Structure Preserving Iterative Solution of Periodic Projected Lyapunov Equations

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Abstract: We discuss the Smith iteration for solving large-scale sparse projected discrete-time periodic Lyapunov equations which arise in periodic state feedback problems and in model reduction of periodic descriptor systems. Two algorithms are presented in this paper. The first one works with the cyclic lifted representation of the corresponding projected discrete-time periodic Lyapunov equations. In this algorithm, the block diagonal structure of the periodic solution is preserved in every iteration step by efficient permutations. The second algorithm works directly with the periodic matrix coefficients. We analyze the cyclic structure of the matrices arising in the iterative computations of the periodic solutions of the projected discrete-time periodic Lyapunov equations. A low-rank version of this method is also presented, which can be used to compute low-rank approximations to the solutions of projected periodic Lyapunov equations. Numerical results are given to illustrate the efficiency and accuracy of the proposed methods.

Keywords: Periodic systems, discrete-time systems, time-varying systems, Lyapunov equation, numerical methods, Smith iteration, control system analysis,

1. INTRODUCTION

Periodic systems and control theory is of interest in various scientific and engineering fields, specially in the aerospace realm, control of industrial processes and communication systems, modeling of periodic time-varying filters and networks, and several man-made and natural phenomena. Simulations and analysis of such systems can be unacceptably expensive and time-consuming when the systems are very large. Hence, model reduction is an efficient tool which helps the scientists and engineers to replace the large periodic models by smaller models which are amenable to fast and efficient simulation and which still preserve the input-output behavior of the original large models as good as possible.

We consider the linear periodic time-varying (LPTV) descriptor system

$$\begin{aligned} E_k x_{k+1} &= A_k x_k + B_k u_k, \\ y_k &= C_k x_k, \qquad k = 0, 1, \ldots, K-1, \end{aligned} \tag{1}$$

where $E_k, A_k \in \mathrm{R}^{n\times n}$, $B_k \in \mathrm{R}^{n\times m}$, $C_k \in \mathrm{R}^{p\times n}$, $x_k \in \mathrm{R}^n$ is the state or descriptor vector, $u_k \in \mathrm{R}^m$ is the control input, and $y_k \in \mathrm{R}^p$ is the output. The coefficient matrices are periodic with a period $K \geq 1$. The matrices E_k are allowed to be singular for all k. Such systems have received a lot of attention in the last 30 years, e.g., in satellite attitude control, helicopter design, harmonic balance methods in nonlinear circuit design, and many more, see, e.g., (Bittanti and Colaneri (2000); Fliege (1994); Lovera et al. (2002); Varga (1999)).

1.1 Preliminaries

Stability analysis and model reduction of LPTV descriptor systems (1) are strongly related to the generalized projected periodic discrete-time algebraic Lyapunov equations (PPDALEs)

$$\begin{aligned} A_k X_k A_k^T - E_k X_{k+1} E_k^T &= Q_l(k) B_k B_k^T Q_l(k)^T, \\ X_k &= Q_r(k) G_k Q_r(k)^T, \end{aligned} \tag{2}$$

where $X_K = X_0$, and $Q_l(k), Q_r(k)$, for $k = 0, 1, \ldots, K-1$, are the spectral projectors onto the k-th left and right deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the eigenvalue at infinity (Benner et al. (2011b); Chu et al. (2007); Stykel (2008)). This type of equations arises in the context of periodic state feedback problems and in model reduction of periodic descriptor systems when the solutions of the noncausal matrix equations associated with the systems (Kuo et al. (2004); Benner et al. (2011a); Chu et al. (2007)) are sought. Note that in that case $Q_l(k) = I_n - P_l(k)$ and $Q_r(k) = I_n - P_r(k)$, where $P_l(k), P_r(k)$ are the spectral projectors onto the k-th left and right deflating subspaces of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ corresponding to the finite eigenvalues.

Assume that the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ are periodic stable (pd-stable), i.e., all finite eigenvalues of the set of periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K}$ lie inside the unit circle. In this case the matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ can be transformed into a periodic Kronecker canonical form

(Sreedhar and Van Dooren (1994); Van Dooren (1979)), for $k = 0, 1, \ldots, K-1$,

$$U_k E_k V_{k+1} = \begin{pmatrix} I & 0 \\ 0 & E_k^b \end{pmatrix}, \; U_k A_k V_k = \begin{pmatrix} A_k^f & 0 \\ 0 & I \end{pmatrix}, \qquad (3)$$

where U_k, V_k are nonsingular, $V_K = V_0$, $A_{k+K-1}^f A_{k+K-2}^f \cdots A_k^f = J_k$ is an $n_f \times n_f$ Jordan matrix corresponding to finite eigenvalues, $E_k^b E_{k+1}^b \cdots E_{k+K-1}^b = N_k$ is an $n_\infty \times n_\infty$ nilpotent Jordan matrix corresponding to infinite eigenvalues, and $n = n_f + n_\infty$. Using (3) the spectral projectors $P_l(k)$ and $P_r(k)$, for $k = 0, 1, \ldots, K-1$, can be represented as

$$P_l(k) = U_k^{-1} \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} U_k, \quad P_r(k) = V_k \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix} V_k^{-1}. \qquad (4)$$

In this paper we discuss the structure preserving Smith iterative method to solve large-scale sparse PPDALEs (2) using their corresponding lifted structure. This contribution builds upon Section-4 in (Benner et al. (2011a)), where the concept for preserving the block diagonal structure of the computed solution at each iteration step is based on (Kressner (2003)).

The rest of the paper is organized as follows. In Section 2, we briefly review the direct methods that have been considered in (Chu et al. (2007); Benner et al. (2011b)) to solve the PPDALEs (2). The computational complexities and the drawbacks of these methods for large-scale sparse problems are also discussed. We then discuss the Smith iterative method to solve the PPDALEs (2) using their corresponding lifted structure and describe briefly the structure preserving technique at each Smith iteration by a choice of an efficient permutation matrix in Section 3. Two algorithms are presented. Low-rank versions of these methods are also presented that can be used to compute low-rank approximations to the solutions of PPDALEs in lifted form with low-rank right-hand side. In Section 4, we report some results of numerical experiments to illustrate the efficiency and accuracy of the proposed methods. A short conclusion is given in Section 5.

2. ANALYSIS OF NUMERICAL ALGORITHMS FOR PPDALES

The numerical solution of (2) has been considered in (Chu et al. (2007)) for time-varying matrix coefficients. The method proposed there extends the periodic Schur method (Bojanczyk et al. (1992); Varga (2004)) and the generalized Schur-Hammarling method (Stykel (2002)) developed for periodic standard and projected generalized Lyapunov equations, respectively. This method is based on an initial reduction of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^{K-1}$ to the generalized periodic Schur form (Kressner (2001); Varga (2004)) and on solving the resulting generalized periodic Sylvester and Lyapunov equations of (quasi)-triangular structure using the recursive blocked algorithms (Granat et al. (2007)). Computing the Kronecker-like canonical forms of the periodic matrix pairs and solving the resulting periodic Sylvester equations are the most computationally expensive tasks in this algorithm (Algorithm 5.1 of Chu et al. (2007)).

An efficient approach which works with the lifted form of (2) has been considered in (Benner et al. (2011b)). The method proposed there works with the cyclic lifted representation of (1) and the corresponding lifted form of (2). Following the work of (Benner et al. (2011b)), the PPDALE (2) is equivalent to the following projected lifted discrete-time algebraic Lyapunov equation (PLDALE)

$$\mathcal{A}\mathcal{X}\mathcal{A}^T - \mathcal{E}\mathcal{X}\mathcal{E}^T = \mathcal{Q}_l \mathcal{B}\mathcal{B}^T \mathcal{Q}_l^T, \quad \mathcal{X} = \mathcal{Q}_r \mathcal{X} \mathcal{Q}_r^T, \qquad (5)$$

where

$$\mathcal{E} = \operatorname{diag}(E_0, E_1, \ldots, E_{K-1}), \; \mathcal{B} = \operatorname{diag}(B_0, B_1, \ldots, B_{K-1}),$$

$$\mathcal{A} = \begin{pmatrix} 0 & \cdots & 0 & A_0 \\ A_1 & & & 0 \\ & \ddots & & \vdots \\ 0 & & A_{K-1} & 0 \end{pmatrix}, \qquad (6)$$

and

$$\begin{aligned} \mathcal{X} &= \operatorname{diag}(X_1, \ldots, X_{K-1}, X_0), \\ \mathcal{Q}_l &= \operatorname{diag}(Q_l(0), Q_l(1), \ldots, Q_l(K-1)), \\ \mathcal{Q}_r &= \operatorname{diag}(Q_r(1), \ldots, Q_r(K-1), Q_r(0)). \end{aligned} \qquad (7)$$

All these methods are suitable for problems of small and medium size. In practice, one should avoid these direct methods for large-scale problems because the computational complexity for solving a Lyapunov equation of the form (2) or (5) using direct methods is at least of order $\mathcal{O}(Kn^3)$, and they require extensive storage. Therefore, we develop iterative methods for such equations, which can exploit the sparse structures of system matrices to generate well approximating solutions (with prescribed tolerance), and have low memory requirements and low computational cost.

3. STRUCTURE PRESERVING SOLUTION OF PLDALES

In the last few years, increasing attention has been devoted to the numerical solution of large-scale sparse Lyapunov equations using iterative approaches, such as the alternating directions implicit (ADI) method (Li and White (2002); Penzl (2000)), the Smith method (Gugercin et al. (2003); Penzl (2000); Smith (1968)), and Krylov subspace methods (Jbilou and Riquet (2006); Simoncini (2007)). For an overview and further references, see Benner et al. (2008). All these methods have also been generalized to projected Lyapunov equations (Stykel (2008); Stykel and Simoncini (2012)). On the other hand, an extension of the Smith method and the Krylov subspace method based on a block Arnoldi algorithm to standard periodic Lyapunov equations has been presented in (Kressner (2003)). Unfortunately, these methods cannot be directly applied to the projected periodic Lyapunov equations. The Smith method for the projected periodic Lyapunov equation has been considered in (Benner et al. (2011a); Hossain (2011)), where the Smith iteration does not preserve the block diagonal structure at every iteration, but the approximate Gramians are block diagonal at each iteration step (see Algorithm 3 of Benner et al. (2011a)). In this paper we show that block diagonal structure at each Smith iteration can be preserved by efficient permutations.

Consider now the PLDALE (5). For nonsingular $\mathcal{A}$, this equation is equivalent to the PLDALE

$$\begin{aligned} \mathcal{X} - (\mathcal{A}^{-1}\mathcal{E})\mathcal{X}(\mathcal{A}^{-1}\mathcal{E})^T &= \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T, \\ \mathcal{X} &= \mathcal{Q}_r\mathcal{X}\mathcal{Q}_r^T. \end{aligned} \tag{8}$$

In this case the relation $\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ holds (Stykel (2008)) and such an equation can be solved by the Smith method (Smith (1968)) given by

$$\begin{aligned} \mathcal{X}_1 &= \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T, \\ \mathcal{X}_i &= \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T + (\mathcal{A}^{-1}\mathcal{E})\mathcal{X}_{i-1}(\mathcal{A}^{-1}\mathcal{E})^T. \end{aligned}$$

Note that $\mathcal{Q}_r$ is the spectral projector onto the invariant subspace of the matrix $\mathcal{A}^{-1}\mathcal{E}$ corresponding to the zero eigenvalues. Then $\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{E} = \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r$ is nilpotent with the nilpotency index ν, where ν is the index of the periodic descriptor system (1). In this case, after ν iterations we obtain

$$\mathcal{X}_\nu = \sum_{i=0}^{\nu-1}(\mathcal{A}^{-1}\mathcal{E})^i\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{B}^T\mathcal{A}^{-T}\mathcal{Q}_r^T((\mathcal{A}^{-1}\mathcal{E})^T)^i = \mathcal{X}. \tag{9}$$

Therefore, the Cholesky factor $\mathcal{R}$ of the solution $\mathcal{X} = \mathcal{R}\mathcal{R}^T$ of (8) and also of the PLDALE (5) takes the form

$$\begin{aligned} \mathcal{R} = [\, &\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}, \ldots, \\ &(\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}]. \end{aligned} \tag{10}$$

Note that the generalized Smith iteration discussed above does not preserve the block diagonal structure at every iteration step in the computation of the Cholesky factor $\mathcal{R}$ (see Algorithm 3 of Benner et al. (2011a), and Algorithm 9.3 of Hossain (2011)), but the approximate Gramian $\mathcal{X}_i = \mathcal{R}_i\mathcal{R}_i^T$ computed there has block diagonal structure at each iteration step, $i = 0, 1, \ldots, \nu-1$. By introducing a cyclic permutation matrix in each iteration step, one can easily preserve the block diagonal structure at every iteration step in the computation of the Cholesky factor $\mathcal{R}$ using relation (10).

Let $\mathcal{P}$ be the cyclic permutation matrix of the form

$$\mathcal{P} = \begin{bmatrix} 0 & \cdots & 0 & I_n \\ I_n & & & 0 \\ & \ddots & & \vdots \\ 0 & & I_n & 0 \end{bmatrix};\ \mathcal{P}_i = \mathcal{P}^i;\ i = 1, 2, \ldots, \nu. \tag{11}$$

We introduce a permutation matrix $\mathcal{P}_i$ for each iteration step i into the computation of (9) where the permutation matrix $\mathcal{P}_i$ changes at each iteration step in a cyclic manner by a forward block-row shift. For an example, suppose that $K = 3$, and $k = 0, 1, 2$. Then for $k = 1$, $\mathcal{P}_1 = \mathcal{P}$ is given by (11). For $k = 2$, we get

$$\mathcal{P}_2 = \mathcal{P}^2 = \begin{bmatrix} 0 & I_n & 0 \\ 0 & 0 & I_n \\ I_n & 0 & 0 \end{bmatrix},$$

which is just a forward shift of the last block-row of $\mathcal{P}$ in (11). Clearly $\mathcal{P}_0$ is the identity matrix of order $n \times K$. One can also prove this using the proposed relation as

$$\mathcal{P}_0 = \mathcal{P}_3 = \mathcal{P}^3 = \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & I_n \end{bmatrix},$$

which is nothing but a forward shift of the last block-row of $\mathcal{P}_2$. One nice property of this permutation matrix is that it satisfies the periodicity property, i.e., $\mathcal{P}_{K+k} = \mathcal{P}_k,\ k = 0, 1, \ldots, K-1$.

Hence (9) has the new form

$$\begin{aligned} \mathcal{X}_\nu = \sum_{i=0}^{\nu-1}(\mathcal{A}^{-1}\mathcal{E})^i\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}_{i+1}\mathcal{P}_{i+1}^T\mathcal{B}^T\mathcal{A}^{-T} \\ \cdot\ \mathcal{Q}_r^T((\mathcal{A}^{-1}\mathcal{E})^T)^i = \mathcal{X}. \end{aligned} \tag{12}$$

Therefore, the Cholesky factor $\mathcal{R}$ has the form

$$\begin{aligned} \mathcal{R} = [\, &\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}, \mathcal{A}^{-1}\mathcal{E}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}^2, \ \ldots, \\ &(\mathcal{A}^{-1}\mathcal{E})^{\nu-1}\mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}\mathcal{P}^\nu\,]. \end{aligned} \tag{13}$$

It can be verified that each factor inside (13) preserves the block diagonal structure analogous to the solution of (8). The computation of this factor is presented in Algorithm 1.

Algorithm 1. Generalized Smith method for PLDALE.
Input: $\mathcal{A}, \mathcal{E}, \mathcal{B}$, spectral projector $\mathcal{Q}_r$, cyclic permutation matrix $\mathcal{P}$.
Output: Low-rank factor R_ν such that $\mathcal{X} = \mathcal{R}_\nu\mathcal{R}_\nu^T$.

$W_1 = \mathcal{Q}_r\mathcal{A}^{-1}\mathcal{B}$
$Z_1 = W_1\mathcal{P}$
$\mathcal{R}_1 = Z_1$
FOR $i = 2, 3, \ldots, \nu$
$W_i = \mathcal{A}^{-1}\mathcal{E}W_{i-1}$
$Z_i = W_i\mathcal{P}_i$
$\mathcal{R}_i = [\mathcal{R}_{i-1},\ Z_i]$
END

If the index ν is unknown, then Algorithm 1 can be stopped as soon as the *residual norm* given by

$$\begin{aligned} \eta(\mathcal{R}_i) = \|&\mathcal{A}\mathcal{R}_i\mathcal{R}_i^T\mathcal{A}^T - \mathcal{E}\mathcal{R}_i\mathcal{R}_i^T\mathcal{E}^T \\ &+\mathcal{Q}_l\mathcal{B}\mathcal{B}^T\mathcal{Q}_l^T\|_F \end{aligned} \tag{14}$$

satisfies the condition $\eta(\mathcal{R}_i) < tol$ with a user-defined tolerance *tol* or a stagnation of residual norms is observed.

Fact 1. In order to guarantee that the second equation in (5) and also in (8) is satisfied in finite precision arithmetic, we have to project W_i onto the image of $\mathcal{Q}_r$ by pre-multiplication with $\mathcal{Q}_r$.

The generalized Smith iteration preserves the block diagonal structure at every iteration step in Algorithm 1. Clearly, at the ith iteration step $\mathcal{R}_i$ has the block diagonal structure $\mathcal{R}_i = \mathrm{diag}(R_{1,i}, \ldots, R_{K-1,i}, R_{0,i})$, where $R_{k,i}$ stand for the periodic Cholesky factors of $X_{k,i} = R_{k,i}R_{k,i}^T$ for different values of k ($k = 0, 1, \ldots, K-1$) at the ith iteration step. Since $\mathcal{X} = \mathcal{R}_\nu\mathcal{R}_\nu^T$, where $\mathcal{R}_\nu = \mathrm{diag}(R_1, \ldots, R_{K-1}, R_0)$, one can easily read off the periodic solutions $X_k = R_kR_k^T$ of (2) from the block diagonal structure of $\mathcal{R}_i$ for different values of k.

4. STRUCTURE PRESERVING SMITH METHOD FOR PPDALES

In fact, the iteration (13) implemented in Algorithm 1 not only proves that the computed Cholesky factors $\mathcal{R}_i$ stay block diagonal at each iteration step i, it also enables

us to rewrite (10) in such a way that one can directly compute the periodic Cholesky factors for different k, $k = 0, 1, \ldots, K-1$. From simple algebraic manipulation of (13), we observe that the periodic matrices E_k, A_k, and B_k appear in a cyclic manner in the computation of the periodic Cholesky factors $R_{k,i}$ in every iteration step i for different values of k, $k = 0, 1, \ldots, K-1$. Observing these cyclic relations and handing them technically, we can compute the periodic Cholesky factors $R_{k,i}$, $k = 0, 1, \ldots, K-1$, $i = 1, 2, \ldots$, directly. We represent some of those computations in the following.

For $i = 1$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,1} &= Q_r(0)A_0^{-1}B_0 \\ R_{1,1} &= Q_r(1)A_1^{-1}B_1 \\ &\vdots \\ R_{K-1,1} &= Q_r(K-1)A_{K-1}^{-1}B_{K-1}. \end{aligned}$$

For $i = 2$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,2} &= A_0^{-1}E_0Q_r(1)A_1^{-1}B_1 \\ R_{1,2} &= A_1^{-1}E_1Q_r(2)A_2^{-1}B_2 \\ &\vdots \\ R_{K-1,2} &= A_{K-1}^{-1}E_{K-1}Q_r(K)A_K^{-1}B_K. \end{aligned}$$

For $i = 3$ and $k = 0, 1, \ldots, K-1$, we get

$$\begin{aligned} R_{0,3} &= A_0^{-1}E_0A_1^{-1}E_1Q_r(2)A_2^{-1}B_2 \\ R_{1,3} &= A_1^{-1}E_1A_2^{-1}E_2Q_r(3)A_3^{-1}B_3 \\ &\vdots \\ R_{K-1,3} &= A_{K-1}^{-1}E_{K-1}A_K^{-1}E_KQ_r(K+1)A_{K+1}^{-1}B_{K+1}, \end{aligned}$$

and so on.

The whole computation is summarized in Algorithm 2. Note that in the above computations and also in Algorithm 2, we use the periodicity of the coefficient matrices and that of the projectors. Here $Q_r(K) = Q_r(0)$, $E_K = E_0$, etc. Clearly then $k = 1, 2, \ldots, K$, and $X_K = X_0 = R_KR_K^T = R_0R_0^T$. It should be also noted that in Algorithm 2, $R_{k,j}$ means the computed R_k at the jth iteration steps. Finally, $R_{k,j}^b$ collocates all these iterative counterparts for an individual k, where $k = 1, 2, \ldots, K$. That means for $k = 1$, we compute

$$R_{1,j}^b = [R_{1,1}, R_{1,2}, \ldots, R_{1,J}],$$

and similarly the others.

For an unknown ν, Algorithm 2 is to be stopped as soon as the *residual norm* given by

$$\begin{aligned} \rho_k \;=\; &\|A_kR_kR_k^TA_k^T - E_kR_{k+1}R_{k+1}^TE_k^T \\ &-Q_l(k)B_kB_k^TQ_l(k)^T\|_F \end{aligned} \tag{15}$$

satisfies the condition $\eta(\rho_k) < tol$ for $k = 0, 1, \ldots, K-1$. We assume that after the Jth iteration we have the exact computation of the periodic Cholesky factors R_k satisfying relation (15) and hence, $X_k = R_kR_k^T$ are the periodic solutions of (2) for $k = 0, 1, \ldots, K-1$.

Algorithm 2. Generalized Smith method for PPDALEs.
Input: (E_k, A_k, B_k), spectral projectors $Q_r(k)$ for $k = 1, \ldots, K$.
Output: Low-rank periodic Cholesky factor R_k such that $X_k = R_kR_k^T$.

```
FOR k = 1 : K
  R_{k,1} = Q_r(k) A_k^{-1} B_k        % note that R_{K,1} = R_{0,1}
END

FOR k = 1 : K
  P_{k,1} = I_n   % initialization of a cyclic matrix
END

FOR j = 2 : ν
  FOR k = 1 : K
    m = mod(j + 1, K)
    P_{k,j} = P_{k,j-1} A_{m+k}^{-1} E_{m+k}
    R_{k,j} = P_{k,j} Q_r(k + m + 1) A_{k+m+1}^{-1} B_{k+m+1}
  END
END

FOR k = 1 : K
  FOR j = 1 : ν
    IF j = 1
      R_{k,j}^b = R_{k,j}
    ELSE
      R_{k,j}^b = [R_{k,j-1}^b   R_{k,j}]
    END
  END
  R_k = RRQR(R_{k,j}^b, τ)
END
```

When the column ranks of the B_k matrices, i.e., m, are big, the situation can arise that $R_{k,j}^b$ may face rank deficiency, because each iteration step of Algorithm 2 will add m more columns to the previous computation of $R_{k,j}^b$. Therefore, we propose the rank-revealing QR decomposition (RRQR) (Golub and Van Loan (1996)) of $R_{k,j}^b$ with tolerance τ to truncate those columns that do not carry any additional information in the subsequent iteration steps. This truncation approach saves memory space and lowers the computational cost for further applications of the approximate solutions, e.g., in model order reduction. In (Benner and Quintana-Ortí (2005)), it is shown that a tolerance $\tau = \sqrt{\epsilon}$, where ϵ is the machine precision, is sufficient to achieve an error of the machine precision magnitude for the solutions X_k of (2) for $k = 0, 1, \ldots, K-1$.

Fact 2. In order to guarantee that the second equation in (2) is satisfied in finite precision arithmetic, we need to project $R_{k,j}$ onto the image of Q_r by pre-multiplication with Q_r.

5. NUMERICAL RESULTS

We consider a periodic discrete-time descriptor system with $n = 10$, $m = 2$, $p = 3$, and period $K = 3$ ($k = 0, 1, 2$) (Chu et al. (2007), Example 1). The system is of index 1 and it is pd-stable with $n_f = 8$ and $n_\infty = 2$ for $k = 0, 1, 2$. The finite eigenvalues of $\{(E_k, A_k)\}_{k=0}^2$ are shown in Fig. 1.

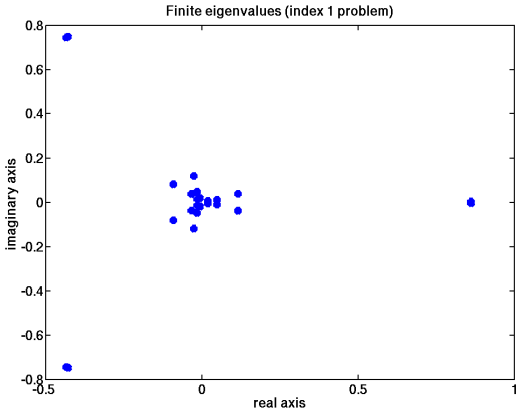

Fig. 1. Finite eigenvalues of $\{(E_k, A_k)\}_{k=0}^2$

For an index 1 problem, Algorithm 1 needs only one iteration. We compute the residuals norm using relation (14) which is zero in that case. Using Algorithm 2 we compute the periodic solutions of (2) directly. Norms of the computed solutions of the periodic Lyapunov equations and the corresponding residuals are shown in Table 1.

Table 1. Norms and relative residuals of periodic Gramians (index-1 problem)

k	$\| X_k \|_F$	ρ_k
0	1.675×10^1	0
1	1.672×10^1	0
2	1.675×10^1	0

We next consider an artificial problem of index 2. With $n = 10,\ m = 2,\ p = 3$, and period $K = 3$, we have, for $k = 0, 1, 2$,

$$E_k = \begin{bmatrix} I_8 & 0 \\ 0 & 0 \end{bmatrix}; \quad A_k = \begin{bmatrix} A_{u_k} & A_{l_k} \\ A_{l_k}^T & 0 \end{bmatrix},$$
$$B_k = \begin{bmatrix} 4 & -1 & s_3+1 & 1 & 0 & -2 & 0 & 1 & 0 & 0 \\ 1 & 0 & s_1+1 & -2 & 1 & -1 & 0 & -13 & 0 & 0 \end{bmatrix}, \tag{16}$$

where $\theta_k := 2\pi k/K$,

$$A_{u_k} = \begin{bmatrix} 1 & 0 & c_1 & s_1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -s_1 & c_1 & 0 & 0 & 0 & 0 \\ c_1 & -s_1 & 1 & 0 & c_2 & s_2 & 0 & 0 \\ s_1 & c_1 & 0 & 1 & -s_2 & c_2 & 0 & 0 \\ 0 & 0 & c_2 & -s_2 & 1 & 0 & c_3 & s_3 \\ 0 & 0 & s_2 & c_2 & 0 & 1 & s_3 & c_3 \\ 0 & 0 & 0 & 0 & c_3 & -s_3 & 1 & 0 \\ 0 & 0 & 0 & 0 & s_3 & c_3 & 0 & 1 \end{bmatrix}, \tag{17}$$

$$A_{l_k} = \begin{bmatrix} 0 & 0 & 2 & 0 & 1 & s_1 & 0 & 1 \\ 0 & 1 & s_3 & -1 & 0 & -1 & 0 & 1 \end{bmatrix},$$

and

$$c_1 = \cos(\theta_k),\ c_2 = 0.2c_1,\ c3 = 0.6c_1,$$
$$s_1 = \sin(\theta_k),\ s2 = 0.2s_1,\ s3 = 0.6s_1.$$

The computed eigenspectrum of the periodic matrix pairs $\{(E_k, A_k)\}_{k=0}^2$ consists of three pairs of finite eigenvalues lying inside the unit circle and two pairs of infinite eigenvalues, i.e., $n_f = 6$ and $n_\infty = 4$ for $k = 0, 1, 2$. The eigenspectrum of the finite eigenvalues of $\{(E_k, A_k)\}_{k=0}^2$

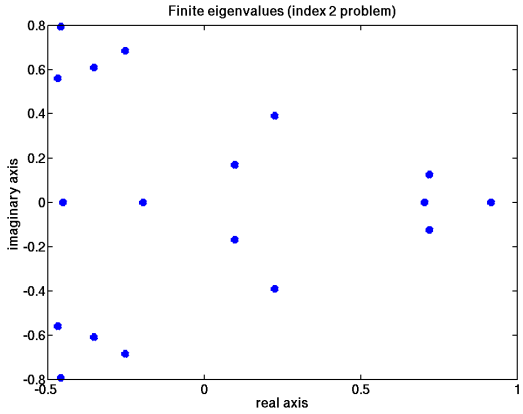

Fig. 2. Finite eigenvalues of $\{(E_k, A_k)\}_{k=0}^2$

is shown in Fig. 2.

Using Algorithm 2 we have computed the solutions of the periodic Lyapunov equations and the corresponding residuals. The computed numerical results in Table 2 show the accuracy of the proposed algorithm. Similar accuracy also holds in the computational results using Algorithm 1.

Table 2. Norms and relative residuals of periodic Gramians (index-2 problem)

k	$\| X_k \|_F$	ρ_k
0	6.5587×10^0	8.881×10^{-14}
1	4.6125×10^0	1.057×10^{-13}
2	6.3544×10^0	1.475×10^{-13}

6. CONCLUSION

In this paper, we have suggested iterative low-rank algorithms based on the Smith iteration for computing low-rank factors of the solutions of generalized projected periodic discrete-time algebraic Lyapunov equations. These factors can be used in a balanced truncation model reduction approach to find a reduced-order model for the periodic discrete-time descriptor system. The proposed algorithms are easily implementable for dimension-varying system matrices in (1).

It remains a task for further work to test the algorithms for real-world problems, and also for higher index problems.

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