

CHARGES FOR LINEARIZED GRAVITY

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ABSTRACT. Maxwell test fields as well as solutions of linearized gravity on the Kerr exterior admit non-radiating modes, i.e. non-trivial time-independent solutions. These are closely related to conserved charges. In this paper we discuss the non-radiating modes for linearized gravity, which may be seen to correspond to the Poincare Lie-algebra. The 2-dimensional isometry group of Kerr corresponds to a 2-parameter family of gauge-invariant non-radiating modes representing infinitesimal perturbations of mass and azimuthal angular momentum. We calculate the linearized mass charge in terms of linearized Newman-Penrose scalars.

1. INTRODUCTION

The black hole stability problem, i.e. the problem of proving dynamical stability for the Kerr family of black hole spacetimes, is one of the central open problems in General Relativity. The analysis of linear test fields on the exterior Kerr spacetime is an important step towards the full non-linear stability problem. For test fields of spin 0, i.e. solutions of the wave equation $\nabla^a \nabla_a \psi = 0$, estimates proving boundedness and decay in time are known to hold. See [20, 14, 2, 42] for references and background.

The field equations for linear test fields of spins 1 and 2 are the Maxwell and linearized gravity¹ equations, respectively. These equations imply wave equations for the Newman-Penrose Maxwell and linearized Weyl scalars. In particular, the Newman-Penrose scalars of spin weight zero satisfy (assuming a suitable gauge condition for the case of linearized gravity) analogs of the Regge-Wheeler equation. These wave equations take the form

$$(\nabla^a \nabla_a + c_s \Psi_2) \psi_s = 0$$

where for spin $s = 1$, $c_1 = 2$, $\psi_1 = \Psi_2^{-1/3} \phi_1$, while for spin $s = 2$, $c_2 = 8$, and $\psi_2 = \Psi_2^{-2/3} \dot{\Psi}_2$. Here $\dot{\Psi}_2$ is the linearized Weyl scalar of spin weight zero. See [1] for details. As these scalars can be used as potentials for the Maxwell and linearized Weyl fields, one may apply the techniques developed in the previously mentioned papers to prove estimates also for the Maxwell and linearized gravity equations. This approach has been applied in the case of the Maxwell field on the Schwarzschild background in [7].

In contrast to the spin-0 case, the spin 1 and 2 field equations on the Kerr exterior admit non-trivial finite energy time-independent solutions. We shall refer to time-independent solutions as non-radiating modes. There is a close relation between gauge-invariant non-radiating modes and conserved charge integrals. For the Maxwell field, there is a two-parameter family of non-radiating, Coulomb type solutions which carry the two conserved electric and magnetic charges. In fact,

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¹Note that linearized gravity is distinct from the massless spin-2 equation. On a type D background, any solution to the massless spin-2 equation is proportional to the Weyl tensor of the spacetime. This fact is referred to as the Buchdahl constraint, cf. [8], see also equation (5.8.2) in [37].

a Maxwell field on the Kerr exterior will disperse exactly when it has vanishing charges. For linearized gravity, however, there are both non-radiating modes corresponding to gauge-invariant conserved charges, and “pure gauge” non-radiating modes. Thus conditions ensuring that a solution of linearized gravity will disperse must be a combination of charge-vanishing and gauge conditions.

From the discussion above, it is clear that in order to prove boundedness and decay for higher spin test fields on the Kerr exterior, it is a necessary step to eliminate the non-radiating modes. Due in part to this additional difficulty, decay estimates for the higher spin fields have been proved only for Maxwell test fields. See [7] for the Schwarzschild case and [3] for the Kerr case. In view of the just mentioned relation between non-radiating modes and charges, an essential step in doing so involves setting conserved charges to zero. In order to make effective use of such charge vanishing conditions, it is necessary to have simple expressions for the charge integrals in terms of the field strengths. The main result of this paper is to provide an expression for the conserved charge corresponding to the linearized mass, in terms of linearized curvature quantities on the Kerr background.

We start by discussing the relation between charges and non-radiating modes for the case of the Maxwell field. Let the symmetric valence-2 spinor ϕ_{AB} be the Maxwell spinor², i.e. a solution of the massless spin-1 (source-free Maxwell) equation

$$\nabla_{A'}^A \phi_{AB} = 0$$

and let $\mathcal{F}_{ab} = \phi_{AB}\epsilon_{A'B'}$ be the corresponding complex self-dual two-form. The Maxwell equation takes the form $d\mathcal{F} = 0$ and hence the charge integral

$$\int_S \mathcal{F}$$

depends only on the homology class of the surface S . Here real and imaginary parts correspond to electric and magnetic charges, respectively. The Kerr exterior, being diffeomorphic to \mathbb{R}^4 with a solid cylinder removed, contains topologically non-trivial 2-spheres, and hence the Maxwell equation on the Kerr exterior admits solutions with non-vanishing charges. In view of the fact that the charges are conserved, it is natural that there is a time-independent solution which “carries” the charge. In Boyer-Lindquist coordinates, this takes the explicit form

$$\phi_{AB} = \frac{c}{(r - ia \cos \theta)^2} \iota_{(A} o_{B)}, \quad (1)$$

where c is a complex number, and ι_A, o_A are principal spinors for Kerr.

In order to prove boundedness and decay for the Maxwell field, it is necessary to make use of the above mentioned facts, see [3]. In particular, one eliminates the non-radiating modes by imposing the charge vanishing condition

$$\int_S \mathcal{F} = 0. \quad (2)$$

Written in terms of the Newman-Penrose scalars ϕ_I , $I = 0, 1, 2$, the charge vanishing condition (2) in the Carter tetrad [45] takes the form [3]

$$\int_{S^2(t,r)} 2V_L^{-1/2} \phi_1 + ia \sin \theta (\phi_0 - \phi_2) d\mu = 0, \quad (3)$$

where $S^2(t, r)$ is a sphere of constant t, r in the Boyer-Lindquist coordinates, $V_L = \Delta / (r^2 + a^2)^2$ and $d\mu = \sin \theta d\theta d\varphi$. This yields a relation between the $\ell = 0, m = 0$ spherical harmonic of ϕ_1 and the $\ell = 1, m = 0$ spherical harmonics with spin weights 1, -1 of ϕ_0, ϕ_2 , respectively.

Next, we consider the spin-2 case. Recall that the Kerr spacetime is a vacuum space of Petrov type D and hence, in addition to the Killing vector fields $\partial_t, \partial_\phi$

²The following discussion is in terms of the 2-spinor formalism, cf. [37, 38]

admits a “hidden symmetry” manifested by the existence of the valence-2 Killing spinor $\kappa_{AB} = \psi \iota_{(A} \iota_{B)}$. Here the scalar ψ is determined up to a constant, which we fix by setting³ $M\psi^{-3} = -\Psi_2$ on a Kerr background. In this situation, one may consider the spin-lowered version

$$\psi_{ABCD}\kappa^{CD}$$

of the Weyl spinor, which is again a massless spin-1 field and hence the complex self-dual two-form

$$\mathcal{M}_{ab} = \psi_{ABCD}\kappa^{CD}\epsilon_{A'B'}$$

satisfies the Maxwell equations $d\mathcal{M} = 0$. The charge for this field defined on any topologically non-trivial 2-sphere in the Kerr exterior is

$$\frac{1}{4\pi i} \int_S \mathcal{M} = M, \quad (4)$$

cf. [32] for a tensorial version (the calculation has been done much earlier in [34], but not in the context of Killing spinors and spin-lowering). Here M is the ADM mass [4] of the Kerr spacetime⁴. The relation between the mass and charge for the spin-lowered Weyl tensor \mathcal{M} is natural in view of the fact that the divergence

$$\xi^{A'A} = \nabla^{A'}{}_{B'}\kappa^{AB}$$

is proportional to ∂_t , see the discussion in [38, Chapter 6].

Note that the charge (4) is in general complex. The imaginary part corresponds to the NUT charge, which is the gravitational analog of a magnetic charge. Details are not discussed in this paper, see [39] for the construction of charge integrals in NUT spacetime.

For linearized gravity on the Kerr background, the non-radiating modes include perturbations within the Kerr family, i.e. infinitesimal changes of mass and axial rotation speed. We denote the parameters for these deformations \dot{M}, \dot{a} . Since M, a are gauge-invariant quantities, it is not possible to eliminate these modes by imposing a gauge condition. A canonical analysis along the lines of [28], see below, yields conserved charges corresponding to the Killing fields $\partial_t, \partial_\phi$, which in turn correspond to the gauge invariant deformations \dot{M}, \dot{a} mentioned above.

The infinitesimal boosts, translations and (non-axial) rotations of the black hole yield further non-radiating modes which are, however, “pure gauge” in the sense that they are generated by infinitesimal coordinate changes. If one imposes suitable regularity⁵ conditions on the perturbations which exclude e.g. those which turn on the NUT charge, a 10-dimensional space of non-radiating modes remains. This is spanned by the 2-dimensional space of non-gauge modes which carry the \dot{M}, \dot{a} charges, together with the “pure gauge” non-radiating modes, and corresponds in a natural way to the Lie algebra of the Poincare group. It can be seen from this discussion that a combination of charge vanishing conditions and gauge conditions allows one to eliminate all non-radiating solutions of linearized gravity.

The constraint equations implied by the Maxwell and linearized gravity equations are underdetermined elliptic systems, and therefore admit solutions of compact support, see [16] and references therein. In particular, one may find solutions of the constraint equations with arbitrarily rapid fall-off at infinity. The corresponding solutions of the Maxwell equations have vanishing charges. For the case of linearized

³This choice has the natural (non vanishing) Minkowski limit $\psi = r$.

⁴Equivalently, the mass parameter in the Boyer-Lindquist form of the Kerr line element.

⁵The Kerr family of line elements may be viewed as part of the type D family of vacuum metrics which includes, among others, the NUT and C-metrics. See section 4.2 for further discussion. The perturbations corresponding e.g. to infinitesimal deformations of the NUT parameter are singular and may thus be excluded by suitable regularity and decay conditions. See [43], [29] for remarks.

gravity, the charges corresponding to \dot{M} , \dot{a} vanish for solutions of the field equations with rapid fall-off at infinity. For such solutions, all non-radiating modes may therefore be eliminated by imposing suitable gauge conditions.

The following discussion may easily be extended to the Einstein-Maxwell equations. Given an asymptotically flat vacuum spacetime (N, g_{ab}) , a solution of the linearized Einstein equations \dot{g}_{ab} (satisfying suitable asymptotic conditions) and a Killing field $\xi^a \partial_a$ we have that the variation of the Hamiltonian current is an exact form, which yields the relation

$$\dot{\mathcal{P}}_{\xi; \infty} = \int_S \dot{\mathbf{Q}}[\xi] - \xi \cdot \Theta. \quad (5)$$

Here, $\mathcal{P}_{\xi; \infty}$ is the Hamiltonian charge at infinity, generating the action of ξ , $\mathbf{Q}[\xi]$ is the Noether charge two-form for ξ , and Θ is the symplectic current three-form, defined with respect to the variation \dot{g}_{ab} . We use a $\dot{}$ to denote variations along \dot{g}_{ab} , thus $\dot{\mathcal{P}}_{\xi; \infty}$ and $\dot{\mathbf{Q}}[\xi]$ denote the variation of the Hamiltonian and the Noether two-form, respectively. The integral on the right hand side of (5) is evaluated over an arbitrary sphere, which generates the second homology class.

For the case of $\xi = \partial_t$, and considering solutions of the linearized Einstein equations on the Kerr background we have, following the discussion above,

$$\dot{M} = \dot{\mathcal{P}}_{\partial_t; \infty}$$

Working with the Carter tetrad, let Ψ_i , $i = 0, \dots, 4$ be the Weyl scalars and let Z^I , $I = 0, 1, 2$ denote the corresponding basis for the space of complex, self-dual two-forms, see section 2 for details. In this paper we shall show that the natural linearization of the spin-lowered Weyl tensor \mathcal{M} is the two-form

$$\dot{\mathcal{M}} = \psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \frac{3}{2} \psi \Psi_2 \dot{Z}^1.$$

As will be demonstrated, see section 5 below, $\dot{\mathcal{M}}$ is closed, and hence the integral

$$\int_S \dot{\mathcal{M}} \quad (6)$$

defines a conserved charge. A charge vanishing condition for the linearized mass, analogous to the one discussed above for the charges of the Maxwell field, may be introduced by requiring that this integral vanishes. The coordinate form of this charge vanishing condition is

$$\int_{S^2(t,r)} (2V_L^{-1/2} \hat{\Psi}_2 + ia \sin \theta \dot{\Psi}_{diff}) (r - ia \cos \theta) d\mu = 0, \quad (7)$$

which should be compared to the corresponding condition for the Maxwell case, cf. (3). Here, $\hat{\Psi}_2$ and $\dot{\Psi}_{diff}$ are suitable combinations of the linearized curvature scalars $\dot{\Psi}_1, \dot{\Psi}_2, \dot{\Psi}_3$ and linearized tetrad.

Let \dot{g}_{ab} be a solution of the linearized Einstein equation on the Kerr background, satisfying suitable asymptotic conditions, and let \dot{M} be the corresponding perturbation of the ADM mass. Letting $S = S^2(t, r)$ and evaluating the limit of (6) as $r \rightarrow \infty$ one finds, in view of the fact that (6) is conserved, the identity

$$\dot{M} = \frac{1}{4\pi i} \int_S \dot{\mathcal{M}}$$

for any smooth 2-sphere S in the exterior of the Kerr black hole. Thus we have the relation

$$\int_S \dot{\mathbf{Q}}[\partial_t] - \partial_t \cdot \Theta = \frac{1}{4\pi i} \int_S \dot{\mathcal{M}} \quad (8)$$

for any surface S in the Kerr exterior. We remark that the left hand side of (8) can be evaluated in terms of the metric perturbation using the expressions for \mathbf{Q} and Θ given in [28, section V]. On the other hand, the right hand side has been

calculated in terms of linearized curvature. It would be of interest to have a direct derivation of the resulting identity.

The canonical analysis following [28] which has been discussed above shows that in addition to the conserved charge corresponding to \dot{M} , equation (5) with $\xi = \partial_\phi$, the angular Killing field, gives a conserved charge integral for linearized angular momentum \dot{a} . If ∂_ϕ is tangent to S , then the term $\partial_\phi \cdot \Theta$ does not contribute in (5). We remark that an expression for \dot{a} for linearized gravity on the Schwarzschild background was given in [30, section 3]. A charge integral for \dot{a} for linearized gravity on the Kerr background will be considered in a future paper.

Remark 1.1. (1) *There are many candidates for a quasi-local mass expression in the literature including, to mention just a few, those put forward by Penrose, Brown and York, and Wang and Yau. See the review of Szabados [41] for background and references. Although as discussed above, cf. equation (4), for a spacetime of type D, there is a quasi-local mass charge, it must be emphasized that for a general spacetime one cannot expect the existence of a quasi-local mass which is conserved, i.e. independent of the 2-surface used in its definition. The same is true for linearized gravity on a general background. Thus the existence of a conserved charge integral for the linearized mass is a feature which is special to linearized gravity on a background with Killing symmetries.*

- (2) *If we consider linearized gravity without sources, on the Minkowski background, the linearized mass must vanish due to the fact that Minkowski space is topologically trivial. This reflects the fact that when viewed as a function on the space of Cauchy data, the ADM mass vanishes quadratically at the trivial data, cf. [10]. On the other hand, by the positive mass theorem, for any non-flat spacetime, asymptotic to Minkowski space in a suitable sense, the ADM mass defined at infinity must be positive.*

This paper is organized as follows. In section 2, we introduce bivector formalism. Conformal Killing Yano tensors and Killing spinors are discussed in section 3. Section 4 deals with conserved charges for spin-2 fields on Minkowski (§4.1) and type D spacetimes (§4.2). The main result, a charge integral in terms of linearized curvature, is derived in section 5, and finally, section 6 contains some concluding remarks.

2. PRELIMINARIES AND NOTATION

Let (N, g_{ab}) be a 4 dimensional Lorentzian spacetime of signature $+- --$, admitting a spinor structure. Although most of the results can be generalized to the electrovac case with cosmological constant, we restrict in this paper to the vacuum case. In particular, we consider test Maxwell fields and linearized gravity on vacuum type D background spacetimes.

Let o_A, ι_A be a spinor dyad, normalized so that $o_A \iota^A = 1$, and let

$$l^a = o^A \bar{o}^{A'}, \quad m^a = o^A \bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}$$

be the corresponding null tetrad, satisfying $l^a n_a = -m^a \bar{m}_a = 1$, the other inner products being zero. The 2-spinor calculus provides a powerful tool for computations in 4-dimensional geometry. The GHP formalism deals with dyad (or equivalently tetrad) components of geometric objects and exploits the simplifications arising by taking into account the action of dyad rescalings and permutations. These formalisms are closely related to the less widely used *bivector formalism* [34, 6, 9, 27] in which the basic quantity is a basis for the 3-dimensional space of complex self-dual two-forms. A two-form Z is called self-dual, if $*Z = iZ$ and anti self-dual, if

$*Z = -iZ$. Given a spinor dyad, a natural choice⁶ is

$$Z_{ab}^0 = 2\bar{m}_{[a}n_{b]} = \iota_{A'}\iota_B\bar{\epsilon}_{A'B'} \quad (9a)$$

$$Z_{ab}^1 = 2n_{[a}l_{b]} - 2\bar{m}_{[a}m_{b]} = -2o_{(A'}\iota_{B)}\bar{\epsilon}_{A'B'} \quad (9b)$$

$$Z_{ab}^2 = 2l_{[a}m_{b]} = o_{A'}o_B\bar{\epsilon}_{A'B'}, \quad (9c)$$

where the notation $2x_{[a}y_{b]} = x_a y_b - y_a x_b$ for anti symmetrization and $2x_{(a}y_{b)} = x_a y_b + y_a x_b$ for symmetrization is used. We use capital latin indices I, J, K taking values in $0, 1, 2$ for the elements in the bivector triad Z^I . The metric g_{ab} induces a triad metric G_{IJ} and its inverse G^{IJ} given by

$$G^{IJ} = Z^I \cdot Z^J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad G_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here, \cdot is the induced inner product on two-form, $Z^I \cdot Z^J = \frac{1}{2}Z^I{}_{ab}Z^{Jab}$. Triad indices are raised and lowered with this metric,

$$Z_0 = Z^2, \quad Z_1 = -\frac{1}{2}Z^1, \quad Z_2 = Z^0.$$

More general we have

Proposition 2.1.

$$Z^J{}_a{}^c Z^K{}_{bc} = \frac{1}{2}G^{JK}g_{ab} + \epsilon^{JKL}Z_{Lab} \quad (10a)$$

$$Z^J{}_{[a}{}^c \bar{Z}^K{}_{b]c} = 0 \quad (10b)$$

$$Z^{Jab} \bar{Z}^K{}_{ab} = 0 \quad (10c)$$

with ϵ^{JKL} the totally antisymmetric symbol fixed by $\epsilon^{012} = 1$.

A real two-form F_{ab} , e.g. the Maxwell field strength, has spinor representation

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \bar{\phi}_{A'B'}\epsilon_{AB}.$$

It is equivalent to the symmetric 2-spinor $\phi_{AB} = \phi_2 o_A o_B - 2\phi_1 o_{(A} \iota_{B)} + \phi_0 \iota_A \iota_B$, where the six real degrees of freedom of F_{ab} are encoded in 3 complex scalars

$$\phi_0 = \phi_{AB} o^A o^B = F_{ab} l^a m^b = F \cdot Z_0$$

$$\phi_1 = \phi_{AB} \iota^A o^B = \frac{1}{2}F_{ab}(l^a n^b - m^a \bar{m}^b) = F \cdot Z_1$$

$$\phi_2 = \phi_{AB} \iota^A \iota^B = F_{ab} \bar{m}^a n^b = F \cdot Z_2.$$

So the real two-form has bivector representation

$$F = \phi_0 Z^0 + \phi_1 Z^1 + \phi_2 Z^2 + \bar{\phi}_0 \bar{Z}^0 + \bar{\phi}_1 \bar{Z}^1 + \bar{\phi}_2 \bar{Z}^2,$$

or in index notation $\phi_I = F \cdot Z_I$ and $F = \phi_I Z^I + \bar{\phi}_I \bar{Z}^I$.

The Weyl tensor is a symmetric 2-tensor over bivector space and has spinor representation

$$-C_{abcd} = \Psi_{ABCD}\bar{\epsilon}_{A'B'}\bar{\epsilon}_{C'D'} + \bar{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

⁶We use the convention of [18], which differs from [27, 17] by a factor of 2 in the middle component and the numbering.

where Ψ_{ABCD} is a completely symmetric 4-spinor. The 10 degrees of freedom of the Weyl tensor are given by 5 complex scalars⁷

$$\begin{aligned}\Psi_0 &= \Psi_{ABCD} o^A o^B o^C o^D = -C_{abcd} l^a m^b l^c m^d = -C \cdot (Z_0, Z_0) \\ \Psi_1 &= \Psi_{ABCD} o^A o^B o^C \iota^D = -C_{abcd} l^a n^b l^c m^d = -C \cdot (Z_0, Z_1) \\ \Psi_2 &= \Psi_{ABCD} o^A o^B \iota^C \iota^D = -C_{abcd} l^a m^b \bar{m}^c n^d = -C \cdot (Z_0, Z_2) = -C \cdot (Z_1, Z_1) \\ \Psi_3 &= \Psi_{ABCD} o^A \iota^B \iota^C \iota^D = -C_{abcd} l^a n^b \bar{m}^c n^d = -C \cdot (Z_2, Z_1) \\ \Psi_4 &= \Psi_{ABCD} \iota^A \iota^B \iota^C \iota^D = -C_{abcd} n^a \bar{m}^b n^c \bar{m}^d = -C \cdot (Z_2, Z_2).\end{aligned}$$

Similarly we could have used the Weyl 2-bivector

$$C_{IJ} = -\frac{1}{4} C_{abcd} Z_I^{ab} Z_J^{cd} = \begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix}$$

which relates to the real Weyl tensor via

$$-C_{abcd} = C_{IJ} Z_{ab}^I \otimes Z_{cd}^J + \bar{C}_{IJ} \bar{Z}_{ab}^I \otimes \bar{Z}_{cd}^J. \quad (11)$$

Because of different conventions and normalisations in the literature [34, 6, 9, 27], we rederive here the equations of structure in bivector formalism. Based on Cartan's equations of structure for tetrad one-forms⁸

$$de^a = -\omega^a_b \wedge e^b \quad \Omega^a_b = d\omega^a_b + \omega^a_c \wedge \omega^c_b, \quad (12)$$

Bianchi identities

$$\Omega^a_b \wedge e^b = 0 \quad d\Omega^a_b = \Omega^a_c \wedge \omega^c_b - \omega^a_c \wedge \Omega^c_b, \quad (13)$$

and definitions of connection one-forms σ_J and curvature two-forms Σ_J in bivector formalism,

$$\omega_{ab} e^a \wedge e^b = -2\sigma_J Z^J - 2\bar{\sigma}_J \bar{Z}^J \quad \Omega_{ab} e^a \wedge e^b = -2\Sigma_J Z^J - 2\bar{\Sigma}_J \bar{Z}^J, \quad (14)$$

we find

Proposition 2.2. *The bivector equations of structure are*

$$dZ^J = -2\epsilon^{JKL} \sigma_K \wedge Z_L \quad \Sigma_J = d\sigma_J + \frac{1}{2} \epsilon_{JKL} \sigma^K \wedge \sigma^L \quad (15)$$

while the Bianchi identities read

$$\Sigma_{[J} \wedge Z_{K]} = 0 \quad d\Sigma_J = -\epsilon_{JKL} \Sigma^K \wedge \sigma^L. \quad (16)$$

Here \wedge is the usual wedge product of one-forms σ^J and two-forms Z^J, Σ^J .

Proof. Expanding the bivectors $Z^J = \frac{1}{2} Z_{ab}^J e^a \wedge e^b$, we find

$$\begin{aligned}dZ^J &= \frac{1}{2} Z_{ab}^J (de^a \wedge e^b - e^a \wedge de^b) = Z_{ab}^J de^a \wedge e^b \\ &= -Z_{ab}^J \omega^a_c e^c \wedge e^b \\ &= Z_{ab}^J (\sigma_K Z^{Ka}_c + \bar{\sigma}_K \bar{Z}^{Ka}_c) \wedge e^c \wedge e^b \\ &= \epsilon^{JKL} Z_{Lbc} \sigma_K \wedge e^c \wedge e^b \\ &= -2\epsilon^{JKL} \sigma_K \wedge Z_L\end{aligned}$$

⁷Due to its symmetries, the Weyl tensor is a symmetric two-tensor over the space of two-forms. The induced inner product is $C \cdot (Z_I, Z_J) = \frac{1}{4} C_{abcd} Z_I^{ab} Z_J^{cd}$.

⁸Connection and curvature are defined by $\omega^a_{b\mu} = e^a_\nu \nabla_\mu e_b^\nu$ and $\Omega^a_{b\mu\nu} = 2e^a_\sigma \nabla_{[\mu} \nabla_{\nu]} e_b^\sigma$, respectively.

where proposition 2.1 has been used in the third step. For the second equation of structure, we plug (14) into (12),

$$-\Sigma_J Z_{ab}^J - \bar{\Sigma}_J \bar{Z}_{ab}^J = -d\sigma_J Z_{ab}^J - d\bar{\sigma}_J \bar{Z}_{ab}^J + (\sigma_J Z_{ac}^J + \bar{\sigma}_J \bar{Z}_{ac}^J) \wedge (\sigma_K Z^{Kc}_b + \bar{\sigma}_K \bar{Z}^{Kc}_b).$$

Since $Z^J \cdot \bar{Z}^K = 0$ and proposition 2.1, the selfdual part reads

$$\Sigma_J Z_{ab}^J = d\sigma_J Z_{ab}^J + \epsilon^{KLJ} Z_{Jab} \sigma_K \wedge \sigma_L.$$

Changing index positions by using $\det G_{JK} = \frac{1}{2}$ gives the 2nd equation of structure. For the first Bianchi identity, look at

$$\begin{aligned} 0 &= d^2 Z^J \\ &= -2\epsilon^{JKL} (d\sigma_K \wedge Z_L - \sigma_K \wedge dZ_L) \\ &= -2\epsilon^{JKL} \left(\Sigma_K \wedge Z_L - \frac{1}{2} \epsilon_{KNM} \sigma^N \wedge \sigma^M \wedge Z_L + \sigma_K \wedge \epsilon_{LNM} \sigma^N \wedge Z^M \right) \\ &= -2\epsilon^{JKL} \underbrace{\Sigma_K \wedge Z_L + \sigma^L \wedge \sigma^J \wedge Z_L - \sigma^J \wedge \sigma^L \wedge Z_L - 2\sigma_L \wedge \sigma^J \wedge Z^L}_{=0} + 2 \underbrace{\sigma_K \wedge \sigma^K \wedge Z^J}_{=0} \end{aligned}$$

where the identity $\epsilon^{IJK} \epsilon_{INM} = \delta_N^J \delta_M^K - \delta_M^J \delta_N^K$ has been used. Finally, the second Bianchi identity is

$$\begin{aligned} d\Sigma_J &= -\epsilon_{JKL} d\sigma^K \wedge \sigma^L \\ &= -\epsilon_{JKL} (\Sigma^K - \epsilon^{KMN} \sigma_M \wedge \sigma_N) \wedge \sigma^L \\ &= -\epsilon_{JKL} \Sigma^K \wedge \sigma^L + \underbrace{\sigma_L \wedge \sigma_J \wedge \sigma^L}_{=0} - \sigma_J \wedge \underbrace{\sigma_L \wedge \sigma^L}_{=0}. \end{aligned}$$

□

Remark 2.3. *Instead of using Cartan equations for the tetrad one could have used the bivector connection form*

$$\omega_{IJa} := \epsilon_{IJK} \sigma_a^K = Z_{[J}^{bc} \nabla_a Z_{I]bc}. \quad (17)$$

For later use it is convenient to write the components of the equations of structure explicitly. The connection one-forms for example can be expressed in terms of NP spin coefficients,

$$\sigma_{0a} = m^b \nabla_a l_b = \tau l_a + \kappa n_a - \rho m_a - \sigma \bar{m}_a \quad (18a)$$

$$\sigma_{1a} = \frac{1}{2} (n^b \nabla_a l_b - \bar{m}^b \nabla_a m_b) = -\epsilon' l_a + \epsilon n_a + \beta' m_a - \beta \bar{m}_a \quad (18b)$$

$$\sigma_{2a} = -\bar{m}^b \nabla_a n_b = -\kappa' l_a - \tau' n_a + \sigma' m_a + \rho' \bar{m}_a. \quad (18c)$$

The middle component σ_{1a} collects all unweighted coefficients and so can be used to define the GHP covariant derivative $\Theta_a \eta = (\nabla_a - p\sigma_{1a} - q\bar{\sigma}_{1a})\eta$. To avoid clutter in the notation, we write $\Gamma := \sigma_0$ and $\sigma_2 = -\Gamma'$, where $'$ is the GHP prime operation[21]. Derivatives of the spinor dyad can now be written in the compact form $\Theta_a o^A = -\Gamma_a \iota^A$ and $\Theta_a \iota^A = -\Gamma'_a o^A$, and the components of the first equations of structure, which we present here for convenience with the usual exterior derivative and with weighted exterior derivative $d^\Theta = d - p\sigma_1 \wedge -q\bar{\sigma}_1 \wedge$, read

$$d^\Theta Z^0 = \Gamma' \wedge Z^1 \quad \Leftrightarrow \quad dZ^0 = -2\sigma_1 \wedge Z^0 + \Gamma' \wedge Z^1 \quad (19a)$$

$$d^\Theta Z^1 = 2\Gamma \wedge Z^0 + 2\Gamma' \wedge Z^2 \quad \Leftrightarrow \quad dZ^1 = 2\Gamma \wedge Z^0 + 2\Gamma' \wedge Z^2 \quad (19b)$$

$$d^\Theta Z^2 = \Gamma \wedge Z^1 \quad \Leftrightarrow \quad dZ^2 = 2\sigma_1 \wedge Z^2 + \Gamma \wedge Z^1. \quad (19c)$$

Note that the middle component can be simplified to $dZ^1 = -h \wedge Z^1$ with the one-form $h = 2(\rho' l + \rho n - \tau' m - \tau \bar{m})$. This fact and a relation between type D curvature Ψ_2 and h will be crucial in the derivation of the conservation law in section 5.

In vacuum, we have for the curvature two-forms $\Sigma_J = C_{JK} Z^K$ and the components of the second equations of structure read

$$\Sigma_0 = C_{0J} Z^J = d^\ominus \Gamma = d\Gamma - 2\sigma_1 \wedge \Gamma \quad (20a)$$

$$\Sigma_1 = C_{1J} Z^J = d\sigma_1 - \Gamma \wedge \Gamma' \quad (20b)$$

$$\Sigma_2 = C_{2J} Z^J = -d^\ominus \Gamma' = -d\Gamma' - 2\sigma_1 \wedge \Gamma'. \quad (20c)$$

Finally the Bianchi identities are

$$d^\ominus \Sigma_0 = -2\Gamma \wedge \Sigma_1 \quad \Leftrightarrow \quad d\Sigma_0 = 2\sigma_1 \wedge \Sigma_0 - 2\Gamma \wedge \Sigma_1 \quad (21a)$$

$$d^\ominus \Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2 \quad \Leftrightarrow \quad d\Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2 \quad (21b)$$

$$d^\ominus \Sigma_2 = -2\Gamma' \wedge \Sigma_1 \quad \Leftrightarrow \quad d\Sigma_2 = -2\sigma_1 \wedge \Sigma_2 - 2\Gamma' \wedge \Sigma_1. \quad (21c)$$

3. CONFORMAL KILLING YANO TENSORS AND KILLING SPINORS

Conformal Killing Yano tensors of rank 2 are two-forms Y_{ab} solving the conformal Killing Yano equation,

$$Y_{a(b;c)} = g_{bc}\xi_a - g_{a(b}\xi_{c)}, \quad \text{where } \xi_a = \frac{1}{3}Y_a{}^b{}_{;b}. \quad (22)$$

It is well known, that the divergence ξ^a is a Killing vector and in case it vanishes, Y_{ab} is called Killing Yano tensor. The symmetrised product $X_{c(a}Y_{b)}{}^c =: K_{ab}$ of Killing Yano tensors X_{ab}, Y_{ab} is a Killing tensor, $\nabla_{(a}K_{bc)} = 0$, which can be used to construct a constant of motion or a symmetry operator for e.g. the scalar wave equation, known as Carter's constant and Carter operator, respectively. By inserting $Y_{ab} = \kappa_{AB}\epsilon_{A'B'} + \bar{\kappa}_{A'B'}\epsilon_{AB}$ into (22) one can show that κ_{AB} and $\bar{\kappa}_{A'B'}$ satisfy the Killing spinor equation

$$\nabla_{A'}(A\kappa_{BC}) = 0 \quad (23)$$

and its complex conjugated version. For the spinor components $\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A} \iota_{B)} + \kappa_0 \iota_A \iota_B$ (or equivalently the self dual bivector components of Y_{ab} , we find the following set of eight scalar equations

$$\flat \kappa_0 = -2\kappa \kappa_1, \quad \delta \kappa_0 = -2\sigma \kappa_1, \quad \flat' \kappa_2 = -2\kappa' \kappa_1, \quad \delta' \kappa_2 = -2\sigma' \kappa_1 \quad (24)$$

$$\begin{aligned} (\delta' + 2\tau')\kappa_0 + 2(\flat + \rho)\kappa_1 &= -2\kappa \kappa_2, & (\flat' + 2\rho')\kappa_0 + 2(\delta + \tau)\kappa_1 &= -2\sigma \kappa_0 \\ (\delta + 2\tau)\kappa_2 + 2(\flat' + \rho')\kappa_1 &= -2\kappa' \kappa_0, & (\flat + 2\rho)\kappa_2 + 2(\delta' + \tau')\kappa_1 &= -2\sigma' \kappa_2, \end{aligned} \quad (25)$$

by projecting (23) into a spinor dyad. Thus, we have three different sets of equations, (22), (23), (24,25), which are equivalent and we will use the most appropriate for the problem at hand.

As spin-s fields are heavily restricted on curved backgrounds (Buchdahl constraint, see equation (5.8.2) in [37]), so are Killing spinors. Consider a Killing spinor $\kappa_{A_1 \dots A_n} = \kappa_{(A_1 \dots A_n)}$ which satisfies the Killing spinor equation of valence n

$$\nabla_{B'}(B\kappa_{A_1 \dots A_n}) = 0. \quad (26)$$

Contracting a second derivative $\nabla^{B'}{}_C$ and symmetrising gives

$$\begin{aligned} 0 &= \nabla^{B'}{}_C \nabla_{|B'}{}^B \kappa_{A_1 \dots A_n} \\ &= -\square_{(BC} \kappa_{A_1 \dots A_n)} \\ &= \Psi_{(BCA_1}{}^D \kappa_{DA_2 \dots A_n)} + \dots + \Psi_{(BCA_n}{}^D \kappa_{A_1 \dots A_{n-1}D)} \\ &= n\Psi_{(BCA_1}{}^D \kappa_{DA_2 \dots A_n)}. \end{aligned}$$

For Killing spinors of valence 1 (satisfying the twistor equation) this yields $0 = \Psi_{ABCD}\kappa^D$ as can be found in [38], eq.(6.1.6). For 2-spinors we find

$$0 = \Psi_{(ABC}{}^D \kappa_{DE)}. \quad (27)$$

TABLE 1. Poincaré isometries and corresponding charges

label	isometry	charge	#
\mathcal{T}_t	time translation	mass	1
\mathcal{T}_i	spatial translations	linear momenta	3
\mathcal{L}_{ij}	rotations	angular momenta	3
\mathcal{L}_{ti}	boosts	center of mass	3

For non trivial κ , this restricts the spacetime to be of Petrov type D, N or O . For a given spacetime of type D in a principal frame (only $\Psi_2 \neq 0$) (27) becomes

$$\begin{aligned} 0 &= \Psi_2 o_{(A} o_B \iota_C \iota_{D)} (\kappa_0 \iota^D \iota_E) + \kappa_1 o^D \iota_E + \kappa_1 \iota^D o_E + \kappa_2 o^D o_E) \\ &= \Psi_2 (C_1 \kappa_0 o_{(A} \iota_B \iota_C \iota_{E)} + C_2 \kappa_2 \iota_{(A} o_B o_C o_{E)}) \end{aligned}$$

with constants C_1, C_2 and it follows $\kappa_0 \equiv 0 \equiv \kappa_2$. The remaining component satisfies the simplified equations (32), which have only one non trivial complex solution, cf. [22] where explicit integration of the conformal Killing Yano equation was done.

4. CONSERVED CHARGES

4.1. Conserved charges for Minkowski spacetime. The Killing spinor equation or conformal Killing Yano equation on Minkowski space has been widely discussed in the literature [38],[32], [25] and the explicit solution in cartesian coordinates is well known,

$$\kappa^{AB} = U^{AB} + 2x^{A'} (A V_{A'}^B) + x^{A'} x^{B'} W_{A'B'}. \quad (28)$$

Here $U^{AB}, W_{A'B'}$ are constant, symmetric spinors and $V_{A'}^B$ a constant complex vector which yield $2 \cdot 6 + 8 = 20$ independent real solutions. Each solution gives a charge when contracted into a spin-2 field, e.g. the linearized Weyl tensor, and integrated over a 2-sphere. In [38, p.99], 10 of these charges are related to a source for linearized gravity in the following sense. Given a divergence free, symmetric energy momentum tensor T_{ab} , one has for each Killing field ξ^b the divergence free current $J_a = T_{ab} \xi^b$. Using linearized Einstein equations

$$\dot{G}_{ab} = \dot{R}_{acb}{}^c - \frac{1}{2} g_{ab} \dot{R}_{cd}{}^{cd} = -8\pi G \dot{T}_{ab} \quad (29)$$

and the conformal Killing Yano equation (22), they showed

$$3 \int_{\partial\Sigma} \dot{R}_{abcd} \star Y^{cd} dx^a \wedge dx^b = 16\pi G \int_{\Sigma} e_{abc}{}^d \dot{T}_{df} \xi^f dx^a \wedge dx^b \wedge dx^c. \quad (30)$$

Here Σ denotes a 3 dimensional hypersurface with boundary $\partial\Sigma$ and e_{abcd} is the Levi-Civita tensor. The left hand side is the charge integral described above, while the right hand side gives the more familiar form of a conserved three-form corresponding to a linearized source and a Killing vector $\xi^a = \frac{1}{3} Y^{ab}{}_{;b}$. Note that it is the dual conformal Killing Yano tensor on the left hand side, which gives the charge associated to the isometry ξ^a . In cartesian coordinates $x^a = (t, x, y, z)$ the Poincaré isometries read

$$\mathcal{T}_a = \frac{\partial}{\partial x^a} \quad \mathcal{L}_{ab} = x_a \frac{\partial}{\partial x^b} - x_b \frac{\partial}{\partial x^a} \quad (31)$$

and the relation to the charges is listed in table 1. The angular momentum around the z -axis is found in the component $\mathcal{L}_{xy} = \partial_\phi$. Explicit expressions for linearized sources generating these charges can be found in [29, eq.27].

The 10 remaining charges cannot be generated this way, since the corresponding conformal Killing Yano tensors have vanishing divergence (they are Killing Yano

TABLE 2. Solutions to the Killing spinor equation on Minkowski spacetime in spherical coordinates.

label	components			combination	divergence	
	$\kappa_0/\sqrt{2}$	κ_1	$\kappa_2/\sqrt{2}$		Re	Im
Ω_m^0	${}_1Y_{1m}$	${}_0Y_{1m}$	${}_{-1}Y_{1m}$	Ω_{-1}^0	0	0
				Ω_0^0	0	0
				Ω_1^0	0	0
				Ω^1	\mathcal{T}_t	0
Ω_m^1	$(t-r) {}_1Y_{1m}$	$t {}_0Y_{1m}$	$(t+r) {}_{-1}Y_{1m}$	$\Omega_1^1 - \Omega_{-1}^1$	\mathcal{T}_x	0
				$i\Omega_1^1 + i\Omega_{-1}^1$	\mathcal{T}_y	0
				Ω_0^1	\mathcal{T}_z	0
				$\Omega_1^2 - \Omega_{-1}^2$	\mathcal{L}_{tx}	\mathcal{L}_{yz}
Ω_m^2	$(t-r)^2 {}_1Y_{1m}$	$(t^2-r^2) {}_0Y_{1m}$	$(t+r)^2 {}_{-1}Y_{1m}$	$i\Omega_1^2 + i\Omega_{-1}^2$	\mathcal{L}_{ty}	\mathcal{L}_{xz}
				Ω_0^2	\mathcal{L}_{tz}	\mathcal{L}_{xy}

tensors). One of these charges corresponds to the NUT parameter⁹, and the remaining nine are three dual linear momenta and six ofam¹⁰. In the expression (28) for a general Killing spinor, they correspond to U and the imaginary part of V . For a metric perturbation, which one might interpret as a potential for the linearized curvature, these 10 additional charges vanish, see [38, §6.5].

To understand the charges as projections into $l=0$ and $l=1$ mode, we rederive the complete set of solutions in spherical coordinates using spin weighted spherical harmonics. A null tetrad for Minkowski spacetime in spherical coordinates (t, r, θ, ϕ) (symmetric Carter tetrad) is given by

$$l^a = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad n^a = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \quad m^a = \frac{1}{\sqrt{2}r} \begin{bmatrix} 0 \\ 0 \\ 1 \\ \frac{i}{\sin\theta} \end{bmatrix},$$

with non vanishing spin coefficients

$$\rho = -\frac{1}{\sqrt{2}r} = -\rho', \quad \beta = \frac{\cot\theta}{2\sqrt{2}r} = \beta'.$$

A general two-form can be expanded

$$\begin{aligned} Y = & +\kappa_2 \frac{r}{2} (dr - dt) \wedge (d\theta + i \sin\theta d\varphi) \\ & -\kappa_1 (dt \wedge dr + ir^2 \sin\theta d\theta \wedge d\varphi) \\ & +\kappa_0 \frac{r}{2} (dr + dt) \wedge (d\theta - i \sin\theta d\varphi) + c.c. \end{aligned}$$

and it is a conformal Killing Yano tensor, if the components κ_i satisfy (24,25). The subset (24) of the Killing spinor equation becomes

$$\begin{aligned} (\partial_t + \partial_r) \kappa_0 &= 0, & \left(\partial_\theta + \frac{i}{\sin\theta} \partial_\varphi - \cot\theta \right) \kappa_0 &= 0, \\ (\partial_t - \partial_r) \kappa_2 &= 0, & \left(\partial_\theta - \frac{i}{\sin\theta} \partial_\varphi - \cot\theta \right) \kappa_2 &= 0, \end{aligned}$$

so $\kappa_0 = f_0(t-r) {}_1Y_{1m}$ and $\kappa_2 = f_1(t+r) {}_{-1}Y_{1m}$ with functions f_i depending on advanced and retarded coordinates only. Finally (25) can be solved for κ_1 , which is only possible for particular functions f_i . The result is given in table 2. Ω^1 is one complex solution, while $\Omega_m^i, i=0,1,2$ represent 3 complex solutions each,

⁹sometimes called dual mass, because of duality rotation from Schwarzschild to NUT, see the appendix of [39]

¹⁰Obstructions for angular momentum, see [31].

($m = 0, \pm 1$). We find the following correspondence to the solutions (28) in cartesian coordinates

$$\Omega_m^0 \leftrightarrow U^{AB}, \quad \Omega^1, \Omega_m^1 \leftrightarrow V_{A'}^A, \quad \Omega_m^2 \leftrightarrow W_{A'B'}.$$

4.2. Conserved charges for type D spacetimes. The vacuum field equations in the algebraically special case of Petrov type D have been integrated explicitly by Kinnersley [36]. An explicit type D line element solving the Einstein-Maxwell equations with cosmological constant is known, from which all type D line elements of this type can be derived by certain limiting procedures, see [40, §19.1.2], see also [15]. The family of type D spacetimes contains the Kerr and Schwarzschild solutions, but also solutions with more complicated topology and asymptotic behaviour, such as the NUT- or C-metrics, and solutions whose orbits of the isometry group are null. In the following, we again restrict to the vacuum case.

A Newman-Penrose tetrad such that the two real null vectors l^a, n^a are aligned with the two repeated principal null directions of a Weyl tensor of Petrov type D is called a principal tetrad. In this case,

$$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4, \quad \kappa = \kappa' = 0 = \sigma = \sigma'$$

and $\Psi_2 \neq 0$. Due to the integrability condition (27), we have $\kappa_0 = 0 = \kappa_2$. Hence, the components (24,25) of the Killing spinor equation simplify to

$$(\mathfrak{p} + \rho)\kappa_1 = 0, \quad (\delta + \tau)\kappa_1 = 0, \quad (\mathfrak{p}' + \rho')\kappa_1 = 0, \quad (\delta' + \tau')\kappa_1 = 0. \quad (32)$$

Comparison with the Bianchi identities

$$(\mathfrak{p} - 3\rho)\Psi_2 = 0, \quad (\delta - 3\tau)\Psi_2 = 0, \quad (\mathfrak{p}' - 3\rho')\Psi_2 = 0, \quad (\delta' - 3\tau')\Psi_2 = 0, \quad (33)$$

shows that $\kappa_1 := \psi \propto \Psi_2^{-1/3}$ is a solution, and in fact up to a constant $\kappa_{AB} = \psi o_{(A} l_{B)}$ is the only solution of the Killing spinor equation.

The divergence $\xi^{AA'} = \nabla^{A'}_B \kappa^{AB}$ is a Killing vector field, which is proportional to a real Killing vector field for all type D spacetimes except for Kinnersley class IIIB, cf. [11]. If $\xi^{AA'}$ is real, the imaginary part of κ_{AB} is a Killing-Yano tensor. Spacetimes satisfying the just mentioned condition are called *generalized Kerr-NUT* spacetimes [19]. The square of the Killing-Yano tensor is a symmetric Killing tensor $K_{ab} = Y_{ac} Y^c_b$ and it follows, that $\eta^a = K^{ab} \xi_b$ is a Killing vector. On a Kerr background, ξ^a and η^a are linearly independent and span the space of isometries, see [26]. In the special case of a Schwarzschild background, η^a vanishes, see also [12] for details.

For Kerr spacetime in Boyer-Lindquist coordinates we find

$$\Psi_2 = -\frac{M}{(r - ia \cos \theta)^3}, \quad \psi \propto r - ia \cos \theta$$

and we set the factor of proportionality to 1, so that the solution

$$\kappa_0 = 0 \quad \kappa_1 = \psi \quad \kappa_2 = 0 \quad (34)$$

reduces to Ω^1 as given in table 2, in the Minkowski limit $M, a \rightarrow 0$. We find $\nabla_b (\psi Z^{1ab}) = 3 (\partial_t)^a$. The Killing spinor with components given by (34) is

$$\kappa_{AB} = -2\psi o_{(A} l_{B)}, \quad (35)$$

We have $\psi Z^1_{ab} = \kappa_{AB} \epsilon_{A'B'}$ and therefore

$$(\partial_t)_a = \frac{1}{3} \nabla^b (\psi Z^1_{ab}) = -\frac{2}{3} \nabla^{B'B} (\kappa_{AB} \epsilon_{A'B'}) = \frac{2}{3} \nabla_{A'}^B \kappa_{AB}.$$

Spin lowering the Weyl spinor using (35) gives the Maxwell field $\psi_{ABCD}\kappa^{CD}$, which has charges proportional to *mass* and *dual mass*, see also [33]. Letting $\mathcal{M}(C, \kappa)$ denote the corresponding closed complex two-form we have

$$\mathcal{M}(C, \kappa) = \psi \Psi_2 Z^1. \quad (36)$$

Evaluating the charge for the Kerr metric yields

$$\frac{1}{4\pi i} \int_{S^2} \mathcal{M}(C, \kappa) = \frac{1}{4\pi i} \int_{S^2} -\frac{M}{(r - ia \cos \theta)^2} (-i)(r^2 + a^2) \sin \theta d\theta \wedge d\varphi = M, \quad (37)$$

where M is the ADM mass while the dual mass is zero.

The closed two-form (36) has been derived much earlier by Jordan, Ehlers and Sachs [34]. We will repeat the derivation here, since this formulation can be generalized to linearized gravity most easily. On a type D background, the curvature forms and the connection simplify to

$$\Sigma_0 = \Psi_2 Z^2 \quad \Sigma_1 = \Psi_2 Z^1 \quad \Sigma_2 = \Psi_2 Z^0 \quad \Gamma = \tau l - \rho m, \quad (38)$$

so the middle Bianchi identity (21b) becomes

$$\begin{aligned} 2d\Sigma_1 &= 2\Psi_2 [(\rho' \bar{m} - \tau' n) \wedge l \wedge m + (\rho m - \tau l) \wedge \bar{m} \wedge n] \\ &= 2\Psi_2 (\rho' l + \rho n - \tau' m - \tau \bar{m}) \wedge Z^1 \\ &= h \wedge \Sigma_1, \end{aligned}$$

where $h = 2(\rho' l + \rho n - \tau' m - \tau \bar{m})$ was used. As noted in [18], the Bianchi identities (33) can be rewritten as $2d\Psi_2 = 3h\Psi_2$ and one obtains

$$d(\Psi_2 Z^1) = d\Sigma_1 = \frac{1}{2} h \wedge \Sigma_1 = \frac{1}{3} d\Psi_2 \wedge Z^1.$$

We finally end up with the *Jordan-Ehlers-Sachs conservation law* [34],

$$d(\Psi_2^{2/3} Z^1) = 0. \quad (39)$$

Using $\psi \propto \Psi_2^{-1/3}$, this is the same result as (36). See also [27], where the conservation law is generalised to spacetimes of Petrov type II. The result for type D backgrounds fit into the picture of Penrose potentials [23] and in the next section we will see that it generalizes to linear perturbations.

5. FACKERELL'S CONSERVATION LAW

We can of course linearize the two-form (36), which would provide a charge for perturbations within the class of type D spacetimes. But more generally, Fackerell [17] derived a closed two-form for arbitrary linear perturbations around a type D background¹¹. Starting from this conservation law, Fackerell and Crossmann derived field equations for perturbations of Kerr-Newmann spacetime. Let us give a shortened derivation in the vacuum case.

When linearizing (with parameter ϵ) the general bivector equations around a type D background in principal tetrad, we have

$$\Gamma = \tau l - \rho m + O(\epsilon) \quad \Gamma' = \tau' n - \rho' \bar{m} + O(\epsilon)$$

and it follows

$$\begin{aligned} d^\theta Z^0 &= -\frac{1}{2} h \wedge Z^0 + O(\epsilon) & d^\theta Z^2 &= -\frac{1}{2} h \wedge Z^2 + O(\epsilon) \\ \Gamma' \wedge \Sigma_0 &= (\tau' m - \rho' l) \wedge \Sigma_1 + O(\epsilon^2) & \Gamma \wedge \Sigma_2 &= (-\tau \bar{m} + \rho n) \wedge \Sigma_1 + O(\epsilon^2). \end{aligned}$$

¹¹One can expect that such a structure for perturbations of algebraically special solutions exists also for other signatures. A classification of the Weyl tensor in Euclidean signature can be found in [35], see also [24], a unified formulation for arbitrary signature is given in [5].

Proof. Since

$$\begin{aligned}\Sigma_0 &= \Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2 \\ \Sigma_1 &= \Psi_1 Z^0 + \Psi_2 Z^1 + \Psi_3 Z^2 \\ \Sigma_2 &= \Psi_2 Z^0 + \Psi_3 Z^1 + \Psi_4 Z^2\end{aligned}$$

and

$$Z^0 = \bar{m} \wedge n \quad Z^1 = n \wedge l - \bar{m} \wedge m \quad Z^2 = l \wedge m$$

we have

$$\begin{aligned}\Gamma' \wedge \Sigma_0 &= (\tau' n + \kappa l - \rho' \bar{m} - \sigma m) \wedge (\Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2) \\ &= \Psi_0 (\kappa l \wedge \bar{m} \wedge n - \sigma m \wedge \bar{m} \wedge n) \\ &\quad - \Psi_1 (\rho' \bar{m} \wedge n \wedge l + \sigma m \wedge n \wedge l + \tau' n \wedge \bar{m} \wedge m + \kappa l \wedge \bar{m} \wedge m) \\ &\quad + \Psi_2 (\tau' n \wedge l \wedge m - \rho' \bar{m} \wedge l \wedge m) \\ &= -\Psi_1 (\rho' \bar{m} \wedge n \wedge l + \tau' n \wedge \bar{m} \wedge m) + \Psi_2 (\tau' n \wedge l \wedge m - \rho' \bar{m} \wedge l \wedge m) + O(\epsilon^2) \\ &= \Psi_1 (-\rho' l + \tau' m) \wedge Z^0 + \Psi_2 (\tau' m - \rho' l) \wedge Z^1 + O(\epsilon^2)\end{aligned}$$

Because $(\tau' m - \rho' l) \wedge Z^2 = 0$ this could be added which yields the result. \square

Now expanding the Bianchi identity (21b), we find $d\Sigma_1 = \frac{1}{2} h \wedge \Sigma_1 + O(\epsilon^2)$ which can be written

$$(d - \frac{1}{2} h \wedge) \Sigma_1 = O(\epsilon^2) \quad (40)$$

In the background, this gives the Jordan-Ehlers-Sachs conservation law (39). For linearized gravity, making use of $3h\Psi_2 = 2d\Psi_2$, we find the identity

$$\begin{aligned}0 &= \psi (d - \frac{1}{2} h \wedge) \dot{\Sigma}_1 - \frac{1}{2} \psi \dot{h} \wedge \Sigma_1 \\ &= d(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \psi \Psi_2 \dot{Z}^1) - \frac{1}{2} \psi \Psi_2 \dot{h} \wedge Z^1 \\ &= d(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \frac{3}{2} \psi \Psi_2 \dot{Z}^1),\end{aligned} \quad (41)$$

were the linearized version of $dZ^1 = -h \wedge Z^1$ is used in the last step. Note, that also

$$0 = d(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2) - \frac{3}{2} \psi \Psi_2 \dot{h} \wedge Z^1 \quad (42)$$

holds, which looks similar to Maxwell equations with a source. We summarize the above discussion by the following

Theorem 5.1. *For linearized gravity on a vacuum type D background in principal tetrad exists a closed two-form*

$$\dot{\mathcal{M}} = \psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \frac{3}{2} \psi \Psi_2 \dot{Z}^1 \quad (43)$$

which can be used to calculate the “linearized mass”. The integral

$$\frac{1}{4\pi i} \int_{S^2} \dot{\mathcal{M}} \quad (44)$$

is conserved, gauge invariant and gives the linearized ADM mass.

The gauge invariance follows already from its relation to the ADM mass, but the integrand itself has interesting behaviour under gauge transformations. Beside infinitesimal changes of coordinates (coordinate gauge), there are infinitesimal Lorentz transformations of the tetrad (tetrad gauge). To discuss the second one, we need some notation. Following [13], introduce 4 real functions N_1, N_2, L_1, L_2

and 6 complex functions $L_3, N_3, M_i, i = 1, \dots, 4$ to relate the linearized tetrad to the background tetrad

$$\begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B = \begin{pmatrix} L_1 & L_2 & L_3 & \bar{L}_3 \\ N_1 & N_2 & N_3 & \bar{N}_3 \\ M_1 & M_2 & M_3 & M_4 \\ \bar{M}_1 & \bar{M}_2 & \bar{M}_4 & \bar{M}_3 \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}. \quad (45)$$

These are 16 d.o.f. at a point, 10 correspond to metric perturbations and 6 are infinitesimal Lorentz transformations (tetrad gauge). The linearized tetrad one-forms have the representation

$$\begin{pmatrix} l_a \\ n_a \\ m_a \\ \bar{m}_a \end{pmatrix}_B = \begin{pmatrix} -N_2 & -L_2 & \bar{M}_2 & M_2 \\ -N_1 & -L_1 & \bar{M}_1 & M_1 \\ \bar{N}_3 & \bar{L}_3 & -\bar{M}_3 & -M_4 \\ N_3 & L_3 & -\bar{M}_4 & -M_3 \end{pmatrix} \begin{pmatrix} l_a \\ n_a \\ m_a \\ \bar{m}_a \end{pmatrix}. \quad (46)$$

It follows

$$\begin{aligned} \dot{Z}^0 &= -(L_1 + M_3)Z^0 + \frac{1}{2}(\bar{M}_1 + N_3)Z^1 - \bar{M}_4\bar{Z}^0 - \frac{1}{2}(\bar{M}_1 - N_3)\bar{Z}^1 + N_1\bar{Z}^2 \\ \dot{Z}^1 &= -(M_2 + \bar{L}_3)Z^0 - \frac{1}{2}(L_1 + N_2 + M_3 + \bar{M}_3)Z^1 - (\bar{M}_1 + N_3)Z^2 \\ &\quad + (L_3 - \bar{M}_2)\bar{Z}^0 - \frac{1}{2}(L_1 + N_2 - M_3 - \bar{M}_3)\bar{Z}^1 + (\bar{N}_3 - M_1)\bar{Z}^2 \\ \dot{Z}^2 &= -\frac{1}{2}(M_2 + \bar{L}_3)Z^1 - (N_2 + \bar{M}_3)Z^2 + L_2\bar{Z}^0 + \frac{1}{2}(M_2 - \bar{L}_3)\bar{Z}^1 - M_4\bar{Z}^2. \end{aligned} \quad (47)$$

Linearization of the tetrad representation of the metric yields

$$h_{ln} = -L_1 - N_2 \quad h_{m\bar{m}} = \bar{M}_3 + M_3 \quad h_{n\bar{m}} = N_3 - \bar{M}_1 \quad h_{lm} = \bar{L}_3 - M_2$$

and therefore $\text{tr}_g h = -2(L_1 + N_2 + M_3 + \bar{M}_3)$. One should also note, that the selfdual components of \dot{Z}^1 in $\dot{\mathcal{M}}$ cancel some of the additional terms, not coming from the linearized Weyl tensor,

$$\dot{\Psi}_1 = -\dot{C} \cdot (Z_0, Z_1) + \frac{3}{2}(\bar{L}_3 + M_2)\Psi_2 \quad (48a)$$

$$\dot{\Psi}_2 = -\dot{C} \cdot (Z_1, Z_1) + (L_1 + N_2 + M_3 + \bar{M}_3)\Psi_2 \quad (48b)$$

$$\dot{\Psi}_3 = -\dot{C} \cdot (Z_2, Z_1) + \frac{3}{2}(N_3 + \bar{M}_1)\Psi_2. \quad (48c)$$

Using these facts, we show

Proposition 5.2. *On a spacetime of Petrov type D, the two-form $\dot{\mathcal{M}}$ is tetrad gauge invariant and changes only with a term χ which is exact, $\chi = \text{d}f$, under coordinate gauge transformations.*

Remark 5.3. *In the work of Fayos et al. [18], a gauge in which $\text{d}(\psi\Psi_2\dot{Z}^1) = 0$ was used. It is not clear from that work whether this gauge condition is compatible with a hyperbolic system of evolution equations for linearized gravity.*

Proof of Proposition 5.2. Let us first look at the coordinate gauge. Under infinitesimal coordinate transformations $x^a \rightarrow x^a + \xi^a$, a tensor field transforms with Lie derivative, $T \rightarrow T - \mathcal{L}_\xi T$. For linearized gravity, we write this as $\dot{T} \rightarrow \dot{T} + \delta\dot{T} = \dot{T} - \mathcal{L}_\xi \dot{T}$. Now look at the middle bivector component Z^1 and use Cartan's identity $\mathcal{L}_\xi \omega = \text{d}(\xi \lrcorner \omega) + \xi \lrcorner \text{d}\omega$, which holds for arbitrary forms ω . It follows for coordinate gauge transformations in $\dot{\mathcal{M}}$,

$$\begin{aligned} \delta\dot{\mathcal{M}} &= -\psi\xi(\Psi_2)Z^1 - \frac{3}{2}\psi\Psi_2[\text{d}(\xi \lrcorner Z^1) + \xi \lrcorner \text{d}Z^1] \\ &= -\frac{3}{2}\psi\Psi_2(\text{d} + h\wedge)(\xi \lrcorner Z^1) \\ &= -\frac{3}{2}\text{d}[\psi\Psi_2(\xi \lrcorner Z^1)] \end{aligned} \quad (49)$$

where $\xi \lrcorner h = \frac{2}{3}\Psi_2^{-1}\xi(\Psi_2)$ and $\xi \lrcorner (h \wedge Z^1) = (\xi \lrcorner h)Z^1 - h \wedge (\xi \lrcorner Z^1)$ was used. The two-form (49) is exact and hence integrates to zero.

A tetrad gauge transformation changes the tetrad (45) as follows,

$$\delta \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B = \begin{pmatrix} A & 0 & \bar{b} & b \\ 0 & -A & \bar{a} & a \\ a & b & i\vartheta & 0 \\ \bar{a} & \bar{b} & 0 & -i\vartheta \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B \quad (50)$$

with a, b complex and A, ϑ real valued. It follows, that the tetrad gauge dependent terms in (48a,48c) cancel the ones in (47). The anti selfdual part in (47) is invariant, as follows from (50). This shows the tetrad gauge invariance of \mathcal{M} and therefore gauge invariance of (44). \square

Finally, to express the charge integral in a form similar to the Maxwell case (3), we need the $\theta\phi$ components of the bivectors,

$$Z_{\theta\phi}^1 = -i(r^2 + a^2) \sin \theta \quad Z_{\theta\phi}^0 = -Z_{\theta\phi}^2 = \frac{a\sqrt{\Delta}}{2} \sin^2 \theta. \quad (51)$$

The charge integral becomes

$$\frac{2i}{\sqrt{\Delta}} \int_{S^2(t,r)} \dot{\mathcal{M}} = \int_{S^2(t,r)} \left(2V_L^{-1/2} \hat{\Psi}_2 + ia \sin \theta \Psi_{diff} \right) (r - ia \cos \theta) d\mu \quad (52)$$

with $V_L = \Delta/(r^2 + a^2)^2$, $d\mu = \sin \theta d\theta d\varphi$ and

$$\hat{\Psi}_2 = \dot{\Psi}_2 - \Psi_2(M_3 + \bar{M}_3) \quad (53a)$$

$$\Psi_{diff} = \dot{\Psi}_1 - \dot{\Psi}_3 - 3\Psi_2(\text{Re}(M_2 - M_1) - i\text{Im}(L_3 + N_3)) . \quad (53b)$$

6. CONCLUSIONS

For each isometry of a given background, there is a conserved charge for the linearized gravitational field. Working in terms of linearized curvature, we derived a linearized mass charge (corresponding to the time translation isometry) for Petrov type D backgrounds, by using Penrose's idea of spin-lowering with a Killing spinor.

A second Killing spinor, corresponding to the axial isometry of Kerr spacetime does not exist, (32). Hence spin lowering cannot be used directly to derive a linearized angular momentum charge, even though a canonical analysis provides one in terms of the linearized metric.

For a Schwarzschild background, gauge conditions are known, which eliminate the gauge dependent non-radiating modes [44, 30]. Understanding these conditions in a geometric way and generalizing them to a Kerr background needs further investigation.

APPENDIX A. COORDINATE EXPRESSIONS

Using a Carter tetrad, the bivectors and connection one-forms in Boyer-Lindquist coordinates are

$$Z_{ab}^1 = \begin{pmatrix} 0 & -1 & -ia \sin \theta & 0 \\ 1 & 0 & 0 & -a \sin^2 \theta \\ ia \sin \theta & 0 & 0 & -i(r^2 + a^2) \sin \theta \\ 0 & a \sin^2 \theta & i(r^2 + a^2) \sin \theta & 0 \end{pmatrix} \quad (54a)$$

$$Z_{ab}^0 = \frac{1}{2\sqrt{\Delta}} \begin{pmatrix} 0 & -ia \sin \theta & \Delta & -i\Delta \sin \theta \\ ia \sin \theta & 0 & \Sigma & -i(r^2 + a^2) \sin \theta \\ -\Delta & -\Sigma & 0 & a\Delta \sin^2 \theta \\ i\Delta \sin \theta & i(r^2 + a^2) \sin \theta & -a\Delta \sin^2 \theta & 0 \end{pmatrix} \quad (54b)$$

$$Z_{ab}^2 = \frac{1}{2\sqrt{\Delta}} \begin{pmatrix} 0 & ia \sin \theta & -\Delta & -i\Delta \sin \theta \\ -ia \sin \theta & 0 & \Sigma & i(r^2 + a^2) \sin \theta \\ \Delta & -\Sigma & 0 & -a\Delta \sin^2 \theta \\ i\Delta \sin \theta & -i(r^2 + a^2) \sin \theta & a\Delta \sin^2 \theta & 0 \end{pmatrix} \quad (54c)$$

$$\sigma_{0a} = \left(0, \frac{ia \sin \theta}{2p\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{2p}, -\frac{i\sqrt{\Delta} \sin \theta}{2p} \right) \quad (55a)$$

$$\sigma_{1a} = \left(\frac{M}{2p\Sigma}, 0, 0, -\frac{Map^2 \sin^2 \theta + ra \sin^2 \theta \Sigma + i \cos \theta (r^2 + a^2) \Sigma}{2\Sigma^2} \right) \quad (55b)$$

$$\sigma_{2a} = \sigma_{0a} \quad (55c)$$

Here, we used

$$p = r - ia \cos \theta, \quad \Sigma = p\bar{p}, \quad \Delta = r^2 - 2Mr + a^2$$

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