I N F O R M A T I K

Superposition and Chaining for Totally Ordered Divisible Abelian Groups

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Abstract

We present a calculus for first-order theorem proving in the presence of the axioms of totally ordered divisible abelian groups. The calculus extends previous superposition or chaining calculi for divisible torsion-free abelian groups and dense total orderings without endpoints. As its predecessors, it is refutationally complete and requires neither explicit inferences with the theory axioms nor variable overlaps. It offers thus an efficient way of treating equalities and inequalities between additive terms over, e. g., the rational numbers within a first-order theorem prover.

Keywords

Automated Theorem Proving, Superposition, Chaining, First-Order Logic, Equality, Totally Ordered Divisible Abelian Groups, Term Rewriting, Variable Elimination

1 Introduction

Most real life problems for an automated theorem prover contain both uninterpreted function and predicate symbols, that are specific for a particular domain, and standard algebraic structures, such as numbers or orderings. General theorem proving techniques like resolution or superposition are notoriously bad at handling algebraical theories involving axioms like associativity, commutativity, or transitivity, since explicit inferences with these axioms lead to an explosion of the search space. To deal efficiently with such structures, it is therefore necessary that specialized techniques are built tightly into the prover.

AC-superposition (Bachmair and Ganzinger [1], Wertz [11]) is a well-known example of such a technique. It incorporates associativity and commutativity into the standard superposition calculus using AC-unification and extended clauses. In this way, inferences with the theory axioms and certain inferences involving variables are rendered unnecessary. Still, reasoning with the associativity and commutativity axioms remains difficult for an automated theorem prover, even if explicit inferences with the AC axioms can be avoided. This is not only due to the NP-completeness of the AC-unifiability problem, but it stems also from the fact that AC-superposition requires an inference between literals $u_1 + \cdots + u_k \approx s$ and $v_1 + \cdots + v_l \approx t$ (via extended clauses) whenever *some* u_i is unifiable with *some* v_j. Consequently, a variable in a sum can be unified with any part of any other sum – in this situation unification is completely unable to limit the search space.

The inefficiency inherent in the theory of associativity and commutativity can be mitigated by integrating further axioms into the calculus. In abelian groups (or even in cancellative abelian monoids) the ordering conditions of the inference rules can be refined in such a way that summands u_i and v_j have to be overlapped only if they are *maximal* with respect to some simplification ordering $\succ$ (Ganzinger and Waldmann [4, 8], Marché [5], Stuber [7]). In this way, the number of variable overlaps can be greatly reduced; however, inferences with unshielded, i. e., potentially maximal, variables remain necessary.

In non-trivial divisible torsion-free abelian groups (e. g., the rational numbers and rational vector spaces), the abelian group axioms are extended by the torsion-freeness axiom $\forall k \in \mathbf{N}^{>0}\ \forall x, y\colon kx \approx ky \Rightarrow x \approx y$, the divisibility axiom $\forall k \in \mathbf{N}^{>0}\ \forall x\ \exists y\colon ky \approx x$, and the non-triviality axiom $\exists y\colon y \not\approx 0$. In such structures every clause can be transformed into an equivalent clause without unshielded variables. Integrating this variable elimination algorithm into cancellative superposition results in a calculus that requires neither extended clauses, nor variable overlaps, nor explicit inferences with the theory axioms. Furthermore, using full abstraction even AC unification can be avoided (Waldmann [10]).

When we want to work with a transitive relation $>$ in a theorem prover, we encounter a situation that is surprisingly similar to the one depicted above. Just as associativity and commutativity, the transitivity axiom is fairly prolific. It allows to derive a new clause whenever the left hand side of a literal $r > s$ overlaps with the right hand side of another literal $s' > t$. As such an overlap is always possible if s or s' is a variable, unification is not an effective filter to control the generation of new clauses. The use of the chaining inference rule makes explicit inferences with the transitivity axiom superfluous (Slagle [6]). Since this inference rule can be equipped with the restriction that the overlapped term s must be maximal with respect to a simplification ordering $\succ$, overlaps with shielded variables become again unnecessary. Only inferences with unshielded, i. e., potentially maximal, variables have to be computed.

Once more, the number of unshielded variables in a clause can be reduced if further axioms are available. In particular, in dense total orderings without endpoints, unshielded variables can be eliminated completely (Bachmair and Ganzinger [3]).

There are two facts that suggest to investigate the combination of the theory of divisible torsion-free abelian groups and the theory of dense total orderings without endpoints. On the one hand, the vast majority of applications of divisible torsion-free abelian groups (and in particular of the rationals or reals) requires also an ordering; so the combined calculus is likely to be much more useful in practice than the DTAG-superposition calculus on which it is based. On the other hand, these two theories are closely related: An abelian group $(G, +, 0)$ can be equipped with a total ordering that is compatible with $+$ if and only if it is torsion-free; furthermore divisibility and compatibility of the ordering imply that the ordering is dense and has no endpoints. One can thus assume that the two calculi fit together rather smoothly. We show in this paper that this is in fact true. The resulting calculus splits again into two parts: The first one is a base calculus, that works on clauses without unshielded variables, but whose rules may produce clauses with unshielded variables. This calculus has the property that saturated sets of clauses are unsatisfiable if and only if they contain the empty clause, but it can not be used to effectively saturate a given set of clauses. The second part of the calculus is a variable elimination algorithm that makes it possible to get rid of unshielded variables, and thus renders the base calculus effective. The integration of these two components happens in essentially the same way as in the equational case (Waldmann [10]).

2 The Base Calculus

2.1 Preliminaries

We work in a many-sorted framework and assume that the function symbol $+$ is declared on a sort G. If t is a term of sort G and $n \in \mathbf{N}$, then nt is an abbreviation for the n-fold sum $t + \cdots + t$; in particular, $0t = 0$ and $1t = t$.

Without loss of generality we assume that the equality relation $\approx$ and the semantic ordering $>$ are the only predicates of our language. Hence a literal is either an equation $t \approx t'$, or a negated equation $t \not\approx t'$, where t and t' have the same sort, or an inequation $t > t'$, or a negated inequation $t \not> t'$, where t and t' have sort G. Occasionally we write $t' < t$ instead of $t > t'$. The symbol $\gtrless$ denotes either $>$ or $<$, the symbol $\gtrsim$ stands for $>$ or $\approx$, the symbol $\sim$ means either $\gtrless$ or $\approx$, and $\dot{\sim}$ denotes $\gtrless$ or $\approx$ or $\not\approx$. The equality symbol is supposed to be symmetric. Multiple occurrences of one of the symbols $\gtrless$, $\sim$, or $\dot{\sim}$ within a single inference rule denote consistently the same relation. A clause is a finite multiset of literals, usually written as a disjunction.

A (Herbrand) interpretation E is a set of equations and inequations. A positive ground literal e is true in E, if $e \in E$; a negative ground literal $\neg\, e$ is true in E, if $e \notin E$. A ground clause C is true in E, if at least one of its literals is true in E; a non-ground clause is true in E, if all its ground instances are true in E. If a clause C is true in E, we also say that E is a model of C, or that E satisfies C.

The clauses

$$\begin{array}{cr}
(x + y) + z \approx x + (y + z) & \text{(Associativity (A))} \\
x + y \approx y + x & \text{(Commutativity (C))} \\
x + 0 \approx x & \text{(Identity (U))} \\
(-x) + x \approx 0 & \text{(Inverse (Inv))} \\
n\ \mathit{divided\text{-}by}_n(x) \approx x & \text{(Divisibility (Div))} \\
a_0 \not\approx 0 & \text{(Non-Triviality (Nt))} \\
x \not> x & \text{(Irreflexivity (Ir))} \\
x \not> y \ \vee\ y \not> z \ \vee\ x > z & \text{(Transitivity (Tr))} \\
x \not> y \ \vee\ x + z > y + z & \text{(Monotonicity (Mon))} \\
x > y \ \vee\ y > x \ \vee\ x \approx y & \text{(Totality (Tot))}
\end{array}$$

plus the equality axioms[1] are the axioms ODAG of totally ordered divisible abelian groups.

The following clauses are consequences of these axioms (for every $\psi \in \mathbf{N}^{>0}$):

$$\begin{array}{cr}
x + z \not\approx y + z \ \vee\ x \approx y & \text{(Cancellation (K))} \\
\psi x \not\approx \psi y \ \vee\ x \approx y & \text{(Torsion-Freeness (T))} \\
x + z \not> y + z \ \vee\ x > y & \text{(>-Cancellation (K}^{>}\text{))} \\
\psi x \not> \psi y \ \vee\ x > y & \text{(>-Torsion-Freeness (T}^{>}\text{))}
\end{array}$$

[1]including the congruence axiom $x \not\approx y \ \vee\ y \not\gtrless z \ \vee\ x \gtrless z$ for the predicate $>$.

We write OTfCAM for the union of the clauses A, C, U, K, T, Ir, Tr, Mon, $\mathrm{K}^{>}$, $\mathrm{T}^{>}$ and the equality axioms.

We denote the entailment relation modulo ODAG by $\models_{\mathrm{ODAG}}$, and the entailment relation modulo OTfCAM by $\models_{\mathrm{OTfCAM}}$. That is, $\{C_1, \ldots, C_n\} \models_{\mathrm{ODAG}} C_0$ if and only if $\{C_1, \ldots, C_n\} \cup \mathrm{ODAG} \models C_0$, and $\{C_1, \ldots, C_n\} \models_{\mathrm{OTfCAM}} C_0$ if and only if $\{C_1, \ldots, C_n\} \cup \mathrm{OTfCAM} \models C_0$.

A function symbol is called free, if it is different from 0 and $+$. A term is called atomic, if it is not a variable and its top symbol is different from $+$. We say that a term t occurs at the top of s, if there is a position $o \in \mathrm{pos}(s)$ such that $s|_o = t$ and for every proper prefix o' of o, $s(o')$ equals $+$; the term t occurs in s below a free function symbol, if there is an $o \in \mathrm{pos}(s)$ such that $s|_o = t$ and $s(o')$ is a free function symbol for some proper prefix o' of o. A variable x is called shielded in a clause C, if it occurs at least once below a free function symbol in C, or if it does not have sort G. Otherwise, x is called unshielded.

A clause C is called fully abstracted, if no non-variable term of sort G occurs below a free function symbol in C. Every clause C can be transformed into an equivalent fully abstracted clause $\mathrm{abs}(C)$ by iterated rewriting

$$C[f(\ldots,t,\ldots)] \quad \to \quad x \not\approx t \,\vee\, C[f(\ldots,x,\ldots)]\,,$$

where x is a new variable and t is a non-variable term of sort G occurring immediately below the free function symbol f in C.

We say that an ACU-compatible ordering $\succ$ has the multiset property, if whenever a ground atomic term u is greater than v_i for every i in a finite non-empty index set I, then $u \succ \sum_{i \in I} v_i$. Every reduction ordering over terms not containing $+$ that is total on ground terms and for which 0 is minimal can be extended to an ordering that is ACU-compatible and has the multiset property (Waldmann [9]).[2]

From now on we will work *only* with ACU-congruence classes, rather than with terms. So *all* terms, equations, substitutions, inference rules, etc., are to be taken modulo ACU, i. e., as representatives of their congruence classes. The symbol $\succ$ will always denote an ACU-compatible ordering that has the multiset property, is total on ground ACU-congruence classes, and satisfies $t \not\succ s[t]_o$ for every term $s[t]_o$.

Let A be a ground literal. Then the largest atomic term occurring on either side of A is denoted by $\mathrm{mt}(A)$. If C is a ground clause, then $\mathrm{mt}(C)$ is the largest atomic term occurring in C.

The balance value of a ground literal A is 3, if $\mathrm{mt}(A)$ occurs on both sides of A, it is 2, if A is an inequation $[\neg]\ s > t$ and $\mathrm{mt}(A)$ occurs only in s, and otherwise it is 1. The ordering $\succ_{\mathrm{L}}$ on literals compares lexicographically

[2] In fact, we use the extended ordering only as a theoretical device; as we work with fully abstracted clauses, the original reduction ordering is sufficient for actual computations.

first the maximal atomic terms of the literals, then the polarities (negative $\succ$ positive), then the kinds of the literals (inequation $\succ$ equation), then the balance values of the literals, then the multisets of all non-zero terms occurring at the top of the literals, and finally the multisets $\{\{s\},\{t\}\}$ (for equations $[\neg]\ s \approx t$) or $\{\{s,s\},\{t\}\}$ (for inequations $[\neg]\ s > t$). The ordering $\succ_{\mathrm{C}}$ on clauses is the multiset extension of the literal ordering $\succ_{\mathrm{L}}$. Both $\succ_{\mathrm{L}}$ and $\succ_{\mathrm{C}}$ are noetherian and total on ground literals/clauses.

2.2 Superposition and Chaining

We present the ground versions of the inference rules of the base calculus *OCInf*. The non-ground versions can be obtained by lifting in a rather straightforward way (see below).

Let us start the presentation of the inference rules with a few general conventions: Every term occurring in a sum is assumed to have sort G. The letters u and v, possibly with indices, denote atomic terms, unless explicitly said otherwise. In an expression like $mu + s$, m is a natural number, s may be zero.

If an inference involves a literal, then it must be maximal in the respective clause (except for the last but one literal in *factoring* inferences). A positive literal that is involved in a *superposition* or *chaining* inference must be strictly maximal in the respective clause. In all *superposition* or *chaining* inferences, the left premise is smaller than the right premise.

Cancellation
$$\frac{C' \lor mu + s \mathrel{\dot\sim} m'u + s'}{C' \lor (m - m')u + s \mathrel{\dot\sim} s'}$$
if $m \geq m' \geq 1$, $u \succ s$, $u \succ s'$.

Equality Resolution
$$\frac{C' \lor u \not\approx u}{C'}$$
if u either equals 0 or does not have sort G.

Inequality Resolution
$$\frac{C' \lor 0 > 0}{C'}$$

Canc. Superposition
$$\frac{D' \lor nu + t \approx t' \qquad C' \lor mu + s \mathrel{\dot\sim} s'}{D' \lor C' \lor ns + mt' \mathrel{\dot\sim} ns' + mt}$$
if $n \geq 1$, $m \geq 1$, $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.[3]

[3] If $\gcd(m,n) > 1$, then the conclusion of this inference can be simplified to $D' \lor C' \lor \psi s + \chi t' \mathrel{\dot\sim} \psi s' + \chi t$, where $\psi = n/\gcd(m,n)$ and $\chi = m/\gcd(m,n)$ (and similarly for the following inference rules). To enhance readability, we leave out this optimization in the sequel.

Canc. Chaining
$$\frac{D' \vee t' \gtrless nu + t \qquad C' \vee mu + s \gtrless s'}{D' \vee C' \vee ns + mt' \gtrless ns' + mt}$$
if $n \geq 1$, $m \geq 1$, $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

Std. Superposition
$$\frac{D' \vee u \approx u' \qquad C' \vee s[u] \mathrel{\dot\sim} s'}{D' \vee C' \vee s[u'] \mathrel{\dot\sim} s'}$$
if u occurs in a maximal atomic subterm of s and does not have sort G, $u \succ u'$, $s[u] \succ s'$.

Canc. Eq. Factoring
$$\frac{C' \vee nu + t \approx t' \vee mu + s \approx s'}{C' \vee mt + ns' \not\approx mt' + ns \vee nu + t \approx t'}$$
if $n \geq 1$, $m \geq 1$, $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

Canc. Ineq. Factoring (I)
$$\frac{C' \vee nu + t \gtrless t' \vee mu + s \gtrless s'}{C' \vee mt + ns' \gtrless mt' + ns \vee mu + s \gtrless s'}$$
if $n \geq 1$, $m \geq 1$, $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

Canc. Ineq. Factoring (II)
$$\frac{C' \vee nu + t \gtrless t' \vee mu + s \gtrless s'}{C' \vee mt' + ns \gtrless mt + ns' \vee nu + t \gtrless t'}$$
if $n \geq 1$, $m \geq 1$, $u \succ s$, $u \succ s'$, $u \succ t$, $u \succ t'$.

Std. Eq. Factoring
$$\frac{C' \vee u \approx v' \vee u \approx u'}{C' \vee u' \not\approx v' \vee u \approx v'}$$
if u, u' and v' do not have sort G, $u \succ u'$, $u \succ v'$.

The inference rules of the calculus *OCInf* do not handle negative inequality literals. We assume that in the beginning of the saturation process every literal $s \not> t$ in an input clause is replaced by the two literals $t > s \vee t \approx s$, which are equivalent to $s \not> t$ by the totality, transitivity and irreflexivity axioms. Note that the inference rules of *OCInf* do not produce any new negative inequality literals.

In the standard superposition calculus, lifting means replacing equality in the ground inference by unifiability. As long as all variables in our clauses are shielded, the situation is similar here: For instance, in the second premise $C' \vee A_1$ of a *cancellative superposition* inference the maximal literal A_1 need no longer have the form $mu + s \mathrel{\dot\sim} s'$ with a unique maximal atomic term u. Rather, it may contain several (distinct but ACU-unifiable) maximal atomic terms u_k with multiplicities m_k, where k ranges over some finite non-empty index set K. We obtain thus $A_1 = \sum_{k \in K} m_k u_k + s \mathrel{\dot\sim} s'$. In the inference rule, the substitution σ that unifies all u_k (and the corresponding terms v_l from the other premise) is applied to the conclusion. Consequently, the *cancellative superposition* rule has now the following form:

$$\frac{D' \vee \sum_{l \in L} n_l v_l + t \approx t' \qquad C' \vee \sum_{k \in K} m_k u_k + s \mathrel{\dot{\sim}} s'}{(D' \vee C' \vee ns + mt' \mathrel{\dot{\sim}} ns' + mt)\sigma}$$

where

(i) $m = \sum_{k \in K} m_k \geq 1$, $n = \sum_{l \in L} n_l \geq 1$.

(ii) σ is a most general ACU-unifier of all u_k and v_l $(k \in K, l \in L)$.

(iii) u is one of the u_k $(k \in K)$.

(iv) $u\sigma \not\preceq s\sigma$, $u\sigma \not\preceq s'\sigma$, $u\sigma \not\preceq t\sigma$, $u\sigma \not\preceq t'\sigma$.

The other inference rules can be lifted in a similar way, again under the condition that all variables in the clauses are shielded. As usual, the *standard superposition* rule is equipped with the additional restriction that the subterm of s that is replaced during the inference is not a variable. For clauses with unshielded variables, lifting would be significantly more complicated; however, as we will combine the base calculus with an algorithm that eliminates unshielded variables, we need not consider this case.

THEOREM 2.1 *The inference rules of the calculus OCInf are sound with respect to* $\models_{\mathrm{ODAG}}$.

DEFINITION 2.2 *Let N be a set of clauses, let $\overline{N}$ be the set of ground instances of clauses in N. An inference is called OCRed-redundant with respect to N if for each of its ground instances with conclusion $C_0\theta$ and maximal premise $C\theta$ we have $\{ D \in \overline{N} \mid D \prec_{\mathrm{C}} C\theta \} \models_{\mathrm{OTfCAM}} C_0\theta$. A clause C is called OCRed-redundant with respect to N, if for every ground instance $C\theta$, $\{ D \in \overline{N} \mid D \prec_{\mathrm{C}} C\theta \} \models_{\mathrm{OTfCAM}} C\theta$.*

2.3 Rewriting on Equations

To prove that the inference system described so far is refutationally complete we have to show that every saturated clause set that does not contain the empty clause has a model. The traditional approach to construct such a model is rewrite-based: First an ordering is imposed on the set of all ground instances of clauses in the set. Starting with an empty interpretation all such instances are inspected in ascending order. If a reductive clause is false and irreducible in the partial interpretation constructed so far, its maximal positive literal is turned into a rewrite rule and added to the interpretation. If the original clause set is saturated and does not contain the empty clause, then the final interpretation is a model of all ground instances, and thus of the original clause set (Bachmair and Ganzinger [2]).

In order to be able to treat cancellative superposition we have modified this scheme in [4] in such a way that the rewrite relation operates on equations rather than on terms. But if we also have to deal with inequations, a further extension is necessary: We need to be able to rewrite inequations *with inequations*; and unlike rewriting with equations, this does of course not produce logically equivalent formulae.

Definition 2.3 *A ground equation or inequation e is called a cancellative rewrite rule with respect to $\succ$, if $\mathrm{mt}(e)$ does not occur on both sides of e.*

We will usually drop the attributes "cancellative" and "with respect to $\succ$", speaking simply of "rewrite rules".

Every rewrite rule has either the form $mu + s \sim s'$, where u is an atomic term, $m \in \mathbf{N}^{>0}$, $u \succ s$, and $u \succ s'$, or the form $u \approx s'$, where $u \succ s'$ and u (and thus s') does not have sort G. This is an easy consequence of the multiset property of $\succ$.

Definition 2.4 *Given a set R of rewrite rules, the four binary relations $\rightarrow_{\gamma,R}$, $\rightarrow_{\delta,R}$, $\rightarrow_{o,R}$, and $\rightarrow_{\kappa}$ on ground equations and inequations are defined (modulo* ACU*) as follows:*[4]

(i) $mu + t \sim t' \;\rightarrow_{\gamma,R}\; s' + t \sim t' + s$,
if $mu + s \approx s'$ *is a rule in* R.

(ii) $t[s] \sim t' \;\rightarrow_{\delta,R}\; t[s'] \sim t'$,
if (i) $s \approx s'$ *is a rule in* R *and (ii)* s *does not have sort* G *or* s *occurs in* t *below some free function symbol.*

(iii) $mu + t \gtrless t' \;\rightarrow_{o,R}\; s' + t \gtrless t' + s$,
if $mu + s \gtrless s'$ *is a rule in* R.

(iv) $u + t \sim u + t' \;\rightarrow_{\kappa}\; t \sim t'$,
$u \approx u \;\rightarrow_{\kappa}\; 0 \approx 0$,
if u *is atomic and different from* 0.

The union of $\rightarrow_{\gamma,R}$, $\rightarrow_{\delta,R}$, $\rightarrow_{o,R}$, and $\rightarrow_{\kappa}$ is denoted by $\rightarrow_{R}$.[5]

[4]While we have the restriction $u \succ s$, $u \succ s'$ for the rewrite rules, there is no such restriction for the (in-)equations to which rules are applied.

[5]As we deal only with ground terms and as there are no non-trivial contexts around (in-)equations, this operation does indeed satisfy the definition of a rewrite relation, albeit in an unorthodox way.

If $e \to_R e'$ using a γ-, δ- or κ-step, then e and e' are equivalent modulo OTfCAM and the applied rewrite rule. If $s \gtrless s' \to_{o,R} t \gtrless t'$, then both $t \gtrless t'$ and $t \approx t'$ imply $s \gtrless s'$ modulo OTfCAM and the applied rewrite rule.

We say that an (in-)equation e is γ-reducible, if $e \to_{\gamma,R} e'$ (analogously for δ, o, and κ). It is called reducible, if it is γ-, δ-, o-, or κ-reducible.

Unlike κ-reducibility, γ-, δ-, and o-reducibility can be extended to terms: A term t is called γ-reducible, if $t \sim t' \to_{\gamma,R} e'$, where the rewrite step takes place at the left-hand side (analogously for δ and o). It is called reducible, if it is γ-, δ- or o-reducible.

LEMMA 2.5 *The relation $\to_R$ is contained in $\succ_{\mathrm{L}}$ and thus noetherian.*

DEFINITION 2.6 *Given a set R of rewrite rules, the relation $\to_R^\circ$ is defined by $\to_R^\circ = (\to_R^* \circ \to_{o,R} \circ \to_R^*)$.*

Given equations $e_1 = s_1 \approx s_1'$ and $e_2 = s_2 \approx s_2'$ and a positive integer ψ, we write ψe_1 for the equation $\psi s_1 \approx \psi s_1'$ and $e_1 + e_2$ for the equation $s_1 + s_2 \approx s_1' + s_2'$. (Analogously, if e_1 and/or e_2 are inequations $s_1 > s_1'$ and $s_2 > s_2'$).

DEFINITION 2.7 *Given a set R of rewrite rules, the set $\mathrm{tr}(R)$ is the set of all (in-)equations $s \sim s'$ for which there exists a derivation $s \sim s' \to_R^* 0 \sim 0$. The truth set $\mathrm{tr}^\circ(R)$ of R is the set of all equations $s \approx s'$ for which there exists a derivation $s \approx s' \to_R^* 0 \approx 0$, and the set of all inequations $s \gtrless s'$ for which there exists a derivation $s \gtrless s' \to_R^\circ 0 \gtrless 0$. The Ψ-truth set $\mathrm{tr}_\Psi^\circ(R)$ of R is the set of all equations or inequations $e = s \sim s'$, such that either $e \in \mathrm{tr}^\circ(R)$ and s does not have sort G, or $\psi s \sim \psi s' \in \mathrm{tr}^\circ(R)$ for some $\psi \in \mathbf{N}^{>0}$.*

All (in-)equations in $\mathrm{tr}_\Psi^\circ(R)$ are logical consequences of the rewrite rules in R and the theory axioms OTfCAM.

2.4 Model Construction

DEFINITION 2.8 *A ground clause $C' \vee e$ is called reductive for e, if e is a cancellative rewrite rule and strictly maximal in $C' \vee e$.*

DEFINITION 2.9 *Let N be a set of (possibly non-ground) clauses that does not contain the empty clause, and let $\overline{N}$ the set of all ground instances of clauses in N. Using induction on the clause ordering we define sets of rules R_C, R_C^Ψ, E_C, and E_C^Ψ, for all clauses $C \in \overline{N}$. Let C be such a clause and assume that R_D, R_D^Ψ, E_D, and E_D^Ψ have already been defined for all $D \in \overline{N}$ such that $C \succ_{\mathrm{C}} D$. Then the set R_C of primary rules and the set R_C^Ψ of secondary rules* *are given by*

$$R_C = \bigcup_{D \prec_C C} E_D \quad \text{and} \quad R_C^\Psi = \bigcup_{D \prec_C C} E_D^\Psi\,.$$

E_C is the singleton set $\{e\}$, if C is a clause $C' \vee e$ such that (i) C is reductive for e, (ii) C is false in $\mathrm{tr}^\circ(R_C^\Psi)$, (iii) C' is false in $\mathrm{tr}^\circ_\Psi(R_C^\Psi \cup \{e\})$, and (iv) $\chi\,\mathrm{mt}(e)$ is $\gamma\delta$-irreducible with respect to R_C^Ψ for every $\chi \in \mathbf{N}^{>0}$. Otherwise, E_C is empty.

If $E_C = \{e\}$, then E_C^Ψ is the set of all rewrite rules $e' \in \mathrm{tr}^\circ_\Psi(R_C^\Psi \cup E_C)$ such that $\mathrm{mt}(e') = \mathrm{mt}(e)$ and e' is $\delta\kappa$-irreducible with respect to R_C^Ψ. Otherwise, E_C^Ψ is empty.

Finally, the sets R_∞ and R_∞^Ψ are defined by

$$R_\infty = \bigcup_{D \in \overline{N}} E_D \quad \text{and} \quad R_\infty^\Psi = \bigcup_{D \in \overline{N}} E_D^\Psi\,.$$

Our goal is to show that, if N is saturated with respect to *OCInf*, then $\mathrm{tr}^\circ(R_\infty^\Psi)$ is a model of the axioms of totally ordered divisible abelian groups and of the clauses in N. To this end, we will first put together some basic properties of R_C^Ψ and R_∞^Ψ.

LEMMA 2.10 *Let $E_C = \{mu + s \gtrless s'\}$. Then the inequation that is obtained by $\delta\kappa$-normalizing $mu + s \gtrless s'$ with respect to R_C^Ψ is contained in E_C^Ψ.*

PROOF. As u is δ-irreducible with respect to R_C^Ψ, the $\delta\kappa$-normalization of $mu + s \gtrless s'$ has the form

$$\begin{array}{c} mu + s \gtrless s' \\ ① \Big\downarrow {\scriptstyle \delta\cup\kappa}\,{}^{*} \\ mu + r \gtrless r' \end{array}$$

Then $u \succ s \succ r$ and $u \succ s' \succ r'$. Starting from $mu + r \gtrless r'$ we can construct a derivation

$$\begin{array}{c} mu + r \gtrless r' \\ ② \Big\downarrow {\scriptstyle o} \\ s' + r \gtrless r' + s \\ ③ \Big\downarrow {\scriptstyle \delta\cup\kappa}\,{}^{*} \\ r' + r \gtrless r' + r \\ ④ \Big\downarrow {\scriptstyle \kappa}\,{}^{*} \\ 0 \gtrless 0 \end{array}$$

where ② uses $mu + s \gtrless s'$ and ③ simulates ①. Hence $mu + r \gtrless r'$ is contained in $\mathrm{tr}^{\circ}_{\Psi}(R^{\Psi}_{C} \cup E_C)$ and thus in E^{Ψ}_{C}. □

2.5 Refutational Completeness of *OCInf*

The relations $\rightarrow_{R^{\Psi}_{C}}$ and $\rightarrow_{R^{\Psi}_{\infty}}$ are in general not confluent, not even in the purely equational case. One can merely show that that $\rightarrow_{R^{\Psi}_{C}}$ is confluent on equations in $\mathrm{tr}(R^{\Psi}_{C})$, that is, that any two derivations starting from an equation e can be joined, provided that there is a derivation $e \rightarrow^{*} 0 \approx 0$. But even this kind of restricted confluence does not hold for inequations, and in particular, not for o-rewriting. We can only prove that two derivations starting from the same inequation can be joined, if one of them leads to $0 > 0$ and if the other one does not use o-steps. This property will be sufficient for our purposes, however.

Definition 2.11 *Let E be a set of equations and/or inequations. We say that the relation $\rightarrow_R$ is partially confluent on E, if for all equations $e_0 \in E$ and e_1, e_2 with $e_1 \leftarrow^{*}_{R} e_0 \rightarrow^{*}_{R} e_2$ there exists an equation e_3 such that $e_1 \rightarrow^{*}_{R} e_3 \leftarrow^{*}_{R} e_2$, and if for all inequations $e'_0 \in E$ and e'_1 with $e'_1 \leftarrow^{*}_{\gamma\delta\kappa,R} e'_0 \rightarrow^{*}_{R} 0 > 0$ or $e'_1 \leftarrow^{*}_{\gamma\delta\kappa,R} e'_0 \rightarrow^{\circ}_{R} 0 > 0$ there is a derivation $e'_1 \rightarrow^{*}_{R} 0 > 0$ or $e'_1 \rightarrow^{\circ}_{R} 0 > 0$, respectively.*

Lemma 2.12 *Let C be a clause in $\overline{N}$. If an inequation $e \in \mathrm{tr}^{\circ}_{\Psi}(R^{\Psi}_{C})$ is $\delta\kappa$-irreducible with respect to R^{Ψ}_{C}, and $\rightarrow_{R^{\Psi}_{C}}$ is partially confluent on $\mathrm{tr}(R^{\Psi}_{C}) \cap \{\, e' \mid \mathrm{mt}(e) \succ \mathrm{mt}(e') \,\}$, then $e \in R^{\Psi}_{C}$. (Analogously for C replaced by ∞.)*

Proof. We will prove the first part of the lemma, the proof of the second one being similar. By the definition of $\mathrm{tr}^{\circ}_{\Psi}(R^{\Psi}_{C})$, an inequation e cannot be in normal form with respect to $\rightarrow_{R^{\Psi}_{C}}$, hence e is different from $0 > 0$. Let $v = \mathrm{mt}(e)$. By assumption, e is $\delta\kappa$-irreducible. We may thus suppose that e has the form $kv + t \gtrless t'$, where $v \succ t$ and $v \succ t'$. By definition of $\mathrm{tr}^{\circ}_{\Psi}(R^{\Psi}_{C})$, there is a derivation $\psi' e \rightarrow^{\circ}_{R^{\Psi}_{C}} 0 \gtrless 0$ for some $\psi' \in \mathbf{N}^{>0}$. During this derivation all occurrences of v are deleted eventually. As e is $\delta\kappa$-irreducible, this can be done only by (possibly several) γ- or o-rewriting steps, using rules in R^{Ψ}_{C}. We distinguish between two cases, depending on whether the primary rules by which these secondary rules have been generated are equations or inequations.

Case 1: $\{mv + s \approx s'\} = E_D \subseteq R_C$.

Then the occurrences of v are deleted using rules $\hat{e}_i = \hat{m}_i v + \hat{r}_i \approx \hat{r}'_i$ and/or $\tilde{e}_j = \tilde{m}_j v + \tilde{r}_j \gtrless \tilde{r}'_j$, and all all $\hat{e}_i$ and $\tilde{e}_j$ are contained in E^{Ψ}_{D}. We may assume without loss of generality that the derivation $\psi' e \rightarrow^{\circ}_{R^{\Psi}_{C}} 0 \gtrless 0$ has the form

$$
\begin{array}{rcl}
& & \psi' kv + \psi' t \gtrless \psi' t' \\
& & \quad \text{①} \Big\downarrow {}^{\gamma\cup o}_{+} \\
e_1 & = & \sum_j \tilde{r}'_j + \sum_i \hat{r}'_i + \psi' t \gtrless \psi' t' + \sum_i \hat{r}_i + \sum_j \tilde{r}_j \\
& & \quad \text{②} \Big\downarrow {}_{*} \\
& & 0 \gtrless 0
\end{array}
$$

where the rewrite steps of ① use the sequence of rules $\hat{e}_i$ and $\tilde{e}_j$, the rewrite steps of ② use rules from R_D^Ψ, and $\sum_i \hat{m}_i + \sum_j \tilde{m}_j = \psi' k$. There exists a $\psi \in \mathbf{N}^{>0}$ and for every i and j an $(R_D^\Psi \cup E_D)$-derivation

$$
\begin{array}{rclcrcl}
\psi \hat{e}_i & = & \psi \hat{m}_i v + \psi \hat{r}_i \approx \psi \hat{r}'_i & \qquad & \psi \tilde{e}_j & = & \psi \tilde{m}_j v + \psi \tilde{r}_j \gtrless \psi \tilde{r}'_j \\
& & \text{③} \Big\downarrow {}^{\gamma}_{+} & & & & {}^{\gamma}_{+} \Big\downarrow \text{⑤} \\
\hat{e}'_i & = & \hat{\chi}_i s' + \psi \hat{r}_i \approx \psi \hat{r}'_i + \hat{\chi}_i s & & \tilde{e}'_j & = & \tilde{\chi}_j s' + \psi \tilde{r}_j \gtrless \psi \tilde{r}'_j + \tilde{\chi}_j s \\
& & \text{④} \Big\downarrow {}_{*} & & & & {}_{\circ} \Big\downarrow \text{⑥} \\
& & 0 \approx 0 & & & & 0 \gtrless 0
\end{array}
$$

starting with $\hat{\chi}_i$- or $\tilde{\chi}_j$-fold application of $mv + s \approx s'$, where $\psi \hat{m}_i = \hat{\chi}_i m$ and $\psi \tilde{m}_j = \tilde{\chi}_j m$.

Let $e_2 = \psi e_1 + \sum_i \hat{e}'_i + \sum_j \tilde{e}'_j$. Then e_2 has a derivation to $0 \gtrless 0$. Cancellation of $\psi \sum_i (\hat{r}_i + \hat{r}'_i) + \psi \sum_j (\tilde{r}_j + \tilde{r}'_j)$ in e_2 yields

$$
e_3 \;=\; (\textstyle\sum_i \hat{\chi}_i + \sum_j \tilde{\chi}_j) s' + \psi \psi' t \gtrless \psi \psi' t' + (\sum_i \hat{\chi}_i + \sum_j \tilde{\chi}_j) s \, .
$$

By partial confluence of $\to_{R_C^\Psi}$ we obtain $e_3 \;\to^*_{R_C^\Psi}\; 0 > 0$. Since $\mathrm{mt}(e_3) \prec v$, rules in $R_C^\Psi \setminus R_D^\Psi$ cannot be used in this derivation, hence $e_3 \;\to^*_{R_D^\Psi}\; 0 > 0$.

On the other hand, $\psi \psi' k = \sum_i \psi \hat{m}_i + \sum_j \psi \tilde{m}_j = m(\sum_i \hat{\chi}_i + \sum_j \tilde{\chi}_j)$, thus we can rewrite $\psi \psi' e$ to e_3 by $(\sum_i \hat{\chi}_i + \sum_j \tilde{\chi}_j)$-fold application of $mv + s \approx s'$. As e is $\delta\kappa$-irreducible with respect to $R_D^\Psi \subseteq R_C^\Psi$, e is contained in $E_D^\Psi \subseteq R_C^\Psi$ by Def. 2.9.

Case 2: Otherwise.

Otherwise, in the derivation $\psi' e \to^\circ_{R_C^\Psi} 0 \gtrless 0$ the occurrences of v are eliminated by o-applications of secondary rules that have been generated by one or more inequations. Let D be the maximal clause such that rules $\hat{e}_i = \hat{m}_i v + \hat{r}_i \gtrless \hat{r}'_i$ in $E_D^\Psi \subseteq R_C^\Psi$ are used in the derivation. Let $\tilde{e}_j = \tilde{m}_j v + \tilde{r}_j \gtrless \tilde{r}'_j$ be the remaining rules in R_D^Ψ used to eliminate the $\psi' k$ occurrences of v. We may thus assume that $\psi' k = \sum_i \hat{m}_i + \sum_j \tilde{m}_j$ and that the derivation $\psi' e \;\to^*_{R_C^\Psi}\; 0 \gtrless 0$ has the form

$$
\begin{array}{rcl}
& & \psi' kv + \psi' t \gtrless \psi' t' \\
& & \quad ⑦ \Big\downarrow {}^{o}_{+} \\
e_1 & = & \sum_j \tilde{r}'_j + \sum_i \hat{r}'_i + \psi' t \gtrless \psi' t' + \sum_i \hat{r}_i + \sum_j \tilde{r}_j \\
& & \quad ⑧ \Big\downarrow {}_{*} \\
& & 0 \gtrless 0
\end{array}
$$

where the rewrite steps of ⑦ use the sequence of rules $\hat{e}_i$ from E_D^Ψ and $\tilde{e}_j$ from R_D^Ψ, and the rewrite steps of ⑧ use rules from R_D^Ψ with maximal term smaller than v.

Let $E_D = \{mv + s \gtrless s'\}$. Then there exists a $\psi \in \mathbf{N}^{>0}$ and for every i an $(R_D^\Psi \cup E_D)$-derivation

$$
\begin{array}{rcl}
\psi \hat{e}_i & = & \psi \hat{m}_i v + \psi \hat{r}_i \gtrless \psi \hat{r}'_i \\
& & \quad ⑨ \Big\downarrow {}^{o}_{*} \\
\hat{e}'_i & = & \sum_l \bar{r}'_{il} + \hat{\chi}_i s' + \psi \hat{r}_i \gtrless \psi \hat{r}'_i + \hat{\chi}_i s + \sum_l \bar{r}_{il} \\
& & \quad ⑩ \Big\downarrow {}_{*} \\
& & 0 \gtrless 0
\end{array}
$$

where ⑨ uses the rule $mv + s \gtrless s' \in E_D$ $\hat{\chi}_i$ times and then the sequence of rules $\bar{e}_{il} = \bar{m}_{il} v + \bar{r}_{il} \gtrless \bar{r}'_{il}$ from R_D^Ψ, hence $\psi \hat{m}_i = \hat{\chi}_i m + \sum_l \bar{m}_{il}$.

Let $e_2 = \psi e_1 + \sum_i \hat{e}'_i$. Then e_2 has a derivation to $0 \gtrless 0$. Cancellation of $\psi \sum_i (\hat{r}_i + \hat{r}'_i)$ in e_2 yields

$$
\begin{array}{rcl}
e_3 & = & \psi \sum_j \tilde{r}'_j + \sum_i \sum_l \bar{r}'_{il} + \sum_i \hat{\chi}_i s' + \psi \psi' t \\
& & \qquad \gtrless \psi \psi' t' + \sum_i \hat{\chi}_i s + \sum_i \sum_l \bar{r}_{il} + \psi \sum_j \tilde{r}_j \, .
\end{array}
$$

By partial confluence of $\rightarrow_{R_C^\Psi}$ we obtain $e_3 \rightarrow^*_{R_C^\Psi} 0 > 0$. Since $\mathrm{mt}(e_3) \prec v$, rules in $R_C^\Psi \setminus R_D^\Psi$ cannot be used in this derivation, hence $e_3 \rightarrow^*_{R_D^\Psi} 0 > 0$.

On the other hand, $\psi \psi' k = \sum_i \psi \hat{m}_i + \sum_j \psi \tilde{m}_j = \sum_i \hat{\chi}_i m + \sum_i \sum_l \bar{m}_{il} + \sum_j \psi \tilde{m}_j$. Hence we can rewrite $\psi \psi' e$ to e_3 by $(\sum_i \hat{\chi}_i)$-fold application of $mv + s \gtrless s'$, ψ-fold application of every $\tilde{e}_j$, and application of every $\bar{e}_{il}$. As e is $\delta\kappa$-irreducible with respect to $R_D^\Psi \subseteq R_C^\Psi$, e is contained in $E_D^\Psi \subseteq R_C^\Psi$ by Def. 2.9. □

LEMMA 2.13 *Let C be a clause in $\overline{N}$. If an inequation $e \in \mathrm{tr}^\circ_\Psi(R_C^\Psi \cup E_C)$ is $\delta\kappa$-irreducible with respect to $R_C^\Psi \cup E_C$, and $\rightarrow_{R_C^\Psi}$ is partially confluent on $\mathrm{tr}(R_C^\Psi) \cap \{\, e' \mid \mathrm{mt}(e) \succ \mathrm{mt}(e') \,\}$, then $e \in R_C^\Psi \cup E_C^\Psi$.*

PROOF. If e is contained in $\mathrm{tr}^{\circ}_{\Psi}(R_C^{\Psi})$, then $e \in R_C^{\Psi}$ by Lemma 2.12. Otherwise, let $E_C = \{nv + s \sim s'\}$ and $e = ku + t \gtrless t'$, such that $u = \mathrm{mt}(e)$. By definition of $\mathrm{tr}^{\circ}_{\Psi}(R_C^{\Psi} \cup E_C)$, there is a derivation

$$\psi ku + \psi t \gtrless \psi t' \ \to^{\circ}_{R_C^{\Psi} \cup E_C}\ 0 \gtrless 0$$

for some $\psi \in \mathbf{N}^{>0}$. During this derivation all occurrences of u are deleted eventually. If u were larger than v, this would be impossible, as u is δ-irreducible with respect to $R_C^{\Psi} \cup E_C$. If u were smaller than v, then $nv + s \sim s'$ could not be used during this derivation, hence e would be contained in $\mathrm{tr}^{\circ}_{\Psi}(R_C^{\Psi})$. Thus $u = v$, and by Def. 2.9, $e \in E_C^{\Psi}$. □

When we have two primary rules $nu + t < t'$ and $mu + s > s'$ derived from two clauses in $\overline{N}$, then the conclusion of cancellative chaining of these two clauses contains the literal $ns + mt' > ns' + mt$. In this literal the maximal term u is eliminated completely. In the proof that $\mathrm{tr}^{\circ}(R_{\infty}^{\Psi})$ is a model, however, we have to deal with secondary rules, and moreover we have to deal with partial overlaps, that is, overlaps where some occurrences of u remain. The following lemma shows how a secondary rule with maximal term u can be represented by means of primary rules with maximal term u.

LEMMA 2.14 *Let C be a clause in $\overline{N}$, let $\to_{R_C^{\Psi}}$ be partially confluent on $\mathrm{tr}(R_C^{\Psi}) \cap \{\, e' \mid \mathrm{mt}(C) \succ \mathrm{mt}(e') \,\}$. Let $C \succeq_{\mathrm{C}} D$, such that $E_D = \{mu + s \gtrless s'\}$ and $ku + r \gtrless r' \in E_D^{\Psi}$. Then there exist rules $m_i u + s_i \gtrless s'_i \in R_D \cup E_D$ and positive integers ψ, χ_i $(1 \le i \le n)$ such that $\psi k = \sum_i \chi_i m_i$, and*

$$e_0 \ = \ \psi r + \textstyle\sum_i \chi_i s'_i \gtrless \psi r' + \sum_i \chi_i s_i \ \to^{*}_{R_C^{\Psi}}\ 0 \gtrless 0\,.$$

PROOF. By definition of E_D^{Ψ} there exists a $\hat{\psi}$ such that $\hat{\psi} ku + \hat{\psi} r \gtrless \hat{\psi} r' \in \mathrm{tr}^{\circ}(R_D^{\Psi} \cup E_D)$. Without loss of generality, we may assume that the derivation has the form

$$\begin{array}{rcl} & & \hat{\psi} ku + \hat{\psi} r \gtrless \hat{\psi} r' \\ & & \text{①}\Big\downarrow{}^{o}_{*} \\ \hat{e} & = & \hat{\chi} s' + \sum_j \hat{\chi}_j r'_j + \hat{\psi} r \gtrless \hat{\psi} r' + \sum_j \hat{\chi}_j r_j + \hat{\chi} s \\ & & \text{②}\Big\downarrow{}_{*} \\ & & 0 \gtrless 0 \end{array}$$

where ① uses χ-fold application of $mu + s \gtrless s'$ $(\chi \ge 0)$ and $\hat{\chi}_j$-fold application of rules $k_j u + r_j \gtrless r'_j \in R_D^{\Psi}$ $(1 \le j \le j_0)$, and $\hat{\psi} k = \hat{\chi} m + \sum_j \hat{\chi}_j k_j$.

By induction, for every $k_j u + r_j \gtrless r'_j \in R_D^\Psi$ there exist positive integers $\tilde{\psi}_j, \tilde{\chi}_{jl}$ $(1 \le j \le l_j)$ such that $\tilde{m}_{jl} u + \tilde{s}_{jl} \gtrless \tilde{s}'_{jl} \in R_D$, $\tilde{\psi}_j k_j = \sum_l \tilde{\chi}_{jl} \tilde{m}_{jl}$, and

$$\tilde{e}_j \;=\; \tilde{\psi}_j r_j + \textstyle\sum_l \tilde{\chi}_{jl} \tilde{s}'_{jl} \gtrless \tilde{\psi}_j r'_j + \sum_l \tilde{\chi}_{jl} \tilde{s}_{jl} \;\to^*_{R_C^\Psi}\; 0 \gtrless 0\,.$$

Let $\check{\psi} = \prod_j \tilde{\psi}_j$. Then $e'_0 = \check{\psi}\hat{e} + \sum_j \check{\psi}\hat{\chi}_j \tilde{\psi}_j^{-1} \tilde{e}_j$ has a derivation to $0 \gtrless 0$, cancellation of $\sum_j \hat{\chi}_j \check{\psi}(r_j + r'_j)$ in e'_0 yields e_0, and the result follows from partial confluence of $\to_{R_C^\Psi}$. □

This lemma allows us to prove the following crucial fact: If the results of the complete chainings of primary rules with maximal term u are in $\mathrm{tr}^\circ(R_C^\Psi)$, and if moreover sufficiently many (small) peaks can be joined, then the result of the partial overlap of secondary rules $\bar{m}u + \bar{s} > \bar{s}'$ and $\bar{n}u + \bar{t} < \bar{t}'$ is itself a secondary rule:

Lemma 2.15 *Let $E_C = \{m_1 u + s_1 \gtrless s'_1\}$. Suppose that for every pair of rules $mu + s > s'$ and $nu + t < t'$ from $R_C \cup E_C$ the inequation $ns + mt' > ns' + mt$ is contained in $\mathrm{tr}^\circ(R_C^\Psi)$. Let $\to_{R_C^\Psi}$ be partially confluent on $\mathrm{tr}(R_C^\Psi) \cap \{\, e' \mid u \succ \mathrm{mt}(e') \,\}$. For $I \cup J$ finite, $i \in I$, $j \in J$, let $e_i = \bar{m}_i u + \bar{s}_i > \bar{s}'_i$ and $\hat{e}_j = \bar{t}'_j > \bar{t}_j + \bar{n}_j u$ be inequations in $R_C^\Psi \cup E_C^\Psi$. Let e be the result of κ-normalizing $\sum_i e_i + \sum_i \hat{e}_i$. Then e is contained in $E_C^\Psi \cup R_C^\Psi$.*

Proof. Let $m^* = \sum_i \bar{m}_i$ and $n^* = \sum_j \bar{n}_j$. Without loss of generality we assume that $\beta = m^* - n^* \ge 0$. Then the κ-normalization of $\sum_i e_i + \sum_j \hat{e}_j$ has the form

$$\begin{array}{c}
\sum_i e_i + \sum_j \hat{e}_j \\
{\small ①}\Big\downarrow{}^{\kappa}_{*} \\
\beta u + \sum_i \bar{s}_i + \sum_j \bar{t}'_j > \sum_i \bar{s}'_i + \sum_j \bar{t}_j \\
{\small ②}\Big\downarrow{}^{\kappa}_{*} \\
e \;=\; \beta u + q > q'
\end{array}$$

By Lemma 2.14, for every i and j there exist rules $m_{ik} u + s_{ik} > s'_{ik}$ and $n_{jl} u + t_{jl} < t'_{jl} \in R_C \cup E_C$, with $\psi \bar{m}_i = \sum_k \mu_{ik} m_{ik}$, $\psi \bar{n}_j = \sum_l \nu_{jl} n_{jl}$,[6] such that there are R_C^Ψ-derivations

$$\begin{array}{c}
e_i^1 \;=\; \psi \bar{s}_i + \sum_k \mu_{ik} s'_{ik} > \psi \bar{s}'_i + \sum_k \mu_{ik} s_{ik} \\
{\small ③}\Big\downarrow{}_{*} \\
0 > 0
\end{array}$$

[6] We assume that ψ is independent of i and j; this is possible since we may take the least common multiple of all values of ψ obtained from Lemma 2.14 for the individual rules.

$$\hat{e}_j^1 \; = \; \psi \bar{t}_j' + \sum_l \nu_{jl} t_{jl} > \psi \bar{t}_j + \sum_l \nu_{jl} t_{jl}'$$

$$\textcircled{4} \Big\downarrow{}^{*}$$

$$0 > 0$$

Furthermore, by assumption, for all i, j, k, and l, there is an R_C^Ψ-derivation

$$e_{ijkl}^2 \; = \; n_{jl} s_{ik} + m_{ik} t_{jl}' > n_{jl} s_{ik}' + m_{ik} t_{jl}$$

$$\textcircled{5} \Big\downarrow{}^{\circ}$$

$$0 > 0$$

Define the inequation e^3 by

$$\begin{aligned}
e^3 \; &= \; \textstyle\sum_{ijkl} \mu_{ik} \nu_{jl} e_{ijkl}^2 + \sum_i \psi m^* e_i^1 + \sum_j \psi m^* \hat{e}_j^1 \\
&= \; \textstyle\sum_{ijkl} \mu_{ik} \nu_{jl} n_{jl} s_{ik} + \sum_{ijkl} \mu_{ik} \nu_{jl} m_{ik} t_{jl}' \\
&\qquad\qquad \textstyle + \sum_i \psi^2 m^* \bar{s}_i + \sum_{ik} \psi m^* \mu_{ik} s_{ik}' \\
&\qquad\qquad \textstyle + \sum_j \psi^2 m^* \bar{t}_j' + \sum_{jl} \psi m^* \nu_{jl} t_{jl} \\
&\qquad \textstyle > \sum_{ijkl} \mu_{ik} \nu_{jl} n_{jl} s_{ik}' + \sum_{ijkl} \mu_{ik} \nu_{jl} m_{ik} t_{jl} \\
&\qquad\qquad \textstyle + \sum_i \psi^2 m^* \bar{s}_i' + \sum_{ik} \psi m^* \mu_{ik} s_{ik} \\
&\qquad\qquad \textstyle + \sum_j \psi^2 m^* \bar{t}_j + \sum_{jl} \psi m^* \nu_{jl} t_{jl}' \\
&= \; \textstyle\sum_{ik} \psi n^* \mu_{ik} s_{ik} + \sum_{jl} \psi m^* \nu_{jl} t_{jl}' \\
&\qquad\qquad \textstyle + \sum_i \psi^2 m^* \bar{s}_i + \sum_{ik} \psi m^* \mu_{ik} s_{ik}' \\
&\qquad\qquad \textstyle + \sum_j \psi^2 m^* \bar{t}_j' + \sum_{jl} \psi m^* \nu_{jl} t_{jl} \\
&\qquad \textstyle > \sum_{ik} \psi n^* \mu_{ik} s_{ik}' + \sum_{jl} \psi m^* \nu_{jl} t_{jl} \\
&\qquad\qquad \textstyle + \sum_i \psi^2 m^* \bar{s}_i' + \sum_{ik} \psi m^* \mu_{ik} s_{ik} \\
&\qquad\qquad \textstyle + \sum_j \psi^2 m^* \bar{t}_j + \sum_{jl} \psi m^* \nu_{jl} t_{jl}'
\end{aligned}$$

By construction, e^3 has an R_C^Ψ-derivation to $0 > 0$ using a combination of all derivations ③, ④, and ⑤. On the other hand, we can cancel $\sum_{ik} \psi n^* \mu_{ik}(s_{ik} + s_{ik}') + \sum_{jl} \psi m^* \nu_{jl}(t_{jl} + t_{jl}')$ in e^3 and then continue as in ② and obtain

$$e^4 \; = \; \psi^2 m^* q + \textstyle\sum_{ik} \psi \beta \mu_{ik} s_{ik}' > \psi^2 m^* q' + \sum_{ik} \psi \beta \mu_{ik} s_{ik}$$

By partial confluence of $\to_{R_C^\Psi}$, $e^4 \; \to_{R_C^\Psi}^* \; 0 > 0$.

Let $m_{ik} u + \tilde{s}_{ik} > \tilde{s}_{ik}'$ be either $m_{ik} u + s_{ik} > s_{ik}'$ (if the latter equation is contained in E_C), or the equation in R_C^Ψ obtained from $m_{ik} u + s_{ik} > s_{ik}'$ by $\delta\kappa$-normalization as in Lemma 2.10 (if the latter equation is contained in $E_D \subseteq R_C$). Then e^4 rewrites using $\delta\kappa$-steps to

$$e^5 \; = \; \psi^2 m^* q + \textstyle\sum_{ik} \psi \beta \mu_{ik} \tilde{s}_{ik}' > \psi^2 m^* q' + \sum_{ik} \psi \beta \mu_{ik} \tilde{s}_{ik}$$

and $e^5 \to^*_{R_C^\Psi} 0 > 0$ by partial confluence of $\to_{R_C^\Psi}$.

On the other hand, $\psi^2 m^* \beta u = \psi\beta \sum_{ik} \mu_{ik} m_{ik} u$, hence $\psi^2 m^* e$ rewrites to e^5 by $\psi\beta\mu_{ik}$-fold o-application of every $m_{ik}u + \tilde{s}_{ik} > \tilde{s}'_{ik}$. Now there are two possibilities: Either $\beta > 0$, then there is at least one o-step in the derivation from $\psi^2 m^* e$ to e^5. Or $\beta = 0$, then both I and J must be non-empty and $e^3 \to^\circ_{R_C^\Psi} 0 > 0$ because of ⑤. Consequently $e^4 \to^\circ_{R_C^\Psi} 0 > 0$ and $e^5 \to^\circ_{R_C^\Psi} 0 > 0$. In both cases, $\psi^2 m^* e \to^\circ_{R_C^\Psi \cup E_C} 0 > 0$. Therefore, $e \in R_C^\Psi \cup E_C^\Psi$ by Lemma 2.13. □

In the model construction, equations $mu + s \approx s'$ as primary rules can produce (in-)equations $n_0 u + t_0 \gtrsim t'_0$ and $n_1 u + t_1 \lesssim t'_1$ as secondary rules. If sufficiently many (small) peaks can be joined, then the result of the partial overlap of such secondary rules is likewise $0 \gtrsim 0$ or a secondary rule:

Lemma 2.16 *Let $E_C = \{mu + s \approx s'\}$, let $\to_{R_C^\Psi}$ be partially confluent on $\mathrm{tr}(R_C^\Psi) \cap \{\, e' \mid u \succ \mathrm{mt}(e') \,\}$. For $I \cup J$ finite, $i \in I$, $j \in J$, let $e_i = n_i u + t_i \gtrsim t'_i$ and $\hat{e}_j = \hat{t}'_j \gtrsim \hat{t}_j + \hat{n}_j u$ be (in-)equations in E_C^Ψ. Let e be the result of κ-normalizing $\sum_i e_i + \sum_j \hat{e}_j$. Then e is contained in $E_C^\Psi \cup R_C^\Psi \cup \{0 \approx 0\}$.*

Proof. Without loss of generality we assume that $\sum_i n_i \geq \sum_j \hat{n}_j$. Then the κ-normalization of $\sum_i e_i + \sum_j \hat{e}_j$ has the form

$$\begin{array}{c}
\sum_i e_i + \sum_j \hat{e}_j \\
\text{①}\Big\downarrow{}^{\kappa}_{*} \\
(\sum_i n_i - \sum_j \hat{n}_j)u + \sum_i t_i + \sum_j \hat{t}'_j \gtrsim \sum_i t'_i + \sum_j \hat{t}_j \\
\text{②}\Big\downarrow{}^{\kappa}_{*} \\
e \;=\; (\sum_i n_i - \sum_j \hat{n}_j)u + q \gtrsim q'
\end{array}$$

Furthermore there exists a $\psi \in \mathbf{N}^{>0}$ and for every $i \in I$ and $j \in J$ derivations

$$\begin{array}{ccc}
\psi n_i u + \psi t_i \gtrsim \psi t'_i & \qquad & \psi \hat{t}'_j \gtrsim \psi \hat{t}_j + \psi \hat{n}_j u \\
\text{③}\Big\downarrow{}^{\gamma}_{+} & & {}^{\gamma}_{+}\Big\downarrow\text{⑤} \\
e'_i \;=\; \chi_i s' + \psi t_i \gtrsim \psi t'_i + \chi_i s & & \hat{e}'_j \;=\; \hat{\chi}_j s + \psi \hat{t}'_j \gtrsim \psi \hat{t}_j + \hat{\chi}_j s' \\
\text{④}\Big\downarrow{}_{*} & & {}_{*}\Big\downarrow\text{⑥} \\
0 \gtrsim 0 & & 0 \gtrsim 0
\end{array}$$

where $\psi n_i = \chi_i m$ and $\psi \hat{n}_j = \hat{\chi}_j m$. Let

$$\begin{aligned} e'' &= \textstyle\sum_i e'_i + \sum_j \hat{e}'_j \\ &= \textstyle\sum_i \chi_i s' + \sum_i \psi t_i + \sum_j \hat{\chi}_j s + \sum_j \psi \hat{t}'_j \\ &\qquad \gtrsim \textstyle\sum_i \psi t'_i + \sum_i \chi_i s + \sum_j \psi \hat{t}_j + \sum_j \hat{\chi}_j s' \end{aligned}$$

Obviously, $e'' \to^*_{R_C^\Psi} 0 \gtrsim 0$ using a combination of all derivations ④ and ⑥. On the other hand, κ-steps as in ② lead from e'' to

$$e''' = (\textstyle\sum_i \chi_i - \sum_j \hat{\chi}_j)s' + \psi q \gtrsim (\sum_i \chi_i - \sum_j \hat{\chi}_j)s + \psi q'$$

By partial confluence of $\to_{R_C^\Psi}$, we obtain $e''' \to^*_{R_C^\Psi} 0 \gtrsim 0$. On the other hand, $(\sum_i \chi_i - \sum_j \hat{\chi}_j)m = \psi(\sum_i n_i - \sum_j \hat{n}_j)$, thus ψe rewrites to e''' by $(\sum_i \chi_i - \sum_j \hat{\chi}_j)$-fold γ-application of $mu + s \approx s'$. Furthermore, if at least one of the e_i or $\hat{e}_j$ is an inequation, then one of the derivations ④ or ⑥ must contain an o-step, hence $e \to^\circ_{R_C^\Psi \cup E_C} 0 > 0$. Hence $e \in E_C^\Psi \cup R_C^\Psi \cup \{0 \approx 0\}$ by Lemma 2.13 or by the corresponding Lemma for the equational case (Waldmann [8]). □

There is one important technical difference between the equational case developed in (Waldmann [8]) and the inequational case that we consider here: In the equational case, one can show that $\to_{R_\infty^\Psi}$ is confluent on $\mathrm{tr}(R_\infty^\Psi)$, and hence that $\mathrm{tr}(R_\infty^\Psi)$ is a model of the theory axioms, without requiring that the set N of clauses is saturated. Saturation is only necessary to prove that $\mathrm{tr}(R_\infty^\Psi)$ is also a model of $\overline{N}$. In the inequational case, such a separation does not work: Proving partial confluence of $\to_{R_\infty^\Psi}$ requires Lemma 2.15, and Lemma 2.15 requires that *cancellative chaining* inferences are redundant. For this reason, the proof that $\to_{R_\infty^\Psi}$ is partially confluent and the proof that $\mathrm{tr}^\circ(R_\infty^\Psi)$ is a model of $\overline{N}$ must be combined within a single induction.

Lemma 2.17 and Corollary 2.18 are copied almost verbatim from (Waldmann [8]).

LEMMA 2.17 *The relation $\to_{R_C^\Psi}$ is partially confluent on the equations in* $\mathrm{tr}^\circ(R_C^\Psi)$ *for every $C \in \overline{N}$. The relation $\to_{R_\infty^\Psi}$ is partially confluent on the equations in* $\mathrm{tr}^\circ(R_\infty^\Psi)$.[7]

COROLLARY 2.18 *For every $C \in \overline{N}$,* $\mathrm{tr}^\circ(R_C^\Psi)$ *and* $\mathrm{tr}^\circ(R_\infty^\Psi)$ *satisfy* ACUKT *and the equality axioms (except the congruence axiom for the predicate $>$).*

In a similar way as Lemma 2.17, we obtain by a rather tedious case analysis over various kinds of critical pairs:

[7]Note that confluence and partial confluence differ only for inequations.

Lemma 2.19 *If for every pair of rules $mu + s > s'$ and $nu + t < t'$ from $E_C \cup R_C$ the inequation $ns + mt' > ns' + mt$ is contained in $\mathrm{tr}^\circ(R_C^\Psi)$, then $\to_{R_C^\Psi \cup E_C^\Psi}$ is partially confluent on $\mathrm{tr}(R_C^\Psi \cup E_C^\Psi)$.*

Proof. Traditionally the confluence of a noetherian relation is established in two steps. First, one proves by induction that the confluence of a noetherian relation follows from local confluence. Second, one shows that local confluence is implied by the convergence of certain critical pairs. In our case, the induction hypothesis is not only needed to show that local confluence implies confluence, but even to prove local confluence. Consequently, we have to embed the analysis of the critical pairs within the inductive confluence proof.

To show that $\to_{R_C^\Psi \cup E_C^\Psi}$ is partially confluent on $\mathrm{tr}(R_C^\Psi \cup E_C^\Psi)$, it suffices to show that it is partially confluent on $\mathrm{tr}(R_C^\Psi \cup E_C^\Psi) \cap \{\, e \mid e_0 \succeq_{\mathrm{L}} e \,\}$ for every $e_0 \in \mathrm{tr}(R_C^\Psi \cup E_C^\Psi)$. We will do this by induction on the size of e_0 with respect to $\succ_{\mathrm{L}}$. We have to show that for any peak

$$\begin{tikzcd}[column sep=small]
 & e \arrow[dl, "R_C^\Psi \cup E_C^\Psi"'] \arrow[dr, "R_C^\Psi \cup E_C^\Psi", "\gamma\delta\kappa"'] & \\
e_1 \arrow[d, "R_C^\Psi \cup E_C^\Psi"', "*"] & & e_2 \\
0 > 0 & &
\end{tikzcd}$$

such that $e_0 \succeq_{\mathrm{L}} e$ there exists a derivation $e_2 \to^*_{R_C^\Psi \cup E_C^\Psi} 0 > 0$, which uses at least one o-step if the derivation from e to $0 > 0$ via e_1 uses any o-steps.
For $e_0 \succ_{\mathrm{L}} e$, this follows directly from the induction hypothesis, so we assume $e_0 = e$.

Case 1: Trivial peaks.

If $e_1 \leftarrow e \to_{\gamma\delta\kappa} e_2$ and both rewrite steps take place at disjoint redexes, then there is obviously an inequation e_3 such that $e_1 \to_{\gamma\delta\kappa} e_3 \leftarrow e_2$, the induction hypothesis can be applied to e_1, and hence there is a derivation $e_2 \to e_3 \to^* 0 > 0$. Note in particular that δ-steps cannot take place at the same redex as a γ- or o-step, hence γ/δ-peaks, δ/γ-peaks, and o/δ-peaks are necessarily trivial.

If $e_1 \leftarrow_\delta e \to_\kappa e_2$ and the redexes are not disjoint, then e_1 can be rewritten to e_2 by duplicating the original δ-step on the other side of the inequation, followed by a κ-step, hence $e_1 \to_\delta e_3 \to_\kappa e_2$ and the derivation from e_2 to $0 > 0$ is obtained by applying the induction hypothesis first to e_1 and then to e_3. In a similar way, κ/κ- and κ/δ-peaks can be handled.

It is easy to check that in all cases the derivation from e_2 to $0 > 0$ uses an o-step whenever the derivation from e via e_1 to $0 > 0$ uses an o-step.

Case 2: γ/κ-peaks, o/κ-peaks.

Suppose that $e_1 \leftarrow_{\gamma o} e \rightarrow_\kappa e_2$. We assume without loss of generality that the γ- or o-step takes place at the greater side of e (with respect to $>$), the other case is proved analogously. Then the peak has the form

$$\begin{tikzcd}
& kv + s > v + s' \arrow[dl, "\textcircled{1}"', "\gamma o"] \arrow[dr, "\textcircled{2}", "\kappa"'] & \\
r' + s > v + s' + r \arrow[d, "\textcircled{3}"', "*"] & & (k-1)v + s > s' \\
0 > 0 & &
\end{tikzcd}$$

where ① uses a rewrite rule $kv + r \gtrsim r' \in E^\Psi_D \subseteq R^\Psi_C \cup E^\Psi_C$. At some step of the derivation ③ the term v must be eventually deleted. As v is γo-reducible, it must be δ-irreducible, so this deletion can happen only by a κ-step or by a γ- or o-step.

Case 2.1: v is deleted by a κ-step.

The deletion of v by a κ-step requires the existence of another occurrence of v on the left-hand side. This occurrence can only be derived from s or s'. We may thus assume that the derivation has the form ④-⑤-⑥.

$$\begin{tikzcd}
& kv + s > v + s' \arrow[dl, "\textcircled{1}"', "\gamma o"] \arrow[dr, "\textcircled{2}", "\kappa"'] & \\
r' + s > v + s' + r \arrow[d, "\textcircled{4}"', "*"] & & (k-1)v + s > s' \arrow[d, "*"', "\textcircled{7}"] \\
r' + t + v > v + t' + r \arrow[d, "\textcircled{5}"', "\kappa"] & & (k-1)v + t + v > t' \arrow[dll, "\gamma o", "\textcircled{8}"'] \\
r' + t > t' + r \arrow[d, "\textcircled{6}"', "*"] & & \\
0 > 0 & &
\end{tikzcd}$$

As the steps ④ take place only at s and s', we can simulate them by ⑦. Finally, we can close the diagram using γ- or o-rewriting ⑧ by $kv + r \gtrsim r$. Note that the derivation ⑦-⑧-⑥ uses an o-step whenever the derivation ①-④-⑤-⑥ uses an o-step.

Case 2.2: v is deleted by a γ- or o-step.

Otherwise, the deletion of v during ③ happens by application of a rule $k_1 v + r_1 \lesssim r_1' \in R_C^\Psi \cup E_C^\Psi$. Such a step requires the presence of $k_1 - 1$ further occurrences of v on the right hand side. As r and r' are smaller than v, these occurrences can only be derived from s or s'. We may thus assume without loss of generality that the derivation has the form ⑨-⑩-⑪:

$$\begin{tikzcd}
 & & kv + s > v + s' \arrow[dll, "①" above, "\gamma o" below] \arrow[dr, "②" above, "\kappa" below] & \\
r' + s > v + s' + r \arrow[d, "⑨" left, "*"] & & & (k-1)v + s > s' \arrow[d, "⑫", "*" left] \\
r' + t > v + (k_1 - 1)v + t' + r \arrow[d, "⑩" left, "\gamma o"] & & & (k-1)v + t > (k_1 - 1)v + t' \\
r_1 + r' + t > t' + r_1' + r \arrow[d, "⑪" left, "*"] & & & \\
0 > 0 & & &
\end{tikzcd}$$

The steps ⑨ take place only at s and s', thus we can simulate them by ⑫.

Consider the two rules $kv + r \gtrsim r'$ and $k_1 v + r_1 \lesssim r_1' \in R_C^\Psi \cup E_C^\Psi$. We can add these two rules, obtaining $kv + r + r_1' \gtrsim k_1 v + r_1 + r'$. By Lemma 2.15 and 2.16, the result of κ-normalizing this (in-)equation is either $0 \approx 0$ or a rule from $R_C^\Psi \cup E_C^\Psi$.

Case 2.2.1: $k > k_1$.

If $k > k_1$, then κ-normalization yields an (in-)equation $(k - k_1)v + q \gtrsim q'$ in $R_C^\Psi \cup E_C^\Psi$:

$$\begin{tikzcd}
kv + r + r_1' \gtrsim k_1 v + r_1 + r' \arrow[d, "\kappa", "*" below] \\
(k - k_1)v + r + r_1' \gtrsim r_1 + r' \arrow[d, "⑬" left, "\kappa"] \\
(k - k_1)v + q \gtrsim q'
\end{tikzcd}$$

From ⑬ it is easy to construct a derivation from $q' + r + r_1' \gtrsim q + r_1 + r'$ using only cancellation steps:

$$\begin{tikzcd}
q' + r + r'_1 \gtrsim q + r_1 + r' \arrow[d, "\text{⑭}"', "\kappa\ *"] \\
0 \gtrsim 0
\end{tikzcd}$$

Let us add the starting (in-)equations of ⑪ and ⑭. Obviously there is a derivation from this inequation to $0 > 0$.

$$\begin{tikzcd}
r_1 + r' + t + q' + r + r'_1 > t' + r'_1 + r + q + r_1 + r' \arrow[d, "*"] \arrow[r, "\kappa", "*"'] & q' + t > t' + q \arrow[dl, "\text{⑮}", "*"'] \\
0 > 0 &
\end{tikzcd}$$

On the other hand, we can cancel $r + r' + r_1 + r'_1$. By the induction hypothesis, there is a derivation ⑮ from $q' + t > t' + q$ to $0 > 0$.

We can now close the diagram above: We use κ-steps ⑯ to cancel $k_1 - 1$ occurrences of v. Then we continue by γ- or o-application of $(k - k_1)v + q \gtrsim q'$ ⑰ and then append derivation ⑮.

$$\begin{tikzcd}
& kv + s > v + s' \arrow[dl, "\text{①}"', "\gamma o"] \arrow[dr, "\text{②}", "\kappa"'] & \\
r' + s > v + s' + r \arrow[d, "\text{⑨}"', "*"] & & (k-1)v + s > s' \arrow[d, "\text{⑫}", "*"'] \\
r' + t > v + (k_1 - 1)v + t' + r \arrow[d, "\text{⑩}"', "\gamma o"] & & (k-1)v + t > (k_1 - 1)v + t' \arrow[d, "\text{⑯}", "\kappa\ *"'] \\
r_1 + r' + t > t' + r'_1 + r \arrow[dd, "\text{⑪}"', "*"] & & (k - k_1)v + t > t' \arrow[d, "\text{⑰}", "\gamma o\ *"'] \\
& & q' + t > t' + q \arrow[dll, "\text{⑮}", "*"'] \\
0 > 0 & &
\end{tikzcd}$$

Routine checking shows that the derivation ⑫-⑯-⑰-⑮ uses an o-step whenever the derivation ①-⑨-⑩-⑪ uses an o-step.

Case 2.2.2: $k < k_1$.

This is essentially the mirror image of Case 2.2.1.

Case 2.2.3: $k = k_1$.

If $k = k_1$, then κ-normalization yields an (in-)equation $q \gtrsim q'$ that is either $0 \approx 0$ or a rule from R_C^Ψ. In any case, $q \gtrsim q'$ has itself a derivation to $0 \gtrsim 0$ (containing at least one o-step if $q \gtrsim q'$ is an inequation).

$$\begin{tikzcd}
kv + r + r_1' \gtrsim k_1 v + r_1 + r' \arrow[d, "\kappa *"] \\
r + r_1' \gtrsim r_1 + r' \arrow[d, "\kappa *", "18"'] \\
q \gtrsim q' \arrow[d, "*", "19"'] \\
0 \gtrsim 0
\end{tikzcd}$$

Let us add the starting (in-)equations of ⑪ and ⑱. Obviously there is a derivation from this inequation to $0 > 0$.

$$\begin{tikzcd}
r_1 + r' + t + r + r_1' > t' + r_1' + r + r_1 + r' \arrow[dd, "*"] \arrow[dr, "\kappa *"] & \\
& t > t' \arrow[dl, "20 *"] \\
0 > 0 &
\end{tikzcd}$$

On the other hand, we can cancel $r + r' + r_1 + r_1'$. By the induction hypothesis, there is a derivation ⑳ from $t > t'$ to $0 > 0$.

The diagram above can now be closed by using κ-steps ㉑ to cancel $k - 1$ occurrences of v, followed by derivation ⑳.

$$
\begin{tikzcd}
 & kv + s > v + s' \arrow[dl, "\text{①}"', "\gamma o"] \arrow[dr, "\text{②}", "\kappa"'] & \\
r' + s > v + s' + r \arrow[d, "\text{⑨}"', "*"] & & (k-1)v + s > s' \arrow[d, "\text{⑫}", "*"'] \\
r' + t > v + (k_1 - 1)v + t' + r \arrow[d, "\text{⑩}"', "\gamma o"] & & (k-1)v + t > (k_1 - 1)v + t' \arrow[d, "\text{㉑}", "\kappa\ *"'] \\
r_1 + r' + t > t' + r'_1 + r \arrow[d, "\text{⑪}"', "*"] & & t > t' \arrow[dll, "*"', "\text{⑳}"] \\
0 > 0 & &
\end{tikzcd}
$$

Routine checking shows that the derivation ⑫-㉑-⑳ uses an o-step whenever the derivation ①-⑨-⑩-⑪ uses an o-step.

Case 3: o/γ-peaks.

Suppose that $e_1 \leftarrow_o e \rightarrow_\gamma e_2$ at overlapping redexes. Without loss of generality both the o- and the γ-step take place at the greater side of e (with respect to $>$), the other case is proved analogously. Then we may assume that the γ-step uses a rule $k_0 v + r_0 \approx r'_0$ and the o-step uses a rule $k_1 v + r_1 > r'_1$.

Case 3.1: $k_1 > k_0$.

If $k_1 > k_0$, then the peak has the form

$$
\begin{tikzcd}
 & k_1 v + s > s' \arrow[dl, "\text{①}"', "o"] \arrow[dr, "\text{②}", "\gamma"'] & \\
r'_1 + s > s' + r_1 \arrow[d, "\text{③}"', "*"] & & (k_1 - k_0)v + r'_0 + s > s' + r_0 \\
0 > 0 & &
\end{tikzcd}
$$

We can add the two rules $k_1 v + r_1 > r'_1$ and $r'_0 \approx k_0 v + r_0$, obtaining $k_1 v + r_1 + r'_0 > k_0 v + r_0 + r'_1$. By Lemma 2.16, the result of κ-normalizing this inequation must be a rule $(k_1 - k_0)v + q > q'$ from $R^\Psi_C \cup E^\Psi_C$.

$$
\begin{tikzcd}
k_1 v + r_1 + r_0' > k_0 v + r_0 + r_1' \arrow[d, "\kappa"', "*"] \\
(k_1 - k_0)v + r_1 + r_0' > r_0 + r_1' \arrow[d, "④", "\kappa"', "*"] \\
(k_1 - k_0)v + q > q'
\end{tikzcd}
$$

From ④ it is easy to construct a derivation from $q' + r_1 + r_0' > q + r_0 + r_1'$ using only cancellation steps:

$$
\begin{tikzcd}
q' + r_1 + r_0' > q + r_0 + r_1' \arrow[d, "⑤", "\kappa"', "*"] \\
0 > 0
\end{tikzcd}
$$

Let us add the starting inequations of ③ and ⑤. Obviously there is a derivation from this inequation to $0 > 0$.

$$
\begin{tikzcd}
r_1' + s + q' + r_1 + r_0' > s' + r_1 + q + r_0 + r_1' \arrow[dd, "*"] \arrow[dr, "\kappa"', "*"] & \\
& s + q' + r_0' > s' + q + r_0 \arrow[dl, "⑥", "*"'] \\
0 > 0 &
\end{tikzcd}
$$

On the other hand, we can cancel $r_1 + r_1'$. By the induction hypothesis, there is a derivation ⑥ from $s + q' + r_0' > s' + q + r_0$ to $0 > 0$.

The diagram above can now be closed by an o-step using $(k_1 - k_0)v + q > q'$ ⑦ followed by derivation ⑥.

$$
\begin{tikzcd}
& k_1 v + s > s' \arrow[dl, "①"', "o"] \arrow[dr, "②", "\gamma"'] & \\
r_1' + s > s' + r_1 \arrow[dd, "③"', "*"] & & (k_1 - k_0)v + r_0' + s > s' + r_0 \arrow[d, "o"', "⑦"] \\
& & s + q' + r_0' > s' + q + r_0 \arrow[dll, "⑥", "*"'] \\
0 > 0 & &
\end{tikzcd}
$$

Note that the derivation ⑦-⑥ contains at least one o-step.

Case 3.2: $k_1 \leq k_0$.

If $k_1 \leq k_0$, then the peak has the form

$$
\begin{tikzcd}
 & k_0 v + s > s' \arrow[dl, "o", "①"'] \arrow[dr, "\gamma"', "②"] & \\
(k_0 - k_1)v + r_1' + s > s' + r_1 \arrow[d, "*", "③"'] & & r_0' + s > s' + r_0 \\
0 > 0 & &
\end{tikzcd}
$$

During the derivation ③, the $k_0 - k_1$ occurrences of v are deleted completely. As v is δ-irreducible, this can happen only by κ-, γ-, or o-steps. We may assume that $k_2 \geq 0$ and $k_3 \geq 0$ further occurrences of v are generated on the left-hand and right-hand side, respectively, such that k_3 occurrences of v are eliminated by κ-steps and the remaining $k_0 - k_1 + k_2 - k_3$ ones are eliminated by γ- or o-steps. Without loss of generality the derivation has the form

$$
\begin{tikzcd}
 & k_0 v + s > s' \arrow[dl, "o", "①"'] \arrow[dr, "\gamma"', "②"] & \\
(k_0 - k_1)v + r_1' + s > s' + r_1 \arrow[d, "*", "④"'] & & r_0' + s > s' + r_0 \arrow[d, "*"', "⑧"] \\
(k_0 - k_1 + k_2)v + r_1' + t > t' + r_1 + k_3 v \arrow[d, "\gamma o *", "⑤"'] & & k_2 v + r_0' + t > t' + r_0 + k_3 v \\
r_2' + k_3 v + r_1' + t > t' + r_1 + k_3 v + r_2 \arrow[d, "\kappa *", "⑥"'] & & \\
r_2' + r_1' + t > t' + r_1 + r_2 \arrow[d, "*", "⑦"'] & & \\
0 > 0 & &
\end{tikzcd}
$$

where the steps ④ take place only at s and s' and ⑤ uses rules $e_i' \in R_C^{\Psi} \cup E_C^{\Psi}$ such that $\sum_i e_i' = (k_0 - k_1 + k_2 - k_3)v + r_2 \gtrsim r_2'$. As the steps ④ take place only at s and s' we can simulate them by ⑧.

Case 3.2.1: $k_2 > k_3$.

We can add the rules $k_1 v + r_1 > r'_1$ and $r'_0 \approx k_0 v + r_0$ and all rules e'_i obtaining $(k_0 + k_2 - k_3)v + r_1 + r_2 + r'_0 > k_0 v + r'_1 + r'_2 + r_0$. By Lemma 2.16, the result of κ-normalizing this inequation must be a rule $(k_2 - k_3)v + q > q'$ from $R_C^\Psi \cup E_C^\Psi$.

$$\begin{tikzcd}
(k_0 + k_2 - k_3)v + r_1 + r_2 + r'_0 > k_0 v + r'_1 + r'_2 + r_0 \arrow[d, "\kappa"', "*"] \\
(k_2 - k_3)v + r_1 + r_2 + r'_0 > r'_1 + r'_2 + r_0 \arrow[d, "\kappa", "*"', "⑨"] \\
(k_2 - k_3)v + q > q'
\end{tikzcd}$$

From ⑨ it is easy to construct a derivation from $q' + r_1 + r_2 + r'_0 > q + r'_1 + r'_2 + r_0$ using only cancellation steps:

$$\begin{tikzcd}
q' + r_1 + r_2 + r'_0 > q + r'_1 + r'_2 + r_0 \arrow[d, "\kappa", "*"', "⑩"] \\
0 > 0
\end{tikzcd}$$

Let us add the starting inequations of ⑦ and ⑩. Obviously there is a derivation from this inequation to $0 > 0$.

$$\begin{tikzcd}
r'_2 + r'_1 + t + q' + r_1 + r_2 + r'_0 > t' + r_1 + r_2 + q + r'_1 + r'_2 + r_0 \arrow[d, "*"] \arrow[dr, "\kappa", "*"'] & \\
& q' + r'_0 + t > t' + r_0 + q \arrow[dl, "⑪", "*"'] \\
0 > 0 &
\end{tikzcd}$$

On the other hand, we can cancel $r_1 + r_2 + r'_1 + r'_2$. By the induction hypothesis, there is a derivation ⑪ from $t + q' + r'_0 > t' + q + r_0$ to $0 > 0$.

The diagram above can now be closed by cancellation of $k_3 v$ ⑫, followed by an o-step using $(k_2 - k_3)v + q > q'$ ⑬ followed by derivation ⑪.

$$
\begin{tikzcd}[column sep=small]
 & k_0v+s>s' \arrow[dl, "1"', "o"] \arrow[dr, "2", "\gamma"'] & \\
(k_0-k_1)v+r_1'+s>s'+r_1 \arrow[d, "4"', "*"] & & r_0'+s>s'+r_0 \arrow[d, "8", "*"'] \\
(k_0-k_1+k_2)v+r_1'+t>t'+r_1+k_3v \arrow[d, "5"', "\gamma o\, *"] & & k_2v+r_0'+t>t'+r_0+k_3v \arrow[d, "12", "\kappa\, *"'] \\
r_2'+k_3v+r_1'+t>t'+r_1+k_3v+r_2 \arrow[d, "6"', "\kappa\, *"] & & (k_2-k_3)v+r_0'+t>t'+r_0 \arrow[d, "13", "o"'] \\
r_2'+r_1'+t>t'+r_1+r_2 \arrow[d, "7"', "*"] & & q'+r_0'+t>t'+r_0+q \arrow[dll, "11", "*"'] \\
0>0 & &
\end{tikzcd}
$$

Case 3.2.2: $k_2 < k_3$.

This is essentially the mirror image of Case 3.2.1.

Case 3.2.3: $k_2 = k_3$.

Again, we add the rules $k_1v + r_1 > r_1'$ and $r_0' \approx k_0v + r_0$ and all rules e_i' obtaining $k_0v + r_1 + r_2 + r_0' > k_0v + r_1' + r_2' + r_0$. By Lemma 2.16, the result $q > q'$ of κ-normalizing this inequation must be a rule from $R_C^\Psi \cup E_C^\Psi$, hence $q > q'$ has itself a derivation to $0 > 0$.

$$
\begin{tikzcd}
k_0v+r_1+r_2+r_0' > k_0v+r_1'+r_2'+r_0 \arrow[d, "\kappa\, *"] \\
r_1+r_2+r_0' > r_1'+r_2'+r_0 \arrow[d, "18"', "\kappa\, *"] \\
q > q' \arrow[d, "19"', "\circ"] \\
0 > 0
\end{tikzcd}
$$

Let us add the starting (in-)equations of 7 and 18. Obviously there is a derivation from this inequation to $0 > 0$.

$$\begin{tikzcd}
r'_2 + r'_1 + t + r_1 + r_2 + r'_0 > t' + r_1 + r_2 + r'_1 + r'_2 + r_0 \arrow[dd, "\circ"] \arrow[dr, "\kappa"', "*"] & \\
& r'_0 + t > t' + r_0 \arrow[dl, "20", "\circ"'] \\
0 > 0 &
\end{tikzcd}$$

On the other hand, we can cancel $r_1 + r_2 + r'_1 + r'_2$. By the induction hypothesis, there is a derivation 20 from $t + r'_0 > t' + r_0$ to $0 > 0$.

The diagram above can now be closed by using κ-steps 21 to cancel $k_2 v$ followed by derivation 20.

$$\begin{tikzcd}
& k_0 v + s > s' \arrow[dl, "1"', "o"] \arrow[dr, "2", "\gamma"'] & \\
(k_0 - k_1)v + r'_1 + s > s' + r_1 \arrow[d, "4"', "*"] & & r'_0 + s > s' + r_0 \arrow[d, "8", "*"'] \\
(k_0 - k_1 + k_2)v + r'_1 + t > t' + r_1 + k_3 v \arrow[d, "5"', "\gamma o\,*"] & & k_2 v + r'_0 + t > t' + r_0 + k_3 v \arrow[d, "21", "\kappa\,*"'] \\
r'_2 + k_3 v + r'_1 + t > t' + r_1 + k_3 v + r_2 \arrow[d, "6"', "\kappa\,*"] & & r'_0 + t > t' + r_0 \arrow[ddll, "20", "\circ"'] \\
r'_2 + r'_1 + t > t' + r_1 + r_2 \arrow[d, "7"', "*"] & & \\
0 > 0 & &
\end{tikzcd}$$

Case 4: γ/γ-peaks, δ/δ-peaks, κ/γ-peaks.

It remains to consider peaks of the form $e_1 \leftarrow_\gamma e \rightarrow_\gamma e_2$ or $e_1 \leftarrow_\delta e \rightarrow_\delta e_2$ or $e_1 \leftarrow_\kappa e \rightarrow_\gamma e_2$. These can be joined in virtually the same way as the corresponding peaks in the equational case. $\square$

Using the same techniques as in (Waldmann [8]) and (Bachmair and Ganzinger [3]) we can now prove the following theorem. Note in particular that in the presence of the totality axiom *cancellative inequality factoring (I)/(II)* inferences are simplifications, hence clauses where the maximal atomic term occurs in two ordering literals do not produce primary rules.

THEOREM 2.20 *Let N be a set of clauses without negative inequality literals and without unshielded variables; suppose that N is saturated up to redundancy and contains the theory axiom* Div, Inv, Nt, *and all ground instances of* Tot. *If all clauses of N, except the ground instances of* Tot, *are fully abstracted, and if N does not contain the empty clause, then we have for every ground clause $C\theta \in \overline{N}$:*

(i) The relation $\rightarrow_{R^{\Psi}_{C\theta}}$ is partially confluent on $\mathrm{tr}(R^{\Psi}_{C\theta})$.

(ii) $\mathrm{tr}^{\circ}(R^{\Psi}_{C\theta})$ *satisfies the axioms* Ir, Tr, Mon, $\mathrm{K}^{>}$, $\mathrm{T}^{>}$, *and the congruence axiom for the predicate $>$.*

(iii) $E_{C\theta} = \emptyset$ if and only if $C\theta$ is true in $\mathrm{tr}^{\circ}(R^{\Psi}_{C\theta})$.

(iv) $C\theta$ is true in $\mathrm{tr}^{\circ}(R^{\Psi}_{\infty})$ *and in* $\mathrm{tr}^{\circ}(R^{\Psi}_{D\theta})$ *for every $D\theta \succ_{\mathrm{C}} C\theta$.*

(v) If $C\theta = C'\theta \vee e\theta$ and $E_{C\theta} = \{e\theta\}$, then $C'\theta$ is false in $\mathrm{tr}^{\circ}(R^{\Psi}_{\infty})$ *and* $\mathrm{tr}^{\circ}(R^{\Psi}_{D\theta})$ *for any $D\theta \succeq_{\mathrm{C}} C\theta$.*

(vi) The relation $\rightarrow_{R^{\Psi}_{C\theta} \cup E^{\Psi}_{C\theta}}$ is partially confluent on $\mathrm{tr}(R^{\Psi}_{C\theta} \cup E^{\Psi}_{C\theta})$.

(vii) $\mathrm{tr}^{\circ}(R^{\Psi}_{C\theta} \cup E^{\Psi}_{C\theta})$ *satisfies the axioms* Ir, Tr, Mon, $\mathrm{K}^{>}$, $\mathrm{T}^{>}$, *and the congruence axiom for the predicate $>$.*

(viii) The relation $\rightarrow_{R^{\Psi}_{\infty}}$ is partially confluent on $\mathrm{tr}(R^{\Psi}_{\infty})$ *and* $\mathrm{tr}^{\circ}(R^{\Psi}_{\infty})$ *satisfies the axioms* Ir, Tr, Mon, $\mathrm{K}^{>}$, $\mathrm{T}^{>}$, *and the congruence axiom for the predicate $>$.*

PROOF. We use induction on the clause ordering $\succ_{\mathrm{C}}$ and assume that (i)–(vii) are already satisfied for all clauses in $\overline{N}$ that are smaller than $C\theta$.

Property (i) is a direct consequence of the fact that $R^{\Psi}_{C\theta}$ is the union of all $R^{\Psi}_{D\theta}$ with $D\theta \prec_{\mathrm{C}} C\theta$: Note that every finite $R^{\Psi}_{C\theta}$-derivation is also an $R^{\Psi}_{D\theta}$-derivation for some $D\theta \in \overline{N}$ with $D\theta \prec_{\mathrm{C}} C\theta$ and that (vi) is satisfied for $D\theta$.

Property (ii) follows from partial confluence. For the transitivity axiom, consider two inequations $r > s$ and $s > t$ in $\mathrm{tr}^{\circ}(R^{\Psi}_{C\theta})$.

$$\begin{tikzcd} r > s \arrow[d, "\textcircled{1}"', "\circ"] & s > t \arrow[d, "\textcircled{2}", "\circ"'] \\ 0 > 0 & 0 > 0 \end{tikzcd}$$

We can combine the derivations ① and ② and obtain a derivation ③:

$$
\begin{tikzcd}
r+s > s+t \arrow[dd, "\textcircled{3}"', "\circ"] \arrow[dr, "\textcircled{4}", "\kappa *"'] & \\
& r > t \arrow[dl, "\textcircled{5}", "\circ"'] \\
0 > 0 &
\end{tikzcd}
$$

On the other hand, we can use κ-steps ④ to cancel s on both sides of the inequation. By partial confluence, there is a derivation ⑤, hence $r > t \in \mathrm{tr}^\circ(R_C^\Psi)$.

For the axiom $\mathrm{T}^>$ we have to show that $\psi s > \psi t \in \mathrm{tr}^\circ(R_{C\theta}^\Psi)$ entails $s > t \in \mathrm{tr}^\circ(R_{C\theta}^\Psi)$. Consider derivation ①:

$$
\begin{tikzcd}
\psi s > \psi t \arrow[ddd, "\textcircled{1}"', "\circ"] \arrow[dr, "\textcircled{2}", "\delta *"'] & \\
& \psi(s'+r) > \psi(t'+r) \arrow[d, "\textcircled{3}", "\kappa *"'] \\
& \psi s' > \psi t' \arrow[dl, "\textcircled{4}", "\circ"'] \\
0 > 0 &
\end{tikzcd}
$$

We can $\delta\kappa$-normalize $\psi s > \psi t$, first by δ-rewriting s to $s' + r$ and t to $t' + r$ ②, then by cancelling ③ the common part ψr. By confluence, there exists a derivation ④. The inequation $\psi s' > \psi t'$ is $\delta\kappa$-irreducible with respect to $R_{C\theta}^\Psi$, hence it is contained in some $E_{D\theta}^\Psi \subseteq R_{C\theta}^\Psi$ by Lemma 2.12. By the construction of $E_{D\theta}^\Psi$ and by the definition of tr°, we get $s' > t' \in E_{D\theta}^\Psi \subseteq R_{C\theta}^\Psi$. It is now easy to construct a derivation from $s > t$ to $0 > 0$, using first $\delta\kappa$-normalization and then $s' > t'$.

The other axioms are proved in a similar way.

The "if" part of (iii) is obvious from the model construction. To prove the "only if" part let us assume that $C\theta$ is false in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$. We distinguish between three cases:

If C contains a variable x such that $x\theta \approx r \in \mathrm{tr}^\circ(R_{C\theta}^\Psi)$ for some term $r \prec x\theta$, then there is a smaller instance of C that is true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$ by the induction hypothesis. As $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$ satisfies the equality axioms, $C\theta$ must also be true in

$\mathrm{tr}^\circ(R_{C\theta}^\Psi)$, contradicting our assumption. Similarly, if $C\theta$ is an instance of the totality axiom and some term occurring in $C\theta$ equals some smaller term, then there is a smaller instance of the totality axiom, and again $C\theta$ must be true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$.

Suppose that $C\theta$ contains a maximal negative literal $\neg\, e_1\theta$. Then there must be an $R_{C\theta}^\Psi$-derivation from $e_1\theta$ to $0 \approx 0$. If the maximal atomic term of $e_1\theta$ occurs on both sides of $e_1\theta$, then there is either a *cancellation* or an *equality resolution* inference from $C\theta$. This inference is an instance of a *cancellation* or *equality resolution* inference from C. By saturation up to redundancy, the conclusion of the inference must be true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$, hence $C\theta$ must also be true, contradicting our assumption.

If the maximal term occurs on only one side, then it must be either γ- or δ-reducible, using a rule $e'' \in E_{D\theta}^\Psi \subseteq R_{C\theta}^\Psi$. Consequently, there is either a *cancellative superposition* or a *standard superposition* inference between $D\theta$ and $C\theta$, and by saturation up to redundancy, the conclusion of the inference must be true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$. From this we can again infer that $C\theta$ is true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$.

It remains to consider the case that $C\theta$ does not contain a maximal negative literal. Then it must contain a maximal positive literal e_1. If the maximal atomic term of $e_1\theta$ occurs on both sides of $e_1\theta$, then there is either a *cancellation* or an *inequality resolution* inference from $C\theta$ and $C\theta$ must be true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$ by saturation.

If the maximal atomic term occurs on only one side of $e_1\theta$, then there are again three possibilities: If $e_1\theta$ is maximal, but not strictly maximal in $C\theta$, then there is either a *cancellative equality factoring*, or a *standard equality factoring*, or a *cancellative inequality factoring* inference from $C\theta$, from which we can conclude that $C\theta$ is true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$.

If $\chi\,\mathrm{mt}(e_1\theta)$ is $\gamma\delta$-reducible for some $\chi \in \mathbf{N}^{>0}$ using a rule $e'' \in E_{D\theta}^\Psi \subseteq R_{C\theta}^\Psi$, then there is either a *cancellative superposition* or a *standard superposition* inference between $D\theta$ and $C\theta$. Once more, $C\theta$ must be true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$.

Otherwise, either $E_{C\theta} = \{e_1\theta\}$ (then there is nothing to show), or $C'\theta$ is true in $\mathrm{tr}_\Psi^\circ(R_{C\theta}^\Psi \cup \{e_1\theta\})$. In this case, $C'\theta = C''\theta \vee e_2\theta$, where the literal $e_2\theta$ is smaller than $e_1\theta$ and is contained in $\mathrm{tr}_\Psi^\circ(R_{C\theta}^\Psi \cup \{e_1\theta\}) \setminus \mathrm{tr}^\circ(R_{C\theta}^\Psi)$. This can happen only if $\mathrm{mt}(e_2\theta) = \mathrm{mt}(e_1\theta)$. Then there is either a *cancellative equality factoring*, or a *standard equality factoring*, or a *cancellative inequality factoring* inference from $C\theta$, from which we can conclude that $C\theta$ is true in $\mathrm{tr}^\circ(R_{C\theta}^\Psi)$.

Property (iv) follows from (iii) and from the fact that rules in $R_\infty^\Psi \setminus R_{C\theta}^\Psi$ cannot be used to disprove a negative equality in $C\theta$.

If $C\theta = C'\theta \vee e\theta$ and $E_{C\theta} = \{e\theta\}$, then $C'\theta$ is obviously false in $\mathrm{tr}^\circ(R^\Psi_{C\theta})$. Let e' be any positive literal in C'. The literal ordering is defined in such a way that there is only one situation in which rules in $R^\Psi_\infty \setminus R^\Psi_{C\theta}$ could be used in a derivation $e'\theta \to^\circ 0 \sim 0$, namely if both e and e' are inequations and $\mathrm{mt}(e'\theta) = \mathrm{mt}(e\theta)$ occurs in both $e'\theta$ and $e\theta$ either only on the greater side or only on the smaller side (with respect to $>$). However, in this case (and in the presence of the totality axiom) *cancellative inequality factoring (I)/(II)* inferences are simplifications, that is the conclusions of both *cancellative inequality factoring* inferences and some sufficiently small instance of the totality axiom imply $C\theta$. As N is saturated up to redundancy, $C\theta$ must be true in $\mathrm{tr}^\circ(R^\Psi_{C\theta})$, hence $E_{C\theta} = \emptyset$. This proves property (v).

Let $mu + s > s'$ and $nu + t < t'$ be rules from $E_{C\theta} \cup R_{C\theta}$. Then there is a *cancellative chaining* inference from the two clauses producing these two rules. As this inference must be redundant, the inequation $ns + mt' > ns' + mt$ is contained in $\mathrm{tr}^\circ(R^\Psi_{C\theta})$. By Lemma 2.19, $\to_{R^\Psi_{C\theta} \cup E^\Psi_{C\theta}}$ is partially confluent on $\mathrm{tr}(R^\Psi_{C\theta} \cup E^\Psi_{C\theta})$, hence property (vi) holds.

Property (vii) follows from (vi) in the same way as property (ii) follows from (i). This completes the inductive proof of properties (i)–(vii).

It remains to prove property (viii): Partial confluence of $\to_{R^\Psi_\infty}$ follows from the fact that R^Ψ_∞ is the union of all $R^\Psi_{C\theta}$ (cf. property (i)), the rest is proved again in the same way as property (ii). □

THEOREM 2.21 *Let N be a set of clauses without negative inequality literals and without unshielded variables; suppose that N is saturated up to redundancy and contains the theory axiom* Div, Inv, Nt, *and all ground instances of* Tot. *Suppose that all clauses of N, except the ground instances of* Tot, *are fully abstracted. Then $N \cup$* ODAG *is unsatisfiable if and only if N contains the empty clause.*

PROOF. If N contains the empty clause, then it is unsatisfiable. Otherwise, $\mathrm{tr}^\circ(R^\Psi_\infty)$ is a model of the equality axioms, of ODAG, and of $\overline{N}$. □

We may assume without loss of generality that the constant a_0 does not occur in non-theory input clauses and that the function symbols $-$ and *divided-by*$_n$ are eliminated eagerly from all non-theory input clauses. In this case, no inferences are possible with the axioms Div, Inv, and Nt. Furthermore, one can show that inferences with instances of the totality axiom Tot are always redundant (analogously to Bachmair and Ganzinger [3]).

3 The Extended Calculus

3.1 Variable Elimination

As we have mentioned in the introduction, the calculus *OCInf* works on clauses without unshielded variables, but its inference rules may produce clauses with unshielded variables. To make it effectively saturate a given set of clauses, it has to be supplemented by a variable elimination algorithm.

In the equational case, every clause with unshielded variables can be transformed into an equivalent clause without unshielded variables. However, in the presence of ordering literals, this does no longer hold.

EXAMPLE 3.1 Consider the clause $C = x > a \vee x \approx b \vee x < c$. This clause is true for every value of x, if either $c > a$ or both $a \approx b$ and $c \approx b$. So C can be replaced by the clause normal form of $c > a \vee (a \approx b \wedge c \approx b)$, that is, by the two clauses $c > a \vee a \approx b$ and $c > a \vee c \approx b$, but C is not equivalent to a single clause without unshielded variables.

For any disjunction of conjunctions of literals F let $\mathrm{CNF}(F)$ be the clause normal form of F (represented as a multiset of clauses).

Let x be a variable of sort G. We define a binary relation $\to_x$ over multisets of clauses by

CancelVar

$$M \cup \{C' \vee mx + s \mathrel{\dot\sim} m'x + s'\} \;\to_x$$
$$M \cup \{C' \vee (m{-}m')x + s \mathrel{\dot\sim} s'\}$$

if $m \geq m' \geq 1$.

ElimNeg

$$M \cup \{C' \vee mx + s \not\approx s'\} \;\to_x$$
$$M \cup \{C'\}$$

if $m \geq 1$ and x does not occur in C', s, s'.

ElimPos

$$M \cup \bigl\{C' \vee \textstyle\bigvee_{i\in I} l_i x + r_i \approx r'_i \vee \bigvee_{j\in J} m_j x + s_j > s'_j \vee \bigvee_{k\in K} n_k x + t_k < t'_k\bigr\} \;\to_x$$
$$M \cup \mathrm{CNF}\bigl(C' \vee \textstyle\bigvee_{j\in J}\bigvee_{k\in K}(n_k s_j + m_j t'_k > n_k s'_j + m_j t_k \vee \bigvee_{i\in I}(l_i s_j + m_j r'_i \approx l_i s'_j + m_j r_i \wedge l_i t_k + n_k r'_i \approx l_i t'_k + n_k r_i))\bigr)$$

if $I \cup J \cup K \neq \emptyset$, $l_i \geq 1$, $m_j \geq 1$, $n_k \geq 1$ and x does not occur in $C', r_i, r'_i, s_j, s'_j, t_k, t'_k$, for $i \in I$, $j \in J$, $k \in K$.

Coalesce

$$M \cup \{C' \vee mx + s \not\approx s' \vee nx + t \mathrel{\dot\sim} t'\} \;\to_x$$
$$M \cup \{C' \vee mx + s \not\approx s' \vee mt + ns' \mathrel{\dot\sim} mt' + ns\}$$

if $m \geq 1$, $n \geq 1$, and x does not occur in s, s', t, t'.

It is easy to show that $\to_x$ is noetherian. We define the relation $\to_{\mathrm{elim}}$ over multisets of clauses in such a way that $M \cup \{C\} \to_{\mathrm{elim}} M \cup M'$ if and only if C contains an unshielded variable x and M' is a normal form of $\{C\}$ with respect to $\to_x$.

The relation $\to_{\mathrm{elim}}$ is again noetherian. For a clause C, $\mathrm{elim}(C)$ denotes some (arbitrary but fixed) normal form of $\{C\}$ with respect to the relation $\to_{\mathrm{elim}}$.

COROLLARY 3.2 *For any C, the clauses in $\mathrm{elim}(C)$ contain no unshielded variables.*

LEMMA 3.3 *For every C, $\{C\} \models_{\mathrm{ODAG}} \mathrm{elim}(C)$ and $\mathrm{elim}(C) \cup \mathrm{Tot} \models_{\mathrm{OTfCAM}} C$. For every ground instance $C\theta$, $\mathrm{elim}(C)\theta \cup \mathrm{Tot} \models_{\mathrm{OTfCAM}} C\theta$.*

3.2 Integration of the Elimination Algorithm

Using the technique sketched so far, every clause C_0 can be transformed into a set of clauses $\mathrm{elim}(C_0)$ that do not contain unshielded variables, follow from C_0 and the axioms of totally ordered divisible abelian groups, and imply C_0 modulo $\mathrm{OTfCAM} \cup \mathrm{Tot}$. Obviously, we can perform this transformation for all initially given clauses *before* we start the saturation process. However, when clauses with unshielded variables are produced during the saturation process, then logical equivalence is not sufficient to eliminate them. We have to require that the transformed set of clauses $\mathrm{elim}(C_0)$ makes the inference ι producing C_0 redundant. Unfortunately, it may happen that the clauses in $\mathrm{elim}(C_0)$ or the instances of the totality axiom needed in Lemma 3.3 are too large, at least for some instances of ι. To integrate the variable elimination algorithm into the base calculus, it has to be supplemented by a case analysis technique.

Let $k \in \{1, 2\}$, let $C_1, \ldots, C_k$ be clauses without unshielded variables and let ι be an *OCInf*-inference

$$\frac{C_k \ \ldots \ C_1}{C_0\sigma}$$

We call the unifying substitution σ that is computed during ι and applied to the conclusion the pivotal substitution of ι. (For ground inferences, the pivotal substitution is the identity mapping.) If the last premise C_1 has the form $C_1' \vee A$ where A is maximal (and the replacement or cancellation takes place at A) then we call $A\sigma$ the pivotal literal of ι.[8] Finally, if u_0 is the atomic term that is cancelled out in ι, or in which some subterm is replaced,[9] then we call $u_0\sigma$ the pivotal term of ι.

[8] In *cancellative inequality factoring* inferences, the pivotal literal is *not* deleted; however, factoring does not produce unshielded variables anyway.

[9] More precisely, u_0 is the maximal atomic subterm of s containing u in *standard superposition* inferences, and the term u in all other inferences.

Two properties of pivotal terms are important for us: First, whenever an inference ι from clauses *without* unshielded variables produces a conclusion *with* unshielded variables, then all these unshielded variables occur in the pivotal term of ι. Second, no atomic term in the conclusion of ι can be larger than the pivotal term of ι.

One can now show that, if the clauses in $\mathrm{elim}(C_0)$ or the instances of the totality axiom needed in Lemma 3.3 are too large to make the *OCInf*-inference ι redundant, then there must be an atomic term in some clause in $\mathrm{elim}(C_0)$ that is unifiable with the pivotal term. If we apply the unifier to the conclusion of the *OCInf*-inference, then the result does no longer contain unshielded variables, and moreover it subsumes the critical instances of ι. Using this result, we can now transform the inference system *OCInf* into a new inference system that operates on clauses without unshielded variables and produces again such clauses. The new system *ODInf* is given by two meta-inference rules:

Eliminating Inference

$$\frac{C_n \quad \ldots \quad C_1}{C'}$$

if the following conditions are satisfied:

(i) $\dfrac{C_n \quad \ldots \quad C_1}{C_0}$ is a *OCInf*-inference.

(ii) $C' \in \mathrm{elim}(C_0)$.

Instantiating Inference

$$\frac{C_n \quad \ldots \quad C_1}{C_0\tau}$$

if the following conditions are satisfied:

(i) $\dfrac{C_n \quad \ldots \quad C_1}{C_0}$ is a *OCInf*-inference with pivotal literal A and pivotal term u.

(ii) $\mathrm{elim}(C_0) \neq \{C_0\}$.

(iii) A literal A_1 with the same polarity as A occurs in some clause in $\mathrm{elim}(C_0)$.

(iv) An atomic term u_1 occurs at the top of A_1.

(v) τ is contained in a minimal complete set of ACU-unifiers of u and u_1.

We define the redundancy criterion for the new inference system in such a way, that an *ODInf*-inference is redundant, if the appropriate instances of its parent *OCInf*-inference are redundant. Then a set of clauses without unshielded variables that is saturated with respect to *ODInf* up to redundancy is also saturated with respect to *OCInf* up to redundancy. *ODInf* can thus be used for effective saturation of a given set of input clauses:

THEOREM 3.4 *Let N_0 be a set of clauses without negative inequality literals and without unshielded variables; let N_0 contain the theory axiom* Div, Inv, Nt, *and all ground instances of* Tot. *Suppose that all clauses of N_0, except the ground instances of* Tot, *are fully abstracted. Let $N_0 \vdash N_1 \vdash N_2 \vdash \dots$ be a fair ODInf-derivation. Let N_∞ be the limit of the derivation. Then $N_0 \cup \mathrm{ODAG}$ is unsatisfiable if and only if N_∞ contains the empty clause.*

4 Conclusions

We have presented a superposition-based calculus for first-order theorem proving in the presence of the axioms of totally ordered divisible abelian groups. It is based on the DTAG-superposition calculus from (Waldmann [10]) and the ordered chaining calculus for dense total orderings without endpoints (Bachmair and Ganzinger [3]), and it shares the essential features of these two calculi: It is refutationally complete, it does not require explicit inferences with the theory clauses, and due to the integrated variable elimination algorithm it does not require variable overlaps. It offers thus an efficient way of treating equalities and inequalities between additive terms over, e. g., the rational numbers within a first-order theorem prover.

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