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On the Bollobás – Eldridge conjecture for bipartite graphs

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Abstract

Let G be a simple graph on n vertices. A conjecture of Bollobás and Eldridge asserts that if $\delta(G) \geq \frac{kn-1}{k+1}$ then G contains any n vertex graph H with $\Delta(H) = k$. We strengthen this conjecture: we prove that if H is bipartite, $3 \leq \Delta(H) = \Delta$ is bounded and n is sufficiently large , then there exists $\beta > 0$ such that if $\delta(G) \geq \frac{\Delta}{\Delta+1}(1-\beta)n$, then $H \subset G$.

Keywords

Extremal Graph Theory, Spanning Subgraphs

1 Introduction

In this paper we will consider only simple graphs. Let us denote by $\delta(F)$ the minimum degree and by $\Delta(F)$ the maximum degree of the graph F. In 1978 the following conjecture was formulated by Bollobás and Eldridge in [4]:

Conjecture 1 (Bollobás-Eldridge) *If G is a simple graph on n vertices with*

$$\delta(G) \geq \frac{kn-1}{k+1}$$

then G contains any n vertex simple graph H with $\Delta(H) = k$.

The simplest special case of Conjecture 1 is $\Delta(H) = 1$, which can be solved easily. Much harder cases of this conjecture have been proved by Hajnal and Corrádi [5], Hajnal and Szemerédi [7], Aigner and Brandt [2] and Alon and Fischer [3], Csaba, Shokoufandeh and Szemerédi [6]. However, the conjecture is wide open for most cases.

In this paper we show that a stronger version of this conjecture is true for all sufficiently large n when H is bipartite and $3 \leq \Delta(H)$ is bounded:

Theorem 2 *Given $\Delta \geq 3$ integer, there exists an n_0 and a $\beta > 0$ real such that for all $n \geq n_0$ the following statement holds: Let H be a simple bipartite graph on n vertices, with $3 \leq \Delta(H) = \Delta$. Then if G is any n vertex simple graph having minimum degree*

$$\delta(G) \geq \frac{\Delta}{\Delta+1}(1-\beta)n$$

then it contains H as a spanning subgraph.

Remark 1 *The case $\Delta(H) = 1$ of Conjecture 1 is easily seen to be tight, while $\Delta(H) = 2$ and $\chi(H) = 2$ – in which case $\delta(G) \geq \frac{n}{2}$ is sufficient – is a special case of El-Zahar's conjecture, which was shown in [1].*

In understanding the proof of the result some familiarity with the Regularity Lemma of Szemerédi [11] will be helpful, although we will give a brief survey on the necessary notions in the second section. In the third section we will formulate and prove another important tool for this graph embedding problem, a modified version of the Blow-up Lemma [8], [9]. A special case of this version (for embedding graphs of maximum degree three) appeared in [6], although it was not stated explicitly there, and perhaps it is not easy to separate the lemma from the main result of that paper. In the fourth section we will prove Theorem 2, and will give another embedding result, too.

2 Notation and Definitions

For a graph G, $V(G)$ and $E(G)$ will denote its vertex-set and edge-set, respectively. For any vertex v, $deg_G(v)$ is the degree of vertex v, $deg_G(v, X)$ is the number of neighbors of v in X, and $e(X, Y)$ is the number of edges between X and Y. $N_G(v)$ is the set of neighbors of v and $N_G(v, X)$ is the set of neighbors of v in X. Throughout the paper we will apply the relation "$\ll$": $a \ll b$, if a is sufficiently smaller, than b.

A bipartite graph G with color-classes A and B and edge-set E will be denoted by $G = (A, B, E)$. The *density* between disjoint sets X and Y is defined as:

$$d(X, Y) = \frac{e(X, Y)}{|X||Y|}.$$

In the proof of Theorem 2, Szemerédi's Regularity Lemma [11], [10] plays a pivotal role. We will need the following definition to state the Regularity Lemma.

Definition 1 (Regularity condition) *Let $\varepsilon > 0$. A pair (A, B) of disjoint vertex-sets in G is ε-regular if for every $X \subset A$ and $Y \subset B$, satisfying*

$$|X| > \varepsilon|A|, \ |Y| > \varepsilon|B|$$

we have

$$|d(X, Y) - d(A, B)| < \varepsilon.$$

This definition implies that regular pairs are highly uniform bipartite graphs; namely, the density of any reasonably large subgraph is almost the same as the density of the regular pair.

We will use the following form of the Regularity Lemma:

Lemma 3 (Degree Form) *For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G = (V, E)$ is any graph and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set V into $\ell + 1$ clusters $V_0, V_1, \ldots, V_\ell$, and there is a subgraph G' of G with the following properties:*

- *$\ell \leq M$,*
- *$|V_0| \leq \varepsilon|V|$,*
- *all clusters V_i, $i \geq 1$, are of the same size $m \left(\leq \lfloor \frac{|V|}{\ell} \rfloor < \varepsilon|V|\right)$,*
- *$deg_{G'}(v) > deg_G(v) - (d + \varepsilon)|V|$ for all $v \in V$,*

- *$G'|_{V_i} = \emptyset$ (V_i is an independent set in G') for all $i \geq 1$,*
- *all pairs (V_i, V_j), $1 \leq i < j \leq \ell$, are ε-regular, each with density either 0 or greater than d in G'.*

Often we call V_0 the *exceptional cluster*. In the rest of the paper we assume that $0 < \varepsilon \ll d \ll 1$.

Definition 2 (Reduced graph) *Apply Lemma 3 to the graph $G = (V, E)$ with parameters ε and d, and denote the clusters of the resulting partition by $V_0, V_1, \ldots, V_\ell$, V_0 being the exceptional cluster. We construct a new graph G_r, the reduced graph of G' in the following way: The non-exceptional clusters of G' are the vertices of the reduced graph G_r (hence $|V(G_r)| = \ell$). We connect two vertices of G_r by an edge if the corresponding two clusters form an ε-regular pair with density at least d*

The following corollary is immediate:

Corollary 4 *Apply Lemma 3 to the n-graph $G = (V, E)$ satisfying $\delta(G) \geq \gamma n$ for some $\gamma > 0$ with parameters ε and d. Denote G_r the reduced graph of G'. Then $\delta(G_r) \geq (\gamma - \theta)\ell$, where $\theta = 2\varepsilon + d$.*

Remark 2 *In our application of Lemma 3 we will assume that all densities equal to d – for a regular pair with density exceeding this number one can randomly discard edges to achieve the desired density without ruining the ε-regularity condition.*

A stronger one-sided property of regular pairs is super-regularity:

Definition 3 (Super-Regularity condition) *Given a graph G and two disjoint subsets of its vertices A and B, the pair (A, B) is (ε, d)-*super-regular*, if it is ε-regular and furthermore,*

$$deg(a) > d|B|, \text{ for all } a \in A,$$

and

$$deg(b) > d|A|, \text{ for all } b \in B.$$

2.1 A rough outline of the proof

Our goal is to embed H into the host graph G. For achieving this goal first we apply the Regularity Lemma to G. Then we distribute (but not embed) the vertices of H among the non-exceptional clusters of G'. It is important to do this distribution evenly and *consistently.* That is, we assign $m+|V_0|/\ell \pm o(n)$ H-vertices to each non-exceptional cluster, and if $(x, y) \in E(H)$ and x is assigned to the cluster V_x and y is assigned to V_y, then $(V_x, V_y) \in E(G_r)$. Then we embed appropriately chosen H-vertices to V_0. After this step we will have m H-vertices assigned to each non-exceptional cluster. For embedding these H-vertices we will apply the modified Blow-up Lemma.

3 Modified Blow-up Lemma

As it was mentioned above, most of H will be embedded by a similar procedure to that of the Blow-up Lemma. Readers familiar with the lemma may observe that unlike in our setup, the Blow-up Lemma applies for a fixed reduced graph which does not depend on the parameters ε and d, and all the edges of that (fixed) reduced graph are super-regular pairs. Besides, as we will see, there will be restrictions for the embedding of certain H-vertices. Hence, we need a stronger statement than the Blow-up Lemma, but that will require several new conditions, and this version below will be more technical. However, the main message have not changed: if certain conditions are satisfied, one can embed bounded degree spanning subgraphs into pseudo-random graphs. In this section we discuss this embedding algorithm, and then prove its correctness.

Given H and G our goal is to find a subgraph of G which is isomorphic to H. Let us denote by $I' \subset V(H)$ a set the elements of which are of distance at least 4 from each other, and $I' > \frac{n}{\Delta^5}$ - the existence of I' can be shown easily by the help of a greedy algorithm. We assume that the vertex set of G is partitioned into clusters $V_0, V_1, \ldots, V_\ell$, and the vertex set of H is partitioned into clusters $L_0, L_1, L_2, \ldots, L_\ell$, and there is a bijective mapping φ between L_0 and V_0. I'_i will denote $L_i \cap I'$.

Lemma 5 (Modified Blow-up lemma) *For all positive integer Δ there exists n_0 and $\varepsilon, d > 0$ such that if $n > n_0$, H and G are two n-graphs, $\Delta(H) = \Delta$,*

$$0 < \varepsilon \ll \varepsilon' \ll \varepsilon'' \ll \varepsilon''' \ll \delta''' \ll \delta'' \ll \delta' \ll d \ll 1,$$

for $1 \leq i < j \leq \ell$ the pair (V_i, V_j) is ε-regular, with density 0 or d, and all *the conditions listed below hold, then H is embeddable to G by a randomized algorithm. These conditions are the following:*

Conditions

C1 $|L_0| = |V_0| \leq K_1 dn$,

C2 $L_0 \subset I'$,

C3 $|L_i| = |V_i| = m$ *for* $1 \leq i \leq \ell$,

C4 $L_i s$ $(1 \leq i \leq \ell)$ *are independent in* H,

C5 $|N(L_0) \cap L_i| \leq K_2 dm$ *for* $1 \leq i \leq \ell$,

C6 *for* $1 \leq i \leq \ell$ *there is a set* $B_i \subset I'_i$ *with* $B_i \subset L_i$ *and* $|B_i| = \delta' m$, $B = \cup_i B_i$ *such that* $||N_H(B) \cap L_i| - |N_H(B) \cap L_j|| < \varepsilon n$ *for* $1 \leq i, j \leq \ell$,

C7 *if* $(x, y) \in E(H)$ *and* $x \in L_i$, $y \in L_j$ $(1 \leq i, j \leq \ell)$ *then* (V_i, V_j) *is* ε*-regular pair with density* d,

C8 *if* $(x, y) \in E(H)$ *and* $x \in L_0$, $y \in L_j$ $(1 \leq j \leq \ell)$ *then* $deg(\varphi(x), V_j) \geq c_1 |V_j| = c_1 m$.

For $1 \leq i \leq \ell$ $v \in V_i$ *is good for* $x \in L_i$, *if for all* y *adjacent to* x, *if* $y \in L_j$, *then*
$deg_G(v, V_j) \geq (d - \varepsilon)m$.

C9 *For* $1 \leq i \leq \ell$ *all* $v \in V_i$ *is good for at least* $c_2 m$ L_i*-vertices.*

Let $E_i \subset V_i$ *be a set of size at most* $\varepsilon'' m$ *for* $1 \leq i \leq \ell$.

C10 *For* $1 \leq i \leq \ell$ *there exists a bijection* $\psi : E_i \rightarrow F_i \subset L_i \cap (I' - B)$ *such that for all* $v \in E_i$
v *is good for* $\psi(v)$.

C11 *Let* $F = \cup F_i$, *then* $|N_H(F) \cap L_i| \leq K_3 \varepsilon'' m$.

Here K_1, K_2, K_3 *and* c_1 *and* c_2 *are positive numbers, which may depend on* Δ, *but not on* ε *and* d.

The elements of B will be called *buffer vertices*, and E_i is the set of *exceptional* G*-vertices* in V_i.

Let us explaine the role of these conditions. We want to embed L_i-vertices to V_i $(0 \leq i \leq \ell)$. First, $x \in L_0$ will be embedded to $\varphi(x) \in V_0$, that is why we need C1 and C2. We have C3 and C4 since L_i will be embedded to V_i

$(1 \le i \le \ell)$. C7 and C8 are so called *consistency conditions*. The meaning of C5-C6 will be clear later, these are measures for the "evenness" of the distribution of H-vertices among the clusters of G. C9 is analogous to C8. We need C10 and C11 since we have to take special care of the exceptional G-vertices, and we want to cover them with such H-vertices for which their neighbors are well-spread among the L_i-clusters.

3.1 The embedding algorithm

From now on we suppose that the requirements of Lemma 5 are satisfied. Since L_0 has already been embedded, we will consider only the vertices of $H-L_0$. Let $n' = |V(H-L_0)|$, we order the vertices of $H-L_0$ into a sequence $\mathcal{S} = (x_1, x_2, \ldots, x_{n'})$ which is almost the order in which $V(H-L_0)$ will be embedded. For each $1 \le i \le \ell$, we have a subset B_i of L_i of size $\delta' m$, the buffer vertices. Recall, that $B = \cup_i B_i$. Let $M = |B|$, and $b_1, b_2, \ldots, b_M$ be the buffer vertices, then they will form the last part of $\mathcal{S}$. The sequence $\mathcal{S}$ starts with the vertices of $N_H(L_0)$, followed by $\{N_H(b_1),\ N_H(b_2), \ldots, N_H(b_M)\}$, the neighbors of the buffers. We let $T_0 = |N_H(L_0)|$ and $T_1 = \sum_{i=1}^{M} |N_H(b_i)|$. Then we add all the other vertices to the sequence, in such a way that the buffer vertices form the tail of $\mathcal{S}$. For technical reasons we assume that $\mathcal{S}$ is ordered evenly according to the L_i lists, i.e., the consecutive segments of length $\delta'' n'$ have the same number of vertices from every list. Later we may place some vertices forward, but then we rearrange $\mathcal{S}$ to maintain this property.

The embedding of the vertices of $H-L_0$ occurs in three separate phases. In the first phase we are going to embed the vertices of $N_H(L_0)$. In the second phase will come the embedding of the next vertices of $\mathcal{S}$ after each other according to their position in the sequence (some reordering is possible), until only buffer vertices are left in $\mathcal{S}$. In the third phase, by a matching procedure we embed the remaining buffer vertices. The phase for embedding $N_H(L_0)$ is a randomized procedure, while the other two are deterministic.

In the next subsection we outline our method for the embedding, with the exception of selecting a vertex to be covered. That will be done in a separate subsection.

3.1.1 Outline

For an unembedded vertex $x \in L_i$ we will denote by $H_{t,x}$ its monotonically shrinking *host set* in V_i at time t. Also, for technical reasons we keep track of another set, $C_{t,x}$. By Z_t we denote the set of occupied vertices (note that $Z_0 = V_0$), and we also maintain a set Bad_t of exceptional pairs in $H-L_0$.

At time 0, we set $C_{0,x} = H_{0,x} = V_i$, where $x \in L_i$, and x does not have any neighbor in L_0. For those vertices having neighbors in L_0 the setup is different. Let x in L_0 have neighbors $y_1 \in L_{i_1}, y_2 \in L_{i_2}, \ldots, y_\Delta \in L_{i_\Delta}$, and $v = \phi(x)$. By virtue of condition C8 we have ensured that v has at least $c_1 m$ neighbors in $V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta}$. These neighborhoods give $C_{0,y_1} = H_{0,y_1}, C_{0,y_2} = H_{0,y_2}, \ldots, C_{0,y_\Delta} = H_{0,y_\Delta}$, respectively.

Recall, that $T_0 = |N_H(L_0)|$ and $T_1 = \sum_{i=1}^{M} |N_H(b_i)|$. We let $T_2 = \delta'' n'$. Given the initial host sets, the embedding algorithm will go as follows:

Phase 0. For $1 \le t \le T_0$ repeat the following steps

Pick an appropriate vertex v_t for $x_t \in N_H(L_0)$ randomly and uniformly from H_{t-1,x_t} using the *Selection Algorithm* of section 3.1.2. Update

$$Z_t = Z_{t-1} \cup \{v_t\}$$

and for all unembedded vertices x_i, with $t < i \le n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} - Z_t$$

Phase 1. For $t \ge T_0 + 1$ repeat the following steps

Step 1. Embed the vertex x_t from the sequence $\mathcal{S}$: using the *Selection Algorithm* choose an appropriate vertex v_t from the set H_{t-1,x_t} as x_t's image.

Step 2. Update

$$Z_t = Z_{t-1} \cup \{v_t\}$$

and for all unembedded vertices x_i, with $t < i \le n'$

$$C_{t,x_i} = \begin{cases} C_{t-1,x_i} \cap N_G(v_t) & \text{if } (x_i, x_t) \in E(H), \\ C_{t-1,x_i} & \text{otherwise,} \end{cases}$$

and

$$H_{t,x_i} = C_{t,x_i} - Z_t$$

Step 3. Exceptional vertices in G

1. If $t \ne T_0 + T_1$ go to step 4.

2. If $t = T_0 + T_1$ then for every cluster V_i form a set E_i containing those uncovered vertices satisfying

$$|\{b : b \in B_i, v \in C_{t,b}\}| < \delta''|B_i|.$$

We will cover them right after the neighbors of the buffer vertices, thereby eliminating a possible objection to embed the buffer vertices in *Phase 2.* We slightly change the ordering of $\mathcal{S}$. From every list $L(V_i)$ we take $|E_i|$ vertices belonging to I' to form the set $\psi(E_i) = F_i$. Let $F = \cup F_i$. We place the vertices of F forward, $x = \psi(v) \in F_i$ will be embedded to $v \in E_i$. The requirements for choosing F have been formulated in C10 and C11. We will maintain the even ordering of $\mathcal{S}$.

Step 4. Exceptional vertices in $H - L_0$

1. If T_2 does not divide t, then go to Step 5.

2. If T_2 divides t, we will find all exceptional unembedded vertices $y \in H - L_0$ such that $|H_{t,y}| \leq (\delta')^2 m$. We again slightly change the order of the remaining vertices in $\mathcal{S}$ by bringing these exceptional vertices forward in $\mathcal{S}$ (including the exceptional buffer vertices) and will maintain the even distribution of vertices assigned to different clusters. This is possible because of the very small number of exceptional vertices we can find in this step.

Step 5. If the unembedded vertices are all buffer vertices, go to *Phase 2.*, otherwise set $t \leftarrow t + 1$ and go back to *Step 1.*

Phase 2. Find a system of distinct representatives of the sets $H_{t,y}$ for all unembedded vertices.

3.1.2 Selection Algorithm

There can be two possible cases.

Case 1. $x_t \notin F$.
As the image of x_t, we will choose some $v_t \in H_{t-1,x_t}$ such that the following conditions are satisfied for every unembedded vertex y with $(x_t, y) \in E(H)$:

$$(d - \varepsilon)|H_{t-1,y}| \leq deg_G(v_t, H_{t-1,y}) \leq (d + \varepsilon)|H_{t-1,y}|, \quad (3)$$

$$(d - \varepsilon)|C_{t-1,y}| \leq deg_G(v_t, C_{t-1,y}) \leq (d + \varepsilon)|C_{t-1,y}|, \quad (4)$$

and

$$(d-\varepsilon)|C_{t-1,y}\cap C_{t-1,y'}| \leq deg_G(v_t, C_{t-1,y}\cap C_{t-1,y'}) \leq (d+\varepsilon)|C_{t-1,y}\cap C_{t-1,y'}|, \quad (5)$$

for at least $(1-\varepsilon')$ portion of the unembedded vertices y' so that y and y' are assigned to the same cluster V_i, and $\{y, y'\} \notin Bad_{t-1}$. The set Bad_t will be formed as the union of Bad_{t-1} and those pairs $\{y, y'\}$ which does not satisfy (5) for v_t. Clearly, at most $\Delta\varepsilon' m$ new vertices will be added to Bad_t.

Case 2. $x_t \in F$.
By the virtue of C10 we will assign $x_t \in L(V_i)$ to an exceptional $v_t \in E_i$ so that for all unembedded $y \in N_H(x_t)$ the following is satisfied:

$$deg_G(v_t, C_{t-1,y}) = deg_G(v_t) \geq (d-\varepsilon)m \geq (d-\varepsilon)|C_{t-1,y}|, \quad (6)$$

and

$$deg_G(v_t, H_{t-1,y}) \geq deg_G(v_t) - 2\Delta\delta' m - |E_i| \geq (d-\varepsilon)m - 2(\Delta+1)\delta' m \geq \frac{d}{2}m. \quad (7)$$

In (7) we use C6 and the fact that for each i $|E_i| \leq \delta' m$. We will prove this in Lemma 9.

3.2 Correctness

We start by proving that *Phase 0* of the algorithm succeeds with high probability. First we show that the Selection Algorithm succeeds for $1 \leq t \leq T_0$ in finding the v_t-vertices.

Lemma 6 *Assuming that Phase 0 succeeds for all t', with $t' < t \leq T_0$ and $H_{t-1,x_t} \geq \delta'' m$, then it succeeds for t.*

Proof. We only need to consider **Case 1** of the *Selection Algorithm*. The selected vertex $v_t \in H_{t-1,x_t}$ should satisfy conditions (3), (4), and (5). By ε-regularity we will have at most $2\varepsilon m$ vertices in H_{t-1,x_t} which do not satisfy (3), and the same holds for (4). For condition (5) we will define a bipartite graph $B = (W_1, W_2, E(B))$. Here $W_1 = H_{t-1,x_t}$, and the elements of W_2 are the sets $C_{t-1,y} \cap C_{t-1,y'}$ for all pairs $\{y, y'\}$ where $(x_t, y) \in E(H)$, y and y' are both assigned to the same cluster, and $\{y, y'\} \notin Bad_{t-1}$. For $v \in W_1$ and $u \in W_2$, we have $(v, u) \in E(B)$ if (5) does not hold for v and the pairs corresponding to u. If we assume that there are more than $\varepsilon' m$ vertices

$v \in W_1$ with $deg_B(v) > \varepsilon'|W_2|$, then there should be a vertex $u \in W_2$ such that

$$deg_B(u) > \varepsilon'^2 m \gg \varepsilon m.$$

But this is a contradiction with the ε-regularity since the pair $\{y_u, y'_u\}$ corresponding to u does not belong to Bad_{t-1} and

$$|C_{t-1,y_u} \cap C_{t-1,y'_u}| \geq (\delta - \varepsilon)^{2\Delta} m \gg \varepsilon m.$$

This in turn implies that H_{t-1,x_t} can contain at most $4\varepsilon m + \varepsilon' m \ll \delta'' m$ vertices which cannot be used to map x_t, proving the succession of *Phase 0*.
∎

Observe, that when we progress to *Phase 1* (after the successful completion of *Phase 0*), the aformentioned proof will work. We will be able to find a vertex to cover if the host sets are not too small.

What is left to show is that for all time t, $1 \leq t \leq T_0$, the host sets do not become too small. Actually, we prove this not just for the host sets for the unembedded $N_H(L_0)$–vertices, but for all unembedded H–vertices.

Lemma 7 *If Phase 0 succeeds for all t, with $t < T_0$ then for all $t' > t$ $H_{t-1,x_{t'}} \geq \delta' m$ with high probability.*

Proof. For $x \in N_H(L_0)$ $|H_{0,x}| \geq c_1 m$. In Lemma 6 we proved that if the algorithm succeeds up to time t and $|H_{t,x}| \geq \delta'' m$, we can find a v_t to embed x. Since no two vertices in $N_H(L_0)$ are adjacent, the only way the host set of x decreases is that we cover some vertices of it by other $N_H(L_0)$-vertices. When deciding which G–vertex to cover by an $N_H(L_0)$–vertex x, the Selection Algorithm will always provide almost all of $H_{t,x}$ as a possibility – a subset of size $\delta'' m$ can be left out. Out of those possibilities we choose the host vertex randomly and uniformly. Recall that according to C5, the at most $\Delta|V_0|$ restricted vertices of $N_H(L_0)$ are distributed among a constant proportion of the clusters, as evenly as possible. We can conclude that only a very small number of vertices ($K_2 dm$) have a restriction to embed them into a set of size at least $c_1 m$ in each cluster. Now, by applying the statement of Lemma 6 one can easily conclude that up to time T_0 all the unembedded $N_H(L_0)$-vertices have a host set of size at least $(c_1 - 2K_2 d)m$.

A vertex y from the rest of H can have Δ neighbors embedded by time T_0. This means its starting host set $H_{0,y}$ may shrink up to Δ times, each time the sizes are multiplied by a number between $d - \varepsilon$ and $d + \varepsilon$. We also lose some places because of the embedded $N_H(L_0)$-vertices. The randomness in the *Selection Algorithm* helps us. The expected number of covered vertices in this host set is proportional to its size. Suppose that y's host set shrinks at time t_1 and the next shrinking is at time t_2 ($t_1, t_2 \leq T_0$). Denote hm the size of y's host set at time t_1. Then we can use Hoeffding's bound between these two shrinkages. Easy calculation shows that with probability $1 - 2e^{2K_2dmh^2}$ the size of y's host set is of size at least $h(1 - 2K_2d)m$ at time $t_2 - 1$. Therefore, after shrinking Δ times, with very high probability (remember, n is larger than some threshold!) the size of y's host set will be at least $(d - \varepsilon)^\Delta(1 - 2K_2d)^\Delta m > d^{\Delta+1}m$. Observing that we have linear number of host sets in a cluster, we get that with high probability, for every unembedded vertex x at time t, $1 \leq t \leq T_1$, $|H_{t,x}| \geq d^{\Delta+1}m$. ∎

One can easily conclude from the above that *Phase 0* succeeds with probability $1 - o(1)$.

For $t > T_0$ we will need a more thorough analysis. At time t for the cluster V_i and a subset of the unembedded vertices $Q_i \subseteq L_i$, we define a bipartite graph $U_t = (V_i, Q_i, E(U_t))$. Here if $x \in Q_i$, $v \in V_i$, and $v \in C_{t,x}$ then $(x, v) \in E(U_t)$.

The following lemma is pivotal for the proof of the correctness of *Phase 1.*

Lemma 8 *For every $1 \leq i \leq \ell$ and $T_0 + 1 \leq t \leq T_0 + T_1$ and any set of unembedded vertices $Q_i \subseteq L_i$ at time t, with $|Q_i| \geq (\delta''')^2m$, if Phase 1 succeeds for all $t' \leq t$, then apart from an exceptional set J of size at most $\varepsilon''m$ the following will hold for every $v \in V_i$:*

$$deg_{U_t}(v) \geq (1 - \varepsilon'')d(V_i, Q_i)|Q_i|.$$

Proof. We use the so called "defect form" of the Cauchy-Schwarz inequality, that states: if for some $p \leq q$

$$\sum_{i=1}^{p} \alpha_i = \frac{p}{q}\sum_{i=1}^{q} \alpha_i + \beta$$

then

$$\sum_{i=1}^{q} \alpha_i^2 \geq \frac{1}{q}\left(\sum_{i=1}^{q} \alpha_i\right)^2 + \frac{\beta^2 q}{p(q-p)}.$$

Assume to the contrary that the lemma is not true, that is, $|J| > \varepsilon''m$. Choose $J_0 \subset J$ with $|J_0| = \varepsilon''m$. Define $\nu(t, x)$ as the number of embedded neighbors of x by time t. Observe that if x has a neighbor in L_0, then $\nu(0, x) \geq 1$, otherwise it is 0. Then

$$|E(U_t)| = \sum_{x \in Q_i} |C_{t,x}| \geq \sum_{x \in Q_i} (d - \varepsilon)^{\nu(t,x)}m. \quad (8)$$

We also have

$$
\begin{aligned}
&\sum_{x\in Q_i}\sum_{x'\in Q_i}|C_{t,x}\cap C_{t,x'}| \\
&\le \sum_{x\in Q_i}\sum_{x'\in Q_i}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}m + |Q_i|m + \Delta^2|Q_i|m + 2\Delta\varepsilon' m^3 \\
&\le \sum_{x\in Q_i}\sum_{x'\in Q_i}(d+\varepsilon)^{\nu(t,x)+\nu(t,x')}m + 4\Delta\varepsilon' m^3 \qquad (9)
\end{aligned}
$$

For each pair $\{x, x'\}$, we can upper-bound $|C_{t,x}\cap C_{t,x'}|$ by m. The diagonal terms ($x = x'$) result in error $|Q_i|m$. For the non-diagonal terms for which $N_H(x)\cap N_H(x')\neq\emptyset$ we have the term $\Delta^2|Q_i|m$. If $\{x, x'\}\in Bad_t$, by **Case 1** of the Selection Algorithm either x or x' can appear in at most $\Delta\varepsilon' m$ bad pairs. Hence there will be at most $\Delta\varepsilon' m^2$ bad pairs associated with the cluster V_i. Using the Cauchy-Schwarz inequality with $p = \varepsilon'' m$, $q = m$ and the variables $\alpha_k = deg_{U_t}(v_k)$, $1 \le k \le m$ with $v_k \in V_i$ and the first $\varepsilon'' m$ values set to degrees in J_0, we have:

$$
\begin{aligned}
|\beta| &= \varepsilon''\sum_{v\in V_i} deg_{U_t}(v) - \sum_{v\in J_0} deg_{U_t}(v) \\
&\ge \varepsilon''\sum_{v\in V_i} deg_{U_t}(v) - \varepsilon''(1-\varepsilon'')d(V_i,Q_i)|Q_i|m \\
&= (\varepsilon'')^2\sum_{v\in V_i} deg_{U_t}(v). \qquad (10)
\end{aligned}
$$

Then using (8), (10) and the Cauchy-Schwarz inequality we get

$$
\begin{aligned}
&\sum_{x\in Q_i}\sum_{x'\in Q_i}|C_{t,x}\cap C_{t,x'}| \\
&= \sum_{v\in V_i}\left(deg_{U_t}(v)\right)^2 \\
&\ge \tfrac{1}{m}\left(\sum_{v\in V_i} deg_{U_t}(v)\right)^2 + (\varepsilon'')^3 d(V_i,Q_i)^2 m|Q_i|^2 \\
&\ge \tfrac{1}{m}\left(\sum_{x\in Q_i}(d-\varepsilon)^{\nu(t,x)}m\right)^2 + (\varepsilon'')^3(d-\varepsilon)^{2\Delta}m|Q_i|^2 \\
&\ge \sum_{x\in Q_i}\sum_{x'\in Q_i}(d-\varepsilon)^{\nu(t,x)+\nu(t,x')}m + (\varepsilon'')^3(d-\varepsilon)^{2\Delta}m|Q_i|^2
\end{aligned}
$$

which is a contradiction to (9), since $|Q_i| \ge (\delta''')^2 m$,

$$(\varepsilon'')^3(d-\varepsilon)^{2\Delta}(\delta''')^2 \gg 4\varepsilon' \gg 4\varepsilon$$

and

$$(d+\varepsilon)^{\nu(t,x)+\nu(t,x')} - (d-\varepsilon)^{\nu(t,x)+\nu(t,x')} \ll 4\varepsilon.$$

▮

As a consequence we will have the following bound on the size of the exceptional sets E_i:

Lemma 9 *In Step 3, for each $1 \le i \le \ell$ we have $|E_i| \le \varepsilon'' m$.*

Proof. Applying the previous lemma with $t = T_0$ and $Q_i = B_i$, which means $|Q_i| \ge (\delta''')^2 m$, we will have

$$(1 - \varepsilon'')d(V_i, Q_i)|Q_i| \ge \frac{d^\Delta}{2}|Q_i| > \delta''|Q_i|$$

and $E_i \subset J$. ▮

Next we will prove a result similar to Lemma 8 for $t > T_0 + T_1$.

Lemma 10 *For every $1 \le i \le \ell$ and $T_0 + T_1 < t \le T$ and any set of unembedded vertices $Q_i \subseteq L_i$ at time t, with $|Q_i| \ge (\delta''')^2 m$, if Phase 1 succeeds for all $t' \le t$, then apart from an exceptional set of size at most $\varepsilon''' m$ the following will hold for every $v \in V_i$:*

$$deg_{U_t}(v) \ge (1 - \varepsilon''')d(V_i, Q_i)|Q_i|.$$

Proof. The proof follows the same line of argument as Lemma 8 with parameter ε''', except those vertices in the neighborhood of F. The inequality in (8) will hold with the same parameters, since for all $x \in N_H(F)$ we have

$$|C_{t,x}| \ge (d - \varepsilon)^{\nu(t,x)} m.$$

Here we used condition C10 and the fact that $\nu(t, x) = 1$ since $x \in I'$.

In (9) there are more bad pairs. More precisely, based on *Step 3* of the embedding algorithm, there will be an additional error term of $2\Delta K_3 \varepsilon'' m^2 |Q_i|$ by condition C11. Using the fact that

$$(\varepsilon''')^3 (d - \varepsilon)^{2\Delta} (\delta''')^2 \gg \varepsilon''$$

we can see that the contradiction still holds. ▮

The following lemma is an easy consequence of Lemmas 8 and 10.

Lemma 11 *For every $1 \le i \le \ell$ and $T_0 < t \le T$ and any set of unembedded vertices $Q_i \subseteq L_i$ at time t, with $|Q_i| \ge \delta''' m$ and a set $A \subset V_i$ with $|A| \ge \delta''' m$, if Phase 1 succeeds for all $t' \le t$ then apart from an exceptional set J of size at most $(\delta''')^2 m$, the following will hold for every $x \in Q_i$:*

$$|A \cap C_{t,x}| \ge \frac{|A|}{2m}|C_{t,x}|.$$

Proof. Let us suppose that the lemma is not true, there exists a set $J \subseteq Q_i$ such that $|J| > (\delta''')^2 m$, and for every $x \in J$ the inequality of the statement does not hold. We again consider the bipartite graph $U_t = U_t(J, V_i)$.

$$\sum_{v \in A} deg_{U_t}(v) = \sum_{x \in J} |A \cap C_{t,x}| < \frac{|A|}{2m} d(J, V_i)|J|m.$$

Applying Lemmas 8 or 10 with J, we get

$$\sum_{v \in A} deg_{U_t}(v) \geq (1 - \varepsilon''')d(J, V_i)|J|(|A| - \varepsilon''' m),$$

which is a contradiction. ∎

In the following lemma we show that the host sets do not become too small.

Lemma 12 *For every $T_0 + 1 \leq t \leq T$ and for every H-vertex y which is unembedded at time t, if Phase 1 succeeds for all $t' \leq t$ then the following holds:*

$$|H_{t,y}| > \delta'' m.$$

Proof. Let Q_i be the set of all the unembedded vertices in V_i at time t, and let $A_t = V_i - Z_t$. Applying Lemma 11 we can see that for all $x \in Q_i$ (except at most $(\delta''')^2 m$ vertices)

$$|H_{t,x}| = |A_t \cap C_{t,x}| \geq \frac{|A_t|}{2m}|C_{t,x}| \geq \frac{\delta'}{4}(d - \varepsilon)^{\Delta} m \gg (\delta')^2 m,$$

if $|A_t| \geq \frac{\delta'}{2} m$. Next we prove this statement. Let us suppose indirectly that there is a T' such that $T_1 + 1 \leq T' < T$ and

$$|A_{T'}| \geq \frac{\delta'}{2} m \text{ but } |A_{T'+1}| < \frac{\delta'}{2} m.$$

We know that at any time t, where T_2 divides t, there are at most $(\delta''')^2 m$ exceptional unembedded vertices. Thus, up to time T' we can find at most

$$\frac{1}{\delta''}(\delta''')^2 m \gg \delta'' m$$

exceptional vertices. This implies that at time T' there are many more than $(\delta' - \delta'')m$ unembedded buffer vertices, thus, on the contrary, $|A_{T'+1}| \gg (\delta' - \delta'')m$. Note, that we also proved that $T \leq \ell m - \ell\delta' m + \ell\delta'' m$. Let us consider now an arbitrary $y \in L(V_i)$ unembedded at time t ($1 \leq t \leq T$), and let $k\delta'' n' = kT_2 \leq t < (k+1)T_2$ for some $0 \leq k \leq T/T_2$. There are two cases to discuss:

Case 1. If y was not among the at most $(\delta''')^2$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^{\Delta} (\delta')^2 m - K,$$

where K is the number of vertices covered in V_i during the period between kT_2 and $(k+1)T_2$. Recall that the sequence $\mathcal{S}$ is as balanced as possible; hence, $K \leq 2\delta'' m$, where $2\delta'' m$ comes from the reordering of $\mathcal{S}$ because of the exceptional vertices of G and H. Also, at time kT_2 we had that $|H_{kT_2,y}| \geq (\delta')^2 m$. These facts imply that in this case the statement of the lemma holds.

Case 2. If y was among the at most $(\delta''')^2$ exceptional vertices of *Step 4*, then

$$|H_{t,y}| \geq \left(\frac{d}{2}\right)^{\Delta} (\delta')^2 m - K',$$

where K' is the number of vertices covered in V_i during the period between $(k-1)T_2$ and $(k+1)T_2$. Now K' can be as big as $(\delta'' + (\delta''')^2)m$, because at time $(k-1)T_2$ at most $(\delta''')^2 m$ exceptional vertices were placed forward. Again, by observing that at time $(k-1)T_2$ we had that $|H_{(k-1)T_2,y}| \geq (\delta')^2 m$, the proof of the lemma is finished. ∎

Now it is easy to show the succession of the *Selection Algorithm* in finding the v_t-vertices. We have just proved that the host sets can never get too small. In Lemma 6 we proved that *Phase 0* succeeds for time t, whenever it succeeds for all t' with $t' < t \leq T_1$ and the host set is big enough. It is easy to check that exactly the same proof works for *Phase 1* and up to time T. Putting these together, we have that *Phase 1* of the algorithm succeeds.

To prove that *Phase 2* of the algorithm succeeds, we will show that for all $1 \leq i \leq \ell$ there is a system of distinct representatives between the unembedded vertices of L_i and the remaining buffer vertices of V_i. Let $Q_i \subset L_i$ denote the set of unembedded vertices assigned to the cluster V_i, and $Y_i \subset V_i$ be the remaining vertices of the cluster V_i, with $M_i = |Q_i| = |Y_i|$. Then by Lemma 12 for every $x \in Q_i$ we will have $H_{T,x} > \delta''' M_i$. Furthermore, for all subsets $S \subset Q_i$, if $|S| \geq \delta''' M_i$ then by Lemma 10

$$\left| \bigcup_{x \in S} H_{T,x} \right| \geq (1 - \delta''') M_i.$$

Finally, for any $v \in Y_i$, since v cannot be exceptional in G, by *Step 3* there are at least $\delta''' M_i$ host sets $H_{T,x}$ containing v implies that for the subsets $S \subset Q_i$ with $|S| \geq (1 - \delta''' M_i)$ we have

$$\left| \bigcup_{x \in S} H_{T,x} \right| = M_i,$$

which in turn implies the existence of the system of distinct representatives. This finishes the proof of Lemma 5. ∎

4 Assigning H to clusters of G_r

The process of embedding will go as follows: First, we apply the degree form of the Regularity Lemma for G with parameters ε and d. As a result we will have a partitioning of the vertex set into the clusters $V_0, V_1, V_2, \dots, V_\ell$. Now our goal will be to find an assignment of H-vertices to the clusters of G_r so as to satisfy conditions C1-C11.

Let us denote the color classes of H by A and B, and suppose that $|A| \leq |B|$. We randomly distribute the A-vertices among the non-exceptional clusters. Then we are going to map the B-vertices to non-exceptional clusters *consistently and evenly.* That is, if $y \in B$ has the neighbors $\{x_1, x_2, \dots, x_\Delta\}$, and the x_is are mapped to the clusters $V_{j_1}, \dots, V_{j_\Delta}$, then y will be mapped to a cluster V_s which is connected to $V_{j_1}, \dots, V_{j_\Delta}$ by regular edges. Besides, we require that the number of mapped A-vertices and B-vertices to all non-exceptional clusters are $\frac{|A|}{\ell} \pm o(n)$ and $\frac{|B|}{\ell} \pm o(n)$, respectively. The assignment of B-vertices will be done by the help of matching.

Still, there is no H-vertex assigned to V_0 (and hence all non-exceptional clusters are over-saturated). For dealing with this problem we first discard some B-vertices (the surplus) from each non-exceptional cluster, these will form L_0, and the H-vertices assigned to V_s give the set L_s for $1 \leq s \leq \ell$. This may not be the final partitioning of H – for satisfying C8 we may have to *switch* between some L_0-vertex and another L_s-vertex. When all the requirements of C1–C11 will be satisfied, the actual embedding can be done by the help of the modified Blow-up lemma.

4.1 Assigning A

We start by assigning the vertices of A to the non-exceptional clusters of G_r. For every vertex $x \in A$ choose a non-exceptional cluster randomly and independently. It is easy to see that this procedure will guarantee an almost even distribution of the vertices of A among the clusters of G_r:

Lemma 13 *Let A_i, $1 \leq i \leq \ell$ denote the set of vertices assigned to V_i after distributing the vertices of A using the above procedure. Then, with high probability $|A_i| = \frac{|A|}{\ell} \pm o(n)$.*

Proof. Applying Chebyshev's inequality gives the proof of the lemma. ∎

Let $B' \subset B$ is a maximal set in which any two vertices are of distance at least 4 from each other. (Note that $|B'|/|B|$ depends on Δ, but not on ε or d.) Now we will argue that an appropriate distribution of A among the clusters of G_r will facilitate an even assignment of the vertices of B' and $B - B'$ to the clusters of G_r. Let V_i be a cluster in G_r, we define the *associated list* $Q(V_i)$ as $\{y : y \in B, x \in A_i, (x, y) \in E(H)\}$, which is the set of B-vertices with a neighbor assigned to the cluster V_i. Let $V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}$ be any Δ clusters of G_r. We define the random variables R and R': $R'(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) = |B' \cap Q(V_{s_1}) \cap Q(V_{s_2}) \cap \ldots \cap Q(V_{s_\Delta})|$ and $R(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) = |(B - B') \cap Q(V_{s_1}) \cap Q(V_{s_2}) \cap \ldots \cap Q(V_{s_\Delta})|$.

We are going to measure the evenness of the distribution of A in terms of these random variables.

Lemma 14 *For any Δ clusters $V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}$ of G_r the following inequalities hold:*

$$\Pr\left[|R(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - \mathrm{E}[R(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta})]| = \Omega(n^{\frac{4}{5}})\right] = o(1),$$

$$\Pr\left[|R'(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - \mathrm{E}[R'(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta})]| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

Proof. Similar to the proof of Lemma 13, again we omit the details. ∎

We need the following simple corollary of the above lemmas.

Corollary 15 *For any two Δ-tuples of clusters $V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}$ and $V'_{s_1}, V'_{s_2}, \ldots, V'_{s_\Delta}$ in G_r the following inequalities hold:*

$$\Pr\left[|R(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - R(V'_{s_1}, V'_{s_2}, \ldots, V'_{s_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1),$$

$$\Pr\left[|R'(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - R'(V'_{s_1}, V'_{s_2}, \ldots, V'_{s_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

In other words, Lemma 13 and Corollary 15 states that most of the possible assignments of A are even assignments.

Let N be a positive integer, depending only on ε. For all s ($1 \leq s \leq \ell$) we randomly divide $B' \cap Q(V_s)$ into N equal sized subsets, getting $Q_1(V_s), Q_2(V_s), \ldots, Q_N(V_s)$. We define a new set of random variables: $R'_p(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) = |Q_p(V_{s_1}) \cap Q_p(V_{s_2}) \cap \ldots \cap Q_p(V_{s_\Delta})|$, for all $1 \leq p \leq N$. Then the following is implied by Lemma 13 and 14:

Corollary 16 *For any two Δ-tuples of clusters $V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}$ and $V'_{s_1}, V'_{s_2}, \ldots, V'_{s_\Delta}$ in G_r and two integers p and q ($1 \leq p, q \leq N$) the following inequalities hold:*

$$\Pr\left[|R'_p(V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta}) - R'_q(V'_{s_1}, V'_{s_2}, \ldots, V'_{s_\Delta})| = \Omega(n^{\frac{4}{5}})\right] = o(1).$$

4.2 Pre-assigning B

In this section we will present a consistent assignment of the vertices in B to the clusters of G_r. As we will see, such assignments can be formulated as special matching problems. (In order to finish the embedding of H into G, some of the H-vertices should be assigned to the exceptional cluster V_0. This will be carried out in another section.)

We repeat the definitions of [6]. For a bipartite graph $F = (V, T, E(F))$ where $|T| = q|V|$ for some positive integer q, $M \subset E(F)$ is a *proportional matching* if every $v \in V$ is adjacent to exactly q vertices in T and every $u \in T$ is adjacent to exactly one V vertex in M. In order to show that F contains a proportional matching we will check the König–Hall conditions, that is, for every subset U of V, its neighborhood in T should satisfy $|N_F(U, T)| \geq q|U|$. One can easily see this from the construction of an auxiliary graph: substitute every $v \in V$ with q instances $v_1, \ldots, v_q$, and if (v, u) ($u \in T$) was an edge, then connect the v_is to u for all $1 \leq i \leq q$. This auxiliary graph has a perfect matching if and only if F has a proportional matching.

Besides this kind of matching we are going to need another kind of matching about which we demand that the "loads of the vertices" are distributed more evenly. We say F *allows a strong proportional matching with respect to* μ ($0 < \mu \ll 1$) if there is a proportional matching in the following bipartite graph F'. Its color classes are V and T'. For every vertex $u \in T$, we add $\frac{\ell}{\mu}$ copies, $u_1, \ldots, u_{\frac{\ell}{\mu}}$, to T'. If $N_F(u) = \{v_1, \ldots, v_s\}$ then we will have the following edges: (u_i, v_i) for $1 \leq i \leq s$, and (u_j, v_i) where $1 \leq i \leq s$ and $s < j \leq \frac{\ell}{\mu}$. In other words, the first s copies of u have degree 1, while the others have the same degree, s. The existence of a strong proportional matching can be checked through the *strong König-Hall conditions*: one can see that for $U \subset V$ $|N_F(U)|(1 - \mu) \leq |N_{F'}(U)|$. Using this fact we can prove the existence of a strong proportional matching, and at the same time the existence of a proportional matching as well. We will see that both these matchings are needed to assign the vertices of I and Q to clusters of G_r.

Recall that G_r is an ℓ–graph with $\delta(G_r) = (1-\frac{1}{\Delta+1})(1-\theta)(1-\beta)\ell$, where $0 < \theta = \varepsilon + d \ll 1$ and $0 < \beta < 1$ are two constants. We will denote $\delta(G_r)$ by δ.

Let us construct a bipartite graph $F = (V(G_r), T, E(F))$. One color class is $V(G_r)$ (the non-exceptional clusters), the other, T is the set of all possible Δ–tuples composed of G_r–clusters. There is an edge between $V_j \in V(G_r)$ and a Δ–tuple $t = (V_{s_1}, V_{s_2}, \ldots, V_{s_\Delta})$ iff for $1 \le i \le \Delta$ $(V_j, V_{s_i}) \in E(G_r)$. Let us denote $(1-\theta)(1-\beta)(1-\mu)$ by $(1-\nu)$ (here μ is the constant for the strong proportional matching).

The following lemma is the cornerstone of our proof.

Lemma 17 *There is a strong proportional matching in F with respect to μ if ν is small enough.*

For proving Lemma 17 we will need the following statement.

Lemma 18 *For $0 \le i \le \Delta - 2$ if ν is small enough, then $\delta^{\Delta-i}(1-\mu) > (i+1)(1-\delta)$.*

Proof. We are going to prove a stronger statement: $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$.

We proceed by induction. First we prove the case $i = \Delta - 2$:

$$(\frac{\Delta}{\Delta+1})^{2}(1-\nu)^{\Delta} > \frac{\Delta-1}{\Delta+1}(1+\nu\Delta),$$

since by multiplying both sides by $\frac{\Delta+1}{\Delta}$ we get the true inequality

$$\frac{\Delta}{\Delta+1}(1-\nu)^{\Delta} > \frac{\Delta-1}{\Delta}(1+\nu\Delta).$$

So now we may assume that $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$. Decreasing i by 1 we have to check the inequality below:

$$(\frac{\Delta}{\Delta+1})^{\Delta-i+1}(1-\nu)^{\Delta} > \frac{i}{\Delta+1}(1+\nu\Delta).$$

Multiplying both sides by $\frac{\Delta+1}{\Delta}$ we get the inequality

$$(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i}{\Delta}(1+\nu\Delta).$$

Now since $(\frac{\Delta}{\Delta+1})^{\Delta-i}(1-\nu)^{\Delta} > \frac{i+1}{\Delta+1}(1+\nu\Delta)$, and the latter is larger than $\frac{i}{\Delta}(1+\nu\Delta)$ for $i < \Delta$, we have finished the proof of the lemma. ∎

We can start proving Lemma 17.

Proof. We will check the strong König–Hall conditions.

- Let $v \in V(G_r)$ be an arbitrary cluster. Then $|N(v,T)|(1-\mu) \geq \delta^{\Delta}(1-\mu)|T|$, therefore it is larger than $(1-\delta)|T|$ by Lemma 18.
- Let $U_i \subset V(G_r)$ is a set of size greater than $i(1-\delta)\ell$ for some $1 \leq i \leq \Delta - 2$. From the minimum degree condition of G_r every i vertex will have a common neighbor in U_i. Now $|N(U_i,T)|(1-\mu) \geq \delta^{\Delta-i}(1-\mu)|T|$, and by Lemma 18 the latter is larger than $(i+1)(1-\delta)$, therefore $|N(U_i,T)|(1-\mu) > (i+1)(1-\delta)$. Notice that by the above argument we can jump up to $|U_{\Delta-2}| > (\Delta-1)(1-\delta)\ell$ (in case $i = \Delta - 2$).
- Assume that $U \subset V(G_r)$ with $|U| = (\Delta-1)(1-\delta)\ell$. Then every $\Delta - 1$ vertex will have a common neighbor in U by the minimum degree condition of G_r. Thus, $|N(U,T)| \geq \delta(1-\mu)|T|$.
- Assume that $U \subset V(G_r)$ with $|U| = \delta(1-\mu)\ell$. We will estimate the number of $(\Delta-1)$–tuples having more than $\frac{1}{\Delta+1}\ell$ U–neighbors. First of all, there are at least $\delta(1-\mu)\ell\delta^{\Delta-1}\binom{\ell}{\Delta-1}$ edges going from U to the set T' of $(\Delta-1)$–tuples. We divide T' into two parts, T_1' and T_2'. In T_1' all the tuples has at most $(1-\delta)\ell$ neighbors in U, while in T_2' it is possible that the tuples are connected to all of U. Denote $\frac{|T_1'|}{|T'|}$ by x, then we will have the following inequality:

  $$x(1-\delta) + (1-x)\delta(1-\mu) \geq \delta^{\Delta}(1-\mu).$$

  That is, $x \leq \frac{(1-\mu)(\delta-\delta^{\Delta})}{2\delta-1}$. It can be shown directly, that this expression is less than 0.7 for $\Delta \geq 3$ if ν is small. Since all the tuples of T_2' has degree larger than $(1-\delta)\ell$ to U, all of them with any other cluster will form a Δ-tuple which is connected to some U-cluster. This is enough for us to conclude that $|N(U,T)|(1-\mu) > \delta\ell$.
- Assume that $U \subset V(G_r)$ with $|U| > \delta\ell$. Now every Δ-tuple will have a U-neighbor, except those having only one neighbor out of U. (This is enough for the existence of a proportional matching.)
- For proving the existence of a strong proportional matching assume that $U \subset V(G_r)$ with $|U| = (1-\omega)\ell$ $(0 \leq \omega < \mu)$. Since every Δ-tuple having more than one neighbor is connected to this U, we have

to consider only the tuples with only one neighbor. Now $|\overline{U}| = \omega\ell$, and clusters from $\overline{U}$ are connected to at most $\omega\ell\binom{\ell}{\Delta}$ tuples which have one neighbor. Hence, the proportion of tuples which are not neighbors of some U-cluster is at most $\mu\omega < \omega$. In other words, $|N(U)| > \frac{|U|}{\ell}|T|$ for any $0 \le \omega < \mu$. ∎

Now we are ready to present the procedure for assigning the vertices in B to G_r-clusters. We start with the vertices in $B - B'$. First let $L_i = A_i$ for $1 \le i \le \ell$. Assume $\mathcal{M}$ denotes the matching provided by Lemma 17 with respect to the graph F. For a cluster V_t, let $\{V_{i_1}, \ldots, V_{i_\Delta}\}$ be one of the Δ-tuples matched to it in $\mathcal{M}$. We will assign the vertices of $(B - B') \cap Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})$ to the cluster V_t by adding them to the set L_t. Using Lemma 15, $|(B - B') \cap Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})|$ is almost the same for all choices of Δ-tuples, which in turn implies that the set L_t for all $V_t \in G_r$ will have almost the same size after the distribution of $B - B'$. Also, note that the construction of $F(G_r)$ and the structure of the proportional matching $\mathcal{M}$ implies that if $x \in B - B'$ is assigned to L_t then the $N_H(x)$-vertices are assigned to neighboring clusters of V_t.

The vertices of B' will be mapped by the help of strong proportional matching, on the same way as we did for $B - B'$. The only difference is that since every Δ-tuple $V_{i_1}, \ldots, V_{i_\Delta}$ has $\frac{\ell}{\mu}$ copies, the elements of $Q(V_{i_1}) \cap \ldots \cap Q(V_{i_\Delta})$ will be distributed randomly among these copies. It is easy to see that the strong proportional matching assigns B'-vertices evenly - we refer to Corollary 16.

We remark that there are other cases to consider: e.g., some of B-vertices can have all their neighbors assigned to $\Delta - 1$ clusters. But it is easy to see that those matchings are easy to find once the harder cases are dealt with. Then mapping such B-vertices can be done in a similar way as we did for others.

4.3 Finishing the assignment

Now we have to make sure that all conditions of Lemma 5 are satisfied. Obviously, some of them are violated at this moment. E.g. C1 and C3, since so far we have not mapped any vertex to V_0 (L_0 is empty). Therefore, there are more H-vertices assigned to every non-exceptional cluster than its size. We will take care of these problems in separate subsections.

4.3.1 Bad vertices in G

Every vertex in a cluster of a super-regular pair has big degree to the other cluster. Our edges in G_r are regular pairs, some vertices may have just a small number of neighbors in the other cluster (this number can be even zero). To avoid problems which can be caused by this, we are going to discard some vertices from the clusters and put them into V_0, this way C9 will be satisfied.

Let $\mathcal{M}$ be the matching provided by Lemma 17. For a fixed cluster $V_t \in V(G_r)$ let $\mathcal{T}$ denote the set of Δ-tuples matched to V_t in $\mathcal{M}$. We say that $v \in V(G_r)$ has small degree to a Δ-tuple, if v has less than $(d-\varepsilon)m$ neighbors in one of the clusters composing that tuple. Let us call a vertex $v \in V_t$ *bad*, if v has small degree to at least half of the Δ-tuples in $\mathcal{T}$.

Lemma 19 *No cluster in G_r can contain more than $2\Delta\varepsilon m$ bad vertices.*

Proof. For a cluster $V_t \in V(G_r)$ which is matched to the Δ-tuples of $\mathcal{T}$, let $\{v_1, \ldots, v_s\}$ denote the set of bad vertices. If $s > 2\Delta\varepsilon m$ then there should be a tuple $\tau \in \mathcal{T}$ to which more than $\Delta\varepsilon m$ vertices of V_t have small degree. Thus to one of the clusters of this triplet there are more than εm vertices with degree less than $(d-\varepsilon)m$, which contradicts the ε-regularity condition. ∎

By removing the $2\Delta\varepsilon m$ bad vertices from every cluster, we can guarantee that all of their remaining vertices have big degrees to at least half of the matched triplets, and overall at most $2\Delta\varepsilon n$ bad vertices will be added to V_0.

4.3.2 Selecting the L_0-vertices

As we mentioned earlier, every cluster has a surplus, that is, more H-vertices are assigned to them than the clustersize m. We will form L_0 by removing a subset of B' vertices from the L_i sets, achieving that $|L_i| = m$ for $1 \le i \le \ell$.

Let $\phi : L_0 \to V_0$ be any bijective mapping. We need to ensure that the assignment of L_0 to V_0 is consistent with $E(H)$; that is, for any $x \in L_0$, with $(x, y_1), (x, y_2), \ldots, (x, y_\Delta) \in E(H)$, if $v = \phi(x)$, $y_1 \in L_{i_1}, y_2 \in L_{i_2}, \ldots, y_\Delta \in L_{i_\Delta}$ then $deg_G(v, V_{i_1}), deg_G(v, V_{i_2}), \ldots, deg_G(v, V_{i_\Delta})$ are all at least $c_1 m$ (condition C8). If this condition does not hold for a pair (x, v), a *switching* will be performed. In the switching operation we first pick a cluster V_s and then locate a vertex x' in L_s such that $(V_{i_1}, V_s), (V_{i_2}, V_s), \ldots, (V_{i_\Delta}, V_s)$ are all edges in $E(G_r)$. Furthermore, if $(x', y_1'), (x', y_2'), \ldots, (x', y_\Delta') \in E(H)$ with $y_1' \in L_{i_1'}, y_2' \in L_{i_2'}, \ldots, y_\Delta' \in L_{i_\Delta'}$ then $deg_G(v, V_{i_1'}), deg_G(v, V_{i_2'}), \ldots, deg_G(v, V_{i_\Delta'})$ are all at least cm. We will see that such x' can always be found among those vertices assigned by the strong proportional matching.

Lemma 20 *For every $x \in L_0$ there exists an x' as required above.*

Proof. It is easy to see that any $v \in V_0$ has degree less than $c_1 m$ to at most $\frac{1-\delta}{1-c_1}$ proportion of all clusters. Let V_s be as above, then a simple calculation shows that the number of clusters in its neighborhood for which v has large degree ($\geq c_1 m$) is at least $(1-\frac{1}{\Delta})(\frac{\Delta-1}{\Delta+1})\ell$. (We recall that $c_1 = c_1(\Delta)$.) Hence these clusters in the common neighborhood will span at least $(1-\frac{1}{\Delta})^\Delta(\frac{\Delta-1}{\Delta+1})^\Delta\binom{\ell}{\Delta}$ Δ-tuples. This expression is larger than $\frac{1}{27}\binom{\ell}{\Delta}$ for $\Delta \geq 3$. Any Δ-tuple τ will allocate ω vertices by the help of the strong proportional matching. This means V_s receives at least ω vertices from every tuple which is connected to it.

Letting $\Omega = \frac{\ell}{\mu}\binom{\ell}{\Delta}\omega$, the number of those vertices assigned to V_s by the strong proportional matching determined by this $\frac{1}{27}$ proportion of all Δ-tuples is at least $\frac{1}{27}\frac{\mu}{\ell}\Omega$. Now, since the common neighborhood of $V_{i_1}, V_{i_2}, \ldots, V_{i_\Delta}$ contains at least $(1-\delta)\ell$ clusters, the number of vertices assigned to them by the strong proportional matching is by far larger than $|V_0|$. Hence we can find an appropriate x' for any x easily. ∎

Remark 3 *It should be pointed out that we can perform this switching procedure in such a way that the neighbors of the switched x's are scattered almost evenly in a constant proportion of the Δ-tuples, and so in a constant proportion of the clusters. Whenever we are looking for an x', we pick V_s first among the possible $(1-\delta)\ell$ clusters randomly, and then $V_{i'_1}, V_{i'_2}, \ldots, V_{i'_\Delta}$ randomly among the Δ-tuples in the common neighborhood of the corresponding V_0-vertex and V_s. It is easy to see that there is a constant K such that no cluster will contain more than $K\frac{|V_0|}{\ell}$ neighbors.*

Note, that by the help of the above remark we have found an assignment of H-vertices which satisfies all requirements of C1–C11. In the rest of the paper we are going to prove that we will be able to find H-vertices according to C10 and C11 so as to cover the exceptional G-vertices of Step 3 of the embedding algorithm.

Lemma 21 *In Step 3 for each $1 \leq i \leq \ell$ and $v \in E_i$ we can find an unembedded $x \in L(V_i)$ to cover v. This $x \in B'$, and its neighbors are assigned to such clusters to which v has degree at least $(d-\varepsilon)m$. Also, the (assigned, but not embedded) neighbors of these xs are well spread among the clusters of G_r, no cluster will have more than $2\Delta\varepsilon'' m$.*

Proof. Denote the proportional matching by $\mathcal{M}$. For a cluster V_i let T_i denote the set of Δ-tuples matched to it in $\mathcal{M}$. Recall that we removed the *bad* vertices from every cluster (Lemma 19). Hence, at time 0 all the vertices had degree more than $(d-\varepsilon)m$ to at least half of the tuples in T_i, i.e., to all clusters of those Δ-tuples.

We are at time T_0+T_1 now, after embedding $N_H(L_0)$ and the neighborhood of the buffer vertices. Note that we have payed attention to embed these H–vertices as evenly as possible. Not just the neighbors of the buffers are well spread, but $N_H(L_0)$ as well. Hence, even at time T_0+T_1 every vertex in the cluster V_i has degrees big enough to at least 50% of the tuples of T_i (here we used the fact that the number of buffer vertices is very small, and $N_H(L_0)$ is embedded randomly). The exceptional sets of the clusters are small, as we showed in Lemma 9, hence, there are enough candidates to choose.

Now we prove that the covering of the G–vertices can be done in such a way, that the neighbors of the embedded vertices will not concentrate in any of the clusters.

First we will show a simple and easy-to-analyze algorithm for the ideal case when every vertex in V_i to be covered is connected to all the Δ-tuples of T_i for all i. Then we will modify it for our more general setup.

The algorithm is as follows: We start by completing all the E_i sets by arbitrary V_i-vertices obtaining equal size sets. Consider the tuples of T_1: $\tau_1,\ldots,\tau_k$. Pick one-one unembedded vertex from B' which is assigned by these tuples, and cover k E_1-vertices with those B'-vertices. Then repeat it for all T_i. When we are ready, we have finished one *round*. It is easy to see that in one round we covered exactly k vertices from each exceptional set, and because all cluster appears in the same number of tuples, after these coverings we have the same number of neighbors in every cluster, this number is then $\Delta k\ell$. Iterating the rounds at the end we will arrive to the situation when no E_i-vertices are left, and for every cluster the embedded vertices have the same number of assigned neighbors, which is $\Delta\varepsilon'' m$.

Let us return to the assumptions of the lemma. The modified algorithm for the general case will be different in two points. We again start by completing the sets to the same size by adding arbitrary vertices from the corresponding cluster . We take the tuples of the matching one by one, as we did previously. But we cannot always find a vertex to be covered for a tuple. In such a case, we take the next tuple. Even so in every round at least half of the tuples will assign a vertex which will cover a G-vertex, and the embedded vertices will have at most $\Delta k\ell$ assigned neighbors in a cluster. Iterate this procedure, and stop, when a set gets empty. If no exceptional vertices are left (vertices from the original E_i sets before adding other vertices), the algorithm stops. If not, then complete the sets to have the same size, and restart. Note, that now every exceptional set is at most half the size of the beginning. Hence, with $O(\log m)$ restarts no exceptional vertices are left uncovered. Since between any two restarts the number of exceptional vertices is cut in half, and the "fastest decreasing" set has speed at most twice that of the slowest, we get no cluster that will have more than $2\Delta\varepsilon''m$ embedded neighbors. ∎

Proof of Theorem 2 Since by the help of Lemmas 13, 14 and Corollaries 15, 16 we can distribute the A-vertices, and then by Lemmas 17, 19, 20 and 21 we can provide that all conditions of Lemma 5 are satisfied, we can embed H to G, and thus Theorem 2 is proved. ∎

Remark 4 *We have made no attempt to optimize on β. Simple but tedious calculation shows that β can be as large as $\frac{1}{\Delta^4}$. On the other hand we think that this is still not the right value for β.*

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