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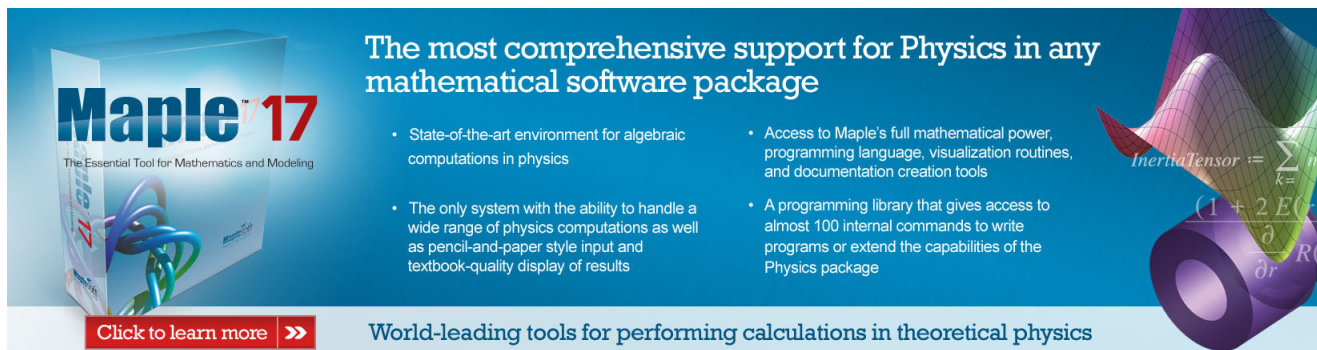
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InertiaTensor := $\sum_{k=0}^n$
 $(1 + 2 E(r))$
 $\frac{\partial}{\partial r} R(r)$

Anisotropic Einstein-aether cosmological models

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We investigate a class of spatially anisotropic cosmological models in Einstein-aether theory with a scalar field in which the self-interaction potential depends on the timelike aether vector field through the expansion and shear scalars. We derive the evolution equations in terms of expansion-normalized variables, which reduce to a dynamical system. We study the local stability of the equilibrium points of the dynamical system corresponding to physically realistic solutions, and find that there are always ranges of values of the parameters of the models for which there exists an inflationary attractor. © 2013 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4802246>]

I. INTRODUCTION

There are various models of early universe cosmology which incorporate a violation of Lorentz invariance, which include Einstein-aether theory^{1,2} and the IR limit of (extended) Horava gravity.³ When the phenomenology of theories with a preferred frame is studied, it is generally assumed that this frame coincides, at least roughly, with the cosmological rest frame defined by the Hubble expansion of the universe. It has recently been shown,⁴ based on a Bayesian analysis and assuming nothing other than the existence of a suitably averaged linear Hubble law, that the standard Cosmic Microwave Background (CMB) rest frame is not the one in which the Hubble flow is most uniform (questioning whether the standard treatment of the cosmic rest frame in which the variance in the Hubble flow can be reduced to a boost at a point is correct).

Einstein-aether theory consists of general relativity (GR) coupled, at second derivative order, to a dynamical timelike unit vector field, the aether. In Horava gravity, the aether vector is assumed to be hypersurface-orthogonal; hence every hypersurface-orthogonal Einstein-aether solution is a Horava solution (most of the solutions studied). In this effective field theory approach, the aether vector field u_a and the metric tensor g_{ab} together determine the local spacetime structure.

The inflationary paradigm⁵ provides one of the simplest ways to describe various aspects of the physics of the early universe in standard cosmology. However, despite its successes some of the fundamental questions still remain to be answered in this paradigm. We shall discuss the late time dynamics of Einstein aether cosmological models. In particular, we explore the impact of Lorentz violation on the inflationary scenario.^{1,2,6–8}

Cosmological models in aether (Lorentz-violating) theories of gravity are currently of interest. A systematic construction of an Einstein-aether gravity theory with a Lorentz violating dynamical field that preserves locality and covariance in the presence of an additional “aether” vector field has been presented (the field equations (FE) are given therein).^{1,9,10}

In an isotropic and spatially homogeneous Friedmann universe with expansion scale factor $a(t)$ and comoving proper time t , the aether field will be aligned with the cosmic frame and is related to the expansion rate of the universe. The Einstein FE are generalised by the contribution of an additional stress tensor for the aether field. If the universe contains a single self-interacting scalar field ϕ (e.g.,

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a scalar inflation which would dominate in any inflationary epoch), with a self-interaction potential V , V can now be a function of ϕ and the expansion rate and even the shear.¹

A. Exponential potentials

Exponential potentials $V_0 e^{-\lambda\phi}$ arise naturally in various higher dimensional frameworks, such as in Kaluza-Klein theories and supergravity.¹¹ Scalar fields with exponential potentials in GR do not yield exponential inflation as with (for example) the harmonic potential.⁵ However, they do lead to a power-law inflation provided the potential is not too steep. The limitation of steep potentials can be alleviated using multiple fields, which can cooperate via Hubble damping to yield assisted inflation.¹² In addition, for sufficiently flat potentials, exponential potentials yield a scaling solution which is a late time attractor.^{13,14}

The dynamical properties of the positive exponential potentials leading to inflation in the Friedmann-Robertson-Walker (FRW) model have been widely studied^{15–17} (also see Appendix A). The classical solution for the scale factor can be written as a power law, $a \propto t^n$, with $n = 2/\lambda^2$; in order to have an inflationary phase with the exponential potential, one requires the steepness parameter λ to satisfy $\lambda^2 < 2$. If the potential is steep, then it does not support inflation. The effective dynamics with an exponential potential have been more widely studied.^{13,14} The equilibrium points corresponding to sources consist of a subset of the (non-oscillatory) Jacobs anisotropic Bianchi I non-vacuum (massless scalar field) solutions.¹⁴

Negative exponential potentials also lead to a rich physics, such as in the Ekpyrotic scenario.¹⁸

II. ANISOTROPIC MODELS

We study a class of spatially homogeneous, anisotropic cosmological models in Einstein-aether theory. Carruthers and Jacobson⁶ and Kanno and Soda⁷ examined Bianchi type I anisotropic solutions in the presence of a positive cosmological constant, with three orthogonal principal directions of expansion, and with the aether tilted in one of the principal directions. They showed that, to linear order in the anisotropy, the system relaxes exponentially to the isotropic, de Sitter solution.

In an anisotropic Einstein-aether model there will be additional terms in the FE:

- The effects on the geometry from the anisotropy (and curvature) of the actual 1-parameter subclass of Bianchi VI_h spatially homogeneous models (with an isotropic curvature; see Eqs. (4)–(5)) considered below.
- The energy momentum tensor of the scalar field, due to the possible dependence of the self-interaction potential V on the Lorentz violating vector field (see Eqs. (14)–(15) below).⁹
- The Einstein FE are generalised by the contribution of an additional stress tensor, S_{ab} , for the aether field which depends on the dimensionless parameters of the aether model (e.g., “the c_i ”). This has the effect of renormalizing some of the parameters in the model (e.g., the gravitational constant G , where we choose units in which $8\pi G = 1$; effectively we set $c_1 + 3c_2 + c_3 = 0$ so that the remaining parameters in the model can be characterized by the constants c^2 and d – see below).⁷
- In anisotropic models, there may be a tilt between the preferred direction of the aether and that of the anisotropy (in an isotropic and spatially homogeneous Friedmann universe the aether field is aligned with the cosmic frame). This adds additional terms to the aether stress tensor S_{ab} , which can be characterized by a hyperbolic tilt angle, $\alpha(t)$, measuring the boost of the aether relative to the rest frame of the homogeneous spatial sections.^{6,7}

In this paper, we are interested in the qualitative features of cosmological models in Einstein aether theory (and in particular in the presence of curvature and shear).

A. The potential

If the universe contains a single self-interacting scalar field ϕ (e.g., a scalar inflation which would dominate in any inflationary epoch), the self-interaction potential V is a function of ϕ but

can now also be a function of the timelike vector field u_a (the aether field);¹ $V = V(\phi, \theta, \sigma)$, where $\theta = 3H = 3\dot{a}/a$ is the expansion rate and σ is the shear scalar, defined by $\sigma^2 \equiv \frac{1}{2}\sigma^{ab}\sigma_{ab}$. The modified stress tensor T_{ab}^ϕ can be written in terms of an effective fluid with density ρ_ϕ and pressure p_ϕ (see below).

In general, we consider a scale invariant potential of the form (cf. Ref. 9),

$$V(\theta, \phi, \sigma) = V_0 \exp[-\lambda\phi] + \sum_{r,s} a_{r,s} \theta^r \sigma^s \exp[(r+s-2)\lambda\phi/2], \quad (1)$$

where V_0, λ ($k \equiv -\lambda$ in Ref. 19 and Appendix A) and $\{a_{r,s}\}$ are constants. The constant V_0 is assumed to be positive. Negative constants $a_{r,s}$ are permitted; however, it might be demanded that the potential $V(\theta, \phi, \sigma)$ is positive definite.

In particular, we shall study the particular potential

$$V(\theta, \phi) = V_0 e^{-\lambda\phi} + a\sqrt{V_0}\theta e^{-\frac{1}{2}\lambda\phi} + b\sqrt{V_0}\sigma e^{-\frac{1}{2}\lambda\phi}, \quad (2)$$

where, for convenience, we have renormalized the constants, dropped the indices on the constants and renamed the second constant b . We could easily consider additional terms with $r = 2, s = 0$ and $r = 0, s = 2$, but these terms are expected to simply renormalize the arbitrary parameters in the resulting model.

We could also study a potential with a single generic term (with arbitrary values for r, s , and non-zero constant $a_{r,s}$).²⁰

B. The model

The evolution equations follow from the FE derived from the Einstein aether action.^{1,2} The energy-momentum conservation law or Klein-Gordon equation is

$$\ddot{\phi} + \theta\dot{\phi} + V_\phi = 0. \quad (3)$$

The generalized Friedmann equation is

$$\theta^2 = 3c^2\sigma^2 + 9d(\dot{\alpha} + \frac{1}{3}\alpha\theta)^2 + 3\rho_\phi - \frac{3}{2}P, \quad (4)$$

where P is the scalar curvature of the homogeneous hypersurfaces, which is always negative in the models discussed here (where we also neglect normal matter). The Raychaudhuri equation governing the evolution of the expansion is given by

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2c^2\sigma^2 + d(\alpha^2(\dot{\theta} + \frac{1}{3}\theta^2) - 3\dot{\alpha}^2 + \alpha\dot{\alpha}\theta + 3\alpha\ddot{\alpha}) - \frac{1}{2}(\rho_\phi + 3p_\phi). \quad (5)$$

Following Ref. 21 (also see Appendix A), we consider a 1-parameter ($m \equiv h - 1$) class of anisotropic cosmological models with metric (A10), which includes Bianchi types III ($m = 0$), V ($m = 1$), VI₀ ($m = -1$), and VI_{*h*} (all other m). The expansion scalar, which determines the volume behavior of the fluid, is given by (A11) and the shear scalar is given by (A13). In the case under consideration here, there is no rotation and no acceleration; i.e.,

$$u_{a;b} = \frac{1}{3}\theta h_{ab} + \sigma_{ab},$$

where $h_{ab} \equiv g_{ab} + u_a u_b$.

Thus, in our model, in the generalized Friedmann equation $-\frac{3}{2}P = \frac{3}{a^2}N$, where $N \equiv m^2 + m + 1 \geq 3/4 > 0$. Using the generalized Friedmann equation, the evolution equation for the shear is

$$c^2\dot{\sigma} = -c^2\sigma\theta + \frac{M}{3\sqrt{3}}(\theta^2 - 3c^2\sigma^2 - 9d(\dot{\alpha} + \frac{1}{3}\alpha\theta)^2 - 3\rho_\phi), \quad (6)$$

where $M \equiv \frac{1-m}{\sqrt{m^2+m+1}}$. The constants M and N are not independent but are related (via m); formally, we can recover the Bianchi type I case by taking $M = N = 0$.

Finally, the evolution equation for the tilt is

$$\ddot{\alpha} + \theta\dot{\alpha} + \frac{1}{9}\alpha(3\dot{\theta} + 2\theta^2) = 0. \quad (7)$$

1. The future evolution of tilt

The evolution equation (the “master” equation in Ref. 7) for the tilt (in the small anisotropy and small tilt limit; i.e., using the Raychaudhuri equation and keeping α , σ to first order only) is given by

$$\ddot{\alpha} + \theta\dot{\alpha} - \frac{1}{9}\alpha[\theta^2 + \frac{27}{2}(\rho_\phi + 3p_\phi)] = 0. \quad (8)$$

From this equation it was shown that the tilt decays ($\alpha \rightarrow 0$) to the future (in the Bianchi type I models) as is expected.⁷ (Note that the possibility that the dimensionless parameters can depend on the cosmological time via the scalar field (i.e., $c_i = c_i(\phi)$) was also considered by Ref. 7, and it was found that inflationary solutions can be obtained even in the absence of a scalar field potential, as in scalar-tensor theories of gravity.) The dynamics of a tilted aether in a Bianchi I cosmological model without the assumption of a small tilt was studied in Ref. 6, and it was found that when the initial hyperbolic tilt angle α (and its time derivative) is sufficiently small, then $\alpha \rightarrow 0$ at late times (consistent with the linearized stability analysis in Ref. 7), but in general (for larger tilt angles) there can be runaway (or singular) behaviour in the anisotropy. It is known that the tilt decays to the future in general *tilted* perfect fluid Bianchi cosmological models (except in the degenerate case in which the tilt becomes *extreme*; but this case must be treated separately here).²²

2. Past evolution

We are also interested in the past evolution of these models. Let us discuss this briefly. Since we expect the spatial curvature not to be dynamically important to the past, as a first step we can study the Bianchi type I models. We need to study the tilt carefully, since the tilt may become extreme to the past (and we may need to parameterize the tilt differently).

In the case of a stiff perfect fluid (or massless scalar field) in GR, the solutions are non-oscillatory to the past, and asymptote towards a one parameter family of sources that correspond to Jacobs anisotropic Bianchi type I non-vacuum solutions.¹³ We shall later find that analogues of the Jacobs anisotropic solutions are sources of the anisotropic Einstein aether models. Let us simply ask whether the tilt could change the qualitative picture. The normalized equations for the tilt angle α are

$$\alpha' = 3\Pi, \quad \Pi' = (q - 2)\Pi + (q - 1)\alpha, \quad (9)$$

where $\Pi \equiv \dot{\alpha}/\theta$. The equilibrium points with $q = 2$ (the analogues of the Jacobs solutions) are not compatible with α non-zero, which implies that there are no sources with non-zero tilt. A local stability analysis of these equilibrium points (with zero tilt) then shows that all of the non-zero eigenvalues (the equilibrium points are non-isolated) of these $q = 2$ Jacobs-like solutions are positive (3 and 1), and hence these equilibrium points are local sources.

3. The case of negligible tilt

Therefore, neglecting the tilt, and assuming the forms of the potential discussed earlier, we obtain the following system of equations, governing the phenomenological Einstein-aether cosmological models under consideration:

$$\theta^2 = 3c^2\sigma^2 + 3\rho_\phi - \frac{3}{2}P, \quad (10)$$

$$\dot{\theta} = -\frac{1}{3}\theta^2 - 2c^2\sigma^2 - \frac{1}{2}(\rho_\phi + 3p_\phi), \quad (11)$$

$$c^2 \dot{\sigma} = -c^2 \sigma \theta + \frac{M}{3\sqrt{3}} (\theta^2 - 3c^2 \sigma^2 - 3\rho_\phi), \quad (12)$$

$$\ddot{\phi} + \theta \dot{\phi} + V_\phi = 0. \quad (13)$$

C. The scalar field

Let us now consider the forms of ρ_ϕ and p_ϕ in the specific class of geometries under consideration here. The Einstein-aether action,¹ which is a generalization of the Einstein-Hilbert action, also depends on the aether field (the timelike vector field u_a). In particular, the matter fields can depend on the aether field. The Lagrangian for the matter (scalar) field is given by

$$L_\phi = \frac{1}{2} \nabla_a \phi \nabla^a \phi - V,$$

where the self-interaction potential V now can depend on scalar quantities constructed from (the covariant derivative of) u_a . Taking variations of $V = V(\phi, \theta, \sigma)$ with respect to the metric and the aether field u_a (see Eq. (13) in Ref. 1) we obtain (for the models under consideration) the effective density ρ_ϕ and pressure p_ϕ ($T_{ab}^\phi \sim \rho_\phi u_a u_b + p_\phi h_{ab}$) of the form (see Appendix B),

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V - V_\theta (\theta + \sqrt{6}\sigma) - V_\sigma (\sigma - \frac{1}{\sqrt{6}}\theta), \quad (14)$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V + V_\theta (\theta + \sqrt{6}\sigma + \sqrt{6} \frac{\dot{\sigma}}{\theta}) + V_\sigma (\sigma - \frac{1}{\sqrt{6}}\theta - \frac{1}{\sqrt{6}} \frac{\dot{\theta}}{\theta}) + \dot{V}_\theta (1 + \sqrt{6} \frac{\sigma}{\theta}) + \dot{V}_\sigma (\frac{\sigma}{\theta} - \frac{1}{\sqrt{6}}). \quad (15)$$

Note that in the Bianchi I case with $V_\sigma = 0$, we have that

$$\rho_\phi = \frac{1}{2} \dot{\phi}^2 + V - V_\theta (\theta + \sqrt{6}\sigma), \quad (16)$$

$$p_\phi = \frac{1}{2} \dot{\phi}^2 - V + \theta V_\theta + \dot{V}_\theta (1 + \sqrt{6} \frac{\sigma}{\theta}). \quad (17)$$

(In the absence of shear ($\sigma = 0$) we obtain Eq. (14) in Ref. 1).

D. Special cases

There are a number of special cases that might be of interest.

- $V = V(\phi, \theta)$, which has non-trivial corrections when the shear is not zero.
- The Bianchi type I subcase with $K = 0$.
- The case $c^2 = 1$

Regarding this last subcase, there are 4 dimensionless free parameters (constants) that define the Einstein aether theory ($c_i; i = 1..4$). Since we have normalized G ($c_1 + 3c_2 + c_3 = 0$), and since there is an invariance in the action in the cosmological application, there are effectively only 2 independent parameters,^{1,23} denoted here by c^2 and d . The arbitrary parameter $d \sim c_1 - c_4$ does not occur in the tilt-free equations above. The final parameter, $c^2 - 1 \sim (c_1 + c_3)$ satisfies $0 \leq c^2 \leq 1$, where $c^2 = 1$ is the corresponding GR value. In Ref. 23 it was shown that $(c_1 + c_3)$ is positive and bounded above; indeed, it is expected that $(c_1 + c_3)$ is very small (i.e., $\sim 10^{-3}$). Therefore, $c^2 \sim 1$ is consistent with Refs. 1 and 23. In addition, since it is not expected that there would be any significant different qualitative behaviour in the late time dynamics, we could assume self-consistently here that $c^2 \sim 1$ to simplify the analysis.

E. Dimensionless variables

We now introduce new expansion-normalized variables and a new time variable as follows:

$$\begin{aligned}\beta &= \sqrt{6} \frac{\sigma}{\theta}, & \frac{dt}{d\tau} &= \frac{3}{\theta}, \\ \Psi &= \frac{\sqrt{3} \dot{\phi}}{\sqrt{2} \theta}, & \Phi &= \sqrt{3V_0} \frac{e^{-\lambda\phi/2}}{\theta}.\end{aligned}\quad (18)$$

For a given potential $V(\theta, \phi, \sigma)$, we can find $3\rho_\phi/\theta^2$ and $3p_\phi/\theta^2$, in terms of β , Ψ , and Φ , and determine the system of differential (evolution) equations in terms of the expansion-normalized variables (see below). We define $K \equiv -\frac{3P}{2\theta^2}$, and since K is always positive the generalized Friedmann equation usually determines a compact region of interest (i.e., β , Ψ , and Φ are bounded). The *deceleration parameter*, q , is defined by the decoupled evolution equation for the expansion

$$\frac{d\theta}{d\tau} = -\theta(1 + q), \quad (19)$$

and inflation is defined by $q < 0$.

III. A CLASS OF ANISOTROPIC MODELS

We are looking for general scale invariant solutions in which

$$V(\theta, \phi, \sigma) = V_0 e^{-\sqrt{6}\bar{\lambda}\phi} + \frac{a}{\sqrt{3}} \sqrt{V_0} \theta e^{-\frac{\sqrt{6}}{2}\bar{\lambda}\phi} + \sqrt{2}b \sqrt{V_0} \sigma e^{-\frac{\sqrt{6}}{2}\bar{\lambda}\phi}, \quad (20)$$

where we have defined $\bar{\lambda} = \lambda/\sqrt{6}$, and we have normalized the constants a and b appropriately.

Let us now consider the class of anisotropic models under consideration (with $B_1 = 1$ and $B_2 = 1$ in Appendix B). The evolution equations are

$$\beta' = (q - 2)\beta - \frac{\sqrt{2}M}{c^2} K, \quad (21)$$

$$\Psi' = (q - 2)\Psi + \frac{3\bar{\lambda}}{2} (2\Phi^2 + a\Phi + b\beta\Phi), \quad (22)$$

$$\Phi' = (q + 1)\Phi - 3\bar{\lambda}\Psi\Phi, \quad (23)$$

where

$$K = \frac{c^2}{2} \beta^2 + \Omega - 1, \quad (24)$$

$$\Omega = \Psi^2 + \Phi^2 - a\Phi\beta + b\Phi, \quad (25)$$

$$q - 2 = \frac{(\frac{a\sqrt{2}M}{c^2}\Phi - 4)K + 3(a + b)\bar{\lambda}\Psi\Phi\beta + 3(a - b)\bar{\lambda}\Psi\Phi + 6\Phi^2 + 3(b - a\beta)\Phi}{b\Phi - 2}. \quad (26)$$

Note that $0 \leq \Omega \leq 1$ and $\frac{c^2}{2}\beta^2 \leq 1$. First we shall make some general comments about the full 3D system. In Sec. IV, we shall comprehensively study the 2D zero curvature case.

A. Equilibrium points with zero curvature ($K = 0$)

1. One parameter family, P_j :

$$\Phi = 0, \quad \Psi^2 + \frac{c^2}{2}\beta^2 = 1, \quad q_{P_j} = 2. \quad (27)$$

Eigenvalues:

$$\lambda_1 = 4 - \sqrt{2}M\beta, \quad \lambda_2 = 3(1 - \bar{\lambda}\Psi), \quad \lambda_3 = 0. \tag{28}$$

Since $-1 < M < 1$, we have that the smallest possible value for λ_1 occurs for $M \rightarrow \pm 1$ when $\Psi = 0$ and $\beta = \pm \frac{\sqrt{2}}{|c|}$. On the other hand, the smallest possible value for λ_2 is obtained for the largest possible value of Ψ . We have that if $|c| > \frac{1}{2}$ and $\bar{\lambda} < 1$, then all of the circle of equilibrium points defined above consists of sources for all allowed values of M . If $\bar{\lambda} > 1$ or $|c| < \frac{M}{2}$, a part of the circle consists of saddles (or even sinks if $\bar{\lambda}$ is large enough, M is non-zero and c is small enough). The vacuum solution $\Psi = 0, M = 0, \beta = \pm \frac{\sqrt{2}}{|c|}$ is always a source. These sources correspond to analogues of the (non-oscillatory) Jacobs anisotropic Bianchi I non-vacuum solutions.¹³

2. Isotropic equilibrium points p_2, p_3 :

$$\Phi = \frac{2(2\bar{\lambda}^2b - b - a\bar{\lambda}^2 \pm \sqrt{b^2 - 2a\bar{\lambda}^2b + a^2\bar{\lambda}^2 + 4 - 4\bar{\lambda}^2})}{4\bar{\lambda}^2b^2 - 4a\bar{\lambda}^2b + a^2\bar{\lambda}^2 + 4}, \quad \beta = 0. \tag{29}$$

For physical solutions we require Φ to be real and positive. To make the linearization (for small values of the parameters a and b) of the two isotropic equilibrium points p_2, p_3 easier we introduce new parameters (A, B) specifically adapted to the points p_2 and p_3 through the relations:

$$a = \frac{1 - \lambda^2 - A^2(1 + B^2)}{A} + B(\lambda^{-1} - 2\lambda), \quad b = \frac{1 - \lambda^2 - A^2(1 + B^2) - 2AB\lambda}{A}, \tag{30}$$

$$A = \frac{2(2\bar{\lambda}^2b - b - a\bar{\lambda}^2 \pm \sqrt{(b - a\bar{\lambda}^2)^2 + (1 - \bar{\lambda}^2)(a^2\bar{\lambda}^2 + 4)})}{\bar{\lambda}^2(2b - a)^2 + 4}, \quad B = \left(\frac{a}{2} - b\right)\bar{\lambda}. \tag{31}$$

The two signs are necessary to cover the whole range of parameter values. We then obtain a new parameterization of the two points, $(p_1, p_2) \rightarrow (\bar{p}_1, \bar{p}_2)$,

$$\bar{p}_2 : \Phi = A, \quad \Psi = AB + \bar{\lambda}, \quad \beta = 0, \quad q_{\bar{p}_2} = -(1 - 3AB\bar{\lambda} - 3\bar{\lambda}^2), \quad (A > 0), \tag{32}$$

$$\bar{p}_3 : \Phi = \frac{\bar{\lambda}^2 - 1}{A(B^2 + 1)}, \quad \Psi = B \frac{\bar{\lambda}^2 - 1}{A(B^2 + 1)} + \bar{\lambda}, \quad \beta = 0, \tag{33}$$

$$q_{\bar{p}_3} = -\frac{A(1 + B^2)(1 - 3\bar{\lambda}^2) + 3B\bar{\lambda}(1 - \bar{\lambda}^2)}{A(1 + B^2)}, \quad (A > 0, \bar{\lambda}^2 > 1), \quad (A < 0, \bar{\lambda}^2 < 1). \tag{34}$$

Eigenvalues:

$$\bar{p}_2 : \lambda_1 = -3(1 - AB\bar{\lambda} - \bar{\lambda}^2), \quad \lambda_2 = -2(1 - 3AB\bar{\lambda} - 3\bar{\lambda}^2), \tag{35}$$

$$\lambda_3 = \frac{-3(1 + A^2 + A^2B^2 - \bar{\lambda}^2)}{1 + A^2 + (AB + \bar{\lambda})^2}, \tag{36}$$

$$\bar{p}_3 : \lambda_1 = -2 \frac{A(1 + B^2)(1 - 3\bar{\lambda}^2) + 3B\bar{\lambda}(1 - \bar{\lambda}^2)}{A(1 + B^2)}, \quad \lambda_2 = -\frac{3(A + AB^2 + B\bar{\lambda})(1 - \bar{\lambda}^2)}{A(1 + B^2)}, \tag{37}$$

$$\lambda_3 = \frac{-3(1 - \bar{\lambda}^2)(1 - \bar{\lambda}^2 + A^2 + A^2B^2)}{(1 - AB\bar{\lambda} - \bar{\lambda}^2)^2 + A^2(1 + B^2 + \bar{\lambda}^2)}. \tag{38}$$

The equilibrium point \bar{p}_2 is an inflationary sink for small $\bar{\lambda}$ and positive A , while \bar{p}_3 is an inflationary sink for small $\bar{\lambda}$ and negative A .

3. Anisotropic points, p_4, p_5 :

$$\Psi = \frac{1}{\bar{\lambda}}, \quad \Phi = -\frac{1}{2}(b\beta + a), \quad \beta = \frac{b^2 - a^2 - ab}{2ba + 2c^2 + b^2} \pm \tag{39}$$

$$\frac{\sqrt{(a^2 + b^2)^2 + 2(2b - a)ac^2\bar{\lambda}^2 + 4(2ab + b^2 + 2c^2)(\bar{\lambda}^2 - 1)}}{\bar{\lambda}(2ba + 2c^2 + b^2)}, \tag{40}$$

$$q_{p_4, p_5} = 2. \tag{41}$$

The non-vacuum solutions corresponding to these equilibrium points also are spatially flat anisotropic solutions with a deceleration parameter equal to 2 (and hence are never inflationary), but with a fraction of the energy density contained in the potential energy of the scalar field.

Eigenvalues:

$$\lambda_1 = 4 - \sqrt{2}M\beta_{p_4}, \quad \lambda_{2,3} = -\frac{3\Phi_{p_4,p_5}(b\beta_{p_4,p_5}^2 - b\beta_{p_4,p_5} + 2\beta_{p_4,p_5}\Phi_{p_4,p_5} + b \pm D)}{2(b\Phi_{p_4,p_5} - 2)}, \quad (42)$$

where

$$D = [(-\beta_{p_4,p_5}(b - 2\Phi_{p_4,p_5}) + b\beta_{p_4,p_5}^2 + b)^2 - 2(b\Phi_{p_4,p_5} - 2)(\beta_{p_4,p_5}\bar{\lambda}^2(b^2 + 2b\Phi_{p_4,p_5} + 4\Phi_{p_4,p_5}^2) + b^2\beta_{p_4,p_5}^3\bar{\lambda}^2 + 4b\beta_{p_4,p_5}^2\bar{\lambda}^2\Phi_{p_4,p_5} + 4(b\bar{\lambda}^2\Phi_{p_4,p_5} + \bar{\lambda}^2\Phi_{p_4,p_5}^2 - \bar{\lambda}^2 + 1))]^{1/2}. \quad (43)$$

The signs of the eigenvalues can be studied. In Sec. IV, we will analyse the equilibrium points in the zero curvature case in detail. There are some sources, which are analogues of the Jacobs solutions studied above.

B. Equilibrium points with curvature

$$p_6 : \quad \Phi = 0, \quad \Psi = 0, \quad \beta = \frac{M}{\sqrt{2c^2}}, \quad (44)$$

$$K = \frac{M^2}{4c^2} - 1, \quad q = \frac{M^2}{2c^2}. \quad (45)$$

Eigenvalues:

$$\lambda_1 = \frac{M^2}{2c^2} + 1, \quad \lambda_2 = \frac{M^2}{2c^2} - 2, \quad \lambda_3 = \frac{M^2}{2c^2} - 2. \quad (46)$$

This equilibrium point is always a saddle when the curvature K is negative.

C. Discussion

There are a number of important invariant sets. From Eqs. (B3)–(B5), we can compute the evolution equations for K . By setting $K = 0$ in this expression we obtain $K' = 0$, so that $K = 0$ is an invariant set of the system. We recall that we can obtain the Bianchi I models (with $K = 0$) when $M = N = 0$. We shall study this 2D invariant set (not necessarily for small values of the parameters a and b) in Sec. IV (where we will use the same notation for the 2D equilibrium points as the 3D zero curvature equilibrium points in this section).

In the invariant set $\Phi = 0$, the scalar potential is dynamically negligible. The isotropic models with $\beta = 0$ (i.e., no shear anisotropy), were studied in Ref. 20.

IV. ANALYSIS OF THE SYSTEM WITH ZERO CURVATURE

When $K = 0$ we obtain the 2D system (we also set $c^2 = 1$),

$$\beta' = (q - 2)\beta, \quad (47)$$

$$\Phi' = (q + 1)\Phi - 3\bar{\lambda}\Psi\Phi, \quad (48)$$

where

$$\Psi^2 = 1 - \frac{1}{2}\beta^2 - \Phi^2 + a\beta\Phi - b\Phi, \quad (49)$$

$$q - 2 = \frac{3\Phi}{b\Phi - 2} [(\bar{\lambda}\Psi - 1)(a + b)\beta + (a - b)\bar{\lambda}\Psi + b(\beta + 1) + 2\Phi]. \quad (50)$$

TABLE I. Equilibrium points of the anisotropic model.

Eq pts	Φ	β
$p_{2,3}$	$\frac{2(2\bar{\lambda}^2 b - b - a\bar{\lambda}^2 \pm Z)}{4\bar{\lambda}^2 b^2 - 4a\bar{\lambda}^2 b + a^2\bar{\lambda}^2 + 4}$	0
$p_{4,5}$	$-\frac{\bar{\lambda} b^3 - b a^2 \bar{\lambda} - 2a\bar{\lambda} \pm b N}{2\bar{\lambda}(2+2ab+b^2)}$	$\frac{\bar{\lambda} b^2 - \bar{\lambda} ab - \bar{\lambda} a^2 \mp N}{\bar{\lambda}(2+2ab+b^2)}$

A. Equilibrium points

We define

$$Z = \sqrt{b^2 - 2ab\bar{\lambda}^2 + a^2\bar{\lambda}^2 + 4 - 4\bar{\lambda}^2}$$

and

$$N^2 = b^4\bar{\lambda}^2 + 2\bar{\lambda}^2 a^2 b^2 + \bar{\lambda}^2 a^4 + 8\bar{\lambda}^2 - 8 + 12ab\bar{\lambda}^2 - 2a^2\bar{\lambda}^2 - 8ab + 4b^2\bar{\lambda}^2 - 4b^2.$$

The equilibrium points that are potentially of interest (i.e., possible future attractors) in the 2D system with zero curvature are given in Table I.

For physical solutions all of Φ , β , and Ψ must be real, and $\Phi \geq 0$. Since $K = 0$, if $\Omega \geq 0$, then Ω and β^2 are bounded. For sufficiently small $\bar{\lambda}$, Z is real. We cannot hope to study the stability of these equilibrium points in all generality. However, we consider some appropriate values of $\bar{\lambda}$ (e.g., for small $\bar{\lambda}$ such as $\bar{\lambda} = \frac{1}{2}$). First, if $a = b = 0$ we will have the same equilibrium points p_i as in the single scalar field model with matter. We will study the stability of the p_i in the two cases $a = 0$ and $b = 0$, separately.

For the point p_1 , where $\Phi = 0$ and $\beta = \pm \frac{\sqrt{2\bar{\lambda}^2 - 2}}{\bar{\lambda}}$, which is only valid for $\bar{\lambda} > 1$, we have that $q = 2$, and hence p_1 is always non-inflationary. p_1 always has one zero eigenvalue. In general, this point will be a source, and we will not consider it further here. Some of the other equilibria may also turn out to be sources for particular ranges of the parameters. Recall that analogues of the Jacobs anisotropic Bianchi I non-vacuum solutions are sources.

1. Point p_2

Φ is always positive for sufficiently small $\bar{\lambda}$. The value of Ψ for p_2 is given by

$$\Psi = \frac{\bar{\lambda}(-ab + 2b^2 + 4 + aZ - bZ)}{4\bar{\lambda}^2 b^2 - 4a\bar{\lambda}^2 b + a^2\bar{\lambda}^2 + 4}.$$

We study the stability in two cases for the values of a, b between -3 and 3 .

The first case is when $a = 0$. For $a = 0$, Φ is real and positive for $\bar{\lambda}^2 < 1$, and also for $\bar{\lambda}^2 > 1$ if $b > 2\sqrt{\bar{\lambda}^2 - 1} > 0$. The eigenvalues μ_i , where $i = 1, 2$ (corresponding to $+, -$), are given by

$$\mu_i = \frac{-3(m_2 m_1 (m_3 + m_4)) \pm \sqrt{2} m_1 m_5 + \sqrt{2}\bar{\lambda}^3 b^2 m_6}{m_2(m_7 \pm m_1 m_8)},$$

where

$$m_1 = \sqrt{b^2 - 4\bar{\lambda}^2 + 4},$$

$$m_2 = \sqrt{\frac{-\bar{\lambda}^2(-4b^2 + 2\bar{\lambda}^2 b^2 + (b^3 + 2)b m_1 - 2 - b^4)}{(\bar{\lambda}^2 b^2 + 1)^2}},$$

$$m_3 = 4b\bar{\lambda}^2 + 4b^3\bar{\lambda}^4 + b^5\bar{\lambda}^4 - b,$$

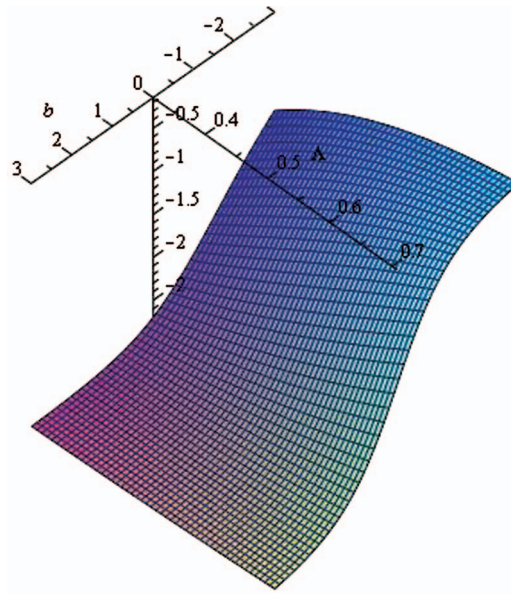


FIG. 1. $a = 0$; μ_1 -eigenvalue of p_2 .

$$m_4 = 4 - b^6\bar{\lambda}^4 + b^2 - 4\bar{\lambda}^2 + 4b^4\bar{\lambda}^6 + 2\bar{\lambda}^2b^6 + 2b^2\bar{\lambda}^2,$$

$$m_5 = b^5\bar{\lambda}^5 + 3b^3\bar{\lambda}^5 - 3b^3\bar{\lambda}^3 - b\bar{\lambda}^3 - b^3\bar{\lambda}^5,$$

$$m_6 = 5 + b^4 - b^4\bar{\lambda}^2 + 5b^2 + 2b^2\bar{\lambda}^4 - 6\bar{\lambda}^2 - 7b^2\bar{\lambda}^2,$$

$$m_7 = 10b^2\bar{\lambda}^2 + b^6\bar{\lambda}^4 + 2b^4\bar{\lambda}^2 + 2b^6\bar{\lambda}^6 + 8b^4\bar{\lambda}^4 + 4 + b^2,$$

$$m_8 = b^5\bar{\lambda}^4 + 2b^3\bar{\lambda}^2 + b.$$

We plot the two eigenvalues μ_1, μ_2 , where the horizontal axes are the values of the parameters $(b, \bar{\lambda})$ (where Λ in Figures 1 and 2 is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2 .

For illustration, we can also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$. We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda} = \frac{1}{2}$ in Figs. 3 and 4 (where the horizontal axes are the values of $-3 \leq b \leq 3$ and the vertical axes are the value of the eigenvalues μ_1, μ_2). As can be seen from the figures, for the parameter $-3 \leq b \leq 3$, we have two negative eigenvalues which implies that p_2 is a sink.

The second case is when $b = 0$. Φ is real and positive if $\bar{\lambda}^2 \leq 1$ and $a < 2$ (including negative values of a) or $\bar{\lambda}^2 > 1$ and $a > 2$. The eigenvalues are given by

$$\mu_i = \frac{3(m_{13} m_{15} \pm m_{14} m_{16} + m_{17})}{m_{14}(48a^2\bar{\lambda}^2 + 12a^4\bar{\lambda}^4 + a^6\bar{\lambda}^6 + 64)},$$

where

$$m_{13} = \sqrt{4 - 4\bar{\lambda}^2 + \bar{\lambda}^2 a^2},$$

$$m_{14} = \sqrt{\frac{\bar{\lambda}^2(4a^2 + a^4\bar{\lambda}^2 - 4a^2\bar{\lambda}^2 + 8a m_{13} + 16)}{(4 + a^2\bar{\lambda}^2)^2}},$$

$$m_{15} = -2a\bar{\lambda}^3 m_{13}(16 + 4a^2 + a^2\bar{\lambda}^2 - 12\bar{\lambda}^2),$$

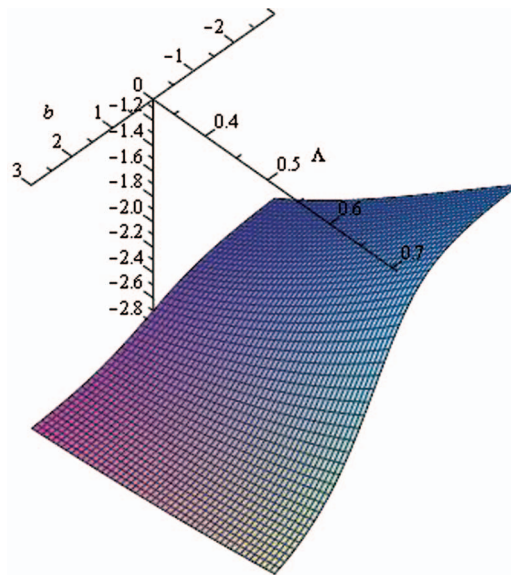


FIG. 2. $a = 0$; μ_2 -eigenvalue of p_2 .

$$m_{16} = -64 - 32a^2\bar{\lambda}^2 - 4a^4\bar{\lambda}^4 + 64\bar{\lambda}^2 + 8a\bar{\lambda}^2 m_{13}(a^2\bar{\lambda}^2 + 4) - 4a^4\bar{\lambda}^6,$$

$$m_{17} = -32a^2\bar{\lambda}^3 - 4a^4\bar{\lambda}^5 a^6\bar{\lambda}^7 + 48a^2\bar{\lambda}^5 - 4a^4\bar{\lambda}^7.$$

We plot the two eigenvalues μ_1, μ_2 in Figs. 5 and 6, where the horizontal axes are the values of the parameters $(a, \bar{\lambda})$ (where Λ in the figures below is equal to $\bar{\lambda}$), and the vertical axes are the values of μ_1, μ_2 .

For illustration, we can also evaluate the two eigenvalues when $\bar{\lambda} = \frac{1}{2}$. We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda} = \frac{1}{2}$ in Figs. 7 and 8 (where the horizontal axes are the values of $-3 \leq a$

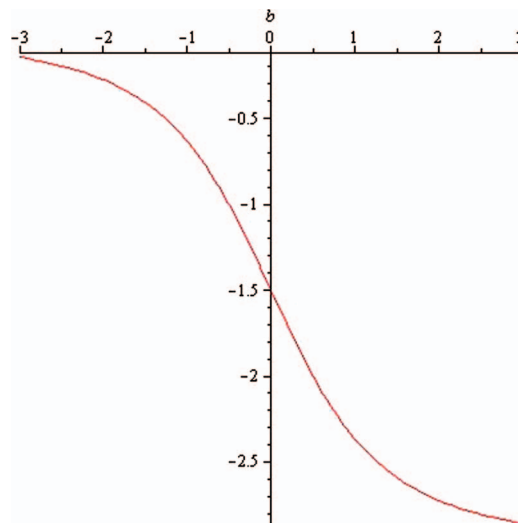


FIG. 3. $\bar{\lambda} = \frac{1}{2}, a = 0$; μ_1 -plot for p_2 .

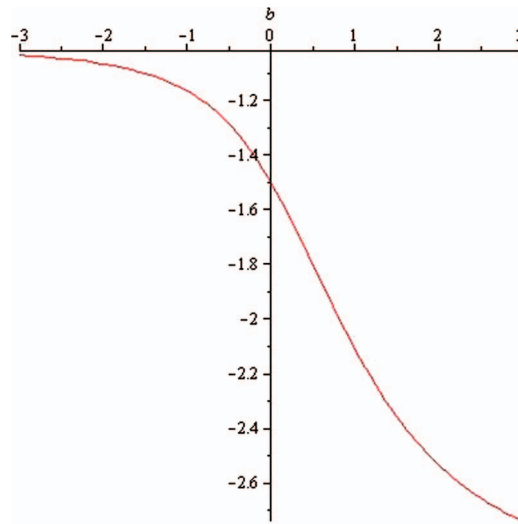


FIG. 4. $\bar{\lambda} = \frac{1}{2}, a = 0; \mu_2$ -plot for p_2 .

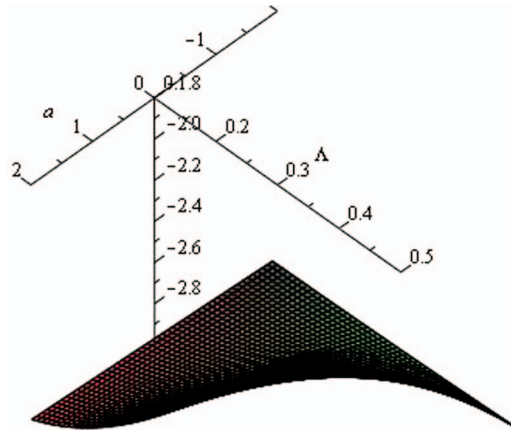


FIG. 5. $b = 0; \mu_1$ -eigenvalues of p_2 .

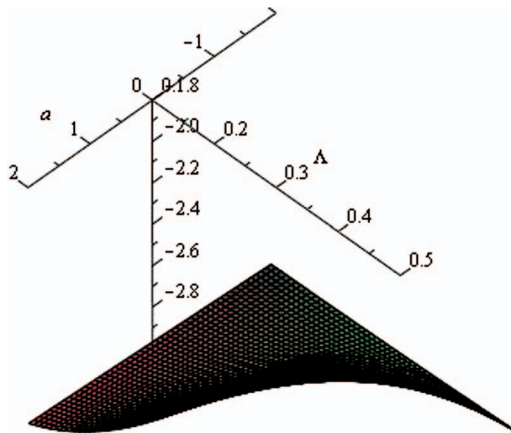


FIG. 6. $b = 0; \mu_2$ -eigenvalues of p_2 .

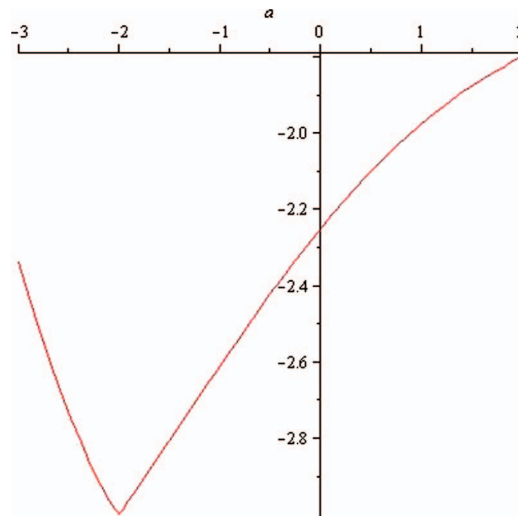


FIG. 7. $\bar{\lambda} = \frac{1}{2}, b = 0$; μ_1 -plot of p_2 .

≤ 2 and the vertical axes are the values of the eigenvalues μ_1, μ_2). As you can see from the Figs. 7 and 8, we always have a sink for the values $-3 \leq a \leq 2$.

The deceleration parameter for p_2 :

In this case, the deceleration parameter can be written in the form

$$q = \frac{-4(4 + b^2) + \bar{\lambda}^2(G + Z C) + Z b}{(X)(S - b Z)},$$

where

$$G = -20ab + 12a^2\bar{\lambda}^2 + 36b^2 - a^2b^2 - 2ab^3 - 16ab^3\bar{\lambda}^2 + 9\bar{\lambda}^2a^2b^2 + \bar{\lambda}^2a^3b - 8a^2 + 8b^4 + 48 + 4\bar{\lambda}^2b^4 - \bar{\lambda}^2a^4,$$

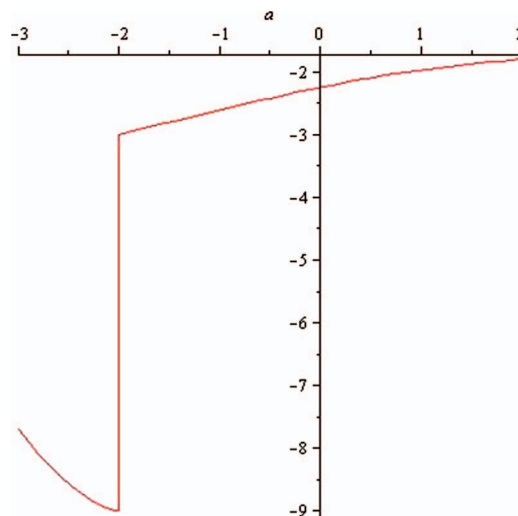
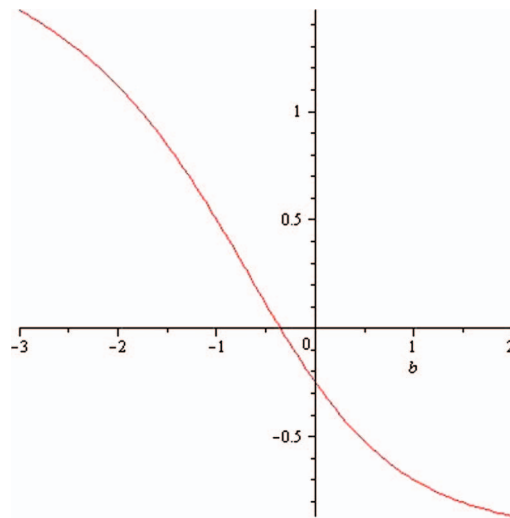


FIG. 8. $\bar{\lambda} = \frac{1}{2}, b = 0$; μ_2 -plot of p_2 .

FIG. 9. $\bar{\lambda} = \frac{1}{2}$, $a = 0$; q for p_2 .

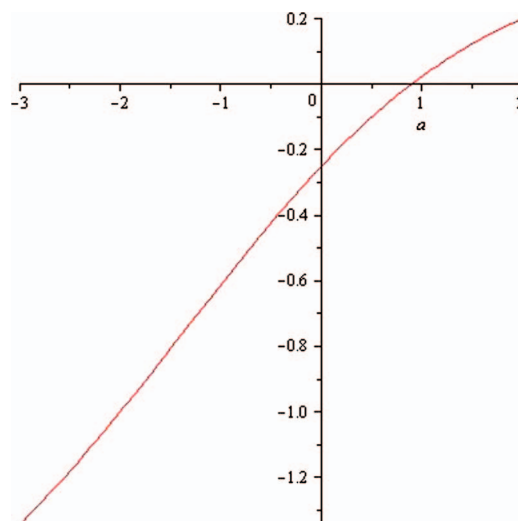
$$C = -8b^3 - 12b^3\bar{\lambda}^2 + 3\bar{\lambda}^2a^3 - 36b + 12a + ba62 + 2ab^2 - 15ba^2\bar{\lambda}^2 - 24\bar{\lambda}^2ab^2,$$

$$S = b^2 + 2b^2\bar{\lambda}^2 - 3ab\bar{\lambda}^2 + 4 + a^2\bar{\lambda}^2,$$

and

$$X = 4\bar{\lambda}^2b^2 - 4a\bar{\lambda}^2b + a^2\bar{\lambda}^2 + 4,$$

and we note that Z is defined earlier. We consider the inflationary behaviour in the same two cases as in the stability analysis above; i.e, when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$ and when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$. We plot the values of q , where the horizontal axis is the value of a or b and the vertical axis is the value of q , in Figs. 9 and 10. For $\bar{\lambda} = \frac{1}{2}$, and the case where $a = 0$, if $b \gtrsim -\frac{1}{2}$, then q is negative; otherwise q is positive. For the second case where $b = 0$, if $a \gtrsim 1$, then q is positive; otherwise q is negative. That is, the model is inflationary (in the latter case) when $a \lesssim 1$.

FIG. 10. $\bar{\lambda} = \frac{1}{2}$, $b = 0$; q for p_2 .

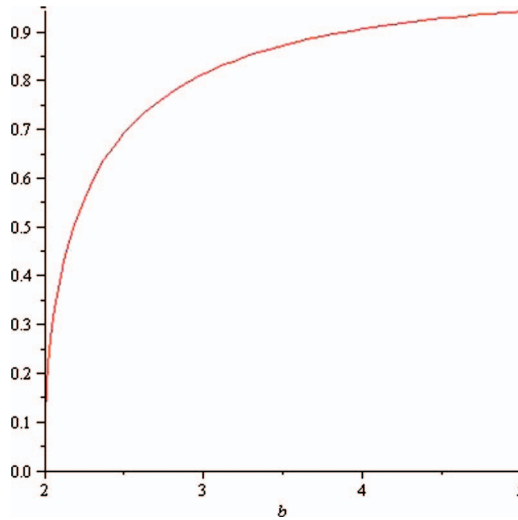


FIG. 11. $\bar{\lambda} = \sqrt{2}, a = 0$; μ_1 -eigenvalues of p_3 .

2. Point p_3

The value of Ψ for p_3 is given by

$$\Psi = \frac{\bar{\lambda}(-ab + 2b^2 + 4 - aZ + bZ)}{4\bar{\lambda}^2b^2 - 4a\bar{\lambda}^2b + a^2\bar{\lambda}^2 + 4}.$$

The value of Φ is given by

$$\Phi = \frac{2(2\bar{\lambda}^2b - b - a\bar{\lambda}^2 - \sqrt{b^2 - 2ab\bar{\lambda}^2 + a^2\bar{\lambda}^2 + 4 - 4\bar{\lambda}^2})}{4\bar{\lambda}^2b^2 - 4a\bar{\lambda}^2b + a^2\bar{\lambda}^2 + 4}.$$

For small $\bar{\lambda}$, Φ is negative; e.g., when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$,

$$\Phi = -\frac{b + 2\sqrt{b^2 + 3}}{b^2 + 4},$$

which is always negative. Indeed, for positive real Φ we must have that $\bar{\lambda} > 1$ and $b > 2\sqrt{\bar{\lambda}^2 - 1} > 0$.

The first case is when $a = 0$: the eigenvalues μ_i are given by

$$\mu_i = \frac{-3(m_2(m_1(m_3 + m_4)) \mp \sqrt{2}m_1 m_5 + \sqrt{2}\bar{\lambda}^3 b^2 m_6)}{m_2(m_7 \mp m_1 m_8)}.$$

We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda}^2 = 2$ in Figs. 11 and 12 (where the horizontal axes are the values of $2 < b \leq 5$ and the vertical axes are the values of the eigenvalues μ_1, μ_2). The signs of the two eigenvalues are different, which implies that p_3 in this case is a saddle.

The second case is when $b = 0$; we have

$$\Phi = \frac{2}{4 + a^2\bar{\lambda}^2} \left(-a\bar{\lambda}^2 - \sqrt{4 + a^2\bar{\lambda}^2 - 4\bar{\lambda}^2} \right).$$

Thus, Φ is real and positive when $\bar{\lambda} > 1, a < \frac{-2}{\bar{\lambda}}\sqrt{\bar{\lambda}^2 - 1} < 0$. The eigenvalues are given by

$$\mu_i = \frac{3(m_{13} m_{15} \mp m_{14} m_{16} + m_{17})}{m_{14}(48a^2\bar{\lambda}^2 + 12a^4\bar{\lambda}^4 + a^6\bar{\lambda}^6 + 64)}.$$

We plot the two eigenvalues μ_1, μ_2 when $\bar{\lambda}^2 = 2$ in Figs. 13 and 14 (where the horizontal axis are the values of $-4 \leq a < -1$ and the vertical axis are the values of the eigenvalues μ_1, μ_2). We always have positive signs for the both eigenvalues μ_1 and μ_2 which means that p_3 is a source in the range of $-4 \leq a < -1$.

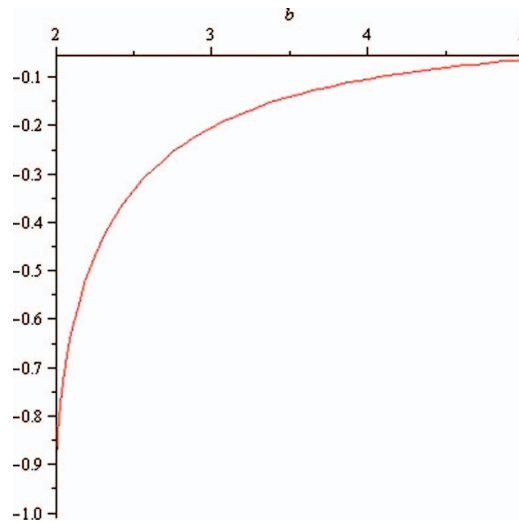


FIG. 12. $\bar{\lambda} = \sqrt{2}, a = 0$; μ_2 -eigenvalues of p_3 .

The deceleration parameter for p_3 :

In this case, the deceleration parameter can be written in the form

$$q = \frac{-4(4 + b^2) + \bar{\lambda}^2(G - Z C) - Z b}{(X)(S + b Z)}.$$

In the two cases considered above, in which $\bar{\lambda}^2 = 2$, p_3 is either a saddle or a source. For illustration, we plot the values of q when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$ and when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$ in Figs. 15 and 16, where the horizontal axis is the value of a or b and the vertical axis is the value of q . When $a = 0$, if $b \gtrsim \frac{1}{2}$, then q is positive (otherwise q is negative). When $b = 0$, if $a \lesssim -1$, then q is positive (otherwise q is negative).

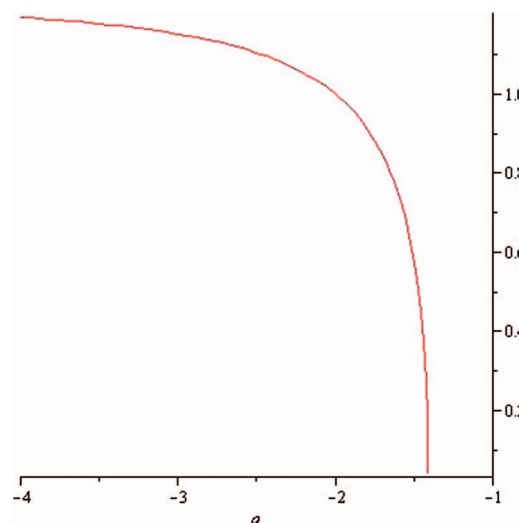


FIG. 13. $\bar{\lambda} = \sqrt{2}, b = 0$; μ_1 -eigenvalues of p_3 .

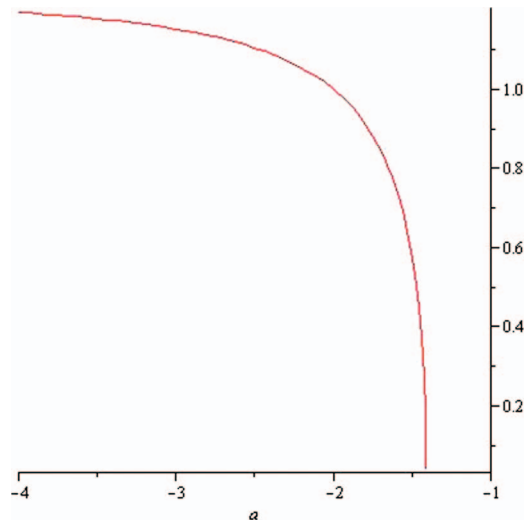


FIG. 14. $\bar{\lambda} = \sqrt{2}, b = 0; \mu_2$ -eigenvalues of p_3 .

3. Point p_4

In this case $\Psi = \frac{1}{\bar{\lambda}}$. The value of Φ is given by

$$\Phi = \frac{-\bar{\lambda}b^3 - ba^2\bar{\lambda} - 2a\bar{\lambda} - bN}{2\bar{\lambda}(2 + 2ab + b^2)},$$

where N was defined earlier (before the table for the equilibrium points). Φ is real if $N > 0$ and it is positive if $b < 0$. We define

$$J_1 = b^{11} - 7b^9 - 32b^7 - 92b^5 - 112b^3 - 32b,$$

$$J_2 = b^{13} - 13b^{11} - 20b^9 - 20b^7 - 160b^5 - 416b^3 - 384,$$

$$J = -2b^{20} - 4200b^{14} + 15b^{22} - 12576b^{12} - 34704b^{10} - b^{24} - 2b^2(1116b^{16} - 189b^{18}),$$

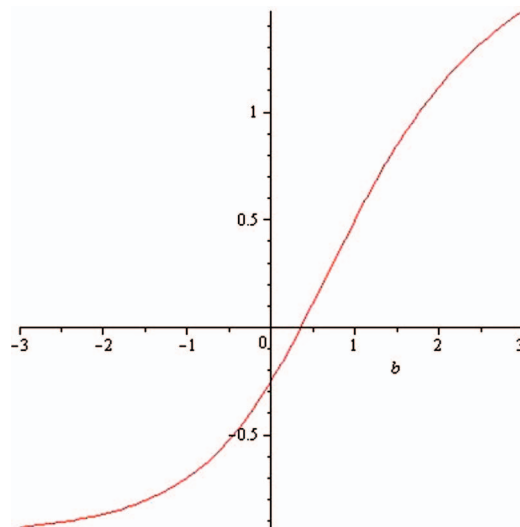
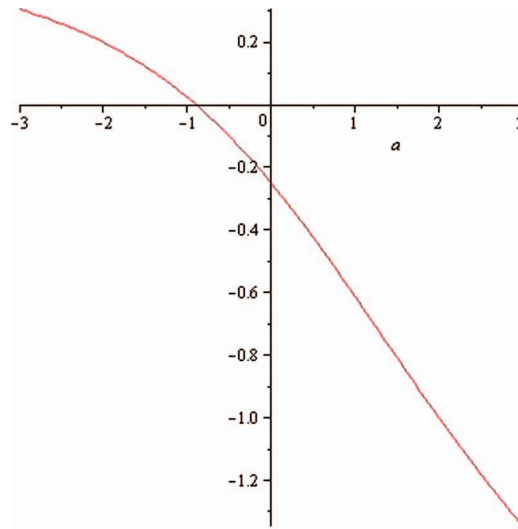


FIG. 15. $\bar{\lambda} = \frac{1}{2}, a = 0; q$ for p_3 .

FIG. 16. $\bar{\lambda} = \frac{1}{2}$, $b = 0$; q for p_3 .

$$\bar{J} = +582784b^{10} + 471552b^8 + 345984b^{12} - 2b^{28} + 127584b^{14},$$

$$F_1 = \bar{J} - 894b^{22} - 124b^{24} + 42b^{26} + 28128b^{16} + 312b^{18},$$

$$J_3 = \sqrt{2b^2(-164864b^2 - 205824b^4 - 61440 - 82752b^8 - 156160b^6 + J)},$$

$$J_4 = \sqrt{-294912 - 983040b^4 - 909312b^2 - 23142b^6 + F_1},$$

$$J_5 = 18b^6 + b^{10} + 48b^4 + 48b^2 + 3b^8,$$

$$J_6 = -3b^{10} - 30b^8 + b^{12} - 80b^6 + 192b^2 + 128.$$

There are also two cases for p_4 . First, when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$, the value of Φ is

$$\Phi = -\frac{1}{2}b \left(\frac{b^2 + \sqrt{b^4 - 12b^2 - 24}}{b^2 + 2} \right),$$

which is real if $b^4 - 12b^2 - 24 > 0$ and Φ is positive for $b < 0$; i.e., we consider $b < -3.75$. The eigenvalues are given by

$$\mu_i = \frac{3b(\sqrt{b^4 - 12b^2 - 24}(J_1) + J_2 \pm \sqrt{b^4 - 12b^2 - 24} J_3 \pm J_4)}{2(b^2 + 2)(\sqrt{b^4 - 12b^2 - 24} J_5 + J_6)}.$$

We note that $\mu_1(b) = \mu_2(-b)$.

We plot the eigenvalues μ_1, μ_2 in Figs. 17 and 18, where the horizontal axis is the real part of the eigenvalues and the vertical axis is the imaginary part of the eigenvalues μ_1 and μ_2 for the range of the parameter $-7 \leq b \leq -4$. As can be seen from these figures, the real parts of μ_1 and μ_2 are both positive in the range $-7 \leq b \leq -4$, so that p_4 is a source.

In the second case, when $b = 0$, the value of Φ is

$$\Phi = -\frac{a}{2},$$

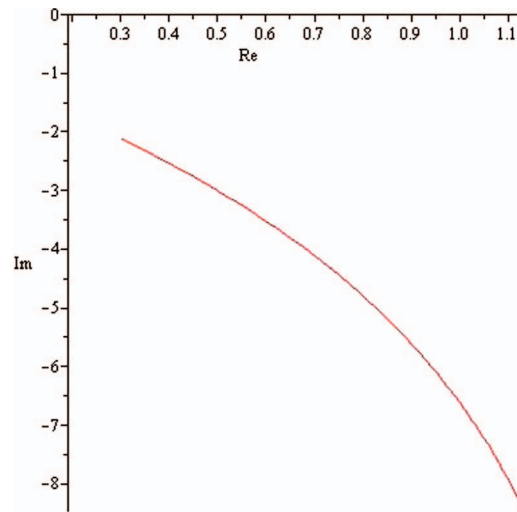


FIG. 17. $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_1 -eigenvalue of p_4 ; Plot $-7 \leq b \leq -4$.

which is valid if $a < 0$. Let now consider the case when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$; the eigenvalues are given by

$$\mu_i = \frac{3}{16}a(-a^3 + a n_2 \pm \bar{n}_2),$$

where

$$n_2 = \sqrt{(a^2 - 6)(a^2 + 4)},$$

$$\bar{n}_2 = \sqrt{2a^6 - 2a^4 n_2 - 20a^2 - 4a^4 + 48 + 2a^2 n_2}.$$

Again we note that $\mu_2(a) = \mu_1(-a)$. We plot the eigenvalues above in Figs. 19 and 20 (where the horizontal axis is the real part of the eigenvalues and the vertical axis is the imaginary part of the eigenvalues for the range of parameter $-3 \leq a < 0$). As can be seen from these figures, the real

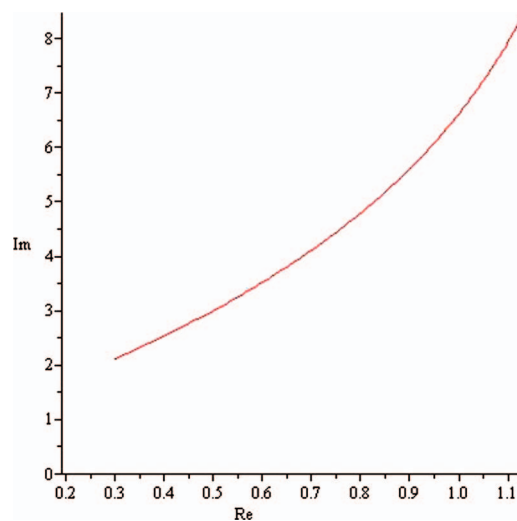


FIG. 18. $\bar{\lambda} = \frac{1}{2}$, $a = 0$; μ_2 -eigenvalue of p_4 ; Plot $-7 \leq b \leq -4$.

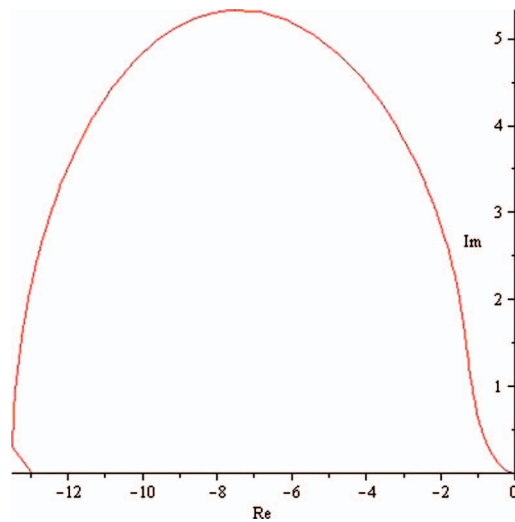


FIG. 19. $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_1 -eigenvalue of p_4 ; Plot for $-3 \leq a < 0$.

part of μ_1 is negative and the real part of μ_2 is positive in the range $-3 \leq a < 0$, which means that in this case p_4 is a saddle.

4. Point p_5

The first case for p_5 is when $a = 0$. The value of Φ is given by

$$\Phi = \frac{-\bar{\lambda}b^3 + b\sqrt{D}}{2\bar{\lambda}(2 + b^2)},$$

where $D \equiv b^4\bar{\lambda}^2 + 8\bar{\lambda}^2 - 8 + 4b^2\bar{\lambda}^2 - 4b^2$. Φ is real if $D > 0$, so that if $\bar{\lambda}^2 > 1$ then D always is positive. But if $\bar{\lambda}^2 < 1$, then we have the condition that

$$b^2 \geq -2 + \frac{2}{\bar{\lambda}^2} + \frac{2}{\bar{\lambda}^2}\sqrt{1 - \bar{\lambda}^4}.$$

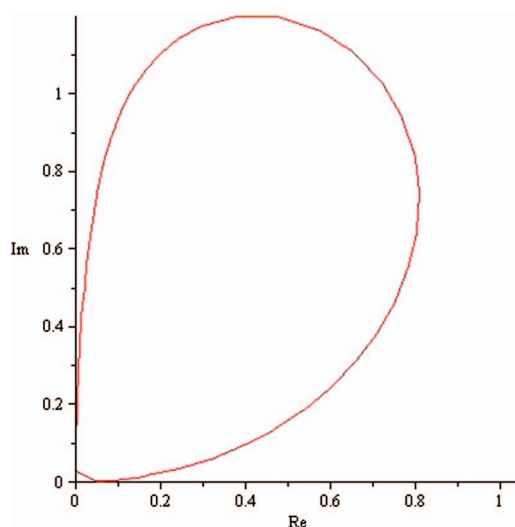


FIG. 20. $\bar{\lambda} = \frac{1}{2}$, $b = 0$; μ_2 -eigenvalue of p_4 ; Plot for $-3 \leq a < 0$

Φ is positive if $-\bar{\lambda}b^3 + b\sqrt{D} > 0$. Then, there are two cases: If $b < 0$, then $\bar{\lambda}^2 < 1$ and $b^2 \geq -2 + \frac{2}{\bar{\lambda}^2} + \frac{2}{\bar{\lambda}^2}\sqrt{1-\bar{\lambda}^4}$, which means that $b < -3.75$. Let us consider the case when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$. The value of Φ is

$$\Phi = \frac{1}{2}b\left(\frac{-b^2 + \sqrt{b^4 - 12b^2 - 24}}{b^2 + 2}\right).$$

The eigenvalues are given by

$$\mu_i = \frac{3\left(J_2 - \sqrt{b^4 - 12b^2 - 24}J_1 \mp J_3\sqrt{b^4 - 12b^2 - 24} \pm J_4\right)}{2(b^2 + 2)(-\sqrt{b^4 - 12b^2 - 24}J_5 + J_6)}.$$

It can be shown that p_5 is a saddle when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$.

In the case $b > 0$, we then have $\bar{\lambda}^2 > 1$. Let us consider the case when $a = 0$ and $\bar{\lambda} = 2$. The value of Φ is

$$\Phi = \frac{1}{2}b\left(\frac{-b^2 + \sqrt{b^4 + 3b^2 + 6}}{b^2 + 2}\right).$$

We define

$$k_1 = 13b^{14} - 169b^2 - 332b^{10} - 620b^8 - 3232b^6 - 8000b^4 - 7680b^2 - 768,$$

$$k_2 = -13b^{12} + 91b^{10} + 488b^8 + 1508b^6 + 1936b^4 + 704b^2,$$

$$k_3 = 589824 + 2359296b^2 + 9535488b^4 + 10665984b^6,$$

$$k_4 = -15224832b^{10} - 3186688b^8 - 11761280b^{12} - 2668608b^{14},$$

$$k_5 = +978b^{28} - 32b^{30} + 1508064b^{16} + 1383072b^{18} + 450120b^{20},$$

$$k_6 = +40428b^{22} - 21822b^{24} - 6004b^{26},$$

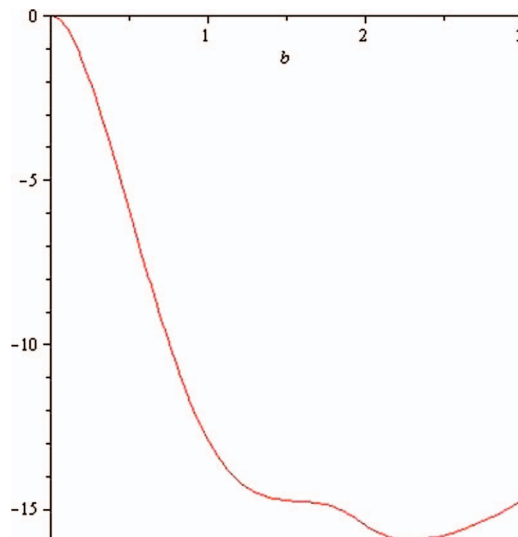


FIG. 21. $\bar{\lambda} = 2$, $a = 0$; the real part of μ_1 -eigenvalue of p_5 ; Plot for $0 < b \leq 3$.

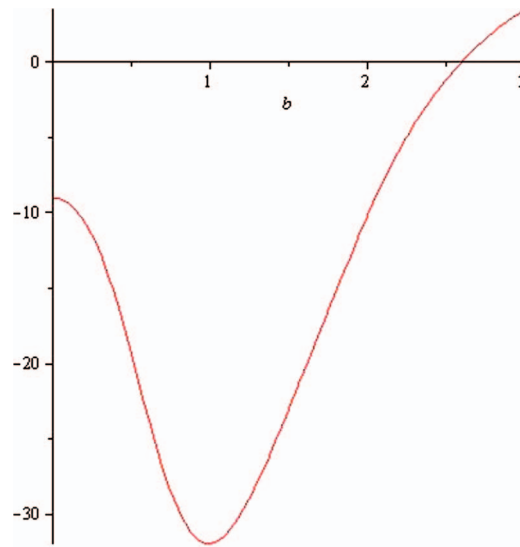


FIG. 22. $\bar{\lambda} = 2, a = 0$; the real part of μ_2 -eigenvalue of p_5 ; Plot for $0 < b \leq 3$.

$$k_7 = \sqrt{k_3 + k_4 + k_5 + k_6},$$

$$k_8 = -20208b^{18} + 39432b^{20} + 17490b^{22} - 1623264b^{14} + 2248b^{24},$$

$$k_9 = -407040b^{16} - 786432b^2 - 3710976b^4 - 9141760b^{10} - 9193472b^6,$$

$$k_{10} = -11950080b^8 - 4576896b^{12} - 786b^{26} + 32b^{28},$$

$$k_{11} = \sqrt{k_8 + k_9 + k_{10}}.$$

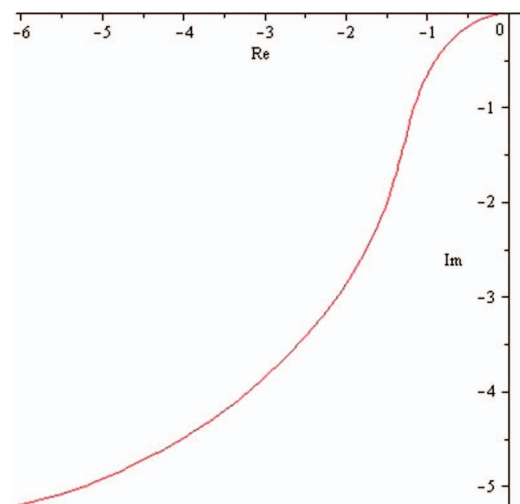


FIG. 23. $\bar{\lambda} = \frac{1}{2}, b = 0$; μ_1 -eigenvalue of p_5 ; Plot $-2 \leq a < 0$.

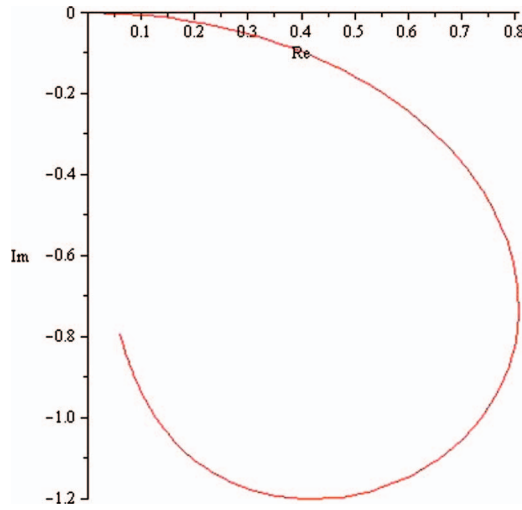


FIG. 24. $\bar{\lambda} = \frac{1}{2}, b = 0$; μ_2 -eigenvalue of p_5 ; Plot $-2 \leq a < 0$.

The eigenvalues are given by

$$\mu_i = \frac{-3 \left(k_1 + \sqrt{b^4 - 12b^2 - 24k_2} \pm k_7 + \mp k_{11} \sqrt{b^4 - 12b^2 - 24} \right)}{2(b^2 + 2)(-\sqrt{b^4 - 12b^2 - 24} J_5 + J_6)}$$

We plot the real part of the eigenvalues in Figs. 21 and 22 (where the vertical axis is the real part of the eigenvalues and the horizontal axis is the value of the parameter in the range $0 < b \leq 3$). As can be seen from Figs. 21 and 22, the real parts of μ_1 and μ_2 are both negative in the range of $0 < b \lesssim 2.55$; therefore, p_5 is a sink. The real part of μ_2 changes sign for $b \simeq 2.55$, so that implies p_5 becomes a saddle. Note that $q > 0$, which implies that p_5 is not inflationary.

The second case for p_5 is when $b = 0$, and the value of Φ is

$$\Phi = -\frac{a}{2},$$

which is positive if $a < 0$. Let us consider the case where $b = 0$ and $\bar{\lambda} = \frac{1}{2}$; the eigenvalues are given by

$$\mu_i = \frac{3}{16} a(-a^3 - a n_2 \pm \bar{n}_6),$$

where

$$n_2 = \sqrt{(a^2 - 6)(a^2 + 4)}, \bar{n}_6 = \sqrt{2a^6 + 2a^4 n_2 - 20a^2 - 4a^4 + 48 - 2a^2 n_2}.$$

We plot the eigenvalues in Figs. 23 and 24 (where the horizontal axis is the real part of the eigenvalues and the vertical axis is the imaginary part of the eigenvalues for the range of the parameter $-2 \leq a < 0$). As can be seen from these figures, the real part of μ_1 is negative in the range $-2 \leq a < 0$ but the real part of μ_2 is positive in the same range. Thus, p_5 is a saddle in this case.

B. Summary

Let us summarize the stability analysis. The point p_2 is a sink for both cases ($a = 0, b = 0$) in the range of values $-3 \leq a, b \leq 2$. However, p_3 is a saddle when $a = 0$ and $\bar{\lambda} = \frac{1}{2}$ for the range $2 < b \leq 5$. p_3 is a source when $b = 0$ and $\bar{\lambda} = \sqrt{2}$ in the range $-4 \leq a \leq -1$. The model is inflationary for p_2 if either $-\frac{1}{2} \lesssim b$ or $a \lesssim 1$. The model is inflationary for p_3 if either $\frac{1}{2} \lesssim b$ or $a \lesssim -1$. Let us next consider the stability of p_4 . When $a = 0$ and $\bar{\lambda} = \frac{1}{2}$, p_4 is a source in the range $-7 \leq b \leq -4$. But p_4 is a saddle when $b = 0$ and $\bar{\lambda} = \frac{1}{2}$. For p_5 , there are restrictions on the ranges

for a and b in order for p_5 to exist. p_5 is a sink when $a = 0$ and $\bar{\lambda} = 2$ in the range of $0 < a \lesssim 2.55$, but $q > 0$ which implies that p_5 is not inflationary.

The equilibrium points correspond to physically realistic solutions for a restricted set of values of the parameters. Therefore, a complete discussion of the cosmological models would include further analysis of the viable (physical) ranges for the parameters and the nature of the equilibrium points in each of these viable ranges. However, for each set of parameters in the ranges discussed, there always exists an inflationary future attractor for sufficiently small $\bar{\lambda}$.

V. CONCLUSIONS

We have investigated cosmological models in the Einstein-aether theory and (extended) Horava gravity in which both the aether vector field and the metric tensor together determine the evolution. We have been especially interested in the possible inflationary behaviour of the models in a class of spatially anisotropic cosmological models. In particular, we have studied scalar field models in which the self-interaction potential, consisting of terms each containing exponentials, depends on the scalar field ϕ and also on the timelike vector field through the expansion rate θ and the shear scalar σ . We derived the evolution equations in the anisotropic models, which consist of the energy-momentum conservation law or Klein-Gordon equation, the generalized Friedmann equation, the Raychaudhuri equation, and the evolution equations for the shear and for the tilt. It was argued that the tilt generally decays to the future (in the anisotropic models under consideration), and we thus concentrated on the inflationary behaviour of models in the case of negligible tilt.

We introduced expansion-normalized variables and obtained the resulting dimensionless evolution equations which reduce to a 3D dynamical system. We studied the behaviour of the models in general, and in the 2D zero curvature case (which includes the Bianchi I type models) in detail. In particular, we studied the local stability of equilibrium points of the 2D dynamical system corresponding to physically realistic solutions with a restricted set of values of the parameters. We concluded that there are always ranges of values for which there exists an inflationary sink.

In this paper, we have primarily been interested in the local stability of equilibrium points of the cosmological dynamical system and, especially, in demonstrating the local existence of physically important inflationary future attractors (sinks). Some of these local sinks may also be global sinks. We have also briefly discussed the past evolution of the Einstein-aether cosmological models. A global analysis of a particular cosmological model (i.e., a specific Bianchi model with a specific self-interaction potential), based on the dynamical systems approach used here, would certainly be possible.¹⁴

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APPENDIX A: REVIEW OF SCALAR FIELD COSMOLOGY

Spatially homogeneous and isotropic FRW scalar field models have been studied in Ref. 15. Spatially homogeneous but anisotropic models of various Bianchi types have been studied in Ref. 16 (especially exact solutions) and Ref. 19. A qualitative analysis of all Bianchi models with $k^2 < 2$ (where $k \equiv -\lambda$), including standard matter satisfying various energy conditions, was presented in Ref. 17. It was found that the power-law inflationary solution is indeed an attractor for all initially expanding Bianchi models (except for a subclass of the Bianchi type IX models which recollapse).

1. Equations

Cosmological models with a minimally coupled scalar field have an effective energy density and pressure given by

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi). \quad (\text{A1})$$

In the models under consideration, the potential of the scalar field is given by

$$V(\phi) = V_0 e^{k\phi}, \quad (\text{A2})$$

where V_0 (>0) and k are constants.

From the Einstein FE we have the Raychaudhuri equation governing the evolution of the expansion

$$\dot{\theta} = -2\sigma^2 - \frac{1}{3}\theta^2 - \dot{\phi}^2 + V(\phi), \quad (\text{A3})$$

and the generalized Friedmann equation

$$\theta^2 = 3\sigma^2 + \frac{3}{2}\dot{\phi}^2 + 3V(\phi) - \frac{3}{2}P, \quad (\text{A4})$$

where σ is the shear scalar, P is the scalar curvature of the homogeneous hypersurfaces, which is always negative except in the Bianchi IX case, and $V(\phi)$ is given by Eq. (A2). The Klein-Gordon equation for the scalar field with an exponential potential is then

$$\ddot{\phi} + \theta\dot{\phi} + kV(\phi) = 0. \quad (\text{A5})$$

We now introduce new expansion-normalized variables and a new time variable as follows:

$$\begin{aligned} \beta &= \sqrt{3}\frac{\sigma}{\theta}, & \frac{dt}{d\tau} &= \frac{3}{\theta}, \\ \Psi &= \frac{\sqrt{6}\dot{\phi}}{2\theta}, & \Phi &= \sqrt{3V_0}\frac{e^{k\phi/2}}{\theta}. \end{aligned} \quad (\text{A6})$$

With these definitions, Eqs. (A3)–(A5) can be rewritten as

$$\Psi' = -\Psi(2 - 2\beta^2 - 2\Psi^2 + \Phi^2) - \frac{\sqrt{6}k}{2}\Phi^2, \quad (\text{A7})$$

$$\Phi' = -\Phi(-1 - 2\beta^2 - 2\Psi^2 + \Phi^2 - \frac{\sqrt{6}k}{2}\Psi), \quad (\text{A8})$$

where $'$ denotes differentiation with respect the new time τ . There is an additional differential equation for β , which depends on the particular Bianchi model under consideration. The equilibrium points of the system have either $\Phi = \Psi = 0$, which corresponds to the massless scalar field case, $\beta^2 + \Psi^2 = 1$, $\Phi = 0$, which represents the Bianchi type I initial (line) singularity, or else (and in all cases of interest here) obey the following relation:

$$\Phi^2 + \Psi^2 = -\frac{\sqrt{6}}{k}\Psi. \quad (\text{A9})$$

2. A class of anisotropic cosmological models

The diagonal form of the Bianchi type VI_h metric is given by²¹

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 e^{2mx} dy^2 + c(t)^2 e^{2x} dz^2, \quad (\text{A10})$$

where $m = h - 1$. If $m = 1$, then the metric is of Bianchi type V, if $m = 0$, then the metric is of Bianchi type III, and if $m = -1$, then the metric is of Bianchi type VI_0 . Thus, we are considering a

1-parameter (m) class of Bianchi models which include Bianchi types III ($m = 0$), V ($m = 1$), VI₀ ($m = -1$), and VI _{h} (all other m).

The expansion scalar (A11), which determines the volume behaviour of the fluid, is given by

$$\theta = \frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c} \quad (\text{A11})$$

(where an over-dot denotes differentiation with respect to the proper time). The shear tensor, σ_{ab} , determines the distortion arising in the fluid flow leaving the volume invariant. The nonzero components of the shear tensor are

$$\begin{aligned} \sigma_{11} &= \frac{a^2}{3} \left(2\frac{\dot{a}}{a} - \frac{\dot{b}}{b} - \frac{\dot{c}}{c} \right), \\ \sigma_{22} &= \frac{b^2 e^{2mx}}{3} \left(2\frac{\dot{b}}{b} - \frac{\dot{a}}{a} - \frac{\dot{c}}{c} \right), \\ \sigma_{33} &= \frac{c^2 e^{2x}}{3} \left(2\frac{\dot{c}}{c} - \frac{\dot{a}}{a} - \frac{\dot{b}}{b} \right), \end{aligned} \quad (\text{A12})$$

and the shear scalar, $\sigma^2 \equiv \frac{1}{2} \sigma^{ab} \sigma_{ab}$, is given by

$$\sigma^2 = \frac{1}{3} \left[\left(\frac{\dot{a}}{a} \right)^2 + \left(\frac{\dot{b}}{b} \right)^2 + \left(\frac{\dot{c}}{c} \right)^2 - \frac{\dot{a}\dot{b}}{ab} - \frac{\dot{a}\dot{c}}{ac} - \frac{\dot{b}\dot{c}}{bc} \right]. \quad (\text{A13})$$

In the case under consideration here, there is no rotation and no acceleration.

For a scalar field with an exponential potential, the Einstein FE can be written down explicitly.²¹ We obtain the generalized Friedmann equation

$$\theta^2 = 3\sigma^2 + \frac{3}{2}\dot{\phi}^2 + 3V_0 e^{k\phi} + \frac{3}{a^2}(m^2 + m + 1). \quad (\text{A14})$$

Note that $m^2 + m + 1 \geq 3/4 > 0$. The Raychaudhuri equation (Eq. (A3)) is

$$\dot{\theta} = -2\sigma^2 - \frac{1}{3}\theta^2 - \dot{\phi}^2 + V_0 e^{k\phi}. \quad (\text{A15})$$

The evolution equation for the shear is

$$\dot{\sigma} = -\sigma\theta + \frac{(1-m)}{3\sqrt{3}\sqrt{m^2+m+1}} (\theta^2 - 3\sigma^2 - \frac{3}{2}\dot{\phi}^2 - 3V_0 e^{k\phi}). \quad (\text{A16})$$

The Klein-Gordon equation for the scalar field (Eq. (A5)) is

$$\ddot{\phi} = -\theta\dot{\phi} - kV_0 e^{k\phi}. \quad (\text{A17})$$

The system of differential equations in the expansion-normalized variables becomes

$$\frac{d\beta}{d\tau} = \beta(q-2) + \frac{1-m}{\sqrt{m^2+m+1}} (1 - \beta^2 - \Psi^2 - \Phi^2), \quad (\text{A18})$$

$$\frac{d\Psi}{d\tau} = \Psi(q-2) - \frac{\sqrt{6}k}{2}\Phi^2, \quad (\text{A19})$$

$$\frac{d\Phi}{d\tau} = \Phi(1+q) + \frac{\sqrt{6}k}{2}\Psi\Phi, \quad (\text{A20})$$

and the decoupled evolution equation for the expansion

$$\frac{d\theta}{d\tau} = -\theta(1+q), \quad (\text{A21})$$

where the deceleration parameter, q , is defined by

$$q = 2\beta^2 + 2\Psi^2 - \Phi^2. \quad (\text{A22})$$

The domain of interest (determined by Eq. (A14)) is the region defined by

$$\beta^2 + \Psi^2 + \Phi^2 \leq 1, \quad (\text{A23})$$

which describes the surface and interior of a sphere in the (reduced) phase space (β, Ψ, Φ) . Inflation in the context of this paper is defined to occur whenever the deceleration parameter is negative; i.e., $q < 0$.

In this brief review we have not included ordinary matter (satisfying the usual energy conditions). However, matter can be easily included (see Ref. 14).

3. Equilibrium points and qualitative behaviour

The equilibrium point

$$\left\{ \beta = 0, \Psi = -k/\sqrt{6}, \Phi = \sqrt{1 - k^2/6} \right\}, \quad (\text{A24})$$

does not exist if $k^2 > 6$ and is part of the non-isolated line of equilibrium points $\beta^2 + \Psi^2 = 1$ when $k^2 = 6$. The point lies on the boundary of the phase space $\beta^2 + \Psi^2 + \Phi^2 = 1$ and hence it corresponds to a model with zero curvature. The point is inflationary if $q = (k^2 - 2)/2 < 0$; i.e., if $k^2 < 2$. The eigenvalues are: $\lambda_1 = \lambda_2 = (k^2 - 6)/2$, $\lambda_3 = k^2 - 2$. If $k^2 < 2$ the point is therefore a sink, and if $2 < k^2 < 6$, then the point is a saddle point. For $k \neq 0$ the exact solution corresponding to this equilibrium point is that of a flat FRW model. Note: $k^2 \leq 2$ is a necessary condition for the homogeneous models under consideration to isotropize,¹⁷ and for $k^2 < 2$ these models will inflate; the power-law inflationary FRW solution is the unique attractor for any initially expanding Bianchi model.

The equilibrium point

$$\left\{ \beta = (1 - m)/(2\sqrt{m^2 + m + 1}), \Psi = 0, \Phi = 0 \right\}, \quad (\text{A25})$$

satisfies the boundary condition, Eq. (A23), for all m , and when $m = -1$ the point is part of the non-isolated line of equilibrium points $\beta^2 + \Psi^2 = 1$. The inflationary condition $q < 0$ is never satisfied and hence this point is non-inflationary. The eigenvalues are: $\lambda_1 = \lambda_2 = -\lambda_3 = -3(m + 1)^2/(2(m^2 + m + 1))$. It is easily seen that this point is a saddle point with a two-dimensional stable manifold. The exact solution corresponding to this point is that of a vacuum Bianchi type VI_h model or one of its degeneracies.

There is also an equilibrium point with all of β, Ψ, Φ non-zero when $k^2 \geq 2$, and the corresponding solution is non-inflationary when the point exists inside the physical phase space given by Eq. (A23). It can be shown that if $k^2 > 2$, all three eigenvalues are negative and hence the equilibrium point is a stable attractor. It is also interesting to note that if $k^2 > 2 + 4(m^2 + 1)/(7m^2 - 2m + 7)$, then the equilibrium point is a focus. The exact solution corresponding to this point is that of a Bianchi type VI_h model or one of its degeneracies. Thus for $k^2 > 2$, this non-isotropic and non-inflationary solution is a stable attractor for the type III and VI_h cases. When $m = 1$, the corresponding isotropic solution represents the future asymptotic attractor for the Bianchi type VII_h models as well as the asymptotic attractor for the Bianchi type V models.

In the full three-dimensional phase space the ring of equilibrium points ($\beta^2 + \Psi^2 = 1, \Phi = 0$) for $k^2 < 6$ is a global source, and for $k^2 > 6$ we find that some part of the ring acts like a source and the remaining part of the ring acts like a saddle. The equilibrium points corresponding to sources (part of the past attractor) consist of a subset of the Jacobs Disc, which correspond to exact self-similar *Jacobs* stiff perfect fluid (or massless scalar field) solutions.¹⁴ Therefore, the models are asymptotic in the past to an analogue of the (non-oscillatory) *Jacobs* anisotropic Bianchi I non-vacuum solution.

APPENDIX B: GENERAL MODEL

Let us consider the scalar potential

$$V(\theta, \phi, \sigma) = V_0 e^{-\sqrt{6}\lambda\phi} + \frac{a}{\sqrt{3}} \sqrt{V_0} \theta e^{-\frac{\sqrt{6}}{2}\lambda\phi} + \sqrt{2}b\sqrt{V_0}\sigma e^{-\frac{\sqrt{6}}{2}\lambda\phi}, \quad (\text{B1})$$

where we have defined $\bar{\lambda} = \lambda/\sqrt{6}$, and we have normalized the constants a and b appropriately. We could also discuss the more general potential $V(\theta, \phi, \sigma) = V_0 \exp[-\lambda\phi] + a_{r,s} \theta^r \sigma^s \exp[(r+s-2)\lambda\phi/2]$.

In general, we have that

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V - V_\theta(\theta + \sqrt{6}B_1\sigma) - V_\sigma(B_2\sigma - \frac{1}{\sqrt{6}}\theta), \quad (\text{B2})$$

and an appropriate expression for p_ϕ in terms of the arbitrary parameters B_1 and B_2 (where $B_1 = 1$ and $B_2 = 1$ for the anisotropic models above). We then obtain

$$\beta' = (q-2)\beta - \frac{\sqrt{2}M}{c^2}K, \quad (\text{B3})$$

$$\Psi' = (q-2)\Psi + \frac{3\bar{\lambda}}{2}(2\Phi^2 + a\Phi + b\beta\Phi), \quad (\text{B4})$$

$$\Phi' = (q+1)\Phi - 3\bar{\lambda}\Psi\Phi, \quad (\text{B5})$$

where

$$K = \frac{c^2}{2}\beta^2 + \Omega - 1, \quad (\text{B6})$$

$$\Omega \equiv \Psi^2 + \Phi^2 - aB_1\Phi\beta - b(B_2-1)\Phi\beta + b\Phi, \quad (\text{B7})$$

$$q-2 = \frac{K}{b\Phi-2}[(aB_1 + (B_2-1)b)\frac{\sqrt{2}M}{c^2}\Phi - 4] + \frac{\Phi}{b\Phi-2}[3(\bar{\lambda}\Psi - 1)(aB_1 + bB_2)\beta + 3(a-b)\bar{\lambda}\Psi + 3b(\beta+1) + 6\Phi], \quad (\text{B8})$$

The system (B3)–(B5), with arbitrary parameters (including B_1 and B_2), can be regarded as a phenomenological system of equations for an anisotropic cosmological model with a particular self-interaction potential $V(\theta, \phi, \sigma)$.

We note that when $K = 0$, from (B3) the equilibrium points must satisfy $\beta = 0$ or $q = 2$. In the latter case we have $\Phi = 0$ and $\Psi^2 + \frac{c^2}{2}\beta^2 = 1$, which will include the cosmological sources. When $\beta = 0$ we will find equilibrium points (with $\Phi = 0$ and Ψ non-zero) corresponding to inflationary sinks.

1. 2D system with zero curvature

From Eqs. (B3)–(B5), we can compute the evolution equation for the auxiliary variable K ; by setting $K = 0$ in this expression we obtain $K' = 0$, so that $K = 0$ is an invariant set of the system. We recall that we can obtain the Bianchi I models (with $K = 0$) when $M = N = 0$. When $K = 0$ we obtain

$$\beta' = (q-2)\beta, \quad (\text{B9})$$

$$\Phi' = (q+1)\Phi - 3\bar{\lambda}\Psi\Phi, \quad (\text{B10})$$

where

$$\Psi^2 = 1 - \frac{c^2}{2}\beta^2 - \Phi^2 + (B_2-1)b\beta\Phi + aB_1\beta\Phi - b\Phi, \quad (\text{B11})$$

$$q-2 = \frac{3\Phi}{b\Phi-2}[(\bar{\lambda}\Psi - 1)(aB_1 + bB_2)\beta + (a-b)\bar{\lambda}\Psi + b(\beta+1) + 2\Phi]. \quad (\text{B12})$$

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