), it is straightforward to show that

$$\text{Im}(u^{ij}{}_{IJ} u_{ij}{}^{KL}) = -\frac{1}{4} p^2 q^2 (p^2 + q^2) \sin 2\alpha D_-^{IJKL}. \quad (87)$$

The expression on the left-hand-side of the above equation is antisymmetric under the exchange of the pair of indices $[IJ]$ and $[KL]$, since from equation (85) this operation is equivalent to complex conjugation of the expression in the bracket. Therefore, it should come as no surprise that the right-hand-side is given solely in terms of D_- . Furthermore,

$$\text{Im}(v^{ijIJ} u_{ij}{}^{KL}) = 4p^3 q^3 \sin^3 2\alpha \delta_{KL}^{IJ} - \frac{1}{2} p q (p^2 + q^2)^2 \sin \alpha C_-^{IJKL} - p^3 q^3 \sin \alpha \cos^2 \alpha D_+^{IJKL}, \quad (88)$$

which is indeed symmetric under the exchange of the pairs of indices $[IJ]$ and $[KL]$ as expected from equation (84).

Using equations (13), (40), (41) and (52) we derive

$$K_{ab}{}^{IJ}K_c{}^{KL}C_+^{IJKL} = -\frac{16}{3}\delta_{cd}^{ab}\sqrt{(21+\xi)(3-\xi)}\hat{\xi}^d = -\frac{16}{3}\delta_{cd}^{ab}\xi^d, \quad (89)$$

$$K_{ab}{}^{IJ}K_c{}^{KL}C_-^{IJKL} = 16S_{abc}, \quad (90)$$

$$K_{ab}{}^{IJ}K_c{}^{KL}D_+^{IJKL} = -\frac{8}{3}(9+\xi)S_{abc} + 8(21+\xi)\hat{\xi}^d\hat{\xi}_{[a}S_{bc]d} + \frac{4}{9}\xi^d\epsilon_{abcdefg}S^{efg}, \quad (91)$$

$$K_{ab}{}^{IJ}K_c{}^{KL}D_-^{IJKL} = -\frac{8}{3}(3-\xi)S_{abc} + 8(21+\xi)\hat{\xi}^d\hat{\xi}_{[a}S_{bc]d} - \frac{16}{3}(21+\xi)\hat{\xi}^d\hat{\xi}_cS_{abd} - \frac{4}{9}\xi^d\epsilon_{abcdefg}S^{efg}. \quad (92)$$

The first two relations are easily seen to be consistent with equations (24) and (35). Observe also that the last expression is *not* fully anti-symmetric in the indices $[abc]$.

With the use of the above relations, the expression for the 3-form potential, (79), reduces to

$$\begin{aligned} A_{mnp} = \frac{1}{18\sqrt{2}}\Delta g_{pq}\hat{g}^{qr} \left\{ 6\left((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3\right)\hat{S}_{mnr} \right. \\ \left. - 18(21+\xi)(f_1 - 2f_2)\hat{\xi}^s\hat{\xi}_{[m}\hat{S}_{nr]s} + 12(21+\xi)f_1\hat{\xi}^s\hat{\xi}_r\hat{S}_{mns} \right. \\ \left. + \sqrt{(21+\xi)(3-\xi)}(f_1 + 2f_2)\hat{\xi}^s\hat{\eta}_{mnrstuv}\hat{S}^{stuv} \right\}, \quad (93) \end{aligned}$$

where f_1, f_2, f_3 are defined in equations (74)–(76). The quantities $\hat{\xi}^m, \hat{S}_{mnp}$ and $\hat{\eta}_{mnpqrst}$ are constructed with the round S^7 vielbein and they are raised/lowered with the round S^7 metric, as emphasised earlier.

Inserting the expression for the metric, equation (81), found from the non-linear metric ansatz,

$$\begin{aligned} A_{mnp} = \frac{1}{18\sqrt{2}[6b_1 - b_2(\xi + 3)](b_1 + 3b_2)} \left\{ 6(b_1 + 3b_2)\delta_p^q - (21+\xi)b_2\hat{\xi}_p\hat{\xi}^q \right\} \\ \times \left\{ 6\left((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3\right)\hat{S}_{mnq} \right. \\ \left. - 18(21+\xi)(f_1 - 2f_2)\hat{\xi}^r\hat{\xi}_{[m}\hat{S}_{nq]r} + 12(21+\xi)f_1\hat{\xi}^r\hat{\xi}_q\hat{S}_{mnr} \right. \\ \left. + \sqrt{(21+\xi)(3-\xi)}(f_1 + 2f_2)\hat{\xi}^r\hat{\eta}_{mnpqrstu}\hat{S}^{stuv} \right\}, \quad (94) \end{aligned}$$

where b_1 and b_2 are defined in equation (73). Expanding out the terms in the expression above gives

$$\begin{aligned} A_{mnp} = \frac{1}{3\sqrt{2}[6b_1 - b_2(\xi + 3)]} \left\{ 6\left((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3\right)\hat{S}_{mnp} \right. \\ \left. - 18(21+\xi)(f_1 - 2f_2)\hat{\xi}^r\hat{\xi}_{[m}\hat{S}_{np]r} \right. \\ \left. + \sqrt{(21+\xi)(3-\xi)}(f_1 + 2f_2)\hat{\xi}^q\hat{\eta}_{mnpqrst}\hat{S}^{rst} \right\} \\ + \frac{\sqrt{2}(21+\xi)}{[6b_1 - b_2(\xi + 3)](b_1 + 3b_2)} [2b_1f_1 + b_2(2f_1 - 4f_2 - f_3)]\hat{\xi}^q\hat{\xi}_p\hat{S}_{mnq}. \quad (95) \end{aligned}$$

Let us consider the coefficient of the term that is not totally antisymmetric in the indices $[mnp]$,

$$2b_1f_1 + b_2(2f_1 - 4f_2 - f_3) = \frac{1}{2}\sin\alpha vcs^2 [(c^3 + v^3s^3) - (c + vs)(c^2 - vcs + v^2s^2)] = 0,$$

where in the first equality we have simply substituted in the definitions of b_1, b_2, f_1, f_2 and f_3 using equations (73)–(76). The vanishing of the non-antisymmetric term even away from the stationary point is expected from the general argument of [1], where it is shown that the 3-form potential as defined by the non-linear flux ansatz is totally antisymmetric by the E_7 properties of $u^{ij}{}_{IJ}$ and v^{ijIJ} . We now have a totally antisymmetric expression for the 3-form potential

$$A_{mnp} = \frac{1}{3\sqrt{2}[6b_1 - b_2(\xi + 3)]} \left\{ 6 \left((3 - \xi)f_1 - 2(9 + \xi)f_2 + 6f_3 \right) \mathring{S}_{mnp} \right. \\ \left. - 18(21 + \xi)(f_1 - 2f_2) \hat{\xi}^r \hat{\xi}_{[m} \mathring{S}_{np]r} \right. \\ \left. + \sqrt{(21 + \xi)(3 - \xi)} (f_1 + 2f_2) \hat{\xi}^q \mathring{\eta}_{mnpqrst} \mathring{S}^{rst} \right\}. \quad (96)$$

This is only defined up to gauge transformations, hence to make a comparison with the known G_2 invariant solution, we calculate its field strength. Using equations (33), (34) and (36), the field strength of the potential above,

$$F_{mnpq} = 4\mathring{D}_{[m} A_{npq]},$$

is

$$F_{mnpq} = \\ = \frac{4\sqrt{2}m_7}{6b_1 - b_2(3 + \xi)} \left\{ (f_3 - 4f_2) \mathring{\eta}_{mnpqrst} \mathring{S}^{rst} \right. \\ \left. - \frac{2(21 + \xi)}{3[6b_1 - b_2(3 + \xi)]} \left[b_2(f_1 - f_2)(\xi - 27) - 3b_1(f_1 - 4f_2) + 9b_2(3f_1 - 4f_2) \right] \hat{\xi}_{[m} \mathring{\eta}_{npq]rstu} \hat{\xi}^r \mathring{S}^{stu} \right. \\ \left. - \frac{4\sqrt{(21 + \xi)(3 - \xi)}}{6b_1 - b_2(3 + \xi)} \left[b_2(f_1 - f_2)(51 - \xi) + 3b_1(f_1 - 4f_2) - 3b_2(17f_1 - 16f_2 - f_3) \right] \mathring{S}_{[mnp} \hat{\xi}_{q]} \right\}. \quad (97)$$

Substituting relations (77), valid at the G_2 stationary point, we get

$$F_{mnpq} = \frac{32\sqrt{2}(f_1 - f_2)}{b_2(15 - \xi)} m_7 \left\{ \mathring{\eta}_{mnpqrst} \mathring{S}^{rst} - \frac{(21 + \xi)}{12(15 - \xi)} \left(\xi - 27 + \frac{18f_1}{f_1 - f_2} \right) \hat{\xi}_{[m} \mathring{\eta}_{npq]rstu} \hat{\xi}^r \mathring{S}^{stu} \right. \\ \left. + \frac{\sqrt{(21 + \xi)(3 - \xi)}}{2(15 - \xi)} \left(\xi - 51 + \frac{18f_1}{f_1 - f_2} \right) \mathring{S}_{[mnp} \hat{\xi}_{q]} \right\}. \quad (98)$$

Using equations (70)–(72), the expression above reduces to

$$F_{mnpq} = \frac{4 \times 3^{-1/4}}{15 - \xi} m_7 \left\{ \mathring{\eta}_{mnpqrst} \mathring{S}^{rst} - \frac{(21 + \xi)(\xi - 27 + 12\sqrt{3})}{12(15 - \xi)} \hat{\xi}_{[m} \mathring{\eta}_{npq]rstu} \hat{\xi}^r \mathring{S}^{stu} \right. \\ \left. + \sqrt{(21 + \xi)(3 - \xi)} \frac{(\xi - 51 + 12\sqrt{3})}{2(15 - \xi)} \mathring{S}_{[mnp} \hat{\xi}_{q]} \right\}. \quad (99)$$

This is in perfect agreement with the flux of the G_2 invariant solution [11] given in equation (48). It is remarkable that there is not only an agreement with the general structure, but also the precise coefficients.

4 The $SU(4)^-$ invariant solution

4.1 The $SU(4)^-$ invariant solution of eleven-dimensional supergravity

The $SU(4)^-$ invariant solution [33] is a compactification of eleven-dimensional supergravity to a maximally symmetric four-dimensional spacetime with the internal space given by a stretched $U(1)$ fibration over CP^3 . In [33], the solution was expressed in terms of structures on CP^3 . Here, in order to compare the $SU(4)^-$ invariant solution with the result given by the non-linear ansätze, we express the $SU(4)^-$ invariant solution in terms of geometrical quantities defined on a round S^7 .

The antiself-dual $SO(8)$ tensor Y_{IJKL}^- satisfying [11]

$$Y_{IJMN}^- Y_{MNKL}^- = 8\delta_{IJ}^{KL} - 8F_{[I}^- [^K F_{J]}^{-L}], \quad (100)$$

$$Y_{IJKL}^- Y_{MNPQ}^- Y_{PQKL}^- = 16Y_{IJPQ}^- \quad (101)$$

preserves $SU(4)^-$. The antisymmetric tensor F_{IJ}^- is an almost complex structure,

$$F_I^- K F_K^- J = -\delta_I^J. \quad (102)$$

Using the properties of Y_{IJKL}^- and F_{IJ}^- , it is straightforward to show that

$$Y_{MIJK}^- F_L^{-M} = Y_{M[IJK}^- F_L^{-M}], \quad (103)$$

$$Y_{MIJK}^- F_L^{-M} = -\frac{1}{4!} \epsilon_{IJKLPQRS} Y_{MPQR}^- F_S^{-M}, \quad (104)$$

$$F_{[IJ}^- F_{KL}^-] = -\frac{1}{4!} \epsilon_{IJKLPQRS} F_{PQ}^- F_{RS}^-. \quad (105)$$

The $SO(8)$ objects can be used to define the $SO(7)$ tensors

$$K_a = \frac{1}{4} K_a^{IJ} F_{IJ}^-, \quad K_{ab} = \frac{1}{4} K_{ab}^{IJ} F_{IJ}, \quad T_{abc} = \frac{1}{16} K_{[ab}^{IJ} K_{c]}^{KL} Y_{IJKL}^-, \quad (106)$$

where K_a^{IJ} and K_{ab}^{IJ} have been defined in equation (11). Using the relations given in appendix C, the following identities hold

$$K_a K_a = 1, \quad K_a K_{ab} = 0, \quad K_{ac} K_{cb} = K_a K_b - \delta_{ab}, \quad (107)$$

$$K_a T_{abc} = 0, \quad T^{acd} T_{bcd} = 4(\delta_b^a - K^a K_b), \quad \epsilon^{abcdefg} K_d K_{eh} T_{hfg} = -6T^{abc}. \quad (108)$$

Furthermore, using equation (9)

$$\mathring{D}_a K_b = -m_7 K_{ab}, \quad (109)$$

$$\mathring{D}_a T_{bcd} = \frac{1}{6} m_7 \epsilon_{abcdefg} T^{efg}. \quad (110)$$

In terms of the tensors K_a and T_{abc} , the internal metric of the $SU(4)^-$ invariant solution is given by⁶

$$g_{mn} = 2^{-1/3} (\mathring{g}_{mn} + \mathring{K}_m \mathring{K}_n), \quad (111)$$

where as before \mathring{g}_{mn} is the round S^7 metric and

$$\mathring{K}_m = \mathring{e}_m^a K_a$$

⁶As before, we have fixed the allowed arbitrary scaling (5) in anticipation of the form of the metric given by the non-linear ansatz.

is defined with respect to the siebenbein on the round S^7 .

Using equations (107) and (109), the Ricci tensor of this metric is given by

$$R_{mn} = \overset{\circ}{R}_{mn} + 2m_7^2 \overset{\circ}{g}_{mn} - 20m_7^2 \overset{\circ}{K}_m \overset{\circ}{K}_n. \quad (112)$$

The expression for the Ricci tensor of the round S^7 metric is given in equation (20).

The internal flux of the $SU(4)^-$ invariant solution is

$$F_{mnpq} = \frac{1}{3} m_7 \overset{\circ}{\eta}_{mnpqrst} \overset{\circ}{T}^{rst}. \quad (113)$$

To verify that the Einstein equations, (16) and (17), are satisfied it is useful to note that

$$F_{mpqr} F^{npqr} = 48 \times 2^{4/3} m_7^2 (\delta_m^n + K_m K^n), \quad (114)$$

where we have used equations (108). On the left-hand side of the above equation, the indices have been raised with inverse of g_{mn} given in equation (111).

Using the expression for the Ricci tensor in the internal direction, (112), and equation (114), it is straightforward to verify that g_{mn} and F_{mnpq} solve the Einstein equations, (16) and (17), with

$$m_4^2 = \frac{16}{3} m_7^2, \quad \mathfrak{f}_{ER}^2 = 32 m_7^2. \quad (115)$$

With the above value for \mathfrak{f}_{ER} , the equation of motion for F_{mnpq} , (18), is also satisfied.

4.2 The $SU(4)^-$ invariant stationary point of gauged supergravity

The $SU(4)^-$ invariant stationary point of maximal gauged supergravity is obtained for a purely pseudoscalar expectation value ϕ_{IJKL} of the form [31]

$$\phi_{IJKL} = \frac{1}{2} i \lambda Y_{IJKL}^-, \quad (116)$$

where Y_{IJKL}^- is an antiself-dual object satisfying the properties presented in equations (100)–(102).

Using equation (101), it is simple to show that for $n > 0$,

$$(Y^- Y^-)^n_{IJKL} = 2^{4(n-1)} (Y^- Y^-)_{IJKL}, \quad (117)$$

where $(Y^- Y^-)_{IJKL}$ denotes a contraction of the form $Y_{IJMN}^- Y_{MNKL}^-$.

As described in section 3.2, it is fairly straightforward to show that

$$u_{IJ}{}^{KL} = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} (Y^- Y^-)^n_{IJKL}, \quad (118)$$

Using equations (117) and (100), the above expression reduces to

$$\begin{aligned} u_{IJ}{}^{KL} &= \delta_{IJ}^{KL} + \sum_{n=1}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} 2^{4n-1} \left(\delta_{IJ}^{KL} - F_{[I}^- [^K F_{J]}^-]^L \right) \\ &= \delta_{IJ}^{KL} + \frac{1}{2} \left(\sum_{n=0}^{\infty} \frac{(2\lambda)^{2n}}{(2n)!} - 1 \right) \left(\delta_{IJ}^{KL} - F_{[I}^- [^K F_{J]}^-]^L \right) \\ &= \delta_{IJ}^{KL} + \frac{1}{2} (\cosh 2\lambda - 1) \left(\delta_{IJ}^{KL} - F_{[I}^- [^K F_{J]}^-]^L \right) \end{aligned} \quad (119)$$

Defining $c = \cosh(2\lambda)$ as before, and observing that the expression is real,

$$u^{IJ}{}_{KL} = \frac{1}{2}(c+1)\delta_{KL}^{IJ} - \frac{1}{2}(c-1)F_{[K}^{-}[F_L^{-]J]}. \quad (120)$$

Similarly,

$$\begin{aligned} v^{IJKL} &= -iY_{IJMN}^- \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!} (Y^- Y^-)^n{}_{MNKL} \\ &= -\frac{1}{2}iY_{IJMN}^- \left(\lambda + \frac{1}{32} \sum_{n=1}^{\infty} \frac{(2\lambda)^{2n+1}}{(2n+1)!} (Y^- Y^-)_{MNKL} \right) \\ &= -\frac{1}{2}iY_{IJMN}^- \left(\lambda + \frac{1}{32} [\sinh(2\lambda) - 2\lambda] (Y^- Y^-)_{MNKL} \right) \\ &= -\frac{1}{4}i \sinh(2\lambda) Y_{IJKL}^-, \end{aligned} \quad (121)$$

where we have used equation (101) in the final equality in the equation above. Defining $s = \sinh(2\lambda)$,

$$v^{IJKL} = -\frac{1}{4}isY_{IJKL}^-. \quad (122)$$

It is simple to verify that the u and v as given in equations (120) and (122) satisfy the E_7 relations, equations (83) and (84).

From the metric ansatz [11],

$$\Delta^{-1}g^{mn} = \left\{ c^2 \mathring{g}^{mn} - s^2 \mathring{K}^m \mathring{K}^n \right\}. \quad (123)$$

Equivalently,

$$\Delta g_{mn} = c^{-2} \left\{ \mathring{g}_{mn} + s^2 \mathring{K}_m \mathring{K}_n \right\}. \quad (124)$$

The $SU(4)^-$ invariant stationary point is given by [31]

$$c^2 = 2, \quad s^2 = 1. \quad (125)$$

Substituting these values into the expression above and taking the determinant of the resulting expression gives

$$\Delta = 2^{-2/3}. \quad (126)$$

Hence, the metric is of the form

$$g_{mn} = 2^{-1/3} \left\{ \mathring{g}_{mn} + \mathring{K}_m \mathring{K}_n \right\}, \quad (127)$$

which agrees with that given in equation (111).

Substituting the expression for u and v given in equations (120) and (122), and the form of the metric given in equation (124) into equation (79), it is simple to show that

$$A_{mnp} = -\frac{1}{\sqrt{2}}(s/c)\mathring{T}_{mnp}, \quad (128)$$

where we have used the first equation in (108). Now, using equation (110), the field strength is simply

$$F_{mnpq} = -\frac{\sqrt{2}}{3}(s/c)m_7\mathring{\eta}_{mnpqrst}\mathring{T}^{rst}. \quad (129)$$

Substituting the values of c and s given in equation (125) gives

$$F_{mnpq} = -\frac{1}{3}m_7\overset{\circ}{\eta}_{mnpqrst}\overset{\circ}{I}^{rst}. \quad (130)$$

Note that the Einstein equations (16) and (17) and the equation of motion for the flux (18) are satisfied regardless of the overall sign of the flux. Thus, again, we have precise agreement with the flux of the $SU(4)^-$ solution given in equation (113).

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A $\text{SO}(7)^\pm$ invariant solutions

For completeness we here reproduce the metric and flux of the $\text{SO}(7)^\pm$ invariant solutions, even though these are simpler than the ones discussed in the text. The relevant solutions can be found in analogy with the general metric and flux of the G_2 invariant family, given in equations (81) and (97), and by restricting the scalar fields in (51) to $\alpha = 0$ and $\alpha = \pi/2$, respectively.

The $\text{SO}(7)^+$ invariant stationary point of maximally gauged supergravity is given by [31]

$$c^2 = \frac{1}{2}(3/\sqrt{5} + 1), \quad s^2 = \frac{1}{2}(3/\sqrt{5} - 1), \quad v = 1. \quad (131)$$

In particular, these imply that f_1, f_2, f_3 as defined in equations (74)–(76) vanish. It immediately follows that

$$F_{mnpq} = 0, \quad (132)$$

as expected. The metric is

$$\Delta g_{mn} = \frac{6 \times 5^{1/4}}{9 - \xi} \left\{ (\mathring{g}_{mn} - \hat{\xi}_m \hat{\xi}_n) + \frac{(9 - \xi)}{30} \hat{\xi}_m \hat{\xi}_n \right\}. \quad (133)$$

This is the solution of [34]; see also [9, 11]. In particular, in reference [11], the solution is given in the form

$$\Delta g_{mn} = \frac{30\gamma^{-1/2}}{9 - \xi} \left\{ (\mathring{g}_{mn} - \hat{\xi}_m \hat{\xi}_n) + \frac{(9 - \xi)}{30} \hat{\xi}_m \hat{\xi}_n \right\}, \quad (134)$$

which agrees with metric (133) for

$$\gamma = 5^{3/2}.$$

Similarly, the $\text{SO}(7)^-$ stationary point is given by

$$c^2 = \frac{5}{4}, \quad s^2 = \frac{1}{4}, \quad v = 0. \quad (135)$$

Since $v = 0$, b_2 as defined in equation (73) vanishes and the metric is given by the round S^7 metric

$$\Delta g_{mn} = c^{-3} \mathring{g}_{mn}. \quad (136)$$

Moreover the flux for the $\text{SO}(7)^-$ family is

$$F_{mnpq} = \frac{\sqrt{2}}{3} (s/c) m_\gamma \mathring{\eta}_{mnpqrst} \mathring{S}^{rst}. \quad (137)$$

This is consistent with the Englert solution [30]; see also [9, 11]. In reference [11], the solution is expressed as

$$\Delta g_{mn} = \gamma^{-1/2} \mathring{g}_{mn}, \quad (138)$$

$$F_{mnpq} = \frac{1}{3\sqrt{2}} \gamma^{-1/6} m_\gamma \mathring{\eta}_{mnpqrst} \mathring{S}^{rst}, \quad (139)$$

which agree with equations (136) and (137) at the stationary point for

$$\gamma^{1/3} = 5/4.$$

B Useful G_2 identities

In this appendix, we derive identities relating the contraction of G_2 invariants C_\pm and D_\pm , adopting the shorthand notation (53) throughout. In deriving these identities it is useful to observe that viewed as E_7 matrices, C_\pm and D_\pm are generators of an $SU(1,1)$ subalgebra of E_7 . This is the unique subalgebra of E_7 that commutes with G_2 [11], cf.

$$\sigma^1 \sim \begin{pmatrix} 0 & C_+ \\ C_+ & 0 \end{pmatrix}, \quad \sigma^2 \sim \begin{pmatrix} 0 & -iC_- \\ iC_- & 0 \end{pmatrix}, \quad \sigma^3 \sim \begin{pmatrix} D_+ & 0 \\ 0 & -D_+ \end{pmatrix}. \quad (140)$$

Thus,

$$\left[\begin{pmatrix} D_+ & 0 \\ 0 & -D_+ \end{pmatrix}, \begin{pmatrix} 0 & C_+ \\ C_+ & 0 \end{pmatrix} \right] \propto \begin{pmatrix} 0 & -iC_- \\ iC_- & 0 \end{pmatrix}, \quad (141)$$

which implies that

$$(C_+C_-C_+ + 4D_+) \propto C_-. \quad (142)$$

Consistency with equation (22) fixes the constant of proportionality:

$$C_+C_-C_+ = -4(C_- + D_+). \quad (143)$$

Similarly,

$$C_-C_+C_- = -4(C_- - D_+). \quad (144)$$

Using equations (22), (23), (143) and (144), it is straightforward to prove the following identities:

$$C_+D_+ = 4C_- + 2D_-, \quad D_+C_+ = 4C_- - 2D_- \quad (145)$$

$$C_-D_+ = 4C_+ + 2D_-, \quad D_+C_- = 4C_+ - 2D_-, \quad (146)$$

$$C_+D_- = 8C_- + 4D_+ + 2D_-, \quad D_-C_+ = -8C_- - 4D_+ + 2D_-, \quad (147)$$

$$C_-D_- = -8C_+ + 4D_+ - 2D_-, \quad D_-C_- = 8C_+ - 4D_+ - 2D_-, \quad (148)$$

$$D_+D_+ = 48 + 8D_+, \quad D_-D_+ = 16C_+ + 16C_- + 4D_-, \quad (149)$$

$$D_+D_- = -16C_+ - 16C_- + 4D_-, \quad D_-D_- = -96 - 16C_+ + 16C_- - 8D_+. \quad (150)$$

C Seven-dimensional Γ -matrix identities

For the reader's convenience, here we give a list of useful Γ -matrix identities, see also the appendices of [5, 10]. The seven-dimensional, Euclidean 8×8 Γ^a -matrices, where a is a seven-dimensional flat index, satisfy

$$\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}. \quad (151)$$

The Clifford algebra admits a Majorana representation, which in our conventions corresponds to a purely imaginary representation of the Γ -matrices. We use a representation in which all Γ -matrices are hermitian and antisymmetric; or, equivalently, in our representation the charge conjugation matrix is the identity matrix. Moreover,

$$\Gamma^{abcdefg} = -i\epsilon^{abcdefg}\mathbf{1}, \quad (152)$$

where

$$\Gamma^{abcdefg} = \Gamma^{[a \dots \Gamma^g]}$$

and $\mathbf{1}$ is the 8×8 identity matrix.

The Γ^a can be regarded as seven out of the eight components of $Spin(8)$ gamma-matrices in a Majorana-Weyl representation. In this way, one can use $SO(8)$ triality to prove the following important relations [5, 10]

$$\Gamma_{[AB}\Gamma_{CD]}^a = \frac{1}{24}\epsilon_{ABCDEFGH}\Gamma_{EF}\Gamma_{GH}^a, \quad (153)$$

$$\Gamma_{[AB}\Gamma_{CD]}^{ab} = \frac{1}{24}\epsilon_{ABCDEFGH}\Gamma_{EF}\Gamma_{GH}^{ab}, \quad (154)$$

$$\Gamma_{[AB}\Gamma_{CD]}^{[a}\Gamma_{CD]}^{bc]} = -\frac{1}{24}\epsilon_{ABCDEFGH}\Gamma_{EF}^{[a}\Gamma_{GH}^{bc]}, \quad (155)$$

$$\Gamma_{[AB}\Gamma_{CD]}^{[a}\Gamma_{CD]}^{bc]} = \frac{1}{24}i\epsilon^{abcdefg}\Gamma_{[AB}\Gamma_{CD]}^{de}\Gamma_{CD]}^{fg}. \quad (156)$$

The uppercase Latin indices are spinor indices and run from 1 to 8.

Further Γ -matrix identities can be proved using the Fierz identity, which in Euclidean seven-dimensions takes the form

$$X_{AB}Y_{CD} = \frac{1}{8}\delta_{BC}(XY)_{AD} - \frac{1}{8}\Gamma_{BC}^a(X\Gamma_{BC}^a Y)_{AD} + \frac{1}{16}\Gamma_{BC}^{ab}(X\Gamma_{BC}^{ab} Y)_{AD} - \frac{1}{48}\Gamma_{BC}^{abc}(X\Gamma_{BC}^{abc} Y)_{AD}, \quad (157)$$

where X and Y are arbitrary 8×8 matrices. The identity above is obtained by noting that

$$\{\delta_{AB}, \Gamma_{AB}^a, \Gamma_{AB}^{ab}, \Gamma_{AB}^{abc}\}$$

span the vector space of 8×8 matrices.

The Fierz identity can be used to show

$$\Gamma_{AB}\Gamma_{CD}^a = \Gamma_{[AB}\Gamma_{CD]}^a - 2\delta_{CD}^{AB}, \quad (158)$$

$$\Gamma_{AB}^a\Gamma_{CD}^{ab} + \Gamma_{CD}^a\Gamma_{AB}^{ab} = 2\Gamma_{[AB}\Gamma_{CD]}^{ab}, \quad (159)$$

$$\Gamma_{AB}^a\Gamma_{CD}^{ab} - \Gamma_{CD}^a\Gamma_{AB}^{ab} = -4\left(\delta_{C[A}\Gamma_{B]D}^b - \delta_{D[A}\Gamma_{B]C}^b\right), \quad (160)$$

$$\Gamma_{AB}^{ab}\Gamma_{CD}^{ab} = 2\Gamma_{AB}^a\Gamma_{CD}^a + 16\delta_{CD}^{AB}, \quad (161)$$

$$\Gamma_{AB}^{c(a}\Gamma_{CD}^{b)c} = \frac{6}{5}\Gamma_{[AB}\Gamma_{CD]}^{c(a}\Gamma_{CD]}^{b)c} - \Gamma_{AB}^{(a}\Gamma_{CD}^{b)} + \frac{1}{5}\delta^{ab}\Gamma_{AB}^c\Gamma_{CD}^c - \frac{8}{5}\delta^{ab}\delta_{CD}^{AB}, \quad (162)$$

$$\Gamma_{AB}^{c[a}\Gamma_{CD}^{b]c} = -\Gamma_{AB}^{[a}\Gamma_{CD]}^{b]} - 2\left(\delta_{C[A}\Gamma_{B]D}^{ab} - \delta_{D[A}\Gamma_{B]C}^{ab}\right), \quad (163)$$

$$\Gamma_{[AB}\Gamma_{CD]}^{ca}\Gamma_{CD]}^{bc} = 5\Gamma_{[AB}\Gamma_{CD]}^b - \delta^{ab}\Gamma_{AB}^c\Gamma_{CD}^c. \quad (164)$$

Furthermore, it is also useful to note that (see appendix of [10])

$$\Gamma_{[AB}\Gamma_{CD]}^{ab}\Gamma_{CD]}^c \Big|_- = \Gamma_{[AB}\Gamma_{CD]}^{[ab}\Gamma_{CD]}^c, \quad (165)$$

$$\Gamma_{[AB}\Gamma_{CD]}^{ab}\Gamma_{CD]}^{cd} \Big|_- = \Gamma_{[AB}\Gamma_{CD]}^{[ab}\Gamma_{CD]}^{cd}, \quad (166)$$

where the vertical bar $|_-$ denotes projection to the antiself-dual part.

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