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THE FLUCTUATION THEOREM IN A SINAI-BILLIARD GEOMETRY: PREDICTIONS AND MEASUREMENTS

Malte Schmick*, Alexander Hasselhuhn*, Qi Liu†, Qi Ouyang† and Mario Markus

** Max-Planck-Institut für molekulare Physiologie*
Postfach 500247, 44202 Dortmund, Germany
e-mail: mario.markus@mpi-dortmund.mpg.de

† Department of Physics,
Peking University,
Beijing 100871, People's Republic of China
e-mail: qi@pku.edu.cn

Abstract

The fluctuation theorem, which quantifies the probability that a system returns energy to its power supply for small times, is shown experimentally and numerically to hold for a deterministic, one-particle, chaotic system. This indicates that the validity of this theorem is not restricted to many-particle systems or stochastic processes, as has been previously assumed.

Introduction

In the last fifteen years a set of closely related formulations of the Fluctuation Theorem (FT) have been proposed. This theorem describes deviations from the Second Law of Thermodynamics for small systems at small time scales. (For

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reviews, see [1, 2, 3]). The FT is of fundamental importance for nanomachines and for biophysical motors [3]. We consider here the formulation

$$\frac{p(J_\tau)}{p(-J_\tau)} = e^{\frac{J_\tau \tau}{\beta_\tau}} \quad , \tag{1}$$

where J_τ is the mean flux of heat, momentum, work, etc. during the time τ. Here, we consider the mean flux of the work W_τ, i.e. the mean power $J_\tau = W_\tau/\tau$. $p(J_\tau)$ is the probability that the power supply enables the system to perform a work W_τ, whereas $p(-J_\tau)$ is the probability that the system sends the energy W_τ back to its power supply.

The FT was first shown to hold for shear-flow simulations [4]. Later on, it was proven by considering time reversible (highly) chaotic Anosov systems consisting of many particles [5]. Thereafter, experimental verfications were reported for: i) fluctuations in turbulent flows of fluids [6, 7, 8]; ii) a brownian particle in an optical trap, which is translated by forces in the order of pico-Newton [9, 10]; and iii) voltage fluctuations of a resistor at currents in the order of pico-Ampere [11].

At the turn of the millenium, several investigations [12, 13, 14, 15, 16] showed that a many-particle system is not necessary for the validity of the FT, provided stochasticity is assumed. Moreover, Feitosa and Menon [17] showed that the validity of the FT is not restricted to processes governed by molecular fluctuations. In fact, their experimental demonstrations were carried out with a granular medium consisting of shaken mm-sized beads. Further extensions of the FT to macroscopic systems were also performed in simulations of chains of coupled nonlinear oscillators [18, 19] and of an earthquake model consisting of chains of blocks connected by springs [20]. Further loosening of the conditions for the FT is found in simulations of a single particle, which is shaken chaotically in a Duffing potential [21]. In the present work we shall present measurements of such a one-particle system. However, since the chaotic shaking in a Duffing potential proved difficult to be implemented experimentally, we shall investigate a periodically driven particle moving chaotically in a Sinai-billiard configuration. We will compare the experimental results with simulations and we will show that the FT, as given by Eq. (1), holds.

1 Experiments

A glass sphere (diameter: 1.2 cm; mass: $m = 2.21$ g) was placed in a box (15 cm x 15 cm; height: 1.5 cm) with 4x4 cylindrical steel pins (diameter: 0.5 cm; height: 1.5 cm), which were fixed equidistantly on the bottom of the box. A steel wall surrounds the box at its edges. The distance between the pins is

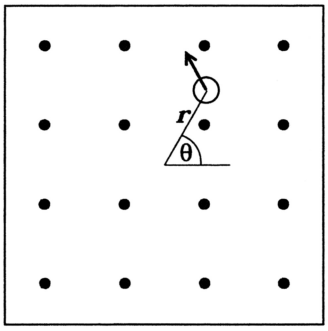

Figure 1: *Scheme of the box that was used in the experiments. The black circles indicate the (fixed) pins and the open circle the (moving) sphere with an exemplary velocity indicated by the arrow.*

$d = 3.75$ cm, while the distance between the pins and the wall of the box is $d/2$. A scheme of the box is shown in Fig. 1. The box with the sphere is placed on a gyratory mixer (GM 1, *Ratek Instruments*, Australia). This gyratory device operates in such a way that a perpendicular line fixed at the center of the box's bottom precesses about verticality with an adjustable frequency f and angle α. A formal description of this movement is given in the following. Given polar coordinates (r, ϕ) on the bottom of the box (see Fig. 1), the laboratory cylindrical coordinates are $\tilde{r}(r, \theta, t) = r\cos(\alpha)$, $\tilde{\theta}(r, \theta, t) = \theta$ and $\tilde{z}(r, \theta, t) = r\sin(\alpha)\cos(2\pi ft + \theta)$. This forced motion in z-direction supplies the sphere with potential energy. The gravitational force $\vec{F}$ acting on the sphere is:

$$\vec{F} = m\,g\,\sin(\alpha)\,[\sin(2\pi ft)\hat{x} + cos(2\pi ft)\hat{y}]\,, \tag{2}$$

with the gravitational acceleration $g = 981\,cm/s^2$. $\hat{x}$ and $\hat{y}$ are the unit vectors on the bottom of the box. For this experiment we chose $f = 0.6\,Hz$ and $\alpha = 5°$. Although the driving of the system is periodical, the well-known divergence of nearby trajectories in this Sinai-billiard geometry causes a chaotic motion of the sphere.

The sphere was monitored by a video equipment every 3.6×10^{-3} s (278 fps). 1.2×10^7 frames (12 hours) were recorded. The position of the sphere was tracked by image analysis and its velocity calculated from the positions in successive frames.

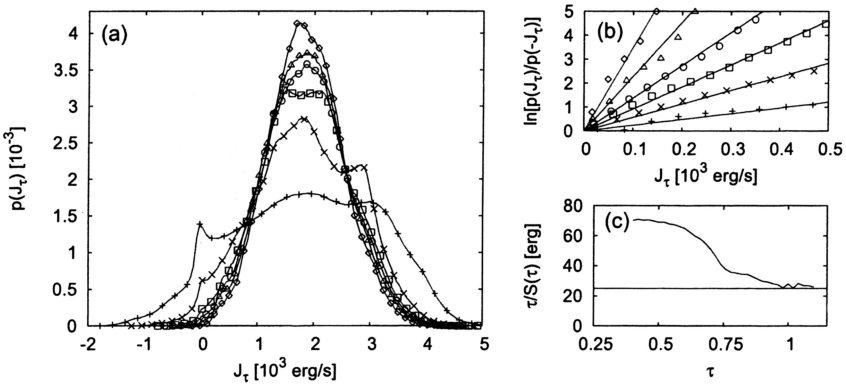

Figure 2: *Experimental results. (a) Probability distribution functions of the mean power* J_τ. $\tau = 0.1\,s$ *(+), 0.4s (×), 0.6s (□), 0.7s (○), 0.8s (△) and 1s (◇). (b) Linearizations obtained from the distribution functions shown in (a). (c)* $\beta_\tau = \tau/S_\tau$, *where* S_τ *are the slopes of the straight lines shown in (b).*

Given a time τ, we determined

$$J_\tau = \frac{1}{\tau}\int_t^{t+\tau} \vec{F}\cdot\vec{v}\,dt \quad , \tag{3}$$

where $\vec{v}$ is the recorded velocity of the sphere and $\vec{F}$ is the gravitational force acting on the sphere. The time dependence of $\vec{F}$, due to the inclination of the box, was determined by registering the rotating motion of the corners of the box.

Returning now to the Introduction, it is useful for the understanding of these investigations to define the terms "system" and "power supply". The system in our case is the sphere; the power supply consists of potential energy provided by the gyratory mixer to the sphere. During time intervals in which $\vec{F}\cdot\vec{v} > 0$, the sphere will move downwards translating potential energy supplied by the mixer into kinetic energy. During time intervals in which $\vec{F}\cdot\vec{v} < 0$, the sphere moves against gravity, i.e. it restores potential energy out of its kinetic energy. The integral in Eq. (3) tells us which of the two processes (consumption or storage of potential energy) predominates in the interval of duration τ.

Fig. 2a shows the measured probability distribution function $p(J_\tau)$. Note that the integral over the negative tail of the curves ($J_\tau < 0$) becomes smaller as τ increases. In other words, flows against gravity become, as expected from the FT, increasingly improbable for increasing times of observation. For a

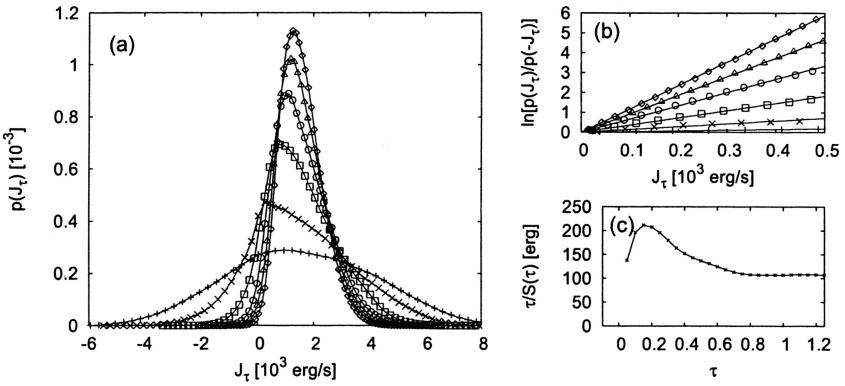

Figure 3: *As Fig. 2, but for simulations using restitution coefficients* $e_p = e_w = 0.7$. $\tau = 0.05\,s$ *(+), 0.25s (×), 0.5s (□), 0.75s (○), 1.0s (△) and 1.25s (◇).*

quantitative check of the FT, as given by Eq. (1), we plotted $\ln[p(J_\tau)/p(-J_\tau)]$ versus J_τ in Fig. 2b. We obtain surprisingly linear relationships, confirming Eq. (1). Furthermore, Fig. 2c shows $\beta_\tau = \tau/S_\tau$ versus τ, where S_τ is the slope of the straight lines in Fig. 2b. In accordance with original derivations of the FT [1, 2, 5], as well as with a number of simulations [16, 20] and experiments [6], Fig. 2c shows that β_τ saturates for large τ to an effective "temperature" $\beta_\infty \approx 25\,\mathrm{erg}$.

2 Simulations

As an approximation, we simulated the system in two dimensions, considering a circle (diameter: 1.2 cm) moving without friction and without rotation. The dimensions of the box and the pins were chosen the same as in the experiments. For the collision of the circle with the pins and with the wall, the component of the velocity (after the collision) perpendicular to a pin or to the wall was multiplied by the restitution coefficient. 4×10^7 snapshots were evaluated.

For the restitution coefficients (e_w for collisions with the wall and e_p for collisions with the pins) we used two parameter sets: i) $e_w = e_p = 0.7$; and ii) $e_w = 1$, $e_p = 0.7$. The set i) was used to mimic the experimental situation, while the set ii) was used to determine the effect of elastic collisions with the wall. For $\alpha = 0$ and $e_p = e_w = 1$ the system is equivalent to the "classical" Sinai-billiard found in the literature. In the latter, collisions with the wall are equivalent to an absent wall and an infinite number of pins behind it. Fig. 3 shows the simulations using parameter set i) and Fig. 4 those using set ii).

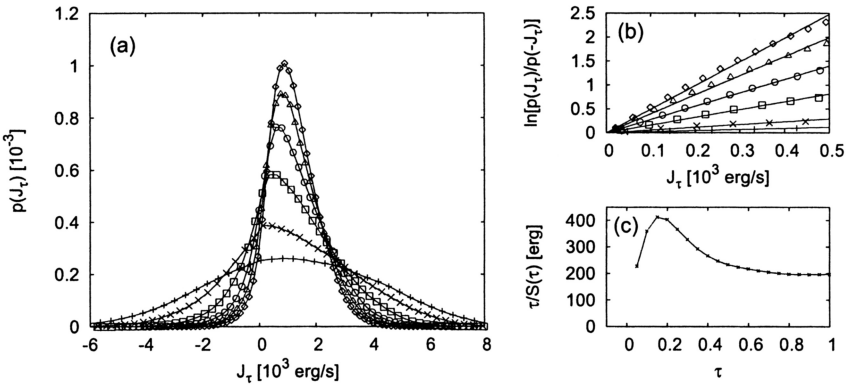

Figure 4: *As Fig. 2, but for simulations using restitution coefficients $e_p = 0.7$ and $e_w = 1$, i.e. the collisions of the sphere with the wall are elastic. $\tau = 0.05\,s$ (+), 0.2s (×), 0.4s (□), 0.6s (○), 0.8s (△) and 1.0s (◇).*

3 Discussion

In order to check the robustness of experimental and numerical results with respect to the number of evaluated points, we reduced this number to one half and obtained invariant distributions $p(J_\tau)$ (Figs. 2a, 3a and 4a). We did not obtain Gaussian distributions $p(J_\tau)$, as reported for other experiments and simulations [6, 10, 15, 20]. However, as shown elsewhere [11, 22, 23], Gaussian distributions are not essential for the validity of the FT. In our distributions, we observe a tendency to an enhanced $p(J_\tau)$ at $J_\tau \approx 0$, which is due to the slowing down of a sphere right after a collision.

In our simulations with inelastic collisions (Fig. 3c) we obtain an effective "temperature" $\beta'_\infty \approx 108\,\mathrm{erg}$. In the simulations with elastic collisions with the wall (Fig. 4c), this "temperature" is $\beta''_\infty \approx 197\,\mathrm{erg}$. For a better understanding of the deviations of β'_∞ and β''_∞ from the experimental value of $\beta_\infty = 25\,\mathrm{erg}$, we determined the mean kinetic energy $\langle E_{kin}\rangle$ in the three cases. For the experiments, $\langle E_{kin}\rangle = 278\,\mathrm{erg}$. In the simulations with inelastic collisions (Fig. 3), $\langle E_{kin}\rangle' = 2100\,\mathrm{erg}$, the deviations from the experimental value being explainable by the omission of friction (besides the omission of rotation and the reduction to two dimensions) in the calculations. Even less dissipation occurs in the simulations considering elastic collisions with the wall (Fig. 4); thus an even larger mean kinetic energy, namely $\langle E_{kin}\rangle'' = 2400\,\mathrm{erg}$, is obtained. As expected, we find a positive correlation between the effective "temperatures" $\beta_\infty < \beta'_\infty < \beta''_\infty$ and the mean kinetic energies $\langle E_{kin}\rangle < \langle E_{kin}\rangle' < \langle E_{kin}\rangle''$.

Nevertheless, one should not overestimate the results concerning effective "temperatures" of a one-particle, macroscopic system. Instead, attention should be paid to the linearity of Figs. 2b, 3b and 4b and the saturation for large τ in Figs. 2c, 3c and 4c.

4 Conclusions

We have shown here, both experimentally and numerically, that it is not necessary to assume very small systems, a large number of particles or stochasticity for the FT to hold. In fact, our macroscopic, deterministic, mechanical system, consisting of only one chaotic particle can be described by the FT, as given by Eq. (1).

The next step is in the hands of mathematicians: the loosening of conditions necessary for the validity of the fluctuation theorem.

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