check the Lopatinski-Shapiro condition for the resulting system on the flat reference domain with an energy argument.

- Use a fixed point argument in $C^{2+\alpha,1+\frac{\alpha}{2}}$ which is non-trivial as the overall system is non-local. In this context ideas of Baconneau and Lunardi [1] are useful.

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Evolution of the Einstein equations on constant mean curvature surfaces

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(joint work with Vincent Moncrief)

This report is concerned with the Einstein equations

$$R_{ab} - \tfrac{1}{2} R g_{ab} = \kappa T_{ab}, \tag{1}$$

where g_{ab} is a pseudo-Riemannian metric on a four-dimensional smooth manifold (spacetime), R_{ab} is its Ricci tensor (with respect to the Levi-Civita connection), and $R = g^{ab} R_{ab}$ is the scalar curvature. On the right-hand side, T_{ab} is the energy-momentum tensor describing the matter content of spacetime.

Here we are interested in solutions to (1) describing *isolated systems*: a compact source surrounded by an asymptotically flat vacuum spacetime. It is useful to introduce [1] a conformally related metric $\tilde{g}_{ab}$ via

$$g_{ab} = \Omega^{-2}\tilde{g}_{ab}. \tag{2}$$

In suitably compactified coordinates, spacetime occupies a finite region, $\tilde{g}_{ab}$ is finite everywhere with respect to these coordinates, and the conformal factor Ω vanishes on the conformal boundary. This can be illustrated by a *Penrose diagram* (Fig. 1).

The standard approach to solving (1) numerically is to foliate spacetime into spacelike hypersurfaces approaching spacelike infinity i^0 (left panel of Fig. 1). These are truncated at a finite distance, where suitable boundary conditions must be imposed such that the resulting initial-boundary value problem is well posed and, ideally, spurious reflections of gravitational radiation are avoided. However, gravitational radiation is only defined unambiguously at future null infinity $\mathscr{I}^+$. Thus it would be very desirable to include $\mathscr{I}^+$ in the computational domain. We do this by considering instead *hyperboloidal* slices that are everywhere spacelike but approach $\mathscr{I}^+$ (right panel of Fig. 1). More specifically, we choose hypersurfaces that have constant mean curvature; in the following $K > 0$.

Throughout we work with the conformal metric in compactified coordinates. Instead of Friedrich's regular conformal field equations [2], we work directly with (a $3+1$ reduction of) the Einstein equations, mainly because these are the equations that numerical relativists have more experience with. The drawback of this approach is that the equations contain terms with negative powers of the conformal factor Ω that become singular at $\mathscr{I}^+$.

In particular, the evolution equation for the traceless part of the momentum π^{ij} conjugate to the induced spatial conformal metric γ_{ij} on the $t = \text{const}$ slices takes the form

$$\partial_t \pi^{\text{tr}\, ij} = -2\tilde{N}\Omega^{-1}(\tfrac{1}{3}K\pi^{\text{tr}\, ij} + \mu_\gamma \text{Hess}\,\Omega^{\text{tr}\, ij}) + (\text{regular}), \tag{3}$$

where tr denotes the tracefree part with respect to γ_{ij}, $\tilde{N}$ is the conformal lapse function, $\mu_\gamma = \sqrt{\det \gamma_{ij}}$, and Hess denotes the Hessian with respect to γ_{ij}. The constraint equations that hold within the $t = \text{const}$ slices are also formally singular at $\mathscr{I}^+$. For instance, the Hamiltonian constraint reads

$$-4\Omega\gamma^{ij}\tilde{\nabla}_i\tilde{\nabla}_j\Omega + 6\gamma^{ij}\Omega_{,i}\Omega_{,j} - \Omega^2\tilde{R} - \tfrac{2}{3}K^2 + \Omega^2\mu_\gamma^{-2}\gamma_{ik}\gamma_{jl}\pi^{\text{tr}\, ij}\pi^{\text{tr}\, kl} = 0, \tag{4}$$

where $\tilde{\nabla}$ is the covariant derivative of γ_{ij} and $\tilde{R}$ is its Ricci scalar. This singular form of the elliptic constraint equations works in our favour here because we can use it in order to determine the leading-order behaviour of the fields near $\mathscr{I}^+$. This will then allow us to evaluate the formally singular terms in (3).

On a given spatial slice, we choose coordinates such that the conformal boundary is given by $x^1 \equiv r = r_+$, where r_+ is a constant. We expand the fields in finite Taylor series about $r = r_+$ and substitute the expansions in the singular elliptic equations. For example, the Hamiltonian constraint (4) yields expressions for the first three radial derivatives of Ω at $\mathscr{I}^+$. A similar procedure is applied to the

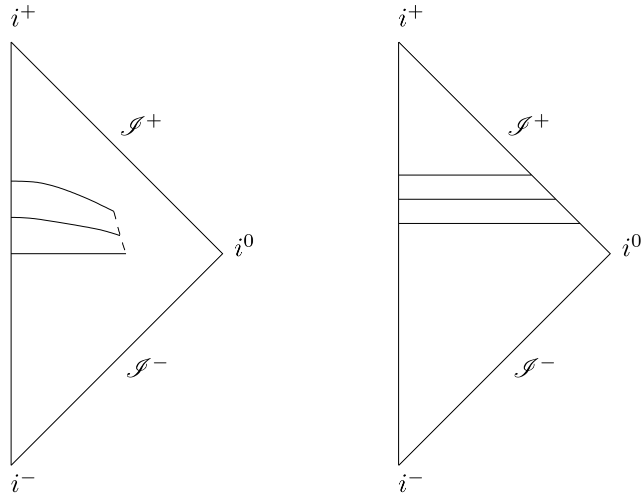

FIGURE 1. Cauchy evolution with timelike boundary (left) vs. hyperboloidal evolution (right). As an example, the Penrose diagram of flat (Minkowski) spacetime is shown. The conformal boundary consists of future and past null infinity ($\mathscr{I}^+$ and $\mathscr{I}^-$), future and past timelike infinity (i^+ and i^-), and spacelike infinity i^0.

momentum constraints. Using this method we obtain manifestly regular expressions for the formally singular terms in (3). We also recover regularity conditions previously derived in [3], namely that the shear and the components $\pi^{\mathrm{tr}\, ri}$ of the traceless momentum must vanish at $\mathscr{I}^+$, and we show that these conditions are preserved under the time evolution. Further details can be found in [4].

A similar scheme to the one described above has been implemented numerically [5] under the assumption that spacetime is axisymmetric so that there are two effective spatial dimensions. The spatial coordinates are chosen such that the two-metric takes on a conformally flat form (quasi-isotropic gauge). The numerical method consists of fourth-order finite differences on a spherical polar grid, the method of lines with a fourth-order Runge-Kutta method for the time integration, and multigrid (FAS) for the elliptic equations, which are solved at each substep of the Runge-Kutta scheme. The regularised form of the evolution equations is used at the outermost grid point at $\mathscr{I}^+$.

As a first test problem, we consider a Schwarzschild black hole. The metric on constant mean curvature hypersurfaces was first derived in [6]. We evolved this spacetime for (at least) 10^3 times the mass of the black hole without any signs of instabilities. The difference between the numerical and the exact solution as well as the residual of the momentum constraints show approximate fourth-order convergence.

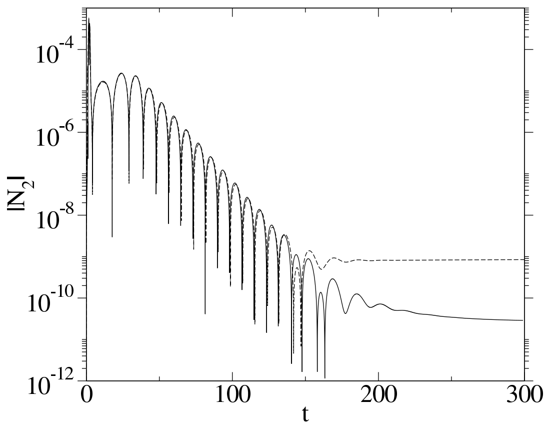

FIGURE 2. Quadrupole contribution ($\ell = 2$) to the Bondi news function at $\mathscr{I}^+$ for a perturbed Schwarzschild black hole (mass $M = 1$). Two different numerical resolutions are shown.

Next we include a gravitational wave perturbation. The Bondi news function [7], an invariant measure of gravitational radiation, is extracted at $\mathscr{I}^+$ (Fig. 2). The quasi-normal mode radiation emitted by the perturbed black hole is clearly visible. The frequency is in agreement with linear perturbation theory. (However we stress that the numerical simulation uses the full nonlinear Einstein equations.) Currently the numerical resolution is insufficient to resolve the power-law decay ('tail') expected at later times. The accuracy and efficiency of the numerical method needs to be improved; this will be the subject of future work.

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