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Abstract

Maxwell test fields as well as solutions of linearized gravity on the Kerr exterior admit non-radiating modes i.e. non-trivial time-independent solutions. These are closely related to conserved charges. In this paper we discuss the non-radiating modes for linearized gravity which may be seen to correspond to the Poincaré Lie algebra. The two-dimensional isometry group of Kerr corresponds to a two-parameter family of gauge-invariant non-radiating modes representing infinitesimal perturbations of mass and azimuthal angular momentum. We calculate the linearized mass charge in terms of linearized Newman–Penrose scalars.

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1. Introduction

The black hole stability problem, i.e. the problem of proving dynamical stability for the Kerr family of black hole spacetimes, is one of the central open problems in General Relativity. The analysis of linear test fields on the exterior Kerr spacetime is an important step towards the full nonlinear stability problem. For test fields of spin 0, i.e. solutions of the wave equation $\nabla^a \nabla_a \Phi = 0$, estimates proving boundedness and decay in time are known to hold. See [20, 14, 2, 43] for references and background.

The field equations for linear test fields of spins 1 and 2 are the Maxwell and linearized gravity[4] equations, respectively. These equations imply wave equations for the Newman–Penrose Maxwell and linearized Weyl scalars. In particular, the Newman–Penrose scalars of spin weight zero satisfy (assuming a suitable gauge condition for the case of linearized gravity) the analogues of the Regge–Wheeler equation. These wave equations take the form

$$(\nabla^a \nabla_a + c_s \Psi_2) \Phi_s = 0,$$

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[4] Note that linearized gravity is distinct from the massless spin-2 equation. On a type D background, any solution to the massless spin-2 equation is proportional to the Weyl tensor of the spacetime. This fact is referred to as the Buchdahl constraint, cf [8], see also equation (5.8.2) in [37].

where for spin $s = 1$, $c_1 = 2$, $\Phi_1 = \Psi_2^{-1/3}\phi_1$, while for spin $s = 2$, $c_2 = 8$ and $\Phi_2 = \Psi_2^{-2/3}\dot{\Psi}_2$. Here, $\dot{\Psi}_2$ is the linearized Weyl scalar of spin weight zero. See [1] for details. As these scalars can be used as potentials for the Maxwell and linearized Weyl fields, one may apply the techniques developed in the previously mentioned papers to prove estimates also for the Maxwell and linearized gravity equations. This approach has been applied in the case of the Maxwell field on the Schwarzschild background in [7].

In contrast to the spin-0 case, the spin 1 and 2 field equations on the Kerr exterior admit non-trivial finite-energy time-independent solutions. We shall refer to time-independent solutions as non-radiating modes. There is a close relation between gauge-invariant non-radiating modes and conserved charge integrals. For the Maxwell field, there is a two-parameter family of non-radiating, Coulomb-type solutions which carry the two conserved electric and magnetic charges. In fact, a Maxwell field on the Kerr exterior will disperse exactly when it has vanishing charges. For linearized gravity, however, there are both non-radiating modes corresponding to gauge-invariant conserved charges and 'pure gauge' non-radiating modes. Thus, conditions ensuring that a solution of linearized gravity will disperse must be a combination of charge-vanishing and gauge conditions.

From the discussion above, it is clear that in order to prove boundedness and decay for higher spin test fields on the Kerr exterior, it is a necessary step to eliminate the non-radiating modes. Due in part to this additional difficulty, decay estimates for the higher spin fields have been proved only for Maxwell test fields. See [7] for the Schwarzschild case and [3] for the Kerr case. In view of the just mentioned relation between non-radiating modes and charges, an essential step in doing so involves setting conserved charges to zero. In order to make effective use of such charge-vanishing conditions, it is necessary to have simple expressions for the charge integrals in terms of the field strengths. The main result of this paper is to provide an expression for the conserved charge corresponding to the linearized mass, in terms of linearized curvature quantities on the Kerr background.

We start by discussing the relation between charges and non-radiating modes for the case of the Maxwell field. Let the symmetric valence-2 spinor ϕ_{AB} be the Maxwell spinor[5], i.e. a solution of the massless spin-1 (source-free Maxwell) equation

$$\nabla_{A'}{}^A \phi_{AB} = 0,$$

and let $\mathcal{F}_{ab} = \phi_{AB}\epsilon_{A'B'}$ be the corresponding complex self-dual 2-form. The Maxwell equation takes the form $d\mathcal{F} = 0$ and hence the charge integral

$$\int_S \mathcal{F},$$

depends only on the homology class of the surface S. Here, the real and imaginary parts correspond to electric and magnetic charges, respectively. The Kerr exterior, being diffeomorphic to $\mathbb{R}^4$ with a solid cylinder removed, contains topologically non-trivial 2-spheres, and hence the Maxwell equation on the Kerr exterior admits solutions with non-vanishing charges. In view of the fact that the charges are conserved, it is natural that there is a time-independent solution which 'carries' the charge. In Boyer–Lindquist coordinates, this takes the explicit form

$$\phi_{AB} = \frac{c}{(r - \mathrm{i}a\cos\theta)^2}\iota_{(A}o_{B)},\tag{1}$$

where c is a complex number, and ι_A, o_A are principal spinors for Kerr.

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[5] The following discussion is in terms of the 2-spinor formalism, cf [37, 38].

In order to prove boundedness and decay for the Maxwell field, it is necessary to make use of the above-mentioned facts, see [3]. In particular, one eliminates the non-radiating modes by imposing the charge-vanishing condition

$$\oint_S \mathcal{F} = 0.   \tag{2}$$

Written in terms of the Newman–Penrose scalars ϕ_I, $I = 0, 1, 2$, the charge-vanishing condition (2) in the Carter tetrad [46] takes the form [3]

$$\int_{S^2(t,r)} 2V_L^{-1/2}\phi_1 + \mathrm{i}a\sin\theta\,(\phi_0 - \phi_2)\,\mathrm{d}\mu = 0,   \tag{3}$$

where $S^2(t, r)$ is a sphere of constant t, r in the Boyer–Lindquist coordinates, $V_L = \Delta/(r^2 + a^2)^2$ and $\mathrm{d}\mu = \sin\theta\mathrm{d}\theta\mathrm{d}\varphi$. This yields a relation between the $\ell = 0$, $m = 0$ spherical harmonic of ϕ_1 and the $\ell = 1$, $m = 0$ spherical harmonics with spin weights 1, -1 of ϕ_0, ϕ_2, respectively.

Next, we consider the spin-2 case. Recall that the Kerr spacetime is a vacuum space of Petrov type D and hence, in addition to the Killing vector fields ∂_t, ∂_ϕ admits a 'hidden symmetry' manifested by the existence of the valence-2 Killing spinor $\kappa_{AB} = -2\psi\,\iota_{(A}o_{B)}$. Here, the scalar ψ is determined up to a constant, which we fix by setting[6] $M\psi^{-3} = -\Psi_2$ on a Kerr background. In this situation, one may consider the spin-lowered version

$$\Psi_{ABCD}\kappa^{CD},$$

of the Weyl spinor, which is again a massless spin-1 field and hence the complex self-dual 2-form

$$\mathcal{M}_{ab} = \Psi_{ABCD}\kappa^{CD}\epsilon_{A'B'},$$

satisfies the Maxwell equations $\mathrm{d}\mathcal{M} = 0$. The charge for this field defined on any topologically non-trivial 2-sphere in the Kerr exterior is

$$\frac{1}{4\pi\mathrm{i}}\int_S \mathcal{M} = M;   \tag{4}$$

cf [32] for a tensorial version (the calculation has been performed much earlier in [34], but not in the context of Killing spinors and spin-lowering). Here, M is the ADM mass [4] of the Kerr spacetime[7]. The relation between the mass and charge for the spin-lowered Weyl tensor $\mathcal{M}$ is natural in view of the fact that the divergence

$$\xi^{A'A} = \nabla^{A'}{}_B\kappa^{AB}$$

is proportional to ∂_t; see section 3.3, in particular, (43) and the discussion in [38, chapter 6].

Note that the charge (4) is in general complex. The imaginary part corresponds to the NUT charge, which is the gravitational analogue of a magnetic charge. Details are not discussed in this paper; see [39] for the construction of charge integrals in NUT spacetime.

For linearized gravity on the Kerr background, the non-radiating modes include perturbations within the Kerr family, i.e. infinitesimal changes of mass and axial rotation speed. We denote the parameters for these deformations $\dot{M}$, $\dot{a}$. Since M, a are gauge-invariant quantities, it is not possible to eliminate these modes by imposing a gauge condition. A canonical analysis along the lines of [28], see below, yields conserved charges corresponding to the Killing fields ∂_t, ∂_ϕ, which in turn correspond to the gauge-invariant deformations $\dot{M}$, $\dot{a}$ mentioned above.

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[6] This choice has the natural (non-vanishing) Minkowski limit $\psi = r$, see section 3.3.

[7] Equivalently, the mass parameter in the Boyer–Lindquist form of the Kerr line element.

The infinitesimal boosts, translations and (non-axial) rotations of the black hole yield further non-radiating modes which are, however, 'pure gauge' in the sense that they are generated by infinitesimal coordinate changes. If one imposes suitable regularity[8] conditions on the perturbations which exclude e.g. those which turn on the NUT charge, then a ten-dimensional space of non-radiating modes remains. This is spanned by the two-dimensional space of non-gauge modes which carry the $\dot{M}$, $\dot{a}$ charges, together with the 'pure gauge' non-radiating modes, and corresponds in a natural way to the Lie algebra of the Poincaré group. It can be seen from this discussion that a combination of charge-vanishing conditions and gauge conditions allows one to eliminate all non-radiating solutions of linearized gravity.

The constraint equations implied by the Maxwell and linearized gravity equations are underdetermined elliptic systems, and therefore admit solutions of compact support; see [16] and references therein. In particular, one may find the solutions of the Maxwell constraint equations with arbitrarily rapid fall-off at infinity. The corresponding solutions of the Maxwell equations have vanishing charges. For the case of linearized gravity, the charges corresponding to $\dot{M}$, $\dot{a}$ vanish for the solutions of the field equations with rapid fall-off at infinity. For such solutions, all non-radiating modes may therefore be eliminated by imposing suitable gauge conditions.

The following discussion may easily be extended to the Einstein–Maxwell equations. Given an asymptotically flat vacuum spacetime (N, g_{ab}), a solution of the linearized Einstein equations $\dot{g}_{ab}$ (satisfying suitable asymptotic conditions) and a Killing field $\xi^a \partial_a$ we have that the variation of the Hamiltonian current is an exact form, which yields the relation

$$\dot{\mathcal{P}}_{\xi;\infty} = \int_S \dot{\mathbf{Q}}[\xi] - \xi \cdot \mathbf{\Theta}. \tag{5}$$

Here, $\mathcal{P}_{\xi;\infty}$ is the Hamiltonian charge at infinity, generating the action of ξ, $\mathbf{Q}[\xi]$ is the Noether charge 2-form for ξ and $\mathbf{\Theta}$ is the symplectic current 3-form, defined with respect to the variation $\dot{g}_{ab}$. We use a ˙ to denote variations along $\dot{g}_{ab}$; thus, $\dot{\mathcal{P}}_{\xi;\infty}$ and $\dot{\mathbf{Q}}[\xi]$ denote the variation of the Hamiltonian and the Noether 2-form, respectively. The integral on the right-hand side of (5) is evaluated over an arbitrary sphere, which generates the second homology class.

For the case of $\xi = \partial_t$, and considering solutions of the linearized Einstein equations on the Kerr background we have, following the discussion above,

$$\dot{M} = \dot{\mathcal{P}}_{\partial_t;\infty}.$$

Working with the Carter tetrad, let Ψ_i, $i = 0, \ldots, 4$, be the Weyl scalars and let Z^I, $I = 0, 1, 2$ denote the corresponding basis for the space of complex, self-dual 2-forms; see section 2 for details. In this paper, we shall show that the natural linearization of the spin-lowered Weyl tensor $\mathcal{M}$ is the 2-form

$$\dot{\mathcal{M}} = \psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \tfrac{3}{2} \psi \Psi_2 \dot{Z}^1.$$

As will be demonstrated, see section 4 below, $\dot{\mathcal{M}}$ is closed, and hence the integral

$$\int_S \dot{\mathcal{M}} \tag{6}$$

defines a conserved charge. A charge-vanishing condition for the linearized mass, analogous to the one discussed above for the charges of the Maxwell field, may be introduced by requiring that this integral vanishes. The coordinate form of this charge-vanishing condition is

$$\int_{S^2(t,r)} \left(2V_L^{-1/2}\widehat{\dot{\Psi}}_2 + \mathrm{i}a\sin\theta\,\dot{\Psi}_{\mathrm{diff}}\right)(r - \mathrm{i}a\cos\theta)\mathrm{d}\mu = 0, \tag{7}$$

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[8] The Kerr family of line elements may be viewed as part of the type D family of vacuum metrics which includes, among others, the NUT and C-metrics. See section 3.3 for further discussion. The perturbations corresponding e.g. to infinitesimal deformations of the NUT parameter are singular and may thus be excluded by suitable regularity and decay conditions. See [44, 29] for remarks.

which should be compared to the corresponding condition for the Maxwell case, cf (3). Here, $\hat{\dot{\Psi}}_2$ and $\dot{\Psi}_{\text{diff}}$ are the suitable combinations of the linearized curvature scalars $\dot{\Psi}_1$, $\dot{\Psi}_2$, $\dot{\Psi}_3$ and linearized tetrad.

Let $\dot{g}_{ab}$ be a solution of the linearized Einstein equation on the Kerr background, satisfying suitable asymptotic conditions, and let $\dot{M}$ be the corresponding perturbation of the ADM mass. Letting $S = S^2(t, r)$ and evaluating the limit of (6) as $r \to \infty$ one finds, in view of the fact that (6) is conserved, the identity

$$\dot{M} = \frac{1}{4\pi\mathrm{i}} \int_S \dot{\mathcal{M}},$$

for any smooth 2-sphere S in the exterior of the Kerr black hole. Thus, we have the relation

$$\int_S \dot{\mathbf{Q}}[\partial_t] - \partial_t \cdot \mathbf{\Theta} = \frac{1}{4\pi\mathrm{i}} \int_S \dot{\mathcal{M}}, \tag{8}$$

for any surface S in the Kerr exterior. We remark that the left-hand side of (8) can be evaluated in terms of the metric perturbation using the expressions for $\mathbf{Q}$ and $\mathbf{\Theta}$ given in [28, section 5]. On the other hand, the right-hand side has been calculated in terms of linearized curvature. It would be of interest to have a direct derivation of the resulting identity.

The canonical analysis following [28] which has been discussed above shows that in addition to the conserved charge corresponding to $\dot{M}$, equation (5) with $\xi = \partial_\phi$, the angular Killing field, gives a conserved charge integral for linearized angular momentum $\dot{a}$. If ∂_ϕ is tangent to S, then the term $\partial_\phi \cdot \mathbf{\Theta}$ does not contribute in (5). We remark that an expression for $\dot{a}$ for linearized gravity on the Schwarzschild background was given in [30, section 3]. A charge integral for $\dot{a}$ for linearized gravity on the Kerr background will be considered in a future paper.

Remark 1.1.

 (1) There are many candidates for a quasi-local mass expression in the literature including, to mention just a few, those put forward by Penrose, Brown and York, and Wang and Yau. See the review of Szabados [42] for background and references. Although as discussed above, cf equation (4), for a spacetime of type D, there is a quasi-local mass charge, it must be emphasized that for a general spacetime one cannot expect the existence of a quasi-local mass which is *conserved*, i.e. independent of the 2-surface used in its definition. The same is true for linearized gravity on a general background. Thus, the existence of a conserved charge integral for the linearized mass is a feature which is special to linearized gravity on a background with Killing symmetries.
 (2) If we consider linearized gravity without sources, on the Minkowski background, the linearized mass must vanish due to the fact that Minkowski space is topologically trivial. This reflects the fact that when viewed as a function on the space of Cauchy data, the ADM mass vanishes quadratically at the trivial data, cf [10]. On the other hand, by the positive mass theorem, for any non-flat vacuum spacetime, asymptotic to Minkowski space in a suitable sense, the ADM mass defined at infinity must be positive.

This paper is organized as follows. In section 2, we introduce the bivector formalism and notation for linearized gravity. Conformal Killing–Yano tensors and Killing spinors are discussed in section 3.1. Section 3.2 deals with conserved charges for spin-2 fields on Minkowski and section 3.3 for type D spacetimes. The main result, a charge integral in terms of linearized curvature, is derived in section 4, and finally, section 5 contains some concluding remarks.

2. Preliminaries and notation

Let (N, g_{ab}) be a four-dimensional Lorentzian spacetime of signature $+ - --$, admitting a spinor structure. Although most of the results can be generalized to the electrovac case with a cosmological constant, we restrict in this paper to the vacuum case. In particular, we consider test Maxwell fields and linearized gravity on vacuum type D background spacetimes.

2.1. Bivector formalism

Let o_A, ι_A be a spinor dyad, normalized so that $o_A \iota^A = 1$, and let

$$l^a = o^A \bar{o}^{A'}, \quad m^a = o^A \bar{\iota}^{A'}, \quad \bar{m}^a = \iota^A \bar{o}^{A'}, \quad n^a = \iota^A \bar{\iota}^{A'}$$

be the corresponding null tetrad, satisfying $l^a n_a = -m^a \bar{m}_a = 1$, with the other inner products being zero. The 2-spinor calculus provides a powerful tool for computations in four-dimensional geometry. The GHP formalism deals with the dyad (or equivalently tetrad) components of geometric objects and exploits the simplifications arising by taking into account the action of dyad rescalings and permutations. These formalisms are closely related to the less widely used *bivector formalism* [34, 6, 9, 27] in which the basic quantity is a basis for the three-dimensional space of complex self-dual 2-forms. A 2-form Z is called self-dual, if $*Z = iZ$ and anti self-dual, if $*Z = -iZ$. Given a spinor dyad, a natural choice[9] is

$$Z_{ab}^0 = 2\bar{m}_{[a} n_{b]} = \iota_A \iota_B \bar{\epsilon}_{A'B'} \tag{9a}$$

$$Z_{ab}^1 = 2n_{[a} l_{b]} - 2\bar{m}_{[a} m_{b]} = -2 o_{(A} \iota_{B)} \bar{\epsilon}_{A'B'} \tag{9b}$$

$$Z_{ab}^2 = 2l_{[a} m_{b]} = o_A o_B \bar{\epsilon}_{A'B'}, \tag{9c}$$

where the notation $2x_{[a} y_{b]} = x_a y_b - y_a x_b$ for anti-symmetrization and $2x_{(a} y_{b)} = x_a y_b + y_a x_b$ for symmetrization is used. We use capital latin indices I, J, K taking values in 0, 1, 2 for the elements in the bivector triad Z^I. The metric g_{ab} induces a triad metric G_{IJ} and its inverse G^{IJ} is given by

$$G^{IJ} = Z^I \cdot Z^J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad G_{IJ} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -\frac{1}{2} & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Here, $\cdot$ is the induced inner product on 2-form, $Z^I \cdot Z^J = \frac{1}{2} Z^I{}_{ab} Z^{Jab}$. Triad indices are raised and lowered with this metric,

$$Z_0 = Z^2, \quad Z_1 = -\frac{1}{2} Z^1, \quad Z_2 = Z^0.$$

More generally, we have

Proposition 2.1.

$$Z^J{}_a{}^c Z^K{}_{bc} = \frac{1}{2} G^{JK} g_{ab} + \epsilon^{JKL} Z_{Lab}, \tag{10a}$$

$$Z^J{}_{[a}{}^c \bar{Z}^K{}_{b]c} = 0, \tag{10b}$$

$$Z^{Jab} \bar{Z}^K{}_{ab} = 0, \tag{10c}$$

with ϵ^{JKL} being the totally antisymmetric symbol fixed by $\epsilon^{012} = 1$.

[9] We use the convention of [18], which differs from [27, 17] by a factor of 2 in the middle component and the numbering.

A real 2-form F_{ab}, e.g. the Maxwell field strength, has the spinor representation

$$F_{ab} = \phi_{AB}\epsilon_{A'B'} + \overline{\phi}_{A'B'}\epsilon_{AB}.$$

It is equivalent to the symmetric 2-spinor $\phi_{AB} = \phi_2 o_A o_B - 2\phi_1 o_{(A}\iota_{B)} + \phi_0 \iota_A \iota_B$, where the six real degrees of freedom of F_{ab} are encoded in three complex scalars

$$\phi_0 = \phi_{AB}o^A o^B = F_{ab}l^a m^b = F \cdot Z_0,$$
$$\phi_1 = \phi_{AB}\iota^A o^B = \tfrac{1}{2}F_{ab}(l^a n^b - m^a \bar{m}^b) = F \cdot Z_1,$$
$$\phi_2 = \phi_{AB}\iota^A \iota^B = F_{ab}\bar{m}^a n^b = F \cdot Z_2.$$

So the real 2-form has a bivector representation

$$F = \phi_0 Z^0 + \phi_1 Z^1 + \phi_2 Z^2 + \overline{\phi}_0 \overline{Z}^0 + \overline{\phi}_1 \overline{Z}^1 + \overline{\phi}_2 \overline{Z}^2,$$

or in index notation $\phi_I = F \cdot Z_I$ and $F = \phi_I Z^I + \overline{\phi}_I \overline{Z}^I$.

The Weyl tensor is a symmetric 2-tensor over bivector space and has the spinor representation

$$-C_{abcd} = \Psi_{ABCD}\overline{\epsilon}_{A'B'}\overline{\epsilon}_{C'D'} + \overline{\Psi}_{A'B'C'D'}\epsilon_{AB}\epsilon_{CD},$$

where Ψ_{ABCD} is a completely symmetric 4-spinor. The ten degrees of freedom of the Weyl tensor are given by five complex scalars[10]

$$\Psi_0 = \Psi_{ABCD}o^A o^B o^C o^D = -C_{abcd}l^a m^b l^c m^d = -C \cdot (Z_0, Z_0),$$
$$\Psi_1 = \Psi_{ABCD}o^A o^B o^C \iota^D = -C_{abcd}l^a n^b l^c m^d = -C \cdot (Z_0, Z_1),$$
$$\Psi_2 = \Psi_{ABCD}o^A o^B \iota^C \iota^D = -C_{abcd}l^a m^b \bar{m}^c n^d = -C \cdot (Z_0, Z_2) = -C \cdot (Z_1, Z_1),$$
$$\Psi_3 = \Psi_{ABCD}o^A \iota^B \iota^C \iota^D = -C_{abcd}l^a n^b \bar{m}^c n^d = -C \cdot (Z_2, Z_1),$$
$$\Psi_4 = \Psi_{ABCD}\iota^A \iota^B \iota^C \iota^D = -C_{abcd}n^a \bar{m}^b n^c \bar{m}^d = -C \cdot (Z_2, Z_2).$$

Similarly, we could have used the Weyl 2-bivector

$$C_{IJ} = -\frac{1}{4}C_{abcd}Z_I^{ab}Z_J^{cd} = \begin{pmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{pmatrix},$$

which relates to the real Weyl tensor via

$$- C_{abcd} = C_{IJ}Z_{ab}^I \otimes Z_{cd}^J + \overline{C}_{IJ}\overline{Z}_{ab}^I \otimes \overline{Z}_{cd}^J. \tag{11}$$

Because of different conventions and normalizations in the literature [34, 6, 9, 27], we rederive here the equations of structure in a bivector formalism. Making use of Cartan's equations of structure for tetrad 1-forms[11]

$$de^a = -\omega_b^a \wedge e^b, \qquad \Omega_b^a = d\omega_b^a + \omega_c^a \wedge \omega_b^c, \tag{12}$$

Bianchi identities

$$\Omega_b^a \wedge e^b = 0, \qquad d\Omega_b^a = \Omega_c^a \wedge \omega_b^c - \omega_c^a \wedge \Omega_b^c, \tag{13}$$

and definitions of connection 1-forms σ_J and curvature 1-forms Σ_J in bivector formalism,

$$\omega_{ab}e^a \wedge e^b = -2\sigma_J Z^J - 2\bar{\sigma}_J \bar{Z}^J, \qquad \Omega_{ab}e^a \wedge e^b = -2\Sigma_J Z^J - 2\bar{\Sigma}_J \bar{Z}^J, \tag{14}$$

we have the following result.

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[10] Due to its symmetries, the Weyl tensor is a symmetric 2-tensor over the space of 2-forms. The induced inner product is $C \cdot (Z_I, Z_J) = \tfrac{1}{4}C_{abcd}Z_I^{ab}Z_J^{cd}$.
[11] Connection and curvature are defined by $\omega_{b\mu}^a = e_\nu^a \nabla_\mu e_b^\nu$ and $\Omega_{b\mu\nu}^a = 2e_\sigma^a \nabla_{[\mu}\nabla_{\nu]}e_b^\sigma$, respectively.

Proposition 2.2. *The bivector equations of structure are*

$$\mathrm{d}Z^J = -2\epsilon^{JKL}\sigma_K \wedge Z_L, \qquad \Sigma_J = \mathrm{d}\sigma_J + \tfrac{1}{2}\epsilon_{JKL}\sigma^K \wedge \sigma^L, \tag{15}$$

while the Bianchi identities read

$$\Sigma_{[J} \wedge Z_{K]} = 0, \qquad d\Sigma_J = -\epsilon_{JKL}\Sigma^K \wedge \sigma^L. \tag{16}$$

Here, $\wedge$ is the usual wedge product of 1-forms σ^J and 2-forms Z^J, Σ^J.

Proof. Expanding the bivectors $Z^J = \tfrac{1}{2}Z^J_{ab}e^a \wedge e^b$, we find

$$\begin{aligned}
\mathrm{d}Z^J &= \tfrac{1}{2}Z^J_{ab}(\mathrm{d}e^a \wedge e^b - e^a \wedge \mathrm{d}e^b) = Z^J_{ab}\mathrm{d}e^a \wedge e^b \\
&= -Z^J_{ab}\omega^a{}_c e^c \wedge e^b \\
&= Z^J_{ab}(\sigma_K Z^{Ka}{}_c + \bar\sigma_K \bar Z^{Ka}{}_c) \wedge e^c \wedge e^b \\
&= \epsilon^{JKL}Z_{Lbc}\sigma_K \wedge e^c \wedge e^b \\
&= -2\epsilon^{JKL}\sigma_K \wedge Z_L,
\end{aligned}$$

where proposition 2.1 has been used in the third step. For the second equation of structure, we plug (14) into (12),

$$-\Sigma_J Z^J_{ab} - \bar\Sigma_J \bar Z^J_{ab} = -\mathrm{d}\sigma_J Z^J_{ab} - \mathrm{d}\bar\sigma_J \bar Z^J_{ab} + \left(\sigma_J Z^J_{ac} + \bar\sigma_J \bar Z^J_{ac}\right) \wedge \left(\sigma_K Z^{Kc}{}_b + \bar\sigma_K \bar Z^{Kc}{}_b\right).$$

Using proposition 2.1, the self-dual part reads

$$\Sigma_J Z^J_{ab} = \mathrm{d}\sigma_J Z^J_{ab} + \epsilon^{KLJ}Z_{Jab}\sigma_K \wedge \sigma_L.$$

Changing index positions on ϵ^{KLJ} and using $\det G_{JK} = \tfrac{1}{2}$ gives the second equation of structure. For the first Bianchi identity, look at

$$\begin{aligned}
0 &= \tfrac{1}{2}\mathrm{d}^2 Z^J \\
&= -\epsilon^{JKL}\left(\mathrm{d}\sigma_K \wedge Z_L - \sigma_K \wedge \mathrm{d}Z_L\right) \\
&= -\epsilon^{JKL}\left(\Sigma_K \wedge Z_L - \tfrac{1}{2}\epsilon_{KNM}\sigma^N \wedge \sigma^M \wedge Z_L + \sigma_K \wedge \epsilon_{LNM}\sigma^N \wedge Z^M\right) \\
&= -\epsilon^{JKL}\Sigma_K \wedge Z_L \underbrace{+ \sigma^L \wedge \sigma^J \wedge Z_L - \sigma^J \wedge \sigma^L \wedge Z_L - 2\sigma_L \wedge \sigma^J \wedge Z^L}_{=0} + \underbrace{\sigma_K \wedge \sigma^K}_{=0} \wedge Z^J,
\end{aligned}$$

where the identity $\epsilon^{IJK}\epsilon_{INM} = \delta^J_N\delta^K_M - \delta^J_M\delta^K_N$ has been used. Finally, the second Bianchi identity is

$$\begin{aligned}
\mathrm{d}\Sigma_J &= -\epsilon_{JKL}\mathrm{d}\sigma^K \wedge \sigma^L \\
&= -\epsilon_{JKL}(\Sigma^K - \epsilon^{KMN}\sigma_M \wedge \sigma_N) \wedge \sigma^L \\
&= -\epsilon_{JKL}\Sigma^K \wedge \sigma^L + \underbrace{\sigma_L \wedge \sigma_J \wedge \sigma^L}_{=0} - \sigma_J \wedge \underbrace{\sigma_L \wedge \sigma^L}_{=0}.
\end{aligned}$$

$\square$

Remark 2.3. Instead of using Cartan equations for the tetrad one could have used the bivector connection form

$$\omega_{IJa} := \epsilon_{IJK}\sigma^K_a = Z^{bc}_{[J}\nabla_a Z_{I]bc}. \tag{17}$$

For later use, it is convenient to write the components of the equations of structure explicitly. The connection 1-forms, for example, can be expressed in terms of NP spin coefficients,

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$$\sigma_{0a} = m^b \nabla_a l_b = \tau l_a + \kappa n_a - \rho m_a - \sigma \bar{m}_a, \tag{18a}$$

$$\sigma_{1a} = \tfrac{1}{2}(n^b \nabla_a l_b - \bar{m}^b \nabla_a m_b) = -\epsilon' l_a + \epsilon n_a + \beta' m_a - \beta \bar{m}_a, \tag{18b}$$

$$\sigma_{2a} = -\bar{m}^b \nabla_a n_b = -\kappa' l_a - \tau' n_a + \sigma' m_a + \rho' \bar{m}_a. \tag{18c}$$

The middle component σ_{1a} collects all unweighted coefficients and so can be used to define the GHP covariant derivative $\Theta_a \eta = (\nabla_a - p\sigma_{1a} - q\bar{\sigma}_{1a})\eta$. To avoid clutter in the notation, we write $\Gamma := \sigma_0$ and $\sigma_2 = -\Gamma'$, where $'$ is the GHP prime operation [21]. The derivatives of the spinor dyad can now be written in the compact form $\Theta_a o^A = -\Gamma_a \iota^A$ and $\Theta_a \iota^A = -\Gamma'_a o^A$, and the components of the first equations of structure, which we present here for convenience with the usual exterior derivative and with weighted exterior derivative $d^\Theta = d - p\sigma_1 \wedge -q\bar{\sigma}_1 \wedge$, read

$$d^\Theta Z^0 = \Gamma' \wedge Z^1 \quad \Leftrightarrow \quad dZ^0 = -2\sigma_1 \wedge Z^0 + \Gamma' \wedge Z^1, \tag{19a}$$

$$d^\Theta Z^1 = 2\Gamma \wedge Z^0 + 2\Gamma' \wedge Z^2 \quad \Leftrightarrow \quad dZ^1 = 2\Gamma \wedge Z^0 + 2\Gamma' \wedge Z^2, \tag{19b}$$

$$d^\Theta Z^2 = \Gamma \wedge Z^1 \quad \Leftrightarrow \quad dZ^2 = 2\sigma_1 \wedge Z^2 + \Gamma \wedge Z^1. \tag{19c}$$

Note that the middle component can be simplified to $dZ^1 = -h \wedge Z^1$ with the 1-form $h = 2(\rho' l + \rho n - \tau' m - \tau \bar{m})$. This fact and a relation between the type D curvature Ψ_2 and h will be crucial in the derivation of the conservation law in section 4.

In vacuum, we have for the curvature 2-forms $\Sigma_J = C_{JK} Z^K$ and the components of the second equations of structure read

$$\Sigma_0 = C_{0J} Z^J = d^\Theta \Gamma = d\Gamma - 2\sigma_1 \wedge \Gamma, \tag{20a}$$

$$\Sigma_1 = C_{1J} Z^J = d\sigma_1 - \Gamma \wedge \Gamma', \tag{20b}$$

$$\Sigma_2 = C_{2J} Z^J = -d^\Theta \Gamma' = -d\Gamma' - 2\sigma_1 \wedge \Gamma'. \tag{20c}$$

Finally, the Bianchi identities are

$$d^\Theta \Sigma_0 = -2\Gamma \wedge \Sigma_1 \qquad \Leftrightarrow \qquad d\Sigma_0 = 2\sigma_1 \wedge \Sigma_0 - 2\Gamma \wedge \Sigma_1, \tag{21a}$$

$$d^\Theta \Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2 \qquad \Leftrightarrow \qquad d\Sigma_1 = -\Gamma' \wedge \Sigma_0 - \Gamma \wedge \Sigma_2, \tag{21b}$$

$$d^\Theta \Sigma_2 = -2\Gamma' \wedge \Sigma_1 \qquad \Leftrightarrow \qquad d\Sigma_2 = -2\sigma_1 \wedge \Sigma_2 - 2\Gamma' \wedge \Sigma_1. \tag{21c}$$

2.2. Linearized gravity

Linearized gravity describes perturbations to first order in some parameter ϵ of a given solution to the field equations of General Relativity. We use an overdot or a subscript $_B$ for quantities of $O(\epsilon)$, with some exceptions which are explained below.

Following [13], we introduce four real functions N_1, N_2, L_1, L_2 and six complex functions $L_3, N_3, M_i, i = 1, \ldots, 4$ to relate the linearized tetrad to the background tetrad

$$\begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B = \begin{pmatrix} L_1 & L_2 & L_3 & \bar{L}_3 \\ N_1 & N_2 & N_3 & \bar{N}_3 \\ M_1 & M_2 & M_3 & M_4 \\ \bar{M}_1 & \bar{M}_2 & \bar{M}_4 & \bar{M}_3 \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}. \tag{22}$$

Here, we use a subscript B instead of a dot for the linearized tetrad. Note that the matrix entries are by definition linearized quantities and we suppress an overdot to avoid clutter. There are 16 degrees of freedom at a point, 10 correspond to metric perturbations and 6 are infinitesimal Lorentz transformations (tetrad gauge). The linearized tetrad 1-forms have the representation

$$
\begin{pmatrix} l_a \\ n_a \\ m_a \\ \overline{m}_a \end{pmatrix}_B = \begin{pmatrix} -N_2 & -L_2 & \overline{M}_2 & M_2 \\ -N_1 & -L_1 & \overline{M}_1 & M_1 \\ \overline{N}_3 & \overline{L}_3 & -\overline{M}_3 & -M_4 \\ N_3 & L_3 & -\overline{M}_4 & -M_3 \end{pmatrix} \begin{pmatrix} l_a \\ n_a \\ m_a \\ \overline{m}_a \end{pmatrix}.
\tag{23}
$$

For the bivectors (9), it follows

$$
\dot{Z}^0 = -(L_1 + M_3)Z^0 + \tfrac{1}{2}(\overline{M}_1 + N_3)Z^1 - \overline{M}_4\overline{Z}^0 - \tfrac{1}{2}(\overline{M}_1 - N_3)\overline{Z}^1 + N_1\overline{Z}^2,
$$

$$
\dot{Z}^1 = -(M_2 + \overline{L}_3)Z^0 - \tfrac{1}{2}(L_1 + N_2 + M_3 + \overline{M}_3)Z^1 - (\overline{M}_1 + N_3)Z^2,
$$

$$
\quad + (L_3 - \overline{M}_2)\overline{Z}^0 - \tfrac{1}{2}(L_1 + N_2 - M_3 - \overline{M}_3)\overline{Z}^1 + (\overline{N}_3 - M_1)\overline{Z}^2,
$$

$$
\dot{Z}^2 = -\tfrac{1}{2}(M_2 + \overline{L}_3)Z^1 - (N_2 + \overline{M}_3)Z^2 + L_2\overline{Z}^0 + \tfrac{1}{2}(M_2 - \overline{L}_3)\overline{Z}^1 - M_4\overline{Z}^2.
\tag{24}
$$

Linearization of the tetrad representation of the metric, $g_{ab} = 2l_{(a}n_{b)} - 2m_{(a}\bar{m}_{b)}$, yields

$$
h_{ln} = -L_1 - N_2, \; h_{m\bar{m}} = \overline{M}_3 + M_3, \; h_{n\bar{m}} = N_3 - \overline{M}_1, \; h_{lm} = \overline{L}_3 - M_2,
$$

and therefore $\mathrm{tr}_g h = -2(L_1 + N_2 + M_3 + \overline{M}_3)$. Linearization of the NP curvature scalars shows

$$
\dot{\Psi}_0 = -\dot{C} \cdot (Z_0, Z_0),
\tag{25a}
$$

$$
\dot{\Psi}_1 = -\dot{C} \cdot (Z_0, Z_1) + \tfrac{3}{2}(\overline{L}_3 + M_2)\Psi_2,
\tag{25b}
$$

$$
\dot{\Psi}_2 = -\dot{C} \cdot (Z_1, Z_1) + (L_1 + N_2 + M_3 + \overline{M}_3)\Psi_2,
\tag{25c}
$$

$$
\dot{\Psi}_3 = -\dot{C} \cdot (Z_2, Z_1) + \tfrac{3}{2}(N_3 + \overline{M}_1)\Psi_2,
\tag{25d}
$$

$$
\dot{\Psi}_4 = -\dot{C} \cdot (Z_2, Z_2).
\tag{25e}
$$

For linearized gravity in a tetrad-based approach, there are gauge degrees of freedom corresponding to infinitesimal changes of the coordinates (coordinate gauge) and of the tetrad (tetrad gauge). Here we only give some basics, which are needed in section 4 and refer to [41] for details. Under infinitesimal coordinate transformations $x^a \rightarrow x^a + \xi^a$, a tensor field $\dot{T}$ transforms as $\dot{T} \rightarrow \dot{T} + \delta\dot{T}$ with

$$
\delta T = -\mathcal{L}_\xi T.
\tag{26}
$$

A tetrad gauge transformation $\dot{e}^a \rightarrow \dot{e}^a + \delta\dot{e}^a$ changes the tetrad (22) as follows,

$$
\delta \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix}_B = \begin{pmatrix} A & 0 & \bar{b} & b \\ 0 & -A & \bar{a} & a \\ a & b & i\vartheta & 0 \\ \bar{a} & \bar{b} & 0 & -i\vartheta \end{pmatrix} \begin{pmatrix} l^a \\ n^a \\ m^a \\ \bar{m}^a \end{pmatrix},
\tag{27}
$$

with a, b being complex and A, ϑ real valued. Here again, the subscript B denotes linearized quantities, δ stands for an infinitesimal tetrad transformation and the matrix entries themselves are linear in the perturbation parameter.

3. Killing spinors and conserved charges

3.1. Conformal Killing–Yano tensors and Killing spinors

Conformal Killing–Yano tensors of rank 2 are 2-forms Y_{ab} solving the conformal Killing–Yano equation,

$$Y_{a(b;c)} = g_{bc}\xi_a - g_{a(b}\xi_{c)}, \text{ where } \xi_a = \tfrac{1}{3}Y_a{}^b{}_{;b}. \tag{28}$$

It is well known that the divergence ξ^a is a Killing vector and in case it vanishes, Y_{ab} is called a Killing–Yano tensor. The symmetrized product $X_{c(a}Y_{b)}{}^c =: K_{ab}$ of Killing–Yano tensors X_{ab} and Y_{ab} is a Killing tensor, $\nabla_{(a}K_{bc)} = 0$, which can be used to construct a constant of motion or a symmetry operator for e.g. the scalar wave equation, known as Carter's constant and Carter operator, respectively. By inserting $Y_{ab} = \kappa_{AB}\epsilon_{A'B'} + \bar{\kappa}_{A'B'}\epsilon_{AB}$ into (28) one can show that κ_{AB} and $\bar{\kappa}_{A'B'}$ satisfy the Killing spinor equation

$$\nabla_{A'(A}\kappa_{BC)} = 0, \tag{29}$$

and its complex conjugated version. For the spinor components $\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_0 \iota_A \iota_B$ (or equivalently the self dual bivector components of Y_{ab}), we find the following set of eight scalar equations:

$$\text{þ}\kappa_0 = -2\kappa\kappa_1, \quad \text{ð}\kappa_0 = -2\sigma\kappa_1, \quad \text{þ}'\kappa_2 = -2\kappa'\kappa_1, \quad \text{ð}'\kappa_2 = -2\sigma'\kappa_1, \tag{30}$$

$$(\text{ð}' + 2\tau')\kappa_0 + 2(\text{þ} + \rho)\kappa_1 = -2\kappa\kappa_2, \quad (\text{þ}' + 2\rho')\kappa_0 + 2(\text{ð} + \tau)\kappa_1 = -2\sigma\kappa_0,$$
$$(\text{ð} + 2\tau)\kappa_2 + 2(\text{þ}' + \rho')\kappa_1 = -2\kappa'\kappa_0, \quad (\text{þ} + 2\rho)\kappa_2 + 2(\text{ð}' + \tau')\kappa_1 = -2\sigma'\kappa_2, \tag{31}$$

by projecting (29) into a spinor dyad. Thus, we have three different sets of equations, (28), (29) and ((30), (31)), which are equivalent and we will use the most appropriate for the problem at hand.

As spin-s fields are heavily restricted on curved backgrounds (see footnote 4), so are Killing spinors. Consider a Killing spinor $\kappa_{A_1...A_n} = \kappa_{(A_1...A_n)}$ which satisfies the Killing spinor equation of valence n,

$$\nabla_{B'(B}\kappa_{A_1...A_n)} = 0. \tag{32}$$

Contracting a second derivative $\nabla^{B'}{}_C$ and symmetrizing gives

$$\begin{aligned}
0 &= \nabla^{B'}{}_{(C}\nabla_{|B'|B}\kappa_{A_1...A_n)} \\
&= -\square_{(BC}\kappa_{A_1...A_n)} \\
&= \Psi_{(BCA_1}{}^D\kappa_{DA_2...A_n)} + \ldots + \Psi_{(BCA_n}{}^D\kappa_{A_1...A_{n-1}D)} \\
&= n\Psi_{(BCA_1}{}^D\kappa_{DA_2...A_n)}.
\end{aligned}$$

For Killing spinors of valence 1 (satisfying the twistor equation), this yields $0 = \Psi_{ABCD}\kappa^D$ as can be found in [38, equation (6.1.6)]. For 2-spinors, we find

$$0 = \Psi_{(ABC}{}^D\kappa_{DE)}. \tag{33}$$

For non-trivial κ_{AB}, this restricts the spacetime to be of Petrov type D, N or O. For a given spacetime of type D in a principal tetrad (only $\Psi_2 \neq 0$), (33) becomes

$$\begin{aligned}
0 &= \Psi_2 o_{(A} o_B \iota_C \iota_D (\kappa_0 \iota^D \iota_{E)} + \kappa_1 o^D \iota_{E)} + \kappa_1 \iota^D o_{E)} + \kappa_2 o^D o_{E)}) \\
&= \Psi_2 (C_1 \kappa_0 o_{(A} \iota_B \iota_C \iota_{E)} + C_2 \kappa_2 \iota_{(A} o_B o_C o_{E)}),
\end{aligned}$$

with constants C_1, C_2 and it follows $\kappa_0 \equiv 0 \equiv \kappa_2$. The remaining component satisfies the simplified equations (39), which have only one non-trivial complex solution (see section 3.3 for details), cf [22] where explicit integration of the conformal Killing–Yano equation was done.

Table 1. Poincaré isometries and corresponding charges.

Label	Isometry	Charge	No.
$\mathcal{T}_t$	Time translation	Mass	1
$\mathcal{T}_i$	Spatial translations	Linear momenta	3
$\mathcal{L}_{ij}$	Rotations	Angular momenta	3
$\mathcal{L}_{ti}$	Boosts	Center of mass	3

3.2. Conserved charges for Minkowski spacetime

The Killing spinor equation or conformal Killing–Yano equation on Minkowski space has been widely discussed in the literature [38, 32, 25] and the explicit solution in Cartesian coordinates is well known,

$$\kappa^{AB} = U^{AB} + 2x^{A'(A}V_{A'}{}^{B)} + x^{A'A}x^{B'B}W_{A'B'}. \tag{34}$$

Here, U^{AB} and $W_{A'B'}$ are constant, symmetric spinors and $V_{A'}{}^{B}$ a constant complex vector which yield $2 \cdot 6 + 8 = 20$ independent real solutions. Each solution gives a charge when contracted into a spin-2 field, e.g. the linearized Weyl tensor, and integrated over a 2-sphere. In [38, p 99], ten of these charges are related to a source for linearized gravity in the following sense. Given a divergence-free, symmetric energy–momentum tensor T_{ab}, one has for each Killing field ξ^b the divergence-free current $J_a = T_{ab}\xi^b$. Using the linearized Einstein equations

$$\dot{G}_{ab} = \dot{R}_{acb}{}^{c} - \tfrac{1}{2}g_{ab}\dot{R}_{cd}{}^{cd} = -8\pi G\dot{T}_{ab}, \tag{35}$$

and the conformal Killing–Yano equation (28), they showed

$$3\int_{\partial\Sigma} \dot{R}_{abcd} * Y^{cd}\mathrm{d}x^a \wedge \mathrm{d}x^b = 16\pi G\int_{\Sigma} e_{abc}{}^{d}\dot{T}_{df}\xi^f\mathrm{d}x^a \wedge \mathrm{d}x^b \wedge \mathrm{d}x^c. \tag{36}$$

Here, Σ denotes a three-dimensional hypersurface with boundary $\partial\Sigma$ and e_{abcd} is the Levi-Civita tensor. The left-hand side is the charge integral described above, while the right-hand side gives the more familiar form of a conserved 3-form corresponding to a linearized source and a Killing vector $\xi^a = \tfrac{1}{3}Y^{ab}{}_{;b}$. Note that it is the dual conformal Killing–Yano tensor on the left-hand side, which gives the charge associated with the isometry ξ^a. In Cartesian coordinates $x^a = (t, x, y, z)$ the Poincaré isometries read

$$\mathcal{T}_a = \frac{\partial}{\partial x^a}, \qquad \mathcal{L}_{ab} = x_a\frac{\partial}{\partial x^b} - x_b\frac{\partial}{\partial x^a}, \tag{37}$$

and the relation to the charges is listed in table 1. The angular momentum around the z-axis is found in the component $\mathcal{L}_{xy} = \partial_\phi$.

Explicit expressions for linearized sources generating these charges can be found in [29, equation (27)].

The ten remaining charges cannot be generated this way, since the corresponding conformal Killing–Yano tensors have vanishing divergence (they are Killing–Yano tensors). One of these charges corresponds to the NUT parameter[12], and the remaining nine are three dual linear momenta and six ofam[13] charges. In the expression (34) for a general Killing spinor, they correspond to U and the imaginary part of V. For a metric perturbation, which one might interpret as a potential for the linearized curvature, these ten additional charges vanish, see [38, section 6.5].

To understand the charges as projections into $l = 0$ and $l = 1$ modes, we rederive the complete set of solutions in spherical coordinates using spin weighted spherical harmonics.

---

[12] Sometimes called dual mass, because of duality rotation from Schwarzschild to NUT, see the appendix of [39].
[13] Obstructions for angular momentum, see [31].

Table 2. Solutions to the Killing spinor equation on Minkowski spacetime in spherical coordinates.

	Components				Divergence	
Label	$\kappa_0/\sqrt{2}$	κ_1	$\kappa_2/\sqrt{2}$	Combination	Re	Im
				Ω^0_{-1}	0	0
Ω^0_m	$_1Y_{1m}$	$_0Y_{1m}$	$_{-1}Y_{1m}$	Ω^0_0	0	0
				Ω^0_1	0	0
Ω^1	0	r	0	Ω^1	$\mathcal{T}_t$	0
				$\Omega^1_1 - \Omega^1_{-1}$	$\mathcal{T}_x$	0
Ω^1_m	$(t-r)\,_1Y_{1m}$	$t\,_0Y_{1m}$	$(t+r)\,_{-1}Y_{1m}$	$i\Omega^1_1 + i\Omega^1_{-1}$	$\mathcal{T}_y$	0
				Ω^1_0	$\mathcal{T}_z$	0
				$\Omega^2_1 - \Omega^2_{-1}$	$\mathcal{L}_{tx}$	$\mathcal{L}_{yz}$
Ω^2_m	$(t-r)^2\,_1Y_{1m}$	$(t^2-r^2)\,_0Y_{1m}$	$(t+r)^2\,_{-1}Y_{1m}$	$i\Omega^2_1 + i\Omega^2_{-1}$	$\mathcal{L}_{ty}$	$\mathcal{L}_{xz}$
				Ω^2_0	$\mathcal{L}_{tz}$	$\mathcal{L}_{xy}$

A null tetrad for Minkowski spacetime in spherical coordinates (t, r, θ, ϕ) (symmetric Carter tetrad) is given by

$$l^a = \frac{1}{\sqrt{2}}[1, 1, 0, 0], \quad n^a = \frac{1}{\sqrt{2}}[1, -1, 0, 0], \quad m^a = \frac{1}{\sqrt{2}r}\left[0, 0, 1, \frac{i}{\sin\theta}\right],$$

with non-vanishing spin coefficients

$$\rho = -\frac{1}{\sqrt{2}r} = -\rho', \qquad \beta = \frac{\cot\theta}{2\sqrt{2}r} = \beta'.$$

A general 2-form can be expanded

$$
\begin{aligned}
Y = & + \kappa_2 \frac{r}{2}(\mathrm{d}r - \mathrm{d}t) \wedge (\mathrm{d}\theta + i\sin\theta\,\mathrm{d}\varphi) \\
& - \kappa_1 (\mathrm{d}t \wedge \mathrm{d}r + ir^2\sin\theta\,\mathrm{d}\theta \wedge \mathrm{d}\varphi) \\
& + \kappa_0 \frac{r}{2}(\mathrm{d}r + \mathrm{d}t) \wedge (\mathrm{d}\theta - i\sin\theta\,\mathrm{d}\varphi) + \text{c.c.},
\end{aligned}
$$

and it is a conformal Killing–Yano tensor, if the components κ_i satisfy (30) and (31). The subset (30) of the Killing spinor equation becomes

$$(\partial_t + \partial_r)\kappa_0 = 0, \qquad \left(\partial_\theta + \frac{i}{\sin\theta}\partial_\varphi - \cot\theta\right)\kappa_0 = 0,$$

$$(\partial_t - \partial_r)\kappa_2 = 0, \qquad \left(\partial_\theta - \frac{i}{\sin\theta}\partial_\varphi - \cot\theta\right)\kappa_2 = 0,$$

so $\kappa_0 = f_0(t-r)\,_1Y_{1m}$ and $\kappa_2 = f_1(t+r)\,_{-1}Y_{1m}$, where the functions f_i depend on advanced and retarded coordinates only and $_sY_{lm}$ are the spin weighted spherical harmonics, see e.g. section 4.15 in [37]. Finally, (31) can be solved for κ_1, which is only possible for particular functions f_i. The result is given in table 2.

There, Ω^1 represents one complex solution of the Killing spinor equation, while Ω^i_m, $i = 0, 1, 2$ represent three complex solutions each ($m = 0, \pm1$). We find the following correspondence to the solutions (34) in Cartesian coordinates:

$$\Omega^0_m \leftrightarrow U^{AB}, \qquad \Omega^1, \Omega^1_m \leftrightarrow V_{A'}{}^A, \qquad \Omega^2_m \leftrightarrow W_{A'B'}.$$

3.3. Conserved charges for type D spacetimes

The vacuum field equations in the algebraically special case of Petrov type D have been integrated explicitly by Kinnersley [36]. An explicit type D line element solving the Einstein–Maxwell equations with a cosmological constant is known, from which all type D line elements of this type can be derived by certain limiting procedures; see [40, section 19.1.2], and also [15]. The family of type D spacetimes contains the Kerr and Schwarzschild solutions, but also solutions with more complicated topology and asymptotic behavior, such as the NUT- or C-metrics, and solutions whose orbits of the isometry group are null. In the following, we again restrict to the vacuum case.

A Newman–Penrose tetrad such that the two real null vectors l^a, n^a are aligned with the two repeated principal null directions of a Weyl tensor of Petrov type D is called a principal tetrad. In this case,

$$\Psi_0 = \Psi_1 = 0 = \Psi_3 = \Psi_4, \qquad \kappa = \kappa' = 0 = \sigma = \sigma'$$

and $\Psi_2 \neq 0$. For convenience, we introduce a new variable ψ, which is related to the non-vanishing curvature scalar via,

$$\psi \propto \Psi_2^{-1/3}. \tag{38}$$

Due to the integrability condition (33), we have $\kappa_0 = 0 = \kappa_2$. Hence, the components (30, 31) of the Killing spinor equation simplify to

$$(\text{þ} + \rho)\kappa_1 = 0, \quad (\eth + \tau)\kappa_1 = 0, \quad (\text{þ}' + \rho')\kappa_1 = 0, \quad (\eth' + \tau')\kappa_1 = 0. \tag{39}$$

Comparison with the Bianchi identities

$$(\text{þ} - 3\rho)\Psi_2 = 0, \quad (\eth - 3\tau)\Psi_2 = 0, \quad (\eth' - 3\rho')\Psi_2 = 0, \quad (\eth' - 3\tau')\Psi_2 = 0 \tag{40}$$

shows that $\kappa_1 := \psi$ is a solution, and in fact is the only non-trivial solution of the Killing spinor equation.

The divergence $\xi^{AA'} = \nabla^{A'}{}_B \kappa^{AB}$ is a Killing vector field, which is proportional to a real Killing vector field for all type D spacetimes except for Kinnersley class IIIB, cf [11]. If $\xi^{AA'}$ is real, then the imaginary part of κ_{AB} is a Killing–Yano tensor. Spacetimes satisfying the just mentioned condition are called *generalized Kerr–NUT* spacetimes [19]. The square of the Killing–Yano tensor is a symmetric Killing tensor $K_{ab} = Y_{ac}Y^c{}_b$ and it follows that $\eta^a = K^{ab}\xi_b$ is a Killing vector. On a Kerr background, ξ^a and η^a are linearly independent and span the space of isometries, see [26]. In the special case of a Schwarzschild background, η^a vanishes, see also [12] for details.

For Kerr spacetime in Boyer–Lindquist coordinates, we find

$$\Psi_2 = -\frac{M}{(r - \mathrm{i}a\cos\theta)^3}, \qquad \psi \propto r - \mathrm{i}a\cos\theta,$$

and we set the factor of proportionality to 1, so that the solution

$$\kappa_0 = 0, \qquad \kappa_1 = \psi, \qquad \kappa_2 = 0, \tag{41}$$

reduces to Ω^1 as given in table 2, in the Minkowski limit $M, a \to 0$. The Killing spinor with components given by (41) is

$$\kappa_{AB} = -2\psi o_{(A} \iota_{B)}. \tag{42}$$

and its divergence is the timelike Killing vector,

$$(\partial_t)_a = \tfrac{1}{3}\nabla^b(\psi Z^1_{ab}) = \tfrac{1}{3}\nabla^{B'B}(\kappa_{AB}\epsilon_{A'B'}) = -\tfrac{1}{3}\nabla_{A'}{}^B \kappa_{AB}. \tag{43}$$

As discussed in the introduction, spin lowering the Weyl spinor using (42) gives the Maxwell field $\Psi_{ABCD}\kappa^{CD}$, which has charges proportional to *mass* and *dual mass*, see also [33]. Letting $\mathcal{M}(C,\kappa)$ denote the corresponding closed complex 2-form we have

$$\mathcal{M}(C,\kappa) = \psi\Psi_2 Z^1. \tag{44}$$

Evaluating the charge for the Kerr metric yields

$$\frac{1}{4\pi\mathrm{i}}\int_{S^2}\mathcal{M}(C,\kappa) = \frac{1}{4\pi\mathrm{i}}\int_{S^2} -\frac{M}{(r-\mathrm{i}a\cos\theta)^2}(-\mathrm{i})(r^2+a^2)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\varphi = M, \tag{45}$$

where M is the ADM mass while the dual mass is zero.

4. Fackerell's conservation law

The closed 2-form (44) has been derived already in 1961 by Jordan *et al* [34]. We will repeat the derivation here, since this formulation can be generalized to linearized gravity most easily. On a type D background, the curvature forms and the connection simplifies to

$$\Sigma_0 = \Psi_2 Z^2, \quad \Sigma_1 = \Psi_2 Z^1, \quad \Sigma_2 = \Psi_2 Z^0, \quad \Gamma = \tau l - \rho m, \tag{46}$$

so the middle Bianchi identity (21*b*) becomes

$$\begin{aligned}
2\mathrm{d}\Sigma_1 &= 2\Psi_2[(\rho'\bar{m}-\tau'n)\wedge l\wedge m + (\rho m-\tau l)\wedge\bar{m}\wedge n] \\
&= 2\Psi_2(\rho'l+\rho n-\tau'm-\tau\bar{m})\wedge Z^1 \\
&= h\wedge\Sigma_1,
\end{aligned}$$

where $h = 2(\rho'l+\rho n-\tau'm-\tau\bar{m})$ is used. As noted in [18], the Bianchi identities (40) can be rewritten as $2\mathrm{d}\Psi_2 = 3h\Psi_2$ and one obtains

$$\mathrm{d}(\Psi_2 Z^1) = \mathrm{d}\Sigma_1 = \tfrac{1}{2}h\wedge\Sigma_1 = \tfrac{1}{3}\mathrm{d}\Psi_2\wedge Z^1,$$

which yields the *Jordan–Ehlers–Sachs conservation law* [34],

$$\mathrm{d}\big(\Psi_2^{2/3}Z^1\big) = 0. \tag{47}$$

Using (38), this is the same result as (44). See also [27], where the conservation law is generalized to the spacetimes of Petrov type II. The result for type D backgrounds fits into the picture of Penrose potentials [23] and it generalizes to the linear perturbations of such backgrounds.

　　One can of course linearize the 2-form (44), which would provide a charge for perturbations within the class of type D spacetimes. But more generally, Fackerell [17] derived a closed 2-form for arbitrary linear perturbations around a type D background[14]. Starting from this conservation law, Fackerell and Crossmann derived field equations for perturbations of Kerr–Newmann spacetime. We summarize the result and give a shortened proof for the vacuum case.

Lemma 4.1. *A series expansion (in ϵ) of the Bianchi identities around a spacetime of Petrov type D yields*

$$\big(\mathrm{d}-\tfrac{1}{2}h\wedge\big)\Sigma_1 = O(\epsilon^2), \tag{48}$$

where $h = 2(\rho'l+\rho n-\tau'm-\tau\bar{m})$.

---

[14] One can expect that such a structure for perturbations of algebraically special solutions exists also for other signatures. A classification of the Weyl tensor in Euclidean signature can be found in [35], see also [24], a unified formulation for arbitrary signature is given in [5].

Proof. Equation (48) is the expansion of the middle Bianchi identity (21*b*). Since

$$\Sigma_0 = \Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2,$$
$$\Sigma_1 = \Psi_1 Z^0 + \Psi_2 Z^1 + \Psi_3 Z^2,$$
$$\Sigma_2 = \Psi_2 Z^0 + \Psi_3 Z^1 + \Psi_4 Z^2,$$

and

$$Z^0 = \bar{m} \wedge n, \qquad Z^1 = n \wedge l - \bar{m} \wedge m, \qquad Z^2 = l \wedge m,$$

we have

$$\begin{aligned}
\Gamma' \wedge \Sigma_0 &= (\tau' n + \kappa l - \rho' \bar{m} - \sigma m) \wedge (\Psi_0 Z^0 + \Psi_1 Z^1 + \Psi_2 Z^2) \\
&= \Psi_0 (\kappa l \wedge \bar{m} \wedge n - \sigma m \wedge \bar{m} \wedge n) \\
&\quad - \Psi_1 (\rho' \bar{m} \wedge n \wedge l + \sigma m \wedge n \wedge l + \tau' n \wedge \bar{m} \wedge m + \kappa l \wedge \bar{m} \wedge m) \\
&\quad + \Psi_2 (\tau' n \wedge l \wedge m - \rho' \bar{m} \wedge l \wedge m) \\
&= -\Psi_1 (\rho' \bar{m} \wedge n \wedge l + \tau' n \wedge \bar{m} \wedge m) + \Psi_2 (\tau' n \wedge l \wedge m - \rho' \bar{m} \wedge l \wedge m) + O(\epsilon^2) \\
&= \Psi_1 (-\rho' l + \tau' m) \wedge Z^0 + \Psi_2 (\tau' m - \rho' l) \wedge Z^1 + O(\epsilon^2) \\
&= (\tau' m - \rho' l) \wedge \Sigma_1 + O(\epsilon^2).
\end{aligned}$$

The last equality holds, because $\Psi_3 (\tau' m - \rho' l) \wedge Z^2 = 0$. A calculation along the same lines (or using the GHP prime operation) yields

$$\Gamma \wedge \Sigma_2 = (-\tau \bar{m} + \rho n) \wedge \Sigma_1 + O(\epsilon^2).$$

Now expanding the Bianchi identity, we find

$$d\Sigma_1 = \tfrac{1}{2} h \wedge \Sigma_1 + O(\epsilon^2).$$

$\square$

Now we state the main result of this paper.

Theorem 4.2. *For linearized gravity on a vacuum type D background in principal tetrad, there exists a closed 2-form*

$$\dot{\mathcal{M}} = \psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \tfrac{3}{2} \psi \Psi_2 \dot{Z}^1, \tag{49}$$

which can be used to calculate the 'linearized mass'. Here, ψ is the coefficient of the Killing spinor (42).

The 2-form $\dot{\mathcal{M}}$ is tetrad gauge invariant and changes only with a term χ which is exact, $\chi = df$, under coordinate gauge transformations. Hence, the integral

$$\frac{1}{4\pi i} \int_{S^2} \dot{\mathcal{M}}, \tag{50}$$

is conserved and gauge invariant. It equals the linearized ADM mass.

Proof. For linearized gravity, making use of (48) and $3h\Psi_2 = 2d\Psi_2$, we find the identity

$$\begin{aligned}
0 &= \psi \left(d - \tfrac{1}{2} h \wedge \right) \dot{\Sigma}_1 - \tfrac{1}{2} \psi \dot{h} \wedge \Sigma_1 \\
&= d(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \psi \Psi_2 \dot{Z}^1) - \tfrac{1}{2} \psi \Psi_2 h \wedge Z^1 \\
&= d \left(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2 + \tfrac{3}{2} \psi \Psi_2 \dot{Z}^1 \right),
\end{aligned} \tag{51}$$

where the linearized version of $dZ^1 = -h \wedge Z^1$ is used in the last step. Note that also

$$0 = d(\psi \dot{\Psi}_1 Z^0 + \psi \dot{\Psi}_2 Z^1 + \psi \dot{\Psi}_3 Z^2) - \tfrac{3}{2} \psi \Psi_2 \dot{h} \wedge Z^1 \tag{52}$$

holds, which looks similar to Maxwell equations with a source.

Now consider the coordinate gauge transformations (26) and use Cartan's identity $\mathcal{L}_\xi \omega = \mathrm{d}(\xi \lrcorner \omega) + \xi \lrcorner \mathrm{d}\omega$, which holds for arbitrary forms ω. It follows for $\dot{\mathcal{M}}$,

$$
\begin{aligned}
\delta \dot{\mathcal{M}} &= -\psi \xi (\Psi_2) Z^1 - \tfrac{3}{2} \psi \Psi_2 [\mathrm{d}(\xi \lrcorner Z^1) + \xi \lrcorner \mathrm{d}Z^1] \\
&= -\tfrac{3}{2} \psi \Psi_2 (\mathrm{d} + h\wedge)(\xi \lrcorner Z^1) \\
&= -\tfrac{3}{2} \mathrm{d}[\psi \Psi_2 (\xi \lrcorner Z^1)],
\end{aligned}
\tag{53}
$$

where $\xi \lrcorner h = \tfrac{2}{3} \Psi_2^{-1} \xi (\Psi_2)$ and $\xi \lrcorner (h \wedge Z^1) = (\xi \lrcorner h) Z^1 - h \wedge (\xi \lrcorner Z^1)$ was used. The 2-form (53) is exact and hence integrates to zero.

The tetrad gauge independence of $\dot{\mathcal{M}}$ can be seen as follows. From (27), we find for (24)

$$
\delta \dot{Z}^1 = -2b Z^0 - 2\bar{a} Z^2.
\tag{54}
$$

This exactly cancels the terms arising from $\dot{\Psi}_1$ and $\dot{\Psi}_3$, as can be seen from (25). $\dot{\Psi}_2$ is tetrad gauge invariant, since the linearized Weyl tensor and $\mathrm{tr}_g h$ do not depend on the tetrad, see (25c). This shows the tetrad gauge invariance of $\dot{\mathcal{M}}$ and therefore gauge invariance of (50). The relation to the ADM mass is discussed in the introduction, see in particular equation (8). $\qquad\square$

Remark 4.3.

(i) Equation (48) is to zeroth order the Jordan–Ehlers–Sachs conservation law (47) and to first order Fackerell's conservation law, $\mathrm{d}\dot{\mathcal{M}} = 0$. In the Minkowski limit, $M, a \to 0$, it reduces to the $l = 0$ Penrose charge with the Killing spinor Ω^1, see table 2.

(ii) In the work of Fayos *et al* [18], a gauge in which $\mathrm{d}(\psi \Psi_2 \dot{Z}^1) = 0$ was used. It is not clear from that work whether this gauge condition is compatible with a hyperbolic system of evolution equations for linearized gravity.

(iii) The closed 2-form, with $\dot{\mathcal{M}}$ in the form (49), has been derived by Fackerell and Crossmann, see [17]. In the present paper, we give a short and simple proof of (48), from which Fackerell's conservation law can be deduced. We also calculate the explicit gauge transformation behavior of $\dot{\mathcal{M}}$, from which the gauge invariance of linearized mass follows. The interpretation of (50) as the linearized ADM mass $\dot{M}$, and also its relation to Penrose's idea of spin lowering are the main results of this paper.

Finally, to express the charge integral in a form similar to the Maxwell case (3), we need the $\theta\phi$ components of the bivectors,

$$
Z^1_{\theta\phi} = -\mathrm{i}(r^2 + a^2) \sin\theta, \qquad Z^0_{\theta\phi} = -Z^2_{\theta\phi} = \frac{a\sqrt{\Delta}}{2} \sin^2\theta.
\tag{55}
$$

The charge integral becomes

$$
\frac{2\mathrm{i}}{\sqrt{\Delta}} \int_{S^2(t,r)} \dot{\mathcal{M}} = \int_{S^2(t,r)} \left(2V_L^{-1/2} \hat{\Psi}_2 + \mathrm{i}a \sin\theta \, \Psi_{\mathrm{diff}} \right)(r - \mathrm{i}a \cos\theta) \mathrm{d}\mu,
\tag{56}
$$

with $V_L = \Delta/(r^2 + a^2)^2$, $\mathrm{d}\mu = \sin\theta \mathrm{d}\theta \mathrm{d}\varphi$ and

$$
\hat{\Psi}_2 = \dot{\Psi}_2 - \Psi_2(M_3 + \bar{M}_3),
\tag{57a}
$$

$$
\Psi_{\mathrm{diff}} = \dot{\Psi}_1 - \dot{\Psi}_3 - 3\Psi_2(\mathrm{Re}(M_2 - M_1) - \mathrm{i}\,\mathrm{Im}(L_3 + N_3)).
\tag{57b}
$$

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5. Conclusions

For each isometry of a given background, there is a conserved charge for the linearized gravitational field. Working in terms of linearized curvature, we derived a linearized mass charge (corresponding to the time translation isometry) for Petrov type D backgrounds, by using Penrose's idea of spin-lowering with a Killing spinor.

A second Killing spinor, corresponding to the axial isometry of Kerr spacetime does not exist, cf (39). Hence, spin lowering cannot be used directly to derive a linearized angular momentum charge, even though a canonical analysis provides one in terms of the linearized metric.

For a Schwarzschild background, gauge conditions are known, which eliminate the gauge-dependent non-radiating modes [45, 30]. Understanding these conditions in a geometric way and generalizing them to a Kerr background needs further investigation.

Acknowledgments

Appendix. Coordinate expressions

For a Kerr spacetime in Boyer–Lindquist coordinates, the bivectors and connection forms in a Carter tetrad are

$$Z^1_{ab} = \begin{pmatrix} 0 & -1 & -ia\sin\theta & 0 \\ 1 & 0 & 0 & -a\sin^2\theta \\ ia\sin\theta & 0 & 0 & -i(r^2+a^2)\sin\theta \\ 0 & a\sin^2\theta & i(r^2+a^2)\sin\theta & 0 \end{pmatrix},$$
(A.1a)

$$Z^0_{ab} = \frac{1}{2\sqrt{\Delta}}\begin{pmatrix} 0 & -ia\sin\theta & \Delta & -i\Delta\sin\theta \\ ia\sin\theta & 0 & \Sigma & -i(r^2+a^2)\sin\theta \\ -\Delta & -\Sigma & 0 & a\Delta\sin^2\theta \\ i\Delta\sin\theta & i(r^2+a^2)\sin\theta & -a\Delta\sin^2\theta & 0 \end{pmatrix},$$
(A.1b)

$$Z^2_{ab} = \frac{1}{2\sqrt{\Delta}}\begin{pmatrix} 0 & ia\sin\theta & -\Delta & -i\Delta\sin\theta \\ -ia\sin\theta & 0 & \Sigma & i(r^2+a^2)\sin\theta \\ \Delta & -\Sigma & 0 & -a\Delta\sin^2\theta \\ i\Delta\sin\theta & -i(r^2+a^2)\sin\theta & a\Delta\sin^2\theta & 0 \end{pmatrix},$$
(A.1c)

$$\sigma_{0a} = \left(0, \frac{ia\sin\theta}{2p\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{2p}, -\frac{i\sqrt{\Delta}\sin\theta}{2p}\right),$$
(A.2a)

$$\sigma_{1a} = \left(\frac{M}{2p^2}, 0, 0, -\frac{Ma\sin^2\theta}{2p^2} - \frac{a+ir\cos\theta}{2p}\right),$$
(A.2b)

$$\sigma_{2a} = \left(0, \frac{\mathrm{i}a\sin\theta}{2p\sqrt{\Delta}}, -\frac{\sqrt{\Delta}}{2p}, \frac{\mathrm{i}\sqrt{\Delta}\sin\theta}{2p}\right).$$ (A.2c)

Here, we used

$$p = r - \mathrm{i}a\cos\theta, \qquad \Sigma = p\bar{p}, \qquad \Delta = r^2 - 2Mr + a^2.$$

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