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Partisan Corroboration, and Shifted Pairing

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Abstract

The Herbrand theorem plays a fundamental role in automated theorem proving methods based on *global variable* or *rigid variable* approaches. The kernel step in procedures based on such methods can be described as the *corroboration* problem (also called the *Herbrand skeleton* problem), where, given a positive integer m, called *multiplicity*, and a quantifier free formula, one seeks for a valid or provable (in classical first-order logic) disjunction of m instantiations of that formula. In logic with equality this problem was recently shown to be undecidable.

The first main contribution of this paper is a logical theorem, that we call the *Partisan Corroboration Theorem*, that enables us to show that, for a certain interesting subclass of Horn formulas, corroboration with multiplicity one can be reduced to corroboration with any given multiplicity.

The second main contribution of this paper is a *finite tree automata* formalization of a technique called *shifted pairing* for proving undecidability results via direct encodings of valid Turing machine computations. We call it the *Shifted Pairing Theorem.*

By using the Partisan Corroboration Theorem, the Shifted Pairing Theorem, and term rewriting techniques in equational reasoning, we improve upon a number of recent undecidability results related to the *corroboration* problem, the *simultaneous rigid E-unification* problem and the *prenex fragment of intuitionistic logic with equality.*

Keywords

logic with equality; Herbrand's theorem; finite tree automata

1 Introduction

We study classical first-order logic with equality but without any other relation symbols. The letters φ and ψ are reserved for quantifier-free formulas. The *signature* of a syntactic object S (a term, a set of terms, a formula, etc.) is the collection of function symbols in S augmented, in the case when S contains no constants, with a constant c. The language of S is the language of the signature of S.

Any syntactic object is *ground* if it contains no variables. A substitution is *ground* if its range is ground, and it is said to be in a given language if the terms in its range are in that language. A set of substitutions is *ground* if each member is ground.

Given a positive integer m, a set of m ground substitutions $\{\theta_1, \ldots, \theta_m\}$ is an *m-corroborator* for φ if the disjunction $\varphi\theta_1 \vee \cdots \vee \varphi\theta_m$ is provable. A ground substitution θ *corroborates* φ if $\{\theta\}$ 1-corroborates φ; such a θ is called a *corroborator* for φ.

One popular form of the classical Herbrand theorem [e.g. Herbrand 1972] is this:

> *An existential formula $\exists\vec{x}\varphi(\vec{x})$ is provable if and only if there exist a positive integer m and an m-corroborator for φ in the language of φ.*

The minimal appropriate number m will be called the *minimum multiplicity for φ*. The minimum multiplicity for a formula may exceed one. Here is a formula for which the minimum multiplicity is two, suggested by Erik Palmgren in a different but similar context; we use '$\approx$' for the formal equality sign.

$$(c \approx c_0 \Rightarrow x \approx c_1) \wedge (c \approx c_1 \Rightarrow x \approx c_0)$$

The Herbrand theorem plays a fundamental role in automated theorem proving methods known as the *rigid variable methods* [Voronkov 1997]. We can identify the following procedure underlying such methods. Let $\exists\vec{x}\varphi(\vec{x})$ be a closed formula that we wish to prove.

> *The principal procedure of rigid variable methods*
> **Step I:** Choose a positive integer m.
> **Step II:** Check if there exists an m-corroborator for φ.
> **Step III:** If Step II succeeds then $\exists\vec{x}\varphi(\vec{x})$ is provable, otherwise increase m and return to Step II.

The kernel of the principal procedure is of course Step II or:

The Corroboration Problem
Instance: A quantifier free formula φ and a positive integer m.
Question: Is the minimum multiplicity for φ bounded by m?

Corroboration for a fixed m is called *m-corroboration*. A detailed discussion of corroboration and related problems is given by Degtyarev, Gurevich & Voronkov [1996]. It is important to us here that corroboration is intimately related to existential intuitionistic provability and simultaneous rigid E-unification [Gallier, Raatz & Snyder 1987]. The first of these problems is easy to formulate:

The Existential Intuitionistic Provability Problem
Instance: An existential formula $\exists\vec{x}\varphi(\vec{x})$.
Question: Is the formula provable in intuitionistic logic with equality?

The second requires auxiliary definitions. A *rigid equation* is an expression $E \vdash^{\mathrm{r}} e$ where E is a finite set of equations and e is an equation. A ground substitution θ *solves* a rigid equation $E \vdash^{\mathrm{r}} e$ if $e\theta$ is a logical consequence of $E\theta$. A system (that is a finite set) of rigid equations is *solvable* if there is one substitution that solves all rigid equations in the system.

The Simultaneous Rigid E-Unification Problem (SREU)
Instance: A system of rigid equations.
Question: Is the system solvable?

The SREU problem has an interesting history [e.g. Degtyarev, Gurevich & Voronkov 1996]. Several false decidability claims have been published until, finally, Degtyarev & Voronkov [1995] proved SREU to be undecidable. Moreover, Plaisted [1995] has shown that the fragment of SREU with ground left-hand sides is already undecidable (the *left-hand side* of a rigid equation $E \vdash^{\mathrm{r}} e$ is E).

It is easy to see that SREU is essentially a special case of 1-corroboration for Horn formulas. Hence, the result of Degtyarev & Voronkov shows that corroboration is undecidable already in this very special case. Voronkov [1997] has suggested the following generalization of the corroboration problem. Let f be a function that assigns a positive integer to every pair (k, φ) where k is a positive integer and φ a formula in our logic. Moreover, it is assumed that $k < l$ implies that $f(k, \varphi) \leq f(l, \varphi)$. Such a function is called a *strategy* for multiplicity. The intended meaning of the first argument of a strategy is the number of times that Step II of the principal procedure has been executed.

The Corroboration Problem with Strategy f
Instance: A quantifier free formula φ and a positive integer k.
Question: Is the minimum multiplicity for φ bounded by $f(k, \varphi)$?

Corroboration with a strategy that does not depend on it arguments, i.e., takes a constant value m for all arguments, is simply m-corroboration. Voda & Komara [1995] have proved that, for each positive integer m, the m-corroboration problem is undecidable. One important conclusion for automated theorem proving, drawn by Voda & Komara, is that there is no m for which one can effectively determine whether m bounds the minimum multiplicity for a given formula. Actually, we had hard time to understand the proof of Voda & Komara until, finally, we convinced ourselves that they have a proof. We wondered if there is a way to derive their result from the Degtyarev–Voronkov theorem. It turns out that indeed there is such a way.

In order to formulate our results, we need to recall a few definitions and give definitions of our own. Recall that a *Horn clause* is a disjunction of negated atomic formulas and at most one non-negated atomic formula; a Horn clause is often represented as a set of its disjuncts. Here we restrict attention to Horn clauses that contain exactly one non-negated atom. A *Horn formula* is a conjunction of Horn clauses. Since the equality sign is the only relation symbol in our logic, every Horn clause ψ is equivalent to an implication $E \Rightarrow s \approx t$ where E is a conjunction of equalities.

We say that a collection of formulas is *constant-disjoint* if there is no constant that occurs in two or more of the given formulas. Call a Horn formula φ *guarded* if, for every variable x that occurs in φ, there exists a clause $E \Rightarrow s \approx t$ in φ where E and s are ground and x occurs in t. Finally, call a corroborator θ of a disjunction φ *partisan* if θ corroborates already one of the disjuncts of φ. Now we are ready to formulate our first result.

Partisan Corroboration Theorem
Every corroborator for a disjunction of constant-disjoint guarded Horn formulas is partisan.

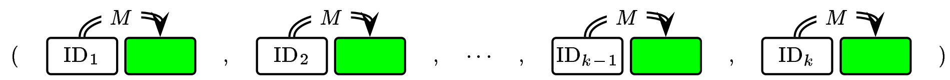
This theorem is proved in Section 3. We believe it is of independent interest. It allows us an easy derivation of Voda & Komara's [1995] result from Degtyarev & Voronkov's [1995] theorem in Section 4. Moreover, we strengthen the theorem of Voda & Komara in several ways. For each m, we effectively reduce SREU to the m-corroboration problem in such a way that the positive-arity part of the signature remains unchanged. In particular, for every m, the monadic (all function symbols are of arity at most one) SREU reduces to monadic m-corroboration; this reduction is of interest because the decidability of monadic SREU is an open problem.

In Section 5 we use *finite tree automata* theory to describe a powerful technique, named *shifted pairing* by Plaisted [1995], for proving undecidability results via encodings of valid Turing machine computations. The main components are two finite tree automata $\mathcal{A}_{\mathrm{mv}}$, $\mathcal{A}_{\mathrm{id}}$ and two ground term rewrite systems Π_1 and Π_2 that are obtained (effectively) from a given Turing machine M. Each term t recognized by $\mathcal{A}_{\mathrm{id}}$ represents a sequence of *IDs* of M:

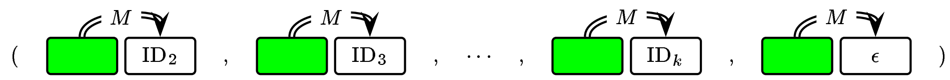
$$(\boxed{\mathrm{ID}_1} , \boxed{\mathrm{ID}_2} , \cdots , \boxed{\mathrm{ID}_{k-1}} , \boxed{\mathrm{ID}_k})$$

Each term s that is recognized by $\mathcal{A}_{\mathrm{mv}}$ represents a sequence of *moves*:

Note that, at this point the consecutive moves are not related, this is where Π_1 and Π_2 come into play. Namely, Π_1 and Π_2 serve the following purpose. If s reduces in Π_1 to t then the *first projection* of s must coincide with t:



Similarly, if s reduces in Π_2 to the "tail" of t, then the *second projection* of s must coincide with the tail of t:



The empty string (ϵ) denotes the successor of any *final* ID of M. The idea is thus, that the systems Π_1 and Π_2 are used to enforce t to encode a *valid computation of* M. The above outline explains the main role of the parameters in the Shifted Pairing Theorem, that is the second main contribution of this paper.

Shifted Pairing Theorem
There are two finite tree automata $\mathcal{A}_{\mathrm{mv}}$ and $\mathcal{A}_{\mathrm{id}}$ and two ground rewrite systems Π_1 and Π_2 such that, it is undecidable whether, given a ground term t_0, $\mathcal{A}_{\mathrm{mv}}$ recognizes a term s and $\mathcal{A}_{\mathrm{id}}$ recognizes a term t, such that s reduces in Π_1 to t and $f(t_0, s)$ reduces in Π_2 to t.

There are some important additional properties on the tree automata and the rewrite systems that are explained in Section 5. The shifted pairing technique, and in particular the Shifted Pairing Theorem that is an improved construction from [Veanes 1997, Gurevich & Veanes 1997], has recently been applied successfully to settle several open decidability questions [Ganzinger, Jacquemard & Veanes 1998, Levy & Veanes 1998, Veanes 1997, Veanes 1998].

In Section 6, we use the Shifted Pairing Theorem to show the undecidability of a fragment of SREU with only two variables and three rigid equations with ground left-hand sides, which constitutes the currently known *least undecidable fragment* of SREU. Using this result and the Partisan Corroboration Theorem, we show, for each positive integer m, the undecidability of m-corroboration when each formula is a conjunction of $3m$ Horn clauses with $2m$ variables and ground negative literals of bounded size.

In Section 7 we obtain some undecidability results related to the prenex fragment of intuitionistic logic with equality and proof search in intuitionistic logic with equality. Finally, in Section 8 we describe the current status of SREU and related results and list some open problems.

2 Preliminaries

We will first establish some notation and terminology. We follow Chang & Keisler [1990] regarding first order languages and structures. For the purposes of this paper it is enough to assume that the first order languages that we are dealing with are languages with equality and contain only function symbols and constants, so we will assume that from here on. We will in general use Σ, possibly with an index, to stand for a signature, i.e., Σ is a collection of function symbols with fixed arities. A function symbol of arity 0 is called a *constant*. We will always assume that Σ *contains at least one constant*.

2.1 Terms and formulas

Terms and formulas are defined in the standard manner and are called Σ-*terms* and Σ-*formulas* respectively whenever we want be precise about the language. We refer to terms and formulas collectively as *expressions*. In the following let X be an expression or a set of expressions or a sequence of such.

We write $\Sigma(X)$ for the *signature of* X: the set of all function symbols that occur in X, $FV(X)$ for the set of all free variables in X and $Con(X)$ for the set of all constants in X. We write $X(x_1, x_2, \ldots, x_n)$ to express that $FV(X) \subseteq \{x_1, x_2, \ldots, x_n\}$. Let $t_1, t_2, \ldots, t_n$ be terms, then $X(t_1, t_2, \ldots, t_n)$ denotes the result of replacing each (free) occurrence of x_i in X by t_i for $1 \leq i \leq n$. By a *substitution* we mean a function from variables to terms. We will use θ to denote substitutions. We write $X\theta$ for $X(\theta(x_1), \theta(x_2), \ldots, \theta(x_n))$.

We say that X is *closed* or *ground* if $FV(X) = \emptyset$. By $\mathcal{T}_\Sigma$ or simply $\mathcal{T}$ we denote the set of all ground Σ-terms. A substitution is called *ground* if its range consists of ground terms.

A closed formula is called a *sentence*. Since there are no relation symbols all the atomic formulas are *equations*, i.e., of the form $t \approx s$ where t and s are terms and '$\approx$' is the formal equality sign.

Atomic formulas and negated atomic formulas are called *positive* and *negative literals* respectively. A *clause* is a disjunction of literals. By a *Horn clause* we mean a clause with exactly one positive literal.[1] A Horn clause can be written as $E \Rightarrow s \approx t$ where E is a conjunction of equations, and s and t are terms. By a *Horn formula* we understand a conjunction of Horn clauses.

2.2 First-order structures

First order structures will (in general) be denoted by capital Gothic letters like $\mathfrak{A}$ and their domains by corresponding capital Roman letters like A. A first order structure in a signature Σ is called a Σ-*structure*. For $f \in \Sigma$ we write $f^{\mathfrak{A}}$ for the interpretation of f in $\mathfrak{A}$.

If $\mathfrak{A}$ is a Σ-structure and $\Sigma' \subseteq \Sigma$ then $\mathfrak{A}{\restriction}\Sigma'$ is the Σ'-structure that is the reduction of $\mathfrak{A}$ to signature Σ'. Let $\mathfrak{A}$ and $\mathfrak{B}$ be Σ-structures, $\mathfrak{A}$ is a *substructure* of $\mathfrak{B}$, in symbols $\mathfrak{A} \subseteq \mathfrak{B}$, if $A \subseteq B$ and for each n-ary $f \in \Sigma$, $f^{\mathfrak{A}} = f^{\mathfrak{B}}{\restriction}A^n$.

For X a sentence or a set of sentences, $\mathfrak{A} \models X$ means that the structure $\mathfrak{A}$ is a *model of* or *satisfies* X according to Tarski's truth definition. A set of sentences is called *satisfiable* if it has a model. If X and Y are (sets of) sentences then $X \models Y$ means that Y is a *logical consequence* of X, i.e., that every model of X is a model of Y. We write $\models X$ to say that X is *valid*, i.e., true in all models.

One easily establishes, by induction on terms and formulas that, if $\mathfrak{A} \subseteq \mathfrak{B}$ then for all quantifier free sentences φ, $\mathfrak{A} \models \varphi$ if and only if $\mathfrak{B} \models \varphi$.

By the *free algebra over* Σ we mean the Σ-structure $\mathfrak{A}$, with domain $\mathcal{T}_\Sigma$, such that for each n-ary $f \in \Sigma$ and $t_1, \ldots, t_n \in \mathcal{T}_\Sigma$, $f^{\mathfrak{A}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$. We let $\mathcal{T}_\Sigma$ also stand for the free algebra over Σ.

Let E be a set of ground equations. Define the equivalence relation $=_E$ on $\mathcal{T}$ by $s =_E t$ if and only if $E \models s \approx t$. By $\mathcal{T}_{\Sigma/E}$ (or simply $\mathcal{T}_{/E}$) we denote the quotient of $\mathcal{T}_\Sigma$ over $=_E$. Thus, for all $s, t \in \mathcal{T}$,

$$\mathcal{T}_{/E} \models s \approx t \quad \Leftrightarrow \quad E \models s \approx t.$$

We call $\mathcal{T}_{/E}$ the *canonical model* of E.

[1] By a Horn clause we mean thus a *strict* Horn clause.

2.3 Term rewriting

In some cases it is convenient to consider a system of ground equations as a rewrite system. We will assume that the reader is familiar with basic notions regarding ground term rewrite systems [e.g. Dershowitz & Jouannaud 1990]. We will only use very elementary properties. In particular, in the next section we will use Birkhoff's [1935] completeness theorem for equational logic. In the case of ground equations it states simply that, given a ground set of equations E and and a ground equation $s \approx t$, $E \models s \approx t$ if and only if s can be reduced to t by using the equations in E as rewrite rules in both directions.

In Section 6 we will use the following property of canonical (or convergent) rewrite systems [e.g. Dershowitz & Jouannaud 1990, Section 2.4]. Let $\mathcal{R}$ be a ground and canonical rewrite system. Then for any two ground terms t and s, the equation $t \approx s$ follows logically from $\mathcal{R}$ (seen as a set of equations) if and only if the normal forms of t and s with respect to $\mathcal{R}$ coincide, i.e.,

$$\mathcal{R} \models t \approx s \quad \Leftrightarrow \quad t{\downarrow_{\mathcal{R}}} = s{\downarrow_{\mathcal{R}}}.$$

Snyder [1989] has given a very simple but useful condition for showing that a ground rewrite system $\mathcal{R}$ is canonical, namely that it is *reduced*: for each rule $s \rightarrow t$ in $\mathcal{R}$, s is irreducible in $\mathcal{R} \setminus \{s \rightarrow t\}$ and t is irreducible in $\mathcal{R}$. We will use this test on several occasions, to show that a ground rewrite system is canonical.

2.4 Finite tree automata

A *finite tree automaton* or *TA* is a quadruple $(\mathcal{Q}, \Sigma, \mathcal{R}, \mathcal{Q}^{\mathrm{f}})$, where

- $\mathcal{Q}$ is a finite set of constants called *states*,
- Σ is a *signature* that is disjoint from $\mathcal{Q}$,
- $\mathcal{R}$ is a set of *rules* of the form $f(q_1, \ldots, q_n) \rightarrow q$, where $f \in \Sigma$ has arity $n \geq 0$ and $q, q_1, \ldots, q_n \in \mathcal{Q}$,
- $\mathcal{Q}^{\mathrm{f}} \subseteq \mathcal{Q}$ is the set of *final states.*

A TA is called *deterministic* or a *DTA* if there are no two different rules in it with the same left-hand side. Terms are also called *trees* and a *forest* is a set of trees. The forest *recognized* by a TA $\mathcal{A} = (\mathcal{Q}, \Sigma, \mathcal{R}, \mathcal{Q}^{\mathrm{f}})$ is the following set that is denoted by $\mathcal{F}(\mathcal{A})$:

$$\{\, t \in \mathcal{T}_\Sigma \mid (\exists q \in \mathcal{Q}^{\mathrm{f}})\; t \xrightarrow{*}_{\mathcal{R}} q \,\}.$$

A forest is *recognizable* or *regular* if it is recognized by some TA. A well-known fact is that every regular forest is recognized by a DTA. Two finite tree automata are called *constant-disjoint* if there is no constant that occurs in both of them.

Example 1 Let $\mathcal{A} = (\{q\}, \Sigma, \mathcal{R}, \{q\})$ be a TA, where

$$\begin{aligned} \mathcal{R} = \ & \{\, c \to q \mid c \text{ is a constant in } \Sigma \,\} \cup \\ & \{\, f(q, \ldots, q) \to q \mid f \text{ is a function symbol in } \Sigma \,\}. \end{aligned}$$

This DTA recognizes the forest $\mathcal{T}_\Sigma$. □

3 Partisan Corroboration Theorem

The following lemma is used in the Partisan Corroboration Theorem; it is actually a consequence of Łoś-Tarski theorem (existential sentences are preserved under extensions). We say that two (sets of) expressions X and Y are *constant-disjoint* if $Con(X) \cap Con(Y) = \emptyset$.

Lemma 2 *Let φ_i for $i \in I$, be pairwise constant-disjoint quantifier free sentences. Then $\models \bigvee_{i \in I} \varphi_i$ implies $\models \varphi_i$ for some $i \in I$.*

Proof. For $i \in I$, let $\Sigma_i = \Sigma(\varphi_i)$ and let $\Sigma = \bigcup_i \Sigma_i$. Assume by contradiction that $\not\models \varphi_i$ for all $i \in I$. Then there is (for each $i \in I$) a Σ_i-structure $\mathfrak{A}_i$ such that $\mathfrak{A}_i \models \neg\varphi_i$. Without loss of generality, take all the A_i's to be pairwise disjoint.

We now construct a Σ-structure $\mathfrak{A}$ such that $\mathfrak{A}_i \subseteq \mathfrak{A}{\restriction}\Sigma_i$ for $i \in I$. First let $A = \bigcup_{i \in I} A_i$. For each $i \in I$ and constant $c \in \Sigma_i$ let $c^{\mathfrak{A}} = c^{\mathfrak{A}_i}$. For each n-ary function symbol f in Σ define $f^{\mathfrak{A}}$ as follows. For all $\vec{a} = a_1, \ldots, a_n \in A$,

$$f^{\mathfrak{A}}(\vec{a}) = \begin{cases} f^{\mathfrak{A}_i}(\vec{a}), & \text{if } \vec{a} \in A_i; \\ a_1, & \text{otherwise.} \end{cases}$$

It is clear that $\mathfrak{A}$ is well-defined because of the disjointness criteria and that $\mathfrak{A}_i \subseteq \mathfrak{A}{\restriction}\Sigma_i$ for $i \in I$. Hence $\mathfrak{A}{\restriction}\Sigma_i \models \neg\varphi_i$, and thus $\mathfrak{A} \models \neg\varphi_i$ for each $i \in I$. But this contradicts that $\models \bigvee_{i \in I} \varphi_i$. ⊠

If we drop the constant-disjointness criterion in Lemma 2, then of course the lemma is false. A simple counterexample is

$$\models c_0 \approx c_1 \vee \neg(c_0 \approx c_1).$$

We will state now some other obvious but useful lemmas. Lemma 3 is an easy corollary of Birkhoff's completeness theorem.

Lemma 3 *Let t and s be ground terms and let E and E' be ground sets of equations such that $Con(E') \cap (Con(E) \cup Con(s)) = \emptyset$. The following is true.*

1. *If $E' \cup E \models t \approx s$ then $E \models t \approx s$.*
2. *If $E \models t \approx s$ then $\Sigma(t) \subseteq \Sigma(E) \cup \Sigma(s)$.*

Proof. Let E, E', s and t be given and assume that $E' \cup E \models t \approx s$. By Birkhoff's [1935] completeness theorem we know that s can be rewritten to t by using $E' \cup E$ as a set of rewrite rules. So there is a sequence of terms $s_0, s_1, \ldots, s_{n-1}, s_n$ where $s_0 = s$, $s_n = t$ and s_i is rewritten to s_{i+1} by using some rule in $E' \cup E$, for $0 \leq i < n$. By induction on i (for $i \leq n$) follows that $\Sigma(s_i) \subseteq \Sigma(E, s)$ and only a rule from E can be used to rewrite s_i. Part 1 follows again by the completeness theorem of Birkhoff and part 2 follows immediately (take $E' = \emptyset$). ⊠

For a finite set E of equations we will write E also for a corresponding conjunction of equations and let the context determine whether a set or a formula is meant.

Lemma 4 *Let t and s be ground terms and E' and E ground sets of equations such that E is finite and $Con(E') \cap (Con(E) \cup Con(s)) = \emptyset$. Then*

$$\mathcal{T}_{/E' \cup E} \models (E \Rightarrow t \approx s) \quad \Rightarrow \quad \models (E \Rightarrow t \approx s).$$

Proof. Let E, E', s and t be given. From $\mathcal{T}_{/E' \cup E} \models (E \Rightarrow t \approx s)$ follows immediately that $\mathcal{T}_{/E' \cup E} \models t \approx s$ and thus $E' \cup E \models t \approx s$. Hence $E \models t \approx s$ by Lemma 3, i.e., $\models (E \Rightarrow t \approx s)$. ⊠

We will use the following definitions. Let φ be a quantifier free formula and m a positive integer. A set of m ground substitutions Θ is an *m-corroborator* for φ if

$$\models \bigvee_{\theta \in \Theta} \varphi\theta.$$

When $\Theta = \{\theta\}$ we say that θ is a *corroborator* for φ or *corroborates* φ. The *m-corroboration* problem is the problem of determining whether a given quantifier free formula has an m-corroborator.

For $x \in FV(\varphi)$, a *guard for x in φ*, if it exists, is a clause

$$E \Rightarrow t \approx s$$

in φ such that E and s are ground and x occurs in t. We say that

$$\bigwedge_{x \in FV(\varphi)} \psi_x$$

is a *guard* of φ if each ψ_x is a guard for x in φ; φ is is called *guarded* if it has a guard.

Intuitively, in the light of the second part of Lemma 3, the notion of a Horn formula being guarded is a sufficient condition to guarantee that if there is a corroborator θ for φ then $\Sigma(\varphi\theta) = \Sigma(\varphi)$.

SREU is, by definition, the 1-corroboration problem for Horn formulas. However, we only need to consider *guarded* Horn formulas. To see that, consider a Horn formula φ; let Σ be its signature and let c be a constant in Σ. For each variable x in φ, let $\mathrm{Gr}_\Sigma(x)$ denote the following Horn clause:

$$\begin{array}{l} \{\, c' \approx c \mid c' \text{ is a constant in } \Sigma \setminus \{c\} \,\} \cup \\ \{\, f(c, \ldots, c) \approx c \mid f \text{ is a function symbol in } \Sigma \,\} \;\; \Rightarrow x \approx c. \end{array}$$

This is a very simple but useful construction that was first used by Degtyarev & Voronkov to enforce certain solutions to be within a given signature. It is easy to see that, for all terms t,

$$\models \mathrm{Gr}_\Sigma(t) \quad \Leftrightarrow \quad t \in \mathcal{T}_\Sigma.$$

Let now ψ be the guarded Horn formula

$$\Big(\bigwedge_{x \in FV(\varphi)} \mathrm{Gr}_\Sigma(x)\Big) \wedge \varphi.$$

From Herbrand's theorem follows that one only needs to consider corroborators in the language of φ, therefore ψ has a corroborator if and only if φ has one.

Example 5 A simple example of a guarded Horn formula is this

$$\begin{array}{rcl} \psi & = & (E_1 \Rightarrow x \approx c_1) \wedge \\ & & (E_2 \Rightarrow y \approx c_2) \wedge \\ & & (\Pi_1 \Rightarrow x \approx y) \wedge \\ & & (\Pi_2 \Rightarrow x \approx t \centerdot y) \end{array}$$

where E_1, E_2, Π_1, Π_2 and t are ground, c_1, c_2 are constants, and '$\centerdot$' is a binary function symbol written in infix notation. A guard of ψ is

$$(E_1 \Rightarrow x \approx c_1) \wedge (E_2 \Rightarrow y \approx c_2).$$

An example of a Horn formula with a common guard for all variables is

$$\begin{array}{rcl} \varphi & = & (E \Rightarrow x \centerdot y \approx c) \wedge \\ & & (\Pi_1 \Rightarrow x \approx y) \wedge \\ & & (\Pi_2 \Rightarrow x \approx t \centerdot y), \end{array}$$

where E, Π_1, Π_2 and t are ground and c is a constant. The guard of φ is

$$E \Rightarrow x \centerdot y \approx c.$$

These formulas are of particular interest for us, see Section 6. $\square$

We say that a corroborator of a disjunction φ is *partisan*, if it corroborates some disjunct of φ. The main result of this section is the following theorem.

Theorem 6 (Partisan Corroboration Theorem) *Every corroborator of a disjunction of constant-disjoint guarded Horn formulas is partisan.*

Proof. Let $\varphi = \bigvee_{i\in I} \varphi_i$ where all the φ_i's are constant-disjoint guarded Horn formulas. Let θ be a corroborator for φ. We must prove that θ corroborates φ_i for some $i \in I$.

We can assume (without loss of generality) that there exist positive integers m and n such that each φ_i has the following form:

$$\varphi_i \quad = \quad \underbrace{\bigwedge_{1\le k\le m} (E_i^k \Rightarrow s_i^k \approx t_i^k)}_{\psi_i} \quad \wedge \quad \bigwedge_{1\le k\le n} (D_i^k \Rightarrow u_i^k \approx v_i^k),$$

where ψ_i is a guard of φ_i, i.e., each E_i^k and s_i^k is ground and $FV(\varphi_i) = FV(\psi_i)$, for all $i \in I$. Let $C_i = \mathit{Con}(\varphi_i)$ for $i \in I$. We have that

$$C_i \cap C_j = \emptyset \quad (\forall i, j \in I,\ i \neq j). \tag{1}$$

Let $\Sigma = \Sigma(\varphi)$. For $i \in I$ let $\mathcal{K}_i$ denote the class of all Σ-structures that satisfy $\varphi_i\theta$, i.e,

$$\mathcal{K}_i = \{\, \Sigma\text{-structure } \mathfrak{A} \mid \mathfrak{A} \models \varphi_i\theta \,\}.$$

From the validity of $\varphi\theta$ follows that each Σ-structure belongs to some $\mathcal{K}_i$.

Let now J be any subset of I such that

$$\models \psi_i\theta \quad (\forall i \in J). \tag{2}$$

So

$$\mathit{Con}(\varphi_i\theta) = C_i \quad (\forall i \in J). \tag{3}$$

To see that, suppose (by contradiction) that $\mathit{Con}(\varphi_i\theta)$ contains some $c \notin C_i$. Clearly, c belongs to some $x\theta$ where x occurs in the guard ψ_i. By the second part of Lemma 3, every constant in $x\theta$ belongs to C_i. This gives the desired contradiction.

If $I = J$ then the theorem follows by (1), (3) and Lemma 2. Assume that $I \neq J$. Below we prove the following statement:

$$\text{If } \not\models \varphi_i\theta \text{ for all } i \in J \text{ then } \models \psi_i\theta \text{ for some } i \in I \setminus J. \tag{4}$$

Let now J be the *maximal* subset of I such that (2) holds. In other words, for all $i \in I \setminus J$, $\not\models \psi_i\theta$. By the contrapositive of (4) we conclude that for some $i \in J$, $\models \varphi_i\theta$ and the theorem follows.

Proof of (4) Assume $\not\models \varphi_i\theta$ for all $i \in J$. Form an equation set D as follows.

- If $J = \emptyset$ let $D = \emptyset$.
- If $J \neq \emptyset$ then there is for each $i \in J$ a clause in $\varphi_i\theta$ that is not valid and by (2) this clause is not in $\psi_i\theta$. In other words, there is a mapping $f : J \to \{1, 2, \ldots, n\}$ such that

$$\not\models (D_i^{f(i)} \Rightarrow u_i^{f(i)} \approx v_i^{f(i)})\theta \quad (\forall i \in J). \tag{5}$$

  Let f be fixed and let $D = \bigcup_{i \in J} D_i^{f(i)}\theta$.

For each mapping $g : I \setminus J \to \{1, 2, \ldots, m\}$ let E_g denote the following set of equations:

$$E_g = \bigcup_{i \in I \setminus J} E_i^{g(i)},$$

and let $\mathfrak{A}_g$ be the canonical model of $D \cup E_g$, i.e.,

$$\mathfrak{A}_g = \mathcal{T}_{/E_g \cup D}.$$

We will now prove the following statement.

(6) Fix $g : I \setminus J \to \{1, 2, \ldots, m\}$. There exists $i \in I \setminus J$ such that $\mathfrak{A}_g \in \mathcal{K}_i$.

Proof. Suppose, by contradiction, that (6) does not hold. (Assume also that $J \neq \emptyset$ or else (6) holds trivially.) Then $\mathfrak{A}_g \in \mathcal{K}_j$ for some $j \in J$. Fix such an appropriate j.

So $\mathfrak{A}_g$ satisfies each clause in $\varphi_j\theta$ and in particular

$$\mathfrak{A}_g \models (D_j^{f(j)} \Rightarrow u_j^{f(j)} \approx v_j^{f(j)})\theta.$$

Let $D' = D_j^{f(j)}\theta$, $u' = u_j^{f(j)}\theta$ and $v' = v_j^{f(j)}\theta$. By (3) follows that

$$Con(D', u', v') \subseteq C_j$$

and

$$\begin{aligned}
Con(E_g, D \setminus D') &= Con(E_g) \cup Con(D \setminus D') \\
&= Con(E_g) \cup \bigcup_{i \in J, i \neq j} Con(D_i^{f(i)}\theta) \\
&\subseteq \bigcup_{i \in I \setminus J} C_i \cup \bigcup_{i \in J, i \neq j} C_i \\
&= \bigcup_{i \in I, i \neq j} C_i.
\end{aligned}$$

So, by (1),

$$Con(D', u', v') \cap Con(E_g, D \setminus D') = \emptyset.$$

It follows, by Lemma 4, that

$$\models (D_j^{f(j)} \Rightarrow u_j^{f(j)} \approx v_j^{f(j)})\theta.$$

But this contradicts (5). ⊠

By using (6) we can now prove (4). Suppose, by contradiction, that there is no $i \in I \setminus J$ such that $\models \psi_i\theta$. Then there is for each $i \in I \setminus J$ a clause in $\psi_i\theta$ that is not valid, i.e., there is a mapping $g : I \setminus J \to \{1, 2, \ldots, m\}$ such that

$$\not\models E_i^{g(i)} \Rightarrow s_i^{g(i)} \approx (t_i^{g(i)}\theta) \quad (\forall i \in I \setminus J).$$

(Note that only the t_i's can be nonground.) Fix such an appropriate g.

By using (6) we know that $\mathfrak{A}_g \in \mathcal{K}_i$ for some $i \in I \setminus J$. Choose such an i. So $\mathfrak{A}_g$ satisfies each clause in $\varphi_i\theta$ and in particular

$$\mathfrak{A}_g \models E_i^{g(i)} \Rightarrow s_i^{g(i)} \approx (t_i^{g(i)}\theta).$$

But, by (3) and (1),

$$Con(E_i^{g(i)}, s_i^{g(i)}) \cap Con(E_g \setminus E_i^{g(i)}, D) = \emptyset.$$

Hence, by Lemma 4,

$$\models E_i^{g(i)} \Rightarrow s_i^{g(i)} \approx (t_i^{g(i)}\theta),$$

which contradicts our choice of g. ⊠

Remark Theorem 6, as well as its proof, remain correct if the disjunction is infinite. We will not use this generalization.

The following example illustrates why the conditions of being constant-disjoint and guarded are important and cannot in general be discarded. In each case there is a counterexample to the theorem.

Example 7 Let us first consider an example where the disjuncts are guarded but not constant-disjoint. Let $\varphi(x)$ be the following guarded Horn formula:

$$(c \approx 0 \Rightarrow x \approx 1) \wedge (c \approx 1 \Rightarrow x \approx 0)$$

where c, 0 and 1 are constants, and let $\varphi_1 = \varphi(x_1)$, $\varphi_0 = \varphi(x_0)$ and $\psi = \varphi_1 \vee \varphi_0$ where x_1 and x_0 are distinct variables. Consider now any ground substitution θ such that $\theta(x_1) = 1$ and $\theta(x_0) = 0$. It is easy to show by case analysis that θ corroborates ψ, i.e., that

$$\begin{aligned} \models \quad & ((c \approx 0 \Rightarrow 1 \approx 1) \wedge (c \approx 1 \Rightarrow 1 \approx 0)) \vee \\ & ((c \approx 0 \Rightarrow 0 \approx 1) \wedge (c \approx 1 \Rightarrow 0 \approx 0)). \end{aligned}$$

However, θ corroborates neither φ_1 nor φ_0.

Let us now consider the case when constant-disjointness is not violated but the disjuncts are not guarded. Let $\varphi_1(y, x_1, y_1)$ be the formula

$$((y \approx 0 \Rightarrow x_1 \approx y_1) \wedge (y \approx y_1 \Rightarrow x_1 \approx 0))$$

and let $\varphi_0(x_0, y_0)$ be the formula

$$((c \approx y_0 \Rightarrow x_0 \approx 1) \wedge (c \approx 1 \Rightarrow x_0 \approx y_0))$$

where c, 0 and 1 are constants and x_1, x_0, y_1, y_0, y distinct variables. Let $\psi = \varphi_1 \vee \varphi_0$. Let θ be a ground substitution such that $\theta(x_1) = 1$, $\theta(x_0) = 0$, $\theta(y) = c$, $\theta(y_1) = 1$ and $\theta(y_0) = 0$. Then $\models \psi\theta$ but $\not\models \varphi_1\theta$ and $\not\models \varphi_0\theta$ (the situation is exactly the same as in the previous case). $\square$

4 From corroboration to m-corroboration

As Degtyarev & Voronkov [1995] have shown, the corroboration problem is undecidable. Shortly after, Voda & Komara [1995] have shown that m-corroboration is undecidable for all multiplicities m. We show that the latter result follows easily from the former result by using the Partisan Corroboration Theorem.

Theorem 8 (Degtyarev–Voronkov) *Corroboration of guarded Horn formulas is undecidable.*

For technical reasons it will be convenient to assume in the following that we have a fixed signature Σ with $\{c_1, c_2, \ldots\}$ as the set of distinct constants in it. Σ may also have other function symbols of arity ≥ 1. Let us also be precise about the variables that we allow in Σ-expressions, by assuming that all variables come from the collection $\{x_1, x_2, \ldots\}$.

For each natural number n, constant c and variable x, let $c^{(n)}$ denote a new constant and let $x^{(n)}$ denote a new variable. We define by induction on any Σ-expression X the corresponding expression $X^{(n)}$ as the one obtained from X by replacing in it each variable x with $x^{(n)}$ and each constant c with $c^{(n)}$. For any substitution θ of Σ-variables with Σ-terms we let $\theta^{(n)}$ denote a substitution that takes the variable $x^{(n)}$ to the term $(x\theta)^{(n)}$. So, for any Σ-expression X and natural number n,

$$(X\theta)^{(n)} = X^{(n)}\theta^{(n)}.$$

The following property is immediate. For any Σ-sentence φ and natural number n,

$$\models \varphi \quad \Leftrightarrow \quad \models \varphi^{(n)}.$$

Theorem 9 *Let φ be a guarded Horn formula and n a positive integer. Then φ has a corroborator if and only if $\bigwedge_{i=1}^{n} \varphi^{(i)}$ has an n-corroborator.*

Proof. The '$\Rightarrow$' direction is immediate. We prove the '$\Leftarrow$' direction as follows. Let $I = \{1, 2, \ldots, n\}$ and let ψ be the formula $\bigwedge_{i \in I} \varphi^{(i)}$. Assume that ψ has an n-corroborator $\{\, \theta_i \mid i \in I \,\}$. So

$$\models \bigvee_{i \in I} (\varphi^{(1)}\theta_i \wedge \cdots \wedge \varphi^{(i)}\theta_i \wedge \cdots \wedge \varphi^{(n)}\theta_i).$$

By using the distributive laws we can costruct an equivalent formula in conjunctive normal form, including as one of the conjuncts the formula $\bigvee_{i \in I} \varphi^{(i)}\theta_i$. Hence

$$\models \bigvee_{i \in I} \varphi^{(i)}\theta_i.$$

Let $X_i = FV(\varphi^{(i)})$ for $i \in I$. Since all the X_i's are pairwise disjoint we can let θ' be a substitution such that $\theta' \restriction X_i = \theta_i \restriction X_i$ for $i \in I$, and it follows that

$$\models \bigvee_{i \in I} \varphi^{(i)}\theta'.$$

From the Partisan Corroboration Theorem 6 follows now that $\models \varphi^{(i)}\theta'$ for some $i \in I$. Fix such an appropriate i. But then, by Lemma 3, the range of $\theta' \restriction X_i$ is $\mathcal{T}_{\Sigma(\varphi^{(i)})}$, and thus there is a substitution θ with range $\mathcal{T}_\Sigma$ such that $\theta^{(i)} \restriction X_i = \theta' \restriction X_i$. Hence $\models \varphi^{(i)}\theta^{(i)}$ and so $\models \varphi\theta$. $\boxtimes$

Theorem 10 (Voda–Komara) *For all $n \geq 1$, n-corroboration is undecidable.*

Proof. The reduction in Theorem 9 is trivially effective. So, if we had a decision procedure (for some n) for finding n-corroborators, we could use it to find corroborators, but this would contradict Theorem 8. ⊠

Assume that we are using an automated theorem proving method that is based on the Herbrand theorem. Roughly, this involves a search for terms, for a given multiplicity m. Voda–Komara theorem tells us that there is no m for which we could effectively decide when to stop our search for such terms in case they do not exist.

By using the fact that SREU is undecidable with ground left-hand sides [Plaisted 1995], (i.e., variables occur only in positive literals in the corresponding Horn formulas), and already in the guarded case with two variables [Veanes 1996], we can sharpen the Voda-Komara theorem as follows.

Corollary 11 *For all $n \geq 1$, n-corroboration is undecidable for guarded Horn formulas with $2n$ variables and ground negative literals.*

By a *monadic* signature or language we mean a signature or language where all function symbols have arity at most one. By *monadic* SREU or corroboration we understand the restriction of that decision problem to monadic languages. The decidability of monadic SREU is currently one of the difficult open problems related to SREU [Gurevich & Voronkov 1997]. An effectively equivalent problem is the decidability of the prenex fragment of intuitionistic logic with equality in monadic languages [Degtyarev & Voronkov 1996*a*]. Some evidence speaks in favor of that the problem is decidable although with very high computational complexity (e.g., many subcases are decidable, see Section 8). From Theorem 9 follows that:

Corollary 12 *If monadic corroboration is undecidable, then so is monadic n-corroboration for any $n > 1$, or equivalently, if monadic n-corroboration is decidable for some $n > 1$ then so is monadic corroboration.*

5 Shifted pairing with finite tree automata

Shifted pairing is a general technique for proving undecidability results. The term shifted pairing was introduced by Plaisted [1995]. A variant of shifted pairing was used already by Hopcroft & Ullman [1979] in establishing the undecidability of the problem of testing nonemptiness of the intersection of two context free languages. Goldfarb's [1981] proof of the undecidability of

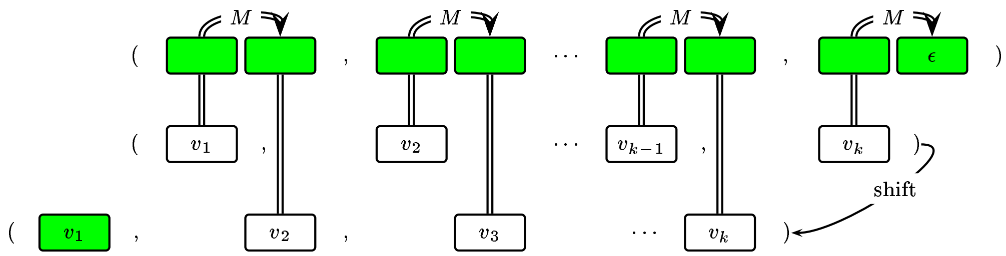


Figure 1: Shifted pairing.

second-order unification uses also similar ideas. Finite tree automata provide a suitable abstraction level for our purposes, for formalizing this technique as a decision problem of finite tree automata. The shifted pairing technique is illustrated in Figure 1. The main result of this section is the Shifted Pairing theorem. In this section we use a binary function symbol '$\centerdot$', and we write it for better readability using infix notation and assume that it associates to the right. For example, if t_1, t_2 and t_3 are terms, then the term $\centerdot(t_1, \centerdot(t_2, t_3))$ is written unambiguously as $t_1 \centerdot t_2 \centerdot t_3$.

Theorem 13 (Shifted Pairing Theorem) *One can effectively construct two constant-disjoint tree automata*

$$\mathcal{A}_{\mathrm{mv}} = (\mathcal{Q}_{\mathrm{mv}}, \Sigma_{\mathrm{mv}}, \mathcal{R}_{\mathrm{mv}}, \{q_{\mathrm{mv}}\}), \quad \mathcal{A}_{\mathrm{id}} = (\mathcal{Q}_{\mathrm{id}}, \Sigma_{\mathrm{id}}, \mathcal{R}_{\mathrm{id}}, \{q_{\mathrm{id}}\}),$$

and two ground and canonical rewrite systems

$$\Pi_1 \subseteq \mathcal{T}_{\Sigma_{\mathrm{mv}}} \times \mathcal{T}_{\Sigma_{\mathrm{id}}}, \quad \Pi_2 \subseteq \mathcal{T}_{\Sigma_{\mathrm{mv}}} \times \mathcal{T}_{\Sigma_{\mathrm{id}}},$$

such that, it is undecidable whether, given $t_0 \in \mathcal{T}_{\Sigma_{\mathrm{id}}}$, there exists $s \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ and $t \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$ such that $s \xrightarrow{*}_{\Pi_1} t$ and $t_0 \centerdot s \xrightarrow{*}_{\Pi_2} t$, where $\centerdot \in \Sigma_{\mathrm{mv}}$.

The rest of this section is devoted to the proof of the Shifted Pairing Theorem.

We consider a fixed deterministic Turing machine M with *initial state* q_0, *final state* q_{f}, a *blank* symbol $\sqcup$. By $\Sigma(M)$ we denote the union of the states and tape symbols of M including the blank symbol. All characters in $\Sigma(M)$ are considered to be *constants*. Moreover, M is only allowed to write a blank when it erases the *last* nonblank symbol on the tape. This means that IDs do not include blanks. However, overwriting the last nonblank symbol on the tape by a blank, means erasing of the last input symbol on the tape. For such a TM M we can assume, without loss of generality, that when M enters the final state then its tape is empty. Given an ID v, we let v^+ denote the following string:

$$v^+ = \begin{cases} \text{successor of } v, & \text{if } v \text{ is nonfinal;} \\ \epsilon, & \text{otherwise.} \end{cases}$$

Note that the final ID of M is the unique one character string q_f and $q_\mathrm{f}^+ = \epsilon$.

5.1 Words and trains

Here we use certain *nonmonadic* terms to represent strings, we call such terms *words*. Similarly, we use certain terms, that we call *trains*, to represent *sequences of strings*. Let c_w and c_t be two distinct constants not in $\Sigma(M)$.

- A term s is called a c_w*-word* if either s is the constant c_w, or s is the term $c \centerdot s'$ for some constant c and c_w-word s'. The *empty* c_w*-word* is simply the constant c_w.
- A term t is called a c_t*-train of* c_w*-words* if either t is the constant c_t, or t is the term $s \centerdot t'$ for some c_w-word s and c_t-train t'. The *empty* c_t*-train* is simply the constant c_t.

We use the following convenient notation for words and trains. A c_w-word

$$c_1 \centerdot c_2 \centerdot \cdots \centerdot c_n \centerdot c_\mathrm{w}$$

is written simply as

$$c_1 c_2 \cdots c_n \centerdot c_\mathrm{w}$$

and is said to *represent* the string $c_1 c_2 \ldots c_n$. When we say that a c_w*-word is in a set* V *of strings*, we mean that the string represented by that c_w-word is in V.

Similarly, a c_t-train

$$(v_1 \centerdot c_\mathrm{w}) \centerdot (v_2 \centerdot c_\mathrm{w}) \centerdot \cdots \centerdot (v_n \centerdot c_\mathrm{w}) \centerdot c_\mathrm{t}$$

is said to *represent* the string sequence

$$(v_1, v_2, \ldots, v_n).$$

In this way one can of course easily represent arbitrary regular sets of strings by corresponding regular forests of words. We use this fact in the Train Lemma, that is our key tool in constructing the two tree automata $\mathcal{A}_\mathrm{mv}$ and $\mathcal{A}_\mathrm{id}$.

Lemma 14 (Train Lemma) *Let* V *be a regular set of strings over a signature* Σ *of constants. Let* c_t *and* c_w *be distinct constants not in* Σ*. Then the set of all* c_t*-trains of* c_w*-words in* V *is recognized by a DTA with one final state.*

Proof. To begin with, let $\mathcal{A}_1 = (\mathcal{Q}_1, \Sigma \cup \{\centerdot, c_\mathrm{w}\}, \mathcal{R}_1, \mathcal{Q}_1^\mathrm{f})$ be a DTA that recognizes the set of all c_w-words in V. Next, let p be a new state and let

$$\mathcal{A} = (\mathcal{Q}_1 \cup \{p\}, \Sigma \cup \{\centerdot, c_\mathrm{w}, c_\mathrm{t}\}, \mathcal{R}, \{p\})$$

where

$$\mathcal{R} = \mathcal{R}_1 \cup \{c_\mathrm{t} \to p\} \cup \{\, q \centerdot p \to p \mid q \in \mathcal{Q}_1^\mathrm{f} \,\}.$$

We prove that $\mathcal{A}$ is a DTA satisfying the claim. Clearly, it *is* a DTA. The rest follows from the equivalence of the following statements for all terms t.

(7) $t \in \mathcal{F}(\mathcal{A})$

(8) t is a term over $\Sigma \cup \{\centerdot, c_\mathrm{w}, c_\mathrm{t}\}$ and $t \xrightarrow{*}_{\mathcal{R}} p$

(9) t is a term over $\Sigma \cup \{\centerdot, c_\mathrm{w}, c_\mathrm{t}\}$ and there exist states $q_1, q_2, \dots, q_n \in \mathcal{Q}_1^\mathrm{f}$, $n \geq 0$, such that

$$t \xrightarrow{*}_{\mathcal{R}_1} q_1 \centerdot q_2 \centerdot \cdots \centerdot q_n \centerdot c_\mathrm{t} \longrightarrow_{\{c_\mathrm{t} \to p\}} q_1 \centerdot q_2 \centerdot \cdots \centerdot q_n \centerdot p \xrightarrow{*}_{\{\, q \centerdot p \to p \mid q \in \mathcal{Q}_1^\mathrm{f} \,\}} p$$

(10) there exist terms $s_1, s_2 \dots, s_n \in \mathcal{F}(\mathcal{A}_1)$, $n \geq 0$, such that $t = s_1 \centerdot s_2 \centerdot \cdots \centerdot s_n \centerdot c_\mathrm{t}$

(11) t is a c_t-train of c_w-words in V.

We show only the implication (8)$\Rightarrow$(9). All the other cases are immediate consequences of the involved definitions. Assume (8). The only rules in $\mathcal{R}$ that involve p are the ones $q \centerdot p \to p$ for $q \in \mathcal{Q}_1^\mathrm{f}$ and the rule $c_\mathrm{t} \to p$.

Hence, any reduction of t in $\mathcal{R}$ to p is either, by induction on the number of rewrite steps in reductions,

1. the rewrite step $t \longrightarrow_{c_\mathrm{t} \to p} p$, and thus $t = c_\mathrm{t}$ and obviously (9) holds,
2. or else a reduction $t \xrightarrow{*}_{\mathcal{R}} q \centerdot p \longrightarrow_{q \centerdot p \to p} p$, for some $q \in \mathcal{Q}_1^\mathrm{f}$. In this case t must be a term $s \centerdot t'$ where $s \xrightarrow{*}_{\mathcal{R}} q$ and $t' \xrightarrow{*}_{\mathcal{R}} p$. But if $s \xrightarrow{*}_{\mathcal{R}} q$ then obviously $s \xrightarrow{*}_{\mathcal{R}_1} q$. Hence $t \xrightarrow{*}_{\mathcal{R}_1} q \centerdot t'$ and (9) follows from the induction hypothesis.

$\boxtimes$

The set of all IDs of M is obviously a regular set of strings.

- A *train of IDs* is a c_t-train of c_w-words representing IDs of M.

The following statement is an immediate consequence of Lemma 14.

(12) *There is a DTA $\mathcal{A}_\mathrm{id} = (\mathcal{Q}_\mathrm{id}, \Sigma_\mathrm{id}, \mathcal{R}_\mathrm{id}, \{q_\mathrm{id}\})$ that recognizes the set of all trains of IDs, where $\Sigma_\mathrm{id} = \Sigma(M) \cup \{\centerdot, c_\mathrm{w}, c_\mathrm{t}\}$.*

5.2 Trains of moves

We now want to represent moves of M in such a way that we can obtain a statement corresponding to (12), but for *moves*. First of all, for technical reasons that are relevant for constant-disjointness of the finite tree automata in Theorem 13, we use a new constant c'_{w} for the empty word and a new constant c'_{t} for the empty train. A naive representation of a move (v, v^+) as the term $(v \centerdot c'_{\mathrm{w}}) \centerdot (v^+ \centerdot c'_{\mathrm{w}})$ does of course not work for several reasons, to mention one: such terms are not recognizable.

Instead, we use the fact that, in a move (v, v^+), the number of symbols in v is either equal to the length of v^+, or it is one less than the length of v^+ (since M can write a new symbol at the end), or one more than the length of v^+ (since M can erase the last tape symbol). Moreover, only a finite substring of an ID is altered by a move. We encode moves by strings of new characters where the i'th character encodes the i'th characters in the components of the move. We now proceed with the formal construction.

Two new constants, denoted by $\langle a, b\rangle$ and $\langle a, b\rangle'$, respectively, are introduced for every pair of constants a and b in $\Sigma(M)$. All these new constants are assumed to be pairwise distinct. Let v be any ID of M and v^+ its successor, say

$$\begin{array}{rcl} v & = & a_1 a_2 \cdots a_m, \\ v^+ & = & b_1 b_2 \cdots b_n. \end{array}$$

Note that $m \geq 1$ and $m-1 \leq n \leq m+1$. The only case when $n = 0$ is when v is the final ID q_{f}. We define $\langle v, v^+\rangle$ as the following string.

$$\langle v, v^+\rangle = \begin{cases} \langle a_1, b_1\rangle\langle a_2, b_2\rangle \cdots \langle a_{n-1}, b_{n-1}\rangle\langle \sqcup, b_n\rangle', & \text{if } m = n-1; \\ \langle a_1, b_1\rangle\langle a_2, b_2\rangle \cdots \langle a_{m-1}, b_{m-1}\rangle\langle a_m, \sqcup\rangle', & \text{if } m = n+1; \\ \langle a_1, b_1\rangle\langle a_2, b_2\rangle \cdots \langle a_{m-1}, b_{m-1}\rangle\langle a_m, b_m\rangle', & \text{if } m = n. \end{cases}$$

we call such a string a *move* also. Intuitively, a blank is added at the end of the shorter of the two strings of a move (in case they differ in length) and the pair of the resulting strings is encoded character by character.

- A *train of moves* is a c'_{t}-train of c'_{w}-words that represent moves.

(13) *There is a DTA $\mathcal{A}_{\mathrm{mv}} = (\mathcal{Q}_{\mathrm{mv}}, \Sigma_{\mathrm{mv}}, \mathcal{R}_{\mathrm{mv}}, \{q_{\mathrm{mv}}\})$ that recognizes the set of all trains of moves, where*

$$\Sigma_{\mathrm{mv}} = \{\, \langle a, b\rangle, \langle a, b\rangle' \mid a, b \in \Sigma(M)\,\} \cup \{\centerdot, c'_{\mathrm{w}}, c'_{\mathrm{t}}\}.$$

Proof. The set of moves is easily seen to be a regular set. For example, the set of all moves corresponding to computation steps that do not change the last tape symbol can be described by the following regular set of strings:

$$V^* V_\delta V^* V'$$

where V_δ is is a certain finite set of three-character or two-character strings constructed from the transition function of M, e.g., if M upon reading the symbol a in state q writes the symbol a', moves right, and enters state q', then $\langle q, a'\rangle\langle a, q'\rangle$ is in V_δ. The set V consists all constants $\langle a, a\rangle$ such that a is an input symbol of M, and V' is the set of all constants $\langle a, a\rangle'$ such that a is an input symbol of M. The other cases are similar. The claim follows now from the Train Lemma 14. ⊠

At this point let $\mathcal{A}_{\mathrm{id}}$ and $\mathcal{A}_{\mathrm{mv}}$ be fixed constant-disjoint DTAs given by (12) and (13).

5.3 Main construction

Given a nonempty train t of moves, say

$$t = (\langle v_1, v_1^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot (\langle v_2, v_2^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot \cdots \centerdot (\langle v_{k-1}, v_{k-1}^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot (\langle v_k, v_k^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot c'_{\mathrm{t}}$$

define the *first projection* of t as the following train of IDs

$$\pi_1(t) = (v_1 \centerdot c_{\mathrm{w}}) \centerdot (v_2 \centerdot c_{\mathrm{w}}) \centerdot \cdots \centerdot (v_{k-1} \centerdot c_{\mathrm{w}}) \centerdot (v_k \centerdot c_{\mathrm{w}}) \centerdot c_{\mathrm{t}}$$

and the *second projection* of t as the following train

$$\pi_2(t) = \begin{cases} (v_1^+ \centerdot c_{\mathrm{w}}) \centerdot (v_2^+ \centerdot c_{\mathrm{w}}) \centerdot \cdots \centerdot (v_{k-1}^+ \centerdot c_{\mathrm{w}}) \centerdot c_{\mathrm{t}}, & \text{if } v_k = q_{\mathrm{f}}; \\ (v_1^+ \centerdot c_{\mathrm{w}}) \centerdot (v_2^+ \centerdot c_{\mathrm{w}}) \centerdot \cdots \centerdot (v_{k-1}^+ \centerdot c_{\mathrm{w}}) \centerdot (v_k^+ \centerdot c_{\mathrm{w}}) \centerdot c_{\mathrm{t}}, & \text{otherwise.} \end{cases}$$

Note that the purpose of taking the second-projection is twofold:

1. to check that the first component of the last move is the final ID, and
2. to return the train consisting of the second components of all the moves.

We say that t is the *shifted pairing* of its first projection if

$$\pi_1(t) = (v_1 \centerdot c_{\mathrm{w}}) \centerdot \pi_2(t)$$

and we refer to v_1 as the *first ID* of t. Recall that q_0 is the initial state of M.

Lemma 15 *Let v_0 be an input string for M. Then M accepts v_0 if and only if there exists a train t of moves with first ID q_0v_0, such that t is the shifted pairing of its first projection.*

Proof. Let v_0 be given and t a train of moves as above, with $v_1 = q_0v_0$. The first projection of t represents the ID sequence

$$(v_1, v_2, \ldots, v_{k-1}, v_k),$$

and, if $v_k = q_\mathrm{f}$ then the second projection of t represents

$$(v_1^+, v_2^+, \ldots, v_{k-1}^+).$$

To say that t is a shifted pairing of it first projection means that $v_k = q_\mathrm{f}$ and

$$\begin{array}{ccccccccccccc} (& v_1 & , & v_2 & , & v_3 & , & \ldots & , & v_{k-1} & , & v_k &) \ = \\ (& q_0v_0 & , & v_1^+ & , & v_2^+ & , & \ldots & , & v_{k-2}^+ & , & v_{k-1}^+ &), \end{array}$$

which is tantamount to saying that the first projection of t represents a valid computation of M with input v_0, i.e., M accepts v_0. The proof of the converse direction is similar. $\boxtimes$

5.3.1 The rewrite systems Π_1 and Π_2

The system Π_1 contains all the following rules:

(14) For all $a, b \in \Sigma(M)$, the rule $\langle a, b\rangle \to a$.

(15) For all $a, b \in \Sigma(M)$ such that $a \neq \sqcup$, the rule $\langle a, b\rangle' \centerdot c'_\mathrm{w} \to a \centerdot c_\mathrm{w}$.

(16) For all $b \in \Sigma(M)$, the rule $\langle \sqcup, b\rangle' \centerdot c'_\mathrm{w} \to c_\mathrm{w}$.

(17) The rule $c'_\mathrm{t} \to c_\mathrm{t}$.

We conclude the following, by first observing from (14)–(17) that Π_1 is reduced.

(18) *The rewrite system Π_1 is canonical and $\Pi_1 \subseteq \mathcal{T}_{\Sigma_\mathrm{mv}} \times \mathcal{T}_{\Sigma_\mathrm{id}}$*

We therefore have the following relationship between Π_1 and the notion of first projection of a train of moves.

Lemma 16 *For all trains s of moves and all trains t of IDs, $s \xrightarrow{*}_{\Pi_1} t$ if and only if $t = \pi_1(s)$.*

Proof. Let s and t be given. By (18) t is irreducible in Π_1 because Σ_{mv} and Σ_{id} do not have any constants in common, and thus $s \xrightarrow{*}_{\Pi_1} t$ if and only if $s{\downarrow_{\Pi_1}} = t$. It remains to check that $s{\downarrow_{\Pi_1}} = \pi_1(s)$, which is straightforward. ⊠

The system Π_2 contains all the following rules:

(19) For all $a, b \in \Sigma(M)$, the rule $\langle a, b\rangle \to b$.

(20) For all $a, b \in \Sigma(M)$ such that $b \neq \sqcup$, the rule $\langle a, b\rangle' \centerdot c'_{\mathrm{w}} \to b \centerdot c_{\mathrm{w}}$.

(21) For all $a \in \Sigma(M)$ such that $a \neq q_{\mathrm{f}}$, the rule $\langle a, \sqcup\rangle' \centerdot c'_{\mathrm{w}} \to c_{\mathrm{w}}$.

(22) The rule $(\langle q_{\mathrm{f}}, \sqcup\rangle' \centerdot c'_{\mathrm{w}}) \centerdot c'_{\mathrm{t}} \to c_{\mathrm{t}}$.

Again, we conclude the following, by first observing from (19)–(22) that Π_2 is reduced.

(23) *The rewrite system* Π_2 *is canonical and* $\Pi_2 \subseteq \mathcal{T}_{\Sigma_{\mathrm{mv}}} \times \mathcal{T}_{\Sigma_{\mathrm{id}}}$

We have also a similar relationship between Π_2 and the second projection of a train of moves, that implies the following.

Lemma 17 *For all trains s of moves and all IDs v, $(v \centerdot c_{\mathrm{w}}) \centerdot s \xrightarrow{*}_{\Pi_2} \pi_1(s)$ if and only if $\pi_1(s) = (v \centerdot c_{\mathrm{w}}) \centerdot \pi_2(s)$.*

Proof. Let s and v be given, say

$$s = (\langle v_1, v_1^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot (\langle v_2, v_2^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot \cdots \centerdot (\langle v_{k-1}, v_{k-1}^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot (\langle v_k, v_k^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot c'_{\mathrm{t}}.$$

So

$$\pi_1(s) = (v_1 \centerdot c_{\mathrm{w}}) \centerdot (v_2 \centerdot c_{\mathrm{w}}) \centerdot \cdots \centerdot (v_{k-1} \centerdot c_{\mathrm{w}}) \centerdot (v_k \centerdot c_{\mathrm{w}}) \centerdot c_{\mathrm{t}}.$$

⇒ Assume that $(v \centerdot c_{\mathrm{w}}) \centerdot s \xrightarrow{*}_{\Pi_2} \pi_1(s)$. This is possible only if, by (23),

$$v_1 = v, \quad (\langle v_i, v_i^+\rangle \centerdot c'_{\mathrm{w}}){\downarrow_{\Pi_2}} = v_{i+1} \centerdot c_{\mathrm{w}} \quad \text{for } 1 \leq i < k, \tag{24}$$

and

$$((\langle v_k, v_k^+\rangle \centerdot c'_{\mathrm{w}}) \centerdot c'_{\mathrm{t}}){\downarrow_{\Pi_2}} = c_{\mathrm{t}}. \tag{25}$$

(25) is possible only if $v_k = q_{\mathrm{f}}$ by using the rule in (22). In (24) only the rules in (19)–(21) can be used and these imply that $v_i^+ = v_{i+1}$ for $1 \leq i < k$. The rest is obvious.

⇐ Assume that $\pi_1(s) = (v \centerdot c_{\mathrm{w}}) \centerdot \pi_2(s)$. Then $v = v_1$, $v_i^+ = v_{i+1}$ for $1 \leq i < k$, and $v_k = q_{\mathrm{f}}$. (24) and (25) follow easily. The rest is obvious.

⊠

5.3.2 Proof of the Shifted Pairing Theorem

Proof. Let M in the above construction be a universal TM. Then the claim in Theorem 13 is a consequence of the equivalence of the following statements. The additional conditions on the rewrite systems Π_1 and Π_2 follow from (18) and (23).

(26) M accepts v_0.

(27) There exists $s \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ such that $\pi_1(s) = (q_0 v_0 \centerdot c_{\mathrm{w}}) \centerdot \pi_2(s)$.

(28) There exists $s \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ such that $(q_0 v_0 \centerdot c_{\mathrm{w}}) \centerdot s \xrightarrow{*}_{\Pi_2} \pi_1(s)$.

(29) There exist $s \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ and $t \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$, such that $s \xrightarrow{*}_{\Pi_1} t$ and $(q_0 v_0 \centerdot c_{\mathrm{w}}) \centerdot s \xrightarrow{*}_{\Pi_2} t$.

(26)$\Leftrightarrow$(27) By Lemma 15 and (13).

(27)$\Leftrightarrow$(28) By Lemma 17.

(28)$\Leftrightarrow$(29) By Lemma 16 and (12).

$\boxtimes$

6 Applications of Partisan Corroboration and Shifted Pairing Theorems

The Shifted Pairing Theorem is used here to give a very elementary undecidability proof of SREU. The latter result is then used, in combination with the Partisan Corroboration Theorem to improve upon the undecidability result of n-corroboration for arbitrary n.

6.1 Undecidability of SREU: minimal case

Consider fixed constant-disjoint DTAs $\mathcal{A}_{\mathrm{mv}} = (\mathcal{Q}_{\mathrm{mv}}, \Sigma_{\mathrm{mv}}, \mathcal{R}_{\mathrm{mv}}, \{q_{\mathrm{mv}}\})$ and $\mathcal{A}_{\mathrm{id}} = (\mathcal{Q}_{\mathrm{id}}, \Sigma_{\mathrm{id}}, \mathcal{R}_{\mathrm{id}}, \{q_{\mathrm{id}}\})$, a binary function symbol f, and ground canonical rewrite systems Π_1 and Π_2 given by the Shifted Pairing Theorem 13. Let q be a new state and $\mathcal{A}$ the tree automaton $(\mathcal{Q}, \Sigma, \mathcal{R}, \mathcal{Q}^{\mathrm{f}})$, where

$$
\begin{aligned}
\mathcal{Q} &= \mathcal{Q}_{\mathrm{mv}} \cup \mathcal{Q}_{\mathrm{id}} \cup \{q\}, \\
\Sigma &= \Sigma_{\mathrm{mv}} \cup \Sigma_{\mathrm{id}}, \\
\mathcal{R} &= \mathcal{R}_{\mathrm{mv}} \cup \mathcal{R}_{\mathrm{id}} \cup \{f(q_{\mathrm{mv}}, q_{\mathrm{id}}) \to q\}, \\
\mathcal{Q}^{\mathrm{f}} &= \{q\}.
\end{aligned}
$$

Obviously, $\mathcal{A}$ is still a *deterministic* tree automaton, because $\mathcal{A}_{\mathrm{mv}}$ and $\mathcal{A}_{\mathrm{id}}$ are constant-disjoint and deterministic. We have the following property as a direct consequence of the constant-disjointness of $\mathcal{A}_{\mathrm{id}}$ and $\mathcal{A}_{\mathrm{mv}}$.

(30) *For all ground terms s and t, $f(s,t) \xrightarrow{*}_{\mathcal{R}} q$ if and only if $s \xrightarrow{*}_{\mathcal{R}_{\mathrm{mv}}} q_{\mathrm{mv}}$ and $t \xrightarrow{*}_{\mathcal{R}_{\mathrm{id}}} q_{\mathrm{id}}$.*

We can now prove the following result.

Theorem 18 *There is an integer n, such that SREU is undecidable under the following restrictions:*

(i) the left-hand sides are ground and have less than n symbols, and

(ii) there are at most two variables each occurring at most three times, and

(iii) there are at most three rigid equations.

Proof. Let $S_{t_0}(x,y)$ be the following system of rigid equations where the rewrite systems $\mathcal{R}$, Π_1 and Π_2 are considered as sets of equations and t_0 is a given ground term over Σ_{id}.

$$S_{t_0}(x,y) = \left\{ \begin{array}{lll} \mathcal{R} & \vdash^{\mathrm{r}} & f(x,y) \approx q \\ \Pi_1 & \vdash^{\mathrm{r}} & x \approx y \\ \Pi_2 & \vdash^{\mathrm{r}} & f(t_0,x) \approx y \end{array} \right.$$

First, we prove that the following statements are equivalent for all substitutions θ:

(31) θ solves $S_{t_0}(x,y)$

(32) i) $\mathcal{R} \models f(x\theta, y\theta) \approx q$, and

ii) $\Pi_1 \models x\theta \approx y\theta$ and $\Pi_2 \models f(t_0, x\theta) \approx y\theta$

(33) i) $f(x\theta, y\theta) \xrightarrow{*}_{\mathcal{R}} q$, and

ii) $x\theta{\downarrow}_{\Pi_1} = y\theta{\downarrow}_{\Pi_1}$ and $f(t_0, x\theta){\downarrow}_{\Pi_2} = y\theta{\downarrow}_{\Pi_2}$

(34) i) $x\theta \xrightarrow{*}_{\mathcal{R}_{\mathrm{mv}}} q_{\mathrm{mv}}$ and $y\theta \xrightarrow{*}_{\mathcal{R}_{\mathrm{id}}} q_{\mathrm{id}}$, and

ii) $x\theta{\downarrow}_{\Pi_1} = y\theta{\downarrow}_{\Pi_1}$ and $f(t_0, x\theta){\downarrow}_{\Pi_2} = y\theta{\downarrow}_{\Pi_2}$

(35) i) $x\theta \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ and $y\theta \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$, and

ii) $x\theta{\downarrow}_{\Pi_1} = y\theta{\downarrow}_{\Pi_1}$ and $f(t_0, x\theta){\downarrow}_{\Pi_2} = y\theta{\downarrow}_{\Pi_2}$

(36) i) $x\theta \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ and $y\theta \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$, and

ii) $x\theta \xrightarrow{*}_{\Pi_1} y\theta$ and $f(t_0, x\theta) \xrightarrow{*}_{\Pi_2} y\theta$

(31)⇔(32) By definition.

(32)⇔(33) The rewrite systems are canonical and q is irreducible in $\mathcal{R}$.

(33)⇔(34) By (30).

(34)⇔(35) Assume (34). (34)(i) implies that $x\theta \in \mathcal{T}_{\Sigma_{\mathrm{mv}} \cup \mathcal{Q}_{\mathrm{mv}}}$ and $y\theta \in \mathcal{T}_{\Sigma_{\mathrm{id}} \cup \mathcal{Q}_{\mathrm{id}}}$. But $x\theta$ cannot include constants from $\mathcal{Q}_{\mathrm{mv}}$ and $y\theta$ cannot include constants from $\mathcal{Q}_{\mathrm{id}}$, or else $x\theta{\downarrow_{\Pi_1}} \neq y\theta{\downarrow_{\Pi_1}}$ because the signature of Π_1 is included in $\Sigma_{\mathrm{id}} \cup \Sigma_{\mathrm{mv}}$. Hence $x\theta \in \mathcal{T}_{\Sigma_{\mathrm{mv}}}$ and $y\theta \in \mathcal{T}_{\Sigma_{\mathrm{id}}}$, and thus (35)(i) holds by (34)(i).

(35)⇔(36) The terms in $\mathcal{F}(\mathcal{A}_{\mathrm{id}})$ are irreducible with respect to Π_1 and Π_2, and $y\theta \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$.

We conclude that $S_{t_0}(x, y)$ is solvable if and only if there exists a term $s \in \mathcal{F}(\mathcal{A}_{\mathrm{mv}})$ and a term $t \in \mathcal{F}(\mathcal{A}_{\mathrm{id}})$ such that $s \xrightarrow{*}_{\Pi_1} t$ and $f(t_0, s) \xrightarrow{*}_{\Pi_2} t$. Hence, solvability of $S_{t_0}(x, y)$ is undecidable by Theorem 13. Consequently SREU is undecidable, and the restrictions (i)–(iii) follow as properties of $S_{t_0}(x, y)$, where n is any integer greater than the number of symbols in $\mathcal{R}$, Π_1 and Π_2.

the left-hand sides of the rigid equations in $S_{t_0}(x, y)$. ⊠

6.1.1 Undecidability proofs of SREU

Degtyarev & Voronkov's [1995] original proof of the undecidability of SREU was by reduction of Baaz's [1993] monadic semi-unification problem. This proof was followed by other proofs by Degtyarev & Voronkov, first by reducing second-order unification to SREU [1996*c*], and then by reducing Hilbert's tenth problem to SREU [1996*b*]. The undecidability of second-order unification was proved by Goldfarb [1981]. Plaisted [1995] reduced Post's Correspondence Problem to SREU. From his proof follows that SREU is undecidable already with ground left-hand sides. Veanes [1996] improved that construction by using the halting problem for Turing machines and showed that two variables and one binary function symbol is enough to obtain undecidability. Here we have shown that, in addition, already three rigid equations suffice for the undecidability.

6.2 Undecidability of m-corroboration: minimal case

Consider the above system $S_{t_0}(x, y)$ of rigid equations and let φ_{t_0} denote the corresponding guarded Horn formula:

$$\begin{aligned}&(\mathcal{R} \Rightarrow f(x, y) \approx q) \wedge \\ &(\Pi_1 \Rightarrow x \approx y) \wedge \\ &(\Pi_2 \Rightarrow f(t_o, x) \approx y).\end{aligned}$$

We have the following result.

Theorem 19 *For all $m \geq 1$, m-corroboration is undecidable already for guarded Horn formulas with ground negative literals, at most $2m$ variables, and at most $3m$ clauses.*

Proof. Let m and t_0 be given and construct the formula $\psi = \bigwedge_{1 \leq i \leq m} \varphi_{t_0}^{(i)}$. By Theorem 9, ψ has an m-corroborator if and only if φ_{t_0} has a corroborator. But corroboration of φ_{t_0}, given a term t_0, is undecidable by Theorem 18. ⊠

7 Relations to intuitionistic logic

The decision problems in intuitionistic logic have not been as thoroughly studied as the corresponding problems in classical logic [Börger, Grädel & Gurevich 1997]. In particular, new results about the *prenex fragment* of intuitionistic logic (i.e., closed prenex formulas that are intuitionistically provable), have been obtained recently by Degtyarev & Voronkov in [1996*b*, 1996*c*, 1996*a*] and Voronkov [1996]. Some of these results are:

1. Decidability, and in particular PSPACE-completeness, of the prenex fragment of intuitionistic logic *without* equality [Degtyarev & Voronkov 1996*a*].

2. Prenex fragment of intuitionistic logic *with* equality but *without* function symbols is PSPACE-complete [Degtyarev & Voronkov 1996*a*]. Decidability of this fragment was proved by Orevkov [1976].

3. Prenex fragment of intuitionistic logic with equality in the language with one unary function symbol is decidable [Degtyarev & Voronkov 1996*a*].

4. $\exists^*$-fragment of intuitionistic logic with equality is undecidable [Degtyarev & Voronkov 1996*b*, Degtyarev & Voronkov 1996*c*].

In some of the above results, the corresponding result has first been obtained for a fragment of SREU with similar restrictions. The undecidability of the $\exists^*$-fragment is improved by Veanes [1996] by showing that already the

5. $\exists\exists$-fragment of intuitionistic logic with equality is undecidable.

We can further improve the latter undecidability result.

Corollary 20 *There is an integer n such that the $\exists\exists$-fragment of intuitionistic logic with equality is undecidable already under the following restrictions:*

1. *The only connectives are $\wedge$ and at most three $\Rightarrow$'s.*
2. *The antecedents of all implications are ground and have less than n symbols.*

Proof. Given a system $S(\vec{x}) = \{ E_i \vdash^{\mathrm{r}} s_i \approx t_i \mid 1 \leq i \leq k \}$ of rigid equations, let $\varphi(\vec{x})$ be the following conjunction of implications:

$$\bigwedge_{1 \leq i \leq k} ((\bigwedge_{e \in E_i} e) \Rightarrow s_i \approx t_i).$$

It can be shown that $\exists\vec{x}\varphi(\vec{x})$ is provable in intuitionistic logic with equality if and only if $S(\vec{x})$ is solvable [Degtyarev & Voronkov 1996*c*]. Thus, the claim follows from Theorem 18. ⊠

In contrast, Degtyarev, Gurevich, Narendran, Veanes & Voronkov [1998*b*] have shown that the

6. $\forall^*\exists\forall^*$-fragment of intuitionistic logic with equality is decidable.

7.1 A remark about proof search in $\mathrm{LJ}^{\approx}$

Proof search in intuitionistic logic with equality is closely connected with SREU, and, unlike in the classical case, the handling of SREU is in fact *unavoidable* in that context [Voronkov 1996]. Voronkov [1996] considers a particular sequent calculus based proof system $\mathrm{LJ}^{\approx}$. In that context a *skeleton* is the structure of a derivation in $\mathrm{LJ}^{\approx}$, and *skeleton instantiation* is the problem of the existence of a derivation of a given formula with a given skeleton. SREU is in fact polynomial time equivalent to skeleton instantiation in $\mathrm{LJ}^{\approx}$ [Voronkov 1996]. We get the following result. (We refer the reader to [Voronkov 1996] for precise definitions.) Corollary 20 and Theorem 18 can be used to exhibit a *fixed* skeleton for which the skeleton instantiation problem in $\mathrm{LJ}^{\approx}$ is undecidable. This improves the undecidability of the skeleton instantiation problem in general [Voronkov 1996]. Such a skeleton is illustrated in Figure 2

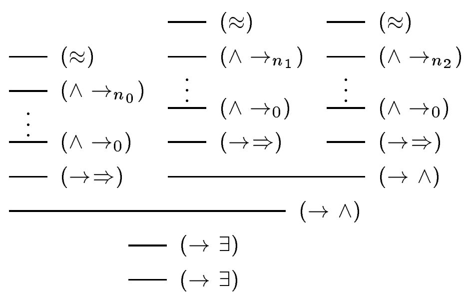


Figure 2: Any derivation in LJ$^{\approx}$ of the formula constructed from the system $S_{t_0}(x, y)$ of rigid equations in Theorem 18, has this skeleton for any t_0. The values of n_0, n_1, and n_2 are fixed integers corresponding to the number of equations in $\mathcal{R}$, Π_1, and Π_2, respectively.

7.2 Other fragments

Decidability problems for other fragments of intuitionistic logic have been studied by Orevkov in [1965, 1976], Mints [1967], Statman [1979], and Lifschitz [1967]. Orevkov [1965] proves that the $\neg\neg\forall\exists$-fragment of intuitionistic logic with function symbols is undecidable. Lifschitz [1967] proves that intuitionistic logic with equality and without function symbols is undecidable, i.e., that the pure constructive theory of equality is undecidable. Orevkov [1976] shows decidability of some fragments (that are close to the prenex fragment) of intuitionistic logic with equality. Statman [1979] proves that the intuitionistic propositional logic is PSPACE-complete.

8 Current status of SREU and open problems

Here we briefly summarize the current status of SREU and mention some open problems. Many related results are already mentioned above. The first decidability proof of rigid E-unification is given by Gallier, Narendran, Plaisted & Snyder [1988]. De Kogel [1995] has presented a simpler proof, without computational complexity considerations. We start with the **solved cases**:

- Rigid E-unification with ground left-hand side is NP-complete [Kozen 1981]. Rigid E-unification in general is NP-complete and there exist finite complete sets of unifiers [Gallier, Narendran, Plaisted & Snyder 1990, Gallier et al. 1988].

- Rigid E-unification with one variable, or, more generally, SREU with one variable and a *fixed* number of rigid equations is P-complete [Degtyarev et al. 1998*b*].

- If all function symbols have arity ≤ 1 (the *monadic* case) then it follows that SREU is PSPACE-hard [Goubault 1994]. If only one unary function symbol is allowed then the problem is decidable [Degtyarev, Matiyasevich & Voronkov 1996]. If only constants are allowed then the problem is NP-complete [Degtyarev, Matiyasevich & Voronkov 1996] assuming that there are at least two constants.

- About the monadic case it is known that if there are more than 1 unary function symbols then SREU is decidable if and only if it is decidable with just 2 unary function symbols [Degtyarev, Matiyasevich & Voronkov 1996].

- If the left-hand sides are ground then the monadic case is decidable [Gurevich & Voronkov 1997]. A more general problem is shown to be decidable in [Ganzinger et al. 1998]. Monadic SREU with one variable is PSPACE-complete [Gurevich & Voronkov 1997].

- The word equation solving [Makanin 1977], which is an extremely hard problem, can be reduced to monadic SREU [Degtyarev, Matiyasevich & Voronkov 1996].

- Monadic SREU is equivalent to a non-trivial extension of word equations [Gurevich & Voronkov 1997].

- Monadic SREU is equivalent to the decidability problem of the prenex fragment of intuitionistic logic with equality with function symbols of arity ≤ 1 [Degtyarev & Voronkov 1996*a*].

- In general SREU is undecidable [Degtyarev & Voronkov 1995]. Moreover, SREU is undecidable under the following restrictions:
  - The left-hand sides of the rigid equations are ground [Plaisted 1995].
  - Furthermore, there are only two variables [Veanes 1996] and three rigid equations with fixed ground left-hand sides.

- SREU with one variable is decidable, in fact EXPTIME-complete [Degtyarev et al. 1998*b*]. Moreover, SREU restricted to rigid equations that either contain one variable, or have a ground left-hand side and a right-hand side that is an equality between two variables, is decidable [Degtyarev, Gurevich, Narendran, Veanes & Voronkov 1998*a*].

- SREU is polynomial time equivalent with second-order unification [Levy 1998, Veanes 1998].

The **unsolved cases** are:

- Decidability of monadic SREU.
- Decidability of SREU with two rigid equations.

Both problems are highly non-trivial. An intriguing problem is also the corroboration problem with a given strategy. In particular, the following open problem is posed by Voronkov [1997]:

- Does there exist a computable strategy f with which the corroboration problem is decidable?

Further problems related to SREU and the Herbrand theorem are discussed in [Voronkov 1998*b*, Voronkov 1998*a*].

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