H_∞-CONTROL OF A CONTINUOUS CRYSTALLIZER

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Abstract: Robustly stabilizing control of an open loop oscillatory crystallization process is considered. The crystallizer is described by a population balance model. From this distributed parameter model an irrational transfer function is obtained which has infinitely many poles and thus represents the infinite-dimensional nature of the system. An infinite-dimensional H_∞ controller synthesis method is applied to solve the weighted mixed sensitivity problem for this transfer function. This procedure results in an irrational controller. For practical implementation, the controller needs to be approximated by a rational transfer function. The effectiveness of the controller is demonstrated in simulations. *Copyright © 2000 IFAC*

Keywords: crystallization, population balance, control, infinite-dimensional system

1. INTRODUCTION

Crystallization is a widely used purification and separation process. Continuous industrial crystallizers have been reported to exhibit undesirable oscillatory behavior which may cause poor product quality (Randolph and Larson, 1988) (Rawlings *et al.*, 1993). Feedback control appears to be a promising way to improve the dynamical properties of such crystallization processes.

Crystal size distribution (CSD) is considered the most important process variable because it basically determines the product quality. Population balance models for crystallization processes describe the dynamic behavior of the CSD. This approach yields models that are distributed with respect to an internal coordinate (e.g. crystal length). Typically, population balance models consist of partial differential and integro-differential equations. The crystallizer model considered here includes the effects of nucleation, crystal growth, fines dissolution and classified product removal. The model exhibits oscillations of the CSD and solute concentration.

Control of oscillatory continuous crystallization processes is an area of active research. Regarding the model on which controller design is based, there are several approaches reported in the literature:

- No explicit plant model (for PID or self-tuning controller design), see e.g. (Redman *et al.*, 1995), (Randolph *et al.*, 1987).
- finite-dimensional model obtained by system identification, see (Eek, 1995), (Rohani *et al.*, 1999).
- Infinite-dimensional population balance model with subsequent reduction to a finite-dimensional approximation (Chiu and Christofides, 1999).

In this article, the idea of "late lumping" is pursued: The controller design is based on the distributed parameter model. The resulting infinite-dimensional controller is approximated by a finite-dimensional transfer function.

Solute concentration in the liquid phase of the crystallizer is considered the only measured variable, i.e. no direct information on the crystal size distribution is available for the controller. The controller acts on the crystallizer by manipulating the solute feed concentration. Approximate controllability of such systems was proven in (Semino

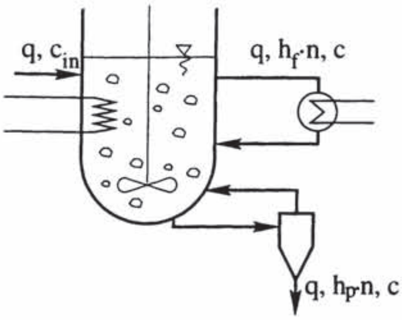

Fig. 1. Continuous Crystallizer

and Ray, 1995). The population balance model is linearized and an irrational transfer function from manipulated input to measured output is obtained. A robust performance problem is formulated based on this transfer function. This leads to an infinite-dimensional H_∞ mixed sensitivity problem, which is solved using a recent H_∞ controller synthesis method for infinite-dimensional single-input-single-output (SISO) plants (Foias *et al.*, 1996). Finally, the effectiveness of the controller is demonstrated in simulations.

This contribution is based on work previously presented in (Vollmer and Raisch, 1999) where control of a more restrictive crystallizer model was considered.

2. DESCRIPTION OF PROCESS AND MODEL

2.1 *The Model*

A crystallizer in continuous mode of operation with fines dissolution and classified product removal is sketched in Figure 1. The crystallization process is modeled under the following assumptions:

- ideal mixing
- isothermal operation
- constant overall volume (liquid + solid)
- nucleation of crystals at negligible size
- size-independent growth rate
- no particle breakage, attrition or agglomeration.

Thus, the population balance equation is obtained as

$$\frac{\partial n(L,t)}{\partial t} = -G(c)\frac{\partial n(L,t)}{\partial L} - \frac{q}{V}\left(h_f(L) + h_p(L)\right) n(L,t)$$
$$n(L,0) = n_0(L), \quad n(0,t) = \frac{B(c)}{G(c)} \qquad (1)$$

with the particle size distribution $n(L,t)$. The classification functions specifying the fines dissolution and product classification are given by

$$h_f(L) = R \cdot (1 - h(L - L_f))$$
$$h_p(L) = 1 + z \cdot h(L - L_p),$$

where $h(L)$ is the unit step function. The dependence of nucleation and growth rates on solute concentration $c(t)$ are expressed by the following empirical power laws:

$$G(c) = k_g\,(c(t) - c_s)^g$$
$$B(c) = k_b\,(c(t) - c_s)^b\,.$$

The mole balance of solute in the liquid phase is

$$M\frac{dc}{dt} = \frac{q(\rho - Mc)}{V} + \frac{\rho - Mc}{\varepsilon}\frac{d\varepsilon}{dt} + \frac{qMc_{in}}{V\varepsilon} - \frac{q\rho}{V\varepsilon}\left(1 + k_v\int_0^\infty (h_p(L) - 1) f(L,t) L^3 dL\right)$$
$$c(0) = c_0 \qquad (2)$$

with void fraction

$$\varepsilon = \frac{V_{liquid}}{V} = 1 - \frac{V_{solid}}{V} = 1 - \frac{\int_0^\infty f(L,t)L^3 dL}{V}.$$

A simulation result of this nonlinear distributed model with parameters referring to a laboratory scale KCl crystallizer is shown in Figures 2 and 3. Parameter values are given in Table 1. The feed concentration is kept constant at $c_f = 4.4\frac{\text{mol}}{\text{l}}$. The simulation starts close to the theoretically computed steady state. It can be seen that the steady state is unstable. The system exhibits sustained oscillations.

Table 1. Parameter Values

feed rate	q	0.05	$\frac{\text{l}}{\text{min}}$
total volume	V	10.5	l
fines removal cut size	L_F	0.2	mm
product class. cut size	L_P	1	mm
fines removal constant	R_1	5	-
product removal constant	R_2	2	-
growth rate constant	k_g	0.0305	$\frac{\text{mm l}}{\text{min mol}}$
growth rate exponent	g	1	-
nucleation rate constant	k_b	$8.36 \cdot 10^9$	$\frac{\text{l}^3}{\text{min mol}^4}$
nucleation rate exponent	b	4	-
KCl crystal density	ρ	1989	$\frac{\text{g}}{\text{l}}$
mole mass KCl	M	74.551	$\frac{\text{g}}{\text{mol}}$
volumetric shape factor	k_v	0.1112	-
saturation concentration	c_s	4.038	$\frac{\text{mol}}{\text{l}}$

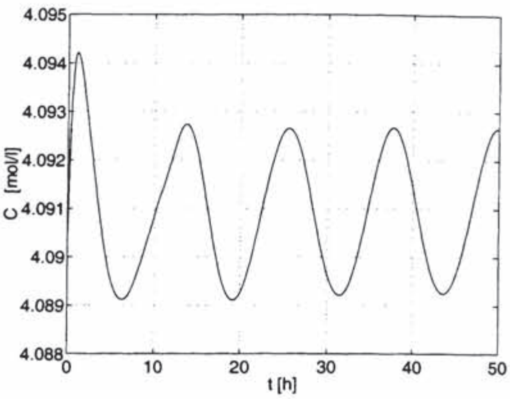

Fig. 2. Solute Concentration (Open Loop)

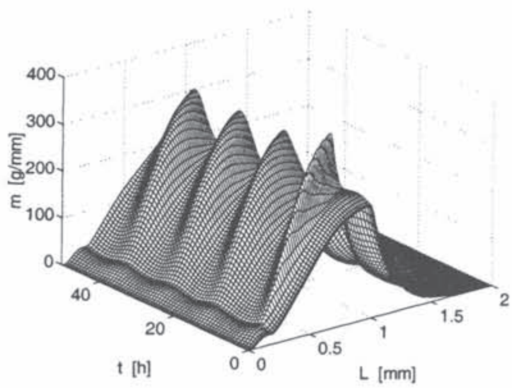

Fig. 3. Mass Density (Open Loop)

2.2 *Derivation of Transfer Function*

The model is linearized with respect to the steady state c_{ss}, $n_{ss}(L)$. Via Laplace-transformation, a transfer function from $\Delta C_f = \mathcal{L}\{c_f - c_{fss}\}$ to $\Delta C = \mathcal{L}\{c - c_{ss}\}$ is obtained:

$$P(s) = \frac{s(R+1+\frac{V}{q}s)^4(z+1+\frac{V}{q}s)^4(1+\frac{V}{q}s)^4}{D(s)}. \quad (3)$$

The denominator $D(s)$ is of the form:

$$D(s) = P_1(s) + e^{-\frac{L_f}{k_g(c_{ss}-c_s)^g}s}P_2(s) + e^{-\frac{L_p}{k_g(c_{ss}-c_s)^g}s}P_3(s) + e^{-\frac{L_p-L_f}{k_g(c_{ss}-c_s)^g}s}P_4(s),$$

where the $P_i(s)$ are polynomials. $D(s)$ is a quasi-polynomial having infinitely many zeros. Therefore, the transfer function $P(s)$ has infinitely many poles which reflects the fact that the system is infinite-dimensional.

3. CONTROLLER DESIGN

3.1 *Theory*

The objective of the controller design is to stabilize the system and to attenuate the effect of disturbances v on the output y. In order to make the controller robust with respect to errors in the model it is designed to guarantee stability and disturbance attenuation not only for the nominal plant model $P(s)$ but for a set of transfer functions containing $P(s)$:

$$\mathcal{P}_m = \{P(s)(1+\Delta_m(s)) : |\Delta_m(j\omega)| < |W_m(j\omega)|, \text{ the number of right half plane poles of } P(s) \text{ and } P(s)(1+\Delta_m(s)) \text{ is the same}\},$$

where $W_m(s)$ is a frequency dependent error bound. That means a multiplicative model error may be present as shown in Figure 4. The effect of the disturbance v on the output y is given by the transfer function

$$S_\Delta(s) = \frac{1}{1+P_\Delta(s)C(s)}$$

and it is required that

$$\|S_\Delta(s)W_d(s)\|_\infty \leq 1 \quad \forall P_\Delta \in \mathcal{P}_m.$$

It can be shown that a controller solves the robust performance problem if

$$\gamma(C) = \left\| \begin{bmatrix} W_d(s)S(s) \\ W_m(s)T(s) \end{bmatrix} \right\|_\infty \leq \frac{1}{\sqrt{2}}.$$

This is a H_∞ mixed sensitivity minimization problem.

The mixed sensitivity problem for infinite-dimensional SISO plants has been solved in (Foias *et al.*, 1996) using operator theoretic methods in the frequency domain. It has been shown that under certain assumptions this problem can be reduced to an eigen-value-eigen-vector problem for a Hankel+Toeplitz type operator for which the solution can be derived from a finite number of linear equations.

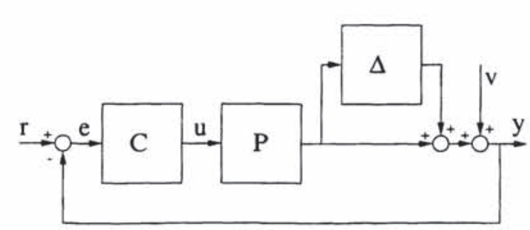

Fig. 4. System with Multiplicative Uncertainty

The theory is applicable if the following assumptions are met:

- The plant transfer function can be decomposed as

$$P(s) = \frac{M_n(s)N_1(s)N_2(s)}{M_d(s)}, \qquad (4)$$

where
  - $M_d \in H_\infty$ is rational inner
  - $M_n \in H_\infty$ is arbitrary inner
  - $N_1 \in H_\infty$ is outer and $N_1^{-1} \in H_\infty$
  - $N_2 \in H_\infty$ is rational outer.

This means in particular that the plant has finitely many unstable poles and for $\omega \to \infty$ the Bode plot of $P(s)$ has a constant roll off rate like a rational transfer function:

$$\frac{\mathrm{d}\lg|P(j\omega)|}{\mathrm{d}\lg\omega} = const.$$

- The weighting functions $W_m(s)$, $W_d(s)$, $W_d^{-1}(s) \in H_\infty$ are rational. Furthermore, $(W_m(s)N_2(s))^{-1} \in H_\infty$. I.e if $P(s)$ is strictly proper $W_m(s)$ has to be improper.

To obtain this factorization of the plant model, the unstable poles of $P(s)$ have to be computed, i.e. the right half plane zeros of a quasi-polynomial have to be determined. This was done by restricting the region within the right half plane where zeros can occur using an algorithm based on (Arunsawatwong, 1996). Then the poles were found by a direct search method.

If the above assumptions are met the minimizing controller is

$$C_{opt}(s) = \frac{E_{\gamma_{opt}}(s)M_d(s)F_{\gamma_{opt}}(s)L(s)}{N_1(s)N_2(s)(1 + M_n(s)F_{\gamma_{opt}}(s)L(s))},$$

with

$$E_\gamma(s) = \frac{W_d(-s)W_d(s)}{\gamma^2} - 1,$$

$$F_\gamma(s) = H_\gamma(s)\prod_{k=1}^{n_1}\frac{s+\eta_k}{s-\eta_k},$$

where $\eta_1 \cdots \eta_{n_1}$ are the poles of $W_d(s)$ and $H_\gamma(s)$ is the stable, minimum-phase transfer function determined by the spectral decomposition

$$H_\gamma(s)H_\gamma(-s) = \left(1 - \left(\frac{W_d(s)W_d(-s)}{\gamma^2} - 1\right)\left(\frac{W_m(s)W_m(-s)}{\gamma^2} - 1\right)\right)^{-1}.$$

$L(s) = \frac{L_1(s)}{L_2(s)}$ is rational and satisfies the following interpolation conditions:

$$\begin{aligned}
0 &= L_1(\beta_k) + M_n(\beta_k)F_\gamma(\beta_k)L_2(\beta_k)\\
0 &= L_1(\alpha_k) + M_n(\alpha_k)F_\gamma(\alpha_k)L_2(\alpha_k)\\
0 &= L_2(-\beta_k) + M_n(\beta_k)F_\gamma(\beta_k)L_1(-\beta_k)\\
0 &= L_2(-\alpha_k) + M_n(\alpha_k)F_\gamma(\alpha_k)L_1(-\alpha_k) \qquad (5)
\end{aligned}$$

where $\beta_1, \cdots, \beta_{2n_1}$ are the poles of $E_\gamma(s)$ and $\alpha_i, \cdots, \alpha_l$ are the unstable poles of the plant $P(s)$. This means that the right half plane zeros of the $M_d(s)$ term in the numerator of C_{opt} are cancelled within the controller and therefore do not cancel the unstable poles of the plant. The largest value for γ such that (5) has a non-trivial solution is the optimal performance cost γ_{opt}. Lower and upper bounds for γ_{opt} can be computed.

The expression for C_{opt} involves the irrational transfer functions M_n and N_1. Therefore, the optimal controller itself is irrational. For practical implementation, it needs to be approximated by a rational transfer function. This was done using a Fourier transform based approximation technique and balanced model reduction (Gu *et al.*, 1989).

The procedure described in this section is easily implementable on a computer. In fact, a Matlab implementation is available (Özbay, 1998) for the computation of C_{opt}, γ_{opt} if $P(s)$ is already decomposed according to (4).

3.2 *Controller Design for the Crystallizer*

For the model described in Section 2, a pair of unstable poles was found at $s_{1/2} = 0.99 \cdot 10^{-4} \pm 0.89 \cdot 10^{-2}$. From equation (3) it can be seen that there are no right half plane zeros. Hence, the factorization of $P(s)$ in (4) is as follows:

$$\begin{aligned}
M_n(s) &= 1\\
M_d(s) &= \frac{(s-s_1)(s-s_2)}{(s+s_1)(s+s_2)}\\
N_1(s) &= \frac{P(s)M_d(s)}{N_2(s)},
\end{aligned}$$

where $N_2(s)$ is any rational stable minimum-phase transfer function of relative degree one. The following weighting functions are chosen:

$$\begin{aligned}
W_d(s) &= \frac{10s+10}{100s+1}\\
W_m(s) &= 5s + 0.5.
\end{aligned}$$

Note that $W_m(s)$ needs to be improper such that $(W_m(s)N_2(s))^{-1} \in H_\infty$. These weights represent the demand of good disturbance attenuation at low frequencies and the possibility of large multiplicative model errors at high frequencies. The optimal performance level is $\gamma_{opt} = 0.786$. Bode plots of the corresponding optimal controller and an 8th order approximation are shown in Figure 5.

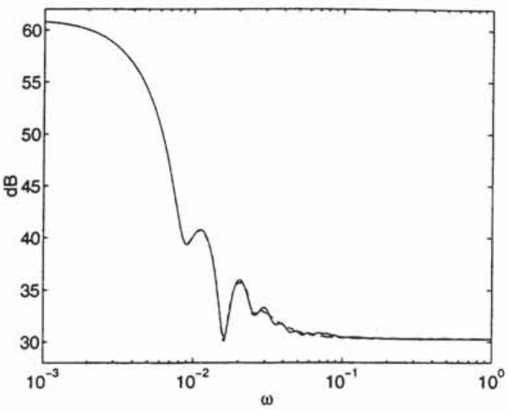

Fig. 5. Bode Magnitude Plot of Irrational Controller Transfer Function (solid line) and 8th Order Approximation (dashed line)

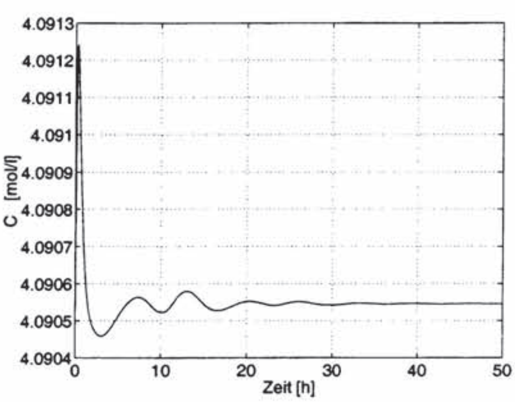

Fig. 6. Solute Concentration (Closed Loop)

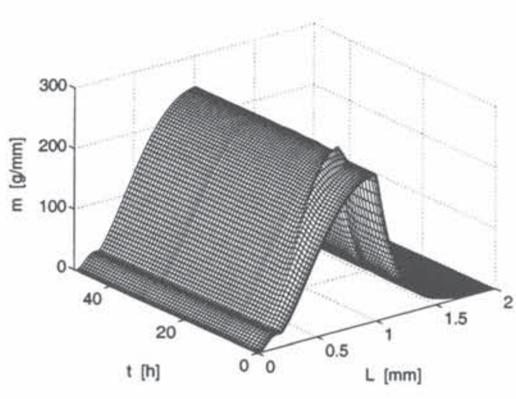

Fig. 7. Mass Density (Closed Loop)

Results from two different closed loop simulations with the full nonlinear population balance model and the reduced order controller are shown in Figures 6 to 9. In the first case (Figures 6 and 7) the initial conditions are the same as in the open loop simulation (see Figures 2, 3). Now, the controller stabilizes the system at the steady state. It can be seen that in the closed loop case the oscillations are eliminated almost completely and the system approaches its steady state very quickly. The second simulation demonstrates the ability of the controller to attenuate the effect of

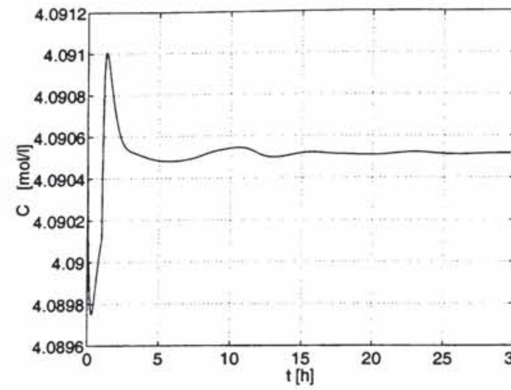

Fig. 8. Solute Concentration (Closed Loop), Effect of Feed Rate Disturbance

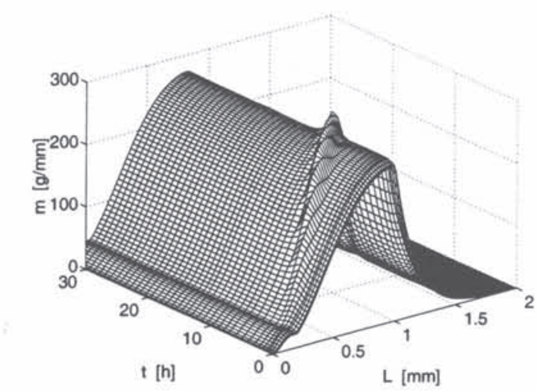

Fig. 9. Mass Density (Closed Loop), Effect of Feed Rate Disturbance

disturbances on the output. The simulation starts at steady state. Then the feed rate q is decreased by 10%, after one hour it is reset to its initial value. The effect of the feed rate disturbance on the solute concentration is presented in Figure 8. The change in feed rate also causes a disturbance in the particle size distribution. This disturbance grows through the particle size range and finally is washed out (see Figure 9).

4. CONCLUSION

In this paper, controller design for a continuous crystallization process was considered. A population balance model of the system showing oscillatory behavior was given. A robust performance problem was formulated reflecting the desire of disturbance rejection at low frequencies and robustness with respect to large multiplicative model uncertainty at high frequencies. The resulting mixed sensitivity problem was solved using recent results from H_∞ theory for infinite-dimensional systems. An infinite-dimensional controller was obtained. For practical implementation, this controller was approximated by a rational, i.e. finite-dimensional transfer function.

Closed loop simulations of the nonlinear population balance model with the approximated 8th order controller were presented to demonstrate the effectiveness of the controller. It does not only stabilize the linearized model for which it was designed but also the original nonlinear model. Disturbance attenuation was shown to be good.

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