

Constraint algebra in loop quantum gravity reloaded. II. Toy model of an Abelian gauge theory: Spatial diffeomorphisms

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(Received 9 December 2012; published 20 August 2013)

In A. Henderson, A. Laddha, and C. Tomlin, preceding article, [Phys. Rev. D **88**, 044028 (2013)], we initiated an approach towards quantizing the Hamiltonian constraint in loop quantum gravity by requiring that it generates an anomaly-free representation of constraint algebra of -shell. We investigated this issue in the case of a toy model of a $(2 + 1)$ -dimensional $U(1)^3$ gauge theory, which can be thought of as a weak coupling limit of Euclidean three-dimensional gravity. However, in paper I, we only focused on the most nontrivial part of the constraint algebra that involves the commutator of two Hamiltonian constraints. In this paper we continue with our analysis and obtain a representation of full constraint algebra in a loop quantized framework. We show that there is a representation of the diffeomorphism group with respect to which the Hamiltonian constraint quantized in paper I is diffeomorphism covariant. Our work can be thought of as a potential first step towards resolving some longstanding issues with the Hamiltonian constraint in canonical loop quantum gravity.

DOI: [10.1103/PhysRevD.88.044029](https://doi.org/10.1103/PhysRevD.88.044029)

PACS numbers: 04.60.Pp

I. INTRODUCTION

A satisfactory definition of Hamiltonian constraint in loop quantum gravity (LQG) [1] remains an open problem. Despite remarkable progress made in the seminal work of Thiemann ([2–4]), it is clear that the current quantization is not satisfactory due to three interrelated issues: (1) Enormous ambiguity in the definition of the continuum Hamiltonian constraint, (2) the absence of a representation of quantum Dirac algebra (referred to as the off-shell closure in [5]), and (3) when the constraint is used in a symmetry reduced sector of loop quantum cosmology, the low energy limit of the theory turns out to be incorrect [6]. Progress in obtaining a satisfactory definition of quantum dynamics in canonical LQG can be achieved by analyzing and overcoming the first two obstacles by taking hints from toy models like loop quantum cosmology.

In [7,8] a new approach was initiated to quantize the Hamiltonian constraint in LQG. This approach is based on the lessons learned in [6,9–11]. The idea in [7] was to look for higher density constraints whose action at finite triangulation was based upon the geometric action of the classical constraints on phase space fields. The continuum limit of finite triangulation constraint is taken *not on* \mathcal{H}_{kin} but on certain distributional subspaces known as habitats [12]. Instead of working with full LQG, in [7] we considered a simple toy model of $(2 + 1)$ -dimensional¹ $U(1)^3$ gauge

theory which can be thought of as a weak coupling limit of Euclidean canonical gravity [13]. In [7] we showed that there exists quantization of (density weight $\frac{5}{4}$) Hamiltonian constraint which satisfied

$$[\hat{H}[N], \hat{H}[M]] = H_{\text{diff}}[q^{-1}(N\widehat{\nabla}M - M\nabla N)]. \quad (1)$$

In this paper we continue the analysis of obtaining a representation of the constraint algebra in the loop quantized $U(1)^3$ gauge theory. Our goal is to obtain a representation of the “Dirac algebra” in the following sense²:

$$\hat{U}(\phi_1)\hat{U}(\phi_2) = \hat{U}(\phi_1 \circ \phi_2), \quad (2)$$

$$\hat{U}(\phi)^{-1}\hat{H}[N]\hat{U}(\phi) = \hat{H}[\phi^*N], \quad (3)$$

$$[\hat{H}[N], \hat{H}[M]] = i\hbar\hat{D}[\hat{\omega}]. \quad (4)$$

In this paper we focus on the (spatial) diffeomorphism covariance of the Hamiltonian constraint. That is, we want to see if there exists a representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} such that (3) is represented without anomaly.

Right at the outset, it appears that the answer will be in the negative, due to background structure which is required to define the quantum Hamiltonian constraint in [7].

Diffeomorphism noncovariance of the quantum shift: The action of the (finite-triangulation) Hamiltonian constraint on a charge network c results in a deformation of the underlying graph $\gamma(c)$ in a neighborhood of vertices of $\gamma(c)$ in the direction of vectors which are themselves

¹As we insisted in [7] and would like to remind the reader again here, although the model we consider is $(2 + 1)$ -dimensional theory, our analysis to a large extent is independent of dimensionality and we believe it goes through rather straightforwardly in $3 + 1$ dimensions. In fact as we argued in [7], some of the technicalities which are present in two spatial dimensions will be absent in three dimensions, thus simplifying the analysis. This should not be too surprising as off-shell closure of Dirac algebra probes the local structure of field theory, even when the theory is topological *on shell*.

²The quotation marks indicate that strictly speaking we are not working with the algebra of constraints but with the crossed product generated by the Hamiltonian constraint and finite diffeomorphisms.

defined using data from c . Given a charge network state $|c\rangle$ and its vertex set $V(c)$, the (regularized) expectation value $\langle c|\hat{E}_i^a \hat{q}^{-1/4}(v)|_\epsilon|c\rangle$ at any vertex $v \in V(c)$ plays the role of this vector, and was referred to as the quantum shift in [7]. The subscript ϵ indicates that the operator implicitly depends on regulating structures which are parametrized by ϵ . As we show below, due to the regularization dependence of the quantum shift vector, it turns out that, given a state $|\phi \cdot c\rangle$ (the diffeomorphic image of c under ϕ), the quantum shift $\langle \phi \cdot c|\hat{E}_i^a \hat{q}^{-1/4}(\phi v)|_\epsilon|\phi \cdot c\rangle$ defined at $\phi(v) \in V(\phi \cdot c)$ is not the pushforward (via ϕ) of $\langle c|\hat{E}_i^a \hat{q}^{-1/4}(v)|_\epsilon|c\rangle$. This is the first obstruction which, unless addressed, will ensure that the Hamiltonian constraint operator will not commute with finite diffeomorphisms.

Noncovariant nature of extraordinary (EO) vertices.—The key feature of the Hamiltonian constraint’s action on kinematical states³ is the creation of so-called extraordinary (EO) vertices. Essentially the idea is the following: Starting with any charge network c , around each of its nondegenerate vertices [vertices at which not all edges emanating from it are charged only in one copy of $U(1)$], we fix, once and for all, a coordinate ball $B(v, \epsilon)$ of radius ϵ and consider a sequence of finite-triangulation Hamiltonian constraint operators $\hat{H}_{T(\delta)}[N]$ with $\delta \leq \delta_0(\epsilon)$, where δ is associated to a coordinate size of simplices in T and $\delta_0(\epsilon)$ ensures that these in the neighborhood of a vertex v , simplices lie within $B(v, \epsilon)$. The action of $\hat{H}_{T(\delta)}[N]$ on $|c\rangle$ creates a linear combination of charge network states, each of which has an EO vertex sitting inside $B(v, \epsilon)$. These EO vertices have several distinguishing properties:

- (1) They are necessarily zero volume.
- (2) By construction they are inside $B(v, \epsilon)$ with v being some nondegenerate and have a “nonzero volume⁴” vertex.
- (3) Their “location” (with respect to the fixed coordinate chart around v) is state dependent and dictated by the so-called quantum shift.
- (4) Given any charge network c' with a vertex v^E satisfying the above three properties, one can always find charge network c with a vertex v such that the action of $\hat{H}_{T(\delta)}(v)$ results in a linear combination of charge network states including $|c'\rangle$.

³By this we mean the Hamiltonian constraint operator at finite triangulation which is densely defined on \mathcal{H}_{kin} .

⁴In the $U(1)^3$ theory in $2+1$ dimensions, the notion of degenerate vertex and zero-volume vertex are not equivalent. As we have defined above, a degenerate vertex is the one on which all incident edges are charged only in one copy of $U(1)$. All such degenerate vertices are necessarily zero volume; however, one could easily have a zero volume vertex which was not degenerate. The Hamiltonian constraint action on a charge network generically created vertices which are degenerate, but in some special cases it creates vertices which are zero volume but nondegenerate. We labeled them type-B EO vertices in [7].

Although the EO vertices are zero volume, the action of the Hamiltonian constraint on such vertices is required to be nontrivial and have a specific form in order to obtain an anomaly-free commutator of two continuum Hamiltonian constraints (for more details we urge the reader to consult [7]). The definition of an EO vertex relies on the vertices lying inside certain prescribed coordinate neighborhoods of nondegenerate vertices. Whence under an arbitrary diffeomorphism, an EO vertex could be dragged outside the the prescribed neighborhood and would no longer be classified as an EO vertex by our prescription. This means that the action of a Hamiltonian constraint on a state will not commute with the action of diffeomorphisms.

Nontrivial density weight of the lapse.—As the Hamiltonian constraint $H(x)$ is a scalar density of weight $\frac{5}{4}$, the lapse function is a scalar density of weight $-\frac{1}{4}$ whence evaluation of lapse at a given point requires an explicit specification of coordinate chart in the neighborhood of the point. Whence the pullback of a lapse by a diffeomorphism which occurs in $\hat{H}[\phi^*N]$ involves Jacobian factor between various coordinate charts and it is not clear how such factors could arise in $\hat{U}(\phi)\hat{H}[N]\hat{U}(\phi^{-1})$. This is yet another potential source of diffeomorphism noncovariance of the Hamiltonian constraint.

In this paper we show that, despite the apparent background dependence of the quantum Hamiltonian constraint, we obtain an anomaly-free representation of the Dirac algebra on \mathcal{V}_{LMI} by defining a new representation of the diffeomorphism group on \mathcal{H}_{kin} (and whence by dual action on \mathcal{V}_{LMI}) such that the continuum Hamiltonian constraint turns out to be diffeomorphism covariant.

The purpose of requiring the Hamiltonian constraint to be diffeomorphism covariant is twofold [2]. On the one hand, this ensures that the quantum constraint algebra is first class, and perhaps equally importantly, the vast amount of ambiguity which persists in the continuum quantum constraint can be reduced by requiring diffeomorphism covariance. This has been explicitly demonstrated in [2] and as we see below, it remains true even in our approach.

This paper is organized as follows. In Sec. II we recall key ideas and results from [7]. That section merely serves to summarize contents of [7] and we do not claim it to be a sufficient prerequisite for understanding all the details in the subsequent sections. We urge the interested reader to consult [7] for a more detailed understanding of the structures involved. In Sec. III we do a sample computation where we take the standard representation of the diffeomorphism group on \mathcal{H}_{kin} and obtain, via dual action, a conjugate representation on \mathcal{V}_{LMI} . We then check if the Hamiltonian constraint operator $\hat{H}[N]$ is covariant under this representation of the diffeomorphism group. As expected, $\hat{H}[N]$ is not covariant and the analysis reveals precisely where the issues mentioned above show up in the computation. In Sec. IV, we define a new representation of

diff(Σ) on \mathcal{V}_{LMI} which essentially ensures that various structures required to define the Hamiltonian constraint operator at finite triangulation behave covariantly with respect to this representation. In Sec. V, which is the main section of the paper, we prove the diffeomorphism-covariance of the continuum Hamiltonian constraint on \mathcal{V}_{LMI} . Together with [7], the results of this section establish a representation of the Dirac algebra for the $U(1)^3$ gauge theory. In Sec. VI we perform a heuristic check on the validity of the new representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} by computing a subset of physical states that capture the topological sector in the quantum theory and argue that the final answer we obtain is the expected one in the sense that we would have arrived at the same answer had we worked with the representation of $\text{Diff}(\Sigma)$ commonly used in LQG.

We end with conclusions where we highlight the unsatisfactory aspects of our work, which is the use of several auxiliary structures, and make some remarks pertaining to the generalization of our work for Euclidean LQG.

II. SUMMARY OF [7] AND SOME NOTATIONAL CHANGES

In this section, we briefly recap the relevant results and notation of [7], where more details can be found when desired. There, a proposal was made for a finite-triangulation Hamiltonian constraint operator $\hat{H}_{T(\delta)}$ on the vector space \mathcal{D} spanned by charge networks $c \equiv (c_1, c_2, c_3)$ (whose completion is the kinematical Hilbert space \mathcal{H}_{kin} of the theory) and its continuum limit ($\delta \rightarrow 0$) was evaluated on a vector space $\mathcal{V}_{\text{LMI}} \subset \mathcal{D}^*$ of distributions spanned by objects of the type

$$(\Psi_{[c]_{(1)}}^{f^{(1)}} | := \sum_{(c'_1, c_2, c_3) \in [c]_{(1)}} f^{(1)}(\bar{V}(c'_1 \cup c_2 \cup c_3)) \langle c'_1, c_2, c_3 |. \tag{5}$$

Here $f^{(1)}: \Sigma^{|V(c)|} \rightarrow \mathbb{C}$ is a smooth function that is symmetric in its arguments,⁵ and the set of arguments $\bar{V}(c'_1 \cup c_2 \cup c_3)$ is a set of vertices [with the same cardinality as the vertex set $V(c)$ of c] in which any WEO pairs (recalled below) of vertices are replaced by the single WEO vertex of the pair. In this work we will omit the parentheses around the $U(1)_i$ labels to slightly simplify the notation; i.e., these states will be written $\Psi_{[c]}^{f^i}$.

$[c]_{(i=1)}$ is a set of charge networks containing a ‘‘parent’’ $c = (c_1, c_2, c_3)$, as well all other charge networks (c'_1, c_2, c_3) in which c_2, c_3 are unaltered, but $c'_1 \neq c_1$, and the nonequality is of a special type, namely, each

(c'_1, c_2, c_3) has at least one vertex which is ‘‘weakly extraordinary (WEO) of type $i = 1$ ’’ with respect to c . We digress briefly to explain the notion of extraordinary and weakly extraordinary vertices.

Roughly speaking, extraordinary vertices of a charge network c' are those produced by the action of a finite-triangulation Hamiltonian constraint operator on a charge network c , and WEO vertices are EO vertices which have additionally been moved by diffeomorphisms which are the identity on c . Given a charge network and the associated coordinate charts based at its vertices, there is a list of criteria (found in [7]) which determines whether a vertex is WEO, EO, or neither. This is largely a topological and charge label-dependent classification, and given an arbitrary charge network c with WEO vertices, it is possible to reconstruct a unique WEO vertex-free charge network \tilde{c} . This comes about as follows. As shown in Appendix B of [7], any WEO vertex v^E in a charge network c is uniquely associated to a vertex $v \in V(c)$. Furthermore all the (maximal analytic extension of) edges beginning at v^E terminate in a three-valent vertex. By erasing each of these edges and adding the corresponding charge to the edge between the above mentioned three-valent vertex and v one reconstructs a WEO vertex-free charge network that we will denote by \tilde{c} throughout this paper.

If the classification scheme determines that a given vertex v^E is EO, then it is uniquely associated to another vertex v , namely, that vertex at which a finite-triangulation Hamiltonian-type operator has acted to produce the pair (v, v^E) . It is helpful to keep in mind a picture of the action of an operator of the type $\hat{E}_i^a \hat{F}_{ab}^j \hat{E}_k^b$ for some fixed $i \neq j \neq k$ at a vertex v of some charge network c . In [7] we have constructed such operators via a ‘‘loop assignment scheme’’ where, roughly speaking, \hat{E}_i^a gives the direction and magnitude of one leg of the loops, \hat{F}_{ab}^j determines that the charge on the attached loops is only nonzero in $U(1)_j$, and \hat{E}_k^b determines the magnitude of those $U(1)_j$ charges via the $U(1)_k$ charges on edges of the underlying state. EO vertices are common apex points of the (charged) loop collections, and hence come in several flavors, and it is necessary in what follows to respect their distinction. To this end, we introduce some additional notation.

Each v^E is first classified as type A or type B; type A EO vertices lie off of the original graph c , and type B vertices lie on c . This distinction is not important below, so we omit this information from our notation, and focus the discussion on type A EO vertices v^E . Let $v_\delta^E(j, k)$ denote an EO vertex from which all outgoing edges are charged in $U(1)_j$,⁶ with the magnitude of those charges being determined by the $U(1)_k$ charges in the underlying charge network. The subscript δ denotes that $v_\delta^E(j, k)$ is located a coordinate distance $\delta|\langle \hat{E}_i^a \rangle|$ (with respect to the coordinate

⁵This assumption was not made in [7]; however, it is invoked here in the interest of pedagogy. The analysis given in the paper can be easily seen to hold when we relax this assumption.

⁶In [7], we denoted this vertex as $v_\delta^E(M, k)$.

system based at the vertex v of the underlying charge network) from v in the direction of $\langle \hat{E}_i^a \rangle$. By the classification scheme detailed in [7], the pair $(v, v_8^E(j, k))$ is unique, and we term it an extraordinary pair.

Weakly extraordinary vertices are again detectable via the classification scheme, and are generated from EO vertices by applying diffeomorphisms (which do not move the underlying state) to states containing EO pairs. That is, these diffeomorphisms only move the loop collections produced by Hamiltonian-type actions.

In [7], the calculation of the continuum limit action $\hat{H}\Psi_{[c]_i}$ (in this work we drop the prime on \hat{H} as an operator on \mathcal{V}_{LMI}) was performed by first writing the Hamiltonian as a sum $H = \sum_i H^i$, where i labels the $U(1)$ index appearing on the curvature F_{ab}^i , and considering the various actions $\hat{H}^i \Psi_{[c]_j}$ for each i, j . The result is as follows: Given a density weight $-\frac{1}{2}$ lapse function N , it was found that

$$\hat{H}^1[N]\Psi_{[c]_1}^{f^1} = \sum_{v \in V(c)} \left(\Psi_{[c]_1}^{f_{v,2}^{1,1}[N]} - \Psi_{[c]_1}^{f_{v,3}^{1,1}[N]} \right), \quad (6)$$

where $f_{v,2}^{1,1}[N]$ is a (generally discontinuous) function which agrees with f^1 at almost all points of $\Sigma^{|V(c)|}$, except when its argument is $\bar{V}(c'_1 \cup c_2 \cup c_3)$ for some $(c'_1, c_2, c_3) \in [c]_1$, and $v \in \bar{V}(c'_1 \cup c_2 \cup c_3)$, in which case it takes the value

$$\begin{aligned} & f_{v,2}^{1,1}[N](\bar{V}(c'_1 \cup c_2 \cup c_3)) \\ &= N(v)\lambda(\vec{n}_v) \sum_{e|b(e)=v} n_e^2 \dot{e}^a(0) \frac{\partial}{\partial v^a} f^1(v, \bar{V}(c'_1 \cup c_2 \cup c_3) \\ & \quad - \{v\}), \end{aligned} \quad (7)$$

where $\dot{e}(0)$ is the vector tangent to edge e at $v = b(e)$, and is assumed to be of unit length in a prescribed coordinate system.

In [7] this function was called $\bar{f}_v^{(1)(1)}$, where the second superscripted (1) refers to the action of \hat{H}^1 , and the bar to the fact that the directional derivative is along $\langle \hat{E}_2^a(v) \rangle_c$ [as opposed to along $\langle \hat{E}_3^a(v) \rangle_c$, which in [7] was denoted by a double bar]. The expressions for $\hat{H}^2 \Psi_{[c]_2}$ and $\hat{H}^3 \Psi_{[c]_3}$ can be obtained by cyclic permutation of the indices in the above equations.

The action of the mixed-index cases were found to be of the form

$$\hat{H}^2[N]\Psi_{[c]_1}^{f^1} = \sum_{v \in V(c)} \left(\Psi_{[c]_1}^{f_{v,1}^{1,2}[N]} - \Psi_{[c]_1}^{f_{v,3}^{1,2}[N]} \right), \quad (8)$$

where again, the functions $f_{v,1}^{1,2}[N], f_{v,3}^{1,2}[N]$ agree with f^1 at all values of $\Sigma^{|V(c)|}$, except when those arguments coincide with $\bar{V}(c'_1 \cup c_2 \cup c_3)$ with $v \in V(c)$, and there is an EO vertex $v_8^E(1, 2) \in \text{supp}(N)$ associated with v , in which case we have

$$\begin{aligned} f_{v,1}^{1,2}[N](\bar{V}(c'_1 \cup c_2 \cup c_3)) &= \left[\sum_{e \in E(c)|b(e)=v} n_e^1 \dot{e}^a(0) \partial_a N(v) \right] \\ & \quad \times f^1(\bar{V}(c'_1 \cup c_2 \cup c_3)), \end{aligned} \quad (9a)$$

$$\begin{aligned} f_{v,3}^{1,2}[N](\bar{V}(c'_1 \cup c_2 \cup c_3)) &= \left[\sum_{e \in E(c)|b(e)=v} n_e^3 \dot{e}^a(0) \partial_a N(v) \right] \\ & \quad \times f^1(\bar{V}(c'_1 \cup c_2 \cup c_3)). \end{aligned} \quad (9b)$$

Recall that n_e^i are charges on the edge $e \in E(c)$ in $U(1)_i$. Similarly,

$$\hat{H}^3[N]\Psi_{[c]_1}^{f^1} = \sum_{v \in V(c)} \left(\Psi_{[c]_1}^{f_{v,2}^{1,3}[N]} - \Psi_{[c]_1}^{f_{v,1}^{1,3}[N]} \right) \quad (10)$$

with [under analogous conditions as stated above, with $v_8^E(1, 2)$ replaced with $v_8^E(1, 3)$]

$$\begin{aligned} f_{v,2}^{1,3}[N](\bar{V}(c'_1 \cup c_2 \cup c_3)) &= \left[\sum_{e \in E(c)|b(e)=v} n_e^2 \dot{e}^a(0) \partial_a N(v) \right] \\ & \quad \times f^1(\bar{V}(c'_1 \cup c_2 \cup c_3)). \end{aligned} \quad (11a)$$

$$\begin{aligned} f_{v,1}^{1,3}[N](\bar{V}(c'_1 \cup c_2 \cup c_3)) &= \left[\sum_{e \in E(c)|b(e)=v} n_e^1 \dot{e}^a(0) \partial_a N(v) \right] \\ & \quad \times f^1(\bar{V}(c'_1 \cup c_2 \cup c_3)). \end{aligned} \quad (11b)$$

The expressions for the remaining $\hat{H}^i[N]\Psi_{[c]_j}^{f^j}$ are cyclic permutations of these. Given these preliminaries, we now embark on a first attempt (and failure) to arrive at a statement of diffeomorphism covariance of this Hamiltonian.

III. NAIVE ATTEMPT

We now quantify the worries laid out in the Introduction regarding why, using the usual representation of the group of semianalytic diffeomorphisms (denoted in this paper by $\text{diff}(\Sigma)$) that is used in loop quantum gravity, the Hamiltonian constraint constructed in [7] is not diffeomorphism covariant. More in detail, in this section we ask the following question. Consider a representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} induced via dual action:

$$(\hat{U}(\phi)' \Psi_{[\bar{c]}_i}^{f^i})(|c\rangle) := \Psi_{[\bar{c]}_i}^{f^i}(\hat{U}(\phi)|c\rangle), \quad (12)$$

where the right-hand side of the above equation is given by using the ‘‘natural’’ unitary representation of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} [14]. We now ask if $\hat{U}(\phi)' \hat{H}^j[N] \hat{U}(\phi)^{-1} \Psi_{[\bar{c]}_i}^{f^i}$ equals $\hat{H}^j[\phi^* N] \Psi_{[\bar{c]}_i}^{f^i}$ for all i, j . As we will see, the answer is no, and the reasons are precisely those which were given in the Introduction.

Readers who are convinced by the arguments given in the Introduction can safely skip this section. However those who wish to follow details in the subsequent sections might find it helpful to peruse the computations done here. Without loss of generality, we restrict attention to $i = 1$

and check the diffeomorphism covariance of $\hat{H}^{j=1,2}[N]$ in the domain defined by $\Psi_{[\tilde{c}]_1}^{f^1}$.

A. Checking diffeomorphism covariance of $\hat{H}^1[N]$ on $\Psi_{[\tilde{c}]_1}^{f^1}$

Given any charge network state $|c_A\rangle$ we would like to see if

$$\begin{aligned} & (\hat{U}(\phi)' \hat{H}^1[N] \hat{U}(\phi)^{-1'} \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) \\ &= (\hat{H}^1[\phi^* N] \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle), \end{aligned} \quad (13)$$

where $\hat{U}(\phi)'$ denotes the natural representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} obtained by dualizing the action of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} . We can deduce this representation as follows:

$$\begin{aligned} (\hat{U}(\phi)' \Psi_{[\tilde{c}]_1}^{f^1})(|c'\rangle) &:= \Psi_{[\tilde{c}]_1}^{f^1}(|\phi \cdot c'\rangle) \\ &= \sum_{c'' \in [\tilde{c}]_1} f^1(\bar{V}(c'')) \delta_{c'', \phi \cdot c'} \\ &= (\Psi_{\phi^{-1} \cdot [\tilde{c}]_1}^{f^1 \circ \phi})(|c'\rangle), \end{aligned} \quad (14)$$

where

$$\phi \cdot [\tilde{c}]_1 = \{\phi \cdot (c'_1, \tilde{c}_2, \tilde{c}_3) | (c'_1, \tilde{c}_2, \tilde{c}_3) \in [\tilde{c}]_1\} \equiv [\phi \cdot \tilde{c}]_1. \quad (15)$$

Whence, the natural representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} is given by

$$\hat{U}(\phi)' \Psi_{[\tilde{c}]_i}^{f^i} = \Psi_{\phi^{-1} \cdot [\tilde{c}]_i}^{f^i \circ \phi}, \quad (16)$$

Let us first evaluate the left-hand side (LHS) of (13),

$$\begin{aligned} \text{LHS} &= (\hat{U}(\phi)' \hat{H}^1[N] \hat{U}(\phi)^{-1'} \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) \\ &= (\hat{H}^1[N] \hat{U}(\phi)^{-1'} \Psi_{[\tilde{c}]_1}^{f^1})(|\phi \cdot c_A\rangle) \\ &= (\hat{H}^1[N] \Psi_{[\phi \cdot \tilde{c}]_1}^{f^1 \circ \phi^{-1}})(|\phi \cdot c_A\rangle) \\ &= \left[\Psi_{[\phi \cdot \tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),2}^{1,1}[N]} - \Psi_{[\phi \cdot \tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),3}^{1,1}[N]} \right] (|\phi \cdot c_A\rangle), \end{aligned} \quad (17)$$

where, in the final line we have assumed (without loss of generality) that the only vertex in $V(\phi \cdot \tilde{c})$ which lies in the support of N is $\phi(v_0)$ with $v_0 \in V(\tilde{c})$. The resulting vertex functions $(f \circ \phi^{-1})_{\phi(v_0),2}^{1,1}[N]$ and $(f \circ \phi^{-1})_{\phi(v_0),3}^{1,1}[N]$ are given by

$$(f \circ \phi^{-1})_{\phi(v_0),2}^{1,1}[N] = \begin{cases} (f^1 \circ \phi^{-1})(\bar{V}(c')), & \text{if } \phi \cdot v_0 \notin \bar{V}(c') \\ \lambda(\vec{n}_{\phi(v_0)}^{\phi \cdot \tilde{c}}) N(\phi(v_0)) V_2^a(\phi(v_0)) \partial_a^{\phi(v_0)} (f^1 \circ \phi^{-1})(\phi(v_0), \bar{V}(c') - \{\phi(v_0)\}), & \text{otherwise.} \end{cases} \quad (18)$$

$(f \circ \phi^{-1})_{\phi(v_0),3}^{1,1}[N]$ is defined similarly with V_2^a replaced by V_3^a .

Whence, assuming $\phi \cdot c_A \in [\phi \cdot \tilde{c}]_1$, we have

$$\text{LHS} = \begin{cases} 0 & \text{if } \phi(v_0) \notin \bar{V}(\phi \cdot c_A) \\ N(\phi(v_0)) [V_2^a(\phi(v_0)) - V_3^a(\phi(v_0))] \partial_a^{\phi(v_0)} (f^1 \circ \phi^{-1})(\phi(v_0), \bar{V}(\phi \cdot \tilde{c}) - \{\phi(v_0)\}) & \text{if } \phi(v_0) \in \bar{V}(\phi \cdot c_A) \end{cases} \quad (19)$$

and if $\phi \cdot c_A \notin [\phi \cdot \tilde{c}]_1$, we have

$$\text{LHS} = 0. \quad (20)$$

On the other hand, the right hand side (RHS) of (13) is given by

$$\text{RHS} = (\hat{H}^1[\phi^* N] \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) = \left[\Psi_{[\tilde{c}]_1}^{f_{v_0,2}^{1,1}[\phi^* N]} - \Psi_{[\tilde{c}]_1}^{f_{v_0,3}^{1,1}[\phi^* N]} \right] |c_A\rangle, \quad (21)$$

where, following the assumption regarding the support of the lapse with respect to the vertex set of $\phi \cdot \tilde{c}$, it is clear that the only vertex in $V(\tilde{c})$ which lies inside the support of $\phi^* N$ is v_0 . As before, we can evaluate the resulting vertex functions, and find

$$f_{v_0,2}^{1,1}[\phi^* N](\bar{V}(c')) = \begin{cases} f^1(V(c')) & \text{if } v_0 \notin \bar{V}(c') \\ (\phi^* N)(v_0) \lambda(\vec{n}_{v_0}^{\tilde{c}}) V_2^a(v_0) \partial_a^{v_0} f^1(v_0, \bar{V}(c') - \{v_0\}) & \text{if } v_0 \in \bar{V}(c'). \end{cases} \quad (22)$$

$f_{v_0,3}^{1,1}[\phi^* N]$ is defined similarly with V_2^a replaced by V_3^a .

Thus, if $c_A \in [\tilde{c}]_1$ we have

$$\text{RHS} = \begin{cases} 0 & \text{if } v_0 \notin \bar{V}(c_A) \\ (\phi^* N)(v_0) \lambda(\vec{n}_{v_0}^{\tilde{c}}) [V_2^a(v_0) - V_3^a(v_0)] \partial_a^v f^1(v_0, \bar{V}(c_A) - \{v_0\}) & \text{if } v_0 \in \bar{V}(c_A) \end{cases} \quad (23)$$

and if $c_A \notin [\tilde{c}]_1$, we have

$$\text{RHS} = 0. \quad (24)$$

We would now like to see if the LHS and RHS of (13) as detailed in (19) to (23) are equal for all $\Psi_{[\tilde{c}]_1}^{f^1}$, N , and $|c_A\rangle$.

Case 1 $c_A \notin [\tilde{c}]_1 \Rightarrow \phi \cdot c_A \notin [\phi \cdot \tilde{c}]_1$. In this case from (20) and (24) we clearly see that LHS = RHS = 0.

Case 2 $c_A \in [\tilde{c}]_1$ but $v_0 \notin \bar{V}(c_A) \Rightarrow \phi \cdot c_A \in [\phi \cdot \tilde{c}]_1$ but $\phi(v_0) \notin \bar{V}(\phi \cdot c_A)$. In this case from the first equation in (19) and in (23) we see that LHS = RHS = 0.

Case 3 $c_A \in [\tilde{c}]_1$ and $v_0 \in \bar{V}(c_A) \Rightarrow \phi \cdot c_A \in [\phi \cdot \tilde{c}]_1$ and $\phi(v_0) \in \bar{V}(\phi \cdot c_A)$. In this case the LHS and RHS are given by the first equations in (19) and (23), respectively. It is clear that for a generic choice of the habitat state $\Psi_{[\tilde{c}]_1}^{f^1}$, the LHS and RHS are not equal for two reasons:

- (i) The LHS involves $N(\phi(v_0))$, whereas RHS involves $(\phi^* N)(v_0) = \left| \frac{d\phi}{dv_0} \right| N(\phi(v_0))$; i.e., the two differ by a Jacobian factor.
- (ii) The LHS involves $\lambda(\vec{n}_{\phi(v_0)}^{\phi \cdot \tilde{c}}) \vec{V}_i(\phi(v_0), \phi \cdot \tilde{c})$ which is not equal to $\lambda(\vec{n}_{v_0}^{\tilde{c}}) \phi_* \vec{V}_i(v_0, \tilde{c})$.

Thus the nontrivial density weight of the lapse and the diffeomorphism noncovariance of the quantum shift are the two reasons why the naive attempt to prove diffeomorphism covariance of $\hat{H}^1[N]$ fails.

B. Checking diffeomorphism covariance: $\hat{H}^2[N] \Psi_{[\tilde{c}]_1}^{f^1}$

In the previous section, we analyzed the behaviour of $\hat{H}^1[N]$ under conjugation by the natural representation of $\text{Diff}(\Sigma)$ on $\Psi_{[\tilde{c}]_1}^{f^1}$ and identified two problems which are responsible for its spatial noncovariance. In this section, we continue along the same route and analyze the diffeomorphism (non)covariance of $\hat{H}^2[N]$ on $\Psi_{[\tilde{c}]_1}^{f^1}$. At the very least, we expect the two culprits identified in the last section to spoil the covariance properties again, but as we will see in this case there is an additional difficulty. The action of $\hat{H}^2[N]$ on charge network states containing EO vertices (which are by definition zero volume) is different from its action on non-EO zero-volume vertices. However, a quick look at the definition of EO vertices reveals that the entire EO structure is diffeomorphism noncovariant: A diffeomorphism can map an EO vertex into a WEO vertex. This transcends into another issue in the continuum

limit, ensuring diffeomorphism noncovariance of $\hat{H}^2[N]$ on $\Psi_{[\tilde{c}]_1}^{f^1}$. We now turn to a detailed analysis of this issue.

Given $\Psi_{[\tilde{c}]_1}^{f^1} \in \mathcal{V}_{\text{LMI}}$ and $|c_A\rangle \in \mathcal{H}_{\text{kin}}$, we once again want to see if

$$\begin{aligned} & (\hat{U}(\phi)' \hat{H}^2[N] \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) \\ &= (\hat{H}^{(2)}[\phi^* N] \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) \end{aligned} \quad (25)$$

$\forall \phi \in \text{diff}(\Sigma)$. We compute

$$\begin{aligned} \text{LHS} &= (\hat{U}(\phi)' \hat{H}^2[N] \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_1}^{f^1})(|c_A\rangle) \\ &= (\hat{U}(\phi)' \hat{H}^2[N] \Psi_{\phi \cdot [\tilde{c}]_1}^{f^1 \circ \phi^{-1}})(|c_A\rangle) \\ &= (\hat{U}(\phi)' \left[\Psi_{\phi \cdot [\tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),1}^{1,2}[N]} - \Psi_{\phi \cdot [\tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),3}^{1,2}[N]} \right])(|c_A\rangle) \\ &= \left[\Psi_{\phi \cdot [\tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),1}^{1,2}[N]} - \Psi_{\phi \cdot [\tilde{c}]_1}^{(f \circ \phi^{-1})_{\phi(v_0),3}^{1,2}[N]} \right](|\phi \cdot c_A\rangle), \end{aligned} \quad (26)$$

where in the third line we have used (8) and assumed (without loss of generality) that $\exists v_0 \in V(\tilde{c})$ such that the only vertex in $V(\phi \cdot \tilde{c})$ which falls inside the support of N is $\phi(v_0)$. In the fourth line we have used (14). The vertex functions in the third and fourth lines of (26) are given in Sec. II, and

$$(f \circ \phi^{-1})_{\phi(v_0),3}^{1,2}[N](\bar{V}(c')) = (f^1 \circ \phi^{-1})(\bar{V}(c')), \quad (27a)$$

$$(f \circ \phi^{-1})_{\phi(v_0),1}^{1,2}[N](\bar{V}(c')) = (f^1 \circ \phi^{-1})(\bar{V}(c')) \quad (27b)$$

$\forall c'$ such that $\bar{V}(c')$ does not contain an EO vertex $\phi(v_0)_{\delta}^E(1, 2)$ of type (1, 2).

If on the other hand, $\bar{V}(c')$ contains an EO vertex, $(\phi \cdot v_0)_{\delta}^E(1, 2)$ (for some δ) then,

$$\begin{aligned} & (f \circ \phi^{-1})_{\phi \cdot v_0,3}^{1,2}[N](\bar{V}(c')) \\ &= (f \circ \phi^{-1})_{\phi \cdot v_0,3}^{1,2}[N]((\phi \cdot v_0)_{\delta}^E(1, 2), \dots) \\ &= [\lambda(\vec{n}_{\phi \cdot v_0}^{\phi \cdot \tilde{c}}) V_1^a(\phi \cdot v_0, \phi \cdot \tilde{c}) (\nabla_a N)(\phi \cdot v_0)] f^1 \\ & \quad \circ \phi^{-1}((\phi \cdot v_0)_{\delta}^E(1, 3), \dots), \\ & (f \circ \phi^{-1})_{\phi \cdot v_0,1}^{1,2}[N](\bar{V}(c')) \\ &= (f \circ \phi^{-1})_{\phi \cdot v_0,1}^{1,2}[N]((\phi \cdot v_0)_{\delta}^E(1, 2), \dots) \\ &= [\lambda(\vec{n}_{\phi \cdot v_0}^{\phi \cdot \tilde{c}}) V_3^a(\phi \cdot v_0, \phi \cdot \tilde{c}) (\nabla_a N)(\phi \cdot v_0)] f^1 \\ & \quad \circ \phi^{-1}((\phi \cdot v_0)_{\delta}^E(1, 1), \dots). \end{aligned} \quad (28)$$

Using the last line in (26), it is easy to see that,

$$\begin{aligned}
 \text{LHS} &= 0 \quad \text{if } \phi \cdot c_A \notin \phi \cdot [c]_1 \\
 &= [(f \circ \phi^{-1})_{\phi \cdot v_0, 1}^{1,2}[N](\vec{V}(c_A)) \\
 &\quad - (f \circ \phi^{-1})_{\phi \cdot v_0, 3}^{1,2}[N](\vec{V}(c_A))] \text{ otherwise.} \quad (29)
 \end{aligned}$$

Whence upon using (28) in (29) we see that if c_A does not contain an EO vertex of type 1, 2 associated to v_0 then

$$\begin{aligned}
 f_{v_0, 3}^{1,2}[\phi^* N](\vec{V}(c')) &= f_1(\vec{V}(c')) \\
 &\quad \text{if there is no EO vertex of type-1, 2 w.r.t } v_0 \text{ in } \vec{V}(c')[\lambda(\vec{n}_{v_0}^{\tilde{c}} V_1^a(\tilde{c}, v_0)(\nabla_a(\phi^* N)))(v_0)] f^1((v_0)_\delta^E(1, 3), \dots) \\
 &\quad \text{if there is an E.O vertex } (v_0)_\delta^E \text{ w.r.t. } v_0, \\
 f_{v_0, 1}^{1,2}[\phi^* N](\vec{V}(c')) &= f_1(\vec{V}(c')) \\
 &\quad \text{if there is no EO vertex of type-1, 2 w.r.t } v_0 \text{ in } \vec{V}(c')[\lambda(\vec{n}_{v_0}^{\tilde{c}} V_3^a(\tilde{c}, v_0)(\nabla_a(\phi^* N)))(v_0)] f^1((v_0)_\delta^E(1, 1), \dots) \\
 &\quad \text{if there is an E.O vertex } (v_0)_\delta^E \text{ w.r.t. } v_0. \quad (32)
 \end{aligned}$$

Thus it is straightforward to see that RHS as defined in (31) is given by

$$\text{RHS} = 0 \quad \text{if } c_A \notin [\tilde{c}]_1 = f_{v_0, 1}^{1,2}[\phi^* N](\vec{V}(c_A)) - f_{v_0, 3}^{1,2}[\phi^* N](\vec{V}(c_A)) \text{ otherwise,} \quad (33)$$

which using (32) further implies that

$$\begin{aligned}
 \text{RHS} &= 0 \quad \text{if there exists no EO vertex of type-I, } j \text{ in } V(c_A) \text{ associated to } v_0 \\
 &= -\lambda(\vec{n}_{v_0}^{\tilde{c}})[V_3^a(\tilde{c}, v_0) f^1((v_0)_\delta^E(1, 1), \dots) - V_1^a(\tilde{c}, v_0) f^1((v_0)_\delta^E(1, 3), \dots)](\nabla_a(\phi^* N))(v_0) \text{ otherwise.} \quad (34)
 \end{aligned}$$

Comparing (34) with (29) we can easily verify that if $c_A \notin [\tilde{c}]_1$ (which is equivalent to $\phi \cdot c_A \notin [\phi \cdot \tilde{c}]_1$) then

$$\text{LHS} = \text{RHS} = 0. \quad (35)$$

However if $c_A \in [\tilde{c}]_1$ then LHS and RHS are only equal for all diffeomorphisms, if there is no WEO vertex of type (1, 2) associated to v_0 in c_A . Otherwise there could exist a diffeomorphism ϕ such that it would map a WEO vertex associated to v_0 to an EO vertex (of the same type) associated to $\phi \cdot v_0$ in which case LHS would be zero [from (30) but RHS would be given by the second line in (34)].

We thus conclude that given a $\Psi_{[c]}^{f_i}$ in the LMI habitat, the action of $\hat{H}[N] = \sum_{j=1}^3 H^{(j)}[N]$ is not covariant under action of spatial diffeomorphisms due to three reasons.

- The quantum shift is not a covariant object in any sense: If two charge networks c_1 and c_2 are diffeomorphic to each other (which means there are infinitely many semianalytic diffeomorphisms which map c_1 to c_2 , there need not exist any diffeomorphism whose pushforward maps $\vec{V}_j(v, c_1)$ to $\vec{V}_j(\phi \cdot v, c_2)$).
- The nontrivial density weight of lapse causes extra Jacobian factors to arise when comparing $\phi^* N$ with $N \circ \phi$.
- The EO structure is a diffeomorphism noncovariant concept unlike the WEO structure.

$$\text{LHS} = 0. \quad (30)$$

We now turn our attention to RHS ($\hat{H}^{(2)}[\phi^* N] \Psi_{[\tilde{c}]}^{f_1}$) $|c_A\rangle$.

$$\text{RHS} = \left[\Psi_{[\tilde{c}]}^{f_{v_0, 1}^{1,2}[\phi^* N]} - \Psi_{[\tilde{c}]}^{f_{v_0, 3}^{1,2}[\phi^* N]} \right] |c_A\rangle, \quad (31)$$

where the vertex functions are once again given by

C. Our strategy

In this section we briefly outline our approach and explain the key ideas that are developed in subsequent sections. As some of the analysis done in later sections is slightly involved, we hope that a reading of this section will give the reader an understanding of the concepts.

Our aim is to show that despite the apparent background dependence of quantum Hamiltonian constraint, we obtain an anomaly-free representation of the Dirac algebra on \mathcal{V}_{LMI} by defining a new representation of the diffeomorphism group on \mathcal{H}_{kin} (and whence by dual action on \mathcal{V}_{LMI}). The basic ideas behind our construction are summarized below.

As shown in [7], and recalled briefly in the Sec. II, the continuum Hamiltonian constraint on the LMI habitat is a sum of three operators given by

$$\begin{aligned}
 \hat{H}[N] \Psi_{[\tilde{c}]}^{f_i} &= (\hat{H}^1[N] + \hat{H}^2[N] + \hat{H}^3[N]) \Psi_{[\tilde{c}]}^{f_i} \\
 \forall i &\in \{1, 2, 3\}, \quad (36)
 \end{aligned}$$

where $\Psi_{[\tilde{c}]}^{f_i}$ is an arbitrary element in \mathcal{V}_{LMI} . We will restrict our analysis to the $i = 1$ case (as the analysis for states in the $i = 2, 3$ sectors is exactly analogous) and prove diffeomorphism covariance of $H[N]$ by showing

$$\begin{aligned}
 \hat{U}(\phi) \hat{H}^1[N] \hat{U}(\phi^{-1}) \Psi_{[\tilde{c}]}^{f_1} &= \hat{H}^1[\phi^* N] \Psi_{[\tilde{c}]}^{f_1}, \\
 \hat{U}(\phi) \hat{H}^j[N] \hat{U}(\phi^{-1}) \Psi_{[\tilde{c}]}^{f_1} &= \hat{H}^j[\phi^* N] \Psi_{[\tilde{c}]}^{f_1} \quad j \in \{2, 3\}. \quad (37)
 \end{aligned}$$

From Eq. (37) it follows that $\hat{H}[N]$ is diffeomorphism covariant on any $\Psi_{[\tilde{c}]_1}^{f^1} \in \mathcal{V}_{\text{LMI}}$. Diffeomorphism covariance of $\hat{H}[N]$ on an arbitrary state in \mathcal{V}_{LMI} is a trivial extension of the above claim.

We now describe the main ideas behind the new representation of $\text{Diff}(\Sigma)$ defined in Sec. IV. As recalled in Sec. II, given any charge network c , there is a unique “undeformed” \tilde{c} associated to it such that the action of the finite-triangulation Hamiltonian constraint on c involves a set of vectors $\vec{V}(v, \tilde{c})$ associated to each vertex $v \in V(\tilde{c})$, and a characterization of which of the vertices $v^E(i, j)$ in c are EO with respect to vertices in \tilde{c} . Whence it is clear that the data set we are dealing with, as far as the definition of the Hamiltonian constraint action on c goes, is $\{\vec{V}(v, \tilde{c}), v^E(i, j) | v \in V(\tilde{c}) \cap V(c), v^E(i, j) \in V(c)\}$. We denote the collection of all such data sets associated to any diffeomorphism-invariant orbit of charge networks by $\mathcal{C}([c]_{\text{diff}}) = \cup_{c' \in [c]_{\text{diff}}} \{\vec{V}(v, \tilde{c}'), v^E(i, j) | v \in V(\tilde{c}'), v^E(i, j) \in V(c')\}$. Intuitively we would like to choose a representation of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} which preserves $\mathcal{C}([c]_{\text{diff}})$; that is, any diffeomorphism should act in such a way that it maps one element of $\mathcal{C}([c]_{\text{diff}})$ to some other element of $\mathcal{C}([c]_{\text{diff}})$. We achieve this objective as follows:

- (i) In each diffeomorphism-invariant orbit $[\tilde{c}]_{\text{diff}}$ of undeformed charge networks, we fix once and for all an “initial” charge network $\tilde{c}^0 \in [\tilde{c}]_{\text{diff}} \equiv [\tilde{c}^0]_{\text{diff}}$, and a set of diffeomorphisms $\{\phi_{\tilde{c}^0, \tilde{c}'}\}_{\tilde{c}' \in [\tilde{c}^0]_{\text{diff}}}$ which maps \tilde{c}^0 to any $\tilde{c}' \in [\tilde{c}^0]_{\text{diff}}$.
- (ii) We also associate to each such \tilde{c}^0 an atlas $\mathcal{U}(\tilde{c}^0)$ on Σ such that each vertex v of \tilde{c}^0 lies in precisely one open set of $\mathcal{U}(\tilde{c}^0)$ and to each $\tilde{c}' \in [\tilde{c}^0]_{\text{diff}}$ we associate an atlas obtained by pushforward⁷ of $\mathcal{U}(\tilde{c}^0)$ by $\phi_{\tilde{c}^0, \tilde{c}'}$.
- (iii) We compute the quantum shift vectors $\{\vec{V}(v, \tilde{c}^0) | v \in V(\tilde{c}^0)\}$ on the vertices of reference charge network \tilde{c}^0 once and for all and *define* quantum shift vectors for any $\tilde{c}' \in [\tilde{c}^0]_{\text{diff}}$ as

$$\vec{V}(v', \tilde{c}') := (\phi_{\tilde{c}^0, \tilde{c}'}^{-1})_* \vec{V}(\phi_{\tilde{c}^0, \tilde{c}'}^{-1}(v'), \tilde{c}^0). \quad (38)$$

Thus, given a $[\tilde{c}]_{\text{diff}}$ with a reference charge network \tilde{c}^0 , the set $\mathcal{C}([c]_{\text{diff}})$ (such that the unique WEO vertex-free charge network associated to c is \tilde{c}), is invariant under the action of diffeomorphisms $\phi_{\tilde{c}^0, \tilde{c}}$. This motivates our new representation which essentially amounts to working with $\cup_{\tilde{c}^0} \cup_{\tilde{c}' \in [\tilde{c}^0]_{\text{diff}}} \phi_{\tilde{c}^0, \tilde{c}'}$ instead of $\text{diff}(\Sigma)$.⁸ We now revisit

⁷By pushforward of the coordinate chart we merely mean the coordinates of the diffeomorphic image of a point in the pushed-forward coordinate chart are the same as the coordinates of the original point in the initial coordinate chart.

⁸At this point we are only trying to motivate our construction of new representation. The details are given in Sec. IV. For example, at this point we have not even shown that the set that we are working with forms a group.

the transformation properties of $\hat{H}[N]$ on \mathcal{V}_{LMI} under this representation of $\text{Diff}(\Sigma)$ and show that it transforms covariantly.

D. Preferred diffeomorphisms: ϕ maps

In this section we explain how we assign to each diffeomorphism-invariant orbit $[\tilde{c}]_{\text{diff}}$ of WEO vertex-free charge networks a set of diffeomorphisms which will be a crucial ingredient in defining a new representation of $\text{Diff}(\Sigma)$ in the quantum theory.

We start with a trivial observation. $[\tilde{c}]_{\text{diff}}$ is a category (in fact a groupoid) with $\tilde{c}' \in [\tilde{c}]_{\text{diff}}$ being the objects and all the diffeomorphisms which map say \tilde{c}' to \tilde{c}'' constitute $\text{Hom}(\tilde{c}', \tilde{c}'')$. Our idea is to work with a subcategory (in fact a subgroupoid) in $[\tilde{c}]_{\text{diff}}$ in defining a representation of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} . Pick a reference charge network \tilde{c}_0 and for all $\tilde{c}' \in [\tilde{c}]_{\text{diff}}$ fix once and for all a diffeomorphism $\phi_{\tilde{c}_0, \tilde{c}'}$ which map \tilde{c}_0 to \tilde{c}' . (We choose $\phi_{\tilde{c}_0, \tilde{c}_0} = \text{Id}$). Now given any $\tilde{c}', \tilde{c}'' \in [\tilde{c}]_{\text{diff}}$ we define a diffeomorphism which maps \tilde{c}' to \tilde{c}'' as

$$\phi_{\tilde{c}', \tilde{c}''} := \phi_{\tilde{c}_0, \tilde{c}''} \circ \phi_{\tilde{c}_0, \tilde{c}'}^{-1}. \quad (39)$$

It is easy to verify that

$$\phi_{\tilde{c}_1, \tilde{c}_2} \circ \phi_{\tilde{c}_2, \tilde{c}_3} = \phi_{\tilde{c}_1, \tilde{c}_3}, \quad \phi_{\tilde{c}_1, \tilde{c}_2} = \phi_{\tilde{c}_2, \tilde{c}_1}^{-1} \quad (40)$$

$\forall \tilde{c}_1, \tilde{c}_2 \in [\tilde{c}]_{\text{diff}}$.

The categorical notions are not essential in understanding the representation of $\text{Diff}(\Sigma)$; however, it is a useful concept to understand the type of structure we are dealing with when we fix a diffeomorphism once and for all between any two charge networks.

We will sometimes refer to these select set of diffeomorphisms as ϕ maps.

E. A diffeomorphism-covariant regularization scheme

Classically, $V_i^a = q^{-1/4} E_i^a$ is a C^∞ densitized vector field. In [7] the quantization of $V_i^a(v)$ at a given point $v \in \Sigma$ involved a choice of regulating structures such that, at finite regularization parametrized by ϵ , a densely defined operator $\hat{V}_i^a(v)|_\epsilon$ on \mathcal{H}_{kin} was obtained. Although this operator is explicitly independent of ϵ due to the density weights of various quantities, it is implicitly dependent on the chosen regulating structures, which can be most easily seen through its spectrum. In particular, this dependence implies that generically, given two charge networks c_1 and c_2 that are diffeomorphic to each other,

$$\phi_* (\langle c | \hat{V}_i^a(v) |_\epsilon | c_1 \rangle) \neq \langle \phi c | \hat{V}_i^a(\phi \cdot v) |_\epsilon | c_2 \rangle, \quad (41)$$

$\forall \phi$ which map c_1 to c_2 . This result implies the following.

Consider $[c_0]_{\text{diff}}$ which is a diffeomorphism-invariant set of charge networks that contains c_0 . The definition of the quantum shift vectors associated to a given c is essentially an assignment of vectors $\{\vec{V}_i(v, c) | v \in V(c), i\{1, 2, 3\}\}$.

Equation (41) implies that, given $c_1, c_2 \in [c_0]_{\text{diff}}$, there is *no* meaningful sense in which we can talk about the quantum shift vectors associated to c_1 being diffeomorphically related to quantum shift vectors associated to c_2 . We term this property, diffeomorphism noncovariance of quantum shift. As we saw in Sec. III, the diffeomorphism noncovariance of the quantum shift in turn implies that the Hamiltonian constraint operator as we have defined it will not be diffeomorphism covariant; i.e., Eq. (3) will not be satisfied. We cure this problem by first taking a cue from the construction of rigging a map for finite diffeomorphisms [14], then defining an alternative (as opposed to the representation currently used in LQG) representation of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} . We show that this leads to a solution to the diffeomorphism noncovariance problem of the quantum shift,⁹ and finally to a diffeomorphism-covariant Hamiltonian constraint operator.

First let us briefly recall the result of the construction of the quantum shift in [7]. At each point $p \in \Sigma$, we fix once and for all a coordinate system $\{x_p\}$ with origin at p . Let \tilde{c} be a WEO vertex-free charge network with a vertex $v \in \Sigma$. The (coordinate-dependent) regularization procedure, detailed in [7], gives

$$\begin{aligned} \hat{V}_j^a(v)|_\epsilon|\tilde{c}\rangle &= \hat{E}_j^a|_\epsilon(v)\hat{q}_\epsilon^{-1/4}(v)|\tilde{c}\rangle = \lambda(\tilde{n}_\epsilon^v)\frac{1}{\pi}\sum_{e_i \cap v} \hat{e}_i^a n_i^j|\tilde{c}\rangle \\ &\equiv \lambda(\tilde{n}_\epsilon^v)\langle E_j^a(v)\rangle_{\tilde{c}}|\tilde{c}\rangle =: V_j^a(v, \tilde{c})|\tilde{c}\rangle, \end{aligned} \quad (42)$$

where we have employed the abstract index notation and the \hat{e}_i^a are unit tangent vectors (with respect to the coordinate system at v with metric δ_{ab}) to the edges of \tilde{c} emanating from v . As it stands, $V_j^a(v, \tilde{c})$ is computed separately for each member \tilde{c} of the diffeomorphism equivalence class $[\tilde{c}]_{\text{diff}}$ of \tilde{c} .

To solve the noncovariance problem stated above, we will modify this construction and compute $V_j^a(v, \tilde{c}_0)$ only in some reference charge network $\tilde{c}_0 \in [\tilde{c}]_{\text{diff}} \equiv [\tilde{c}_0]_{\text{diff}}$. The result will be transported to vertices of other charge nets in the equivalence class by the set of relevant ϕ maps which were defined in Sec. III D.

We define the quantum shift vectors $V_j^a(v, \tilde{c}) \quad \forall v := \phi_{\tilde{c}_0, \tilde{c}} \cdot v_0 | v \in V(\tilde{c}), \tilde{c} \in [\tilde{c}_0]_{\text{diff}}$ via pushforward with respect to the $\varphi_{\tilde{c}_0, \tilde{c}}$:

$$\vec{V}_j(v, \tilde{c}) := (\phi_{\tilde{c}_0, \tilde{c}})_* \vec{V}_j(v_0, \tilde{c}_0). \quad (43)$$

IV. REPRESENTATION OF $\text{Diff}(\Sigma)$ ON THE LMI HABITAT

As we saw in Sec. III A, there is a natural representation of a diffeomorphism group on \mathcal{V}_{LMI} . It is given by

⁹This means that we can in a precise sense talk about a map between quantum shifts associated to two charge networks which are diffeomorphic to each other.

$$\hat{U}(\phi)' \Psi_{[c]_i}^{f^i} = \psi_{[\phi \cdot c]_i}^{f^i \circ \phi^{-1}}. \quad (44)$$

However as we saw in Sec. III B, this representation is not the one that will lead us to a nonanomalous Dirac algebra on \mathcal{V}_{LMI} , as $\hat{U}(\phi)$ generically maps an EO vertex to a WEO vertex. Keeping this in mind, we define a new representation of $\text{Diff}(\Sigma)$ on \mathcal{H}_{kin} which in turn leads to a novel representation of the diffeomorphism group on \mathcal{V}_{LMI} . We will see that this representation has some desirable properties.

- (1) Given $[\tilde{c}_0]_{\text{diff}}$, and the collection of vectors $\cup_{\tilde{c}' \in [\tilde{c}_0]_{\text{diff}}} \{\vec{V}_i(v, \tilde{c}') | v \in V(\tilde{c}')\}$, the new representation preserves this set. More precisely, $(\phi_{\tilde{c}_0, \tilde{c}'})_* \times (\vec{V}_i(v, \tilde{c}_0)) = \vec{V}_i(v', \tilde{c}')$ for all $v \in V(\tilde{c}_0)$ such that $v' = \phi_{\tilde{c}_0, \tilde{c}'}(v) \in V(\tilde{c}')$.
- (2) It preserves the EO structure associated to charge nets. (As we will see below, this will be achieved by making coordinate charts around a given vertex “state dependent.”)

A. An alternative representation of $\text{diff}(\Sigma)$

1. Preliminaries

Definitions: Let \tilde{c} be a WEO vertex-free (signified by the tilde) charge network with vertex set $V(\tilde{c}) =: \{v_1, \dots, v_I, \dots, v_{|V(\tilde{c})}|\}$, and let $\delta < \delta_0(\tilde{c})$ be an admissible small parameter with respect to each of the coordinate systems based at the points of $V(\tilde{c})$, as detailed in [7] [roughly, the bound $\delta_0(\tilde{c})$ guarantees that the finite-triangulation Hamiltonian-type deformations at “finesness” δ that are performed on \tilde{c} are “local enough” so that one can actually classify these so-called EO vertices which are formed by the action of Hamiltonian constraint]. We define the *i*th δ -cilium at the vertex v_I , denoted $\sigma_i^I(\delta, \tilde{c})$, as a straight-line arc of coordinate length $\delta|\langle \hat{E}_i^a \rangle_{\tilde{c}}|$, directed along the quantum shift $V_i^a(v_I, \tilde{c})$, with one end at v_I , which goes into the definition of the curvature loop appearing in the Hamiltonian action.

The δ -ciliated graph determined by \tilde{c} , denoted $\gamma_{\sigma(\delta, \tilde{c})}(\tilde{c})$, is given by the union of the graph $\gamma(\tilde{c})$ underlying \tilde{c} and the set of δ -cilia [see Fig. 1]:

$$\gamma_{\sigma(\delta, \tilde{c})}(\tilde{c}) := \gamma(\tilde{c}) \cup \bigcup_{i=1}^3 \bigcup_{I=1}^{|V(\tilde{c})|} \sigma_i^I(\delta, \tilde{c}). \quad (45)$$

Now consider the diffeomorphism equivalence class $[\tilde{c}]_{\text{diff}}$. We choose once and for all a preferred element of $\tilde{c}_0 \in [\tilde{c}]_{\text{diff}} \equiv [\tilde{c}_0]_{\text{diff}}$ to represent the equivalence class, and in the neighborhood of each $v \in V(\tilde{c}_0)$ a fixed coordinate chart $\{x_v\}^{\tilde{c}_0}$. For all $\tilde{c} \in [\tilde{c}_0]_{\text{diff}}$, we choose once and for all a preferred collection of coordinate charts $\{x_v\}^{\tilde{c}}$ in the neighborhood of each of its vertices $v' = \phi_{\tilde{c}_0, \tilde{c}}(v)$ which is obtained by a pushforward of $\{x_v\}^{\tilde{c}_0}$ using $\phi_{\tilde{c}_0, \tilde{c}}$:

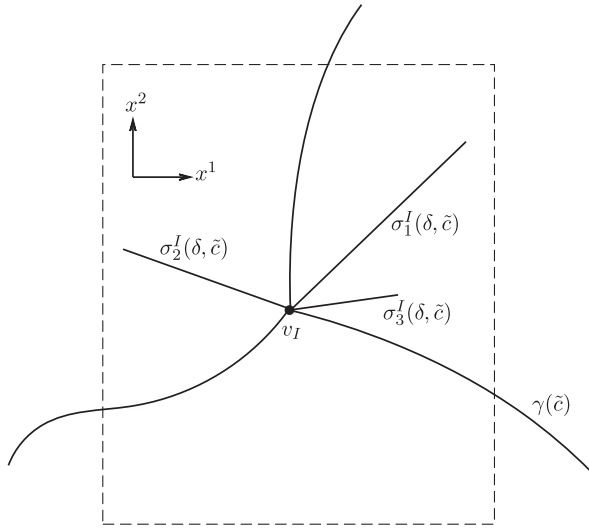


FIG. 1. The δ -ciliated graph $\gamma_{\sigma(\delta, \tilde{c})}(\tilde{c})$ determined by \tilde{c} in the neighborhood of the vertex v_I with respect to the coordinate system $\{x_{v'}\} = \{x^1, x^2\}$.

$$\{x_{v'}\}^{\tilde{c}} = (\phi_{\tilde{c}_0, \tilde{c}})_* \{x_{v'}\}^{\tilde{c}_0}, \quad \forall v' \in V(\tilde{c}), \quad (46)$$

$$v \in V(\tilde{c}_0) \text{ such that } v' = \phi_{\tilde{c}_0, \tilde{c}}(v).$$

This means that given a $v_0 \in V(\tilde{c}_0)$, $\sigma_i(v_0, \delta, \tilde{c}_0)$ which is a linear curve (in parameter $t \in [0, \delta]$) with respect to the coordinate chart $\{x_{v'}\}^{\tilde{c}_0}$, beginning at v with its tangent at v being $\vec{V}_i(v, \tilde{c}_0)$ gets mapped to a linear curve $\sigma_i(v', \delta, \tilde{c})$ (in parameter $t \in [0, \delta]$) with respect to the coordinate chart $\{x_{v'}\}^{\tilde{c}}$, beginning at v with its tangent at v being $\vec{V}_i(v', \tilde{c}) := (\phi_{\tilde{c}_0, \tilde{c}})_*[\vec{V}_i(v, \tilde{c}_0)]$. Whence, we get

$$\begin{aligned} \phi_{\tilde{c}_0, \tilde{c}}(\gamma_{\sigma(\delta, \tilde{c}_0)}(\tilde{c}_0)) &= \phi_{\tilde{c}_0, \tilde{c}}\left(\gamma(\tilde{c}_0) \cup \bigcup_{i=1}^3 \bigcup_{I=1}^{|\tilde{c}_0|} \sigma_i^I(\delta, \tilde{c}_0)\right) \\ &= \gamma(\tilde{c}) \cup \bigcup_{i=1}^3 \phi_{\tilde{c}_0, \tilde{c}}\left(\bigcup_{I=1}^{|\tilde{c}_0|} \sigma_i^I(\delta, \tilde{c}_0)\right) \\ &= \gamma(\tilde{c}) \cup \bigcup_{i=1}^3 \left(\bigcup_{I=1}^{|\tilde{c}_0|} \sigma_i^I(\delta, \tilde{c})\right) \\ &= \gamma_{\sigma(\delta, \tilde{c})}(\tilde{c}). \end{aligned} \quad (47)$$

Whence by adapting the coordinate charts around vertices to charge networks, we preserve the EO nature of a vertex.¹⁰ As we will see, this new ingredient will turn out to be crucial in obtaining an anomaly-free constraint algebra.

2. A new representation

We are now ready to define a new representation of $\text{diff}(\Sigma)$ on \mathcal{V}_{LMI} via a new representation on \mathcal{H}_{kin} . For charge networks \tilde{c} with no WEO vertices, we have that

¹⁰We are indebted to Madhavan Varadarajan who explained to us the use of state-dependent coordinate charts in regularization of quantum constraints.

$$\hat{U}(\phi)|\tilde{c}\rangle := |\phi_{\tilde{c}, \phi\tilde{c}} \cdot \tilde{c}\rangle. \quad (48)$$

If a charge network c_1 has any WEO vertices, then, as shown in the Appendix, there is a unique WEO vertex-free charge network $\tilde{c}(c_1)$ associated to it and $\tilde{c}(c_1)$ can be recovered from c_1 by performing a certain surgery. Using this fact we then define

$$\hat{U}(\phi)|c\rangle := |\phi_{\tilde{c}(c), \phi \cdot \tilde{c}(c)} \cdot c\rangle. \quad (49)$$

Clearly this defines a representation:

$$\begin{aligned} \hat{U}(\phi')\hat{U}(\phi)|c\rangle &= |\phi_{\phi_{\tilde{c}, \phi\tilde{c}} \cdot c, \phi' \cdot \phi_{\tilde{c}, \phi\tilde{c}} \cdot c} \phi_{\tilde{c}, \phi\tilde{c}} \cdot c\rangle \\ &= |\phi_{\phi \cdot \tilde{c}, \phi' \cdot \phi \cdot \tilde{c}} \phi_{\tilde{c}, \phi\tilde{c}} \cdot c\rangle \\ &= |\phi_{\tilde{c}, (\phi' \circ \phi) \cdot \tilde{c}} \cdot c\rangle \\ &= \hat{U}(\phi' \circ \phi)|c\rangle. \end{aligned} \quad (50)$$

The action of the $\hat{U}(\phi)$ on \mathcal{V}_{LMI} descends from the action on \mathcal{H}_{kin} via

$$\begin{aligned} \langle c|\hat{U}(\phi)' := (\hat{U}(\phi^{-1})|c\rangle)^\dagger &= |\phi_{\tilde{c}(c), \phi^{-1} \cdot \tilde{c}(c)} \cdot c\rangle^\dagger \\ &= \langle \phi_{\tilde{c}(c), \phi^{-1} \cdot \tilde{c}(c)} \cdot c|. \end{aligned} \quad (51)$$

It is easy to see that using the above the equation we have

$$\begin{aligned} \langle \Psi_{[\tilde{c}]_1}^{f^1}|\hat{U}(\phi)' &= \sum_{(c'_1, c_2, c_3) \in [\tilde{c}]_1} f^1(\vec{V}(c'_1 \cup c_2 \cup c_3)) \\ &\quad \times \langle \phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}} \cdot (c'_1, c_2, c_3)|, \end{aligned} \quad (52)$$

where by \tilde{c} we mean the WEO vertex-free charge network $\tilde{c}(c'_1, \tilde{c}_2, \tilde{c}_3)$ underlying $(c'_1, \tilde{c}_2, \tilde{c}_3)$ which is exactly \tilde{c} for all $(c'_1, \tilde{c}_2, \tilde{c}_3) \in [\tilde{c}]$.

Now note that the vertex set $\vec{V}(c'_1 \cup c_2 \cup c_3)$ transforms *equivariantly* under diffeomorphisms:

$$\begin{aligned} \phi_{\tilde{c}, \phi\tilde{c}}(\vec{V}(c'_1 \cup c_2 \cup c_3)) \\ = \vec{V}(\phi_{\tilde{c}, \phi\tilde{c}}(c'_1) \cup \phi_{\tilde{c}, \phi\tilde{c}}c_2 \cup \phi_{\tilde{c}, \phi\tilde{c}}c_3). \end{aligned} \quad (53)$$

We can now define a (dual) representation on \mathcal{V}_{LMI} based on the representation given in (51) as follows:

$$\hat{U}(\phi)'\Psi_{[\tilde{c}]_i}^{f^i} := \Psi_{\phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}} \cdot [\tilde{c}]_i}^{f^i \circ \phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}}} \quad (54)$$

$\forall i$. The above definition is justified by the following lemma.

Lemma: $\Psi_{[\tilde{c}]_i}^{f^i}(\hat{U}(\phi)|c_A\rangle) = \Psi_{\phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}} \cdot [\tilde{c}]_i}^{f^i \circ \phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}}}|c_A\rangle$ for all $|c_A\rangle \in \mathcal{H}_{\text{kin}}$.

Proof: We give the proof for $i = 1$. Without loss of generality, we assume that all WEO vertices of c_A are of type 1, as otherwise both sides are trivially zero.

Compute

$$\Psi_{[\tilde{c}]_1}^{f^1}(\hat{U}(\phi)|c_A\rangle) = \sum_{c' \in [\tilde{c}]_1} f^1(\vec{V}(c')) \delta_{c', \phi_{\tilde{c}, \phi\tilde{c}} \cdot c_A} \quad (55)$$

where $c' = (c'_1, \tilde{c}_2, \tilde{c}_3) \in [\tilde{c}]_1$, and \tilde{c}_A is the WEO vertex-free charge network associated to c_A . We have that

$$\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \notin [\tilde{c}]_1 \Leftrightarrow \phi \cdot \tilde{c}_A \neq \tilde{c}, \quad (56)$$

and in this case, $\Psi_{[\tilde{c}]_1}^{f^1}(\hat{U}(\phi)|c_A) = 0$. On the other hand, if $\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \in [\tilde{c}]_1 (\Leftrightarrow \phi \cdot \tilde{c}_A = \tilde{c})$, then

$$\begin{aligned} \Psi_{[\tilde{c}]_1}^{f^1}(\hat{U}(\phi)|c_A) &= f^1(\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)) \\ &= f^1(\bar{V}(\phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}} \cdot c_A)), \end{aligned} \quad (57)$$

whereas the RHS is given by

$$\begin{aligned} \sum_{c' \in [\tilde{c}]_1} f^1(\bar{V}(c')) \delta_{c', \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A}} \\ = \sum_{c' \in \phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}}[\tilde{c}]_1} f^1 \circ \phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}}(\bar{V}(c')) \delta_{c', c_A}. \end{aligned} \quad (58)$$

Whence if $c_A \notin \phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}} \cdot [\tilde{c}]_1 (\Leftrightarrow \tilde{c}_A \neq \phi^{-1} \cdot \tilde{c})$, then the right-hand side of (55) vanishes. On the other hand if $c_A \in \phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}} \cdot [\tilde{c}]_1 (\Leftrightarrow \tilde{c}_A = \phi^{-1} \cdot \tilde{c})$, then

$$\text{RHS} = f^1 \circ \phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}}(\bar{V}(c_A)) = f^1(\bar{V}(\phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}} \cdot c_A)). \quad (59)$$

In the second equality we have used the fact that \bar{V} is a diffeomorphism-equivariant map on the set of vertices. This proves the lemma.

It is straightforward to verify that (54) defines a representation:

$$\begin{aligned} \hat{U}(\phi)' \Psi_{[\tilde{c}]_1}^{f^i} &:= \Psi_{\phi_{\tilde{c}, \phi^{-1} \cdot \tilde{c}}[\tilde{c}]_1}^{f^i \circ \phi_{\phi^{-1} \cdot \tilde{c}, \tilde{c}}}, \\ \hat{U}(\phi_1)' \hat{U}(\phi_2)' \Psi_{[\tilde{c}]_1}^{f^1} &= \hat{U}(\phi_1)' \Psi_{\phi_{\tilde{c}, \phi_2^{-1} \cdot \tilde{c}}[\tilde{c}]_1}^{f^1 \circ \phi_{\phi_2^{-1} \cdot \tilde{c}, \tilde{c}}}, \end{aligned} \quad (60)$$

$$\begin{aligned} \hat{U}(\phi_1)' \hat{U}(\phi_2)' \Psi_{[\tilde{c}]_1}^{f^1} &= \left(\Psi_{[\phi_2^{-1} \phi_1^{-1} \tilde{c}]_1}^{f^1 \circ \alpha_{\tilde{c}, \phi_1^{-1} \tilde{c}}^{-1} \circ \alpha_{\phi_1^{-1} \tilde{c}, \phi_2^{-1} \phi_1^{-1} \tilde{c}}^{-1}} \right) \\ &= \left(\Psi_{[(\phi_1 \circ \phi_2)^{-1} \tilde{c}]_1}^{f^1 \circ \alpha_{\tilde{c}, (\phi_1 \circ \phi_2)^{-1} \tilde{c}}^{-1}} \right) \\ &= \left(\Psi_{[\tilde{c}]_1}^{f^1} \right) | \hat{U}(\phi_1 \circ \phi_2)', \end{aligned} \quad (61)$$

so that we indeed have a representation.

We now point out a rather interesting property of this representation. Although \mathcal{V}_{LMI} is a subspace of distributions on \mathcal{H}_{kin} , there is a canonical choice of inner product on this space.¹¹ Given $\Psi_{[\tilde{c}']_i}^{f^i_1}$, $\Psi_{[\tilde{c}'']_j}^{f^j_2}$, the inner product is defined as

$$\left(\Psi_{[\tilde{c}']_i}^{f^i_1}, \Psi_{[\tilde{c}'']_j}^{f^j_2} \right) := \delta_{\tilde{c}', \tilde{c}''} \bar{f}_1^i(V(\tilde{c}'))^* \bar{f}_2^j(V(\tilde{c}'')) \delta_{i,j}. \quad (62)$$

Note that this inner product is not positive definite unless we restrict ourselves to ‘‘basis’’ states $\Psi_{[\tilde{c}]_i}^{f^i}$ in \mathcal{V}_{LMI} which are such that $|f^i(V(\tilde{c}))|^2 > 0$. In any case, it is clear that the representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} is unitary with respect to (62) as

$$\begin{aligned} (\hat{U}(\phi) \Psi_{[\tilde{c}']_i}^{f^i_1}, \hat{U}(\phi) \Psi_{[\tilde{c}'']_j}^{f^j_2}) \\ = \left(\Psi_{\phi_{\tilde{c}', \phi^{-1} \cdot \tilde{c}'}[\tilde{c}']_i}^{f^i_1 \circ \phi_{\phi^{-1} \cdot \tilde{c}', \tilde{c}'}} , \Psi_{\phi_{\tilde{c}'', \phi^{-1} \cdot \tilde{c}''}[\tilde{c}'']_j}^{f^j_2 \circ \phi_{\phi^{-1} \cdot \tilde{c}'', \tilde{c}''}} \right) \\ = \delta_{\phi \cdot \tilde{c}', \phi \cdot \tilde{c}''} (f^i_1 \circ \phi_{\phi^{-1} \cdot \tilde{c}', \tilde{c}'})(V(\phi^{-1} \tilde{c}'))^* (f^j_2 \circ \phi_{\phi^{-1} \cdot \tilde{c}'', \tilde{c}''}) \\ \times (V(\phi^{-1} \tilde{c}'')) \delta_{i,j} \\ = \delta_{\phi \cdot \tilde{c}', \phi \cdot \tilde{c}''} f^i_1(V(\tilde{c}'))^* f^j_2(V(\tilde{c}'')) \delta_{i,j} \\ = \left(\Psi_{[\tilde{c}']_i}^{f^i_1}, \Psi_{[\tilde{c}'']_j}^{f^j_2} \right), \end{aligned} \quad (63)$$

where we have used the fact that $\phi_{\tilde{c}, \phi \cdot \tilde{c}}|_{\tilde{c}} = \phi|_{\tilde{c}} \quad \forall \phi$.

V. DIFFEOMORPHISM COVARIANCE

A. $\hat{H}^1[N] \Psi_{[c]_1}^{f^1}$

In this section we revisit the diffeomorphism-covariance of $\hat{H}^1[N]'$ on $\Psi_{[\tilde{c}]_1}^{f^1}$ in light of the new representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} involving $\phi_{\tilde{c}_0, \tilde{c}}$ maps.

Whence our aim is to check if

$$\begin{aligned} (\hat{U}(\phi)' \hat{H}^1[N]' \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_1}^{f^1})|c_A \rangle \\ = (\hat{H}^1[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N]' \Psi_{[\tilde{c}]_1}^{f^1})|c_A \rangle \end{aligned} \quad (64)$$

$\forall \Psi_{[\tilde{c}]_1}^{f^1} \in \mathcal{V}_{\text{LMI}}$ and $|c_A \rangle \in \mathcal{H}_{\text{kin}}$.

Note that on the right-hand side, we expect the lapse to be pulled back by the diffeomorphism $\phi_{\tilde{c}, \phi \cdot \tilde{c}}$ (which given a \tilde{c} and a ϕ is fixed once and for all) and not by ϕ as warranted by the new representation of the diffeomorphism group.

Before proceeding with the computation, we outline our setup which will also help us in clarifying our (often confusing) notations. We denote the reference charge network in $[\tilde{c}]_{\text{diff}}$ by \tilde{c}^0 . The WEO vertex-free state underlying c_A will be denoted by \tilde{c}_A . Given a vertex v_A in \tilde{c} , we will denote the corresponding (image under diffeomorphism $\phi_{\tilde{c}, \tilde{c}^0}$) vertex in \tilde{c}^0 as v_A^0 .

Without loss of generality we assume that the only WEO vertices which belong to $V(c_A)$ are of type 1, as otherwise both sides are trivially zero.

¹¹Note that on the Lewandowski Marolf habitat defined in [12] no such canonical choice exists.

We first compute the LHS using (51), (54), and (6)

$$\begin{aligned}
\text{LHS} &= \left(\hat{U}(\phi)' \hat{H}^1[N] \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_1}^{f^1} \right) |c_A\rangle \\
&= \left(\hat{H}^1[N] \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_1}^{f^1} \right) | \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \rangle \\
&= \left(\hat{H}^1[N] \Psi_{\phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1}^{f^1 \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}}} \right) | \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \rangle \\
&= \sum_{v \in V(\tilde{c})} \left(\Psi_{\phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1}^{(f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N]} - \Psi_{\phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1}^{(f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N]} \right) \\
&\quad | \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \rangle. \tag{65}
\end{aligned}$$

Further analysis of the above equation can be divided into the following two (mutually exclusive and exhaustive) cases:

- Case 1: $\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \notin \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1 \Leftrightarrow \phi \cdot \tilde{c}_A \neq \phi \cdot \tilde{c} \Leftrightarrow \tilde{c}_A \neq \tilde{c}$. In this case it is clear that LHS = 0.
- Case 2: $\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \in \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1 \Leftrightarrow \phi \cdot \tilde{c}_A = \phi \cdot \tilde{c} \Leftrightarrow \tilde{c}_A = \tilde{c}$. In this case,

$$\begin{aligned}
\text{LHS} &= \sum_{v \in V(\tilde{c})} \left((f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A})) \right. \\
&\quad \left. - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A})) \right). \tag{66}
\end{aligned}$$

For the sake of pedagogy, we now assume that the only vertex in $V(\phi \cdot \tilde{c}) = V(\phi \cdot \tilde{c}_A)$ which falls inside the support of N is a vertex v_A . As the Hamiltonian constraint action is linearly distributed over vertices, there is no loss of generality in this assumption.

In this case Case 2 gets further subdivided into the following two cases:

Case 2a: $v_A \notin \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$, and in this case,

$$\begin{aligned}
\text{LHS} &= \sum_{v \in V(\tilde{c})} \left((f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)) \right. \\
&\quad \left. - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)) \right) \\
&= \left((f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A)) \right. \\
&\quad \left. - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A)) \right) \\
&= (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^1 (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A)) - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^1 \\
&\quad \times (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)) = 0, \tag{67}
\end{aligned}$$

where in the second line we have used the assumption stated above and in the third line, we have used the defining property of $f^{1,1}$ functions,

$$f_{v,2}^{1,1}[N](\bar{V}(c')) = f^1(\bar{V}(c')) \tag{68}$$

if $v \notin \bar{V}(c')$.

The only case where LHS is nontrivial is given by case 2b: $v_A \in \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$. [Recall that $v_A \in V(\phi \cdot \tilde{c}_A)$ by definition.]

In this case, we can start off with the second line in (67) and evaluate the LHS,

$$\begin{aligned}
\text{LHS} &= \left((f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A)) \right. \\
&\quad \left. - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (\bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A)) \right) \\
&= (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] (v_A, \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A) \\
&\quad - \{v_A\}) - (f \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}})^{1,1}[N] \\
&\quad \times (v_A, \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot v_A) - \{v_A\}). \tag{69}
\end{aligned}$$

Here without loss of generality we have assumed that the first argument of $f^1 \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}}$ is v_A .

Notice that as $\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \in \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1$, the Lemma (A.1) in the Appendix tells us that $\tilde{c}_A = \tilde{c}$. We can now use (7) in the above equation along with the fact that $\tilde{c}_A = \tilde{c}$ and get

$$\begin{aligned}
\text{LHS} &= N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) \lambda(\bar{n}_{v_A}^{\phi \cdot \tilde{c}}) [V_2^a(v_A, \tilde{c}) - V_3^a(v_A, \tilde{c})] \\
&\quad \times \frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} (\phi_{\phi \cdot \tilde{c}, \tilde{c}}^* f)(v_A, \{., \dots\}). \tag{70}
\end{aligned}$$

Recall that the components of quantum-shift $V_i^a(v_A, \tilde{c})$ are evaluated in the coordinate chart $\{x_{v_A}\}^{\phi \cdot \tilde{c}}$ which is centered at v_A and is obtained by the pushforward of $\{x_{v_A}\}^{\tilde{c}^0}$ centered at a vertex $v_A^0 \in V(\tilde{c}^0)$. Thus components of $\bar{V}(v_A, \tilde{c})$ in $\{x_{v_A}\}^{\tilde{c}}$ are equal to the components of $\bar{V}(v_A^0, \tilde{c}^0)$ in the coordinate system $\{x_{v_A^0}\}^{\tilde{c}^0}$. Using this, the above equation simplifies to

$$\begin{aligned}
\text{LHS} &= N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) \lambda(\bar{n}_{v_A^0}^{\tilde{c}^0}) [V_2^{a'}(v_A^0, \tilde{c}^0) - V_3^{a'}(v_A^0, \tilde{c}^0)] \\
&\quad \times \frac{\partial}{\partial (x_{v_A^0}^{\tilde{c}^0})^{a'}} (\phi_{c^0, \phi \cdot \tilde{c}}^* \circ \phi_{\phi \cdot \tilde{c}, \tilde{c}}^* f)(v_A^0, \{., \dots\}), \\
\text{LHS} &= N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) \lambda(\bar{n}_{v_A^0}^{\tilde{c}^0}) [V_2^{a'}(v_A^0, \tilde{c}^0) - V_3^{a'}(v_A^0, \tilde{c}^0)] \\
&\quad \times \frac{\partial}{\partial (x_{v_A^0}^{\tilde{c}^0})^{a'}} (\phi_{c^0, \tilde{c}}^* f)(v_A^0, \{., \dots\}). \tag{71}
\end{aligned}$$

In the above equations we have also explicitly displayed the dependence of a density-weighted lapse on a coordinate system.

We now evaluate the RHS in (64)

$$\text{RHS} = \left(\hat{H}^1[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N] \Psi_{[\tilde{c}]_1}^{f^1} \right) |c_A\rangle. \tag{72}$$

As the only vertex in $V(\phi \cdot \tilde{c}_A)$ which is inside the support of N is v_A , it implies that the only vertex in $V(\tilde{c}_A)$ which falls inside the support of $(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N$ is $\phi^{-1} \cdot v_A$.

As before we analyze two cases (Case 1 and Case 2) separately.

Case 1 Recall that case 1 corresponds to $\tilde{c}_A \neq \tilde{c}$ in which case it is easy to see that

$$\begin{aligned} \text{RHS} &= \sum_{v \in V(\tilde{c})} \left(\Psi_{[\tilde{c}]_1}^{f_v^{1,2}[\phi^*N]} - \Psi_{[\tilde{c}]_1}^{f_v^{1,3}[\Phi^*N]} \right) |c_A\rangle = 0 \\ &= \text{LHS}. \end{aligned} \quad (73)$$

Case 2 This is the complementary case where $\tilde{c}_A = \tilde{c}$.

While analyzing the LHS in case 2, we specialized to the situation where the only vertex in $V(\phi \cdot \tilde{c}) = V(\phi \cdot \tilde{c}_A)$ which is inside the support of N is v_A . Clearly this implies that the only vertex in $V(\tilde{c}_A) = V(\tilde{c})$ which lies in the support of ϕ^*N is $\phi^{-1} \cdot v_A$. In this case, the RHS is given by

$$\text{RHS} = (f_{\phi^{-1} \cdot v_A}^{1,2}(\bar{V}(c_A)) - f_{\phi^{-1} \cdot v_A}^{1,3}(\bar{V}(c_A))). \quad (74)$$

As in the case of the evaluation of the LHS, this case can be further analyzed by looking at to subcases, case-2a and case-2b, separately.

Case 2a: $v_A \notin \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$.

As \bar{V} is a diffeomorphism equivariant map, we have

$$\phi_{\phi \cdot \tilde{c}_A, \tilde{c}_A} \cdot v_A \notin \bar{V}(c_A) \Rightarrow \phi^{-1} \cdot v_A \notin \bar{V}(c_A).$$

The second line in the above equation needs and explanation.

As $v_A \in V(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$ but $v_A \notin \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$, it is clear that $v_A \in V(\phi \cdot \tilde{c}_A)$. But on $V(\phi \cdot \tilde{c}_A)$, $\phi_{\phi \cdot \tilde{c}_A, \tilde{c}_A} = \phi^{-1}$, which is used in the second line of the above equation. However, if $\phi^{-1} \cdot v_A \notin \bar{V}(c_A)$ we have

$$\begin{aligned} \text{RHS} &= (f_{\phi^{-1} \cdot v_A}^{1,2}(\bar{V}(c_A)) - f_{\phi^{-1} \cdot v_A}^{1,3}(\bar{V}(c_A))) \\ &= (f(\bar{V}(c_A)) - f(\bar{V}(c_A))) = 0. \end{aligned} \quad (75)$$

Whence even in this case we get

$$\text{LHS} = \text{RHS}$$

we are finally left with the final and only nontrivial case, case 2b.

Case 2b: $v_A \in \bar{V}(\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A)$.

An argument similar to the one given above (75) leads us to conclude that $\phi^{-1} \cdot v_A \in \bar{V}(c_A)$. Whence in this case, the RHS is given by

$$\begin{aligned} \text{RHS} &= (f_{\phi^{-1} \cdot v_A}^{1,2}(\bar{V}(c_A)) - f_{\phi^{-1} \cdot v_A}^{1,3}(\bar{V}(c_A))) \\ &= (f_{\phi^{-1} \cdot v_A}^{1,2}(\phi^{-1} \cdot v_A, \bar{V}(c_A) - \{\phi^{-1} \cdot v_A\}) \\ &\quad - f_{\phi^{-1} \cdot v_A}^{1,3}(\phi^{-1} \cdot v_A, \bar{V}(c_A) - \{\phi^{-1} \cdot v_A\})), \\ \text{RHS} &= (\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(\phi^{-1} v_A, \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}_A}) \lambda(\vec{n}_{\phi^{-1} v_A}^{\tilde{c}_A}) \\ &\quad \times [V_2^a(\phi^{-1} v_A, \tilde{c}) - V_3^a(\phi^{-1} v_A, \tilde{c})] \\ &\quad \times \left(\frac{\partial}{\partial (x_{\phi^{-1} \cdot v_A}^{\tilde{c}_A})^a} f \right) (\phi^{-1} v_A, \dots). \end{aligned} \quad (76)$$

Once again [in exact analogy with the way we arrived at (71)] we can use the following three observations to “pull back” the above equation to (v_A^0, \tilde{c}^0) :

(1) $\vec{V}_2(\phi^{-1} \cdot v_A, \tilde{c})$ is obtained by pushforward of $\vec{V}_2(v_A^0, \tilde{c}^0)$ using $(\phi_{\tilde{c}^0, \tilde{c}})^*$, it implies that the (ordered set of) components $V^{a''}(\phi^{-1} v_A, \tilde{c})$ in the preferred coordinate system $\{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}} := (\phi_{\tilde{c}^0, \tilde{c}})^* \{x_{v_A^0}\}^{\tilde{c}^0}$ centered at $\phi^{-1} \cdot v_A$ are the same as the components $V^a(v_A^0, \tilde{c}^0)$ of $\vec{V}(v_A^0, \tilde{c}^0)$ in the coordinate system $\{x_{v_A^0}\}^{\tilde{c}^0}$.

(2) We also have, by construction $\lambda(\vec{n}_{\phi^{-1} v_A}^{\tilde{c}_A}) = \lambda(\vec{n}_{v_A^0}^{\tilde{c}^0})$.

$$\begin{aligned} (3) \quad &\frac{\partial}{\partial (x_{\phi^{-1} \cdot v_A}^{\tilde{c}})^a} f(\phi^{-1} v_A, \dots) \\ &= \left((\phi_{\tilde{c}^0, \tilde{c}})^* \frac{\partial}{\partial (x_{v_A^0}^{\tilde{c}^0})^a} \right) f(\phi^{-1} v_A, \dots) \\ &= \frac{\partial}{\partial (x_{v_A^0}^{\tilde{c}^0})^a} (\phi_{\tilde{c}^0, \tilde{c}}^* f)(v_A^0, \dots) \quad \forall a. \end{aligned} \quad (77)$$

Whence

$$\begin{aligned} \text{RHS} &= (\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(\phi^{-1} v_A, \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}_A}) \lambda(\vec{n}_{v_A^0}^{\tilde{c}^0}) \\ &\quad \times [V_2^a(v_A^0, \tilde{c}^0) - V_3^a(v_A^0, \tilde{c}^0)] \left(\frac{\partial}{\partial (x_{v_A^0}^{\tilde{c}^0})^a} \phi_{\tilde{c}^0, \tilde{c}}^* f \right) \\ &\quad \times (v_A^0, \dots). \end{aligned} \quad (78)$$

We can now compare the above equation with (71) and see that the only possible source of mismatch arises from the evaluation of lapse. The dependence of lapse in (71) and (78) are respectively given by

$$\begin{aligned} &N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) \\ &= N(v_A, (\phi_{\tilde{c}^0, \phi \cdot \tilde{c}})^* \{x_{v_A^0}\}^{\tilde{c}^0}), \\ &(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(\phi^{-1} v_A, \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}_A}) \\ &= N(v_A, (\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}_A}) \\ &= N(v_A, \phi_{\tilde{c}, \phi \cdot \tilde{c}}^* \{x_{v_A^0}\}^{\tilde{c}^0}) \\ &= N(v_A, (\phi_{\tilde{c}^0, \phi \cdot \tilde{c}})^* \{x_{v_A^0}\}^{\tilde{c}^0}). \end{aligned}$$

Thus even for Case 2b we see that the LHS equals the RHS. Whence we conclude that

$$(\hat{U}(\phi)\hat{H}^1[N]\hat{U}(\phi^{-1}))\Psi_{[\tilde{c}_1]}^{f_1} = \hat{H}^1[(\phi_{\tilde{c},\phi\cdot\tilde{c}})^*N]\Psi_{[\tilde{c}_1]}^{f_1}. \quad (79)$$

B. $\hat{H}^2[N]\Psi_{[\tilde{c}_1]}^{f_1}$

In this section we would like to show that

$$\begin{aligned} & (\hat{U}(\phi)\hat{H}^2[N]\hat{U}(\phi^{-1})\Psi_{[\tilde{c}_1]}^{f_1})(|c_A\rangle) \\ &= (\hat{H}^2[(\phi_{\tilde{c},\phi\cdot\tilde{c}})^*N]\Psi_{[\tilde{c}_1]}^{f_1})(|c_A\rangle) \end{aligned} \quad (80)$$

$\forall |c_A\rangle \in \mathcal{H}_{\text{kin}}, \forall \phi \in \text{Diff}(\Sigma)$ and $\forall N$.

Once again without loss of generality we assume that the only WEO vertices which belong to $V(c_A)$ are of "type 1" [i.e. all the edges incident on any WEO vertex is only charged under $U(1)_1$], as otherwise both sides are trivially zero.

We first compute the LHS using (51), (54), and (8)

$$\begin{aligned} \text{LHS} &= (\hat{U}(\phi)\hat{H}^2[N]\hat{U}(\phi^{-1})\Psi_{[\tilde{c}_1]}^{f_1})|c_A\rangle \\ &= (\hat{H}^2[N]\hat{U}(\phi^{-1})\Psi_{[\tilde{c}_1]}^{f_1})|\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A\rangle \\ &= (\hat{H}^2[N]\Psi_{\phi_{\tilde{c},\phi\cdot\tilde{c}}^*[\tilde{c}_1]}^{f_1 \circ \phi_{\phi\cdot\tilde{c},\tilde{c}}})|\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A\rangle \\ &= \sum_{v \in V(\tilde{c})} \left(\Psi_{\phi_{\tilde{c},\phi\cdot\tilde{c}}^*[\tilde{c}_1]}^{(f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v,1}}[N] - \Psi_{\phi_{\tilde{c},\phi\cdot\tilde{c}}^*[\tilde{c}_1]}^{(f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v,3}}[N] \right) \\ &\quad \times |\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A\rangle. \end{aligned} \quad (81)$$

Further analysis of the above equation can be divided into the following two (mutually exclusive and exhaustive) cases exactly as in the previous section.

$$\begin{aligned} \text{Case 1: } \phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A &\notin \phi_{\tilde{c},\phi\cdot\tilde{c}} \cdot [\tilde{c}_1] \Leftrightarrow \phi \cdot \tilde{c}_A \neq \phi \cdot \tilde{c} \\ &\Leftrightarrow \tilde{c}_A \neq \tilde{c}. \end{aligned} \quad (82)$$

In this case it is clear that LHS = 0.

$$\begin{aligned} \text{Case 2: } \phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A &\in \phi_{\tilde{c},\phi\cdot\tilde{c}} \cdot [\tilde{c}_1] \Leftrightarrow \phi \cdot \tilde{c}_A = \phi \cdot \tilde{c} \\ &\Leftrightarrow \tilde{c}_A = \tilde{c}. \end{aligned} \quad (83)$$

In this case,

$$\begin{aligned} \text{LHS} &= \sum_{v \in V(\tilde{c})} ((f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v,2}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \\ &\quad - (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v,3}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A}))). \end{aligned} \quad (84)$$

For the sake of pedagogy, and without any loss in generality, we again assume (this assumption was also made in the previous section) that the only vertex in $V(\phi \cdot \tilde{c}) = V(\phi \cdot \tilde{c}_A)$ which falls inside the support of N is a vertex v_A .

In this case Case 2 gets further subdivided into the following two complementary cases:

Case 2a: $\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A$ does not contain an EO vertex $(v_A)_\delta^E$ of type (1, 2) in the neighborhood of v_A .

In this case the vertex functions are unchanged, and

$$\begin{aligned} \text{LHS} &= (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v_A,2}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \\ &\quad - (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v_A,3}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \\ &= (f(\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) - f(\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A}))) \\ &= 0. \end{aligned} \quad (85)$$

Case 2b: $\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A} \cdot c_A$ contains an EO vertex $(v_A)_\delta^E$ of type (1, 2) in the neighborhood of v_A for some δ .

In this case we can use (9a) and (9b) to get

$$\begin{aligned} \text{LHS} &= (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v_A,2}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) - (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})^{1,2}_{v_A,3}[N](\bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \\ &= \left[\lambda(\tilde{n}_{\tilde{c}}^{v_A})V_1^a(v_A, \phi \cdot \tilde{c}) \frac{\partial}{\partial(x_{v_A}^{\phi\cdot\tilde{c}})^a} N(v_A, \{x_{v_A}\}^{\phi\cdot\tilde{c}}) (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})((v_A)_\delta^E(1, 3), \bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \right. \\ &\quad \left. - \{(v_A)_\delta^E(1, 2)\} - \lambda(\tilde{n}_{\tilde{c}}^{v_A})V_3^a(v_A, \phi \cdot \tilde{c}) \nabla_a N(v_A, \{x_{v_A}\}^{\phi\cdot\tilde{c}}) (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})((v_A)_\delta^E(1, 1), \bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) - \{(v_A)_\delta^E(1, 2)\} \right], \end{aligned} \quad (86)$$

$$\begin{aligned} \text{LHS} &= \left[\lambda(\tilde{n}_{\tilde{c}}^{v_A})V_1^a(v_A, \phi \cdot \tilde{c}) \frac{\partial}{\partial(x_{v_A}^{\phi\cdot\tilde{c}})^a} N(v_A, \{x_{v_A}\}^{\phi\cdot\tilde{c}}) (f \circ \phi_{\phi\cdot\tilde{c},\tilde{c}})((v_A)_\delta^E(1, 3), \bar{V}(\phi_{\tilde{c}_A,\phi\cdot\tilde{c}_A})) \right. \\ &\quad \left. - \{(v_A)_\delta^E(1, 2)\} - \lambda(\tilde{n}_{\tilde{c}}^{v_A})V_3^a(v_A, \phi \cdot \tilde{c}) \nabla_a N(v_A, \{x_{v_A}\}^{\phi\cdot\tilde{c}}) f((\phi^{-1} \cdot v_A)_\delta^E(1, 1), \bar{V}(c_A)) - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\} \right], \end{aligned} \quad (87)$$

where in the last line we have used the key property of our new representation. If $(v_A)_\delta^E(1, 2)$ is an EO vertex in $V(\phi_{\tilde{c},\phi\cdot\tilde{c}} \cdot c_A)$ which is associated to v_A [which is in turn a vertex in $V(\phi \cdot \tilde{c}_A)$] then,

$$\phi_{\phi\cdot\tilde{c},\tilde{c}} \cdot (v_A)_\delta^E(1, 2) = (\phi^{-1} \cdot v_A)_\delta^E(1, 2), \quad (88)$$

that is, $\phi_{\phi\cdot\tilde{c},\tilde{c}}$ maps it to an EO vertex in $V(\phi_{\phi\cdot\tilde{c},\tilde{c}})$ which is associated to $\phi^{-1} \cdot v_A \in V(\tilde{c})$.

We now analyze the RHS and show that in all three cases (case 1, case 2a, case 2b) it matches the LHS answers given above.

$$\text{RHS} = (\hat{H}^2[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N] \Psi_{[\tilde{c}_1]}^{f^1})|_{c_A}. \quad (89)$$

It is clear that in the first case, case 1, as $\tilde{c} \neq \tilde{c}_A$ clearly

$$\text{RHS} = 0.$$

Now consider case 2a.

$$\text{Case 2a: } \phi_{\tilde{c}_A, \phi \tilde{c}_A} \dot{c}_A \in \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}_1] \Leftrightarrow \tilde{c} = \tilde{c}_A. \quad (90)$$

But $\phi_{\tilde{c}_A, \phi \tilde{c}_A} \cdot c_A$ does not contain an EO vertex $(v_A)_\delta^E$ of type (1, 2) in the neighborhood of v_A . Where v_A is the only vertex of $\phi \cdot \tilde{c}$ which lies inside the support of N . Obviously this implies that the only vertex of \tilde{c} which lies inside the support of $\phi_{\tilde{c}, \phi \cdot \tilde{c}}^* N$ is $\phi^{-1} \cdot v_A$.

Now notice that as as the EO structure associated to any charge network c is preserved under the $\phi_{\tilde{c}, \phi \cdot \tilde{c}}$ for any

diffeomorphism ϕ [as demonstrated in Eq. (47)], c_A does not contain an EO vertex of type (1, 2) in the neighborhood of $\phi^{-1} \cdot v_A$, whence in this case

$$\begin{aligned} \text{RHS} &= f_{\phi^{-1} \cdot v_A, 2}^{1,2} [(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N](\bar{V}(c_A)) \\ &\quad - f_{\phi^{-1} \cdot v_A, 3}^{1,2} [(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N](\bar{V}(c_A)) = 0. \end{aligned} \quad (91)$$

Recall that even the LHS was trivial in this case.

We now turn to the remaining case, case 2b, for which the LHS was nontrivial. For the benefit of the reader, we recall the conditions defining this case again.

Case 2b: $\phi_{\tilde{c}_A, \phi \tilde{c}_A} \cdot c_A$ contains an EO vertex $(v_A)_\delta^E$ of type (1, 2) in the neighborhood of v_A for some δ .

Once again, using Eq. (47) we see that c_A contains an EO vertex $(\phi^{-1} \cdot v_A)_\delta^E$ of type (1, 2) in the neighborhood of $\phi^{-1} \cdot v_A$ for the same δ . Hence in this case, the RHS is given by

$$\begin{aligned} \text{RHS} &= f_{\phi^{-1} \cdot v_A, 2}^{1,2} [(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N](\bar{V}(c_A)) - f_{\phi^{-1} \cdot v_A, 3}^{1,2} [(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N](\bar{V}(c_A)) \\ &= \left[\lambda(\tilde{n}_{\tilde{c}}^{\phi^{-1} \cdot v_A}) V_1^a(\phi^{-1} \cdot v_A, \tilde{c}) \left(\frac{\partial}{\partial (x_{\phi^{-1} \cdot v_A}^{\tilde{c}})^a} \right) (\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(\phi^{-1} \cdot v_A, \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}}) f^1((\phi^{-1} \cdot v_A)_\delta^E(1, 3), \bar{V}(c_A)) \right. \\ &\quad \left. - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\} - \lambda(\tilde{n}_{\tilde{c}}^{\phi^{-1} \cdot v_A}) V_3^a(\phi^{-1} \cdot v_A, \tilde{c}) \left(\frac{\partial}{\partial (x_{\phi^{-1} \cdot v_A}^{\tilde{c}})^a} \right) (\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(\phi^{-1} \cdot v_A, \{x_{\phi^{-1} \cdot v_A}\}^{\tilde{c}}) f^1((\phi^{-1} \cdot v_A)_\delta^E \right. \\ &\quad \left. \times (1, 1), \bar{V}(c_A) - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\} \right]. \end{aligned} \quad (92)$$

We can use

$$\lambda(\tilde{n}_{\tilde{c}}^{\phi^{-1} \cdot v_A}) = \lambda(\tilde{n}_{\phi \cdot \tilde{c}}^{v_A}), \quad V_i^a(\phi^{-1} \cdot v_A, \tilde{c}) = V_i^a(v_A, \phi \cdot \tilde{c}) \quad \forall a, i, \quad \frac{\partial}{\partial (x_{\phi^{-1} \cdot v_A}^{\tilde{c}})^a} = (\phi_{\phi \cdot \tilde{c}, \tilde{c}})^* \frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} \quad (93)$$

to simplify (92)

$$\begin{aligned} \text{RHS} &= \left[\lambda(\tilde{n}_{\phi \cdot \tilde{c}}^{v_A}) V_1^a(v_A, \phi \cdot \tilde{c}) \left(\frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} \right) (\phi_{\phi \cdot \tilde{c}, \tilde{c}}^* \phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) f^1((\phi^{-1} \cdot v_A)_\delta^E(1, 3), \bar{V}(c_A)) \right. \\ &\quad \left. - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\} - \lambda(\tilde{n}_{\phi \cdot \tilde{c}}^{v_A}) V_3^a(v_A, \phi \cdot \tilde{c}) \left(\frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} \right) (\phi_{\phi \cdot \tilde{c}, \tilde{c}}^* \phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}), \right. \\ &\quad \left. f^1((\phi^{-1} \cdot v_A)_\delta^E(1, 1), \bar{V}(c_A) - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\}) \right] \\ \text{RHS} &= \lambda(\tilde{n}_{\phi \cdot \tilde{c}}^{v_A}) \left[V_1^a(v_A, \phi \cdot \tilde{c}) \left(\frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} \right) N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) f^1((\phi^{-1} \cdot v_A)_\delta^E(1, 3), \bar{V}(c_A) - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\}) \right. \\ &\quad \left. - V_3^a(v_A, \phi \cdot \tilde{c}) \left(\frac{\partial}{\partial (x_{v_A}^{\phi \cdot \tilde{c}})^a} \right) N(v_A, \{x_{v_A}\}^{\phi \cdot \tilde{c}}) f^1((\phi^{-1} \cdot v_A)_\delta^E(1, 1), \bar{V}(c_A) - \{(\phi^{-1} \cdot v_A)_\delta^E(1, 2)\}) \right]. \end{aligned} \quad (94)$$

On comparing (94) with (87) we conclude that even in this case case 2b

$$\text{LHS} = \text{RHS.}$$

Whence, we finally have

$$\hat{U}(\phi)' \hat{H}^{(2)}[N]' \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}_1]}^{f^1} = \hat{H}^{(2)}[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N]' \Psi_{[\tilde{c}_1]}^{f^1} \quad (95)$$

$\forall \phi$.

One can similarly show that

$$\hat{U}(\phi)' \hat{H}^{(3)}[N]' \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_i}^{f^1} = \hat{H}^{(3)}[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N]' \Psi_{[\tilde{c}]_i}^{f^1} \quad (96)$$

$\forall \phi$.

Using (79), (95), and (96) we see that

$$\hat{U}(\phi)' \hat{H}[N]' \hat{U}(\phi^{-1})' \Psi_{[\tilde{c}]_i}^{f^1} = \hat{H}[(\phi_{\tilde{c}, \phi \cdot \tilde{c}})^* N]' \Psi_{[\tilde{c}]_i}^{f^1} \quad (97)$$

$\forall \phi$.

It is straightforward to generalize this result to $\Psi_{[\tilde{c}]_i}^{f^i} \quad \forall i$.

The above result, in conjunction with (61) and the result of [7] shows that we have a representation of Dirac algebra on \mathcal{V}_{LMI} in the loop quantized $(2+1)$ -dimensional $U(1)^3$ theory.

VI. SPECTRUM OF THE THEORY

The new representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} which was a crucial ingredient in establishing the diffeomorphism covariance of $H[N]$ required us to introduce certain auxiliary structures. The two key ingredients were (1) reference charge networks for each gauge orbit $[\tilde{c}]_{\text{diff}}$ and (2) reference diffeomorphisms.

In this section we analyze the spectrum of the theory in order to probe the viability of new representation for $\text{Diff}(\Sigma)$. Let us try to find the simplest class of states in \mathcal{V}_{LMI} which are solutions to $\hat{H}[N]$. Consider a class of states of the form

$$|\Phi\rangle = \sum_{I=1}^N \sum_{m=1}^3 [a_I^{(m)} \Psi_{[\tilde{c}]_m}^{(f_I)^m}], \quad (98)$$

where $1 \leq N < \infty$. This is a fairly large class of states in which we look for states which satisfy

$$\sum_{i=1}^3 \hat{H}^i[N]|\Phi\rangle = 0 \quad (99)$$

$\forall N$. The resulting equation can be written in a condensed form as

$$\sum_{i=1}^3 \hat{H}^i[N]|\Phi\rangle = \sum_{I=1}^N \sum_{v \in V(\tilde{c}_I)} \sum_{i,j,k=1}^3 a_I^i \left[\epsilon_{ijk} \Psi_{[\tilde{c}]_i}^{(f_I)^{ij}} + \epsilon_{ijk} (\Psi_{[\tilde{c}]_i}^{(f_I)^{ij}} - \Psi_{[\tilde{c}]_i}^{(f_I)^{ij}}) \right]. \quad (100)$$

From here it is easy to see that, in the class of states given in (98) there is a subset obtained by choosing $f_I = \text{constant}$ and $a_I^1 = a_I^2 = a_I^3 \quad \forall I$ which lie in the kernel of the Hamiltonian constraint. This result is not completely expected *a priori* as action of $H^j[N]$ on $\Psi_{[\tilde{c}]_k}^{f^k}$ is not trivial even when the vertex function f^k is taken to be constant when $j \neq k$. The antisymmetry in the internal

indices in the Hamiltonian constraint (which is rather neatly encoded in this expression) is responsible for the fact that $\sum_{i=1}^3 \Psi_{[\tilde{c}]_i}^{f^i = \text{const}}$ lie in the kernel of the Hamiltonian constraint.

As the set $[\tilde{c}]_i$ is diffeomorphism (intact homeomorphism) invariant, we can see that (formally) identifying all the Habitat states which are related by diffeomorphisms will yield distributions on \mathcal{H}_{kin} of the type

$$\begin{aligned} |[\Phi]\rangle &= \sum_{i=1}^3 \sum_{[\tilde{c}']_i | \tilde{c}' = \phi \cdot \tilde{c}} \Psi_{[\tilde{c}']_i}^{f^i = \text{const}} = \sum_{i=1}^3 \sum_{\tilde{c}' = \phi \cdot \tilde{c}} \Psi_{[\tilde{c}']_i}^{f^i = \text{const}} \\ &= \sum_{i=1}^3 \sum_{\tilde{c}' = \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot \tilde{c}} \Psi_{[\tilde{c}']_i}^{f^i = \text{const}}. \end{aligned} \quad (101)$$

Whence the sum over diffeomorphisms reduces to summing over the ϕ maps as each $[\tilde{c}]_i$ has a unique vertex-free state \tilde{c} associated to it. We see this as a hint that as far as the spectrum of the theory is concerned, summing over all diffeomorphisms might be equivalent to summing over the selected set of diffeomorphisms as dictated by the new representation.

VII. CONCLUSION AND OUTLOOK

In this paper, we continued our construction of a representation of Dirac algebra in quantum $U(1)^3$ gauge theory which was initiated in [7]. We considered the Hamiltonian constraint $\hat{H}[N]$ defined in [7] and constructed a representation of the diffeomorphism group on the LMI habitat \mathcal{V}_{LMI} such that $\hat{U}(\phi)$, $\hat{H}[N]$ satisfy the off-shell closure condition. In contrast to the original Hamiltonian constraint of Thiemann (constructed remarkably for the case of four-dimensional LQG) where diffeomorphism covariance followed as a result of (i) a diffeomorphism-covariant choice of (state-dependent) triangulation, and (ii) by assigning state-dependent neighborhoods to each vertex of the spin network such that this assignment was diffeomorphism invariant, we had to introduce several new ingredients not the least of which was a new representation of the diffeomorphism group. From the point of view of \mathcal{H}_{kin} this representation is completely *ad hoc* (and not even unitary); however, this is no longer relevant to us as in our scheme, the ‘‘kinematical’’ arena (the space on which quantum constraints are defined) is played out by \mathcal{V}_{LMI} . It is here where this representation is unitary with respect to the canonical inner product. In addition to the new representation for $\text{Diff}(\Sigma)$ we also introduced a notion of state-dependent atlas on Σ . Roughly speaking the idea is to fix an atlas $\mathcal{U}(\Sigma, \tilde{c}_0)$ for each reference charge net \tilde{c}_0 (one reference charge network associated to each diffeomorphism-invariant orbit $[\tilde{c}]_{\text{diff}}$ of WEO vertex-free charge networks) and then for any $\tilde{c}' \in [\tilde{c}]_{\text{diff}}$ we defined an atlas $\mathcal{U}(\Sigma, \tilde{c}')$ associated to \tilde{c}' by pushing forward $\mathcal{U}(\Sigma, \tilde{c}_0)$ using $\phi_{\tilde{c}_0, \tilde{c}'}$. The new representation together

with the state dependence of coordinate charts ensured that extraordinariness of a vertex is a diffeomorphism-invariant notion. This was crucial in establishing diffeomorphism covariance of the Hamiltonian constraint.

The use of new representation of the diffeomorphism group may seem worrisome as the canonical representation used so far in LQG has been analyzed in great detail and whose solution lead to generalized knot classes. In order to analyze the validity of the new representation we considered solving the Hamiltonian constraint in \mathcal{V}_{LMI} and ask if the states obtained by ‘‘formally’’ averaging over all diffeomorphisms would agree with states obtained by averaging over the preferred set. As we saw, for the $(2 + 1)$ -dimensional theory, these results do in fact match for a subspace of kernel that we computed in Sec. VI. This merely represents a small check on the validity of the new representation of $\text{Diff}(\Sigma)$ on \mathcal{V}_{LMI} . The issue however needs further investigation. As we have seen, the requirement that a Hamiltonian constraint be diffeomorphism covariant on \mathcal{V}_{LMI} is quite a stringent requirement and certainly reduces the vast amount of ambiguity which was present in quantization of $\hat{H}[N]$ presented in [7]. The main source of ambiguity in the definition of $\hat{H}[N]$ was in the determination of quantum shift vectors. As the definition of quantum shift is regularization dependent, in principle one can associate to each WEO vertex-free charge network \tilde{c} a different regularization scheme for computing the quantum shift. However, as we saw above, diffeomorphism covariance of $\hat{H}[N]$ requires determination of quantum shift *only* on reference charge nets in each diffeomorphism-invariant orbit. For any other charge net the quantum shift vectors are uniquely determined via pushforwards.

Perhaps the most unsatisfactory part of our construction is that our final definition of quantum constraints (or finite transformations generated by them) depends on various auxiliary structures. We list them below.

- (1) The choice of reference charge network \tilde{c}_0 in each orbit of WEO vertex-free charge nets.
 - (2) The choice of the subcategory in $[\tilde{c}]_{\text{diff}}$ or equivalently choice of set of diffeomorphisms $\phi_{\tilde{c}_0, \tilde{c}'}$ associated to each diffeomorphism-invariant orbit.
 - (3) The choice of coordinate atlas $\mathcal{U}(\Sigma, \tilde{c}_0)$ for each \tilde{c}_0 .
- These structures can also be thought of as the data parametrizing quantization ambiguities which are input in the definition of quantum constraints.

A key open question is if the use of this auxiliary structures is viable. The final answer to this question can only be obtained by looking at expectation value of observables in the physical Hilbert space which should not depend on any *ad hoc* intermediate structures.

We believe that the work we have done here admits a possibility of generalization to Euclidean quantum gravity. An extremely important aspect to keep in mind here is that the geometric action of Hamiltonian constraint in a $SU(2)$

case can also be understood in terms of phase-space-dependent diffeomorphism on the dynamical fields [15]. In light of this result one could seek a quantization of Hamiltonian constraint in $SU(2)$ theory with the key lesson being provided by Eq. (100). The structure of internal indices shows a tempting possibility of how the extension to $SU(2)$ may be possible.

In any event we believe that some of the lessons we have learnt here as well as in [7] together with the lessons learned in [8,16] will have implications in defining quantum dynamics in canonical loop quantum gravity.

ACKNOWLEDGMENTS

We are indebted to Madhavan Varadarajan for sharing his results in [16] with us prior to publication and also for explaining to us the use of state-dependent coordinate charts as a tool in constructing a finite triangulation Hamiltonian constraint. A.L. is also indebted to Abhay Ashtekar for sharing his insights regarding $SU(2)$ theory and his constant encouragement. We thank Miguel Campiglia for many discussions over the course of this work and for support. The work of A.L. is supported by a Ramanujan fellowship of the Department of Science and Technology. The work of C.T. is supported by NSF Grant No. PHY-0748336, a Penn State Mebus fellowship, and a DAAD research grant.

APPENDIX

In this appendix we prove a lemma which is used crucially in Sec. VA.

Lemma A.1: Let \tilde{c}_A, \tilde{c} be WEO vertex-free states and let $c_A \in [\tilde{c}_A]_i, c \in [\tilde{c}]_i$. If for any diffeomorphism ϕ

$$\phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A} \cdot c_A \in \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_i \quad (\text{A1})$$

then $\tilde{c}_A = \tilde{c}$.

Proof: Without loss of generality we assume that $i = 1$. We will also assume that \tilde{c} has only one WE vertex. That is, \exists a $v_0 \in V(\tilde{c})$ such that (v_0, v'_0) is the WEO pair in c with v'_0 being a WEO vertex of type 1.

We also recall some notations from Sec. II.

- (1) As $c_A \in [\tilde{c}_A]_1$

$$c_A = (c_{A1}, \tilde{c}_{A2}, \tilde{c}_{A3}). \quad (\text{A2})$$

- (2) Any $c' \in \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot [\tilde{c}]_1$ is of the form

$$c' = (\phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot c'_1, \phi \cdot \tilde{c}_2, \phi \cdot \tilde{c}_3). \quad (\text{A3})$$

Hence we have

$$\tilde{c}_{Ai} = \tilde{c}_i \quad \text{for } i = 2, 3. \quad (\text{A4})$$

Thus we have the following:

$$\begin{aligned} \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A}(c_{A1}, \tilde{c}_2, \tilde{c}_3) &= \phi_{\tilde{c}, \phi \cdot \tilde{c}}(c_1, \tilde{c}_2, \tilde{c}_3), \\ (\tilde{c}_{A1}, \tilde{c}_2, \tilde{c}_3) &\in [\tilde{c}]_{\text{diff}}, \quad \tilde{c}_{A1} \neq \tilde{c}_1. \end{aligned} \quad (\text{A5})$$

As $\gamma(\tilde{c}_{A1} \cup \tilde{c}_2 \cup \tilde{c}_3) \subset \gamma(c_{A1} \cup \tilde{c}_2 \cup \tilde{c}_3)$, \exists a $(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) | \gamma(\tilde{c}_1 \cup \tilde{c}_2 \cup \tilde{c}_3) | \subset \gamma(c_1 \cup \tilde{c}_2 \cup \tilde{c}_3)$ such that

$$\begin{aligned} \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A}(\tilde{c}_{A1}, c_{A2}, c_{A3}) &= \phi \cdot (\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \\ &= \phi_{\tilde{c}, \phi \cdot \tilde{c}}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3), \end{aligned} \quad (\text{A6})$$

where (as a trivial consequence of the above equation) we have

- (1) $(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ is gauge invariant.
- (2) $(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3)$ has no WEO vertex.

But from the above lemma we know that there is a unique charge network contained associated to $(c_1, \tilde{c}_2, \tilde{c}_3)$ which

satisfies the above two conditions, and that is \tilde{c} . Whence we have

$$\begin{aligned} \phi_{\tilde{c}_A, \phi \cdot \tilde{c}_A}(\tilde{c}_{A1}, \tilde{c}_2, \tilde{c}_3) &= \phi \cdot (\tilde{c}_{A1}, \tilde{c}_2, \tilde{c}_3) = \phi_{\tilde{c}, \phi \cdot \tilde{c}}(\tilde{c}_1, \tilde{c}_2, \tilde{c}_3) \\ &= \phi_{\tilde{c}, \phi \cdot \tilde{c}} \cdot \tilde{c} = \phi \cdot \tilde{c}. \end{aligned} \quad (\text{A7})$$

Hence

$$\tilde{c}_A = \tilde{c} \quad (\text{A8})$$

QED.

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