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Reply to Comment on 'Wigner function for a particle in an infinite lattice'

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Abstract. In a recent paper (2012 *New J. Phys.* **14** 103009), we proposed a definition of the Wigner function for a particle on an infinite lattice. Here we argue that the criticism to our work raised by Bizarro is not substantial and does not invalidate our proposal.

In [1], we proposed the following definition for the discrete Wigner function for an infinite-dimensional Hilbert space, such as that of a particle moving on an infinite lattice:

$$W(m,k) \equiv \mathrm{tr}\,[\rho A(m,k)] = \frac{1}{2\pi}\sum_n \langle n|\rho|m-n\rangle \mathrm{e}^{-\mathrm{i}(2n-m)k} \qquad (1)$$

with the indefinite sums here and in the following running over all the integers. This definition fulfills all the fundamental properties required from a Wigner representation of the quantum state of the system. Moreover, it allowed us to generalize the concept of negativity to the states of the particle on the lattice while keeping the positive character of discretized Gaussian distributions in a consistent way with the continuum results.

Bizarro [2] has criticized the definition above, with the objection that it does not obey what he considers the defining requirements of a Wigner function for this system. However,

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although the Wigner function offers a full description of the physical state of the system, it is not a physical object in itself. In fact, different phase space representations exist which describe the state completely and exhibit various mathematical properties.

A list of formal properties that uniquely define the Wigner function was originally established by Wigner and co-worker [3] for the continuous case, precisely in an attempt to single out the definition of this function from other phase space pseudo-probability distributions using a set of physically sound requirements.

In an earlier work, Bizarro [4] mapped these defining properties of the continuum Wigner function to the problem of canonically conjugated phase and angular momentum variables, analogous to the system we describe. With this mapping, he concluded that the unique Wigner function for such a system reads, instead [5, equation (57)]

$$W_{\mathrm{B}}(m,k) \equiv \frac{1}{2\pi}\sum_{n}\langle m-n|\rho|m+n\rangle \mathrm{e}^{-\mathrm{i}2nk}+\frac{1}{2\pi^2}\sum_{n,n'}\frac{(-1)^{n'}}{n'+\frac{1}{2}}\langle m+n'-n|\rho|m+n'+n+1\rangle \mathrm{e}^{-\mathrm{i}(2n+1)k}. \tag{2}$$

An identical Wigner function was independently derived by Rigas *et al* [6], who also determined that the only states for which (2) is positive over the whole phase space are the localized states in the $\{|m\rangle\}$ basis [7]. At the time the paper was written, we were unaware of the work by Bizarro [4], and therefore unintentionally omitted the proper reference.

However, the particular expression of the defining constraints, and thus the uniqueness of Bizarro's definition for the Wigner function is determined not only by a physically meaningful set of properties, but also by his particular (but non-unique) choice for the phase space.

Bizarro's criticism refers, in particular, to the following two requirements:

(i) Having the correct space probability distribution as marginal

$$\int_{-\pi}^{\pi} \mathrm{d}k\, W_{\mathrm{B}}(m,k) = \langle m|\rho|m\rangle. \tag{3}$$

(ii) Transforming under translations as

$$\langle n|\rho'|m\rangle = \langle n-m_0|\rho|m-m_0\rangle \Leftrightarrow W'_{\mathrm{B}}(m,k) = W_{\mathrm{B}}(m-m_0,k). \tag{4}$$

In contrast, our definition satisfies

(i) $$\int_{-\pi}^{\pi} \mathrm{d}k\, W(2n,k) = \langle n|\rho|n\rangle, \quad \int_{-\pi}^{\pi} \mathrm{d}k\, W(2n+1,k) = 0, \tag{5}$$

(ii) $$\langle n|\rho'|n'\rangle = \langle n-n_0|\rho|n'-n_0\rangle \Leftrightarrow W'(m,k) = W(m-2n_0,k), \tag{6}$$

while the remaining conditions in [4] ([2, (4, 11)]) are satisfied in exactly the same form as for Bizarro's definition.

Equations (3) and (4) are not among the fundamental requirements from the axiomatic definition of the Wigner function in the continuum [3, 8]. They are instead the transcription of some of these requirements to a particular phase space definition for the discrete system at hand.

The origin of the translation property (4) is the requirement of Galilean invariance, which following [8] requires commutativity of the diagram

$$\begin{array}{ccc} \rho & \xrightarrow{T^{n_0}} & T^{n_0}\rho\left(T^{\dagger}\right)^{n_0} \\ \downarrow & & \downarrow \\ W_{\rho} & \xrightarrow{V(n_0)} & W_{\rho_{\mathrm{T}}}, \end{array} \tag{7}$$

where T^{n_0} represents a spatial translation by n_0 lattice units, ρ_{T} denotes the translated state and $V(n_0)$ is the corresponding transformation in phase space. In other words, the requirement that the Wigner function for the translated state can be recovered by directly transforming the Wigner function in phase space.

Since ρ and W live in different spaces, obviously they may transform differently. Equation (4) is obtained only if one additionally requires that spatial translations act on the space-like variable of phase space, m, in exactly the same way as on the real line. With our choice of phase space, it is easy to check that the action of space translations in phase space reads

$$W(m,k) \xrightarrow{V(n_0)} W(m-2n_0,k) \tag{8}$$

and that the fundamental transformation requirement (7) is indeed satisfied by definition (1).[4]

On the other hand, equation (3) derives from the requirement that integration over any of the phase space variables can be interpreted as a marginal probability. This requirement is also fulfilled by our definition. Indeed, when integrating over k, our Wigner function yields a valid probability distribution (positive and normalized), which actually gives the correct probabilities for every point in the lattice, $P(n) \equiv \langle n|\rho|n\rangle = \int_{-\pi}^{\pi} W(2n,k)\,\mathrm{d}k$, and assigns zero probability to the half points $P(n+\frac{1}{2})=0$.

The doubling of phase space points is not an unusual feature, having been discussed before for finite-dimensional systems (see, for instance, the phase space constructions in [9–11]). The freedom in the definition of the phase space is also evident from the variety of Wigner function constructions proposed for such systems (e.g. [12–15]).

In our case, the particular structure of the phase space is indeed useful to make evident the particularities due to the discreteness of the system. For instance, with our prescription the even m points in phase space contain the direct discretization of the continuum Wigner function, which can then be easily recovered and compared with the complete representation.

Our definition is thus a valid Wigner function, fulfilling all the fundamental requirements and allowing the complete description of the system, as we discussed in [1]. In particular, the Wigner function (2) can be reconstructed by[5]

$$W_{\mathrm{B}}(n,k) = W(2n,-k) + \frac{2}{\pi}\sum_{n'} \frac{(-1)^{n'-n}}{2n'+1-2n} W(2n'+1,-k). \tag{9}$$

For some purposes, it might be more useful to employ our description, instead of using the compact phase space on [4]. For instance, with the definition (2), even and odd distances in the real lattice have a very different representation. This is evident from the Wigner representation (2) for a superposition of two deltas, whose structure depends dramatically on whether the separation between both terms is even or odd, as can be seen in figure 2 of [5]. This is true even for arbitrarily large separations. In contrast, with our definition the Wigner function for a superposition of two deltas has a similar aspect, independent of the parity of their distance, and it admits a closed description (see [1, figure 5 and equation (24)]). Also, for discrete Gaussian states and their superposition, it is possible to compute a closed form of the Wigner function (1). Moreover, using (1) it is easy to distinguish between negative contributions

[4] Momentum translations induce a similar transformation law, with $W(m,k) \to W(m,k-k_0)$ when applying a momentum displacement k_0.

[5] Notice the typo in [2] for the same relation.

to the Wigner function coming from the continuum equivalent of the state and those originating from the structure of the phase space.

In conclusion, as we have argued, Bizarro's comment [2] is not of substantial nature and does not invalidate our work.

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