MINIMAL HÖLDER REGULARITY IMPLYING FINITENESS OF INTEGRAL MENGER CURVATURE

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Abstract. We study two families of integral functionals indexed by a real number $p > 0$. One family is defined for 1-dimensional curves in $\mathbb{R}^3$ and the other one is defined for m-dimensional manifolds in $\mathbb{R}^n$. These functionals are described as integrals of appropriate integrands (strongly related to the Menger curvature) raised to power p. Given $p > m(m+1)$ we prove that $C^{1,\alpha}$ regularity of the set (a curve or a manifold), with $\alpha > \alpha_0 = 1 - \frac{m(m+1)}{p}$ implies finiteness of both curvature functionals ($m = 1$ in the case of curves). We also show that α_0 is optimal by constructing examples of C^{1,α_0} functions with graphs of infinite integral curvature.

1. Introduction

Geometric curvature energies are functionals defined on submanifolds of an Euclidean space, whose values can be considered as "total curvature" of the set. Desired features of such energies are regularization and self-avoidance effects - i.e. finiteness of the energy implies higher regularity of the set and excludes self-intersections. These properties make curvature energies extremely useful in modeling long, entangled objects like DNA molecules, protein structures or polymer chains; see for example the paper by Banavar et al. [1] or the book by Sutton and Balluffi [17] and the references therein. In this paper we study two families of such functionals.

Let S_L be a circle of length L. Given $p \in [1, \infty]$ and an arc-length parametrization $\Gamma : S_L \to \mathbb{R}^3$ of some curve $\gamma \subseteq \mathbb{R}^3$ (by *arc-length* we mean that $|\Gamma'| = 1$ a.e.), we define

$$\mathcal{M}_p(\gamma) := \int_{S_L} \int_{S_L} \int_{S_L} R^{-p}(\Gamma(x), \Gamma(y), \Gamma(z))\, dx\, dy\, dz\,,$$

where $R^{-1}(a, b, c)$ is the *Menger curvature* of a triple of points (a, b, c), i.e. the inverse of the radius of the smallest circle passing through a, b and c (cf. Definition 2.1).

For an m-dimensional manifold $\Sigma \subseteq \mathbb{R}^n$ we set

$$\mathcal{E}_p(\Sigma) = \int_{\Sigma^{m+2}} \mathcal{K}(\mathbf{x}_0, \ldots, \mathbf{x}_{m+1})^p \, d\mathcal{H}^m_{\mathbf{x}_0} \cdots d\mathcal{H}^m_{\mathbf{x}_{m+1}}\,, \qquad \Sigma^{m+2} = \underbrace{\Sigma \times \cdots \times \Sigma}_{(m+2)\text{ times}},$$

where $\mathcal{K}$ is an appropriately defined analogue of the Menger curvature in higher dimensions (see Definition 2.2).

The functionals $\mathcal{M}_p$ and $\mathcal{E}_p$ occurred to serve very well as geometric curvature energies. If $p > m(m + 2)$ then finiteness of $\mathcal{M}_p(\gamma)$ ($m = 1$ in this case) or $\mathcal{E}_p(\Sigma)$ implies lack of self-intersections and $C^{1,\alpha}$ regularity, with $\alpha = 1 - \frac{m(m+2)}{p}$ (see [11] for the $\mathcal{M}_p$ case and [4]

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for the $\mathcal{E}_p$ case). In this paper we prove that implications in the inverse direction also hold if we assume a little more about α.

Theorem.

- *If* $\alpha > 1 - \frac{2}{p}$ *and* $\Gamma \in C^{1,\alpha}$ *is injective then* $\mathcal{M}_p(\gamma)$ *is finite.*
- *If* $\alpha > 1 - \frac{m(m+1)}{p}$ *and* Σ *is compact and* $C^{1,\alpha}$ *regular then* $\mathcal{E}_p(\Sigma)$ *is finite.*

Moreover, $\alpha_0 = 1 - \frac{m(m+1)}{p}$ *is the minimal Hölder exponent above which we have finite energy.*

Note that there is a gap between $1 - \frac{m(m+2)}{p}$ and $1 - \frac{m(m+1)}{p}$. To show that our estimates are sharp, in §5 we construct a concrete example of a $C^{1,1-2/p}$ function $F : \mathbb{R} \to \mathbb{R}$ for which $\mathcal{M}_p(\mathrm{graph}(F))$ is infinite. Then in §6, using essentially the same construction, we also cook up an example of a function $G : \mathbb{R}^m \to \mathbb{R}$ for which $\mathcal{E}_p(G([0,1]^m))$ is infinite.

Recently Blatt [2] showed that finiteness of $\mathcal{M}_p(\gamma)$ is exactly equivalent to the condition that γ lies in a Sobolev-Slobodeckij space $W^{1+s,p}$, where $s = 1 - \frac{2}{p}$. For $\alpha > 1 - \frac{2}{p}$ we have $C^{1,\alpha} \subseteq W^{1+s,p} \subseteq C^{1,1-3/p}$, so the result by Blatt generalizes [11] and also some results of this paper. Nevertheless, our method is simpler and more geometrical and we apply it also in higher dimensions.

Besides $\mathcal{M}_p$, in [13], [10] and [11] two other functionals.

$$\mathcal{U}_p := \int_{S_L} \left(\inf_{\{s,t \in S_L \setminus \{u\} \ | \ s \neq t\}} R(\Gamma(s), \Gamma(t), \Gamma(u)) \right)^{-p} du$$

and

$$\mathcal{I}_p := \int_{S_L} \int_{S_L} \left(\inf_{s \in S_L \setminus \{t,u\}} R(\Gamma(s), \Gamma(t), \Gamma(u)) \right)^{-p} du\, dt \,,$$

were examined. In all cases, finiteness of the above functionals (for sufficiently large p) implies existence of injective arc-length parametrization of γ and assures its higher regularity.

Yet another example of a curvature energy for curves is the tangent-point energy defined as

$$\mathcal{E}_q(\Gamma) := \int_0^L \int_0^L \frac{ds\, dt}{r^q(\Gamma(t), \Gamma(s))},$$

where Γ is an arc-length parametrization of the curve, and $r(\Gamma(t), \Gamma(s))$ is the radius of the unique circle passing through $\Gamma(s)$ and tangent to the curve at $\Gamma(t)$. Properties of curves with finite $\mathcal{E}_q$ energies for $q > 2$ were investigated in [16] (in C^2 case) and in [12] (in continuous case). A similar tangent-point energy in higher dimensions was studied in [15], where the authors once again establish self-avoidance and smoothing effects.

The functional $\mathcal{M}_2$ proved to be useful also in harmonic analysis. Using roughly the same formula, one can define $\mathcal{M}_p$ for any Borel set E. David and Léger [5] showed that 1-dimensional sets with finite $\mathcal{M}_2$ total curvature are 1-rectifiable. This was a crucial step in the proof of Vitushkin's conjecture and allowed to fully characterize removable sets of bounded analytical functions. Surveys of Mattila [8] and Tolsa [18] explain in more detail the connection between these subjects.

A close analogue of the energy $\mathcal{E}_p$, defined for 2-dimensional, non-smooth surfaces in $\mathbb{R}^3$ was also studied by Strzelecki and von der Mosel in [14]. The authors proved that one can use their notion of total curvature to impose topological constraints in variational problems. They proved existence of area minimizing surfaces in a given isotopy class under the constraint of bounded curvature.

Lerman and Whitehouse in [6] and in [7] suggested a whole class of curvature energies for higher dimensional objects. However, the integrands in their definitions scale differently and it seems that these energies can not serve our needs. Nevertheless, the authors proved [7, Theorems 1.2 and 1.3] that their integral curvatures can be used to characterize d-dimensional rectifiable measures thereby establishing a link between the theory of geometric energies and uniform rectifiablility in the sense of David and Semmes [3].

Remark 1.1. We shall use a letter C to denote a general constant, whose value may change from line to line even in one series of transformations.

2. Preliminaries

In the section we introduce the notation that will be used throughout the paper. We also explain the relations between two types of energies we consider.

We use standard symbols for commonly used notions, therefore $\mathbb{B}(x,r)$ stands for the ball with radius r centered at x, while we write $\mathbb{B}_r$ if the origin is the center of the ball and $\mathbb{B}_r^m$, when we additionally want to emphasize the dimension of the ball. We denote m-dimensional Hausdorff measure by $\mathcal{H}^m$ and $T_a\Sigma$ denotes the vector space tangent to Σ at the point a. The symbol S_L stands for the circle of length L, i.e. $S_L = \mathbb{R}/L\mathbb{Z}$.

For a tuple $T = (\mathbf{x}_0, \dots, \mathbf{x}_k)$ of $(k+1)$ points in $\mathbb{R}^n$, we write $\triangle T$ to denote the convex hull of the set $\{\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_k\}$, i.e. the smallest convex subset of $\mathbb{R}^n$ which contains all the points $\mathbf{x}_0$, $\dots$, $\mathbf{x}_k$. Typically $\triangle T$ will just be k-simplex (a triangle for $k = 2$ and a tetrahedron for $k = 3$).

As we mentioned in the introduction, Menger curvature of three points is a reciprocal of the radius of the smallest circle passing through those points. The expression given below, can be treated as an equivalent definition of the radius of the curvature of three distinct points.

Definition 2.1. Let x_0,x_1,x_2 be three points in $\mathbb{R}^n$. We define Radius of Menger curvature of x_0,x_1,x_2 as

$$R(x_0,x_1,x_2) = \frac{|x_1 - x_0||x_2 - x_0||x_2 - x_1|}{4\mathcal{H}^2(\triangle(x_0,x_1,x_2))} .$$

We can use the above formula to compare Menger curvature and its higher dimensional generalizations [1]. In this paper we use integral curvature functional $\mathcal{E}_p$ defined in [4] whose integrand is the p-th power of the discrete curvature $\mathcal{K}$.

Definition 2.2. Let $T = (\mathbf{x}_0, \dots, \mathbf{x}_{m+1})$ be an $(m+2)$-tuple of points in $\mathbb{R}^n$. The *discrete curvature* of T is given by the formula

$$\mathcal{K}(T) = \frac{\mathcal{H}^{m+1}(\triangle T)}{\operatorname{diam}(T)^{m+2}} .$$

Now we introduce the definition of our functional.

Definition 2.3. Let $\Sigma \subseteq \mathbb{R}^n$ be some m-dimensional subset of $\mathbb{R}^n$. We define the *p-energy* of Σ by the formula

$$\mathcal{E}_p(\Sigma) = \int_{\Sigma^{m+2}} \mathcal{K}(\mathbf{x}_0, \dots, \mathbf{x}_{m+1})^p \; d\mathcal{H}^m_{\mathbf{x}_0} \cdots d\mathcal{H}^m_{\mathbf{x}_{m+1}} .$$

[1] Natural generalization of Menger curvature of three points would be the inverse of the radius of the sphere passing through four points. However this definition is not rewarding for integral curvature energies - see [14, Appendix B]

The quantity $\mathcal{K}(T)$ should be seen as a generalization of the Menger curvature to higher dimensions. It behaves in the same way as R^{-1} under scaling, i.e.

$$\mathcal{K}(\lambda T) = \lambda^{-1}\mathcal{K}(T)\,.$$

Notice that we always have $R^{-1}(x,y,z) > \mathcal{K}(x,y,z)$. Furthermore, for a class of roughly regular triangles $T = (x,y,z)$ (i.e. satisfying $\mathfrak{h}_{\min}(T) \geq \eta\,\mathrm{diam}(T)$, where $\mathfrak{h}_{\min}(T)$ is the minimal height of T and $\eta \in (0,1]$ is some fixed number) the two quantities $\mathcal{K}(T)$ and $R^{-1}(T)$ are comparable up to a constant depending only on η. However they are not comparable, when considered on the family of all triangles so we cannot infer finiteness of $\mathcal{M}_p(\gamma)$ from finiteness of $\mathcal{E}_p(\gamma)$.

The definition of $\mathcal{K}$ is based on another notion of discrete curvature $\mathcal{K}_{\mathrm{SvdM}}$ introduced by Strzelecki and von der Mosel in [14] for 4-tuples of points (tetrahedrons). Yet again, the quantities $\mathcal{K}(T)$ and $\mathcal{K}_{\mathrm{SvdM}}(T)$ are comparable for the class of roughly regular tetrahedrons, i.e. such that $\mathfrak{h}_{\min}(T) \geq \eta\,\mathrm{diam}(T)$ for some fixed $\eta \in (0,1]$.

In the proofs we will use the Jones' β-numbers. For a set $E \subset \mathbb{R}^n$, for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$ we define

$$\beta_E^m(x,r) := \inf_H \sup_{y \in E \cap \mathbb{B}(x,r)} \frac{d(y, x+H)}{r},$$

where the infimum is taken over all m-hyperplanes H in the Grassmannian $G(m,n)$. The quantity $\beta_E^m(x,t)$ measures, in a scale invariant way, how well the set is approximated by hyperplanes in the ball $\mathbb{B}(x,t)$. We omit the indices E and m if the choice of the set and its dimension is clear from the context. The relations between total Menger curvature ($\mathcal{M}_2$) and double integral of β-numbers for rectifiable curves and Ahlfors regular sets were investigated by Peter Jones, who never published those results, however they are presented in Herve Pajot's book [9, chapter 3].

3. Injective $C^{1,\alpha}$ curves have finite $\mathcal{M}_p$ curvature

The main step of the proof of the finiteness of $\mathcal{M}_p(\gamma)$ for $C^{1,\alpha}$ curves with $\alpha > 1 - 2/p$, is to show that one can control the angle between secants on small arcs on the curves. This leads to the estimation of Menger curvature.

Theorem 3.1. *If $\Gamma : S_L \to \mathbb{R}^3$ is an injective arc-length parametrization of γ and $\Gamma \in C^{1,\alpha}(S_L)$, where $\alpha > 1 - \frac{2}{p}$ for some $p > 2$, then $\mathcal{M}_p(\gamma)$ is finite.*

We start with an easy lemma, which shows that the parametrization is bi-lipschitz.

Lemma 3.2. *If $\Gamma : S_L \to \mathbb{R}^3$ is injective arc-length parametrization of γ, such that*

$$|\Gamma(x) - \Gamma(y)| \leq C|x-y|^{\alpha},$$

then there exists a constant $\lambda = \lambda(C,\alpha,L)$, such that

$$|\Gamma(x) - \Gamma(y)| > \lambda|x-y|$$

For the convenience of the reader we repeat the proof, which can be found in [10].

Proof. First we show that Γ is uniformly locally bi-lipschitz, i.e. for given $c_1 \in (0,1)$

$$\exists_{\delta>0} \text{ such that } \forall_{x,y\in S_L}\ |x-y| < \delta \ \Rightarrow\ |\Gamma(x) - \Gamma(y)| > c_1|x-y|$$

We take $\delta := \left(\frac{1-c_1}{C}\right)^{1/\alpha}$, then for all $|x-y|<\delta$

$$|\Gamma_1'(x)-\Gamma_1'(y)| < |\Gamma'(x)-\Gamma'(y)| < C|x-y|^\alpha < 1-c_1.$$

Without loss of generality we can assume that $\Gamma'(x)=[1,0,0]$ and for t satisfying $|x-t|<\delta$ we have

$$\Gamma_1'(x) - |\Gamma_1'(x)-\Gamma_1'(t)| > c_1 \quad \text{and} \quad \text{thus} \quad \Gamma_1'(t) > c_1.$$

Using absolute continuity of Γ we get

$$|\Gamma(x)-\Gamma(y)| = \Big|\int_x^y \Gamma'(t)dt\Big| \geq \Big|\int_x^y \Gamma_1'(t)dt\Big| > c_1|x-y|$$

To finish the proof it is enough to notice that the set

$$A_\delta := \{(x,y)\in S_L \mid |x-y|\geq\delta\}$$

is compact, thus injectivity and continuity of Γ implies that $\inf_{A_\delta}|\Gamma(x)-\Gamma(y)| = a > 0$, therefore for $|x-y|\geq\delta$ we have

$$|\Gamma(x)-\Gamma(y)| \geq \frac{a}{L}|x-y|.$$

□

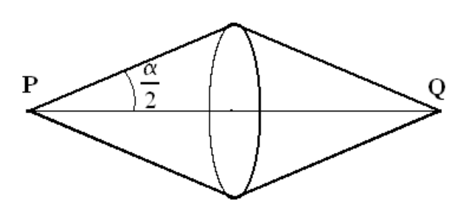
To formulate next the lemma we need to introduce geometric objects we will use. Let $C^+(P,\vec{v},\alpha)$ be a "half cone" with a vertex P, an axis parallel to the given vector $\vec{v}$ and an opening angle α

$$C^+(P,\vec{v},\alpha) := \left\{P + x : |\sphericalangle(\vec{v},x)| < \frac{\alpha}{2}\right\}.$$

Intersection of two half-cones with vertices P and Q and common axis PQ and opening angle α (see fig. 1) will be denoted as follows

$$D(P,Q,\alpha) := C^+(P,\vec{PQ},\alpha)\cap C^+(Q,\vec{QP},\alpha)$$

Fig. 1



Now we are ready to formulate and prove the lemma.

Lemma 3.3. *Let $\Gamma : S_L \to \mathbb{R}^3$ be injective, and*

$$|\Gamma'(x)-\Gamma'(y)| \leq C|x-y|^\alpha \quad \textit{for all} \quad x,y\in S_L$$

Then for any $x,y\in S_L$ satisfying $\frac{5}{2}C|x-y|^\alpha < 1$ the following inclusion holds

$$\Gamma([x,y]) \subset D(\Gamma(x),\Gamma(y),\eta|x-y|^\alpha),$$

where η is a constant which depends only on C.

Proof of Lemma 3.3. Let $x, y, z \in S_L$ be such that $x < z < y$. We are going to estimate the angle between two secant lines: one passing through $\Gamma(x)$ and $\Gamma(y)$ and the other one passing through $\Gamma(x)$ and $\Gamma(z)$. First, note that for any two unit vectors $u, v \in S^{n-1}$ forming a small angle $\sphericalangle(u,v) = \theta$ we have $\theta \simeq |u - v|$. Hence, it suffices to estimate the difference $\frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|}$ and $\frac{\Gamma(x)-\Gamma(z)}{|\Gamma(x)-\Gamma(z)|}$. Let us calculate

$$\left|\frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|} - \frac{\Gamma(x)-\Gamma(z)}{|\Gamma(x)-\Gamma(z)|}\right| \le \left|\frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|} - \Gamma'(x)\right| + \left|\frac{\Gamma(x)-\Gamma(z)}{|\Gamma(x)-\Gamma(z)|} - \Gamma'(z)\right| + |\Gamma'(x) - \Gamma'(z)| \,.$$

We have

$$\begin{aligned}\left|\Gamma'(x) - \frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|}\right| &\le \left|\Gamma'(x) - \frac{\Gamma(x)-\Gamma(y)}{|x-y|}\right| + \left|\frac{\Gamma(x)-\Gamma(y)}{|x-y|} - \frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|}\right| \\ &\le 2\left|\Gamma'(x) - \frac{\Gamma(x)-\Gamma(y)}{|x-y|}\right| ,\end{aligned}$$

where the last inequality holds, because the distance between vector $v = \frac{\Gamma(x)-\Gamma(y)}{|x-y|}$ and its projection onto the unit sphere $\frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|}$ is not greater than the distance between v and any arbitrarily chosen unit vector (recall that Γ is an arc-length parametrization, so $|\Gamma'| \equiv 1$).

$$\begin{aligned}\left|\Gamma'(x) - \frac{\Gamma(x)-\Gamma(y)}{|x-y|}\right| &= \left|\Gamma'(x) - \frac{1}{y-x}\int_{[x,y]} \Gamma'(s)\ ds\right| \\ &\le \frac{1}{y-x}\int_{[x,y]} |\Gamma'(x) - \Gamma'(s)|\ ds \le \frac{1}{y-x}\int_{[x,y]} C(s-x)^{\alpha}\ ds \\ &\le C(y-x)^{\alpha} \,.\end{aligned}$$

Combining the above inequalities we obtain

$$\left|\frac{\Gamma(x)-\Gamma(y)}{|\Gamma(x)-\Gamma(y)|} - \frac{\Gamma(x)-\Gamma(z)}{|\Gamma(x)-\Gamma(z)|}\right| \le 2C|x-y|^{\alpha} + 2C|x-z|^{\alpha} + C|x-z|^{\alpha} \le 5C|x-y|^{\alpha} \,.$$

This implies that $\Gamma[(x,y)]$ is included in the cone

$$C\left(\Gamma(x), \Gamma(y) - \Gamma(x), 2\arcsin\left(\tfrac{5}{2}C|x-y|^{\alpha}\right)\right) \,.$$

Analogously

$$\Gamma[(x,y)] \in C\left(\Gamma(y), \Gamma(x) - \Gamma(y), 2\arcsin\left(\tfrac{5}{2}C|x-y|^{\alpha}\right)\right) \,.$$

Thus

$$\Gamma[(x,y)] \in D\left(\Gamma(x), \Gamma(y), \eta|x-y|^{\alpha}\right) \,.$$

□

The lemma proven above gives an estimation for Jones β-numbers; it is easy to notice that there exists R_0, such that for $r < R_0$ we have

$$\beta(x,r) < r^{1+\alpha}.$$

In case of plane curves it is possible to use the estimation to prove the finiteness of $\mathcal{M}_p$, (one can modify the reasoning from [9]), but it is not clear if it is possible to adapt it to curves in $\mathbb{R}^3$ and even in 2-space the arguing is long and complicated. Here, Lemma 3.3 gives us additional information. We not only know that the curve is close to a line in a small ball, but we can also point out the line. This information makes the proof of the finiteness of $\mathcal{M}_p$ much easier, as the following lemma holds.

Lemma 3.4. *Let $\Gamma : S_L \to \mathbb{R}^3$ be an injective, arc-length parametrization of a curve γ and let $p > 2$ and $\alpha > 1 - \frac{2}{p}$. If there exist constants η and $\varepsilon > 0$ such that $\eta\varepsilon^\alpha < \frac{\pi}{2}$ and for each $|x - y| < \varepsilon$ we have*

$$\Gamma((x, y)) \subset D(\Gamma(x), \Gamma(y), \eta|x - y|^\alpha)\,, \tag{1}$$

then $\mathcal{M}_p(\gamma) < \infty$.

Proof of Lemma 3.4. We start with a simple geometric observation. If $S \in D(P, Q, \beta)$ and $0 < \beta < \frac{\pi}{2}$ then

$$c(P, Q, S) = \frac{1}{R(P, Q, S)} \leq \frac{2\sin\beta}{|PQ|}.$$

Thus, from (1), for $|x - y| < \varepsilon$ and $z \in [x, y]$ we have

$$c(\Gamma(x), \Gamma(y), \Gamma(z)) \leq \frac{2\sin(\eta|x - y|^\alpha)}{|\Gamma(x) - \Gamma(y)|} \leq \frac{2\eta|x - y|^\alpha}{|\Gamma(x) - \Gamma(y)|}$$

As we know from Lemma 3.2, Γ is bi-lipschitz, hence there exists a constant $d > 0$ such that

$$c(\Gamma(x), \Gamma(y), \Gamma(z)) \leq d|x - y|^{\alpha-1}. \tag{2}$$

Now we are ready to estimate the triple integral

$$\begin{aligned}
\mathcal{M}_p(\gamma) &= \int_{S_L}\int_{S_L}\int_{S_L} c^p(\Gamma(x), \Gamma(y), \Gamma(z))\; dx\; dy\; dz \\
&\leq C\int_{S_L}\int_{\{y\in S_L\ |\ |x-y|<\varepsilon\}}\int_{[x,y]} c^p(\Gamma(x), \Gamma(y), \Gamma(z))\; dx\; dy\; dz \\
&+ \int_{S_L}\int_{\{y\in S_L\ |\ |x-y|>\varepsilon\}}\int_{S_L} c^p(\Gamma(x), \Gamma(y), \Gamma(z))\; dx\; dy\; dz \\
&\overset{(2)}{\leq} C\int_{S_L}\int_{\{y\in S_L\ |\ |x-y|<\varepsilon\}} |x - y|^{p(\alpha-1)} \cdot |x - y|\; dx\; dy \\
&+ \int_{S_L}\int_{\{y\in S_L\ |\ |x-y|>\varepsilon\}}\int_{S_L} \varepsilon^p\; dx\; dy\; dz \\
&\leq C\int_{S_L}\int_{\{y\in S_L\ |\ |x-y|<\varepsilon\}} |x - y|^{p\alpha-p+1}\; dx\; dy + \text{const} < \infty\,,
\end{aligned}$$

as $\alpha > 1 - \frac{2}{p}$. □

The proof of Theorem 3.1 follows immediately from the Lemmas 3.3 and 3.4.

Proof of Theorem 3.1. As Γ satisfies assumption of Lemma 3.3, we know that there exists $\varepsilon > 0$ such that if $|x - y| \leq \varepsilon$ then

$$\Gamma([x, y]) \subset D(\Gamma(x), \Gamma(y), \mathrm{d}|x - y|^\alpha)\,,$$

where d is a constant which depends only on Hölder constant C. Thus using Lemma 3.4 we obtain the thesis. □

4. Manifolds of class $C^{1,\alpha}$ have finite integral curvature

In this section we prove a counterpart of Theorem 3.1 for m-dimensional submanifolds of $\mathbb{R}^n$.

Theorem 4.1. *Let $p > m(m+1)$ be some number and let $\Sigma \subseteq \mathbb{R}^n$ be a compact manifold of class $C^{1,\alpha}$. If $\alpha > 1 - \frac{m(m+1)}{p}$ then $\mathcal{E}_p(\Sigma)$ is finite.*

Lemma 4.2. *Let $\Sigma \subseteq \mathbb{R}^n$ be a compact manifold of class $C^{1,\alpha}$ for some $\alpha \in (0,1)$. Then there exist constants $R = R(\Sigma) > 0$ and $C = C(\Sigma) > 0$ such that for each $a \in \Sigma$ and each $r \le R$*

$$\beta(a,r) \le C r^{\alpha}\,.$$

Proof. Since Σ is compact, we can find a radius $R > 0$ and a constant $C > 0$ such that for each $a \in \Sigma$ there exists a function $f_a \in C^{1,\alpha}(T_a\Sigma \cap \mathbb{B}_{2R}, T_a\Sigma^{\perp})$ such that

$$\Sigma \cap \mathbb{B}(a,R) \subseteq a + \operatorname{graph}(f_a)\,,$$
$$f_a(0) = 0\,, \quad Df_a(0) = 0\,,$$
$$\text{and} \quad \forall x,y \in \mathbb{B}^m_{2R} \quad |Df_a(x) - Df_a(y)| \le C|x-y|^{\alpha}\,.$$

Fix some $a \in \Sigma$ and a radius $r \le R$. Let $b \in \Sigma \cap \mathbb{B}(a,r)$. Since $\Sigma \cap \mathbb{B}(a,R)$ is the graph of f, there exists a point $x \in T_a\Sigma$ such that $b = a + x + f_a(x)$. By the fundamental theorem of calculus we have

$$\begin{aligned}
|f_a(x)| = |f_a(x) - f_a(0)| &= \left| \int_0^1 \tfrac{d}{dt} f_a(tx)\, dt \right| \\
&\le |x| \sup_{y \in T_a\Sigma \cap \mathbb{B}_{|x|}} |Df_a(y) - Df_a(0)| \\
&\le C|x|^{1+\alpha} \le C|b-a|^{1+\alpha}\,.
\end{aligned}$$

Note that $|f_a(x)|$ is just the distance of b from the affine plane $a + T_a\Sigma$. Hence

$$\sup_{b \in \Sigma \cap \mathbb{B}(a,r)} \operatorname{dist}(b, a + T_a\Sigma) \le C r^{1+\alpha}$$

and we obtain

$$\begin{aligned}
\beta(a,r) &= \frac{1}{r} \inf_{H \in G(n,m)} \Big(\sup_{b \in \Sigma \cap \mathbb{B}(a,r)} \operatorname{dist}(b, a+H) \Big) \\
&\le \frac{1}{r} \sup_{b \in \Sigma \cap \mathbb{B}(a,r)} \operatorname{dist}(b, a + T_a\Sigma) \le C r^{\alpha}\,.
\end{aligned}$$

□

Lemma 4.3. *Let $\Sigma \subseteq \mathbb{R}^n$ be an m-dimensional manifold. Choose $m+2$ points $\mathbf{x}_0,\ldots,\mathbf{x}_{m+1}$ of Σ and set $T = \triangle(\mathbf{x}_0,\ldots,\mathbf{x}_{m+1})$ and $d = \operatorname{diam}(T)$. There exists a constant $C = C(m,n)$ such that*

$$\mathcal{H}^{m+1}(T) \le C\beta(\mathbf{x}_0, d) d^{m+1}\,,$$

hence

$$\mathcal{K}(\mathbf{x}_0,\ldots,\mathbf{x}_{m+1}) \le C\frac{\beta(\mathbf{x}_0,d)}{d}\,.$$

Note that we assumed Σ to be a manifold but the proof works also for an arbitrary set Σ of Hausdorff dimension m or even for any set.

Proof. If the affine space $\operatorname{aff}\{\mathbf{x}_0, \ldots, \mathbf{x}_{m+1}\}$ is not $(m+1)$-dimensional then $\mathcal{H}^{m+1}(T) = 0$ and there is nothing to prove. Hence, we can assume that T is an $(m+1)$-dimensional simplex. The measure $\mathcal{H}^{m+1}(T)$ can be expressed by the formula

$$\mathcal{H}^{m+1}(T) = \frac{1}{m+1} \operatorname{dist}(\mathbf{x}_{m+1}, \operatorname{aff}\{\mathbf{x}_0, \ldots, \mathbf{x}_m\}) \mathcal{H}^m(\triangle(\mathbf{x}_0, \ldots, \mathbf{x}_m)) .$$

In the same way, one can express the measure $\mathcal{H}^m(\triangle(\mathbf{x}_0, \ldots, \mathbf{x}_m))$, so certainly

$$\mathcal{H}^{m+1}(T) \leq \frac{1}{(m+1)!} d^{m+1} .$$

Hence, if $\beta(\mathbf{x}_0, d) = 1$, then there is nothing to prove, so we can assume that $\beta(\mathbf{x}_0, d) < 1$.

Due to compactness of the Grassmannian $G(n, m)$ we can find an m-plane $H \in G(n, m)$ such that

$$\sup_{y \in \Sigma \cap \mathbb{B}(\mathbf{x}_0, d)} \operatorname{dist}(y, \mathbf{x}_0 + H) = d\beta(\mathbf{x}_0, d) . \tag{3}$$

Set $h = d\beta(\mathbf{x}_0, d) < d$. Without loss of generality we can assume that $\mathbf{x}_0$ lies at the origin. Let us choose an orthonormal coordinate system $v_1, \ldots, v_n$ such that $H = \operatorname{span}\{v_1, \ldots, v_m\}$. Because of (3) in our coordinate system we have

$$T \subseteq [-d, d]^m \times [-h, h]^{n-m} .$$

Of course T lies in some $(m+1)$-dimensional section of the above product. Let

$$\begin{aligned} V &:= \operatorname{aff}\{\mathbf{x}_0, \ldots, \mathbf{x}_{m+1}\} = \operatorname{span}\{\mathbf{x}_1, \ldots, \mathbf{x}_{m+1}\} , \\ Q(a, b) &:= [-a, a]^m \times [-b, b]^{n-m} , \\ Q &:= Q(d, h) \\ \text{and} \quad P &:= V \cap Q . \end{aligned}$$

Note that all of the sets V, Q and P contain T. Choose another orthonormal basis $w_1, \ldots, w_n$ of $\mathbb{R}^n$, such that $V = \operatorname{span}\{w_1, \ldots, w_{m+1}\}$. Set

$$S := \{x \in V^{\perp} : |\langle x, w_i \rangle| \leq h \text{ for } i = 1, \ldots, m+1\} .$$

Observe that S is just the cube $[-h, h]^{n-m-1}$ placed in the orthogonal complement of V and that $\operatorname{diam} S = 2h\sqrt{n-m-1} =: 2Ah$, where $A = A(n, m) = \operatorname{diam}([0, 1]^{n-m-1})$. In this setting we have

$$P \times S = P + S \subseteq Q(d + 2hA, h + 2hA) . \tag{4}$$

Recall that $h = d\beta(\mathbf{x}_0, d) < d$. We obtain the following estimate

$$\begin{aligned} \mathcal{H}^n(T \times S) &\leq \mathcal{H}^n(P \times S) \leq \mathcal{H}^n(Q(d + 2hA, h + 2hA)) \\ &\leq (2d + 4hA)^m (2h + 4hA)^{n-m} \\ &\leq (2d + 4d\beta(\mathbf{x}_0, d)A)^m (2d\beta(\mathbf{x}_0, d) + 4d\beta(\mathbf{x}_0, d)A)^{n-m} \\ &\leq (2 + 4A)^n d^n \beta(\mathbf{x}_0, d)^{n-m} . \end{aligned}$$

On the other hand we have

$$\begin{aligned} \mathcal{H}^n(T \times S) &= \mathcal{H}^{m+1}(T)\mathcal{H}^{n-m-1}(S) = \mathcal{H}^{m+1}(T) 2^{n-m-1} h^{n-m-1} \\ &= 2^{n-m-1} \mathcal{H}^{m+1}(T) d^{n-m-1} \beta(\mathbf{x}_0, d)^{n-m-1} . \end{aligned}$$

Hence

$$2^{n-m-1}\mathcal{H}^{m+1}(T)d^{n-m-1}\beta(\mathbf{x}_0,d)^{n-m-1} \le (2+4A)^n d^n \beta(\mathbf{x}_0,d)^{n-m} \iff$$
$$\iff \mathcal{H}^{m+1}(T) \le (2+4A)^n 2^{-(n-m-1)} d^{m+1}\beta(\mathbf{x}_0,d)\,.$$

We may set $C = C(n,m) = (2+4A)^n 2^{-(n-m-1)}$. □

Now we can prove the main result of this section.

Proof of Theorem 4.1. Let

$$\mu = \underbrace{\mathcal{H}^m \otimes \cdots \otimes \mathcal{H}^m}_{m+1}\,.$$

If $T = (x_0, x_1, \ldots, x_{m+1}) \in \Sigma^{m+2}$, we shall write $T = (x_0, \bar{x})$. Using Lemma 4.3 we obtain

$$\begin{aligned}\mathcal{E}_p(\Sigma) &= \int_{\Sigma^{m+2}} \mathcal{K}(x_0,\bar{x})^p \; d\mathcal{H}^m(x_0)\; d\mu(\bar{x}) \\ &\le \int_{\Sigma}\int_{\Sigma^{m+1}} \left(\frac{\beta(x_0, \operatorname{diam}(x_0,\ldots,x_{m+1}))}{\operatorname{diam}(x_0,\ldots,x_{m+1})}\right)^p \; d\mathcal{H}^m(x_0) d\mu(\bar{x})\,.\end{aligned}$$

For $x_0 \in \Sigma$ and $k \in \mathbb{N}$ we define the sets

$$\Sigma_k(x_0) := \{(x_1,\ldots,x_{m+1}) \in \Sigma^{m+1} : \operatorname{diam}(x_0,\ldots,x_{m+1}) \in (2^{-k-1}, 2^{-k}]\}\,.$$

Choose $K_0 \in \mathbb{Z}$ such that $2^{-K_0} \ge 2\operatorname{diam}(\Sigma)$. Now we can write

$$\mathcal{E}_p(\Sigma) \le \int_\Sigma \sum_{k=K_0}^{\infty} \int_{\Sigma_k(x_0)} \left(\frac{\beta(x_0,\operatorname{diam}(x_0,\bar{x}))}{\operatorname{diam}(x_0,\bar{x})}\right)^p \; d\mathcal{H}^m(x_0)d\mu(\bar{x})\,.$$

Fix some small number $\varepsilon > 0$. Since Σ is compact, we can find a radius $R > 0$ and a constant $C > 0$ such that for each $a \in \Sigma$ there exists a function $f_a \in C^{1,\alpha}(T_a\Sigma \cap \mathbb{B}_{2R}, T_a\Sigma^{\perp})$ such that

$$\Sigma \cap \mathbb{B}(a,R) \subseteq a + \operatorname{graph}(f_a)\,,$$

$$f_a(0) = 0\,, \quad Df_a(0) = 0$$

$$\text{and} \quad \forall x,y \in \mathbb{B}^m_{2R} \quad |f_a(x) - f_a(y)| \le \varepsilon|x-y|\,.$$

For $r < R$ and $x_0 \in \Sigma$ we have the following estimate

$$\mathcal{H}^m(\Sigma \cap \mathbb{B}(x_0,r)) \le (1+\varepsilon)^m \mathcal{H}^m((x_0 + T_{x_0}\Sigma)\cap \mathbb{B}(x_0,r)) = (1+\varepsilon)^m\omega_m r^m\,.$$

Choose $k_0 \in \mathbb{Z}$ such that $2^{-k_0} \le R$. Then for each $k \ge k_0$ we have

$$\mathcal{H}^{m(m+1)}(\Sigma_k) \le (\omega_m(1+\varepsilon)^m 2^{-km})^{m+1}\,.$$

Of course we have

$$\begin{aligned}&\int_\Sigma \sum_{k=K_0}^{k_0-1} \int_{\Sigma_k(x_0)} \left(\frac{\beta(x_0,\operatorname{diam}(x_0,\bar{x}))}{\operatorname{diam}(x_0,\bar{x})}\right)^p \; d\mathcal{H}^m(x_0)d\mu(\bar{x}) \\ &\qquad\qquad \le \mathcal{H}^m(\Sigma)(k_0 - K_0)\omega_m^{m+1}(1+\varepsilon)^{m(m+1)}2^{-K_0 m(m+1)}2^{pk_0} < \infty\,,\end{aligned}$$

so to show that $\mathcal{E}_p(\Sigma)$ is finite it suffices to estimate the sum from k_0 to ∞. Using Lemma 4.2 we can write

$$\int_\Sigma \sum_{k=k_0}^\infty \int_{\Sigma_k(x_0)} \left(\frac{\beta(x_0, \operatorname{diam}(x_0,\bar{x}))}{\operatorname{diam}(x_0,\bar{x})}\right)^p \, d\mathcal{H}^m(x_0) d\mu(\bar{x})$$
$$\le C\mathcal{H}^m(\Sigma) \sum_{k=k_0}^\infty (\omega_m(1+\varepsilon)^m)^{m+1} 2^{-km(m+1)} \left(\frac{2^{-k\alpha}}{2^{-k-1}}\right)^p$$
$$= C'(m,p,\Sigma,\varepsilon) \sum_{k=k_0}^\infty 2^{-km(m+1)-kp\alpha+kp} \,.$$

This sum is finite if and only if

$$-m(m+1) - p\alpha + p < 0 \qquad \iff \qquad \alpha > 1 - \frac{m(m+1)}{p} \,.$$

□

5. Construction of a $C^{1,1-2/p}$ curve with infinite $\mathcal{M}_p$ energy

In this section we shall prove the following theorem.

Theorem 5.1. *Let $p > 2$ and set $\alpha = 1 - \frac{2}{p}$. There exists a function $F \in C^{1,\alpha}(\mathbb{R})$ such that $\mathcal{M}_p(\operatorname{graph}(F)) = \infty$.*

The construction of our function is based on the van der Waerden saw. Let

$$\tilde{f}(x) = \begin{cases} 2x & \text{for } x \in \left[0, \frac{1}{2}\right] \\ 2-2x & \text{for } x \in \left(\frac{1}{2}, 1\right] \end{cases} \qquad \text{and} \qquad f_0(x) = \tilde{f}(x - [x]) \,.$$

For $\alpha \in (0,1)$ we define a sequence of functions

$$f_k(x) = \frac{f_0(N^k x)}{N^{\alpha k}} \,,$$

where $N \in \mathbb{N}$ is a fixed number whose value be determined later on. We set

$$f(x) = \sum_{k=0}^\infty f_k(x) \,. \tag{5}$$

Finally we define

$$F(x) = \int_0^x f(t)\, dt = \sum_{k=0}^\infty \int_0^x f_k(t)\, dt \,. \tag{6}$$

First we show that F is $C^{1,\alpha}$.

Lemma 5.2. *The function $F : \mathbb{R} \to \mathbb{R}$ defined by (6) is of class $C^{1,\alpha}$. Moreover we have*

$$\forall x,y \in \mathbb{R} \quad |F'(x) - F'(y)| \le \left(\frac{2N}{N^{1-\alpha}-1} + \frac{2N^\alpha}{N^\alpha - 1}\right) |x-y|^\alpha \,.$$

Proof. Since $F'(x) = f(x)$, it suffices to indicate that $f : \mathbb{R} \to \mathbb{R}$ is Hölder continuous. Note that f is periodic with period 1, so it is enough to show that $|f(x) - f(y)| \lesssim |x-y|^\alpha$ only for $x \in [0,1]$ and $y \in [0,1]$.

Fix two numbers $x \in [0,1]$ and $y \in [0,1]$ such that $x < y$. Let $h = y - x$ and let $l \in \mathbb{N}$ be such that

$$\frac{1}{N^l} \le h \le \frac{1}{N^{l-1}} .$$

We can express x and h as infinite sums

$$x = \sum_{j=0}^{\infty} \frac{x_j}{N^j} \qquad \text{and} \qquad y - x = h = \sum_{j=l}^{\infty} \frac{h_j}{N^j} ,$$

where $x_j, h_j \in \{0, 1, \ldots, N-1\}$ for each $j \in \mathbb{N}$. Now, we calculate

$$\begin{aligned} |f(y) - f(x)| = |f(x+h) - f(x)| &= \left| \sum_{k=0}^{\infty} \frac{1}{N^{\alpha k}} \left(f_0\Big(\sum_{j=0}^{\infty} \tfrac{N^k x_j}{N^j} \Big) - f_0\Big(\sum_{j=0}^{\infty} \tfrac{N^k x_j + N^k h_j}{N^j} \Big) \right) \right| \\ &\le \sum_{k=0}^{\infty} \frac{1}{N^{\alpha k}} \left| f_0\Big(\sum_{j=\max(k+1,l)}^{\infty} \tfrac{N^k x_j}{N^j} \Big) - f_0\Big(\sum_{j=\max(k+1,l)}^{\infty} \tfrac{N^k x_j + N^k h_j}{N^j} \Big) \right| . \end{aligned}$$

Hence, using the fact that f_0 is Lipschitz continuous with Lipschitz constant 2 we obtain

$$\begin{aligned} |f(y) - f(x)| &\le \sum_{k=0}^{l-1} \frac{2}{N^{\alpha k}} \sum_{j=l}^{\infty} \frac{N^k h_j}{N^j} + \sum_{k=l}^{\infty} \frac{2}{N^{\alpha k}} \sum_{j=k+1}^{\infty} \frac{N^k h_j}{N^j} \\ &\le 2|y-x| \sum_{k=0}^{l-1} N^{k(1-\alpha)} + 2 \sum_{k=l}^{\infty} \frac{N^k}{N^{\alpha k}} \sum_{j=k+1}^{\infty} \frac{N-1}{N^j} \\ &= 2|y-x| \frac{N^{l(1-\alpha)} - 1}{N^{1-\alpha} - 1} + \sum_{k=l}^{\infty} \frac{2}{N^{\alpha k}} \le \frac{2}{N^{\alpha l}} \frac{N}{N^{1-\alpha} - 1} + \frac{2}{N^{\alpha l}} \frac{N^\alpha}{N^\alpha - 1} \\ &\le \left(\frac{2N}{N^{1-\alpha} - 1} + \frac{2N^\alpha}{N^\alpha - 1} \right) |y-x|^\alpha . \end{aligned}$$

□

Recall that $\alpha = 1 - \frac{2}{p}$. Now we shall prove that the p-integral curvature $\mathcal{M}_p(\mathrm{graph}(F))$ is infinite.

Lemma 5.3. *The function $F : \mathbb{R} \to \mathbb{R}$ defined by* (6) *satisfies $\mathcal{M}_p(\mathrm{graph}(F)) = \infty$.*

Proof. The graph of F is not a closed curve, thus we express the $\mathcal{M}_p$ energy in the following way

$$\mathcal{M}_p(\mathrm{graph}(F)) := \int_0^L \int_0^L \int_0^L R^{-p}(\Gamma(x), \Gamma(y), \Gamma(z))\, dx\, dy\, dz ,$$

where Γ is an arc-length parametrization of $\mathrm{graph}(F)$ and $L = \mathcal{H}^1(\mathrm{graph}(F))$. It is easy to notice that

$$\mathcal{M}_p(\mathrm{graph}(F)) > \int_0^1 \int_0^1 \int_0^1 R^{-p}((x, F(x)), (y, F(y)), (z, F(z)))\, dx\, dy\, dz$$

We denote by $\vec{t}$ a point of the graph given by argument t; i.e. $\vec{t} = (t, F(t))$. Since F is a Lipschitz function, for any two points of the graph we have

$$|t-s| \le \|\vec{t}-\vec{s}\| = \sqrt{(t-s)^2 + \big(F(t)-F(s)\big)^2} \le C(\alpha)|t-s|\,.$$

Let us start with the following estimation of the Menger curvature of three points of the graph. If $0 \le x \le z \le y \le 1$ we have

$$\frac{1}{R(\vec{x},\vec{y},\vec{z})} = \frac{4\mathcal{H}^2\big(\triangle(\vec{x},\vec{y},\vec{z})\big)}{\|\vec{x}-\vec{y}\|\|\vec{y}-\vec{z}\|\|\vec{x}-\vec{z}\|} \ge \frac{2h}{\|\vec{x}-\vec{y}\|^2} = \frac{\sin\sphericalangle(\vec{x}-\vec{y},\vec{x}-\vec{z})}{\|\vec{x}-\vec{y}\|^2}\|\vec{x}-\vec{z}\|\,,$$

where h denotes the height of the triangle $\triangle(\vec{x},\vec{y},\vec{z})$ which is perpendicular to $\vec{x}-\vec{y}$. In order to find a lower bound for the above expression we estimate tangent of the angle between $\vec{x}-\vec{y}$ and $\vec{x}-\vec{z}$

$$|\tan(\sphericalangle(\vec{x}-\vec{y},\vec{x}-\vec{z}))| = \frac{\left|\frac{F(y)-F(x)}{y-x} - \frac{F(z)-F(x)}{z-x}\right|}{1+\frac{F(y)-F(x)}{y-x}\frac{F(z)-F(x)}{z-x}} \ge C(\alpha)\left|\frac{F(y)-F(x)}{y-x} - \frac{F(z)-F(x)}{z-x}\right|\,.$$

Therefore if $\sphericalangle(\vec{x}-\vec{y},\vec{x}-\vec{z}) \le \frac{\pi}{3}$ we have

$$\frac{1}{R(\vec{x},\vec{y},\vec{z})} \ge \frac{C(\alpha)\left|\frac{F(y)-F(x)}{y-x} - \frac{F(z)-F(x)}{z-x}\right|(z-x)}{(y-x)^2}\,.$$

We will prove that the energy is infinite even when we consider much smaller domain of integration. For $k \in \mathbb{N}$ and $m \in \{0,1,\dots,N-1\}$ we define the following intervals

$$\begin{aligned}
X_{k,m} &= \left[\frac{m}{N^k}, \frac{m}{N^k} + \frac{1}{16}\frac{1}{N^k}\right]\,,\\
Y_{k,m} &= \left[\frac{m+1/2}{N^k} - \frac{1}{16}\frac{1}{N^k}, \frac{m+1/2}{N^k}\right]\,,\\
Z_{k,m} &= \left[\frac{m}{N^k} + \frac{1}{4}\frac{1}{N^k}, \frac{m}{N^k} + \frac{1}{4N^k} + \frac{1}{16N^k}\right]\,.
\end{aligned}$$

Of course, for sufficiently large N, $\sphericalangle(\vec{x}-\vec{y},\vec{x}-\vec{z}) \le \frac{\pi}{3}$ and we have

$$\mathcal{M}_p(\mathrm{graph}(F)) \ge \sum_{k=1}^{\infty}\sum_{m=0}^{N^k}\int_{X_{k,m}}\int_{Y_{k,m}}\int_{Z_{k,m}} R^{-p}(\vec{x},\vec{y},\vec{z})\;dx\;dy\;dz\,,$$

For our purposes it is enough to carry the calculation only for $x \in X_{k,m}$, $y \in Y_{k,m}$ and $z \in Z_{k,m}$. Recall that we have

$$F(x) = \sum_{k=0}^{\infty}\int_0^x f_k(t)\;dt\,.$$

For convenience we denote $F_k(x) = \int_0^x f_k(t)$ and we notice that

$$\left|\frac{F(y)-F(x)}{y-x} - \frac{F(z)-F(x)}{z-x}\right| = \left|\sum_{k=0}^{\infty}\frac{F_k(y)-F_k(x)}{y-x} - \frac{F_k(z)-F_k(x)}{z-x}\right|\,.$$

We need to divide the above sum into two parts which behave differently.

- For $n \le k$ there are two possibilities. Either

$$\frac{F_n(y)-F_n(x)}{y-x} = \frac{1}{y-x}\int_x^y \left(\frac{1}{N^n}\right)^\alpha f(N^n t)\, dt$$
$$= \frac{1}{y-x}\int_x^y \left(\frac{1}{N^n}\right)^\alpha 2N^n t\, dt = 2\left(\frac{1}{N^n}\right)^{\alpha-1}(x+y)$$

and

$$\frac{F_n(z)-F_n(x)}{z-x} = 2\left(\frac{1}{N^n}\right)^{\alpha-1}(x+z)$$

or

$$\frac{F_n(y)-F_n(x)}{y-x} = \frac{1}{y-x}\int_x^y \left(\frac{1}{N^n}\right)^\alpha 2(1-N^n t)\, dt$$
$$= 2\left(\frac{1}{N^n}\right)^\alpha - 2\left(\frac{1}{N^n}\right)^{\alpha-1}(x+y)$$

and

$$\frac{F_n(z)-F_n(x)}{z-x} = 2\left(\frac{1}{N^n}\right)^\alpha - 2\left(\frac{1}{N^n}\right)^{\alpha-1}(x+z)\,.$$

Denoting by

$$\delta_n(x,z,y) := \frac{F_n(y)-F_n(x)}{y-x} - \frac{F_n(z)-F_n(x)}{z-x},$$

in both cases we have

$$|\delta_n(x,z,y)| = 2(N^n)^{1-\alpha}(y-z)\,.$$

Now, taking N sufficiently large:

$$\left|\sum_{n=1}^{k}\delta_n(x,z,y)\right| \ge \frac{1}{2}(N^k)^{1-\alpha}(y-z) \ge \frac{1}{2}(N^k)^{1-\alpha}\frac{3}{16}\frac{1}{N^k} = \frac{3}{32}N^{-k\alpha}\,. \tag{7}$$

- For $n > k$ we notice that

$$F_n(y)-F_n(x) = \int_x^y f_n(t)\, dt \le [(y-x)2N^n]\frac{1}{4N^n}\left(\frac{1}{N^n}\right)^\alpha + \frac{1}{2N^n}\left(\frac{1}{N^n}\right)^\alpha$$
$$\le (y-x)\frac{1}{2}\left(\frac{1}{N^n}\right)^\alpha + \frac{1}{2}\left(\frac{1}{N^n}\right)^{\alpha+1}$$

and

$$F_n(y)-F_n(x) \ge \frac{1}{2}(y-x)\left(\frac{1}{N^n}\right)^\alpha - \frac{1}{4}\left(\frac{1}{N^n}\right)^{\alpha+1}\,.$$

As $y-x \ge \frac{3}{8}\frac{1}{N^k}$ we have

$$\frac{1}{2}\left(\frac{1}{N^n}\right)^\alpha - \frac{2}{3}N^k\left(\frac{1}{N^n}\right)^{\alpha+1} \le \frac{F_n(y)-F_n(x)}{y-x} \le \frac{1}{2}\left(\frac{1}{N^n}\right)^\alpha + \frac{4}{3}\left(\frac{1}{N^n}\right)^{\alpha+1}N^k\,.$$

Analogously

$$\frac{1}{2}\left(\frac{1}{N^n}\right)^\alpha - \frac{4}{3}N^k\left(\frac{1}{N^n}\right)^{\alpha+1} \le \frac{F_n(z)-F_n(x)}{z-x} \le \frac{1}{2}\left(\frac{1}{N^n}\right)^\alpha + \frac{8}{3}N^k\left(\frac{1}{N^n}\right)^{\alpha+1}\,.$$

Thus there exists a constant (which does not depend on k, n and N), such that

$$|\delta_n(x,z,y)| < CN^k \left(\frac{1}{N^n}\right)^{\alpha+1} .$$

We can choose N large enough to estimate the geometric series from above by its first term

$$\left|\sum_{n=k+1}^{\infty} \delta_n(x,z,y)\right| < \sum_{n=k+1}^{\infty} CN^n \left(\frac{1}{N^n}\right)^{\alpha+1} < 2CN^k \left(\frac{1}{N^{k+1}}\right)^{\alpha+1} ,$$

enlarging N if necessary we get

$$\left|\sum_{n=k+1}^{\infty} \delta_n(x,z,y)\right| < \frac{1}{32}\left(\frac{1}{N^k}\right)^{\alpha} . \tag{8}$$

Putting both estimates (7) and (8) together we obtain

$$\begin{aligned}\left|\sum_{n=0}^{\infty} \delta_n(x,z,y)\right| &\geq \left|\sum_{n=0}^{k} \delta_n(x,z,y)\right| - \left|\sum_{n=k+1}^{\infty} \delta_n(x,z,y)\right| \\ &\geq \frac{3}{32}\left(\frac{1}{N^k}\right)^{\alpha} - \frac{1}{32}\left(\frac{1}{N^k}\right)^{\alpha} \geq \frac{1}{16}\left(\frac{1}{N^k}\right)^{\alpha} .\end{aligned}$$

Thus for each $k, m \in \mathbb{N}$ and $x \in X_{k,m}$, $y \in X_{k,m}$, $z \in Z_{k,m}$ we have

$$\left|\frac{F(y)-F(x)}{y-x} - \frac{F(x)-F(z)}{x-z}\right| \geq \frac{1}{16}\left(\frac{1}{N^k}\right)^{\alpha} . \tag{9}$$

Using (7) and (8) we can estimate the integral

$$\begin{aligned}\mathcal{M}_p(\mathrm{graph}(F)) &\geq \sum_{k=1}^{\infty}\sum_{m=0}^{N^k-1} \int_{X_{k,m}}\int_{Y_{k,m}}\int_{Zk,m} R^{-p}(\vec{x},\vec{y},\vec{z})\, dx\, dy\, dz \\ &\geq \sum_{m=0}^{N^k-1} \int_{X_{k,m}}\int_{Y_{k,m}}\int_{Zk,m} C(\alpha)\left(\frac{\frac{1}{16}N^{-k\alpha}(z-x)}{(y-x)^2}\right) \\ &\geq C(\alpha)\sum_{k=1}^{\infty}\sum_{m=0}^{N^k-1}\left(\frac{\frac{1}{16}N^{-k\alpha}\frac{3}{16}N^{-k}}{\frac{1}{4}N^{-2k}}\right)^p |X_{k,m}|\cdot|Y_{k,m}|\cdot|Z_{k,m}| \\ &\geq C(\alpha)\sum_{k=1}^{\infty}\sum_{m=0}^{N^k-1}(N^k)^{(1-\alpha)p-3} \geq C(\alpha)\sum_{k=1}^{\infty}(N^k)^{p-p\alpha-2} .\end{aligned}$$

Hence, the energy of the graph of F is infinite whenever

$$p - p\alpha - 2 \geq 0 \qquad \Longleftrightarrow \qquad \alpha \leq 1 - \frac{2}{p} .$$

$\square$

6. Higher dimensional case

Here we establish an analogue of Theorem 5.1 for the energy $\mathcal{E}_p$.

Theorem 6.1. *Let $p > m(m+1)$ and set $\alpha = 1 - \frac{m(m+1)}{p}$. There exists a manifold Σ of class $C^{1,\alpha}$ such that $\mathcal{E}_p(\Sigma)$ is infinite.*

Our construction is based on the same idea as the construction presented in §5. Let $N \in \mathbb{N}$ be a big natural number and let $F : \mathbb{R} \to \mathbb{R}$ be defined by (6). We define

$$G : \mathbb{R}^m \to \mathbb{R}^{m+1} \quad \text{by the formula} \quad G(x^1, \dots, x^m) = (x^1, \dots, x^m, F(x^1))$$

and we set $\Sigma = G([0,1]^m)$. From Lemma 5.2 it follows that Σ is a $C^{1,\alpha}$ manifold.

To perform the proof of Theorem 6.1 we need to introduce some additional notation. By $(\mathbf{e}_1, \dots, \mathbf{e}_m)$ we denote the standard basis of $\mathbb{R}^m$. We adopt the convention to typeset points and vectors in $\mathbb{R}^m$ with the bold font $\mathbf{x}, \mathbf{y}, \mathbf{z}$ etc. and to number the components with a superscript, so we shall have $\mathbf{x} = (x^1, \dots, x^m)$. Also, vectors and points in $\mathbb{R}^{m+1}$ will always be marked with an arrow $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}}$ etc. and we silently assume that $\mathbf{x}$ and $\vec{\mathbf{x}}$ always satisfy

$$\pi_{\mathbb{R}^m}(\vec{\mathbf{x}}) = \pi_{\mathbb{R}^m}(x^1, \dots, x^{m+1}) = (x^1, \dots, x^m) = \mathbf{x}\,.$$

Let U and V be any m-dimensional subspaces of $\mathbb{R}^{m+1}$. We define

$$Q_U = \pi_{U^\perp} = \mathrm{id} - \pi_U \quad \text{and} \quad \sphericalangle(U, V) := \|\pi_U - \pi_V\| = \|Q_U - Q_V\|\,.$$

Let $\varepsilon > 0$ be some small constant - its value will be fixed later on. Let $A \in (0,1)$ be such that $|F'(t)| \le \varepsilon$ whenever $t < A$. For $n \in \mathbb{N}$ we set

$$J_n = N^{-n}\mathbb{Z}^m \cap [0, A)^m = \left\{ \left(\tfrac{k_1}{N^n}, \dots, \tfrac{k_m}{N^n} \right) : \forall i \in \{1, \dots, m\}\ k_i \in \{0, 1, \dots, \lceil AN^n - 1 \rceil\} \right\}.$$

For any $\mathbf{x} \in J_n$ we define

$$\mathbf{x}_0^{(\mathbf{x})} = \mathbf{x}\,, \quad \mathbf{x}_k^{(\mathbf{x})} = \mathbf{x} + \frac{\mathbf{e}_k}{2N^n} \quad \text{for } k = 1, \dots, m \quad \text{and} \quad \mathbf{x}_{m+1}^{(\mathbf{x})} = \mathbf{x} + \frac{\mathbf{e}_1}{4N^n}\,.$$

Of course $\mathbf{x}_i^{(\mathbf{x})}$ depends on n but we do not highlight it in our notation. Next, we choose a small number $\delta \in (0, \frac{1}{16})$ and we define

$$U(\mathbf{x}) = \mathbb{B}^m\big(\mathbf{x}_0^{(\mathbf{x})}, \tfrac{\delta}{N^n}\big) \times \dots \times \mathbb{B}^m\big(\mathbf{x}_{m+1}^{(\mathbf{x})}, \tfrac{\delta}{N^n}\big)\,.$$

Note that

$$|J_n| = \lceil AN^n \rceil^m \ge A^m N^{nm}\,, \tag{10}$$

$$\forall \mathbf{x} \in J_{n_1}\ \forall \mathbf{y} \in J_{n_2} \quad \mathbf{x} \ne \mathbf{y} \ \Rightarrow\ U(\mathbf{x}) \cap U(\mathbf{y}) = \emptyset \tag{11}$$

$$\text{and} \quad \forall \mathbf{x} \in J_n \quad \mathcal{H}^{m(m+2)}(U(\mathbf{x})) = \left(\frac{\omega_m \delta^m}{N^{nm}} \right)^{m+2}. \tag{12}$$

Let $T = (\mathbf{z}_0, \dots, \mathbf{z}_{m+1}) \in (\mathbb{R}^m)^{m+2}$ and $G(T) = (G(\mathbf{z}_0), \dots, G(\mathbf{z}_{m+1})) = (\vec{\mathbf{z}}_0, \dots, \vec{\mathbf{z}}_{m+1})$. We define

$$\begin{aligned}
\mathfrak{fc}(T) &= \triangle\{\vec{\mathbf{z}}_0, \dots, \vec{\mathbf{z}}_m\} && \text{- the face of } G(T) \text{ spanned by } \vec{\mathbf{z}}_0, \dots, \vec{\mathbf{z}}_m\,,\\
\mathfrak{p}(T) &= \operatorname{span}\{\vec{\mathbf{z}}_1 - \vec{\mathbf{z}}_0, \dots, \vec{\mathbf{z}}_m - \vec{\mathbf{z}}_0\} && \text{- the vector space containing } \mathfrak{fc}(T) - \vec{\mathbf{z}}_0\\
\text{and} \quad \mathfrak{h}(T) &= \operatorname{dist}(\vec{\mathbf{z}}_{m+1}), \vec{\mathbf{z}}_0 + \mathfrak{p}(T)) && \text{- the height of } G(T) \text{ lowered from } \vec{\mathbf{z}}_{m+1}\,.
\end{aligned}$$

Finally we define

$$\mathcal{K}_G(T) = \mathcal{K}(G(T)) = \mathcal{K}(G(\mathbf{z}_0), \ldots, G(\mathbf{z}_{m+1}))\,.$$

Note that

$$|JG(x)| = \sqrt{\det(DG(x)^T DG(x))} \geq 1\,,$$

so we have

$$\mathcal{E}_p(\Sigma) \geq \int_{[0,1]^{m(m+2)}} \mathcal{K}_G(\mathbf{z}_0, \ldots, \mathbf{z}_{m+1})^p \, d\mathbf{z}_0 \cdots \, d\mathbf{z}_{m+1}\,. \tag{13}$$

Proposition 6.2. *There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, if $\mathbf{x} \in J_n$ and $T = (\mathbf{z}_0, \ldots, \mathbf{z}_{m+1}) \in U(\mathbf{x})$, then we have*

$$\mathfrak{h}(T) \geq CN^{-n(1+\alpha)} \qquad \text{and} \qquad \mathcal{K}_G(T) \geq \tilde{C} N^{n(1-\alpha)}\,.$$

Using Proposition 6.2 we can finish the proof of the main theorem.

Proof of Theorem 6.1. Using (10), (11) and (12) together with (13) we get

$$\begin{aligned}
\mathcal{E}_p(\Sigma) &\geq \sum_{n=n_0}^{\infty} \sum_{\mathbf{x} \in J_n} \int_{U(\mathbf{x})} \mathcal{K}_G(T)^p \, d\mathbf{z}_0 \cdots \, d\mathbf{z}_{m+1} \\
&\geq \tilde{C} \sum_{n=n_0}^{\infty} A^m N^{nm} \frac{\omega_m^{m+2} \delta^{m(m+2)}}{N^{nm(m+2)}} N^{n(1-\alpha)p} \\
&= \hat{C}(\delta, m) \sum_{n=n_0}^{\infty} N^{n(m - m(m+2) + (1-\alpha)p)}\,,
\end{aligned}$$

which is infinite if and only if

$$m - m(m+2) + (1-\alpha)p \geq 0 \quad \iff \quad p \geq \frac{m(m+1)}{1-\alpha} \quad \iff \quad \alpha \leq 1 - \frac{m(m+1)}{p}\,.$$

In particular, for $\alpha = 1 - \frac{m(m+1)}{p}$ we have $\mathcal{E}_p(\Sigma) = \infty$. □

Now we only need to prove Proposition 6.2.

Proof of Proposition 6.2. Recall that $T = (\mathbf{z}_0, \ldots, \mathbf{z}_{m+1}) \in U(\mathbf{x})$ and $\mathbf{x} \in J_n$, so T and $\mathbf{x}$ depend on n. Note that the vectors $\mathbf{z}_i - \mathbf{z}_0$ for $i = 1, \ldots, m$ satisfy

$$\begin{aligned}
(1 - 2\delta)\frac{1}{2N^n} \leq |\mathbf{z}_i - \mathbf{z}_0| &\leq (1 + 2\delta)\frac{1}{2N^n} \\
\text{and} \quad |\langle \mathbf{z}_i - \mathbf{z}_0, \mathbf{z}_j - \mathbf{z}_0 \rangle| &\leq \frac{3\delta}{(2N^n)^2}\,.
\end{aligned}$$

Hence, decreasing δ we can make the vectors $\mathbf{z}_i - \mathbf{z}_0$ for $i = 1, \ldots, m$ form a roughly orthogonal basis of $\mathbb{R}^m$. Observe that $\varepsilon > 0$ can be chosen as small as we want and recall that we have $|F'| \leq \varepsilon$. Therefore, we can find ε such that for each n we have (see [4, §1.3] for the proof)

$$\sphericalangle(\mathfrak{p}(T), \mathbb{R}^m) = \|\pi_{\mathfrak{p}(T)} - \pi_{\mathbb{R}^m}\| = \|Q_{\mathfrak{p}(T)} - Q_{\mathbb{R}^m}\| \leq \frac{1}{2}\,. \tag{14}$$

Note that the vector $\mathbf{z}_{m+1} - \mathbf{z}_0$ lies in $\mathbb{R}^m$, so it can be expressed as

$$\mathbf{z}_{m+1} - \mathbf{z}_0 = \sum_{i=1}^{m} \zeta_i (\mathbf{z}_i - \mathbf{z}_0)\,,$$

for some $\zeta_1, \ldots, \zeta_m$. Observe that when we decrease δ to zero, the point $\mathbf{z}_{m+1}$ approaches the midpoint $\frac{1}{2}(\mathbf{z}_0 + \mathbf{z}_1)$, so ζ_1 converges to $\frac{1}{2}$ and all the other ζ_i for $i = 2, \ldots, m$ converge to 0. The values $|\zeta_1 - \frac{1}{2}|$ and $|\zeta_i|$ can be bounded above independently of the scale we are working in (i.e. independently of the choice of n) and also independently of the choice of T in $U(\mathbf{x})$. Hence

$$\left|\zeta_1 - \frac{1}{2}\right| \le \Xi(\delta) \quad \text{and} \quad |\zeta_i| \le \Xi(\delta) \quad \text{for } i = 2, \ldots, m\,, \quad \text{where } \ \Xi(\delta) \xrightarrow{\delta \to 0} 0 \tag{15}$$

and $\Xi(\delta)$ is a term depending only on δ and m.

For $i = 1, \ldots, m+1$ we set

$$\vec{\mathbf{z}}_i = G(\mathbf{z}_i) \quad \text{and we define} \quad \vec{\mathbf{w}} = \vec{\mathbf{z}}_0 + \sum_{i=1}^{m} \zeta_i (\vec{\mathbf{z}}_i - \vec{\mathbf{z}}_0) \in \vec{\mathbf{z}}_0 + \mathfrak{p}(T)\,.$$

Then

$$\mathbf{z}_{m+1} = \pi_{\mathbb{R}^m}(\vec{\mathbf{z}}_{m+1}) = \pi_{\mathbb{R}^m}(\vec{\mathbf{w}}) \quad \text{and} \quad (\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}) \perp \mathbb{R}^m$$
$$\text{and also} \quad Q_{\mathfrak{p}(T)}(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}) = Q_{\mathfrak{p}(T)}(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{z}}_0)\,.$$

Therefore, using the angle estimate (14) we obtain

$$\begin{aligned} \mathfrak{h}(T) &= \operatorname{dist}(\vec{\mathbf{z}}_{m+1}, \vec{\mathbf{z}}_0 + \mathfrak{p}(T)) = |Q_{\mathfrak{p}(T)}(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{z}}_0)| \\ &= |Q_{\mathfrak{p}(T)}(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}})| = |Q_{\mathbb{R}^m}(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}) - (Q_{\mathbb{R}^m} - Q_{\mathfrak{p}(T)})(\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}})| \\ &\ge |\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}| - \frac{1}{2}|\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}| = \frac{1}{2}|\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}|\,. \end{aligned} \tag{16}$$

Hence, to get a lower bound on $\mathfrak{h}(T)$ it suffices to estimate $|\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}|$ from below. We calculate

$$\begin{aligned} |\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}| &= \left| \big(\mathbf{z}_{m+1}, F(z^1_{m+1})\big) - \Big(\mathbf{z}_{m+1}, F(z^1_0) + \sum_{i=1}^{m} \zeta_i (F(z^1_i) - F(z^1_0))\Big) \right| \\ &= \left| F(z^1_{m+1}) - F(z^1_0) + \sum_{i=1}^{m} \zeta_i (F(z^1_i) - F(z^1_0)) \right| \\ &\ge \big| F(z^1_0 + \zeta_1(z^1_1 - z^1_0)) - F(z^1_0) - \zeta_1(F(z^1_1) - F(z^1_0)) \big| \\ &\quad - \Bigg| \bigg(F\Big(z^1_0 + \zeta_1(z^1_1 - z^1_0) + \sum_{i=2}^{m} \zeta_i (z^1_i - z^1_0)\Big) - F(z^1_0 + \zeta_1(z^1_1 - z^1_0)) \bigg) \\ &\qquad - \sum_{i=2}^{m} \zeta_i (F(z^1_i) - F(z^1_0)) \Bigg|\,. \end{aligned} \tag{17}$$

Using (15) one can estimate the term

$$\big| F(z^1_0 + \zeta_1(z^1_1 - z^1_0)) - F(z^1_0) - \zeta_1(F(z^1_1) - F(z^1_0)) \big|$$

exactly the same way as in the one dimensional case (cf. (9) with z^1_0, z^1_1 and $z^1_0 + \zeta_1(z^1_1 - z^1_0)$ playing the roles of x, y and z respectively and with $\frac{z-x}{y-x} = \zeta_1$) and then obtain

$$\big| F(z^1_0 + \zeta_1(z^1_1 - z^1_0)) - F(z^1_0) - \zeta_1(F(z^1_1) - F(z^1_0)) \big| \ge C|z^1_1 - z^1_0|^{1+\alpha} \ge \tilde{C} N^{-n(1+\alpha)}\,. \tag{18}$$

We estimate the remaining terms using the mean value theorem for F. First we have

$$F\big(z_0^1 + \zeta_1(z_1^1 - z_0^1) + \sum_{i=2}^{m} \zeta_i(z_i^1 - z_0^1)\big) - F(z_0^1 + \zeta_1(z_1^1 - z_0^1)) = F'(\xi_{m+1}) \sum_{i=2}^{m} \zeta_i(z_i^1 - z_0^1)\,, \tag{19}$$

where ξ_{m+1} is some number satisfying

$$z_{m+1}^1 - \frac{2m\delta}{N^n} < \xi_{m+1} < z_{m+1}^1 + \frac{2m\delta}{N^n}$$

In the same way we obtain

$$\sum_{i=2}^{m} \zeta_i(F(z_i^1) - F(z_0^1)) = \sum_{i=2}^{m} F'(\xi_i)\zeta_i(z_i^1 - z_0^1)\,, \tag{20}$$

where $\xi_i \in (z_0^1 - \frac{2\delta}{N^n}, z_0^1 + \frac{2\delta}{N^n})$. Note that

$$|\xi_{m+1} - \xi_i| \le C|z_{m+1}^1 - z_i^1| \le C' N^{-n}\,,$$

$$|z_i^1 - z_0^1| \le \bar{C}\delta N^{-n} \quad \text{and} \quad |F'(\xi_i) - F'(\xi_{m+1})| \le \hat{C}|\xi_i - \xi_{m+1}|^{\alpha} \le \tilde{C} N^{-n\alpha}\,.$$

Putting (19) and (20) together we get

$$\begin{aligned} \Bigg| \bigg(F(z_0^1 + \zeta_1(z_1^1 - z_0^1) + \sum_{i=2}^{m} \zeta_i(z_i^1 - z_0^1)) - F(z_0^1 + \zeta_1(z_1^1 - z_0^1)) \bigg) \\ - \sum_{i=2}^{m} \zeta_i(F(z_i^1) - F(z_0^1)) \Bigg| = \Bigg| \sum_{i=2}^{m} (F'(\xi_i) - F'(\xi_{m+1}))\zeta_i(z_i^1 - z_0^1) \Bigg| \le \hat{C}\delta N^{-n(1+\alpha)}\,. \end{aligned} \tag{21}$$

Plugging (18) and (21) into (17) we obtain

$$|\vec{\mathbf{z}}_{m+1} - \vec{\mathbf{w}}| \ge \tilde{C} N^{-n(1+\alpha)} - \hat{C}\delta N^{-n(1+\alpha)}\,.$$

Since we have a freedom in choosing δ, we can make $\hat{C}\delta$ as small as we want, so recalling (16) the first part of Proposition 6.2 is proven. The second part follows from a simple calculation

$$\begin{aligned} \mathcal{K}_G(T) &= \frac{\mathcal{H}^{m+1}(\triangle(G(T)))}{\operatorname{diam}(G(T))^{m+2}} = \frac{\mathfrak{h}(T)\mathcal{H}^m(\mathfrak{fc}(T))}{(m+1)\operatorname{diam}(G(T))^{m+2}} \\ &\ge C\frac{\mathfrak{h}(T)\mathcal{H}^m(\pi_{\mathbb{R}^m}(\mathfrak{fc}(T)))}{\operatorname{diam}(T)^{m+2}} \ge \tilde{C}\frac{N^{-n(1+\alpha)}N^{-nm}}{N^{-n(m+2)}} = \tilde{C}N^{n(1-\alpha)}\,. \end{aligned}$$

□

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