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Testing the nonlinear flux *Ansatz* for maximal supergravity

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We put to test the recently proposed nonlinear flux *Ansatz* for maximal supergravity in 11 dimensions, which gives the seven-dimensional flux in terms of the scalars and pseudoscalars of maximal $N = 8$ supergravity, by considering a number of nontrivial solutions of gauged supergravity for which the higher-dimensional solutions are known. These include the G_2 and $SU(4)^-$ invariant stationary points. The examples considered constitute a very nontrivial check of the *Ansatz*, which it passes with remarkable success.

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I. INTRODUCTION

Recently [1], a simple nonlinear flux *Ansatz* giving the seven-dimensional components of the 3-form potential of 11-dimensional supergravity [2] in terms of the scalars and pseudoscalars of maximal (gauged) $N = 8$ supergravity [3] has been proposed. This result arose from an attempt to understand the embedding of a recently discovered continuous family of inequivalent maximal ($N = 8$) gauged supergravities in four dimensions [4]. The emergence of this new family of theories follows from the electric-magnetic duality of the ungauged $N = 8$ theory [5] and can thus be understood in terms of the freedom to rotate between how one chooses to define electric and magnetic vector fields [6]. The inequivalence of the resulting theories is confined to the gauged theory, because in the ungauged theory electric-magnetic duality renders all such theories equivalent. From an 11-dimensional perspective, the electric vector fields arise from the off-diagonal elfbein components (graviphoton), while the magnetic vector fields emerge from particular components of the 3-form potential.

A standard method by which new theories are obtained in supergravity is reducing a higher dimensional theory on some group manifold or coset space. The problem of determining whether a coset space reduction is consistent, in the sense that every solution of the lower-dimensional theory can be uplifted to a solution of the higher-dimensional theory, is a subtle one. In fact, the expectation is that such reductions are, in general, inconsistent [7]. A notable exception to this expectation is the consistency of the seven-sphere reduction of 11-dimensional supergravity [8,9]. Central to this result is a local SU(8) invariant reformulation of the 11-dimensional theory [10], which in the reduction on a seven-torus T^7 immediately reduces to the E_7 invariant theory of Cremmer and Julia [5], without the need to dualize tensors to scalars. This reformulation necessitates the introduction of new SU(8) covariant objects in 11 dimensions. The most significant such object is the generalized vielbein, which arises from the study of the supersymmetry transformation of the graviphoton and replaces the siebenbein in the reformulated theory. The intimate connection between the electric vector fields of the four dimensional theory and the graviphoton leads naturally to the nonlinear metric *Ansatz* [11],

$$\Delta^{-1} g^{mn}(x,y) = \frac{1}{8} K^{mIJ}(y) K^{nKL}(y)[(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})](x), \tag{1}$$

whereby the seven-dimensional metric g_{mn} is given in terms of the scalar and pseudoscalar fields of $N = 8$ supergravity, via the $E_{7(7)}$ matrix components $u(x)$ and $v(x)$ (see (49) below), and of the Killing vectors $K^m(y)$ on the seven-sphere S^7 [where the 11-dimensional coordinates are split as $z^M = (x^\mu, y^m)$]. We note that the above formula has been subjected to numerous tests and has proven its usefulness in other contexts, such as the AdS/CFT correspondence [11–18]. The proof of consistency in Ref. [8] also furnishes a formula for the 4-form field strength, modulo a subtlety that is resolved in Ref. [9]. However, this formula appears to be too cumbersome for practical applications.

The remarkable result of Ref. [1] is that there exists an object analogous to the generalized vielbein that arises from the supersymmetry transformation of the components of the 3-form potential from which the magnetic vector fields of the four-dimensional theory arise. This new generalized vielbein now replaces the components of the 3-form potential A_{mnp} along the seven directions in the local SU(8) invariant reformulation of the 11-dimensional theory. Furthermore, it leads to a nonlinear flux *Ansatz*, which complements the nonlinear metric *Ansatz* above [1],

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$$\sqrt{2}K^{pIJ}(y)K^{qKL}(y)[(u^{ij}{}_{IJ}+v^{ijIJ})(u_{ij}{}^{KL}+v_{ijKL})](x)A_{mnp}(x,y)=-iK_{mn}{}^{IJ}(y)K^{qKL}(y)[(u^{ij}{}_{IJ}-v^{ijIJ})(u_{ij}{}^{KL}+v_{ijKL})](x). \tag{2}$$

While this nonlinear flux *Ansatz* takes a surprisingly simple form, it is not a formula that can be found by asserting consistency with previous results. This is a major difference with the corresponding result for the $AdS_7 \times S^4$ compactification of maximal supergravity, where the nonlinear *Ansätze* can be directly substituted into the higher-dimensional field equations [19]; such a direct substitution is not possible for the $AdS_4 \times S^7$ compactification. Let us also mention that there exist partial results and uplift formulae for truncated versions of the maximal theory where the scalar sector is much simpler (see for example Refs. [20–27] and references therein). However, the formula above cannot be guessed from these. Its derivation is critically dependent on an analysis of the manifestly local formulation of 11-dimensional supergravity. Last but not least we wish to point out that in comparison with other theories, $N = 8$ supergravity is distinguished by an astonishingly rich variety of stationary points [28][1] that can now be explored by means of the new formula.

The nonlinear flux *Ansatz* given in Eq. (5.11) of Ref. [1] differs from the above expression by the factor of $\sqrt{2}$ on the left-hand side. In fact, as already pointed out there, this overall factor can so far not be determined from intrinsically Kaluza-Klein theoretic considerations matching the 11-dimensional 3-form potential with the four-dimensional gauge field. This is in marked contrast to the graviphoton, for which general Kaluza-Klein theory gives its precise relation to the four dimensional gauge field, by matching the non-Abelian interaction with the commutator of two Killing vector fields. However, as we will show here, this factor is universally and unambiguously the same for all solutions, and does follow by explicitly computing the 3-form potential using the above *Ansatz* for solutions of gauged supergravity for which the higher dimensional uplift solution is known (the factor $\sqrt{2}$ is most easily checked for the Englert solution [30]).

The aim of this paper, then, is to test the nonlinear flux *Ansatz* for a number of solutions of gauged supergravity. In order to make the paper self-contained we begin, in Sec. II, by describing the main conventions and definitions that are required, summarizing various known results. In addition, in Appendix C, we list some important Γ-matrix identities, most of which already appear in the appendices of Refs. [5,10], while some are new.

In Sec. III, we begin by describing the 11-dimensional G_2 invariant solution of Ref. [11]. Then, we consider the G_2 invariant stationary point of $N = 8$ supergravity [31,32] that uplifts to the aforementioned solution, rederiving the E_7 matrix components u and v that are parametrized by the scalars and pseudoscalars. These components are essential inputs in the nonlinear *Ansätze*. We calculate the 3-form potential using the nonlinear flux *Ansatz*, Eq. (2), verifying its total antisymmetry as is expected from the general argument in Ref. [1]. The field strength of this potential is then derived for the G_2 family of solutions. Substituting the G_2 stationary point values yields the flux of the G_2 invariant solution of the 11-dimensional theory with precise agreement.

As our next test, we consider the $SU(4)^-$ invariant solution of Ref. [33] in Sec. IV. We rewrite this solution of 11-dimensional supergravity in terms of geometric quantities defined on the seven-sphere. As in Sec. III, we derive the 3-form potential using the nonlinear flux *Ansatz* and confirm that at the stationary point [31,32] the associated field strength matches precisely with that of the 11-dimensional solution.

Furthermore, in Appendix A, we give the metric and the flux calculated from the nonlinear *Ansätze* with the scalars of the $SO(7)^\pm$ invariant family of maximal gauged supergravity [31,32]. These examples are simple enough for the reader to immediately match with the known $SO(7)^+$ [34] and $SO(7)^-$ [30] solutions of 11-dimensional supergravity and are thus included mainly for the reader's convenience.

II. PRELIMINARIES

In this paper, we follow the conventions of Ref. [10]. The bosonic field equations of 11-dimensional supergravity [2] read[2]

$$R_{MN} = \frac{1}{72} g_{MN} F^2_{PQRS} - \frac{1}{6} F_{MPQR} F_N{}^{PQR}, \tag{3}$$

$$E^{-1}\partial_M(EF^{MNPQ}) = \frac{\sqrt{2}}{1152} i\eta^{NPQR_1\ldots R_4 S_1\ldots S_4} F_{R_1\ldots R_4} F_{S_1\ldots S_4}, \tag{4}$$

where E is the determinant of the elfbein $E_M{}^A$. We note that solutions to these combined equations are only determined up to an overall constant scaling,

$$g_{MN} \to \lambda g_{MN}, \qquad F_{MNPQ} \to \lambda^{3/2} F_{MNPQ}. \tag{5}$$

Such a rescaling must be taken into account when comparing the various solutions given in the literature with the ones constructed from the nonlinear *Ansätze* (1) and (2).

[1] Whereas, for instance, maximal gauged supergravity in seven dimensions has only one nontrivial stationary point besides the trivial vacuum [29].

[2] Note that for consistency with Ref. [11], we use a negative curvature convention, i.e., $[D_M, D_N]V^P = -R^P{}_{QMN}V^Q$. Hence, the scalar curvature of a sphere is negative.

We emphasize that the normalization of all solutions is thus completely fixed by (1) and (2), once the trivial vacuum solution has been specified.

We are interested in solutions of the above equations that are obtained via a compactification to a four-dimensional maximally symmetric spacetime. The most general *Ansatz* for the elfbein that is consistent with this requirement is of the warped form,[3]

$$E_M{}^A(x,y) = \begin{pmatrix} \Delta^{-1/2}(y)\mathring{e}_\mu{}^\alpha(x) & 0 \\ 0 & e_m{}^a(y) \end{pmatrix}, \tag{6}$$

where x^μ are coordinates on the four-dimensional spacetime and y^m are coordinates on the compact seven-dimensional space; $\mathring{e}_\mu{}^\alpha(x)$ is the vierbein of the maximally symmetric four-dimensional spacetime and $e_m{}^a(y)$ is the siebenbein of the compact space. In particular, we assume the siebenbein to be that of a deformed round seven-sphere with the deformation parametrized by a matrix $S_a{}^b(y)$

$$e_m{}^a(y) = \mathring{e}_m{}^b(y) S_b{}^a(y), \tag{7}$$

where $\mathring{e}_m{}^a(y)$ is the siebenbein on the round seven-sphere with inverse radius m_7, and where

$$\Delta(y) \equiv \det S_a{}^b(y). \tag{8}$$

The presence of the warp factor in (6) is required by consistency with the supersymmetry transformation rules of the fields that would correspond with those of the maximal theory upon reduction to four dimensions [34].

The eight Killing spinors of S^7, η^I satisfy

$$\left(\mathring{D}_m + \frac{1}{2} i m_7 \mathring{e}_m{}^a \Gamma_a\right)\eta^I = 0, \tag{9}$$

where $\mathring{D}_m$ is the covariant derivative on the round seven-sphere and the Γ^a matrices are flat, Euclidean, antisymmetric and purely imaginary. In a Majorana representation of the Clifford algebra in seven Euclidean dimensions, the charge conjugation matrix that is used to define spinor conjugates, or raise and lower spinor indices, can be chosen to be the identity matrix. Here we make such a choice. Furthermore, it is useful to choose Killing spinors that are orthonormal,

$$\bar{\eta}^I \eta^J = \delta^{IJ}, \qquad \eta^I \bar{\eta}^I = \mathbf{1}, \tag{10}$$

where on the right-hand side of the second equation, $\mathbf{1}$ denotes the identity matrix with spinor indices.

These spinors can be used to define a set of vectors and 2-forms as follows:

$$K_a{}^{IJ} = i\bar{\eta}^I \Gamma^a \eta^J, \qquad K_{ab}{}^{IJ} = \bar{\eta}^I \Gamma^{ab} \eta^J, \tag{11}$$

$$K_m{}^{IJ} = \mathring{e}_m{}^a K_a{}^{IJ}, \qquad K_{mn}{}^{IJ} = \mathring{e}_m{}^a \mathring{e}_n{}^b K_{ab}{}^{IJ}. \tag{12}$$

In the following we will adopt the rule that the curved indices on Killing vectors and their derivatives are always lowered and raised with the *round* seven-sphere metric $\mathring{g}_{mn}$ and its inverse. It is now straightforward to show that

$$K_{ab}{}^{IJ} K_c{}^{IJ} = 0, \qquad K^{aIJ} K_b{}^{IJ} = 8\delta^a_b, \qquad K^{abIJ} K_{cd}{}^{IJ} = 16\delta^{ab}_{cd}. \tag{13}$$

Assuming the four-dimensional spacetime to be maximally symmetric implies that the only nonzero components of the field strength F_{MNPQ} are $F_{\mu\nu\rho\sigma}$ and F_{mnpq}. Following Ref. [35], we parametrize $F_{\mu\nu\rho\sigma}$ as follows,

$$F_{\mu\nu\rho\sigma} = i\mathfrak{f}_{\rm FR}\,\eta_{\mu\nu\rho\sigma}, \tag{14}$$

where $\eta_{\mu\nu\rho\sigma}$ is the alternating tensor in four dimensions.[4] The Bianchi identities imply that the Freund-Rubin parameter $\mathfrak{f}_{\rm FR}$ is a constant. Beware that switching to flat indices introduces y dependence,

$$F_{\alpha\beta\gamma\delta} = i\mathfrak{f}_{\rm FR}\Delta^2 \eta_{\alpha\beta\gamma\delta}. \tag{15}$$

Given an elfbein of the form given in Eq. (6) and using Eq. (14), it is fairly straightforward to show that the 11-dimensional Eqs. (3) and (4) reduce to [11],

$$R_\mu{}^\nu = \left(\frac{2}{3}\mathfrak{f}_{\rm FR}^2\Delta^4 + \frac{1}{72}F^2_{mnpq}\right)\delta^\nu_\mu, \tag{16}$$

$$R_m{}^n = -\frac{1}{6}F_{mpqr}F^{npqr} + \left(\frac{1}{72}F^2_{mnpq} - \frac{1}{3}\mathfrak{f}_{\rm FR}^2\Delta^4\right)\delta^n_m, \tag{17}$$

$$\mathring{D}_q(\Delta^{-1}F^{mnpq}) = \frac{1}{24}\sqrt{2}\mathfrak{f}_{\rm FR}\,\mathring{\eta}^{mnpqrst}F_{qrst}, \tag{18}$$

where seven-dimensional indices $m, n, p, \ldots$ are raised (lowered) with

$$g^{mn} = e_a{}^m e_b{}^n \delta^{ab} (g_{mn} = e_m{}^a e_n{}^b \delta_{ab}),$$

except in cases where the object is denoted with a circle $\circ$ on top, in which case indices are raised (lowered) with $\mathring{g}^{mn}$ ($\mathring{g}_{mn}$) analogously defined. Hence, $\mathring{\eta}_{mnpqrst}$ is the alternating tensor corresponding to the round seven-sphere metric $\mathring{g}_{mn}$ and its indices are raised with $\mathring{g}^{mn}$.

As is well known, the four-dimensional spacetime must be AdS_4. We choose to parametrize its radius such that

$$R_{\mu\nu} = 3m_4^2 g_{\mu\nu}. \tag{19}$$

Furthermore, for an S^7 of inverse radius m_7,

$$\mathring{R}_{mn} = -6m_7^2 \mathring{g}_{mn}. \tag{20}$$

[3]In general, of course, $e_m{}^a$ can also have x dependence. But here we are considering compactifications.

[4]Note that the conventions used in this paper are such that $\eta_{\mu\nu\rho\sigma}\eta^{\mu\nu\rho\sigma} = +4!$.

Thus, in our conventions, the S^7 compactification [36] is given by

$$m_4 = 2m_7, \qquad \mathfrak{f}_{\rm FR} = \pm 3\sqrt{2} m_7. \tag{21}$$

We repeat that the normalization of *all* solutions away from the trivial AdS_4 vacuum is fixed by the nonlinear *Ansätze*. Thus, they are all expressed in terms of a single dimensionful parameter m_7.

III. THE G_2 INVARIANT SOLUTION

A. The G_2 invariant solution of 11-dimensional supergravity

In order to write out the G_2 invariant solution, we must first define the geometrical quantities, respectively preserving the $SO(7)^+$ and $SO(7)^-$ subgroups of $SO(8)$ whose common subgroup is $G_2 = SO(7)^+ \cap SO(7)^-$. These are given in terms of the following self-dual C_+^{IJKL} and antiself-dual C_-^{IJKL} $SO(8)$ tensors, respectively, which satisfy the identities [34,37]

$$C_+^{IJMN} C_+^{MNKL} = 12\delta_{KL}^{IJ} + 4C_+^{IJKL}, \tag{22}$$

$$C_-^{IJMN} C_-^{MNKL} = 12\delta_{KL}^{IJ} - 4C_-^{IJKL}. \tag{23}$$

These tensors will also appear below in the parametrization of the scalar and pseudoscalar expectations in $N = 8$ supergravity.

The self-dual tensor C_+ can be used to define $SO(7)^+$ invariant quantities [34]

$$\xi_a = \frac{1}{16} C_+^{IJKL} K_{ab}{}^{IJ} K_b{}^{KL}, \tag{24}$$

$$\xi_{ab} = -\frac{1}{16} C_+^{IJKL} K_a{}^{IJ} K_b{}^{KL}, \tag{25}$$

$$\xi = \delta^{ab} \xi_{ab}. \tag{26}$$

These quantities satisfy the nontrivial identities [34]

$$\xi_a \xi_a = (21 + \xi)(3 - \xi), \tag{27}$$

$$\xi_{ab} = \frac{1}{6}(3 + \xi)\delta_{ab} - \frac{1}{6(3 - \xi)} \xi_a \xi_b, \tag{28}$$

$$\mathring{D}_c \xi_{ab} = \frac{1}{3} m_7 (\delta_{ab} \xi_c - \xi_{(a} \delta_{b)c}), \tag{29}$$

$$\mathring{D}_a \xi = 2m_7 \xi_a, \tag{30}$$

$$\mathring{D}_a \xi_b = m_7 (3 - \xi)\delta_{ab} - \frac{m_7}{3 - \xi} \xi_a \xi_b. \tag{31}$$

Hence, the variable ξ lies in the range $-21 < \xi < 3$, with the endpoints corresponding to the north and south poles of the seven sphere. Alternatively, in terms of the unit vector

$$\hat{\xi}_a = \frac{1}{\sqrt{(21 + \xi)(3 - \xi)}} \xi_a, \tag{32}$$

the last two equations become [34]

$$\mathring{D}_a \xi = 2m_7 \sqrt{(21 + \xi)(3 - \xi)} \hat{\xi}_a, \tag{33}$$

$$\mathring{D}_a \hat{\xi}_b = m_7 \sqrt{\frac{3 - \xi}{21 + \xi}} (\delta_{ab} - \hat{\xi}_a \hat{\xi}_b). \tag{34}$$

The antiself-dual C_-^{IJKL} can similarly be used to define the $SO(7)^-$ invariant tensor (alias "the parallelizing torsion" on S^7),

$$S_{abc} = \frac{1}{16} C_-^{IJKL} K_{[ab}^{IJ} K_{c]}^{KL}, \tag{35}$$

which satisfies the relations

$$\mathring{D}_a S_{bcd} = \frac{1}{6} m_7 \epsilon_{abcdefg} S^{efg}, \tag{36}$$

$$S^{[abc} S^{d]ef} = \frac{1}{4} \epsilon^{abcd[e}{}_{gh} S^{f]gh}, \tag{37}$$

$$S^{a[bc} S^{de]f} = \frac{1}{6} \epsilon^{bcde(a}{}_{gh} S^{f)gh}, \tag{38}$$

$$S^{abe} S_{cde} = 2\delta_{cd}^{ab} + \frac{1}{6} \epsilon^{ab}{}_{cdefg} S^{efg}. \tag{39}$$

These relations have been derived in Refs. [37,38]. There is a potential ambiguity in the sign of terms with S on the right-hand side of the equations above, which is fixed by requiring that C_- is antiself-dual and satisfies Eq. (23) (see Eqs. (3.6)–(3.17) of Ref. [37]). Equation (36) is derived using the Γ-matrix identity (C6) and (9).

The relations (24), (25), and (35) can be inverted to give the $SO(8)$ tensors in terms of the $SO(7)^+$ and $SO(7)^-$ geometric quantities [9,34,38],

$$\begin{aligned} C_+^{IJKL} = &-\frac{1}{12}(9 + \xi) K_a{}^{[IJ} K_a{}^{KL]} \\ &+ \frac{1}{4}(21 + \xi) \hat{\xi}^a \hat{\xi}^b K_a{}^{[IJ} K_b{}^{KL]} \\ &+ \frac{1}{12}\sqrt{(21 + \xi)(3 - \xi)} \hat{\xi}^a K_{ab}{}^{[IJ} K_b{}^{KL]}, \end{aligned} \tag{40}$$

$$C_-^{IJKL} = \frac{1}{2} S_{abc} K_{ab}{}^{[IJ} K_c{}^{KL]}. \tag{41}$$

In Appendix B, we explain that C_+, iC_- together with their symmetrized product iD_+ generate the $SU(1,1)$ algebra in E_7 [11], which commutes with G_2. This fact can be used to derive the relations listed in Eqs. (B6)–(B11).

In terms of the $SO(7)^\pm$ invariant tensors defined above, the G_2 invariant solution of 11-dimensional supergravity is given by the following expressions. In the uncompactified dimensions, it is the usual AdS_4 metric, while the metric in the internal seven-dimensional space is given by [11]

$$g_{mn} = 6^{2/3}\gamma^{-1/9}(15-\xi)^{-1/3} \times \left\{(\overset{\circ}{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{1}{36}(15-\xi)\hat{\xi}_m\hat{\xi}_n\right\}, \tag{42}$$

where γ is an arbitrary positive constant and the index on $\hat{\xi}_m$ is raised with metric $\overset{\circ}{g}{}^{mn}$. The determinant of this metric is

$$\det(g_{mn}) = \Delta^2 \det(\overset{\circ}{g}_{mn}), \tag{43}$$

where

$$\Delta = 6^{4/3}\gamma^{-7/18}(15-\xi)^{-2/3}. \tag{44}$$

The internal flux (4-form field strength) is

$$F_{mnpq} = \frac{4\sqrt{6/5}}{15-\xi}\gamma^{-1/6}m_7\left\{\overset{\circ}{\eta}_{mnpqrst}\overset{\circ}{S}{}^{rst} - \frac{(21+\xi)(\xi-27\pm12\sqrt{3})}{12(15-\xi)}\hat{\xi}_{[m}\overset{\circ}{\eta}_{npq]rstu}\hat{\xi}^r\overset{\circ}{S}{}^{stu} + \sqrt{(21+\xi)(3-\xi)}\frac{(\xi-51\pm12\sqrt{3})}{2(15-\xi)}\overset{\circ}{S}_{[mnp}\hat{\xi}_{q]}\right\}. \tag{45}$$

The $\pm$ ambiguity in the expression above arises from the arbitrariness in the sign of the Freund-Rubin parameter $\mathfrak{f}_{\mathrm{FR}}$ [11]. As shown there, this solution has $N=1$ residual supersymmetry.

The solution given above solves the Einstein equations for any value of the constant γ [see Eq. (5)]. However, the nonlinear metric *Ansatz* gives the solution with a particular value for γ. In anticipation of this fact, and for ease of comparison later, we choose

$$\gamma^{-1/3} = \frac{5}{6\sqrt{3}}. \tag{46}$$

Hence,

$$g_{mn} = 3^{1/6}10^{1/3}(15-\xi)^{-1/3} \times \left\{(\overset{\circ}{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{1}{36}(15-\xi)\hat{\xi}_m\hat{\xi}_n\right\} \tag{47}$$

and

$$F_{mnpq} = \frac{4\times3^{-1/4}}{15-\xi}m_7\left\{\overset{\circ}{\eta}_{mnpqrst}\overset{\circ}{S}{}^{rst} - \frac{(21+\xi)(\xi-27\pm12\sqrt{3})}{12(15-\xi)}\hat{\xi}_{[m}\overset{\circ}{\eta}_{npq]rstu}\hat{\xi}^r\overset{\circ}{S}{}^{stu} + \sqrt{(21+\xi)(3-\xi)}\frac{(\xi-51\pm12\sqrt{3})}{2(15-\xi)}\overset{\circ}{S}_{[mnp}\hat{\xi}_{q]}\right\}. \tag{48}$$

B. The G_2 invariant stationary point of gauged supergravity

The 70 scalars and pseudoscalars of the $N=8$ supergravity theory that parametrize an element of the coset space $E_7/SU(8)$ can be described by an element in the fundamental representation of E_7 as follows [5]:

$$\mathcal{V} = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ijIJ} \\ v^{ijIJ} & u^{ij}{}_{IJ} \end{pmatrix}. \tag{49}$$

Note that complex conjugation is represented by a respective lowering/raising of indices.

Using an SU(8) transformation, the E_7 matrix $\mathcal{V}$ can be brought into a symmetric gauge of the form

$$\mathcal{V} = \exp\Phi \equiv \exp\begin{pmatrix} 0 & \phi_{IJKL} \\ \phi^{IJKL} & 0 \end{pmatrix}. \tag{50}$$

Once this gauge is fixed, the distinction between $i, j, \ldots$ and $I, J, \ldots$ indices may be safely ignored, as we shall do so hereafter. For a G_2 invariant configuration, the most general vacuum expectation value that ϕ_{IJKL} can take may be parametrized as follows [31,32]:

$$\phi_{IJKL} \equiv \phi_{IJKL}(\lambda,\alpha) = \frac{1}{2}\lambda(C_+^{IJKL}\cos\alpha + iC_-^{IJKL}\sin\alpha), \tag{51}$$

where λ and α take a particular value for each stationary point consistent with this configuration. The self-dual C_+^{IJKL}, antiself-dual C_-^{IJKL} and

$$D_\pm^{IJKL} = \frac{1}{2}(C_+^{IJMN}C_-^{MNKL} \pm C_-^{IJMN}C_+^{MNKL}) \tag{52}$$

form a basis of G_2 invariant objects in E_7. In the remainder we will often keep the index structure implicit for brevity, so

$$A\cdot B \equiv (A\cdot B)^{IJKL} \equiv A^{IJMN}B^{MNKL}. \tag{53}$$

Given ϕ of the form above, the components of the $E_7/SU(8)$ coset elements $u^{IJ}{}_{KL}$ and v^{IJKL} can be written in terms of the G_2 invariant $C_\pm$ and $D_\pm$. Given the structure of the matrix Φ, it is not too difficult to see that

$$\mathcal{V} = \sum_{n=0}^{\infty}\frac{1}{n!}\Phi^n = \begin{pmatrix} \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n}}{(2n)!}(\varphi\varphi^*)^n & \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n+1}}{(2n+1)!}\varphi(\varphi^*\varphi)^n \\ \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n+1}}{(2n+1)!}\varphi^*(\varphi\varphi^*)^n & \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n}}{(2n)!}(\varphi^*\varphi)^n \end{pmatrix}, \tag{54}$$

where

$$\begin{aligned}\varphi &= \cos\alpha C_+ + i\sin\alpha C_-,\\ \varphi^* &= \cos\alpha C_+ - i\sin\alpha C_-.\end{aligned} \tag{55}$$

Of course, the order in which the tensors appear is now important and indicative of the index structure of the terms. For example, $C_+C_- = D_+ + D_-$, while $C_-C_+ = D_+ - D_-$. The matrix above is clearly compatible with the structure of the matrix given in the defining Eq. (49).

Consider

$$\varphi\varphi^* = (\cos\alpha C_+ + i\sin\alpha C_-)(\cos\alpha C_+ - i\sin\alpha C_-). \tag{56}$$

Using Eqs. (22) and (23), we find that

$$\varphi\varphi^* = 12 + 4\tilde{\Theta}, \tag{57}$$

where we have omitted a δ symbol in the first term above for brevity and

$$\tilde{\Theta} = \cos^2\alpha C_+ - \sin^2\alpha C_- - \frac{1}{4} i\sin 2\alpha D_-. \tag{58}$$

We notice that $\tilde{\Theta}$ has the rather convenient property that

$$\tilde{\Theta}^2 = 12 + 4\tilde{\Theta}. \tag{59}$$

One can simply verify the above equation using Eqs. (B6)–(B11). Now, define a new quantity,

$$\Theta = \frac{1}{8}(\tilde{\Theta} + 2), \tag{60}$$

which has been chosen so that

$$\Theta^2 = \Theta. \tag{61}$$

From Eq. (57), we have

$$\varphi\varphi^* = 12 + 4\tilde{\Theta} = 4 + 32\Theta. \tag{62}$$

Comparing the components of Eqs. (49) and (54) gives that

$$u_{IJ}{}^{KL} = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!}(\varphi\varphi^*)^n \tag{63}$$

and

$$v^{IJKL} = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n+1}}{(2n+1)!}\varphi^*(\varphi\varphi^*)^n, \tag{64}$$

where, as before, indices have been suppressed on the right-hand side of the above equations.

First, consider $u_{IJ}{}^{KL}$.

$$\begin{aligned} u_{IJ}{}^{KL} &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!}(\varphi\varphi^*)^n = \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!}(4 + 32\Theta)^n \\ &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!} \sum_{p=0}^{n} \binom{n}{p} 4^{n-p}(32\Theta)^p, \end{aligned}$$

where we have used Eq. (62) in the second equality and applied the binomial theorem in the third equality. Using the property satisfied by Θ, Eq. (61), the previous expression can be rewritten as follows:

$$\begin{aligned} u_{IJ}{}^{KL} &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!}\left\{4^n + \sum_{p=1}^{n}\binom{n}{p}4^{n-p}(32)^p\Theta\right\} \\ &= \sum_{n=0}^{\infty} \frac{(\lambda/2)^{2n}}{(2n)!}\left\{4^n + \left[\sum_{p=0}^{n}\binom{n}{p}4^{n-p}(32)^p - 4^n\right]\Theta\right\} \\ &= \sum_{n=0}^{\infty} \frac{\lambda^{2n}}{(2n)!}\{1 + [3^{2n} - 1]\Theta\}. \end{aligned}$$

Identifying the above expressions as the Taylor expansions of the cosh function simplifies the expression to

$$\begin{aligned} u_{IJ}{}^{KL} &= \cosh\lambda + (\cosh 3\lambda - \cosh\lambda)\Theta \\ &= \cosh\lambda + \frac{1}{8}(\cosh 3\lambda - \cosh\lambda)(\tilde{\Theta} + 2) \\ &= \cosh^3\lambda + \frac{1}{2}\cosh\lambda\sinh^2\lambda\tilde{\Theta}, \end{aligned}$$

where we have used Eq. (60) in the second equality and well-known multiple angle identities for hyperbolic functions in the final equality. Defining

$$p = \cosh\lambda, \qquad q = \sinh\lambda \tag{65}$$

and substituting for $\tilde{\Theta}$ using Eq. (58) gives

$$\begin{aligned} &u_{IJ}{}^{KL}(\lambda, \alpha) \\ &\quad = p^3\delta_{IJ}^{KL} + \frac{1}{2}pq^2\cos^2\alpha C_+^{IJKL} \\ &\qquad - \frac{1}{2}pq^2\sin^2\alpha C_-^{IJKL} - \frac{1}{8}ipq^2\sin 2\alpha D_-^{IJKL} \end{aligned} \tag{66}$$

or, equivalently,

$$\begin{aligned} &u^{IJ}{}_{KL}(\lambda, \alpha) \\ &\quad = p^3\delta^{IJ}_{KL} + \frac{1}{2}pq^2\cos^2\alpha C_+^{IJKL} \\ &\qquad - \frac{1}{2}pq^2\sin^2\alpha C_-^{IJKL} + \frac{1}{8}ipq^2\sin 2\alpha D_-^{IJKL} \end{aligned} \tag{67}$$

for the complex conjugate.

The derivation of v^{IJKL} is essentially the same as that of $u_{IJ}{}^{KL}$. Starting from Eq. (64),

$$\begin{aligned} v^{IJKL} &= \varphi^* \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n+1}}{(2n+1)!}(\varphi\varphi^*)^n \\ &= \frac{1}{2}\varphi^*\sum_{n=0}^{\infty}\frac{\lambda^{2n+1}}{(2n+1)!}\{1 + [3^{2n} - 1]\Theta\} \\ &= \frac{1}{2}\varphi^*\left\{\sinh\lambda + \left(\frac{1}{3}\sinh 3\lambda - \sinh\lambda\right)\Theta\right\}, \end{aligned}$$

where the second equality is a direct application of the results derived above. Substituting for φ^* and Θ using Eqs. (55) and (60), respectively, the above expression simplifies to

$$v^{IJKL} = \frac{1}{48}(\cos\alpha C_+ - i\sin\alpha C_-) \times \Big\{2(\sinh 3\lambda + 9\sinh\lambda) + (\sinh 3\lambda - 3\sinh\lambda) \times \Big(\cos^2\alpha C_+ - \sin^2\alpha C_- - \frac{i}{4}\sin 2\alpha D_-\Big)\Big\}, \quad (68)$$

where we have substituted for $\tilde{\Theta}$ using Eq. (58). Expanding out the bracket above and using Eqs. (B6)–(B11), one can simply show that the above expression reduces to

$$v^{IJKL}(\lambda,\alpha) = q^3(\cos^3\alpha + i\sin^3\alpha)\delta^{IJ}_{KL} + \frac{1}{2}p^2 q\cos\alpha C_+^{IJKL} - \frac{1}{2}ip^2 q\sin\alpha C_-^{IJKL} - \frac{1}{8}q^3\sin 2\alpha(\sin\alpha + i\cos\alpha)D_+^{IJKL}, \quad (69)$$

where we have used well-known multiple angle identities for hyperbolic functions and definitions (65).

The G_2 invariant stationary point is given by[5] [31]

$$c^2 = (p^2+q^2)^2 = \frac{1}{5}(3+2\sqrt{3}), \quad (70)$$

$$s^2 = (2pq)^2 = \frac{2}{5}(\sqrt{3}-1), \quad (71)$$

$$v^2 = \cos^2\alpha = \frac{1}{4}(3-\sqrt{3}), \quad (72)$$

where $c = \cosh(2\lambda)$ and $s = \sinh(2\lambda)$. We also define the following useful combinations

$$b_1(\lambda,\alpha) = c^3 + v^3 s^3, \quad b_2(\lambda,\alpha) = csv(c+vs), \quad (73)$$

$$f_1(\lambda,\alpha) = p^2q^2(p^2+q^2)\sin\alpha\cos\alpha = \frac{1}{4}s^2cv\sqrt{1-v^2}, \quad (74)$$

$$f_2(\lambda,\alpha) = p^3q^3\sin\alpha\cos^2\alpha = \frac{1}{8}s^3v^2\sqrt{1-v^2}, \quad (75)$$

$$f_3(\lambda,\alpha) = pq(p^2+q^2)^2\sin\alpha = \frac{1}{2}sc^2\sqrt{1-v^2}. \quad (76)$$

At the G_2 invariant stationary point we have the following simplifying relations

$$b_1 = 3b_2, \qquad f_3 = 4(2f_1 - f_2), \quad (77)$$

which will be useful below.

[5] The parametrization of ϕ_{IJKL} used in this paper coincides with that defined in Refs. [31,32] by taking $\lambda \to \frac{1}{2\sqrt{2}}\lambda$. Thus, the values of c, s and v coincide precisely with those given in Ref. [31].

C. Derivation of the 3-form potential

In this section we derive the potential of the 11-dimensional G_2 invariant solution using the nonlinear flux *Ansatz* (2), showing that its field strength coincides with the expression found in Ref. [11].

We identify the expression multiplying the potential in Eq. (2) as

$$8\Delta^{-1}g^{pq} = K^{pIJ}K^{qKL}(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL}) \quad (78)$$

via the nonlinear metric *Ansatz* (1). Using this, Eq. (2) takes the form

$$A_{mnp} = -\frac{1}{8\sqrt{2}}(i\Delta g_{pq})K_{mn}{}^{IJ}K^{qKL} \times (u^{ij}{}_{IJ} - v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL}). \quad (79)$$

From the metric *Ansatz* [11]

$$g^{mn} = \frac{\Delta}{6}\{[6b_1 - b_2(\xi+3)]\mathring{g}^{mn} + b_2(21+\xi)\hat{\xi}^m\hat{\xi}^n\}. \quad (80)$$

Equivalently,

$$\Delta g_{mn} = \Big\{\frac{6}{6b_1 - b_2(\xi+3)}(\mathring{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{1}{b_1+3b_2}\hat{\xi}_m\hat{\xi}_n\Big\}. \quad (81)$$

Substituting the G_2 invariant stationary point values given in Eqs. (70)–(72) and using (77) gives

$$g_{mn} = 3^{1/6}10^{1/3}(15-\xi)^{-1/3} \times \Big\{(\mathring{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{1}{36}(15-\xi)\hat{\xi}_m\hat{\xi}_n\Big\}, \quad (82)$$

which coincides with the metric given in Eq. (47).

In order to simplify the right-hand side of Eq. (79), recall that $u^{ij}{}_{IJ}$ and v^{ijIJ} are components of E_7 matrices. In particular, they satisfy the relations [3]

$$u^{ij}{}_{IJ}u_{ij}{}^{KL} - v_{ijIJ}v^{ijKL} = \delta^{KL}_{IJ}, \quad (83)$$

$$u^{ij}{}_{IJ}v_{ijKL} - v_{ijIJ}u^{ij}{}_{KL} = 0. \quad (84)$$

These relations can be explicitly verified for the components of the $E_7/SU(8)$ coset element given in Eqs. (67) and (69) by using identities (B6)–(B11). Now, the expression for the E_7 matrix components in Eq. (79)

$$\begin{aligned}(u^{ij}{}_{IJ} - v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL}) &= u^{ij}{}_{IJ}u_{ij}{}^{KL} - v^{ijIJ}v_{ijKL} + u^{ij}{}_{IJ}v_{ijKL} - v^{ijIJ}u_{ij}{}^{KL},\\ &= u^{ij}{}_{IJ}u_{ij}{}^{KL} - (u^{ij}{}_{KL}u_{ij}{}^{IJ} - \delta^{IJ}_{KL})\\ &\quad + u^{ij}{}_{KL}v_{ijIJ} - v^{ijIJ}u_{ij}{}^{KL}.\end{aligned}$$

Recalling that

$$u_{ij}{}^{IJ} = (u^{ij}{}_{IJ})^*, \qquad v_{ijIJ} = (v^{ijIJ})^*, \quad (85)$$

the above expression reduces to

$$(u^{ij}{}_{IJ} - v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL}) = \delta^{IJ}_{KL} + 2iIm(u^{ij}{}_{IJ}u_{ij}{}^{KL} - v^{ijIJ}u_{ij}{}^{KL}), \tag{86}$$

where the last term is the imaginary part of the expression in the bracket.

Using Eqs. (B6)–(B11), it is straightforward to show that

$$\mathrm{Im}(u^{ij}{}_{IJ}u_{ij}{}^{KL}) = -\frac{1}{4}p^2q^2(p^2+q^2)\sin 2\alpha D_-^{IJKL}. \tag{87}$$

The expression on the left-hand side of the above equation is antisymmetric under the exchange of the pair of indices $[IJ]$ and $[KL]$, since from Eq. (85) this operation is equivalent to complex conjugation of the expression in the bracket. Therefore, it should come as no surprise that the right-hand side is given solely in terms of D_-. Furthermore,

$$\mathrm{Im}(v^{ijIJ}u_{ij}{}^{KL}) = 4p^3q^3\sin^3 2\alpha\delta^{IJ}_{KL} - \frac{1}{2}pq(p^2+q^2)^2 \times \sin\alpha C_-^{IJKL} - p^3q^3\sin\alpha\cos^2\alpha D_+^{IJKL}, \tag{88}$$

which is indeed symmetric under the exchange of the pairs of indices $[IJ]$ and $[KL]$ as expected from Eq. (84).

Using Eqs. (13), (40), (41), and (52) we derive

$$K_{ab}{}^{IJ}K_c{}^{KL}C_+^{IJKL} = -\frac{16}{3}\delta^{ab}_{cd}\sqrt{(21+\xi)(3-\xi)}\hat{\xi}^d = -\frac{16}{3}\delta^{ab}_{cd}\xi^d, \tag{89}$$

$$K_{ab}{}^{IJ}K_c{}^{KL}C_-^{IJKL} = 16S_{abc}, \tag{90}$$

$$K_{ab}{}^{IJ}K_c{}^{KL}D_+^{IJKL} = -\frac{8}{3}(9+\xi)S_{abc} + 8(21+\xi)\hat{\xi}^d\hat{\xi}_{[a}S_{bc]d} + \frac{4}{9}\xi^d\epsilon_{abcdefg}S^{efg}, \tag{91}$$

$$K_{ab}{}^{IJ}K_c{}^{KL}D_-^{IJKL} = -\frac{8}{3}(3-\xi)S_{abc} + 8(21+\xi)\hat{\xi}^d\hat{\xi}_{[a}S_{bc]d} - \frac{16}{3}(21+\xi)\hat{\xi}^d\hat{\xi}_cS_{abd} - \frac{4}{9}\xi^d\epsilon_{abcdefg}S^{efg}. \tag{92}$$

The first two relations are easily seen to be consistent with Eqs. (24) and (25). Observe also that the last expression is *not* fully anti-symmetric in the indices $[abc]$.

With the use of the above relations, the expression for the 3-form potential, (79), reduces to

$$A_{mnp} = \frac{1}{18\sqrt{2}}\Delta g_{pq}\mathring{g}^{qr}\{6((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3)\mathring{S}_{mnr} - 18(21+\xi)(f_1-2f_2)\hat{\xi}^s\hat{\xi}_{[m}\mathring{S}_{nr]s} + 12(21+\xi)f_1\hat{\xi}^s\hat{\xi}_r\mathring{S}_{mns} + \sqrt{(21+\xi)(3-\xi)} \times (f_1+2f_2)\hat{\xi}^s\mathring{\eta}_{mnrstuv}\mathring{S}^{tuv}\}, \tag{93}$$

where f_1, f_2, f_3 are defined in Eqs. (74)–(76). The quantities $\hat{\xi}^m$, $\mathring{S}_{mnp}$ and $\mathring{\eta}_{mnpqrst}$ are constructed with the round S^7 vielbein and they are raised or lowered with the round S^7 metric, as emphasized earlier.

Inserting the expression for the metric, Eq. (81), found from the nonlinear metric *Ansatz*,

$$A_{mnp} = \frac{1}{18\sqrt{2}[6b_1 - b_2(\xi+3)](b_1+3b_2)}\Big\{6(b_1+3b_2)\delta^q_p - (21+\xi)b_2\hat{\xi}_p\hat{\xi}^q\Big\}\{6((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3)\mathring{S}_{mnq} - 18(21+\xi)(f_1-2f_2)\hat{\xi}^r\hat{\xi}_{[m}\mathring{S}_{nq]r} + 12(21+\xi)f_1\hat{\xi}^r\hat{\xi}_q\mathring{S}_{mnr} + \sqrt{(21+\xi)(3-\xi)}(f_1+2f_2)\hat{\xi}^r\mathring{\eta}_{mnqrstu}\mathring{S}^{stu}\Big\}, \tag{94}$$

where b_1 and b_2 are defined in Eq. (73). Expanding out the terms in the expression above gives

$$A_{mnp} = \frac{1}{3\sqrt{2}[6b_1 - b_2(\xi+3)]}\Big\{6((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3)\mathring{S}_{mnp} - 18(21+\xi)(f_1-2f_2)\hat{\xi}^r\hat{\xi}_{[m}\mathring{S}_{np]r} + \sqrt{(21+\xi)(3-\xi)}(f_1+2f_2)\hat{\xi}^q\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}\Big\} + \frac{\sqrt{2}(21+\xi)}{[6b_1 - b_2(\xi+3)](b_1+3b_2)}[2b_1f_1 + b_2(2f_1 - 4f_2 - f_3)]\hat{\xi}^q\hat{\xi}_p\mathring{S}_{mnq}. \tag{95}$$

Let us consider the coefficient of the term that is not totally antisymmetric in the indices $[mnp]$,

$$2b_1f_1 + b_2(2f_1 - 4f_2 - f_3) = \frac{1}{2}\sin\alpha vcs^2[(c^3 + v^3s^3) - (c+vs)(c^2 - vcs + v^2s^2)] = 0,$$

where in the first equality we have simply substituted in the definitions of b_1, b_2, f_1, f_2 and f_3 using Eqs. (73)–(76). The vanishing of the nonantisymmetric term even away from the stationary point is expected from the general argument of

Ref. [1], where it is shown that the 3-form potential as defined by the nonlinear flux *Ansatz* is totally antisymmetric by the E_7 properties of $u^{ij}{}_{IJ}$ and v^{ijIJ}. We now have a totally antisymmetric expression for the 3-form potential

$$A_{mnp} = \frac{1}{3\sqrt{2}[6b_1 - b_2(\xi+3)]}\Big\{6((3-\xi)f_1 - 2(9+\xi)f_2 + 6f_3)\mathring{S}_{mnp} - 18(21+\xi)(f_1-2f_2)\hat{\xi}^r\hat{\xi}_{[m}\mathring{S}_{np]r} + \sqrt{(21+\xi)(3-\xi)}(f_1+2f_2)\hat{\xi}^q\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}\Big\}. \tag{96}$$

This is only defined up to gauge transformations, hence to make a comparison with the known G_2 invariant solution, we calculate its field strength. Using Eqs. (33), (34), and (36), the field strength of the potential above,

$$F_{mnpq} = 4\mathring{D}_{[m}A_{npq]},$$

is

$$F_{mnpq} = \frac{4\sqrt{2}m_7}{6b_1 - b_2(3+\xi)}\Big\{(f_3-4f_2)\mathring{\eta}_{mnpqrst}\mathring{S}^{rst} - \frac{2(21+\xi)}{3[6b_1-b_2(3+\xi)]}[b_2(f_1-f_2)(\xi-27) - 3b_1(f_1-4f_2) + 9b_2(3f_1-4f_2)]\hat{\xi}_{[m}\mathring{\eta}_{npq]rstu}\hat{\xi}^r\mathring{S}^{stu} - \frac{4\sqrt{(21+\xi)(3-\xi)}}{6b_1-b_2(3+\xi)}[b_2(f_1-f_2)(51-\xi) + 3b_1(f_1-4f_2) - 3b_2(17f_1-16f_2-f_3)]\mathring{S}_{[mnp}\hat{\xi}_{q]}\Big\}. \tag{97}$$

Substituting relations (77), valid at the G_2 stationary point, we get

$$F_{mnpq} = \frac{32\sqrt{2}(f_1-f_2)}{b_2(15-\xi)}m_7\Big\{\mathring{\eta}_{mnpqrst}\mathring{S}^{rst} - \frac{(21+\xi)}{12(15-\xi)}\Big(\xi - 27 + \frac{18f_1}{f_1-f_2}\Big)\hat{\xi}_{[m}\mathring{\eta}_{npq]rstu}\hat{\xi}^r\mathring{S}^{stu} + \frac{\sqrt{(21+\xi)(3-\xi)}}{2(15-\xi)}\Big(\xi - 51 + \frac{18f_1}{f_1-f_2}\Big)\mathring{S}_{[mnp}\hat{\xi}_{q]}\Big\}. \tag{98}$$

Using Eqs. (70)–(72), the expression above reduces to

$$F_{mnpq} = \frac{4\times 3^{-1/4}}{15-\xi}m_7\Big\{\mathring{\eta}_{mnpqrst}\mathring{S}^{rst} - \frac{(21+\xi)(\xi-27+12\sqrt{3})}{12(15-\xi)}\hat{\xi}_{[m}\mathring{\eta}_{npq]rstu}\hat{\xi}^r\mathring{S}^{stu} + \sqrt{(21+\xi)(3-\xi)}\frac{(\xi-51+12\sqrt{3})}{2(15-\xi)}\mathring{S}_{[mnp}\hat{\xi}_{q]}\Big\}. \tag{99}$$

This is in perfect agreement with the flux of the G_2 invariant solution [11] given in Eq. (48). It is remarkable that there is not only an agreement with the general structure, but also the precise coefficients.

IV. THE SU(4)$^-$ INVARIANT SOLUTION

A. The SU(4)$^-$ invariant solution of 11-dimensional supergravity

The SU(4)$^-$ invariant solution [33] is a compactification of 11-dimensional supergravity to a maximally symmetric four-dimensional spacetime with the internal space given by a stretched U(1) fibration over CP^3. In Ref. [33], the solution was expressed in terms of structures on CP^3. Here, in order to compare the SU(4)$^-$ invariant solution with the result given by the nonlinear *Ansätze*, we express the SU(4)$^-$ invariant solution in terms of geometrical quantities defined on a round S^7.

The antiself-dual SO(8) tensor Y^-_{IJKL}, satisfying [11]

$$Y^-_{IJMN}Y^-_{MNKL} = 8\delta^{KL}_{IJ} - 8F^{-[K}_{[I}F^{-L]}_{J]}, \tag{100}$$

$$Y^-_{IJKL}Y^-_{MNPQ}Y^-_{PQKL} = 16Y^-_{IJPQ} \tag{101}$$

preserves SU(4)$^-$. The antisymmetric tensor F^-_{IJ} is an almost complex structure,

$$F^-_I{}^K F^-_K{}^J = -\delta^J_I. \tag{102}$$

Using the properties of Y^-_{IJKL} and F^-_{IJ}, it is straightforward to show that

$$Y^-_{MIJK}F^-_L{}^M = Y^-_{M[IJK}F^-_{L]}{}^M, \tag{103}$$

$$Y^-_{MIJK}F^-_L{}^M = -\frac{1}{4!}\epsilon_{IJKLPQRS}Y^-_{MPQR}F^-_S{}^M, \tag{104}$$

$$F^-_{[IJ}F^-_{KL]} = -\frac{1}{4!}\epsilon_{IJKLPQRS}F^-_{PQ}F^-_{RS}. \tag{105}$$

The SO(8) objects can be used to define the SO(7) tensors

$$K_a = \frac{1}{4}K_a^{IJ}F^-_{IJ}, \qquad K_{ab} = \frac{1}{4}K_{ab}^{IJ}F^-_{IJ}, \qquad T_{abc} = \frac{1}{16}K^{IJ}_{[ab}K^{KL}_{c]}Y^-_{IJKL}, \tag{106}$$

where K_a^{IJ} and K_{ab}^{IJ} have been defined in Eq. (11). Using the relations given in Appendix C, the following identities hold

$$K_aK_a = 1, \qquad K_aK_{ab} = 0, \qquad K_{ac}K_{cb} = K_aK_b - \delta_{ab}, \tag{107}$$

$$K_aT_{abc} = 0, \qquad T^{acd}T_{bcd} = 4(\delta^a_b - K^aK_b), \qquad \epsilon^{abcdefg}K_dK_{eh}T_{hfg} = -6T^{abc}. \tag{108}$$

Furthermore, using Eq. (9)

$$\mathring{D}_aK_b = -m_7K_{ab}, \tag{109}$$

$$\mathring{D}_aT_{bcd} = \frac{1}{6}m_7\epsilon_{abcdefg}T^{efg}. \tag{110}$$

In terms of the tensors K_a and T_{abc}, the internal metric of the $SU(4)^-$ invariant solution is given by[6]

$$g_{mn} = 2^{-1/3}(\mathring{g}_{mn} + \mathring{K}_m\mathring{K}_n), \tag{111}$$

where as before $\mathring{g}_{mn}$ is the round S^7 metric and

$$\mathring{K}_m = \mathring{e}_m{}^aK_a$$

is defined with respect to the siebenbein on the round S^7.

Using Eqs. (107) and (109), the Ricci tensor of this metric is given by

$$R_{mn} = \mathring{R}_{mn} + 2m_7^2\mathring{g}_{mn} - 20m_7^2\mathring{K}_m\mathring{K}_n. \tag{112}$$

The expression for the Ricci tensor of the round S^7 metric is given in Eq. (20).

The internal flux of the $SU(4)^-$ invariant solution is

$$F_{mnpq} = \frac{1}{3}m_7\mathring{\eta}_{mnpqrst}\mathring{T}^{rst}. \tag{113}$$

To verify that the Einstein Eqs. (16) and (17), are satisfied it is useful to note that

$$F_{mpqr}F^{npqr} = 48\times 2^{4/3}m_7^2(\delta^n_m + K_mK^n), \tag{114}$$

where we have used Eq. (108). On the left-hand side of the above equation, the indices have been raised with inverse of g_{mn} given in Eq. (111).

Using the expression for the Ricci tensor in the internal direction, (112) and (114), it is straightforward to verify that g_{mn} and F_{mnpq} solve the Einstein Eqs. (16) and (17), with

$$m_4^2 = \frac{16}{3}m_7^2, \qquad \mathfrak{f}_{\mathrm{FR}}^2 = 32m_7^2. \tag{115}$$

With the above value for $\mathfrak{f}_{\mathrm{FR}}$, the equation of motion for F_{mnpq}, (18), is also satisfied.

B. The $SU(4)^-$ invariant stationary point of gauged supergravity

The $SU(4)^-$ invariant stationary point of maximal gauged supergravity is obtained for a purely pseudoscalar expectation value ϕ_{IJKL} of the form [31]

$$\phi_{IJKL} = \frac{1}{2}i\lambda Y^-_{IJKL}, \tag{116}$$

where Y^-_{IJKL} is an antiself-dual object satisfying the properties presented in Eqs. (100)–(102).

Using Eq. (101), it is simple to show that for $n > 0$,

$$(Y^-Y^-)^n{}_{IJKL} = 2^{4(n-1)}(Y^-Y^-)_{IJKL}, \tag{117}$$

where $(Y^-Y^-)_{IJKL}$ denotes a contraction of the form $Y^-_{IJMN}Y^-_{MNKL}$.

As described in Sec. III B, it is fairly straightforward to show that

$$u_{IJ}{}^{KL} = \sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n}}{(2n)!}(Y^-Y^-)^n{}_{IJKL}, \tag{118}$$

Using Eqs. (117) and (100), the above expression reduces to

$$\begin{aligned} u_{IJ}{}^{KL} &= \delta^{KL}_{IJ} + \sum_{n=1}^{\infty}\frac{(\lambda/2)^{2n}}{(2n)!}2^{4n-1}\Big(\delta^{KL}_{IJ} - F^{-[K}_{[I}F^{-L]}_{J]}\Big) \\ &= \delta^{KL}_{IJ} + \frac{1}{2}\Big(\sum_{n=0}^{\infty}\frac{(2\lambda)^{2n}}{(2n)!} - 1\Big)\Big(\delta^{KL}_{IJ} - F^{-[K}_{[I}F^{-L]}_{J]}\Big) \\ &= \delta^{KL}_{IJ} + \frac{1}{2}(\cosh 2\lambda - 1)\Big(\delta^{KL}_{IJ} - F^{-[K}_{[I}F^{-L]}_{J]}\Big) \end{aligned} \tag{119}$$

Defining $c = \cosh(2\lambda)$ as before, and observing that the expression is real,

$$u^{IJ}{}_{KL} = \frac{1}{2}(c+1)\delta^{IJ}_{KL} - \frac{1}{2}(c-1)F^{-[I}_{[K}F^{-J]}_{L]}. \tag{120}$$

[6] As before, we have fixed the allowed arbitrary scaling (5) in anticipation of the form of the metric given by the nonlinear *Ansatz*.

Similarly,

$$v^{IJKL} = -iY^-_{IJMN}\sum_{n=0}^{\infty}\frac{(\lambda/2)^{2n+1}}{(2n+1)!}(Y^-Y^-)^n_{MNKL} = -\frac{1}{2}iY^-_{IJMN}\left(\lambda + \frac{1}{32}\sum_{n=1}^{\infty}\frac{(2\lambda)^{2n+1}}{(2n+1)!}(Y^-Y^-)_{MNKL}\right)$$
$$= -\frac{1}{2}iY^-_{IJMN}\left(\lambda + \frac{1}{32}[\sinh(2\lambda) - 2\lambda](Y^-Y^-)_{MNKL}\right) = -\frac{1}{4}i\sinh(2\lambda)Y^-_{IJKL}, \tag{121}$$

where we have used Eq. (101) in the final equality in the equation above. Defining $s = \sinh(2\lambda)$,

$$v^{IJKL} = -\frac{1}{4}isY^-_{IJKL}. \tag{122}$$

It is simple to verify that the u and v as given in Eqs. (120) and (122) satisfy the E_7 relations, Eqs. (83) and (84).

From the metric *Ansatz* [11],

$$\Delta^{-1}g^{mn} = \{c^2\mathring{g}^{mn} - s^2\mathring{K}^m\mathring{K}^n\}. \tag{123}$$

Equivalently,

$$\Delta g_{mn} = c^{-2}\{\mathring{g}_{mn} + s^2\mathring{K}_m\mathring{K}_n\}. \tag{124}$$

The $SU(4)^-$ invariant stationary point is given by [31]

$$c^2 = 2, \qquad s^2 = 1. \tag{125}$$

Substituting these values into the expression above and taking the determinant of the resulting expression gives

$$\Delta = 2^{-2/3}. \tag{126}$$

Hence, the metric is of the form

$$g_{mn} = 2^{-1/3}\{\mathring{g}_{mn} + \mathring{K}_m\mathring{K}_n\}, \tag{127}$$

which agrees with that given in Eq. (111).

Substituting the expression for u and v given in Eqs. (120) and (122), and the form of the metric given in Eq. (124) into Eq. (79), it is simple to show that

$$A_{mnp} = -\frac{1}{\sqrt{2}}(s/c)\mathring{T}_{mnp}, \tag{128}$$

where we have used the first equation in (108). Now, using Eq. (110), the field strength is simply

$$F_{mnpq} = -\frac{\sqrt{2}}{3}(s/c)m_7\mathring{\eta}_{mnpqrst}\mathring{T}^{rst}. \tag{129}$$

Substituting the values of c and s given in Eq. (125) gives

$$F_{mnpq} = -\frac{1}{3}m_7\mathring{\eta}_{mnpqrst}\mathring{T}^{rst}. \tag{130}$$

Note that the Einstein Eqs. (16) and (17) and the equation of motion for the flux (18) are satisfied regardless of the overall sign of the flux. Thus, again, we have precise agreement with the flux of the $SU(4)^-$ solution given in Eq. (113).

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APPENDIX A: SO(7)$^{\pm}$ INVARIANT SOLUTIONS

For completeness we here reproduce the metric and flux of the $SO(7)^{\pm}$—invariant solutions, even though these are simpler than the ones discussed in the text. The relevant solutions can be found in analogy with the general metric and flux of the G_2 invariant family, given in Eqs. (81) and (97), and by restricting the scalar fields in (51) to $\alpha = 0$ and $\alpha = \pi/2$, respectively.

The $SO(7)^+$ invariant stationary point of maximally gauged supergravity is given by [31]

$$c^2 = \frac{1}{2}(3/\sqrt{5}+1), \quad s^2 = \frac{1}{2}(3/\sqrt{5}-1), \quad v = 1. \tag{A1}$$

In particular, these imply that f_1, f_2, f_3 as defined in Eqs. (74)–(76) vanish. It immediately follows that

$$F_{mnpq} = 0, \tag{A2}$$

as expected. The metric is

$$\Delta g_{mn} = \frac{6\times 5^{1/4}}{9-\xi}\left\{(\mathring{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{(9-\xi)}{30}\hat{\xi}_m\hat{\xi}_n\right\}. \tag{A3}$$

This is the solution of Ref. [34]; see also Refs. [9,11]. In particular, in Ref. [11], the solution is given in the form

$$\Delta g_{mn} = \frac{30\gamma^{-1/2}}{9-\xi}\left\{(\mathring{g}_{mn} - \hat{\xi}_m\hat{\xi}_n) + \frac{(9-\xi)}{30}\hat{\xi}_m\hat{\xi}_n\right\}, \tag{A4}$$

which agrees with metric (A3) for

$$\gamma = 5^{3/2}.$$

Similarly, the $SO(7)^-$ stationary point is given by

$$c^2 = \frac{5}{4}, \qquad s^2 = \frac{1}{4}, \qquad v = 0. \tag{A5}$$

Since $v = 0$, b_2 as defined in Eq. (73) vanishes and the metric is given by the round S^7 metric

$$\Delta g_{mn} = c^{-3}\mathring{g}_{mn}. \tag{A6}$$

Moreover the flux for the $SO(7)^-$ family is

$$F_{mnpq} = \frac{\sqrt{2}}{3}(s/c)m_7\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}. \tag{A7}$$

This is consistent with the Englert solution [30]; see also Refs. [9,11]. In Ref. [11], the solution is expressed as

$$\Delta g_{mn} = \gamma^{-1/2} \mathring{g}_{mn}, \tag{A8}$$

$$F_{mnpq} = \frac{1}{3\sqrt{2}} \gamma^{-1/6} m_7 \mathring{\eta}_{mnpqrst} \mathring{S}^{rst}, \tag{A9}$$

which agree with Eqs. (A6) and (A7) at the stationary point for

$$\gamma^{1/3} = 5/4.$$

APPENDIX B: USEFUL G_2 IDENTITIES

In this Appendix, we derive identities relating the contraction of G_2 invariants $C_\pm$ and $D_\pm$, adopting the shorthand notation (53) throughout. In deriving these identities it is useful to observe that viewed as E_7 matrices, $C_\pm$ and D_+ are generators of an SU(1, 1) subalgebra of E_7. This is the unique subalgebra of E_7 that commutes with G_2 [11], cf.

$$\sigma^1 \sim \begin{pmatrix} 0 & C_+ \\ C_+ & 0 \end{pmatrix}, \qquad \sigma^2 \sim \begin{pmatrix} 0 & -iC_- \\ iC_- & 0 \end{pmatrix}, \qquad \sigma^3 \sim \begin{pmatrix} iD_+ & 0 \\ 0 & -iD_+ \end{pmatrix}. \tag{B1}$$

Thus,

$$\left[\begin{pmatrix} iD_+ & 0 \\ 0 & -iD_+ \end{pmatrix}, \begin{pmatrix} 0 & C_+ \\ C_+ & 0 \end{pmatrix} \right] \propto \begin{pmatrix} 0 & -iC_- \\ iC_- & 0 \end{pmatrix}, \tag{B2}$$

which implies that

$$(C_+C_-C_+ + 4D_+) \propto C_-. \tag{B3}$$

Consistency with Eq. (22) fixes the constant of proportionality,

$$C_+C_-C_+ = -4(C_- + D_+). \tag{B4}$$

Similarly,

$$C_-C_+C_- = -4(C_- - D_+). \tag{B5}$$

Using Eqs. (22), (23), (B4), and (B5), it is straightforward to prove the following identities:

$$C_+D_+ = 4C_- + 2D_-, \qquad D_+C_+ = 4C_- - 2D_- \tag{B6}$$

$$C_-D_+ = 4C_+ + 2D_-, \qquad D_+C_- = 4C_+ - 2D_-, \tag{B7}$$

$$\begin{aligned} C_+D_- &= 8C_- + 4D_+ + 2D_-, \\ D_-C_+ &= -8C_- - 4D_+ + 2D_-, \end{aligned} \tag{B8}$$

$$\begin{aligned} C_-D_- &= -8C_+ + 4D_+ - 2D_-, \\ D_-C_- &= 8C_+ - 4D_+ - 2D_-, \end{aligned} \tag{B9}$$

$$\begin{aligned} D_+D_+ &= 48 + 8D_+, \\ D_-D_+ &= 16C_+ + 16C_- + 4D_-, \end{aligned} \tag{B10}$$

$$\begin{aligned} D_+D_- &= -16C_+ - 16C_- + 4D_-, \\ D_-D_- &= -96 - 16C_+ + 16C_- - 8D_+. \end{aligned} \tag{B11}$$

APPENDIX C: SEVEN-DIMENSIONAL Γ-MATRIX IDENTITIES

For the reader's convenience, here we give a list of useful Γ-matrix identities, see also the appendices of Refs. [5,10]. The seven-dimensional, Euclidean 8×8 Γ^a matrices, where a is a seven-dimensional flat index, satisfy

$$\{\Gamma^a, \Gamma^b\} = 2\delta_{ab}. \tag{C1}$$

The Clifford algebra admits a Majorana representation, which in our conventions corresponds to a purely imaginary representation of the Γ matrices. We use a representation in which all Γ matrices are Hermitian and antisymmetric or, equivalently, in our representation the charge conjugation matrix is the identity matrix. Moreover,

$$\Gamma^{abcdefg} = -i\epsilon^{abcdefg}\mathbf{1}, \tag{C2}$$

where

$$\Gamma^{abcdefg} = \Gamma^{[a} \ldots \Gamma^{g]}$$

and $\mathbf{1}$ is the 8×8 identity matrix.

The Γ^a can be regarded as seven out of the eight components of Spin(8) gamma matrices in a Majorana-Weyl representation. In this way, one can use SO(8) triality to prove the following important relations [5,10]

$$\Gamma^a_{[AB}\Gamma^b_{CD]} = \frac{1}{24}\epsilon_{ABCDEFGH}\Gamma^a_{EF}\Gamma^b_{GH}, \tag{C3}$$

$$\Gamma^a_{[AB}\Gamma^{ab}_{CD]} = \frac{1}{24}\epsilon_{ABCDEFGH}\Gamma^a_{EF}\Gamma^{ab}_{GH}, \tag{C4}$$

$$\Gamma^{[a}_{[AB}\Gamma^{bc]}_{CD]} = -\frac{1}{24}\epsilon_{ABCDEFGH}\Gamma^{[a}_{EF}\Gamma^{bc]}_{GH}, \tag{C5}$$

$$\Gamma^{[a}_{[AB}\Gamma^{bc]}_{CD]} = \frac{1}{24}i\epsilon^{abcdefg}\Gamma^{de}_{[AB}\Gamma^{fg}_{CD]}. \tag{C6}$$

The uppercase latin indices are spinor indices and run from 1 to 8.

Further Γ-matrix identities can be proved using the Fierz identity, which in Euclidean seven-dimensions takes the form

$$X_{AB}Y_{CD} = \frac{1}{8}\delta_{BC}(XY)_{AD} - \frac{1}{8}\Gamma^a_{BC}(X\Gamma^a Y)_{AD} + \frac{1}{16}\Gamma^{ab}_{BC}(X\Gamma^{ab}Y)_{AD} - \frac{1}{48}\Gamma^{abc}_{BC}(X\Gamma^{abc}Y)_{AD}, \tag{C7}$$

where X and Y are arbitrary 8×8 matrices. The identity above is obtained by noting that

$$\{\delta_{AB}, \Gamma^a_{AB}, \Gamma^{ab}_{AB}, \Gamma^{abc}_{AB}\}$$

span the vector space of 8×8 matrices.

The Fierz identity can be used to show

$$\Gamma^a_{AB}\Gamma^a_{CD} = \Gamma^a_{[AB}\Gamma^a_{CD]} - 2\delta^{AB}_{CD}, \tag{C8}$$

$$\Gamma^a_{AB}\Gamma^{ab}_{CD} + \Gamma^a_{CD}\Gamma^{ab}_{AB} = 2\Gamma^a_{[AB}\Gamma^{ab}_{CD]}, \tag{C9}$$

$$\Gamma^a_{AB}\Gamma^{ab}_{CD} - \Gamma^a_{CD}\Gamma^{ab}_{AB} = -4(\delta_{C[A}\Gamma^b_{B]D} - \delta_{D[A}\Gamma^b_{B]C}), \tag{C10}$$

$$\Gamma^{ab}_{AB}\Gamma^{ab}_{CD} = 2\Gamma^a_{AB}\Gamma^a_{CD} + 16\delta^{AB}_{CD}, \tag{C11}$$

$$\Gamma^{c(a}_{AB}\Gamma^{b)c}_{CD} = \frac{6}{5}\Gamma^{c(a}_{[AB}\Gamma^{b)c}_{CD]} - \Gamma^{(a}_{AB}\Gamma^{b)}_{CD} + \frac{1}{5}\delta^{ab}\Gamma^c_{AB}\Gamma^c_{CD} - \frac{8}{5}\delta^{ab}\delta^{AB}_{CD}, \tag{C12}$$

$$\Gamma^{c[a}_{AB}\Gamma^{b]c}_{CD} = -\Gamma^{[a}_{AB}\Gamma^{b]}_{CD} - 2(\delta_{C[A}\Gamma^{ab}_{B]D} - \delta_{D[A}\Gamma^{ab}_{B]C}), \tag{C13}$$

$$\Gamma^{ca}_{[AB}\Gamma^{bc}_{CD]} = 5\Gamma^a_{[AB}\Gamma^b_{CD]} - \delta^{ab}\Gamma^c_{AB}\Gamma^c_{CD}. \tag{C14}$$

Furthermore, it is also useful to note that (see Appendix of Ref. [10])

$$\Gamma^{ab}_{[AB}\Gamma^c_{CD]}|_- = \Gamma^{[ab}_{[AB}\Gamma^{c]}_{CD]}, \tag{C15}$$

$$\Gamma^{ab}_{[AB}\Gamma^{cd}_{CD]}|_- = \Gamma^{[ab}_{[AB}\Gamma^{cd]}_{CD]}, \tag{C16}$$

where the vertical bar $|_-$ denotes projection to the antiself-dual part.

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