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The Dimension of C^1 Splines of Arbitrary Degree on a Tetrahedral Partition

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Abstract

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§1. Introduction

Splines spaces are of particular interest in Approximation Theory and Computer Aided Geometric Design. For splines in one variable there exists an almost completely developed theory (cf. de Boor [5], Nürnberger [16], Prautzsch, Boehm & Paluszny [20], Schumaker [21]). On the other hand, much less is known for bivariate and trivariate splines (cf. Chui [8], Zeilfelder & Seidel [27]), i.e., splines which are defined on triangulations and tetrahedral partitions, respectively. The main reason for this is that such spaces have a more complex structure than univariate spline spaces, and even basic problems for these spaces are sometimes difficult to solve.

One such basic question in the multivariate spline theory is to determine the dimension of the spaces. In order to develop efficient approximation and interpolation methods (cf. Nürnberger & Zeilfelder [18], and, for instance, the approaches from Davydov & Zeilfelder [10], Haber, Zeilfelder, Davydov & Seidel [12], Nürnberger, Schumaker & Zeilfelder [17], Nürnberger & Zeilfelder [19]), it is important to know the dimension of these spaces. This problem is easy to solve for continuous multivariate splines. The situation is completely different and stands in striking contrast to univariate theory if we consider smooth multivariate splines.

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In this case, the problem of determining the dimension of splines on given partitions becomes a complex task when the degree of the splines is low. For bivariate splines on given triangulations the most general results are lower and upper bounds on the dimension (cf. Schumaker [22,23]). Moreover, the dimension is known for splines on uniform partitions (cf. Chui & Wang [9]), on arbitrary triangular cells (cf. Schumaker [24]), and for certain degrees (cf. Alfeld, Piper & Schumaker [2], Hong [13], Ibrahim & Schumaker [14]). For trivariate splines bounds on the dimension of the spaces are difficult to obtain, in general, and it has been recognized that even for splines defined on arbitrary tetrahedral (half) cells an exact dimension count would require at least some knowledge on the dimension of bivariate spline spaces of arbitrary degree on triangulations (cf. Alfeld, Schumaker & Sirvent [3, Example 25], Alfeld, Schumaker & Whiteley [4, Remark 66]).

There are only few papers on the dimension of trivariate splines. Early results known from the finite element literature (cf. Ženišek [28]) deal with certain subspaces (which are now called super spline spaces) of splines with relatively high degree. For splines of low degrees, results are known mainly for C^1 splines. For instance, Alfeld [1] developed a local Hermite interpolation method using trivariate C^1 quintic super splines on tetrahedral partitions, where all the tetrahedra are split into four subtetrahedra (trivariate Clough-Tocher split). Quintic C^1 super splines on a uniform type partition and on certain classes of tetrahedral partitions were investigated in connection with local interpolation methods of Schumaker & Sorokina [25], and Lai & Le Méhauté [15]. Farin & Worsey [26] generalized the bivariate Clough-Tocher element for C^1 cubic splines by splitting each tetrahedron into 12 subtetrahedra. For an application of this method in the context of so-called A-patches, see Bajaj, Bernardini & Xu [7]. As a byproduct of these methods the dimension of the spaces on the resulting tetrahedral partition was determined.

In this paper, we investigate the problem of determining the dimension of trivariate C^1 splines of arbitrary degree on a uniform type tetrahedral partition Δ, where no tetrahedron is splitted. The partitions Δ are obtained as a natural generalization of the four-directional mesh in the bivariate spline theory. Roughly speaking, given a uniform cube partition, each cube Q is subdivided into 24 tetrahedra which have the center of Q as a common vertex (see Figure 3). The dimension of the trivariate C^1 splines on Δ is determined by constructing a suitable minimal determining set $\mathcal{M}$ for the spaces. For doing this, we use a well-known characterization of the C^1 smoothness across the common triangular faces of two neighboring polynomial pieces in Bernstein-Bézier representation (cf. de Boor [6], Farin [11]). We first give minimal determining sets for C^1 splines on a particular tetrahedral cell, i.e., one cube which is subdivided into 24 tetrahedra. Then, we construct step by step a minimal determining set $\mathcal{M}$ for the whole spline space. This is done by considering the tetrahedra of the partition in an appropriate order (see the proof of Theorem 6.1 in Section 6). Here, in each step the remaining degrees of freedom are determined. In this way, we obtain explicit formulae for the dimension of the C^1 spline spaces of arbitrary degree (Theorem 3.1 and Corollary 3.2). The proof of this result is complex.

The paper is organized as follows. In Section 2 we give some preliminaries on trivariate splines, their piecewise Bernstein-Bézier representation, minimal determining sets, and smoothness conditions. In Section 3, we define a uniform tetrahedral partition Δ and state our main results. We give explicit formulae for the dimension of C^1 spline spaces of arbitrary degree on Δ. In Section 4, we introduce some notation needed for the subsequent investigations and we rewrite the smoothness conditions of the spaces in a convenient form. Section 5 contains minimal determining sets for C^1 splines on a special tetrahedral cell which consists of 24 tetraedra. These results are used in Section 6 where we define a suitable minimal determining set for the spline spaces and prove our main results. The paper concludes with some remarks.

§2. Trivariate Splines, Bernstein-Bézier Representation and MDS

We briefly recall some notation well-known in multivariate spline theory (cf. Alfeld, Schumaker & Whiteley [4], de Boor [6], Chui [8], Farin[11]). For any integer q, we call

$$\mathcal{P}_q = \operatorname{span}\{x^i y^j z^k : \ i,j,k \geq 0, \ i+j+k \leq q\}$$

the $\binom{q+3}{3}$ dimensional space of trivariate polynomials of total degree q. Given a (non-degenerate) tetrahedron $T = [v_0, v_1, v_2, v_3]$ in $\mathbb{R}^3$ with vertices v_0, v_1, v_2, and v_3, the linear polynomials $\lambda_\nu \in \mathcal{P}_1, \ \nu = 0, \ldots, 3$, with the interpolation property

$$\lambda_\nu(v_\mu) = \delta_{\nu,\mu}, \qquad \mu = 0, \ldots, 3,$$

are called the barycentric coordinates w.r.t. T. Every polynomial $p \in \mathcal{P}_q$ can be written in its Bernstein-Bézier representation as

$$p = \sum_{i+j+k+\ell=q} a_{i,j,k,\ell} \ B^{q,T}_{i,j,k,\ell}, \tag{2.1}$$

where

$$B^{q,T}_{i,j,k,\ell} = \tfrac{q!}{i!j!k!\ell!} \ \lambda_0^i \lambda_1^j \lambda_2^k \lambda_3^\ell \in \mathcal{P}_q, \qquad i+j+k+\ell = q,$$

are the Bernstein polynomials of degree q w.r.t. T.

Each Bernstein-Bézier coefficient $a_{i,j,k,\ell}$ of p is associated with the domain point

$$\xi^T_{i,j,k,l} = (iv_0 + jv_1 + kv_2 + \ell v_3)/q,$$

and the set of domain points in T is denoted by

$$\mathcal{D}_{q,T} = \{\xi^T_{i,j,k,\ell} : \ i+j+k+\ell = q\}.$$

We call a set of tetrahedra Δ a tetrahedral partition of a finite domain $\Omega \subseteq \mathbb{R}^3$ if the intersection of any two different tetrahedra from Δ is a common vertex, common edge or common triangle, and the union of all tetrahedra from Δ is equal to Ω.

Given a tetrahedral partition Δ of a domain Ω and $r \in \{-1, \ldots, q-1\}$, we set

$$\mathcal{S}^r_q(\Delta) = \{s \in C^r(\Omega) : s|_T \in \mathcal{P}_q \text{ for all tetrahedra } T \in \Delta\}$$

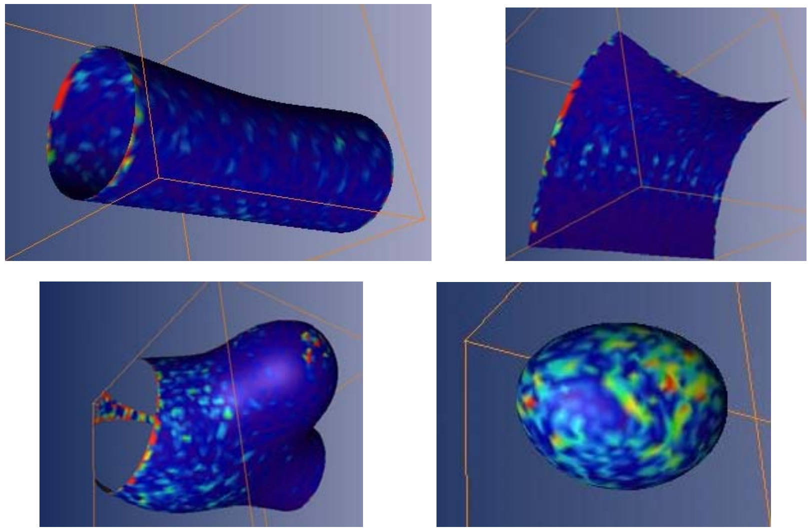

Fig. 1. Isosurfaces with different isovalues of a trivariate spline.

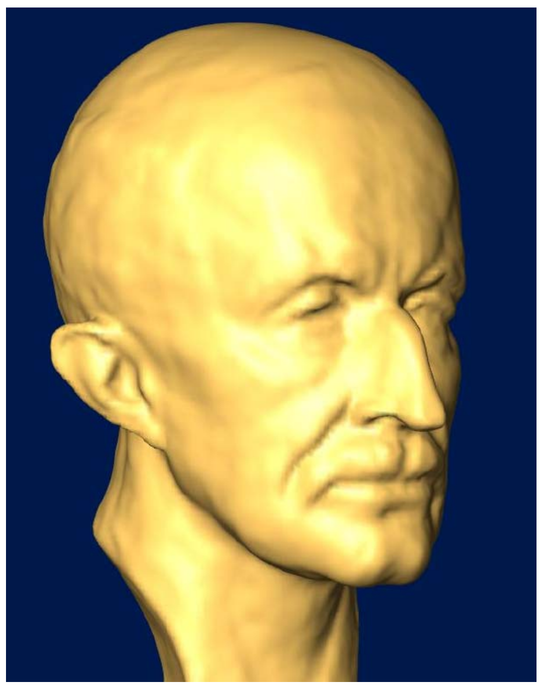

Fig. 2. An isosurface of a trivariate spline approximating volume data representing the boundary of a human head. The data in this example were obtained from a bust of Max Planck.

for the space of trivariate C^r splines of degree q w.r.t. Δ. Figure 1 shows an example of a trivariate spline s. Here, we visualized different isosurfaces, i.e., for different real values of c (called isovalues), we show the surfaces, where s satisfies $s = c$. An other example of an isosurface of a trivariate spline is given in Figure 2. This spline is obtained by applying an approximation method of the authors which is currently under development.

The coefficients

$$a_{i,j,k,\ell} = a_{i,j,k,\ell}(s) = a_{i,j,k,\ell}(s|_T), \qquad i+j+k+\ell = q,$$

of $s \in \mathcal{S}_q^0(\Delta)$ in the representation (2.1) of its polynomial pieces $s|_T$, $T \in \Delta$, are uniquely associated with the domain points in Ω which we denote by

$$\mathcal{D}_{q,\Delta} = \bigcup_{T \in \Delta} \mathcal{D}_{q,T}.$$

Given a vertex $v \in \Delta$ and $T_1 = [v, \ldots], \ldots, T_{n_v} = [v, \ldots]$ the tetrahedra in Δ with common vertex v, for $m \in \{0, \ldots, q\}$ we call

$$\mathcal{R}^m(v) = \bigcup_{\ell=1}^{n_v} \{\xi_{q-m,i,j,k}^{T_\ell} : \ i+j+k = m\}$$

the ring with distance m around v. Moreover, for $m \in \{0, \ldots, q\}$, the set

$$\mathcal{D}^m(v) = \bigcup_{\ell=0}^{m} \mathcal{R}^\ell(v)$$

is called the m-disk of v.

Following Alfeld, Piper & Schumaker [2], we call $\mathcal{M} \subseteq \mathcal{D}_{q,\Delta}$ a determining set (DS) for a linear subspace $\mathcal{S}$ of $\mathcal{S}_q^0(\Delta)$, if setting the coefficients $a_\xi(s)$, $\xi \in \mathcal{M}$ of a spline $s \in \mathcal{S}$ to zero, implies that $s \equiv 0$. A determining set $\mathcal{M}$ is called minimal determining set (MDS) for $\mathcal{S}$, if no determining set for $\mathcal{S}$ with fewer elements than $\mathcal{M}$ exists. Equivalently, $\mathcal{M}$ is a minimal determining set, if setting the coefficients $a_\xi(s)$, $\xi \in \mathcal{M}$, of a spline $s \in \mathcal{S}$ to arbitrary values, all its coefficients $a_\xi(s)$, $\xi \in \mathcal{D}_{q,\Delta}$ are uniquely determined, i.e., s is uniquely determined. If $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$, then it is obvious that $\#(\mathcal{M})$ coincides with the dimension, i.e., the degrees of freedom, of $\mathcal{S}$ (here, and throughout the paper we denote by $\#$ the cardinality of a finite set).

Given an arbitrary tetrahedral partition Δ, the dimension of $\mathcal{S}_q^0(\Delta)$, $q \geq 1$, is easy to determine (cf. Alfeld, Schumaker & Sirvent [3, Theorem 10]). In this case, it is obvious that $\mathcal{D}_{q,\Delta}$ is a minimal determining set for $\mathcal{S}_q^0(\Delta)$, and a straight forward computation shows that

$$\dim \mathcal{S}_q^0(\Delta) = \binom{q-1}{3} T_\Delta + \binom{q-1}{2} F_\Delta + (q-1) E_\Delta + V_\Delta, \qquad q \geq 1, \tag{2.2}$$

where T_Δ is the number of tetrahedra of Δ, F_Δ is the number of triangular faces of Δ, E_Δ is the number of edges of Δ, and V_Δ is the number of vertices of Δ (here, and in the following we set $\binom{i}{j} = 0$, if $i < j$). The problem of determing the dimension of trivariate splines becomes more difficult if we consider subspaces $\mathcal{S}$ of $\mathcal{S}_q^0(\Delta)$ possessing smoothness conditions.

In the following, we are interested in C^1 splines, i.e., we consider the subspaces $\mathcal{S} = \mathcal{S}_q^1(\Delta)$, $q \geq 2$ (where Δ is the tetrahedral partition described in the next section). In order to construct minimal determing sets for these spaces, we use the well-known smoothness conditions connected with the piecewise Bernstein-Bézier representation of the splines (cf. de Boor [6], Chui [8], Farin [11]). Let $T = [v_0, v_1, v_2, v_3]$, $\tilde{T} = [v_0, v_1, v_2, \tilde{v}_3] \in \Delta$, be two different tetrahedra of Δ, and suppose that $s \in \mathcal{S}_q^0(\Delta)$ is given in its piecewise representation (2.1) with coefficients

$$a_{i,j,k,\ell} = a_{i,j,k,\ell}(s) = a_{i,j,k,\ell}(s|_T), \text{ and } \tilde{a}_{i,j,k,\ell} = \tilde{a}_{i,j,k,\ell}(s) = a_{i,j,k,\ell}(s|_{\tilde{T}}),$$

i.e., $a_{i,j,k,\ell} = \tilde{a}_{i,j,k,\ell}$, $i + j + k + \ell = q$. Then s is C^1 across the common triangular face $T \cap \tilde{T} = [v_0, v_1, v_2]$, if and only if for all $i + j + k = q - 1$,

$$\tilde{a}_{i,j,k,1} = a_{i+1,j,k,0}\ \lambda_0(\tilde{v}_3) + a_{i,j+1,k,0}\ \lambda_1(\tilde{v}_3) + a_{i,j,k+1,0}\ \lambda_2(\tilde{v}_3) + a_{i,j,k,1}\ \lambda_3(\tilde{v}_3), \qquad (2.3)$$

where λ_ν, $\nu = 0, \ldots, 3$, are the barycentric coordinates w.r.t. T. Examples for these linear constraints are shown in Figure 7. It is well known that the trivariate conditions (2.3) are lower dimensional conditions if some of the involved barycentric coordinates vanish at $\tilde{v}_3$. These are called degenerate cases - Figure 7 (left) shows such an example, where two barycentric coordinates are zero, and hence the smoothness conditions degenerate to conditions as in the univariate case. Differentiability conditions for the non-degenerate case are shown in Figure 7 (right).

§3. Main Results

In the remainder of this paper we consider a tetrahedral partition Δ of the unit cube $\Omega = [0,1] \times [0,1] \times [0,1] \subseteq \mathbb{R}^3$ which is obtained as follows. Using $n+1$ parallel planes in each of the three space dimensions we first subdivide Ω into n^3 subcubes,

$$Q_{(i,j,k)} = [\tfrac{i-1}{n}, \tfrac{i}{n}] \times [\tfrac{j-1}{n}, \tfrac{j}{n}] \times [\tfrac{k-1}{n}, \tfrac{k}{n}], \qquad i, j, k = 1, \ldots, n.$$

We let $\mathcal{F}^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$, be the six square faces of $Q_{(i,j,k)}$, where we use the following ordering: left ($\ell = 1$), front ($\ell = 2$), bottom ($\ell = 3$), right ($\ell = 4$), back ($\ell = 5$), top ($\ell = [6]$). Each subcube $Q_{(i,j,k)}$, $i, j, k = 1, \ldots, n$, is split into six (Egyptian) pyramids $\mathcal{P}^{[\ell]}_{(i,j,k)}$ by connecting its midpoint

$$v_{(i,j,k)} = (\tfrac{2i-1}{2n}, \tfrac{2j-1}{2n}, \tfrac{2k-1}{2n})$$

with the vertices of the face $\mathcal{F}^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$ (see Figure 3, right). Finally, we insert both diagonals in each of the faces $\mathcal{F}^{[\ell]}_{(i,j,k)}$, denote their intersection point by $w^{[\ell]}_{(i,j,k)}$, and connect $v_{(i,j,k)}$ with $w^{[\ell]}_{(i,j,k)}$, $\ell = 1, \ldots, 6$. This subdivides each pyramid $\mathcal{P}^{[\ell]}_{(i,j,k)}$ into four tetrahedra, and we obtain the tetrahedral partition $\Delta_{(i,j,k)}$ of each subcube $Q_{(i,j,k)}$ which consists of 24 tetrahedra. The tetrahedral partition Δ of Ω is then defined as

$$\Delta = \bigcup_{i,j,k \,\in\, \{1,\ldots,n\}} \Delta_{(i,j,k)}.$$

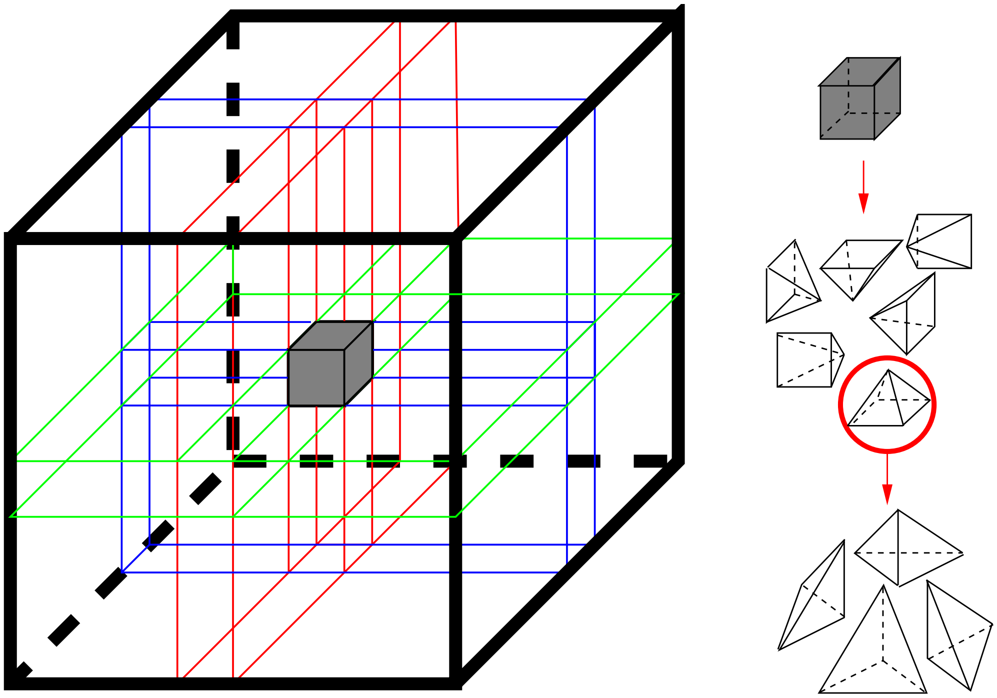

Fig. 3. The uniform tetrahedral partition Δ is obtained by subdividing each subcube into 24 tetrahedra.

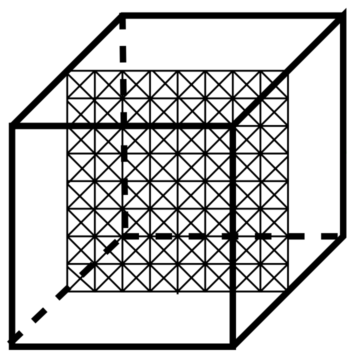

Fig. 4. The intersections of Δ with planes parallel to the three unit planes are four-directional meshes.

Figure 3 illustrates the construction of Δ. The intersection of any plane E defined by the equation $x = \frac{i}{n},\ i = 0, \ldots, n$, $y = \frac{j}{n},\ j = 0, \ldots n$, and $z = \frac{k}{n},\ k = 0, \ldots n$, respectively, with Δ is a the four-directional mesh (sometimes called a Δ^2 triangulation) of the intersecting square domain $E \cap \Omega$ (see Figure 4). Therefore, the tetrahedral decomposition Δ is a natural generalization of the four-directional mesh to a trivariate partition.

It is easy to see that for this uniform tetrahedral partition Δ, we have

$$T_\Delta = 24\ n^3,\ F_\Delta = 48\ n^3 + 12\ n^2,\ E_\Delta = 29\ n^3 + 18\ n^2 + 3\ n, \tag{3.1}$$

for the number of tetraeder T_Δ, triangular faces F_Δ, and edges E_Δ of Δ, respectively.

q	dim $\mathcal{S}_q^1(\triangle)$	dim $\mathcal{S}_q^0(\Delta)$	dim $\mathcal{S}_q^{-1}(\Delta)$
2	$3\ n^2+9\ n+4$	$34\ n^3+24\ n^2+6\ n+1$	$240\ n^3$
3	$6\ n^3+24\ n^2+18\ n+4$	$111\ n^3+54\ n^2+9\ n+1$	$480\ n^3$
4	$39\ n^3+66\ n^2+27\ n+4$	$260\ n^3+96\ n^2+12\ n+1$	$840\ n^3$
5	$120\ n^3+132\ n^2+36\ n+4$	$505\ n^3+150\ n^2+15\ n+1$	$1344\ n^3$
6	$273\ n^3+222\ n^2+45\ n+4$	$870\ n^3+216\ n^2+18\ n+1$	$2016\ n^3$
7	$522\ n^3+336\ n^2+54\ n+4$	$1379\ n^3+294\ n^2+21\ n+1$	$2880\ n^3$
8	$891\ n^3+474\ n^2+63\ n+4$	$2056\ n^3+384\ n^2+24\ n+1$	$3960\ n^3$
9	$1404\ n^3+636\ n^2+72\ n+4$	$2925\ n^3+486\ n^2+27\ n+1$	$5280\ n^3$

Tab. 1. Comparison of dimensions of spline spaces for low degrees.

Hence, (2.2) and some elementary computations imply that

$$\dim\ \mathcal{S}_q^0(\Delta) = (4q^2+1)\ qn^3 + 6\ q^2n^2 + 3\ qn + 1,\ \ q \geq 1. \tag{3.2}$$

The below investigations show that more complex arguments are needed to determine the degrees of freedom of smooth splines. In Section 6, we prove the following main result on the dimension of $\mathcal{S}_q^1(\triangle)$, $q \geq 2$, where Δ is the above uniform tetrahedral partition.

Theorem 3.1. *The dimension of* $\mathcal{S}_q^1(\Delta)$ *is given by*

$$3\ n^2+9\ n+4, \qquad \text{if } q=2, \tag{3.3}$$

and

$$(4q^3-24q^2+53q-45)\ n^3+6\ (2q^2-7q+7)\ n^2+9\ (q-1)\ n+4, \qquad \text{if } q \geq 3. \tag{3.4}$$

Using the result of Theorem 3.1, we compute the dimensions of the spline spaces $\mathcal{S}_q^1(\triangle)$, $q \in \{2,\ldots,9\}$ and compare these numbers with the dimensions of the continuous and non-continuous spline spaces w.r.t. Δ (see Table 1).

In the following we give some alternative formulae for the dimension of the smooth spline spaces. For doing this, in addition to the above notation let V_I be the number of interior vertices of $\triangle$, V_B the number of boundary vertices of $\triangle$, F_I the number of interior triangular faces of $\triangle$, and E_I the number of interior edges of $\triangle$. The next corollary result follows immediately from Theorem 3.1 and some elementary computations using that

$$V_I = 5n^3-6n^2+3n-1,\ V_B = 12n^2+2,$$

(3.1), and the Euler type formulas

$$V_B = 2\ T_\Delta - F_I + 2,\ T_\Delta = V_I - E_I + F_I + 1,$$

which are satisfied for any tetrahedral partition.

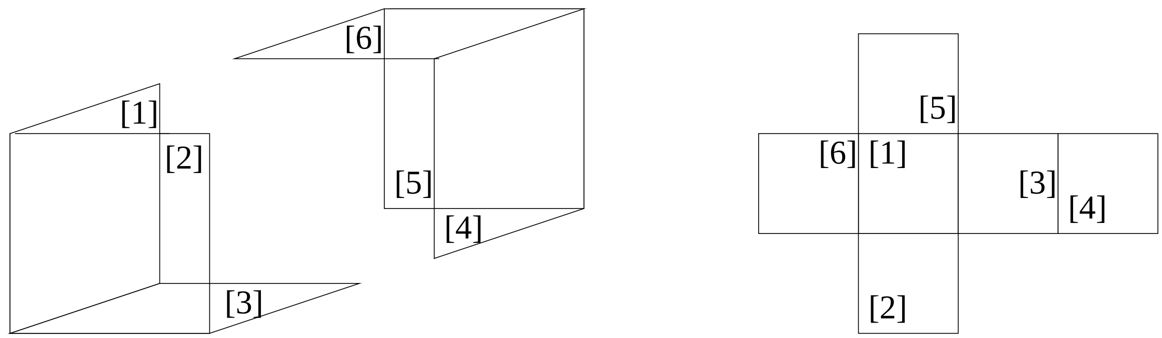

Fig. 5. Cutting of the cubes $Q_{(i,j,k)}$.

Corollary 3.2. *The dimension of* $\mathcal{S}^1_q(\Delta)$ *is given by*

$$\tfrac{1}{8}\Big(7\ T_\triangle - 2\ F_\triangle - 8\ E_\triangle + 32\ V_\triangle\Big) = \tfrac{1}{8}\Big(24\ V_I + 14\ V_B - 5\ T_\triangle + 28\Big)$$
$$= \tfrac{1}{8}\Big(9\ F_I + 57\ V_I - 23\ E_I + 79\Big), \qquad \textit{if } q = 2, \tag{3.5}$$

and

$$\tfrac{1}{12}\Big((2q^3 - 36q^2 + 175q - 219)\ T_\triangle + 12\ (q^2 - 8q + 12)\ F_\triangle + 12\ (3q - 7)\ E_\triangle + 48\ V_\triangle\Big)$$
$$= \tfrac{1}{12}\Big(36\ (q-1)\ V_I + 12\ (q^2 - 2q + 2)\ V_B$$
$$+ (2q^3 - 12q^2 + 19q - 15)\ T_\triangle - 12\ (2q^2 - 7q + 3)\Big)$$
$$= \tfrac{1}{12}\Big((2q^3 - 5q + 9)\ F_I + (2q^3 + 12q^2 + 7q - 3)\ V_I$$
$$- (2q^3 + 12q^2 - 29q + 33)\ E_I + 2q^3 + 12q^2 + 7q + 45\Big)\Big), \qquad \textit{if } q \geq 3. \tag{3.6}$$

§4. Notation and C^1 Smoothness Conditions

For proving our main result (Theorem 3.1) we construct an appropriate MDS $\mathcal{M}$ for the spline spaces (see Section 6). For defining $\mathcal{M}$ we use the notation which we introduce in this section. In addition, we use these to rewrite the smoothness conditions (2.3) for the splines on Δ in a convenient form. It generalizes a similar notation used in Davydov & Zeilfelder [10] for bivariate splines on the four-directional mesh to the trivariate setting.

For all $i, j, k \in \{1, \ldots, n\}$ we set here for the ring with distance $m \in \{0, \ldots, q\}$ around the midpoint $v_{(i,j,k)}$ of $Q_{(i,j,k)}$,

$$\begin{aligned} \mathcal{R}^m(v_{(i,j,k)}) = \{\xi \in \mathcal{D}_{q,\Delta_{(i,j,k)}} :\ \xi = \xi^{m,[\rho,\sigma,\tau]}_{(i,j,k)} = v_{(i,j,k)} + \tfrac{(\rho-m,\sigma-m,\tau-m)}{2qn}&, \\ \rho \in \{0, 2m\} \text{ and } \sigma, \tau \in \{0, \ldots, 2m\},\ \sigma + \tau \text{ even, or}& \\ \sigma \in \{0, 2m\} \text{ and } \rho, \tau \in \{0, \ldots, 2m\},\ \rho + \tau \text{ even, or}& \\ \tau \in \{0, 2m\} \text{ and } \rho, \sigma \in \{0, \ldots, 2m\},\ \rho + \sigma \text{ even}\}.& \end{aligned} \tag{4.1}$$

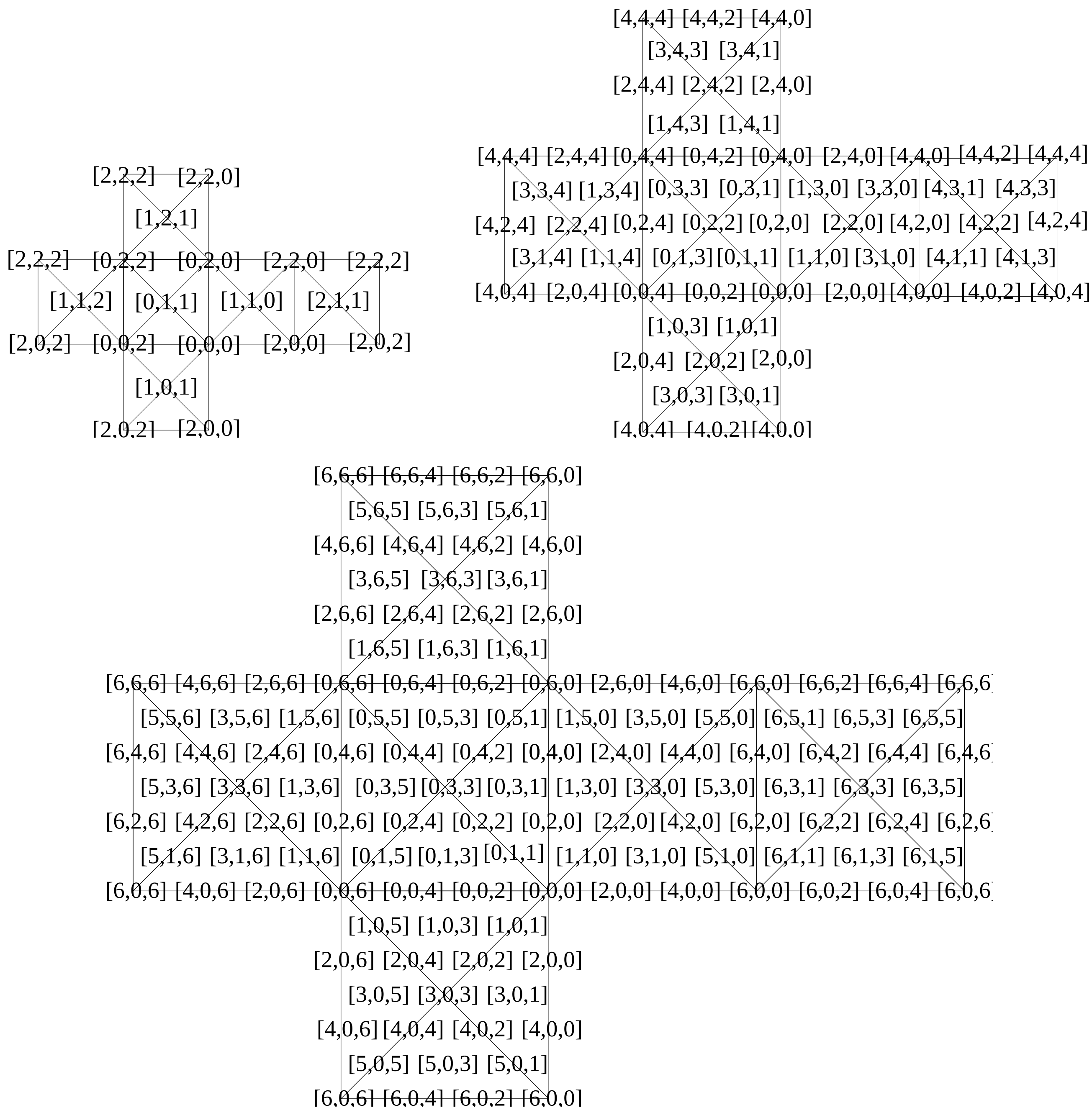

Fig. 6. Indexing the domain points within a cube for $q = 3$. The figure shows the cases $m = 1$ (top, left), $m = 2$ (top, right) and $m = 3$ (bottom).

As it can be seen from this, the cases $\rho = 0$ and $\rho = 2m$ include the domain points from $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[1]}_{(i,j,k)}$ and $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[4]}_{(i,j,k)}$, respectively. Analogously, the cases $\sigma = 0$ and $\sigma = 2m$ include the domain points from $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[2]}_{(i,j,k)}$ and $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[5]}_{(i,j,k)}$, respectively, and $\tau = 0$ and $\tau = 2m$ include the domain points from $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[3]}_{(i,j,k)}$

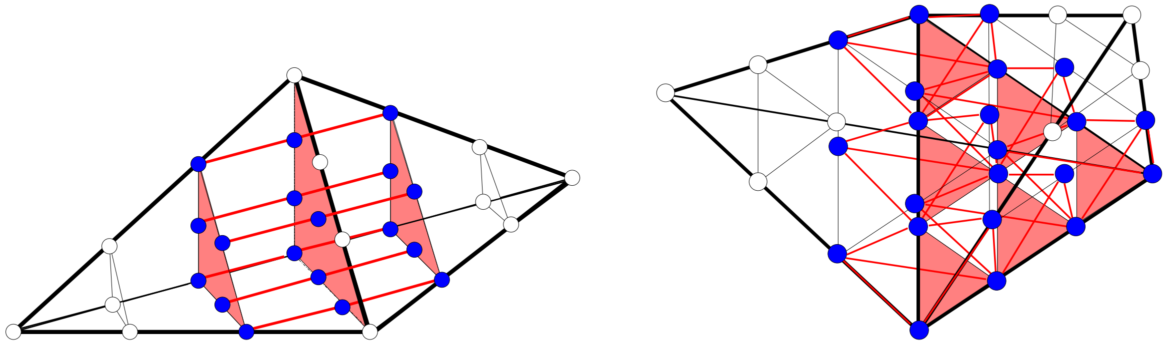

Fig. 7. The smoothness conditions given by equation (4.4) (left) and (4.7) (right) for the case $q = 3$. To the left smoothness conditions are shown which degenerate to univariate smoothness conditions (3 coefficients are therefore involved in each condition), while to the right the non-degenerate case is shown, where 5 coefficients are involved in each condition.

and $\mathcal{R}^m(v_{(i,j,k)}) \cap \mathcal{P}^{[6]}_{(i,j,k)}$, respectively. An example for the indexing is shown in Figure 6, where we use the cutting of the cubes as shown in Figure 5 (here, $[\ell]$ denotes the index of the corresponding face $\mathcal{F}^{[\ell]}_{(i,j,k)}$).

For the coefficients $a_\xi = a_\xi(s)$, $\xi \in \mathcal{D}_{q,\Delta}$, of a spline $s \in \mathcal{S}^1_q(\triangle)$, $q \geq 2$, we now set

$$a_\xi = a^{m,[\rho,\sigma,\tau]}_{(i,j,k)}, \qquad \text{if } \xi = \xi^{m,[\rho,\sigma,\tau]}_{(i,j,k)}.$$

By continuity of s at the triangles of the faces $\mathcal{F}^{[1]}_{(i,j,k)}$ and $\mathcal{F}^{[4]}_{(i-1,j,k)}$, $i \neq 1$, $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[5]}_{(i,j-1,k)}$, $j \neq 1$, and $\mathcal{F}^{[3]}_{(i,j,k)}$ and $\mathcal{F}^{[6]}_{(i,j,k-1)}$, $k \neq 1$, $i,j,k \in \{1,\ldots,n\}$, respectively, we have

$$a_\alpha = a_\beta, \tag{4.2}$$

where

$$(\alpha,\beta) \in \bigcup_{\substack{\sigma,\tau\in\{0,\ldots,2q\} \\ \sigma+\tau \text{ even}}} \{(\xi^{q,[2q,\sigma,\tau]}_{(i-1,j,k)}, \xi^{q,[0,\sigma,\tau]}_{(i,j,k)}), (\xi^{q,[\sigma,2q,\tau]}_{(i,j-1,k)}, \xi^{q,[\sigma,0,\tau]}_{(i,j,k)}), (\xi^{q,[\sigma,\tau,2q]}_{(i,j,k-1)}, \xi^{q,[\sigma,\tau,0]}_{(i,j,k)})\}. \tag{4.3}$$

In the following we use (4.1) to rewrite the smoothness conditions (2.3) for $s \in \mathcal{S}^1_q(\triangle)$, $q \geq 2$, in a convenient form which is used in the proceeding sections. First, we describe the smoothness conditions of s across the common triangles of the faces of adjacent cubes in Ω. Then, we consider the smoothness conditions of s across the common triangular faces of tetrahedra inside the six pyramids of a cube. These smoothness conditions are of the same type (see Figure 7 (left), where we symbolize the domain points associated with the involved Bernstein-Bézier coefficients by black dots). Finally, we rewrite the smoothness conditions of s across the common triangular faces of tetrahedra from different pyramids of a cube (see Figure 7 (right), where we symbolize the domain points associated with the involved Bernstein-Bézier coefficients by black dots).

The smoothness conditions of s across the triangular faces of $\mathcal{F}^{[1]}_{(i,j,k)}$ and $\mathcal{F}^{[4]}_{(i-1,j,k)}$, $i \neq 1$, $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[5]}_{(i,j-1,k)}$, $j \neq 1$, and $\mathcal{F}^{[3]}_{(i,j,k)}$ and $\mathcal{F}^{[6]}_{(i,j,k-1)}$, $k \neq 1$, $i,j,k \in \{1,\dots,n\}$, respectively, are given as

$$a_\alpha = \tfrac{1}{2}\,(a_\beta + a_\gamma), \tag{4.4}$$

where

$$\begin{aligned}(\alpha,\beta,\gamma) \in \bigcup_{\substack{\sigma,\tau\in\{1,\dots,2q-1\}\\ \sigma+\tau \text{ even}}} \{&(\xi^{q,[0,\sigma,\tau]}_{(i,j,k)}, \xi^{q-1,[0,\sigma-1,\tau-1]}_{(i,j,k)}, \xi^{q-1,[2q-2,\sigma-1,\tau-1]}_{(i-1,j,k)}),\\ &(\xi^{q,[\sigma,0,\tau]}_{(i,j,k)}, \xi^{q-1,[\sigma-1,0,\tau-1]}_{(i,j,k)}, \xi^{q-1,[\sigma-1,2q-2,\tau-1]}_{(i,j-1,k)}),\\ &(\xi^{q,[\sigma,\tau,0]}_{(i,j,k)}, \xi^{q-1,[\sigma-1,\tau-1,0]}_{(i,j,k)}, \xi^{q-1,[\sigma-1,\tau-1,2q-2]}_{(i,j,k-1)})\}.\end{aligned} \tag{4.5}$$

The smoothness conditions of s across the common triangular faces of the four tetrahedra in $\mathcal{P}^{[\ell]}_{(i,j,k)}$, $i,j,k \in \{1,\dots,n\}$, $\ell \in \{1,\dots,6\}$, are determined by equation (4.4), where

$$\begin{aligned}(\alpha,\beta,\gamma) \in \bigcup_{m=1}^{q} \bigcup_{\substack{\rho\in\{0,2m\}\\ \sigma\in\{1,\dots,2m-1\}}} \Big(&\{(\xi^{m,[\rho,\sigma,\sigma]}_{(i,j,k)}, \xi^{m,[\rho,\sigma-1,\sigma+1]}_{(i,j,k)}, \xi^{m,[\rho,\sigma+1,\sigma-1]}_{(i,j,k)}),\\ &(\xi^{m,[\sigma,\rho,\sigma]}_{(i,j,k)}, \xi^{m,[\sigma-1,\rho,\sigma+1]}_{(i,j,k)}, \xi^{m,[\sigma+1,\rho,\sigma-1]}_{(i,j,k)}),\\ &(\xi^{m,[\sigma,\sigma,\rho]}_{(i,j,k)}, \xi^{m,[\sigma-1,\sigma+1,\rho]}_{(i,j,k)}, \xi^{m,[\sigma+1,\sigma-1,\rho]}_{(i,j,k)})\}\\ &\bigcup\\ &\{(\xi^{m,[\rho,\sigma,2m-\sigma]}_{(i,j,k)}, \xi^{m,[\rho,\sigma-1,2m-\sigma-1]}_{(i,j,k)}, \xi^{m,[\rho,\sigma+1,2m-\sigma+1]}_{(i,j,k)}),\\ &(\xi^{m,[2m-\sigma,\rho,\sigma]}_{(i,j,k)}, \xi^{m,[2m-\sigma-1,\rho,\sigma-1]}_{(i,j,k)}, \xi^{m,[2m-\sigma+1,\rho,\sigma+1]}_{(i,j,k)}),\\ &(\xi^{m,[\sigma,2m-\sigma,\rho]}_{(i,j,k)}, \xi^{m,[\sigma-1,2m-\sigma-1,\rho]}_{(i,j,k)}, \xi^{m,[\sigma+1,2m-\sigma+1,\rho]}_{(i,j,k)})\}\Big).\end{aligned} \tag{4.6}$$

The conditions determined by (4.4), where (4.6) and $m \in \{1,\dots,q\}$ is fixed, involve Bernstein-Bézier coefficients where the associated domain points are on the ring $\mathcal{R}^m(v_{(i,j,k)})$, only. In addition, there are smoothness conditions involving Bernstein-Bézier coefficients associated with domain points on the rings $\mathcal{R}^m(v_{(i,j,k)})$ and $\mathcal{R}^{m-1}(v_{(i,j,k)})$, simultaneously. These are the smoothness conditions of s across the common triangular faces of tetrahedra inside of different pyramids $\mathcal{P}^{[\ell]}_{(i,j,k)}$, $\mathcal{P}^{[\ell']}_{(i,j,k)}$, $\ell \neq \ell'$, in $Q_{i,j,k}$.

The smoothness conditions of s across the common triangular face of $\mathcal{P}^{[1]}_{(i,j,k)}$ and $\mathcal{P}^{[2]}_{(i,j,k)}$, $\mathcal{P}^{[1]}_{(i,j,k)}$ and $\mathcal{P}^{[3]}_{(i,j,k)}$, $\mathcal{P}^{[2]}_{(i,j,k)}$ and $\mathcal{P}^{[3]}_{(i,j,k)}$, $\mathcal{P}^{[1]}_{(i,j,k)}$ and $\mathcal{P}^{[5]}_{(i,j,k)}$, $\mathcal{P}^{[1]}_{(i,j,k)}$ and $\mathcal{P}^{[6]}_{(i,j,k)}$, $\mathcal{P}^{[3]}_{(i,j,k)}$ and $\mathcal{P}^{[5]}_{(i,j,k)}$, $\mathcal{P}^{[2]}_{(i,j,k)}$ and $\mathcal{P}^{[4]}_{(i,j,k)}$, $\mathcal{P}^{[3]}_{(i,j,k)}$ and $\mathcal{P}^{[4]}_{(i,j,k)}$, $\mathcal{P}^{[2]}_{(i,j,k)}$ and $\mathcal{P}^{[6]}_{(i,j,k)}$, $\mathcal{P}^{[4]}_{(i,j,k)}$ and $\mathcal{P}^{[5]}_{(i,j,k)}$, $\mathcal{P}^{[4]}_{(i,j,k)}$ and $\mathcal{P}^{[6]}_{(i,j,k)}$, and $\mathcal{P}^{[5]}_{(i,j,k)}$ and $\mathcal{P}^{[6]}_{(i,j,k)}$, $i,j,k \in \{1,\dots,n\}$, respectively, are given as

$$a_\alpha = (a_\beta + a_\gamma) - \tfrac{1}{2}(a_\zeta + a_\eta), \tag{4.7}$$

where

$$(\alpha,\beta,\gamma,\zeta,\eta)\in\bigcup_{m=1}^{q}\bigcup_{\substack{\sigma\in\{0,\dots,2(m-1)\}\\ \sigma\ \text{even}}}$$

$$\Big(\{(\xi^{m-1,[0,0,\sigma]}_{(i,j,k)},\xi^{m,[1,0,\sigma+1]}_{(i,j,k)},\xi^{m,[0,1,\sigma+1]}_{(i,j,k)},\xi^{m,[0,0,\sigma]}_{(i,j,k)},\xi^{m,[0,0,\sigma+2]}_{(i,j,k)}),$$
$$(\xi^{m-1,[0,\sigma,0]}_{(i,j,k)},\xi^{m,[1,\sigma+1,0]}_{(i,j,k)},\xi^{m,[0,\sigma+1,1]}_{(i,j,k)},\xi^{m,[0,\sigma,0]}_{(i,j,k)},\xi^{m,[0,\sigma+2,0]}_{(i,j,k)})),$$
$$(\xi^{m-1,[\sigma,0,0]}_{(i,j,k)},\xi^{m,[\sigma+1,1,0]}_{(i,j,k)},\xi^{m,[\sigma+1,0,1]}_{(i,j,k)},\xi^{m,[\sigma,0,0]}_{(i,j,k)},\xi^{m,[\sigma+2,0,0]}_{(i,j,k)})\}$$
$$\bigcup$$
$$\{(\xi^{m-1,[0,2m-2,\sigma]}_{(i,j,k)},\xi^{m,[1,2m,\sigma+1]}_{(i,j,k)},\xi^{m,[0,2m-1,\sigma+1]}_{(i,j,k)},\xi^{m,[0,2m,\sigma]}_{(i,j,k)},\xi^{m,[0,2m,\sigma+2]}_{(i,j,k)}),$$
$$(\xi^{m-1,[0,\sigma,2m-2]}_{(i,j,k)},\xi^{m,[1,\sigma+1,2m]}_{(i,j,k)},\xi^{m,[0,\sigma+1,2m-1]}_{(i,j,k)},\xi^{m,[0,\sigma,2m]}_{(i,j,k)},\xi^{m,[0,\sigma+2,2m]}_{(i,j,k)}),$$
$$(\xi^{m-1,[\sigma,2m-2,0]}_{(i,j,k)},\xi^{m,[\sigma+1,2m,1]}_{(i,j,k)},\xi^{m,[\sigma+1,2m-1,0]}_{(i,j,k)},\xi^{m,[\sigma,2m,0]}_{(i,j,k)},\xi^{m,[\sigma+2,2m,0]}_{(i,j,k)})\}$$
$$\bigcup$$
$$\{(\xi^{m-1,[2m-2,0,\sigma]}_{(i,j,k)},\xi^{m,[2m,1,\sigma+1]}_{(i,j,k)},\xi^{m,[2m-1,0,\sigma+1]}_{(i,j,k)},\xi^{m,[2m,0,\sigma]}_{(i,j,k)},\xi^{m,[2m,0,\sigma+2]}_{(i,j,k)}),$$
$$(\xi^{m-1,[2m-2,\sigma,0]}_{(i,j,k)},\xi^{m,[2m,\sigma+1,1]}_{(i,j,k)},\xi^{m,[2m-1,\sigma+1,0]}_{(i,j,k)},\xi^{m,[2m,\sigma,0]}_{(i,j,k)},\xi^{m,[2m,\sigma+2,0]}_{(i,j,k)}),$$
$$(\xi^{m-1,[\sigma,0,2m-2]}_{(i,j,k)},\xi^{m,[\sigma+1,1,2m]}_{(i,j,k)},\xi^{m,[\sigma+1,0,2m-1]}_{(i,j,k)},\xi^{m,[\sigma,0,2m]}_{(i,j,k)},\xi^{m,[\sigma+2,0,2m]}_{(i,j,k)})\}$$
$$\bigcup$$
$$\{(\xi^{m-1,[2m-2,2m-2,\sigma]}_{(i,j,k)},\xi^{m,[2m,2m-1,\sigma+1]}_{(i,j,k)},\xi^{m,[2m-1,2m,\sigma+1]}_{(i,j,k)},\xi^{m,[2m,2m,\sigma]}_{(i,j,k)},\xi^{m,[2m,2m,\sigma+2]}_{(i,j,k)}),$$
$$(\xi^{m-1,[2m-2,\sigma,2m-2]}_{(i,j,k)},\xi^{m,[2m,\sigma+1,2m-1]}_{(i,j,k)},\xi^{m,[2m-1,\sigma+1,2m]}_{(i,j,k)},\xi^{m,[2m,\sigma,2m]}_{(i,j,k)},\xi^{m,[2m,\sigma+2,2m]}_{(i,j,k)}),$$
$$(\xi^{m-1,[\sigma,2m-2,2m-2]}_{(i,j,k)},\xi^{m,[\sigma+1,2m,2m-1]}_{(i,j,k)},\xi^{m,[\sigma+1,2m-1,2m]}_{(i,j,k)},\xi^{m,[\sigma,2m,2m]}_{(i,j,k)},\xi^{m,[\sigma+2,2m,2m]}_{(i,j,k)})\}\Big). \tag{4.8}$$

§5. Minimal Determing Sets for C^1 Splines on a Tetrahedral Cell

In this section, we consider the spaces $\mathcal{S}^1_q(\Delta_{(1,1,1)}),\ q\ge 2$, where $\Delta_{(1,1,1)}$ is the tetrahedral cell obtained from subdividing the cube $Q=Q_{(1,1,1)}$ into 24 tetrahedra (see Section 3). In the following, we give two different MDS for these spaces. The first MDS for $\mathcal{S}^1_q(\Delta_{(1,1,1)}),\ q\ge 2$, is denoted by $\widetilde{\mathcal{M}}_Q$. Computing the cardinality of $\widetilde{\mathcal{M}}_Q$, we determine the dimension of $\mathcal{S}^1_q(\Delta_{(1,1,1)}),\ q\ge 2$. The second MDS denoted by $\mathcal{M}_Q$ is more complex than $\widetilde{\mathcal{M}}_Q$ since it possesses fewer symmetries. On the other hand, we need $\mathcal{M}_Q$ for the construction of the MDS $\mathcal{M}$ for $\mathcal{S}^1_q(\triangle),\ q\ge 2$, (see Section 6), and therefore for the proof of our main result (Theorem 3.1).

Throughout this section, for short we set

$$v=v_{(1,1,1)},\qquad \text{and}\qquad \xi^{m,[\rho,\sigma,\tau]}=\xi^{m,[\rho,\sigma,\tau]}_{(1,1,1)}\in\mathcal{R}^m(v).$$

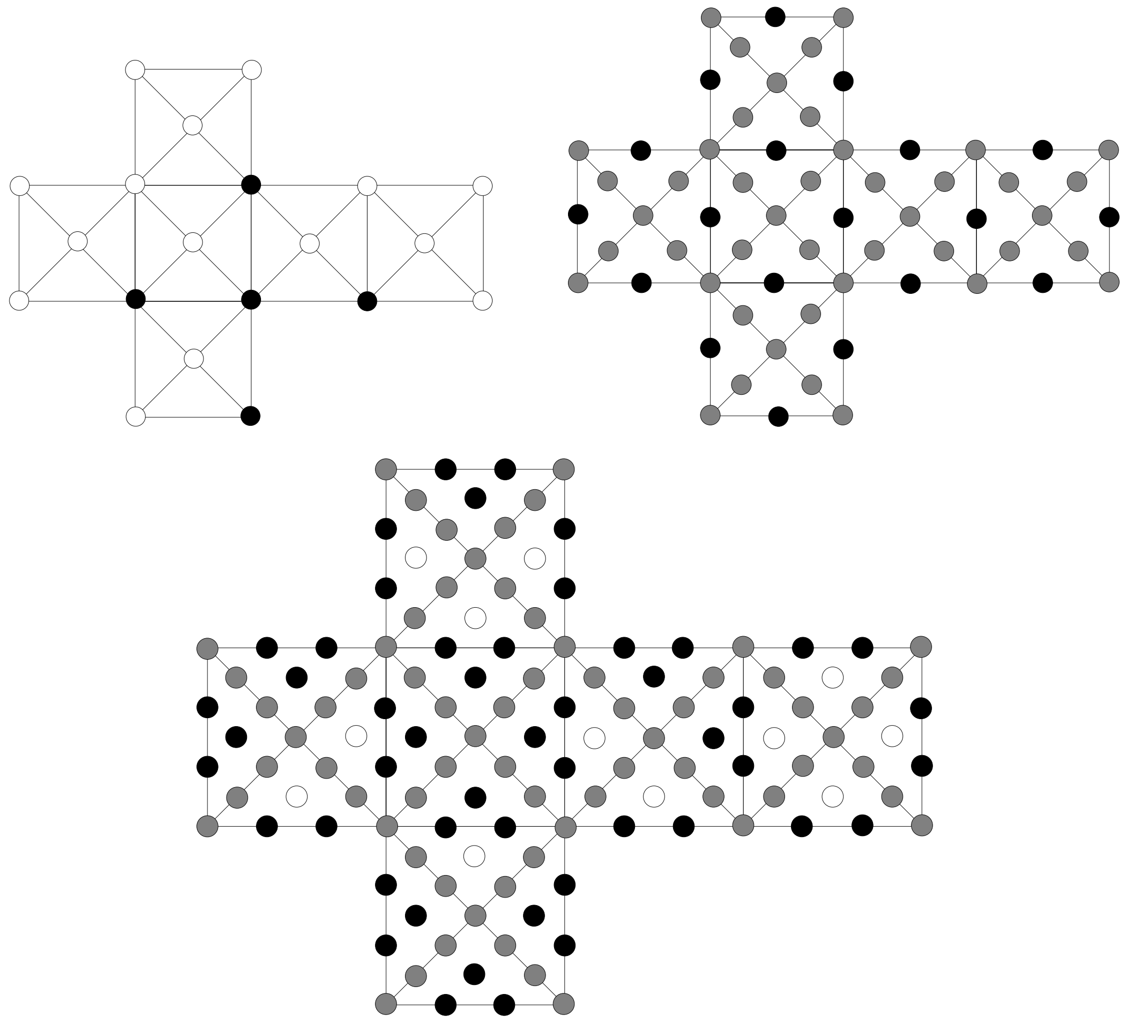

Fig. 8. The choice of points for $\widetilde{\mathcal{M}}_Q$ in the case $q = 3$. The figure shows the rings $\mathcal{R}^m(v)$, $m = 1, 2$, (top) and $m = 3$ (bottom), where the points in $\widetilde{\mathcal{M}}_Q$ are marked by black circles.

In the following, we define $\widetilde{\mathcal{M}}_Q$ which is a subset of the q-disk $\mathcal{D}^q(v)$ of v. For doing this, we define auxiliarly sets which we denote by $\Lambda^m(v)$, $\Theta^m(v)$, $m = 2, \dots, q$, and $\mathcal{D}_1$.

For $m \in \{2, \dots, q\}$, we set

$$\Lambda^m(v) = \bigcup_{\rho \in \{0, 2m\}} \bigcup_{\substack{\sigma \in \{0, \dots, 2m\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \{\xi^{m,[\rho,\sigma,\tau]}, \xi^{m,[\tau,\rho,\sigma]}, \xi^{m,[\sigma,\tau,\rho]}\}.$$

The points of the sets $\Lambda^m(v)$ are marked by grey circles in Figure 8.

Moreover, we set

$$\Theta^m(v) = \begin{cases} \emptyset, & \text{if } m = 2, \\ \displaystyle\bigcup_{\substack{\rho \in \{2, \dots, 2(m-1)-2\} \\ \rho \text{ even}}} \bigcup_{\substack{\sigma \in \{0, 2m-1\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \{\xi^{m,[\rho+1,\sigma+1,\tau]}, \xi^{m,[\sigma+1,\rho+1,\tau]}, \xi^{m,[\sigma+1,\tau,\rho+1]}\}, & \\ & \text{if } m \in \{3, \dots, q\}. \end{cases}$$

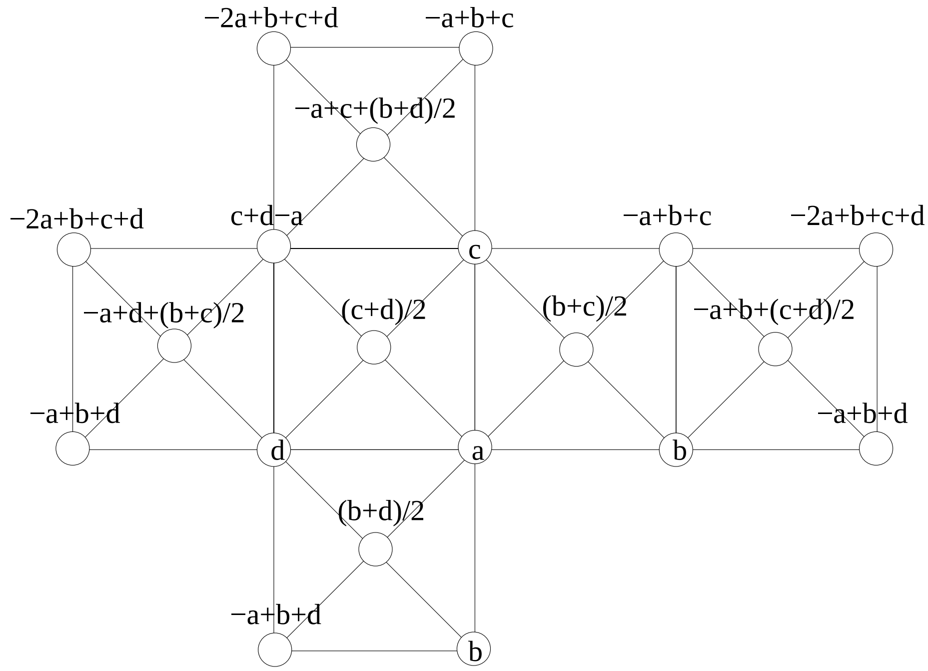

Fig. 9. Computation of the Bernstein-Bézier coefficients associated with domain points in $\mathcal{R}^1(v)$.

The points of the set $\Theta^m(v)$ are marked by white circles in the figure at the bottom of Figure 8 (case $m = 3$). In addition, we let

$$\mathcal{D}_1 = \{\xi^{1,[0,0,0]}, \xi^{1,[2,0,0]}, \xi^{1,[0,2,0]}, \xi^{1,[0,0,2]}\}. \tag{5.1}$$

The set $\widetilde{\mathcal{M}}_Q$ is obtained by adding the points of the rings $\mathcal{R}^m(v)$ to $\mathcal{D}_1$ which are not contained in $\Lambda^m(v) \cup \Theta^m(v)$, $m = 2, \ldots, q$. Formally, we define

$$\widetilde{\mathcal{M}}_Q = \mathcal{D}_1 \cup \bigcup_{m=2}^{q} \Big(\mathcal{R}^m(v) \setminus (\Lambda^m(v) \cup \Theta^m(v))\Big). \tag{5.2}$$

For instance in the case $q = 3$ the set $\widetilde{\mathcal{M}}_Q$ is given as the union of points marked by black circles in Figure 8. Again, we use here the cut of the cube as illustrated in Figure 5. Note that similar as in Figure 6 some domain points appear more than once in Figure 8.

Theorem 5.1. *The set $\widetilde{\mathcal{M}}_Q$ is a minimal determing set for $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geq 2$.*

Proof: Let arbitrary coefficients $a_\xi = a_\xi(s)$, $\xi \in \widetilde{\mathcal{M}}_Q$, of a spline $s \in \mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geq 2$, be given. We have to show that all coefficients of s, i.e., $a_\xi = a_\xi(s)$, $\xi \in \mathcal{D}^q(v)$, are uniquely determined, while all smoothness conditions of the space are satisfied.

The coefficients $a_\xi(s)$, $\xi \in \mathcal{D}^1(v)$ are uniquely determined. This follows from some elementary computations using the 24 smoothness conditions determined by (4.6) and (4.8), where $m = 1$. We illustrate these computations in Figure 9, where we set

$$a = a_\xi(s),\ \xi = \xi^{1,[0,0,0]},\ b = a_\xi(s),\ \xi = \xi^{1,[2,0,0]},\ c = a_\xi(s),\ \xi = \xi^{1,[0,2,0]},$$

and

$$d = a_\xi(s), \qquad \xi = \xi^{1,[0,0,2]},$$

and compute the remaining coefficients from $\mathcal{R}^1(v)$. Note that

$$a_\xi(s) = (-a + b + c + d)/2, \qquad \text{if } \xi = \xi^{0,[0,0,0]}.$$

We claim that the coefficients $a_\xi(s)$, $\xi \in \mathcal{D}^m(v)$ are uniquely determined for $m \in \{2, \dots, q\}$. To show this we use induction.

We first consider the case $m = 2$. Since

$$\bigcup_{\rho \in \{0,4\}} \Big(\{\xi^{2,[\rho,0,2]}, \xi^{2,[\rho,2,0]}, \xi^{2,[\rho,2,4]}, \xi^{2,[\rho,4,2]}\} \cup \{\xi^{2,[2,\rho,0]}, \xi^{2,[2,\rho,4]}\} \Big) \subseteq \widetilde{\mathcal{M}}_Q,$$

it follows from (4.6), where $m = 2$, that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{\rho \in \{0,4\}} \bigcup_{\sigma,\tau \in \{1,3\}} \{\xi^{2,[\rho,\sigma,\tau]}, \xi^{2,[\sigma,\rho,\tau]}, \xi^{2,[\sigma,\tau,\rho]}\},$$

and

$$\xi \in \bigcup_{\rho \in \{0,4\}} \{\xi^{2,[\rho,2,2]}, \xi^{2,[2,\rho,2]}, \xi^{2,[2,2,\rho]}\}.$$

Note that it is a well-known fact from bivariate spline theory (cf. Schumaker [24]) that the coefficients associated with the latter set are not overdetermined.

As we have seen above, the coefficients of s associated with points from $\mathcal{D}^1(v)$ are uniquely determined, and hence it follows from (4.8), where $m = 2$, that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{\rho,\sigma,\tau \in \{0,4\}} \{\xi^{2,[\rho,\sigma,\tau]}\}.$$

Note that the three smoothness conditions determined by (4.8) involving a coefficient associated with the latter set are all satisfied, and we get here, for instance,

$$a_{\xi^{2,[0,0,0]}} = a_{\xi^{2,[2,0,0]}} + a_{\xi^{2,[0,2,0]}} + a_{\xi^{2,[0,0,2]}} - 2\ a_{\xi^{1,[0,0,0]}}.$$

We conclude that the coefficients $a_\xi(s)$, $\xi \in \mathcal{D}^2(v)$ are uniquely determined.

Let us assume that we have already shown that the coefficients $a_\xi(s)$, $\xi \in \mathcal{D}^{m-1}(v)$, where $m \in \{3, \dots, q\}$ are uniquely determined. We now prove that the coefficients $a_\xi(s)$, $\xi \in \mathcal{R}^m(v)$ are uniquely determined.

By definition, we have

$$\bigcup_{\substack{\rho \in \{2,\dots,2(m-1)-2\} \\ \rho \text{ even}}} \ \bigcup_{\substack{\sigma \in \{0,2m-1\} \\ \tau \in \{\sigma+1, 2m-1-\sigma\}}} \{\xi^{m,[\rho+1,\sigma,\tau]}, \xi^{m,[\sigma,\rho+1,\tau]}, \xi^{m,[\sigma,\tau,\rho+1]}\} \subseteq \widetilde{\mathcal{M}}_Q.$$

Since the coefficients associated with domain points in $\mathcal{R}^{m-1}(v)$ are uniquely determined by induction hypothesis, it follows from (4.8) using m and $\sigma \in \{2,\ldots,2(m-1)-2\}$, σ even, that the coefficients

$$a_\xi(s), \qquad \xi \in \Theta^m(v),$$

are uniquely determined. Moreover,

$$\bigcup_{\substack{\rho\in\{0,2m\}\\ \sigma\in\{1,\ldots,2m-1\}}} \Big(\{\xi^{m,[\rho,\sigma-1,\sigma+1]}, \xi^{m,[\rho,\sigma+1,\sigma-1]}, \xi^{m,[\sigma-1,\rho,\sigma+1]}, \xi^{m,[\sigma+1,\rho,\sigma-1]},$$
$$\xi^{m,[\sigma-1,\sigma+1,\rho]}, \xi^{m,[\sigma+1,\sigma-1,\rho]}, \xi^{m,[\rho,\sigma-1,2m-\sigma-1]}, \xi^{m,[\rho,\sigma+1,2m-\sigma+1]},$$
$$\xi^{m,[2m-\sigma-1,\rho,\sigma-1]}, \xi^{m,[2m-\sigma+1,\rho,\sigma+1]}, \xi^{m,[\sigma-1,2m-\sigma-1,\rho]}, \xi^{m,[\sigma+1,2m-\sigma+1,\rho]}\}\Big)$$
$$\setminus \quad \Theta^m(v)$$

is a subset of $\widetilde{\mathcal{M}}_Q$, and therefore the smoothness conditions in (4.6) using m and $\sigma \in \{1,\ldots,2m-1\}$ imply that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{\rho\in\{0,2m\}} \bigcup_{\substack{\sigma\in\{1,\ldots,2m-1\}\\ \tau\in\{\sigma,2m-\sigma\}}} \{\xi^{m,[\rho,\sigma,\tau]}, \xi^{m,[\tau,\rho,\sigma]}, \xi^{m,[\sigma,\tau,\rho]}\}.$$

Note that it is a well-known fact from bivariate spline theory (cf. Schumaker [24]) that the coefficients $a_\xi(s)$, where

$$\xi \in \bigcup_{\rho\in\{0,2m\}} \{\xi^{m,[\rho,m,m]}, \xi^{m,[m,\rho,m]}, \xi^{m,[m,m,\rho]}\}.$$

are not overdetermined. In addition, it follows from the induction hypothesis that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{\rho,\sigma,\tau\in\{0,2(m-1)\}} \{\xi^{m-1,[\rho,\sigma,\tau]}\}.$$

Therefore, (4.8) for m and $\sigma \in \{0, 2(m-1)\}$ implies that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{\rho,\sigma,\tau\in\{0,2m\}} \{\xi^{m,[\rho,\sigma,\tau]}\}.$$

Note that the three smoothness conditions determined by (4.8) involving a coefficient associated with the latter set are all satisfied, and we get here, for instance,

$$a_{\xi^{m,[0,0,0]}} = a_{\xi^{m,[2,0,0]}} + a_{\xi^{m,[0,2,0]}} + a_{\xi^{m,[0,0,2]}} - 2\ a_{\xi^{m-1,[0,0,0]}}.$$

It follows from the choice of $\widetilde{\mathcal{M}}_Q$ that the coefficients $a_\xi(s)$, $\xi \in \mathcal{R}^m(v)$ are uniquely determined, and therefore the coefficients $a_\xi(s)$, $\xi \in \mathcal{D}^m(v)$ are uniquely determined.

q	dim $\mathcal{S}_q^1(\Delta_{(1,1,1)})$	dim $\mathcal{S}_q^0(\Delta_{(1,1,1)})$	dim $\mathcal{S}_q^{-1}(\Delta_{(1,1,1)})$
2	16	65	240
3	52	175	480
4	136	369	840
5	292	671	1344
6	548	1105	2016
7	916	1695	2880
8	1432	2465	3960
9	2116	3439	5280

Tab. 2. Comparison of dimensions for splines on the tetrahedral cell $\Delta_{(1,1,1)}$.

The case $m = q$ shows the assertion, and the proof of the theorem is complete. $\square$

The next result is obtained from counting the number of points in the minimal determining set $\widetilde{\mathcal{M}}_Q$ for $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geq 2$, defined in (5.2). In Table 2 we compare the dimension of these spaces with the dimensions of continuous and non-continuous splines on the same tetrahedral cell.

Theorem 5.2. *The dimension of* $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geq 2$, *is given by*

$$4\ (q^3 - 3q^2 + 5q - 2). \tag{5.3}$$

Proof: Let us denote by d_m the number of points in

$$\widetilde{\mathcal{M}}_Q \cap \mathcal{D}^m(v), \qquad m = 2, \ldots, q.$$

It easily follows from the definition of $\widetilde{\mathcal{M}}_Q$ and the proof of Theorem 5.1 that the set $\mathcal{R}^m(v) \cap \widetilde{\mathcal{M}}_Q$ contains exactly

$$12\ (m-1) + 12\ (m-2) + 24\ \binom{m-2}{2}$$

points, where $m \in \{2, \ldots, q\}$. Therefore, the recurrence relation

$$d_m = d_{m-1} + 24\ m - 36 + 24\ \binom{m-2}{2} \tag{5.4}$$

is satisfied for $m \in \{3, \ldots, q\}$. Since $d_2 = 16$, it follows from induction and some elementary computations that $d_q = 4\ (q^3 - 3q^2 + 5q - 2),\ q \geq 2$. Since d_q coincides with the number of points in $\widetilde{\mathcal{M}}_Q$, the proof of the theorem is complete. $\square$

We proceed by defining a subset $\mathcal{M}_Q$ of $\mathcal{D}^q(v)$ which is different from $\widetilde{\mathcal{M}}_Q$, and we show that $\mathcal{M}_Q$ is a MDS for $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geq 2$. In contrast to $\widetilde{\mathcal{M}}_Q$, the sets $\mathcal{M}_Q \cap \mathcal{R}^m(v),\ m = 2, \ldots, q$, possess fewer symmetries, and therefore it is more complex to describe $\mathcal{M}_Q$. On the other hand, since we can now use that $\widetilde{\mathcal{M}}_Q$ is a MDS for $\mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geq 2$, (Theorem 5.1), the proof of the proceeding theorem becomes simpler than the proof of Theorem 5.1. Here,

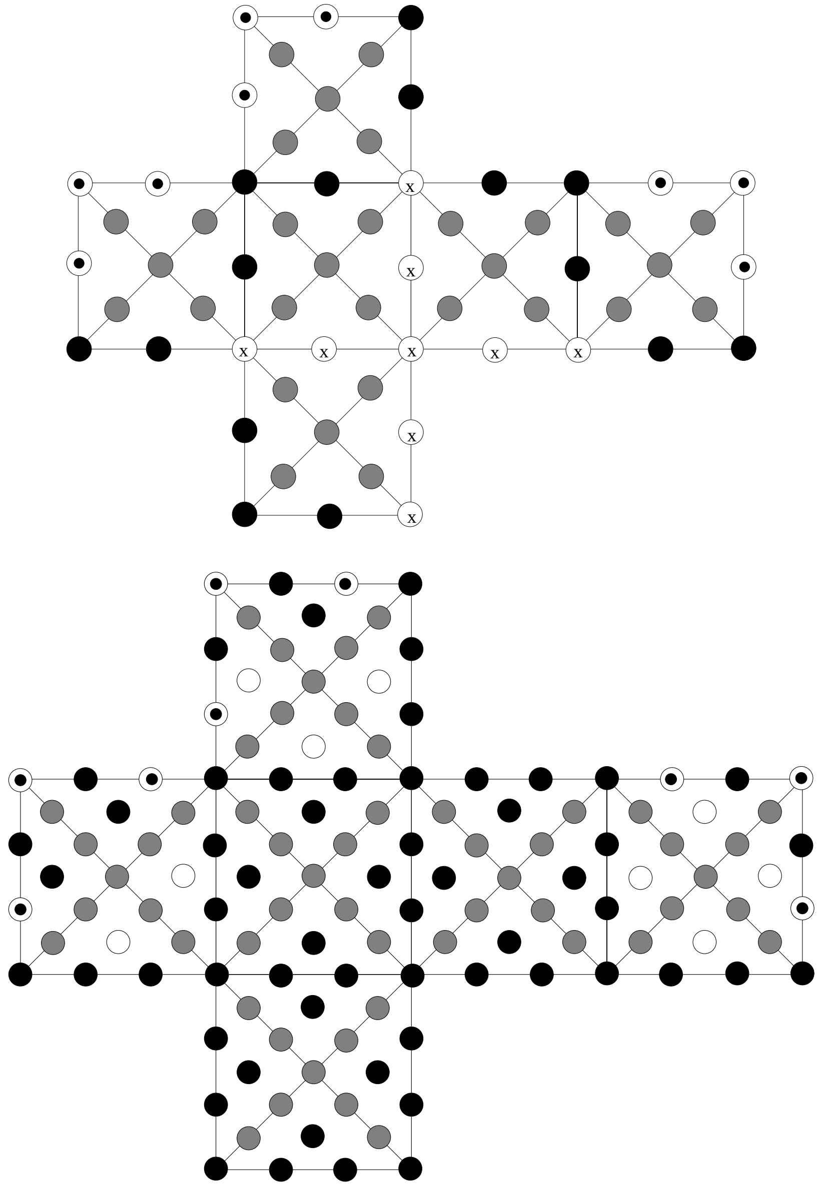

Fig. 10. The choice of points for $\mathcal{M}_Q$ in the case $q = 3$. The figure shows the rings $\mathcal{R}^2(v)$ (top) and $\mathcal{R}^3(v)$ (bottom), where the points of $\mathcal{M}_Q$ are marked by black circles.

we only have to check that the number of points in $\mathcal{M}_Q$ coincides with (5.3) while $\mathcal{M}_Q$ is a DS.

For defining $\mathcal{M}_Q$, we use some auxiliarly sets which we denote by $\Psi^m(v)$, $\Xi^m(v)$, $\Upsilon^m(v)$, and $\Phi^m(v)$, $m = 2, \ldots, q$.

For $m \in \{2, \ldots, q\}$, we set

$$\Psi^m(v) = \bigcup_{\rho \in \{0,2m\}} \bigcup_{\substack{\sigma \in \{1,\ldots,2m-1\} \\ \tau \in \{\sigma, 2m-\sigma\}}} \{\xi^{m,[\rho,\sigma,\tau]}, \xi^{m,[\sigma,\rho,\tau]}, \xi^{m,[\sigma,\tau,\rho]}\}$$

and

$$\Xi^m(v) = \{\xi^{m,[2,2m,2m]}, \xi^{m,[2m,2,2m]}, \xi^{m,[2m,2m,2]}, \xi^{m,[2m,2m,2m]}\}.$$

The definition of $\Psi^m(v)$ is similar to the definition of $\Lambda^m(v)$, but slightly different. The points of the set $\Psi^m(v)$ are marked by grey circles in Figure 10, while the points of $\Xi^m(v)$ are marked by a dot. In addition, we set

$$\Upsilon^m(v) = \begin{cases} \emptyset, \quad \text{if } m = 2, \\ \bigcup\limits_{\substack{\rho \in \{2,\ldots,2(m-1)-2\} \\ \rho \text{ even}}} \{\xi^{m,[\rho+1,1,2m]}, \xi^{m,[1,\rho+1,2m]}, \xi^{m,[1,2m,\rho+1]}, \\ \quad \xi^{m,[\rho+1,2m,1]}, \xi^{m,[2m,\rho+1,1]}, \xi^{m,[2m,1,\rho+1]}, \\ \quad \xi^{m,[\rho+1,2m,2m-1]}, \xi^{m,[2m,\rho+1,2m-1]}, \xi^{m,[2m,2m-1,\rho+1]}\}, \\ \quad \text{if } m \in \{3,\ldots,q\}, \end{cases}$$

and

$$\Phi^m(v) = \begin{cases} \emptyset, \quad \text{if } m = q, \\ \bigcup\limits_{\substack{\rho \in \{0,\ldots,2m\} \\ \rho \text{ even}}} \{\xi^{m,[\rho,0,0]}, \xi^{m,[0,\rho,0]}, \xi^{m,[0,0,\rho]}\}, \qquad \text{if } m \in \{2,\ldots,q-1\}. \end{cases}$$

The definition of $\Upsilon^m(v)$ is similar to the definition of $\Theta^m(v)$, but different. The points of the set $\Upsilon^m(v)$ are marked by white circles in Figure 10, while the points in $\Phi^m(v)$ are marked with a cross.

The set $\mathcal{M}_Q$ is obtained by adding the points of the rings $\mathcal{R}^m(v)$ which are not contained in $\Psi^m(v) \cup \Xi^m(v) \cup \Upsilon^m(v) \cup \Phi^m(v)$, $m = 2,\ldots,q$. Formally, we define

$$\mathcal{M}_Q = \bigcup_{m=2}^{q} \Big(\mathcal{R}^m(v) \setminus (\Psi^m(v) \cup \Xi^m(v) \cup \Upsilon^m(v) \cup \Phi^m(v))\Big). \tag{5.5}$$

For instance in the case $q = 3$ the set $\mathcal{M}_Q$ is given as the union of points marked by black circles in Figure 10. Again, we use here the cut of the cube as illustrated in Figure 5. Note that similar as in Figure 6 some domain points appear more than once in Figure 10. In contrast to the construction of $\widetilde{\mathcal{M}}_Q$, no point from $\mathcal{D}^1(v)$ is contained in $\mathcal{M}_Q$. Moreover, these sets are not nested for different degrees. By this we mean, for instance, that the set $\mathcal{D}^2(v) \cap \mathcal{M}_Q$ consists of 16 points in the case $q = 2$, while for $q = 3$ the set $\mathcal{D}^2(v) \cap \mathcal{M}_Q$ consists of 9 points, only (see Figures 11 and 10, respectively).

Theorem 5.3. *The set $\mathcal{M}_Q$ is a minimal determing set for $\mathcal{S}^1_q(\triangle_{(1,1,1)})$, $q \geq 2$.*

Proof: We first show that the number of points in $\mathcal{M}_Q$ denoted by c_q coincides with the dimension of $\mathcal{S}^1_q(\triangle_{(1,1,1)})$, i.e., according to Theorem 5.2 we have to show that

$$c_q = 4\ (q^3 - 3q^2 + 5q - 2), \qquad q \geq 2. \tag{5.6}$$

This is certainly true for $q = 2$, since in this case, we have

$$\mathcal{M}_Q = \mathcal{R}^2(v) \setminus (\Psi^2(v) \cup \Xi^2(v)) = \bigcup_{\substack{\sigma,\tau \in \{0,\ldots,4\} \\ \sigma,\tau \text{ even},\ (\sigma,\tau) \neq (2,2)}} \{\xi^{2,[0,\sigma,\tau]}, \xi^{2,[\sigma,0,\tau]}, \xi^{2,[\sigma,\tau,0]}\}.$$

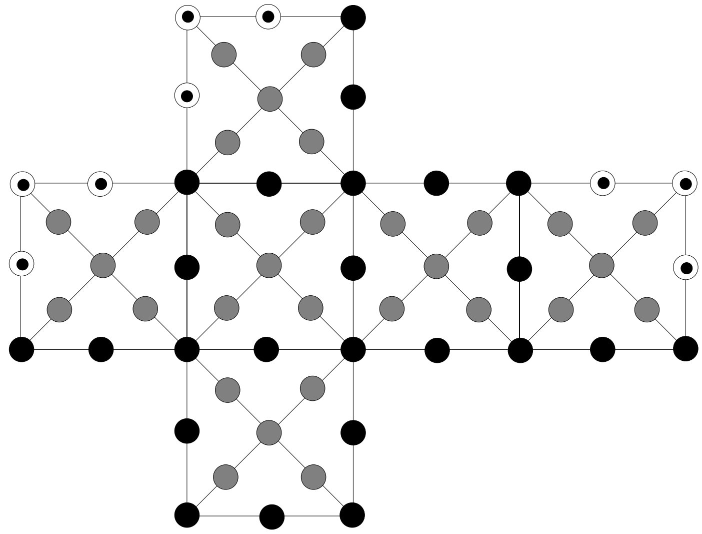

Fig. 11. The choice of points for $\mathcal{M}_Q$ in the case $q = 2$. The figure shows the ring $\mathcal{R}^2(v)$, where the points of $\mathcal{M}_Q$ are marked by black circles.

(See Figure 11, where the points from $\mathcal{M}_Q$ are marked by black circles.)

Moreover, it can be seen from the choice of points in $\mathcal{M}_Q$ that

$$\begin{aligned} c_q &= \Big(c_{q-1} - \#(\Phi^{q-1}(v))\Big) + \#\Big(\mathcal{R}^q(v) \cap \mathcal{M}_Q\Big) \\ &= c_{q-1} - (3q-2) + (12q-4) - 4 + 15\ (q-2) + 24\ \tbinom{q-2}{2} \\ &= c_{q-1} + 24\ q - 36 + 24\ \tbinom{q-2}{2}, \end{aligned}$$

where $q \in \{3, \ldots, q\}$, and therefore a comparison with (5.4) implies that (5.6).

It remains to show that $\mathcal{M}_Q$ is a determining set for $\mathcal{S}_q^1(\triangle_{(1,1,1)})$, $q \geq 2$, i.e., we have to show that for any spline $s \in \mathcal{S}_q^1(\triangle_{(1,1,1)})$, with $a_\xi(s) = 0$, $\xi \in \mathcal{M}_Q$, it follows that $s \equiv 0$. We prove this claim by induction on q.

Let $q = 2$, and $s \in \mathcal{S}_2^1(\Delta_{(1,1,1)})$ be given such that $a_\xi(s) = 0$, where

$$\xi \in \bigcup_{\substack{\sigma,\tau \in \{0,\ldots,4\} \\ \sigma,\tau \text{ even},\ (\sigma,\tau) \neq (2,2)}} \{\xi^{2,[0,\sigma,\tau]}, \xi^{2,[\sigma,0,\tau]}, \xi^{2,[\sigma,\tau,0]}\}.$$

It follows from (4.6) where $m = 2$ and $\sigma \in \{1,2,3\}$ that $a_\xi(s) = 0$ if

$$\xi \in \Big(\bigcup_{\sigma,\tau \in \{1,3\}} \{\xi^{2,[0,\sigma,\tau]}, \xi^{2,[\sigma,0,\tau]}, \xi^{2,[\sigma,\tau,0]}\}\Big) \cup \{\xi^{2,[4,1,1]}, \xi^{2,[1,4,1]}, \xi^{2,[1,1,4]}\},$$

and

$$\xi \in \{\xi^{2,[0,2,2]}, \xi^{2,[2,0,2]}, \xi^{2,[2,2,0]}\}.$$

The smoothness conditions in (4.8) for $m=2$ and $\sigma \in \{0,2\}$ imply that $a_\xi(s)=0,\ \xi \in \mathcal{D}_1$ (for the definition of $\mathcal{D}_1$, see (5.1)), and therefore it follows from the arguments given in the beginning of the proof of Theorem 5.1 that

$$a_\xi(s)=0, \qquad \xi \in \mathcal{D}^1(v).$$

Using the conditions in (4.8) where $m=2$ and $\sigma \in \{0,2\}$, we get $a_\xi(s)=0$, if

$$\xi \in \{\xi^{2,[4,1,3]}, \xi^{2,[1,4,3]}, \xi^{2,[1,3,4]}, \xi^{2,[4,1,3]}, \xi^{2,[1,4,3]}, \xi^{2,[1,3,4]}\}.$$

Hence, the conditions determined in (4.6) imply $a_\xi(s)=0$, if

$$\xi \in \{\xi^{2,[2,2,4]}, \xi^{2,[2,4,2]}, \xi^{2,[4,2,2]}\},$$

and

$$\xi \in \{\xi^{2,[2,4,4]}, \xi^{2,[3,3,4]}, \xi^{2,[4,2,4]}, \xi^{2,[4,3,3]}, \xi^{2,[4,4,2]}, \xi^{2,[3,4,3]}\}.$$

Finally, it follows from the remaining condition determined by (4.8) that $a_\xi(s)=0$, if $\xi=\xi^{2,[4,4,4]}$, and hence $s \equiv 0$.

Let us assume that we have already shown that the above claim holds true for $q-1$, and let a spline $s \in \mathcal{S}_q^1(\triangle_{(1,1,1)}),\ q \geq 3$, be given which satisfies $a_\xi(s)=0,\ \xi \in \mathcal{M}_Q$.

It follows from the conditions determined in (4.6) where $m=q$ and $\sigma \in \{1,\ldots,2q-1\}$ that $a_\xi(s)=0$, if

$$\xi \in \bigcup_{\substack{\sigma \in \{1,\ldots,2q-1\} \\ \tau \in \{\sigma, 2q-\sigma\}}} \{\xi^{q,[0,\sigma,\tau]}, \xi^{q,[\sigma,0,\tau]}, \xi^{q,[\sigma,\tau,0]}\},$$

and

$$\xi \in \{\xi^{q,[2q,1,1]}, \xi^{q,[1,2q,1]}, \xi^{q,[1,1,2q]}\}.$$

The smoothness condition determined in (4.8) where $m=q$ and $\sigma \in \{0,\ldots,2(q-1)\},\ \sigma$ even, imply that

$$a_\xi(s)=0, \qquad \xi \in \Phi^{q-1}(v).$$

Therefore, it follows from the choice of points in $\mathcal{M}_Q$ and the induction hypothesis that

$$a_\xi(s)=0, \qquad \xi \in \mathcal{D}^{q-1}(v).$$

The smoothness conditions determined in (4.8) where $m=q$ and $\sigma \in \{0,\ldots,2(q-1)\},\ \sigma$ even, now imply that $a_\xi(s)=0$, if

$$\xi \in \bigcup_{\substack{\rho \in \{2,\ldots,2(q-1)\} \\ \rho \text{ even}}} \ \bigcup_{\sigma \in \{0,2q-1\}} \{\xi^{q,[\rho+1,\sigma+1,2q-\sigma]}, \xi^{q,[\sigma+1,\rho+1,2q-\sigma]}, \xi^{q,[\sigma+1,2q-\sigma,\rho+1]}\}.$$

Hence, (4.6) gives $a_\xi(s)=0$, if

$$\xi \in \Xi^q(v) \setminus \{\xi^{q,[2q,2q,2q]}\}.$$

It follows from the choice of $\mathcal{M}_Q$ and the smoothness conditions determined in (4.8) where $m = q$ and $\sigma \in \{2, \ldots, 2(q-1)-2\}$, σ even, that $a_\xi(s) = 0$, if

$$\xi \in \bigcup_{\substack{\rho\in\{2,\ldots,2(q-1)-2\} \\ \rho \text{ even}}} \{\xi^{q,[\rho+1,2q,2q-1]}, \xi^{q,[2q,\rho+1,2q-1]}, \xi^{q,[2q,2q-1,\rho+1]}\}.$$

Therefore, the smoothness conditions determined in (4.6) where $m = q$ and $\sigma \in \{1, \ldots, 2q-1\}$ imply that $a_\xi(s) = 0$, if

$$\xi \in \bigcup_{\substack{\sigma\in\{1,\ldots,2q-1\} \\ \tau\in\{\sigma,2q-\sigma\}}} \{\xi^{q,[2q,\sigma,\tau]}, \xi^{q,[\sigma,2q,\tau]}, \xi^{q,[\sigma,\tau,2q]}\}.$$

Finally, the remaining condition determined in (4.8) where $m = q$ gives $a_\xi(s) = 0$, if $\xi = \xi^{q,[2q,2q,2q]}$, and hence $s \equiv 0$. This completes the proof of the theorem. $\square$

§6. Minimal Determining Set for $\mathcal{S}_q^1(\triangle)$, and Proof of the Main Result

We construct a MDS $\mathcal{M}$ for $\mathcal{S}_q^1(\triangle)$, $q \geq 2$, where Δ is the tetrahedral partition defined in Section 3. For doing this, we use the results from the previous section. In particular, we use that $\mathcal{M}_Q$ is a MDS for $\mathcal{S}_q^1(\Delta_{(1,1,1)})$, $q \geq 2$. Counting the number of points in $\mathcal{M}$, we establish the explicit formula for the dimension of these spaces given in Theorem 3.1.

The set $\mathcal{M}$ is obtained as the union of some subsets of $\mathcal{D}_{q,\Delta_{(i,j,k)}}$, $i,j,k \in \{1, \ldots, n\}$, whose definition requires some auxiliarly sets which we denote by $\mathcal{A}_{(i,j,k)}$, $\mathcal{B}_{(i,j,k)}$, and $\mathcal{C}_{(i,j,k)}$.

Let

$$\begin{aligned}
\mathcal{A}_{(i,j,k)} = &\Big(\bigcup_{\rho\in\{0,1\}} \bigcup_{\substack{\sigma,\tau\in\{0,\ldots,2(q-\rho)\} \\ \sigma+\tau \text{ even}}} \{\xi_{(i,j,k)}^{q-\rho,[0,\sigma,\tau]}\}\Big) \\
&\cup \Big(\bigcup_{\rho\in\{0,2q\}} \bigcup_{\substack{\sigma\in\{0,\ldots,2(q-1)\} \\ \sigma \text{ even}}} \{\xi_{(i,j,k)}^{q,[1,\sigma+1,\rho]}, \xi_{(i,j,k)}^{q,[1,\rho,\sigma+1]}\}\Big) \\
&\cup \Big(\bigcup_{\substack{\sigma\in\{0,2q\} \\ \tau\in\{\sigma,2q-\sigma\}}} \{\xi_{(i,j,k)}^{q,[2,\sigma,\tau]}\}\Big), \\
\mathcal{B}_{(i,j,k)} = &\Big(\bigcup_{\rho\in\{0,1\}} \bigcup_{\substack{\sigma,\tau\in\{0,\ldots,2(q-\rho)\} \\ \sigma+\tau \text{ even}}} \{\xi_{(i,j,k)}^{q-\rho,[\sigma,0,\tau]}\}\Big) \\
&\cup \Big(\bigcup_{\rho\in\{0,2q\}} \bigcup_{\substack{\sigma\in\{0,\ldots,2(q-1)\} \\ \sigma \text{ even}}} \{\xi_{(i,j,k)}^{q,[\sigma+1,1,\rho]}, \xi_{(i,j,k)}^{q,[\rho,1,\sigma+1]}\}\Big) \\
&\cup \Big(\bigcup_{\substack{\sigma\in\{0,2q\} \\ \tau\in\{\sigma,2q-\sigma\}}} \{\xi_{(i,j,k)}^{q,[\sigma,2,\tau]}\}\Big),
\end{aligned}$$

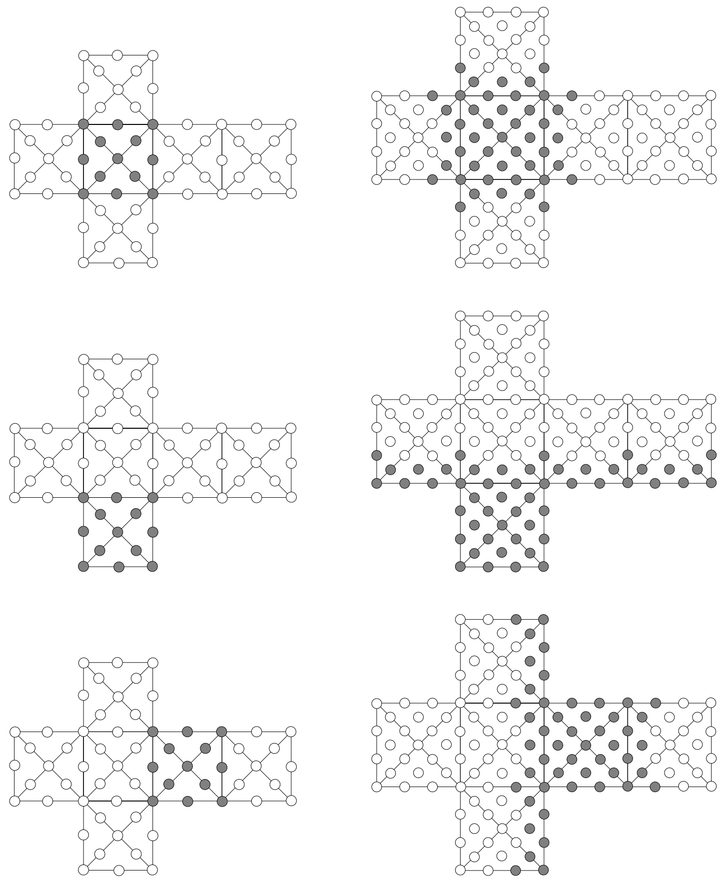

Fig. 12. The sets $\mathcal{A}_{(i,j,k)}$ (top), $\mathcal{B}_{(i,j,k)}$ (middle), and $\mathcal{C}_{(i,j,k)}$ (bottom), in the case $q = 3$.

and

$$\mathcal{C}_{(i,j,k)} = \Big(\bigcup_{\rho \in \{0,1\}} \bigcup_{\substack{\sigma,\tau \in \{0,\ldots,2(q-\rho)\} \\ \sigma+\tau \text{ even}}} \{\xi^{q-\rho,[\sigma,\tau,0]}_{(i,j,k)}\} \Big) \cup \Big(\bigcup_{\rho \in \{0,2q\}} \bigcup_{\substack{\sigma \in \{0,\ldots,2(q-1)\} \\ \sigma \text{ even}}} \{\xi^{q,[\rho,\sigma+1,1]}_{(i,j,k)}, \xi^{q,[\sigma+1,\rho,1]}_{(i,j,k)}\} \Big) \cup \Big(\bigcup_{\substack{\sigma \in \{0,2q\} \\ \tau \in \{\sigma, 2q-\sigma\}}} \{\xi^{q,[\sigma,\tau,2]}_{(i,j,k)}\} \Big).$$

It can be seen form this definition that the set $\mathcal{A}_{(i,j,k)}$ contains all the domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ where the associated Bernstein-Bézier coefficients have influence on the C^1 continuity across the face $\mathcal{F}^{[1]}_{(i,j,k)}$. Similarly, the sets $\mathcal{B}_{(i,j,k)}$ and $\mathcal{C}_{(i,j,k)}$, respectively, contain all the domain points from $\mathcal{D}_{q,\Delta_{(i,j,k)}}$ where the associated Bernstein-Bézier coefficients have influence on the C^1 continuity across the faces $\mathcal{F}^{[2]}_{(i,j,k)}$ and $\mathcal{F}^{[3]}_{(i,j,k)}$, respectively. Figure 12 shows an example for the sets $\mathcal{A}_{(i,j,k)}$, $\mathcal{B}_{(i,j,k)}$, and $\mathcal{C}_{(i,j,k)}$, where the points from these sets are marked by grey circles.

Moreover, we set

$$\mathcal{M}_{(i,j,k)} = \{\xi \in \mathcal{D}_{q,\Delta_{(i,j,k)}} : \xi - (\tfrac{i-1}{n}, \tfrac{j-1}{n}, \tfrac{k-1}{n}) \in \mathcal{M}_Q\}, \tag{6.1}$$

where $Q = Q_{(1,1,1)}$ and $\mathcal{M}_Q$ is defined in (5.5). Hence, $\mathcal{M}_{(1,1,1)} = \mathcal{M}_Q$, and $\mathcal{M}_{(i,j,k)}$, $(i,j,k) \neq (1,1,1)$, is simply a "shifted" version of $\mathcal{M}_Q$.

Finally, we define

$$\begin{aligned}
\mathcal{M} = \quad & \mathcal{M}_{(1,1,1)} \\
\cup & \bigcup_{i\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,1,1)} \setminus \mathcal{A}_{(i,1,1)}) \cup (\mathcal{M}_{(1,i,1)} \setminus \mathcal{B}_{(1,i,1)}) \cup (\mathcal{M}_{(1,1,i)} \setminus \mathcal{C}_{(1,1,i)}) \Big) \\
\cup & \bigcup_{i,j\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,j,1)} \setminus (\mathcal{A}_{(i,j,1)} \cup \mathcal{B}_{(i,j,1)})) \cup (\mathcal{M}_{(i,1,j)} \setminus (\mathcal{A}_{(i,1,j)} \cup \mathcal{C}_{(i,1,j)})) \\
& \qquad \cup (\mathcal{M}_{(1,i,j)} \setminus (\mathcal{B}_{(1,i,j)} \cup \mathcal{C}_{(1,i,j)})) \Big) \\
\cup & \bigcup_{i,j,k\in\{2,\ldots,n\}} \Big((\mathcal{M}_{(i,j,k)} \setminus (\mathcal{A}_{(i,j,k)} \cup \mathcal{B}_{(i,j,k)} \cup \mathcal{C}_{(i,j,k)}) \Big).
\end{aligned} \tag{6.2}$$

Theorem 6.1. *The set $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_q^1(\triangle)$, $q \geq 2$.*

Proof: Let arbitrary coefficients $a_\xi = a_\xi(s)$, $\xi \in \mathcal{M}$, of a spline $s \in \mathcal{S}_q^1(\triangle)$, $q \geq 2$, be given. We have to show that all coefficients of s, i.e., $a_\xi = a_\xi(s)$, $\xi \in \mathcal{D}_{q,\triangle}$, are uniquely determined, while all smoothness conditions of the space are satisfied.

Our method of proof is to show inductively that the coefficients $a_\xi(s)$, $\xi \in Q_{(i,j,k)} \cap \mathcal{D}_{q,\triangle}$ are uniquely determined for $i,j,k \in \{1,\ldots,n\}$, where we consider the cubes $Q_{(i,j,k)}$ in an appropriate order. This natural order is as follows. First, we consider the cases $(i,j,k) = (i,1,1)$, $i = 1,\ldots,n$. Then, we consider the cases $(i,j,k) = (1,j,1)$, $j = 2,\ldots,n$, and $(i,j,k) = (1,1,k)$, $k = 2,\ldots,n$. We proceed by considering the cases $(i,j,k) = (i,j,1)$, $i,j = 2,\ldots,n$, $(i,j,k) = (i,1,k)$, $i,k = 2,\ldots,n$, and $(i,j,k) = (1,j,k)$, $j,k = 2,\ldots,n$. Finally, we consider the cases (i,j,k), $i,j,k = 2,\ldots,n$.

First, it follows from Theorem 5.3 and

$$\mathcal{M}_{(1,1,1)} \subseteq \mathcal{M},$$

that the coefficients $a_\xi(s)$, where

$$\xi \in Q_{(1,1,1)} \cap \mathcal{D}_{q,\triangle}$$

are uniquely determined.

We proceed by considering the cube $Q_{(2,1,1)}$. This cube has exactly one face in common with $Q_{(1,1,1)}$, namely the face $\mathcal{F}^{[4]}_{(1,1,1)} = \mathcal{F}^{[1]}_{(2,1,1)}$. It follows from the conditions determined by (4.3), (4.5), (4.6), and (4.8), that the coefficients

$$a_\xi(s), \qquad \xi \in \mathcal{A}_{(2,1,1)}$$

are uniquely determined. Note that the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(2,1,1)}) \cup \mathcal{R}^{q-1}(v_{(2,1,1)})\Big) \cap \mathcal{P}^{[1]}_{(2,1,1)}$$

are determined by (4.3) and (4.5), respectively, and the smoothness conditions (4.6) involving these points are automatically satisfied, which follows from an elementary computation similarly as in Lai & LeMéhauté [15], since the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(1,1,1)}) \cup \mathcal{R}^{q-1}(v_{(1,1,1)})\Big) \cap \mathcal{P}^{[4]}_{(1,1,1)}$$

satisfy the same type of conditions. Since

$$\mathcal{M}_{(2,1,1)} \setminus \mathcal{A}_{(2,1,1)} \subseteq \mathcal{M},$$

and the coefficients $a_\xi(s)$, $\xi \in \mathcal{A}_{(2,1,1)}$, are uniquely determined, we obtain from arguing along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,1,1)}$, that $a_\xi(s)$ is uniquely determined if

$$\xi \in Q_{(2,1,1)} \cap \mathcal{D}_{q,\triangle},$$

while all the smoothness conditions determined by (4.5) and (4.6) involving coefficients associated with domain points in $Q_{(2,1,1)}$ are satisfied. It follows from induction, the choice of points in $\mathcal{M}$, and a similar argument that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{i \in \{2,\ldots,n\}} \Big(Q_{(i,1,1)} \cup Q_{(1,i,1)} \cup Q_{(1,1,i)}\Big) \cap \mathcal{D}_{q,\triangle}.$$

Next, we consider the cube $Q_{(2,2,1)}$. This cube has exactly two faces in common with some cubes considered before, namely the faces $\mathcal{F}^{[4]}_{(1,2,1)} = \mathcal{F}^{[1]}_{(2,2,1)}$ and $\mathcal{F}^{[5]}_{(2,1,1)} = \mathcal{F}^{[2]}_{(2,2,1)}$. It follows from the conditions determined by (4.3), (4.5), (4.6), and (4.8), that the coefficients $a_\xi(s)$, where

$$\xi \in \mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}$$

are uniquely determined. In particular, the coefficients $a_\xi(s)$, where

$$\xi \in \bigcup_{\substack{\rho \in \{0,\ldots,2(q-1)\} \\ \rho \text{ even}}} \{\xi^{q-1,[0,0,\rho]}_{(2,2,1)}\},$$

are uniquely determined, which follows from the C^1 smoothness of s at the edge with endpoints $(\frac{1}{n}, \frac{1}{n}, 0)$ and $(\frac{1}{n}, \frac{1}{n}, \frac{1}{n})$. Note that the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(2,2,1)}) \cup \mathcal{R}^{q-1}(v_{(2,2,1)})\Big) \cap (\mathcal{P}^{[1]}_{(2,2,1)} \cup \mathcal{P}^{[2]}_{(2,2,1)})$$

are uniquely determined by (4.3) and (4.5), respectively, and, repeating the above arguments, the smoothness conditions (4.6) involving these points are automatically satisfied, since the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(1,2,1)}) \cup \mathcal{R}^{q-1}(v_{(1,2,1)})\Big) \cap \mathcal{P}^{[4]}_{(1,2,1)},$$

or

$$\xi \in \Big(\mathcal{R}^q(v_{(2,1,1)}) \cup \mathcal{R}^{q-1}(v_{(2,1,1)})\Big) \cap \mathcal{P}^{[5]}_{(2,1,1)}$$

satisfy the same type of conditions. Since

$$\mathcal{M}_{(2,2,1)} \setminus (\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}) \subseteq \mathcal{M},$$

and the coefficients $a_\xi(s)$, $\xi \in \mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}$, are uniquely determined, we obtain from arguing along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,2,1)}$, that $a_\xi(s)$ is uniquely determined if

$$\xi \in Q_{(2,2,1)} \cap \mathcal{D}_{q,\triangle},$$

while all the smoothness conditions determined by (4.5) and (4.6) involving coefficients associated with domain points in $Q_{(2,2,1)}$ are satisfied. It follows from induction, the choice of points in $\mathcal{M}$, and a similar argument that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{i,j\in\{2,\ldots,n\}} \Big(Q_{(i,j,1)} \cup Q_{(i,1,j)} \cup Q_{(1,i,j)}\Big) \cap \mathcal{D}_{q,\triangle}.$$

Finally, we consider the cube $Q_{(2,2,2)}$. This cube has exactly three common faces with some cubes considered before, namely the faces $\mathcal{F}^{[4]}_{(1,2,2)} = \mathcal{F}^{[1]}_{(2,2,2)}$, $\mathcal{F}^{[5]}_{(2,1,2)} = \mathcal{F}^{[2]}_{(2,2,2)}$, and $\mathcal{F}^{[6]}_{(2,2,1)} = \mathcal{F}^{[3]}_{(2,2,2)}$. It follows from the conditions determined by (4.3), (4.5), (4.6), and (4.8), that the coefficients $a_\xi(s)$, where

$$\xi \in \mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}$$

are uniquely determined. In particular, the coefficients $a_\xi(s)$, where

$$\xi \in \bigcup_{\substack{\rho\in\{0,\ldots,2(q-1)\}\\ \rho \text{ even}}} \{\xi^{q-1,[\rho,0,0]}_{(2,2,2)}, \xi^{q-1,[0,\rho,0]}_{(2,2,2)}, \xi^{q-1,[0,0,\rho]}_{(2,2,2)}\},$$

are uniquely determined, which follows from the C^1 smoothness of s at the edges with endpoints $(\frac{1}{n},\frac{1}{n},\frac{1}{n})$ and $(\frac{2}{n},\frac{1}{n},\frac{1}{n})$, $(\frac{1}{n},\frac{1}{n},\frac{1}{n})$ and $(\frac{1}{n},\frac{2}{n},\frac{1}{n})$, $(\frac{1}{n},\frac{1}{n},\frac{1}{n})$ and $(\frac{1}{n},\frac{1}{n},\frac{2}{n})$, respectively. Note that the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(2,2,2)}) \cup \mathcal{R}^{q-1}(v_{(2,2,2)})\Big) \cap (\mathcal{P}^{[1]}_{(2,2,2)} \cup \mathcal{P}^{[2]}_{(2,2,2)} \cup \mathcal{P}^{[3]}_{(2,2,2)})$$

are uniquely determined by (4.3) and (4.5), respectively, and, repeating the above arguments, the smoothness conditions (4.6) involving these points are automatically satisfied, since the coefficients $a_\xi(s)$, where

$$\xi \in \Big(\mathcal{R}^q(v_{(1,2,2)}) \cup \mathcal{R}^{q-1}(v_{(1,2,2)})\Big) \cap \mathcal{P}^{[4]}_{(1,2,2)},$$

or

$$\xi \in \Big(\mathcal{R}^q(v_{(2,1,2)}) \cup \mathcal{R}^{q-1}(v_{(2,1,2)})\Big) \cap \mathcal{P}^{[5]}_{(2,1,2)},$$

or

$$\xi \in \Big(\mathcal{R}^q(v_{(2,2,1)}) \cup \mathcal{R}^{q-1}(v_{(2,2,1)})\Big) \cap \mathcal{P}^{[6]}_{(2,2,1)},$$

satisfy the same type of conditions. Since

$$\mathcal{M}_{(2,2,2)} \setminus (\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}) \subseteq \mathcal{M},$$

and the coefficients $a_\xi(s)$, $\xi \in \mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,1)} \cup \mathcal{C}_{(2,2,2)}$, are uniquely determined, we obtain from arguing along the lines of the proof of Theorem 5.3 using the definition of $\mathcal{M}_{(2,2,2)}$, that $a_\xi(s)$ is uniquely determined if

$$\xi \in Q_{(2,2,2)} \cap \mathcal{D}_{q,\triangle},$$

while all the smoothness conditions determined by (4.5) and (4.6) involving coefficients associated with domain points in $Q_{(2,2,2)}$ are satisfied. It follows from induction, the choice of points in $\mathcal{M}$, and a similar argument that $a_\xi(s)$ is uniquely determined if

$$\xi \in \bigcup_{i,j,k \in \{2,\ldots,n\}} Q_{(i,j,k)} \cap \mathcal{D}_{q,\triangle}.$$

This completes the proof of the theorem. □

Counting the number of points in $\mathcal{M}$, we now obtain the result stated in Theorem 3.1.

Proof of Theorem 3.1: It follows from Theorem 5.2 that the set $\mathcal{M}_{(1,1,1)}$ contains

$$4\ (q^3 - 3q^2 + 5q - 2)$$

points, and it is clear that this is also the number of points in every set $\mathcal{M}_{(i,j,k)}$ defined in (6.1). Since the cardinality of $\mathcal{A}_{(2,1,1)} \cap \mathcal{M}_{(2,1,1)}$ is $4\ (q^2 - 2q + 3)$, if $q \geq 3$, and 11, if $q = 2$, it follows that the set $\mathcal{M}_{(2,1,1)} \setminus \mathcal{A}_{(2,1,1)}$ contains $4\ (q^3 - 4q^2 + 7q - 5)$ points, if $q \geq 3$, and 5 points, if $q = 2$. The same number of points are chosen in $\mathcal{M}$ for the cubes $Q_{(i,1,1)}, Q_{(1,i,1)}$, and $Q_{(1,1,i)}$ $i = 2, \ldots, n$. Therefore, these cubes contribute a total number of

$$4\ (q^3 - 3q^2 + 5q - 2) + 12\ (n-1)(q^3 - 4q^2 + 7q - 5)$$

points, if $q \geq 3$, and

$$16 + 15\ (n-1)$$

points, if $q = 2$. The cardinality of $(\mathcal{A}_{(2,2,1)} \cup \mathcal{B}_{(2,2,1)}) \cap \mathcal{M}_{(2,2,1)}$ is $8q^2 - 19q + 23$, if $q \geq 3$, and 15, if $q = 2$. The same number of points are chosen in $\mathcal{M}$ for the cubes $Q_{(i,j,1)}, Q_{(i,1,j)}$, and $Q_{(1,i,j)},\ i,j = 2,\ldots,n$. Therefore, these cubes contribute a total number of

$$3\ (n-1)^2 \left(4\ (q^3 - 3q^2 + 5q - 2) - 8q^2 + 19q - 23\right) = 3\ (n-1)^2(4q^3 - 20q^2 + 39q - 31)$$

points, if $q \geq 3$, and

$$3\ (16-15)(n-1)^2 = 3\ (n-1)^2$$

points, if $q = 2$. The cardinality of $(\mathcal{A}_{(2,2,2)} \cup \mathcal{B}_{(2,2,2)} \cup \mathcal{C}_{(2,2,2)}) \cap \mathcal{M}_{(2,2,2)}$ is $12q^2 - 33q + 37$, if $q \geq 3$, and 16, if $q = 2$. The same number of points are chosen in $\mathcal{M}$ for the cubes $Q_{(i,j,k)},\ i,j,k = 2,\ldots,n$. Therefore, these cubes contribute a total number of

$$(n-1)^3 \left(4\ (q^3 - 3q^2 + 5q - 2) - 12q^2 + 33q - 37\right) = (n-1)^3(4q^3 - 24q^2 + 53q - 45)$$

points, if $q \geq 3$, and no further point, if $q = 2$. Therefore, the total number of points in $\mathcal{M}$ is equal to

$$\begin{aligned} &4\ (q^3 - 3q^2 + 5q - 2) + 12\ (n-1)(q^3 - 4q^2 + 7q - 5) + \\ &\quad 3\ (n-1)^2(4q^3 - 20q^2 + 39q - 31) + (n-1)^3(4q^3 - 24q^2 + 53q - 45), \end{aligned}$$

for $q \geq 3$, and

$$16 + 15\ (n-1) + 3\ (n-1)^2,$$

for $q = 2$. An elementary computation now shows that these numbers coincide with the numbers given in (3.3) and (3.4), respectively. The proof of Theorem 3.1 is complete. $\square$

Remark 6.2. In particular, our results show that the space $\mathcal{S}_2^1(\Delta)$ does not possess enough degrees of freedom to provide good approximation properties, in general. Surprisingly, we observe that for the uniform partition Δ the situation is similar to the case of quadratic C^1 splines on the three-directional mesh, and therefore different to the case of quadratic C^1 splines on the four-directional mesh. Moreover, we note that the dual basis $\{B_\xi \in \mathcal{S}_q^1(\triangle),\ \xi \in \mathcal{M}\}$, where $a_\xi(B_\xi) = 1$, and $a_{\xi'}(B_\xi) = 0,\ \xi' \neq \xi$ resulting from the MDS $\mathcal{M}$ from Theorem 6.1 is non-local, in general. The question of constructing a local basis for splines of small degree is under investigation.

Remark 6.3. The results of this paper can be extended to more general domains as in Figure 13 for which the inductive arguments given in the proof of Theorem 6.1 also work.

Remark 6.4. In Alfeld, Schumaker & Whiteley [4, Theorem 4] a formula for the dimension for C^1 splines of degree ≥ 8 on generic tetrahedral partitions was given. The numbers given in Theorem 3.1 and Corollary 3.2 do not coincide with these dimensions and therefore we conclude that Δ, the tetrahedral partition defined in Section 3, is non-generic for C^1 splines, in general. Moreover, we note that in Alfeld, Schumaker & Whiteley [4, Example 7 & 8] as well as in Alfeld, Schumaker & Sirvent [3, Example 26] the dimension of splines on particular cells were computed. These cells are different from the cell in Section 5.

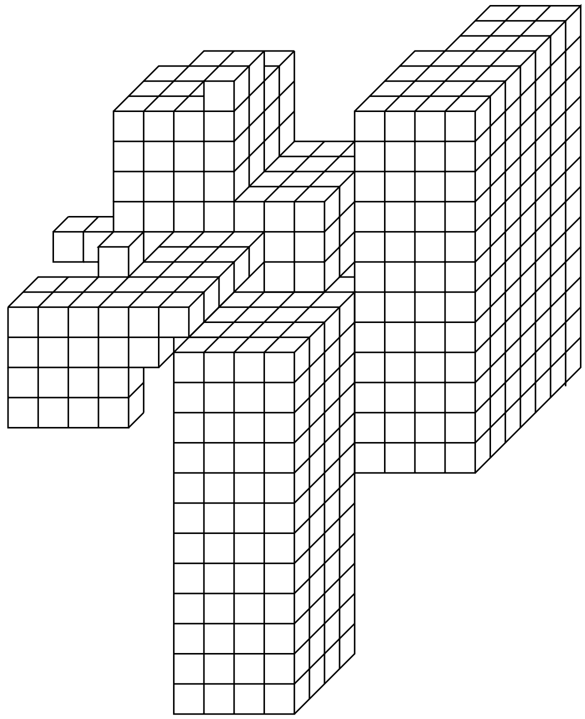

Fig. 13. A more general domain Ω than the unit cube decomposed in uniform cubes.

Remark 6.5. In general, trivariate splines posses much more degrees of freedom than tensor spline spaces of the same degree, which can be considered as subspaces with certain super-smoothness conditions across the interior triangular faces of Δ. For instance, it easy to see that the triquadratic C^1 tensor spline space $\mathcal{S}_2^1 \bigotimes \mathcal{S}_2^1 \bigotimes \mathcal{S}_2^1 \subseteq \mathcal{S}_6^1(\triangle)$, where $\mathcal{S}_2^1$ is the univariate spline space w.r.t. the knots $\frac{i}{n}$, $i = 0, \ldots, n$, possesses $n^3 + 12\, n^2 + 6\, n + 8$ degrees of freedom, which is much smaller than the dimension of $\mathcal{S}_6^1(\triangle)$, i.e., $273\, n^3 + 222\, n^2 + 45\, n + 4$. Similarly, the subspace $\mathcal{S}_3^1 \bigotimes \mathcal{S}_3^1 \bigotimes \mathcal{S}_3^1 \subseteq \mathcal{S}_9^1(\triangle)$, where $\mathcal{S}_3^1$ is the univariate spline space w.r.t. the knots $\frac{i}{n}$, $i = 0, \ldots, n$, possesses $8n^3 + 24n^2 + 24n + 8$ degrees of freedom, while the complete space $\mathcal{S}_9^1(\triangle)$ has a much larger dimension, namely $1404\, n^3 + 636\, n^2 + 72\, n + 4$.

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