Published for SISSA by Springer

Received: August 2, 2013
Revised: November 27, 2013
Accepted: January 6, 2014
Published: February 18, 2014

Generalised geometry from the ground up

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Abstract: Extending previous work on generalised geometry, we explicitly construct an $E_{7(7)}$-valued vielbein in eleven dimensions that encompasses the would be scalar bosonic degrees of freedom after reduction to four dimensions of $D = 11$ supergravity, by identifying new "generalised vielbeine" in eleven dimensions associated with the dual 6-form potential and the dual graviton. By maintaining full on-shell equivalence with the original theory at every step our construction furnishes the non-linear ansatz for the dual (magnetic) 7-form flux for any non-trivial compactification of $D = 11$ supergravity, complementing the known non-linear ansätze for the metric and the 4-form flux. A preliminary analysis of the generalised vielbein postulate for the new vielbein components reveals tantalising hints of new structures beyond $D = 11$ supergravity and ordinary space-time covariancxe, and also points to the possible $D = 11$ origins of the embedding tensor. We discuss the extension of these results to $E_{8(8)}$.

Keywords: Supergravity Models, String Duality

ArXiv ePrint: 1307.8295

doi:10.1007/JHEP02(2014)075

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1 Introduction

Despite the fact that maximal $D = 11$ supergravity [1] has been known and much studied for more than three decades it is still not clear what the most efficient formulation of the theory is, especially in view of the appearance of exceptional duality symmetries under dimensional reduction and the relation of this theory to the non-perturbative formulation of string theory, also known as M-theory. Indeed, the recent discovery [2] of a new structure in $D = 11$ supergravity, a new "generalised vielbein,"[1] is evidence of this, a development which was was triggered by the discovery of new SO(8) gauged supergravities in [4]. The new generalised vielbein was found in the context of the SU(8) invariant reformulation of $D = 11$ supergravity proposed a long time ago [3]. In this reformulation the non-gravitational degrees of freedom of the theory are used to extend the local (tangent space) symmetry by replacing the local Lorentz group SO(1,10) by SO(1,3) $\times$ SU(8), where the second factor coincides with the denominator of the duality coset $E_{7(7)}/SU(8)$ that appears upon the reduction of $D = 11$ supergravity to four dimensions [5]. Similar "generalised

[1]Here we follow the terminology, coined in ref. [3], for the term "generalised vielbein".

vielbeine" had been found in a reformulation of $D = 11$ supergravity appropriate for the reduction to three dimensions [6, 7].

The SU(8) reformulation is based on a $4 + 7$ split of eleven-dimensional space-time, where the fields are packaged in terms of objects that transform under local SU(8) transformations in eleven dimensions by combining the gravitational and matter degrees of freedom into single structures. Moreover, it is shown that the supersymmetry transformations of $D = 11$ supergravity can be written in terms of these objects in a way that makes the local SO(1,3) $\times$ SU(8) symmetry manifest. One particular SU(8) covariant object in this reformulation is the generalised vielbein, which replaces the vielbein along the seven internal directions. The new generalised vielbein found in [2] encompasses the 3-form along the 7-dimensional directions. A clear advantage of the SU(8) reformulation is that it immediately yields the duality manifest Cremmer-Julia theory [5] upon toroidal reduction. Moreover, it is also the appropriate framework in which to analyse the S^7 compactification of $D = 11$ supergravity to maximally gauged supergravity in four dimensions [8]. In fact, it is only within this framework that it has been possible to prove the consistency of the S^7 reduction [9, 10], and to arrive at a workable formula for the full non-linear ansatz for the 3-form field (4-form flux) [2, 11]. Indeed, one of the new results of the present work is that we can now also derive the non-linear ansatz for the dual 6-form field, details of which are given elsewhere [12].

Somewhat independently of these earlier results, more recent attempts in viewing the fields of $D = 10$ and $D = 11$ supergravities in a unified way have centred on *generalised geometry*, again pointing to the importance of duality symmetries in the *unreduced* theory. Generalised geometry as originally proposed in [13, 14] is based on an extension of the tangent space to include p-forms associated to the winding of branes sourcing $(p+1)$-forms, which ultimately allows for diffeomorphisms and gauge transformations to be combined in an enlarged symmetry group. In the most conservative applications of these ideas in the context of $D = 11$ supergravity [15, 16] the tangent space is enlarged to include windings of M2-branes, M5-branes and KK-branes. Meanwhile there are also proposals whereby the base space is also extended so that the fields depend not only on conventional coordinates, but also on winding coordinates [17–19] (in fact, the association of new coordinates with central charges is an old idea).[2] A characteristic feature distinguishing these attempts from the earlier work of [3, 6, 7] is that one usually has to impose restrictions on the coordinate dependence of the fields in order to realise the desired geometric structures, whence the relation to the original $D = 11$ supergravity becomes obscured.

An early proposal for extending space-time, arising from the E_{11} conjecture [20], is made in [17], where it is suggested that there exists an extension of $D = 11$ supergravity via a non-linear realisation of the semi-direct product of E_{11} and its first fundamental representation $L(\Lambda_1)$. In this picture the fields are obtained from a level expansion of the E_{11} algebra, while the coordinate dependence is controlled by $L(\Lambda_1)$ (thus in principle extending eleven-dimensional space-time to a space of infinitely many dimensions).

[2] In the mathematics literature [13, 14], generalised geometry has been exclusively used to refer to the extension of the tangent space and not the base manifold.

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Quite separately from the E_{11} conjecture, the non-linear realisation method [21–23] gives a prescription for determining explicitly a given duality coset element in a particular representation. In order to test the E_{11} proposal in the context of its finite-dimensional $E_{7(7)}$ subgroup, and motivated by [24], this method was applied by Hillmann [18] to advocate an "exceptional geometry"[3] for $D = 11$ supergravity. In this picture one considers a $4 + 56$-dimensional geometry where the dynamics of the fields in the 56-dimensional part is given by an $E_{7(7)}/SU(8)$ coset element. When these fields only depend on seven internal coordinates, and the dependence on the space-time coordinates is dropped, this dynamics reproduces the dynamics of the fields in $D = 11$ supergravity with components along the seven-dimensional directions assuming a duality relation between the 4-form field strength and 7-form field strength. Moreover, the supersymmetry transformation of the coset element is postulated to give rise to the fermionic degrees of freedom, in particular the gravitino along the 7-dimensional directions, which reproduces the supersymmetry transformations in the 7-dimensional part *à la* [3] if dependence of the fields is again restricted to seven internal coordinates. As we will argue, however, focusing attention only on the "internal" part of the geometry, as is also done in other approaches to generalised geometry, may be too restrictive as this assumption is not even respected for the simplest non-trivial compactifications, as we will illustrate in appendix C of this paper.

In the approach of [19], the $D = 11$ theory is viewed in a $7 + 4$ split of space-time, with the four dimensions considered as "internal". In particular a sector of the theory is considered that contains fields along the 4-dimensional directions, which would correspond to internal directions in the usual way that the $SL(5,\mathbb{R})$ duality symmetry appears. Furthermore, the fields are taken to depend on the 4-dimensional directions and time. This sector of the theory is then formulated in terms of an $SL(5,\mathbb{R})$ "generalised metric" which arises from the membrane duality arguments of [25]. As in [18], the formulation is based on a group theory element, but here the extension of the tangent space is considered in a generalised geometric language along the lines of [13–16] and it is shown that the diffeomorphisms and 3-form gauge transformations in this sector of the theory are unified. However, unlike previous versions of generalised geometry, in ref. [19] the base space is also extended so that the 4-dimensional space is enlarged to a 10-dimensional generalised space. This is to be viewed as the M-theory analogue of double field theory [26–28] (see also [29–31] for early related work). This analysis was extended to SO(5,5) in [32] and to $E_{6(6)}$ and $E_{7(7)}$ in [33], where the generalised metrics are found by a truncation of the E_{11} non-linear realisation outlined above. The case of $E_{8(8)}$ is considered in [34], where an $E_{8(8)}$ matrix is found in terms of components of the vielbein, 3-form, 6-form and a new field in the same representation as the putative dual gravity field. However, the direct link to $D = 11$ supergravity is lost, owing to the presence of this new field.

A key aim of generalised geometry is to unify diffeomorphisms and gauge transformations in a single generalised diffeomorphism (as already proposed for $E_{8(8)}$ in [7]). Thereby, generalised geometry extends the notions of Lie derivative and bracket in a way that incorporates gauge transformations of form fields. If one considers an extension of the space-time

[3] In fact, this term was already used in [7].

coordinates as well, then the closure of the generalised transformations requires that the fields satisfy a duality covariant constraint, the section condition [35–37]. This constraint can be viewed as a duality covariant restriction that allows one to reduce from the extended space to the usual number of dimensions. Furthermore, it has been shown [36, 38] that generalised geometric formulations of eleven-dimensions can be realised in a more geometric setting akin to Riemannian geometry. Again considering a sector of $D = 11$ supergravity, or equivalently considering space-times that are a warped product of Minkowski space and a d-dimensional manifold, in ref. [36] it is shown that the dynamics can be written in terms of a generalised Ricci scalar that is defined in terms of an associated generalised connection. Moreover, it is shown that this structure also extends to the fermionic sector [38]. The relevant redefinitions and fermionic representations for $\mathrm{O}(d, d)$ and E_n were already discussed in [20, 39–42].

While these approaches to generalised geometry propose a radical reinterpretation, and, if successful, would amount to a genuine extension of $D = 11$ supergravity, we return here to the viewpoint [3] that it is the theory itself that points at directions in which progress can be made (a view supported by the fact that, in 35 years, no true field theory extension of $D = 11$ supergravity has ever been found). Thus our approach remains grounded in $D = 11$ supergravity such that at every stage of the construction the resulting structures remain *on-shell* equivalent to the full $D = 11$ supergravity both in the bosonic and the fermionic sectors and such that at no point is any truncation or constraint on the coordinate dependence of the fields required.

In this paper, we demonstrate explicitly how a judicious analysis of the supersymmetry variations of fields in the $\mathrm{SU}(8)$ reformulation of $D = 11$ supergravity leads to new structures. In particular, we find two other "generalised vielbeine" and show that together with the other two known from the literature, these generalised vielbeine are to be viewed as components of an $\mathrm{E}_{7(7)}$ matrix in eleven dimensions.[4] In addition, we embark on an understanding of these new structures and the consistency relations that they satisfy. It is known [3] that the original generalised vielbein satisfies generalised vielbein postulates, which constrain its derivative along the four and seven-dimensional directions. These consistency requirements are a crucial ingredient in understanding the relation between the maximal gauged supergravity in four dimensions and the $D = 11$ theory and in proving the consistency of the reduction to the former [9, 10]. We present similar generalised vielbein postulates satisfied by the other generalised vielbeine. Recall that in Riemannian geometry the vielbein postulate is the requirement that the vielbein be covariantly constant, which gives an equivalence between the affine and spin connections that are defined on two different bundles. The fact that the generalised vielbeine satisfy analogous relations with more general connections is strong indication of the emergence of structures beyond Riemannian geometry. Furthermore, at a more practical level, we expect that a deeper study of these relations will lead to the exciting possibility of understanding the higher dimensional origins of the embedding tensor [43–45], which is the most efficient way of understanding gauged supergravities in any dimension.

[4] Such structures are already evident in ref. [7], where it is shown that $D = 11$ supergravity contains a 36×248 matrix that is part of a full $\mathrm{E}_{8(8)}$ matrix in eleven dimensions.

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Another bonus of these results is that they give a non-linear ansatz for the dual 6-form potential. Indeed, by considering the Englert solution [46] as a simple example and showing that not only is the 6-form field non-zero in this case, but that it has non-vanishing mixed components, we argue that this will generically be the case for all compactifications with non-vanishing flux. We believe this is an important point that one must bear in mind in any study involving dual fields in the context of $D = 11$ supergravity, or any truncation thereof. The results presented here are relevant for the $4+7$ split of the eleven-dimensional theory corresponding to the $E_{7(7)}$ duality group. As emphasised before, our analysis is based on the fermionic sector, in contrast to the mainly bosonic approach in the generalised geometry literature. Furthermore, eleven dimensional dualisation of the fields plays an important role in this story. Finally, we also outline how similar structures can be constructed for cases relevant for other duality groups, in particular for $E_{8(8)}$.

In section 2, after a brief review of $D = 11$ supergravity and the supersymmetry transformations satisfied by its fields, we motivate the importance of dualisation of fields in eleven dimensions in any attempt to understand duality symmetries from a higher dimensional point of view. In particular, we emphasise the significance of the supersymmetry transformations of dual fields in the context of this work. We derive the supersymmetry transformation of the six-form potential dual to the three-form potential of $D = 11$ supergravity in section 2.1. Furthermore, we highlight the problems associated with a consistent *covariant* and *Lorentz invariant* formulation of dual gravity in general, but also in the context of eleven dimensions in section 2.2.

Working within the context of the SU(8) invariant reformulation of ref. [3], in section 3 we construct an $E_{7(7)}$ matrix in eleven dimensions that encompasses the bosonic degrees of freedom of the eleven-dimensional theory. In particular, in section 3.2, we demonstrate the existence of another generalised vielbein in addition to the two previously known in the literature [2, 3]. We argue in section 3.3 that these generalised vielbeine must form the components of a single $E_{7(7)}$ valued object in eleven dimensions — a 56-bein, and conclude that the missing component must be related to a dual gravity field. We construct this final generalised vielbein by insisting that it too transform as an $E_{7(7)}$ object. Furthermore, we show that the vector fields whose supersymmetry transformations give the generalised vielbeine can themselves be combined into a **56**-plet of $E_{7(7)}$.

Section 4 is devoted to a preliminary analysis of the generalised vielbein postulates satisfied by the new generalised vielbeine given in ref. [2] and in section 3. The new generalised vielbein postulates give rise to as yet unknown connections associated with p-form gauge transformations. Finally, in section 5, we briefly discuss how one can implement a similar construction for the $3+8$ split of eleven dimensions, which would be relevant for the $E_{8(8)}$ duality group and also for the $5+6$ split relevant for the $E_{6(6)}$ duality group.

2 $D = 11$ supergravity and duality

The lagrangian of eleven-dimensional supergravity [1] in the notation and conventions of [3] and to second order in fermions is

$$\begin{aligned}L = &- \frac{1}{2}R - \frac{1}{2}\overline{\Psi}_M \tilde{\Gamma}^{MNP} D_N \Psi_P - \frac{1}{48} F^{MNPQ} F_{MNPQ} \\ &- \frac{1}{(12)^3\sqrt{2}} i\epsilon^{MNPQRSTUVWX} F_{MNPQ} F_{RSTU} A_{VWX} \\ &- \frac{\sqrt{2}}{192} F_{MNPQ} \left(\overline{\Psi}_R \tilde{\Gamma}^{MNPQRS} \Psi_S + 12 \overline{\Psi}^M \tilde{\Gamma}^{NP} \Psi^Q \right),\end{aligned} \tag{2.1}$$

where the four-form field strength is

$$F_{MNPQ} = 4!\, \partial_{[M} A_{NPQ]}$$

and D_M is the covariant derivative defined with respect to the metric

$$g_{MN} = E_M{}^A E_N{}^B \delta_{AB}. \tag{2.2}$$

The eleven-dimensional $\tilde{\Gamma}$ matrices[5] satisfy

$$\{\tilde{\Gamma}_A, \tilde{\Gamma}_B\} = 2\delta_{AB}, \qquad \tilde{\Gamma}^{A_1 \dots A_{11}} = -i\epsilon^{A_1 \dots A_{11}}, \tag{2.3}$$

where on the right hand side of the second equation we have suppressed a 32-dimensional identity matrix. In this convention, the supersymmetry transformations of $D = 11$ supergravity take the form

$$\delta E_M{}^A = \frac{1}{2} \bar{\varepsilon} \tilde{\Gamma}^A \Psi_M, \tag{2.4}$$

$$\delta A_{MNP} = -\frac{\sqrt{2}}{8} \bar{\varepsilon} \tilde{\Gamma}_{[MN} \Psi_{P]}, \tag{2.5}$$

$$\delta \Psi_M = D_M \varepsilon + \frac{\sqrt{2}}{288} \left(\tilde{\Gamma}_M{}^{ABCD} - 8 E_M{}^A \tilde{\Gamma}^{BCD} \right) \varepsilon F_{ABCD}. \tag{2.6}$$

The appearance of exceptional global symmetries [5] upon the toroidal reduction of the eleven-dimensional theory to dimensions $D \leq 6$ requires the Hodge dualisation of all field strengths whose degrees are greater than or equal to $\frac{1}{2}D$. It should be emphasised [47] that this is a particular choice that is designed to maximise the global symmetry obtained under reduction. Other dualisations will lead to other global symmetries in the reduced theory. One can understand this choice by noting that the most obvious way in which the enhanced symmetry in the reduced theory manifests itself is the observation that the scalars in the reduced theory parametrise a coset whose numerator is the global symmetry group, while the denominator is the local symmetry group, which, in general, corresponds to the maximal compact subgroup of the global group. In the reduction to dimensions $D \geq 6$, the scalar sector is clear and cannot be changed by a process of dualisation. However, this

[5] We put a tilde in order to distinguish these Γ-matrices with lower dimensional Γ-matrices to be introduced below, cf. (3.4).

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is not true for $D \leq 5$, where one can increase the number of scalars by dualising higher degree field strengths. Maximising the number of scalars in the reduced theory maximises the global symmetry obtained under reduction [47].

The requirement for the dualisation of certain fields in the reduced theory for the manifestation of a larger symmetry group can be understood from an eleven-dimensional perspective by the need to include dual fields in eleven dimensions. Thus, there seems to be an intimate connection between dualisation of fields in the reduced theory and dualisation in the full eleven-dimensional theory. Indeed, this relation is explicitly demonstrated in ref. [34], where the bosonic sector of the eleven-dimensional theory is reduced to three dimensions. In the process of writing the scalar sector of the reduced theory as a non-linear coset sigma model with coset $\mathrm{E}_{8(8)}/\mathrm{SO}(16)$, one finds a precise relation between the three-dimensional dual scalar fields ϕ_m and ψ^{mn} associated with the graviphoton $B_\mu{}^m$ and the three-form component $A_{\mu mn}$, respectively, and the purported eleven-dimensional duals $h_{m_1 \dots m_8, n}$ and $A_{m_1 \dots m_6}$, respectively. Therefore, given that our aim here is to understand the role of the four-dimensional global symmetry group $\mathrm{E}_{7(7)}$ in eleven dimensions, it is natural that we should consider the dualisation of eleven-dimensional fields.

The dualisation of a form field can be understood on-shell as simply the Hodge dualisation of the field strength of the form field. However, the dualisation of gravity poses a difficult challenge. Meanwhile, the need for the dualisation of gravity is apparent not only from the perspective of the discussion in this paper, and other papers concerning the higher-dimensional origins of duality symmetries, but also from the fact that $D = 11$ supergravity has solutions, such as the Kaluza-Klein monopole, that are expected to source the dual gravity field [48]. Nevertheless, the elevation of gravitational duality to the non-linear level encounters a no-go theorem [49], which can only be evaded by a loss of either locality or Lorentz invariance. However, what is pertinent in this paper is the coupling of gravity to matter, in particular the 3-form of $D = 11$ supergravity and its 6-form dual, and, moreover, the supersymmetry transformation of a candidate dual gravity field. In this case, the dualisation of gravity becomes problematic even at the linearised level [50]. The supersymmetry transformations can be made to close in the presence of a linearised dual gravity field if one takes a linearised approximation where one has only *global* supersymmetry. However, the supersymmetry transformation is no longer consistent with the equations of motion, or in other words the dual graviton does not carry the same degrees of freedom as the graviton even in a flat background [50]. Furthermore, it is argued in [50] that, under assumptions of locality and Lorentz invariance, it is not possible to dualise a linearised graviton field coupled to matter. Nevertheless, we find that the completion of the $\mathrm{E}_{7(7)}$ matrix in eleven dimensions requires the existence of a field with the same representation as a dual gravity field. Moreover, we explicitly give the supersymmetry transformation of this field up to an undetermined constant in section 3.3. We should stress that our results are not in conflict with the no-go theorems of [49, 50] as we will become apparent later.

2.1 Dualisation of the three-form potential

The relevance of a six-form potential dual to the three-form potential within the context of $D = 11$ supergravity was discussed soon after the eleven-dimensional theory was found [51, 52]. Later, however, it was argued [53] that such a potential is to be thought of as being sourced by a non-perturbative object — M5-brane — in a conjectured M-theory that goes beyond the supergravity theory. As such, the incorporation [54, 55] of a six-form potential in the eleven-dimensional theory, including its supersymmetry transformation [55] (see also [51]), has been considered previously.

Our interest in the six-form potential in this work will be limited to the form of its supersymmetry transformation. The six-form potential dual of the three-form potential is introduced by considering its equation of motion, which can be simply derived from lagrangian (2.1):

$$dF_{(7)} = \frac{7!\sqrt{2}}{2}\, F_{(4)} \wedge F_{(4)} - \frac{\sqrt{2}}{8} d \star X, \tag{2.7}$$

where

$$F_{(7)} = \star F_{(4)} \tag{2.8}$$

and

$$X^{MNPQ} = \overline{\Psi}_R \tilde{\Gamma}^{MNPQRS} \Psi_S + 12 \overline{\Psi}^M \tilde{\Gamma}^{NP} \Psi^Q. \tag{2.9}$$

Observing that

$$F_{(4)} \wedge F_{(4)} = \frac{3!}{7!}\, d(A_{(3)} \wedge F_{(4)}) \tag{2.10}$$

gives

$$d\left(F_{(7)} - 3\sqrt{2}\, A_{(3)} \wedge F_{(4)} + \frac{\sqrt{2}}{8} \star X\right) = 0. \tag{2.11}$$

Hence, there exists locally a six-form potential $A_{(6)}$ such that

$$F_{(7)} = dA_{(6)} + 3\sqrt{2}\, A_{(3)} \wedge F_{(4)} - \frac{\sqrt{2}}{8} \star X. \tag{2.12}$$

Equivalently, in terms of components

$$\begin{aligned} F_{M_1 \dots M_7} = 7! D_{[M_1} A_{M_2 \dots M_7]} + 7! \frac{\sqrt{2}}{2} A_{[M_1 \dots M_3} D_{M_4} A_{M_5 \dots M_7]} \\ - \frac{\sqrt{2}}{192} i \epsilon_{M_1 \dots M_{11}} \left(\overline{\Psi}_R \tilde{\Gamma}^{M_8 \dots M_{11} RS} \Psi_S + 12 \overline{\Psi}^{M_8} \tilde{\Gamma}^{M_9 M_{10}} \Psi^{M_{11}} \right). \end{aligned} \tag{2.13}$$

As should be familiar to the reader, in this process we have interchanged the equations of motion and Bianchi identities. Thus, the Bianchi identity satisfied by $F_{(7)}$ is equivalent to the equation of motion of $A_{(3)}$. For our applications it is best to think of the above equation as a definition of potential $A_{(6)}$ in terms of the usual eleven-dimensional fields. Therefrom, we can find the supersymmetry transformation of $A_{(6)}$.

Let us begin with an ansatz of the form

$$\delta A_{M_1 \dots M_6} = \alpha\, \bar{\varepsilon} \tilde{\Gamma}_{[M_1 \dots M_5} \Psi_{M_6]} + \beta\, \bar{\varepsilon} \tilde{\Gamma}_{[M_1 M_2} \Psi_{M_3} A_{M_4 M_5 M_6]}. \tag{2.14}$$

Now consider a supersymmetry variation of equation (2.13). To fix the coefficients it suffices to consider terms of the form $D_M\epsilon$. Hence, we concentrate on such terms:

$$
\begin{aligned}
&i\epsilon_{M_1\dots M_7}{}^{N_1\dots N_4} D_{N_1}\delta A_{N_2N_3N_4} - \frac{7!\sqrt{2}}{2}\, A_{[M_1M_2M_3} D_{M_4}\delta A_{M_5M_6M_7]} - 7!\, D_{[M_1}\delta A_{M_2\dots M_7]} \\
&\qquad + \frac{\sqrt{2}}{96} i\epsilon_{M_1\dots M_{11}}\left(\delta\overline{\Psi}_R\tilde{\Gamma}^{M_8\dots M_{11}RS}\Psi_S + 12\delta\overline{\Psi}^{M_8}\tilde{\Gamma}^{M_9M_{10}}\Psi^{M_{11}}\right) + \ldots = 0,
\end{aligned}
\tag{2.15}
$$

where we have used the relation

$$
\overline{\psi}\,\tilde{\Gamma}^{A_1}\cdots\tilde{\Gamma}^{A_n}\chi = (-1)^n\,\overline{\chi}\,\tilde{\Gamma}^{A_n}\cdots\tilde{\Gamma}^{A_1}\psi. \tag{2.16}
$$

Substituting the supersymmetry transformations of the relevant fields using equations (2.5), (2.6) and (2.14) into equation (2.15) gives

$$
\begin{aligned}
&7!(1/8-\beta) D_{[M_1}\bar{\varepsilon}\tilde{\Gamma}_{M_2M_3}\Psi_{M_4}A_{M_5M_6M_7]} \\
&\qquad - 7!\alpha D_{[M_1}\bar{\varepsilon}\tilde{\Gamma}_{M_2\dots M_6}\Psi_{M_7]} + \frac{\sqrt{2}}{96} i\,\epsilon_{M_1\dots M_{11}} D_P\bar{\varepsilon}\tilde{\Gamma}^{M_8\dots M_{11}PQ}\Psi_Q + \cdots = 0.
\end{aligned}
\tag{2.17}
$$

Using the fact that

$$
\tilde{\Gamma}^{P_1\dots P_6} = -i/5!\;\epsilon^{P_1\dots P_6 Q_1\dots Q_5}\tilde{\Gamma}_{Q_1\dots Q_5} \tag{2.18}
$$

the above equation simplifies to

$$
\left(\beta - \frac{1}{8}\right) D_{[M_1}\bar{\varepsilon}\tilde{\Gamma}_{M_2M_3}\Psi_{M_4}A_{M_5M_6M_7]} + \left(\alpha + \frac{3}{6!\sqrt{2}}\right) D_{[M_1}\bar{\varepsilon}\tilde{\Gamma}_{M_2\dots M_6}\Psi_{M_7]} + \cdots = 0. \tag{2.19}
$$

Hence, the supersymmetry transformation of the 6-form dual is

$$
\delta A_{M_1\dots M_6} = -\frac{3}{6!\sqrt{2}}\bar{\varepsilon}\tilde{\Gamma}_{[M_1\dots M_5}\Psi_{M_6]} + \frac{1}{8}\bar{\varepsilon}\tilde{\Gamma}_{[M_1M_2}\Psi_{M_3}A_{M_4M_5M_6]}. \tag{2.20}
$$

A complete proof of the consistency of this relation with transformations (2.4)–(2.6) requires use of the Rarita-Schwinger equation for Ψ_M. For this reason, and because duality can anyway be implemented only at the level of the equations of motion, the above supersymmetry transformation rules are jointly valid *on-shell* only. Nevertheless we should emphasise that, apart from this restriction, all formulae are valid at the full non-linear level, that is, we can simultaneously incorporate the 3-form and the 6-form into the full $D=11$ theory.

2.2 Dualisation of gravity

Unlike the dualisation of the 3-form potential of $D=11$ supergravity, the dualisation of gravity is only possible at the linearised level, where the eleven-dimensional metric is expanded according to

$$
g_{MN} = \eta_{MN} + h_{MN} + \mathcal{O}(h^2). \tag{2.21}
$$

In linearised general relativity, the dual graviton can either be formulated from Hodge dualising the Riemann two-form [48, 56] or by Hodge dualising an index of the Einstein tensor [20]. The generalisation of either approach at the non-linear level is obstructed by a no-go theorem [49, 57], which can only be evaded (if it can be evaded at all) by abandoning either locality or Lorentz invariance or both. As shown in previous work [20, 48, 56, 58, 59] (see also [60, 61]), the field formally dual to the linearised metric h_{MN} is a mixed symmetry tensor $h_{M_1\dots M_8|N}$ that belongs to the (8,1) representation of $\mathrm{GL}(11,\mathbb{R})$ (with Dynkin label [1000000100]) and obeys the constraint

$$h_{[M_1\dots M_8|N]} = 0. \tag{2.22}$$

The dual graviton field thus belongs to a non-trivial Young tableau representation, and this feature is one main source of difficulty. At the linear level the gravitational analog of equation (2.13) is

$$Y_{M_1\dots M_9|N} = 9!\,\partial_{[M_1} h_{M_2\dots M_9]|N}, \tag{2.23}$$

where

$$Y_{M_1\dots M_9|N} = \frac{1}{2}\epsilon_{M_1\dots M_9}{}^{PQ}\omega_{N\,PQ} \tag{2.24}$$

with the associated linearised spin connection $\omega_{M\,NP} = 2\kappa\partial_{[N}h_{P]M}$. Note that we do not need to distinguish between curved and flat indices since we are working to linear order. The irreducibility constraint (2.22) is equivalent to $\omega_M{}^M{}_N = 0$.[6] It is now straightforward to show that the fields h_{MN} and $h_{M_1\dots M_8|N}$ form a dual pair in the sense that the Bianchi identity for one implies the (linearised) equation of motion for the other.

As shown in [49] it is not possible to elevate the duality relation (2.24) to the interacting theory if one insists on locality and Lorentz invariance of the dual formulation. These difficulties are also reflected in the impossibility of extending the duality between h_{MN} and $h_{M_1\dots M_8|N}$ to the incorporation of matter, even if gravity is kept linear [50]. The question of extending the gravitational duality to supergravity was studied in [50], though in terms of a simpler example, as well as in unpublished work of the same authors.[7] The most general ansatz for the supersymmetry variation of the dual graviton compatible with the constraint (2.22) reads

$$\delta h_{M_1\dots M_8|N} \propto \bar{\varepsilon}\tilde{\Gamma}_{M_1\dots M_8}\Psi_N - \bar{\varepsilon}\tilde{\Gamma}_{N[M_1\dots M_7}\Psi_{M_8]} - C_0\eta_{N[M_1}\bar{\varepsilon}\tilde{\Gamma}_{M_2\dots M_7}\Psi_{M_8]} \tag{2.25}$$

with an undetermined constant C_0. The second term in the expression above is required so that it cancels the first term upon antisymmetrisation over all indices, as required by constraint (2.22). For $C_0 \neq 0$, the last term on the right hand side leads to a breaking of $\mathrm{GL}(11,\mathbb{R})$ covariance to SO(1,10). It is clear that further restrictions must be imposed

[6] The constraint (2.22) appears naturally in all approaches based on E_{10} [62] and E_{11} [20], but one can also perform the dualisation without imposing it. In this case (2.24) must be replaced by

$$Y_{M_1\dots M_9|N} = \frac{1}{2}\epsilon_{M_1\dots M_9}{}^{PQ}\left(\omega_{N\,PQ} - 2\eta_{NP}\omega_R{}^R{}_Q\right)\ .$$

[7] We would like to thank A. Kleinschmidt for discussions.

at this point. In particular, we must restrict to *global supersymmetry* ($\partial_M \varepsilon = 0$) from the outset, otherwise the supersymmetry algebra cannot close on $h_{M_1 \dots M_8|N}$ even at the linearised level. This is easily seen by noting that the putative parameter

$$\Lambda_{M_1 \dots M_8} = \bar{\varepsilon}_1 \tilde{\Gamma}_{M_1 \dots M_8} \varepsilon_2, \tag{2.26}$$

which would be one of the gauge transformation parameters associated with the field $h_{M_1 \dots M_8|N}$ [49] is *symmetric* under interchange of ε_1 and ε_2, hence it cannot appear in the commutator of two local supersymmetries; instead, the commutator would lead to new transformations that cannot be interpreted as gauge transformations in the sense of [49]. As a consequence it does not appear possible even at the linearised level in this "dual supergravity" to consistently incorporate the gauge transformations necessary to remove unphysical helicity degrees of freedom.[8]

Regardless of these problems, we will find that the form of supersymmetry transformation (2.25) does not coincide with the supersymmetry transformation (3.37) that we find for the putative dual gravity field, even though it too satisfies constraint (2.22). The presence or otherwise of form fields do not affect these conclusions.

The difficulties outlined in this section, in our view, point to the core problem of properly understanding the duality symmetries beyond their explicit realisation in dimensionally reduced maximal supergravity: that is, the problem of *dualising Einstein's theory at the non-linear level*. We note that there does exist a covariant non-linear formulation [63] of gravity containing the metric field *and* its dual, where the metric appears via a topological term following [64, 65], but the putative supersymmetric extension of this proposal is expected to encounter the same problems as described above. We interpret these difficulties as another indication that a proper understanding of M-theory and the role of $D = 11$ supergravity in this context will require the abandonment of conventional notions of covariance and space-time.

3 Generalised geometry from eleven dimensions

In this section we demonstrate by explicit construction how the bosonic degrees of freedom of $D = 11$ supergravity can be assembled into $E_{7(7)}$-valued objects. In particular the vector degrees of freedom can be combined into a **56**-plet of $E_{7(7)}$, and the scalar fields into an $E_{7(7)}$-valued 56-bein $\mathcal{V}$ in eleven dimensions, thus completing the construction of [3]. These results finally establish the relation between the old work of [3] with more recent constructions, where the existence of a generalised vielbein is usually postulated *ad hoc* (usually with further constraints). They also link up with the original construction performed in [5] for the T^7 truncation of $D = 11$ supergravity, but with the crucial difference that the present results are valid *in eleven dimensions.*

[8] One can nevertheless investigate the closure of the *global* supersymmetry algebra, which yields a non-zero value for C_0 (A. Kleinschmidt, private communication).

3.1 SU(8) reformulation of $D = 11$ supergravity

In ref. [3], the eleven-dimensional theory is formulated in a manifestly SU(8) invariant manner. The eleven-dimensional space-time is split into a four-dimensional space-time and a seven-dimensional space. Hence the eleven-dimensional space-time coordinates and tangent coordinates are split as

$$z^M = (x^\mu, y^m), \qquad z^A = (x^\alpha, y^a), \tag{3.1}$$

respectively. Furthermore, in an upper triangular gauge the elfbein takes the form

$$E_M{}^A = \begin{pmatrix} \Delta^{-1/2} e'_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \tag{3.2}$$

where

$$\Delta = \det(e_m{}^a). \tag{3.3}$$

Correspondingly, the eleven-dimensional gamma-matrices are decomposed in the following way

$$\tilde{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbf{1}, \qquad \tilde{\Gamma}_a = \gamma_5 \otimes \Gamma_a, \tag{3.4}$$

where γ_α and Γ_a satisfy the four and seven-dimensional Clifford algebras, respectively, and

$$\gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4.$$

In particular,

$$\Gamma^{a_1 \ldots a_7} = -i\epsilon^{a_1 \ldots a_7}\mathbf{1}. \tag{3.5}$$

The essence of the SU(8) invariant reformulation of the theory is in defining new fields with chiral SU(8) indices [3, 5][9]

$$\varphi_\mu{}^A = \frac{1}{2}(1+\gamma_5)\,\varphi'_{\mu\bar{A}}, \qquad \varphi_{\mu\,A} = \frac{1}{2}(1-\gamma_5)\,\varphi'_{\mu\bar{A}}, \tag{3.6}$$

$$\chi^{ABC} = (1+\gamma_5)\chi'_{\bar{A}\bar{B}\bar{C}}, \qquad \chi_{ABC} = (1-\gamma_5)\chi'_{\bar{A}\bar{B}\bar{C}}, \tag{3.7}$$

where the indices on the right hand side are denoted with a bar to emphasise the fact that they are not chiral SU(8) indices, but SO(7) spinor indices. We shall not make this distinction where the index type is clear from the context. Fields φ' and χ' are related to the original fields in the following manner[10]

$$\varphi'_\mu = \Delta^{-1/4}(i\gamma_5)^{-1/2} e'_\mu{}^\alpha \left(\Psi_\alpha - \frac{1}{2}\gamma_5\gamma_\alpha\Gamma^a\Psi_a\right), \tag{3.8}$$

$$\chi'_{ABC} = \frac{3}{4}\sqrt{2}i\Delta^{-1/4}(i\gamma_5)^{-1/2}\Psi_{a[A}\Gamma^a_{BC]}. \tag{3.9}$$

[9] For these we will use capital Roman letters $A, B, \ldots$, the same as for flat indices in eleven dimensions. There should nevertheless arise no confusion as it should be clear from the context which kind of index is meant.

[10] Note that $(i\gamma_5)^{1/2} = \frac{1}{\sqrt{2}}(1 + i\gamma_5)$, while $(i\gamma_5)^{-1/2} = \frac{1}{\sqrt{2}}(1 - i\gamma_5)$.

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Similarly, the supersymmetry transformation parameter is redefined as follows

$$\epsilon^A = \frac{1}{2}(1+\gamma_5)\,\Delta^{1/4}(i\gamma_5)^{-1/2}\epsilon_{\bar{A}}, \qquad \epsilon_A = \frac{1}{2}(1-\gamma_5)\,\Delta^{1/4}(i\gamma_5)^{-1/2}\epsilon_{\bar{A}}. \tag{3.10}$$

For the spin two degrees of freedom it then follows directly that

$$\delta e'_\mu{}^\alpha = \frac{1}{2}\bar{\epsilon}^A\gamma^\alpha\varphi_{\mu A} \;+\; \text{h.c..} \tag{3.11}$$

3.2 New generalised vielbeine

In the formulation of [3], the local SU(8) symmetry is an enlargement of the local SO(7) symmetry of the tangent space of the seven-dimensional space. As such all fields with SO(7) tangent space indices are replaced in the reformulated theory with new fields carrying SU(8) indices. An important example of this is the generalised vielbein[11]

$$e^m_{AB} = i\Delta^{-1/2}\Gamma^m_{AB} \equiv i\Delta^{-1/2}e^m{}_a\Gamma^a_{AB}, \tag{3.12}$$

which replaces the siebenbein in the reformulated theory. This generalised vielbein is found [3] by considering the supersymmetry transformation:

$$\delta B_\mu{}^m = \frac{\sqrt{2}}{8}e^m_{AB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] \;+\; \text{h.c.} \tag{3.13}$$

with $\gamma'_\mu \equiv e'_\mu{}^\alpha\gamma_\alpha$. Recently, it was found [2] that the supersymmetry transformation of a component of the 3-form A,

$$B_{\mu mn} = A_{\mu mn} - B_\mu{}^p A_{pmn}$$

leads to another generalised vielbein:

$$\delta B_{\mu mn} = \frac{\sqrt{2}}{8}e_{mnAB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] + \text{h.c.}, \tag{3.14}$$

where

$$e_{mnAB} = -\frac{\sqrt{2}}{12}i\Delta^{-1/2}\left(\Gamma_{mnAB} + 6\sqrt{2}A_{mnp}\Gamma^p_{AB}\right) \tag{3.15}$$

with $\Gamma_{mn} \equiv e_m{}^a e_n{}^b\Gamma_{ab}$. Importantly, both generalised vielbeine transform in the same way under a supersymmetry transformation,

$$\delta e^m_{AB} = -\sqrt{2}\Sigma_{ABCD}e^{mCD} - 2\Lambda^C{}_{[A}e^m_{B]C}, \tag{3.16}$$

$$\delta e_{mnAB} = -\sqrt{2}\Sigma_{ABCD}e_{mn}{}^{CD} - 2\Lambda^C{}_{[A}e_{mnB]C} \tag{3.17}$$

with the complex self-dual SU(8) tensor

$$\Sigma_{ABCD} = \bar{\varepsilon}_{[A}\chi_{BCD]} + \frac{1}{4!}\epsilon_{ABCDEFGH}\bar{\varepsilon}^E\chi^{FGH}, \tag{3.18}$$

[11] In fact, the generalised vielbein should be dressed with a general SU(8) matrix, Φ. However, here and in what follows we fix Φ, which corresponds to a partial gauge-fixing of the local SU(8) symmetry (see ref. [3]).

and where

$$\Lambda^B{}_A = \frac{1}{8}\,\bar{\varepsilon}\gamma_5\Gamma_{ab}\Psi^a\,\Gamma^b_{AB} + \frac{1}{8}\,\bar{\varepsilon}\gamma_5\Gamma_a\Psi_b\,\Gamma^{ab}_{AB} + \frac{1}{16}\,\bar{\varepsilon}\Gamma_{ab}\Psi_c\,\Gamma^{abc}_{AB} \tag{3.19}$$

parametrises a field dependent local SU(8) rotation in eleven dimensions.

The generalised vielbeine e^m_{AB} and e_{mnAB} give rise to non-linear ansätze for the internal metric [66] and flux [2], which pass some very non-trivial tests [11]. The ansätze are obtained by comparing the supersymmetry transformations that lead to these vielbeine, (3.13) and (3.14), with the four-dimensional ungauged supergravity supersymmetry transformations [2, 67][12]

$$\delta A_\mu{}^{IJ} = -\frac{1}{2}(u_{ij}{}^{IJ} + v_{ijIJ})\left[2\sqrt{2}\bar{\varepsilon}^i\varphi^j_\mu + \bar{\varepsilon}_k\gamma'_\mu\chi^{ijk}\right] + \text{h.c.}, \tag{3.20}$$

$$\delta A_{\mu IJ} = -\frac{1}{2}i(u_{ij}{}^{IJ} - v_{ijIJ})\left[2\sqrt{2}\bar{\varepsilon}^i\varphi^j_\mu + \bar{\varepsilon}_k\gamma'_\mu\chi^{ijk}\right] + \text{h.c.}, \tag{3.21}$$

for the 28 electric vectors $A_\mu{}^{IJ}$ and the 28 magnetic vectors $A_{\mu IJ}$, respectively. Here $i, j, k, \ldots$ are SU(8) indices, while $I, J, K, \ldots$ are $\mathrm{SL}(8,\mathbb{R})$ indices (which are to be considered as SO(8) indices after gauging). Moreover, since the linear Kaluza-Klein ansatz for vector fields is exact, $B_\mu{}^m$ and $B_{\mu mn}$ are related to $A_\mu{}^{IJ}$ and $A_{\mu IJ}$, respectively via the 28 S^7 Killing vectors $K^{mIJ}(y)$ and 28 two-forms $K^{mnIJ} = \mathring{e}_a{}^m\,\mathring{e}_b{}^n\,\overline{\eta}^I\Gamma^{ab}\eta^J$, where η^I are the S^7 Killing spinors, and $\mathring{e}_a{}^m$ is the inverse siebenbein on S^7. This gives a relation between the generalised vielbeine and the scalars of the four-dimensional gauged supergravity. The non-linear ansätze obtained in this way are highly non-trivial and there is no purely bosonic argument available to derive them otherwise. Indeed the non-linear metric ansatz is part of the consistency proof of the S^7 reduction [9]. Meanwhile, the recently discovered non-linear flux ansatz is shown [11] to be an efficient way to analytically find the internal flux associated to not only a four-dimensional maximally gauged supergravity critical point, but even a whole family.

The generalised vielbeine, (3.12) and (3.15), were found by considering the supersymmetry transformation of fields that under reduction would correspond to vector fields, *viz.* $B_\mu{}^m$ and $B_{\mu mn}$. In the maximally gauged theory these vector fields each give rise to 28 vector fields, accounting for the 56 vector fields of which 28 appear in the gauged theory lagrangian. However, in the ungauged theory these vector fields only account for 28 of the 56 vector fields. The other 28 vector fields are the duals of these fields in 4-dimensions. We can view these dual fields as coming from the reduction of fields that are the dualisations of the eleven-dimensional fields. Therefore, we next turn to the supersymmetry transformation of the 6-form potential in eleven dimensions that is dual to the 3-form gauge potential, with the aim of extracting from it another set of vector components with an associated generalised vielbein. Consider the following components of the 6-form:

$$B_{\mu m_1\ldots m_5} = A_{\mu m_1\ldots m_5} - B_\mu{}^p A_{pm_1\ldots m_5}.$$

[12] See also equation (7.10) in [5].

The supersymmetry transformation of the 6-form, equation (2.20), can now be used to show that

$$\delta\left(B_{\mu m_1\ldots m_5} - \frac{\sqrt{2}}{4} B_{\mu[m_1 m_2} A_{m_3 m_4 m_5]}\right) = \frac{\sqrt{2}}{8} e_{m_1\ldots m_5\, AB}\left[2\sqrt{2}\bar{\varepsilon}^A \varphi_\mu^B + \bar{\varepsilon}_C \gamma'_\mu \chi^{ABC}\right] + \text{h.c.} \tag{3.22}$$

with the associated new generalised vielbein

$$\begin{aligned} e_{m_1\ldots m_5\, AB} = \frac{1}{6!\sqrt{2}} i\Delta^{-1/2}\Big[&\Gamma_{m_1\ldots m_5\, AB} + 60\sqrt{2} A_{[m_1 m_2 m_3}\Gamma_{m_4 m_5]\, AB} \\ &- 6!\sqrt{2}\Big(A_{p m_1\ldots m_5} - \frac{\sqrt{2}}{4} A_{p[m_1 m_2} A_{m_3 m_4 m_5]}\Big)\Gamma^p_{AB}\Big]. \end{aligned} \tag{3.23}$$

This new vielbein depends not only on the metric and 3-form along the seven internal directions, but also on the 6-form potential $A_{m_1\ldots m_6}$.

Using the identities listed in appendix B, one can show that the supersymmetry transformation of this generalised vielbein takes the same form as for the other generalised vielbeine, i.e.

$$\delta e_{m_1\ldots m_5\, AB} = -\sqrt{2}\Sigma_{ABCD} e_{m_1\ldots m_5}{}^{CD} - 2\Lambda^C{}_{[A} e_{m_1\ldots m_5\, B]C}. \tag{3.24}$$

Remarkably all generalised vielbein components transform in exactly the same way under local supersymmetry, and with the *same* compensating SU(8) rotation. We emphasise again that all formulae are valid in eleven dimensions, and at the full non-linear level. Furthermore at no point was it necessary to truncate or impose any restriction on the coordinate dependence. It is now straightforward to derive the non-linear ansatz for the 6-form field $A_{m_1\ldots m_6}$ by substituting the relevant expressions in terms of S^7 Killing vectors and the four-dimensional fields on the left hand side of (3.23), and then projecting out the last component on the right hand side. A detailed discussion will, however, be given elsewhere.

3.3 Generalised vielbeine and $\mathbf{E}_{7(7)}$

The similarity of the transformations for the generalised vielbeine e^m_{AB}, e_{mnAB} and $e_{m_1\ldots m_5 AB}$ suggests that these are components of a *single* object in *eleven* dimensions, namely a 56-bein

$$\mathcal{V}(z) \equiv \left(\mathcal{V}^{\mathtt{MN}}{}_{AB}(z), \mathcal{V}_{\mathtt{MN}\, AB}(z)\right) \tag{3.25}$$

and its complex conjugate $\left(\mathcal{V}^{\mathtt{MN}}{}_{AB}(z), \mathcal{V}_{\mathtt{MN}\, AB}(z)\right)^* \equiv \left(\mathcal{V}^{\mathtt{MN}\, AB}(z), \mathcal{V}_{\mathtt{MN}}{}^{AB}(z)\right)$, where indices $\mathtt{M}, \mathtt{N} = 1, \ldots, 8$ are associated with the SL(8,$\mathbb{R}$) subgroup of $\mathrm{E}_{7(7)}$. As we discuss elsewhere [12] the normalization of the matrix elements in $\mathcal{V}$ can be chosen such that $\mathcal{V}(z) \in \mathrm{E}_{7(7)}/\mathrm{SU}(8)$, but here we will stick with the normalization adopted in previous work. Accordingly, we proceed from the following identification of this new object with the generalised vielbeine obtained so far[13]

$$\begin{aligned}\mathcal{V}^{m8}{}_{AB} &\equiv e^m_{AB}, \qquad \mathcal{V}_{mn\,AB} \equiv e_{mnAB}\\ \mathcal{V}^{mn}{}_{AB} &\equiv \frac{1}{5!}\Delta\epsilon^{mnp_1\ldots p_5}e_{p_1\ldots p_5AB}\end{aligned} \tag{3.26}$$

in accordance with the decomposition

$$\mathbf{56} \;\to\; \mathbf{28}\oplus\overline{\mathbf{28}} \;\to\; \mathbf{7}\oplus\mathbf{21}\oplus\overline{\mathbf{21}}\oplus\overline{\mathbf{7}} \tag{3.27}$$

of the **56** representation of $\mathrm{E}_{7(7)}$ under its SL(8,$\mathbb{R}$) and GL(7,$\mathbb{R}$) subgroups. Dropping the compensating SU(8) rotation the supersymmetry variations obtained in the foregoing section are then all consistent with the formula

$$\delta\mathcal{V}^{\mathtt{MN}}{}_{AB} = -\sqrt{2}\Sigma_{ABCD}\mathcal{V}^{\mathtt{MN}\,CD}\ , \quad \delta\mathcal{V}_{\mathtt{MN}\,AB} = -\sqrt{2}\Sigma_{ABCD}\mathcal{V}_{\mathtt{MN}}{}^{CD} \tag{3.28}$$

which upon reduction to four dimensions precisely coincides with the variation of the 56-bein in $N=8$ supergravity. Because the theory by construction is invariant under local SU(8) in eleven dimensions, this confirms that the vielbein components identified up to here are indeed part of an $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset element $\mathcal{V}(z^M)$ *in eleven dimensions*.

The 56 vectors can likewise be assembled into a single object of the form $(\mathcal{B}_\mu{}^{\mathtt{MN}}, \mathcal{B}_{\mu\,\mathtt{MN}})$. With the identifications obtained so far, we define

$$\mathcal{B}_\mu{}^{m8} \equiv B_\mu{}^m\ , \quad \mathcal{B}_{\mu\,mn} \equiv B_{\mu mn}\ , \quad \mathcal{B}_\mu{}^{mn} \equiv \frac{1}{5!}\Delta\epsilon^{mnp_1\ldots p_5}\left(B_{\mu p_1\ldots p_5} - \frac{\sqrt{2}}{4}B_{\mu[p_1p_2}A_{p_3p_4p_5]}\right). \tag{3.29}$$

The remaining 'missing' component

$$\mathcal{B}_{\mu\,m8} \equiv \frac{1}{7!}\Delta\epsilon^{n_1\ldots n_7}\mathcal{B}_{\mu n_1\ldots n_7,m} \tag{3.30}$$

will be given in equation (3.36) below. Now, the results for the supersymmetry variations of the vectors introduced above can be summarised by the following simple transformation formulae

$$\begin{aligned}\delta\mathcal{B}_\mu{}^{\mathtt{MN}} &= \frac{\sqrt{2}}{8}\mathcal{V}^{\mathtt{MN}}{}_{AB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] + \text{h.c.},\\ \delta\mathcal{B}_{\mu\,\mathtt{MN}} &= \frac{\sqrt{2}}{8}\mathcal{V}_{\mathtt{MN}\,AB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] + \text{h.c.}\end{aligned} \tag{3.31}$$

complementing the supersymmetry transformations (3.28) of $\mathcal{V}$. These transformations now have exactly the same form as the ones for the corresponding variations of the $D=4$ fields, but they are here valid in eleven dimensions. Note also that the distribution of the 28 *physical* spin-one degrees of freedom between these 56 vectors depends on the given compactification. By comparing these with the variations (3.20) and (3.21) and substituting

[13]The extra factor of Δ in the second line, and in (3.32) below, is necessary in order to maintain the form of the supersymmetry variation given in (3.28).

the identifications (3.26) we can now in principle derive non-linear ansätze for all $D = 11$ fields *and their duals* !

The last missing seven components (3.30) corresponding to the $\overline{\mathbf{7}}$ in the decomposition (3.27), whose existence we had already anticipated above, turn out to be related, not unexpectedly, to dual gravity. In order to identify them and to complete the $E_{7(7)}$ matrix, we note that these components of the matrix $\mathcal{V}(z)$ must be of the form

$$\mathcal{V}_{m8\,AB} = \frac{1}{7!}\Delta\epsilon^{n_1\ldots n_7}e_{n_1\ldots n_7,mAB}. \tag{3.32}$$

This component is found by insisting that $e_{m_1\ldots m_7,nAB}$ transform as

$$\delta e_{m_1\ldots m_7,nAB} = -\sqrt{2}\Sigma_{ABCD}e_{m_1\ldots m_7,n}{}^{CD} - 2\Lambda^C{}_{[A}e_{m_1\ldots m_7,nB]C}, \tag{3.33}$$

just as the other components of $\mathcal{V}$. A calculation now shows that the correct expression is given by

$$\begin{aligned} e_{m_1\ldots m_7,nAB} = -\frac{2}{9!}i\Delta^{-1/2}\Big[&(\Gamma_{m_1\ldots m_7}\Gamma_n)_{AB} + 126\sqrt{2}\;A_{n[m_1m_2}\Gamma_{m_3\ldots m_7]AB} \\ &+ 3\sqrt{2}\times 7!\Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{4}A_{n[m_1m_2}A_{m_3m_4m_5}\Big)\Gamma_{m_6m_7]AB} \\ &+ \frac{9!}{2}\Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{12}A_{n[m_1m_2}A_{m_3m_4m_5}\Big)A_{m_6m_7]p}\Gamma^p{}_{AB}\Big]. \end{aligned} \tag{3.34}$$

The coefficients must take the specific values that appear in the definition (3.34).[14] In other words, the form of the supersymmetry variation and the compensating SU(8) rotation uniquely fixes all coefficients.

In accordance with our previous findings we would expect this generalised vielbein to come from the supersymmetry variation of a vector

$$\delta\mathcal{B}_{\mu m_1\ldots m_7,n} = \frac{\sqrt{2}}{8}e_{m_1\ldots m_7,nAB}\left[2\sqrt{2}\bar{\varepsilon}^A\varphi_\mu^B + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\right] + \text{h.c.}. \tag{3.35}$$

The form of this vector suggests a relation with dual gravity. Ignoring difficulties related to the non-linear extension of dual gravity in eleven dimensions, we find that indeed the above generalised vielbein comes from the supersymmetry transformation of

$$\begin{aligned} \mathcal{B}_{\mu m_1\ldots m_7,n} \equiv B_{\mu m_1\ldots m_7,n} - B_{\mu[m_1\ldots m_5}A_{m_6m_7]n} + c\,5!(2\sqrt{2})B_{[\mu m_1\ldots m_5}B_{m_6m_7]n} \\ + \frac{\sqrt{2}}{12}B_{\mu[m_1m_2}A_{m_3\ldots m_5}A_{m_6m_7]n}, \end{aligned} \tag{3.36}$$

[14] Of course, an overall rescaling by a real constant is allowed.

if and only if the supersymmetry transformation of the new field $B_{\mu m_1\dots m_7,n}$ (*not* $\mathcal{B}_{\mu m_1\dots m_7,n}$!) is

$$\begin{aligned}
\delta B_{\mu m_1\dots m_7,n} = &-\frac{1}{9!}\left(\bar{\varepsilon}\tilde{\Gamma}_{\mu m_1\dots m_7}\Psi_n - 8\bar{\varepsilon}\tilde{\Gamma}_n\tilde{\Gamma}_{[\mu m_1\dots m_6}\Psi_{m_7]}\right) + c\bar{\varepsilon}\tilde{\Gamma}_{[\mu m_1\dots m_4}\Psi_{m_5}A_{m_6 m_7]n} \\
&+\frac{\sqrt{2}}{3}\bar{\varepsilon}\tilde{\Gamma}_{[\mu m_1}\Psi_{m_2}\left(A_{m_3\dots m_7]n} + \frac{\sqrt{2}}{12}A_{m_3\dots m_5}A_{m_6 m_7]n}\right) \\
&- c\,5!\bar{\varepsilon}\tilde{\Gamma}_{[\mu m_1}\Psi_{m_2}\left(A_{m_3\dots m_7]n} + \frac{\sqrt{2}}{4}A_{m_3\dots m_5}A_{m_6 m_7]n}\right),
\end{aligned} \tag{3.37}$$

where c is an undetermined constant.

The indeterminacy encoded in the constant c can be viewed as a consequence of the fact that there is no contribution $\propto B_\mu{}^p h_{pm_1\dots m_7,n}$ in the definition of the field $\mathcal{B}_{\mu m_1\dots m_7,n}$, unlike for the other components of the vector fields. In fact, the structure of the first two terms on the right hand side of (3.37) is partly determined by requiring the absence of terms involving $B_\mu{}^n$ in its variation under local supersymmetry.

We see that the first two terms on the right hand side of equation (3.37) disagree with the eleven-dimensional ansatz (2.25), even though the representation constraint (2.22) is trivially satisfied for all terms on the right hand side by virtue of Schouten's identity as applied to seven dimensions. Nevertheless, the above result is valid *at the full non-linear level*. Equally important, the supersymmetry algebra is expected to close properly on-shell on all components of the 56-bein $\mathcal{V}$, because our theory is physically equivalent on-shell to the original $D = 11$ supergravity (although with a suitable re-interpretation of the symmetries). There appears to be no immediate contradiction with the no-go theorems of [49, 50] because we have abandoned general covariance and Lorentz invariance in eleven dimensions in the course of our construction. However, the disagreement does seem to suggest that the supersymmetry transformation (3.37) is only valid for the particular components given and is not to be regarded as part of a covariant expression for the supersymmetry transformation of a dual gravity field, at least not in a simple way.

Let us now return to the question of how these results relate to more recent studies of generalised geometry. The central object there is an element of the duality coset under consideration and is usually also referred to as the "generalised vielbein." This generalised vielbein, which *a priori* could be different from the one identified here is constructed using a non-linear realisation [21–23], which is a group theoretic method for computing a coset element in a particular representation of the numerator group. For the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ duality coset, the non-linear realisation gives a coset element in the fundamental **56** representation of $\mathrm{E}_{7(7)}$, that is uniquely decomposed under its GL(7,$\mathbb{R}$) subgroup as described in equation (3.27) [18]. In order to compare this construction with the 56-bein derived here, rewrite the 56-bein components as follows

$$\begin{aligned}
\mathcal{V}_{m8\,AB} &= \mathcal{V}_{m8}{}^{a}\Gamma_{a\,AB} + \mathcal{V}_{m8\,ab}\Gamma^{ab}_{AB} + i\mathcal{V}_{m8}{}^{ab}\Gamma_{ab\,AB} + i\mathcal{V}_{m8\,a}\Gamma^{a}_{AB},\\
\mathcal{V}^{mn}{}_{AB} &= \mathcal{V}^{mn}{}_{ab}\Gamma^{ab}_{AB} + i\mathcal{V}^{mn\,ab}\Gamma_{ab\,AB} + i\mathcal{V}^{mn}{}_{a}\Gamma^{a}_{AB},\\
\mathcal{V}_{mn\,AB} &= i\mathcal{V}_{mn}{}^{ab}\Gamma_{ab\,AB} + i\mathcal{V}_{mn\,a}\Gamma^{a}_{AB},\\
\mathcal{V}^{m8}{}_{AB} &= i\mathcal{V}^{m8}{}_{a}\Gamma^{a}_{AB},
\end{aligned} \tag{3.38}$$

where the precise coefficients of the Γ-matrices on the right hand side can be computed from the definition of $\mathcal{V}$, equations (3.26) and (3.32), and the definitions of the generalised vielbeine, equations (3.12), (3.15), (3.23) and (3.34). Forming a 4×4 block matrix with the coefficients on the right hand side as is suggested by the structure of the equations above one finds that the form of this matrix is precisely the same as that found in ref. [18] (see the matrix labelled $\mathcal{R}(\mathcal{V})$ on the top of page 21 in ref. [18]). Of course, the precise numerical factors are different, but this is due to differing conventions. What is important is the form of each element and the precise factors of Δ, which agree. Furthermore, this matrix agrees with the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset element also calculated by non-linear realisation in [33], up to an overall Δ (equation (127) of ref. [33]), which is due to the fact that in [33] the $\mathrm{E}_{7(7)}$ algebra is taken to be embedded in E_{11}.

While the triangular structure evident in (3.38) has been known for a long time to emerge in the reduction to four dimensions [5], the new feature here is that all relations displayed are now valid in eleven dimensions. In particular, and as with the first two generalised vielbeine, by comparing transformations (3.22) and (3.35) to (3.20) and (3.21) one can now construct a non-linear ansatz also for the dual field $A_{m_1\ldots m_6}$. As is demonstrated in appendix C for the Englert solution [46], the six-form potential is expected to be generically non-zero for any compactification other than the torus reduction of [5]. The new non-linear flux ansatz would, in principle, give $A_{m_1\ldots m_6}$ from the expectation values of the four-dimensional scalars. In particular, it would reproduce $A_{m_1\ldots m_6}$ of the Englert solution given in appendix C.

4 Generalised Vielbein postulate

In the SU(8) invariant reformulation of $D = 11$ supergravity the generalised vielbein e^m_{AB} satisfies a number of consistency relations, collectively referred to as the *generalised vielbein postulate*. These are differential relations for the action of the $D = 11$ derivatives on the vielbeine. For the seven internal directions, they read

$$\partial_m e^n_{AB} + \mathcal{Q}^C_{m[A}e^n_{B]C} + \mathcal{P}_{mABCD}e^{nCD} = 0. \tag{4.1}$$

The $\mathrm{E}_{7(7)}$ connection coefficients $\mathcal{Q}^A_{mB}$ and $\mathcal{P}_{mABCD}$,[15] are of the form

$$\mathcal{Q}^A_{mB} = \frac{1}{2}(e^p{}_a\partial_m e_{p\,b})\Gamma^{ab}_{AB} + \frac{\sqrt{2}}{14}ife_{ma}\Gamma^a_{AB} - \frac{\sqrt{2}}{48}e^a_m F_{abcd}\Gamma^{bcd}_{AB}, \tag{4.2}$$

$$\mathcal{P}_{mABCD} = -\frac{3}{4}(e^p{}_a\partial_m e_{p\,b})\Gamma^a_{[AB}\Gamma^b_{CD]} + \frac{\sqrt{2}}{56}ife_m{}^a\Gamma_{ab[AB}\Gamma^b_{CD]} + \frac{\sqrt{2}}{32}e^a_m F_{abcd}\Gamma^b_{[AB}\Gamma^{cd}_{CD]}, \tag{4.3}$$

[15] These coefficients are denoted by $\mathcal{B}^A_{m\,B}$ and $\mathcal{A}_{mABCD}$ in [3].

where

$$f = -\frac{1}{24} i\eta^{\alpha\beta\gamma\delta} F_{\alpha\beta\gamma\delta}. \tag{4.4}$$

Note that the partial derivative ∂_m can be traded for a background covariant derivative $\mathring{D}_m$ appropriate for the S^7 compactification as explained in [3]. Let us now take a look at how (4.1) generalises to the new vielbein components identified in this paper. In doing so, we will not aim for completeness, as much further work is obviously required to penetrate the structures exhibited here.

It takes a bit of algebra to check that, in fact, the new generalised vielbeine do satisfy analogous relations. More precisely, we find

$$\partial_p e_{mnAB} + \Xi_{p|mn|q} e^q_{AB} + \mathcal{Q}_p^C{}_{[A} e_{mnB]C} + \mathcal{P}_{pABCD} e_{mn}{}^{CD} = 0, \tag{4.5}$$

$$\begin{aligned} \partial_p e_{m_1 \dots m_5 AB} + \Xi_{p|m_1 \dots m_5|q} e^q_{AB} + \Xi_{p|m_1 \dots m_5}{}^{qr} e_{qrAB} & \\ + \mathcal{Q}_p^C{}_{[A} e_{m_1 \dots m_5 B]C} + \mathcal{P}_{pABCD} e_{m_1 \dots m_5}{}^{CD} &= 0, \end{aligned} \tag{4.6}$$

$$\begin{aligned} \partial_p e_{m_1 \dots m_7, nAB} + \Xi_{p|m_1 \dots m_7, n}{}^{qr} e_{qrAB} + \Xi_{p|m_1 \dots m_7, n}{}^{q_1 \dots q_5} e_{q_1 \dots q_5 AB} & \\ + \mathcal{Q}_p^C{}_{[A} e_{m_1 \dots m_7, nB]C} + \mathcal{P}_{pABCD} e_{m_1 \dots m_7, n}{}^{CD} &= 0, \end{aligned} \tag{4.7}$$

where

$$\Xi_{p|mn|q} \equiv \partial_p A_{mnq} - \frac{1}{4!} F_{pmnq}, \tag{4.8}$$

$$\begin{aligned} \Xi_{p|m_1 \dots m_5|q} \equiv\; & \partial_p A_{qm_1 \dots m_5} + \frac{\sqrt{2}}{48} F_{p[qm_1 m_2} A_{m_3 \dots m_5]} \\ & - \frac{\sqrt{2}}{2} \left(\partial_p A_{[qm_1 m_2} - \frac{1}{4!} F_{p[qm_1 m_2} \right) A_{m_3 \dots m_5]} - \frac{1}{7!} F_{pqm_1 \dots m_5}, \end{aligned} \tag{4.9}$$

$$\Xi_{p|m_1 \dots m_5}{}^{qr} \equiv \frac{1}{\sqrt{2}} \Xi_{p|[m_1 m_2|m_3} \delta^{qr}_{m_4 m_5]}, \tag{4.10}$$

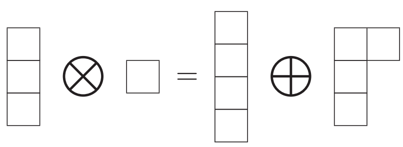
$$\Xi_{p|m_1 \dots m_7, n}{}^{qr} \equiv -\Xi_{p|[m_1 \dots m_5||n|} \delta^{qr}_{m_6 m_7]}, \tag{4.11}$$

$$\Xi_{p|m_1 \dots m_7, n}{}^{q_1 \dots q_5} \equiv \Xi_{p|[m_1 m_2||n|} \delta^{q_1 \dots q_5}_{m_3 \dots m_7]}. \tag{4.12}$$

As was to be expected from the explicit dependence of the new vielbein components on the 3-form and 6-form potentials, there appear terms which are *not gauge invariant*. However, closer inspection of these expressions now reveals a truly remarkable feature, not at all obvious nor to be expected from (3.15), (3.23) and (3.34): they all vanish upon antisymmetrisation, and therefore precisely correspond to the Young tableaux that are eliminated by projecting onto the gauge invariant field strengths upon acting with a derivative on the 3-form or 6-form potential! More specifically, we have

$$\partial_m A_{npq} = \frac{1}{4!} F_{mnpq} + \Xi_{m|np|q} \tag{4.13}$$

corresponding to the Young tableau decomposition

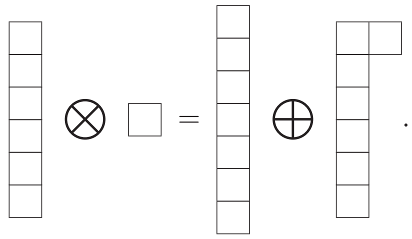


(4.14)

and similarly for the 7-form field strength

$$\partial_{m_1} A_{m_2 \dots m_7} = \frac{1}{7!} F_{m_1 \dots m_7} + \Xi_{m_1|m_2 \dots m_6|m_7} \tag{4.15}$$

corresponding to



(4.16)

In ref. [24], a version of the vielbein postulate was given *with* Christoffel symbols along the internal directions included, so the above findings motivate a similar interpretation of the Ξ symbols as *generalised connections* along the remaining directions of the $E_{7(7)}$ vielbein (3.25), in accordance with the decomposition (3.27). More precisely, this symbol would be of the form $\Xi^{\mathtt{MN}}_{\mathtt{PQ}\,\mathtt{RS}}$, where the SL(8,$\mathbb{R}$) index pairs can appear either in the upper or the lower position. Because the gauge invariant field strengths are part of the connection coefficients $\mathcal{Q}^{A}_{mB}$ and $\mathcal{P}_{mABCD}$ the above decompositions should thus be regarded on a par with the corresponding decomposition of the usual vielbein derivative, *viz.*

$$\partial_M E_N{}^A = \omega_{MN}{}^A + \Gamma_{MN}{}^A \tag{4.17}$$

into a piece covariant with respect to general coordinate transformations, and a non-gauge invariant piece, thus extending ordinary geometry so as to comprise the p-form fields of $D = 11$ supergravity. This interpretation is further supported by the fact that, like the standard connections, the above objects are *not* gauge invariant under the respective 2-form and 5-form gauge transformations, in line with the interpretation of the latter as new coordinate transformations, while the non-gauge covariant part of the variation drops out in the difference of two such connections, again in complete analogy with usual affine connections.

Moreover, note that the generalised vielbein e^m_{AB} is absent from the generalised vielbein postulate for $e_{m_1 \dots m_7, nAB}$, equation (4.7), or equivalently

$$\Xi_{p|m_1 \dots m_7, n|q} \equiv 0. \tag{4.18}$$

For this term to be non-zero, it would be required for it to contain undifferentiated 3-form or 6-form potentials, which would introduce non-gauge invariances beyond what would be expected from connection components. Therefore, the vanishing of the above term is desirable from this perspective.

The appearance of non-gauge invariant expressions for the 3-form and 6-form gauge fields may appear strange at first sight, because all investigations of their role in supergravity and superstring theory have so far focused exclusively on the gauge invariant (p+1)-form field strengths. In this regard it is noteworthy that the level expansion of the E_{10} algebra gives rise to an infinite tower of so-called 'gradient representations', which have been tentatively associated with the (time derivative of the) spatial gradients of the 3-from and

6-form fields [62], and where there is likewise no antisymmetrisation in the spatial indices. In string theory, the gauge invariant field strengths are associated to $D(p-1)$-branes, widely considered a key ingredient towards a better understanding of non-perturbative string theory. Our partial results above again underline the necessity of coming to grips with *non-trivial Young tableau representations*, which in the level expansion of E_{10} constitute the vast majority of representations [68].

For the derivatives along the space-time directions, we have similar relations which now also involve the vector fields $B_\mu{}^m$, $B_{\mu mn}$ and $B_{\mu m_1\dots m_5}$. The components e^m_{AB} are already known to satisfy the following equation [3]

$$\mathcal{D}_\mu e^m_{AB} + \frac{1}{2}\partial_n B_\mu{}^n e^m_{AB} + \partial_n B_\mu{}^m e^n_{AB} + \mathcal{Q}^C_{\mu\,[A} e^m_{B]C} + \mathcal{P}_{\mu ABCD} e^{mCD} = 0, \tag{4.19}$$

where $\mathcal{D}_\mu \equiv \partial_\mu - B_\mu{}^n\partial_n$ and the $\mathrm{E}_{7(7)}$ connection coefficients

$$\mathcal{Q}^A_{\mu\,B} = -\frac{1}{2}\left[e^m{}_a\partial_m B_\mu{}^n e_{nb} - (e^p{}_a\mathcal{D}_\mu e_{p\,b})\right]\Gamma^{ab}_{AB} - \frac{\sqrt{2}}{12}\Delta^{-1/2} e'_\mu{}^\alpha\Big(F_{\alpha abc}\Gamma^{abc}_{AB} - \eta_{\alpha\beta\gamma\delta}F^{\beta\gamma\delta a}\Gamma_{aAB}\Big), \tag{4.20}$$

$$\begin{aligned}\mathcal{P}_{\mu ABCD} = \frac{3}{4}\left[e^m{}_a\partial_m B_\mu{}^n e_{nb} - (e^p{}_a\mathcal{D}_\mu e_{p\,b})\right]\Gamma^a_{[AB}\Gamma^b_{CD]} - \frac{\sqrt{2}}{8}\Delta^{-1/2} e'_\mu{}^\alpha F_{abc\alpha}\Gamma^a_{[AB}\Gamma^{bc}_{CD]} \\ - \frac{\sqrt{2}}{48}\Delta^{-1/2} e'_{\mu\,\alpha}\eta^{\alpha\beta\gamma\delta}F_{a\beta\gamma\delta}\Gamma_{b[AB}\Gamma^{ab}_{CD]}.\end{aligned} \tag{4.21}$$

In analogy with this, the derivative of the new generalised vielbeine along the space-time directions satisfy

$$\begin{aligned}\mathcal{D}_\mu e_{mnAB} + \frac{1}{2}\partial_p B_\mu{}^p e_{mnAB} + 2\partial_{[m} B_{|\mu|}{}^p e_{n]pAB} + 3\partial_{[m} B_{|\mu|np]} e^p_{AB} \\ + \mathcal{Q}^C_{\mu\,[A} e_{mnB]C} + \mathcal{P}_{\mu ABCD} e_{mn}{}^{CD} = 0,\end{aligned} \tag{4.22}$$

$$\begin{aligned}\mathcal{D}_\mu e_{m_1\dots m_5AB} + \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\dots m_5AB} - 5\partial_{[m_1} B_{|\mu|}{}^p e_{m_2\dots m_5]pAB} + \frac{3}{\sqrt{2}}\partial_{[m_1} B_{|\mu|m_2m_3} e_{m_4m_5]AB} \\ - 6\partial_{[m_1}\Big(B_{|\mu|m_2\dots m_5p]} - \frac{\sqrt{2}}{4}B_{|\mu|m_2m_3}A_{m_4m_5p]}\Big)e^p_{AB} + \mathcal{Q}^C_{\mu\,[A} e_{m_1\dots m_5B]C} + \mathcal{P}_{\mu ABCD} e_{m_1\dots m_5}{}^{CD} = 0,\end{aligned} \tag{4.23}$$

$$\begin{aligned}\Big\{\mathcal{D}_\mu e_{m_1\dots m_7,nAB} + \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\dots m_7,nAB} - 7\partial_{m_1} B_\mu{}^p e_{pm_2\dots m_7,nAB} - \partial_n B_\mu{}^p e_{m_1\dots m_7,pAB} \\ + 3\partial_{[n} B_{|\mu|m_1m_2]} e_{m_3\dots m_7AB} - 6\partial_{[n}\left(B_{|\mu|m_1\dots m_5]} - \frac{\sqrt{2}}{4}B_{|\mu|m_1m_2}A_{m_3m_4m_5]}\right)e_{m_6m_7AB} \\ + \mathcal{Q}^C_{\mu\,[A} e_{m_1\dots m_7,nB]C} + \mathcal{P}_{\mu ABCD} e_{m_1\dots m_7,n}{}^{CD}\Big\}_{[m_1\dots m_7]} = 0.\end{aligned} \tag{4.24}$$

Note in particular that the vector fields that enter the 4-dimensional generalised vielbein postulate are precisely the $\mathrm{E}_{7(7)}$ covariant vector fields (3.29) that give rise to the generalised vielbeine.

In refs [12] and [72] we come back to relations (4.22)–(4.24) and further investigate their role with regard to the embedding tensor formalism [43, 44] and the $D = 4$ gaugings studied in [45, 67]. As in the above relations, where we are dealing with a **56** of vector fields, there as well the gauged theory is formulated in terms of a doubled set of 56 vector fields, such that the 28 physical components are selected, together with the non-abelian gauge group, by the embedding tensor, whose $D = 11$ origins are expected to be hidden in the above relations. This is of particular interest in view of the recent work on the vacuum structure of maximal gauged supergravities in four dimensions, which has turned out to be far richer than originally expected [69, 70] (see [71] and references therein for more recent work on this).

5 Outlook: generalisation to $\mathrm{E}_{8(8)}$ and $\mathrm{E}_{6(6)}$

The results of this paper clearly point to an underlying structure of which we have so far only seen a small part. In fact, similar results exist for other reductions of $D = 11$ supergravity, most notably the one corresponding to the 3+8 decomposition of the theory, where the relevant group is $\mathrm{E}_{8(8)}$ and where, finally, the dual gravity field enters with full force, giving rise to eight physical scalar degrees of freedom. For this case partial results have been known for a long time [6, 7].

In this section we briefly sketch how our construction generalises to $\mathrm{E}_{8(8)}$ and also the simpler case of $\mathrm{E}_{6(6)}$. In the former case, some of the relevant vielbeine have already been identified in [6, 7], and the existence of a corresponding $\mathrm{E}_{8(8)}$-valued 248-bein in eleven dimensions is proved in [7], although in a more indirect manner. So let us consider this case first. To this aim, we perform a $3 + 8$ split of $D = 11$. More specifically, the fields of the theory that give rise to scalar and vector degrees of freedom in a conventional reduction to three dimensions are, respectively, g_{mn} and A_{mnp}, and $B_\mu{}^m$ and $B_{\mu mn}$, where now $\mu = 0, 1, 2$ is a 3-dimensional space-time index and $m, n, p = 3, \ldots, 10$ are the 8-dimensional spatial indices. As before, the field $B_\mu{}^m$ is the off-diagonal component of the elfbein in the 3+8 split, while g_{mn} is the metric in 8-dimensional directions. $B_{\mu mn}$ is related to the eleven-dimensional 3-form by the field redefinition $B_{\mu mn} = A_{\mu mn} - B_\mu{}^p A_{mnp}$, as before. As is well known, in three dimensions vector fields are dual to scalars, so these fields account for the $248 - 120 = 128$ scalars that parametrise the $\mathrm{E}_{8(8)}/\mathrm{SU}(8)$ coset. It is shown in [7] that the supersymmetry transformations of $B_\mu{}^m$ and $B_{\mu mn}$ in the SO(16) reformulation of $D = 11$ supergravity [6] lead to two generalised vielbeine

$$e^m{}_{\mathcal{A}} \quad \text{and} \quad e_{mn\mathcal{A}},$$

where $\mathcal{A} = 1, \ldots, 248$ is an $\mathrm{E}_{8(8)}$ index. In analogy with (3.26) these generalised vielbeine can be combined into a 36×248 matrix, which can be thought of as being part of a 248×248 $\mathrm{E}_{8(8)}$ matrix. In fact, the existence of such an $\mathrm{E}_{8(8)}$ matrix in eleven dimensions was inferred in [7] by indirect group theoretic arguments. The results of this paper can now be used to give a more explicit description of this matrix.

We will describe this construction elsewhere, but let us nevertheless outline the calculation that needs to be done. In order to enlarge the 36×248 matrix we must consider a component of the eleven-dimensional 6-form $B_{\mu m_1 \ldots m_5} \sim B_\mu^{npq}$. The supersymmetry variation of this field leads to 56 further components, which add another 56×248 chunk to the generalised vielbein. Finally, the dual gravity field will give 64 further components from $B_{\mu m_1 \ldots m_7, n} \sim B_\mu{}^m{}_n$, which in total give a 156×248 matrix containing scalars coming from the reduction of the metric, 3-form, 6-form and dual gravity. Since these account for all of the scalar degrees of freedom, the completion of this matrix to an $\mathrm{E}_{8(8)}$ matrix will not introduce any new degrees of freedom.[16] In other words, the full $\mathrm{E}_{8(8)}$ matrix is completely determined by this 156×248 submatrix. However, there remains the interesting question of where the extra components come from. The GL(8,$\mathbb{R}$) decomposition

$$\mathbf{248} \longrightarrow \overline{\mathbf{8}} + \mathbf{28} + \overline{\mathbf{56}} + \mathbf{64} + \mathbf{56} + \overline{\mathbf{28}} + \mathbf{8} \tag{5.1}$$

suggests that three more fields are required in eleven dimensions in order to give rise to the remaining $56 + 28 + 8$ vector fields in the dimensionally reduced theory.

In addition, one can consider the status of the generalised vielbein postulate in this case. Decomposing[17]

$$\mathcal{A} = ([IJ], A),$$

where $I, J = 1, \ldots, 16$ and $A, B, \ldots = 1, \ldots, 128$ are now SO(16) vector and chiral spinor indices, respectively, the generalised vielbein $e^m{}_{\mathcal{A}}$ satisfies [6]

$$\mathcal{D}_\mu e^m_{IJ} + \partial_n B_\mu{}^n e^m_{IJ} + \partial_n B_\mu{}^m e^n_{IJ} + 2\mathcal{Q}_{\mu K[I} e^m_{J]K} + \Gamma^{IJ}_{AB} \mathcal{P}_\mu{}^A e^m_B = 0, \tag{5.2}$$

$$\partial_m e^n_{IJ} + 2\mathcal{Q}_{mK[I} e^n_{J]K} + \Gamma^{IJ}_{AB} \mathcal{P}_m{}^A e^n_B = 0, \tag{5.3}$$

where $\Gamma^I_{A\dot{A}}$ is a Spin(16) gamma-matrix and the $\mathrm{E}_{8(8)}$ connection components are defined in [6]. The remaining components e^m_A satisfy similar relations [6]. Analogously, $e_{mn\mathcal{A}}$ found in [7] satisfies

$$\mathcal{D}_\mu e_{mnIJ} + \partial_p B_\mu{}^p e_{mnIJ} + 2\partial_{[m} B_\mu{}^p e_{n]pIJ} + 18\sqrt{2}\partial_{[m} B_{np]\mu} e^p_{IJ} + 2\mathcal{Q}_{\mu K[I} e_{mnJ]K} + \Gamma^{IJ}_{AB} \mathcal{P}_\mu{}^A e_{mnB} = 0, \tag{5.4}$$

$$\partial_p e_{mnA} + 6\sqrt{2}\left(\partial_p A_{mnq} - \frac{1}{4!} F_{pmnq}\right) e^q_A + \frac{1}{4}\mathcal{Q}_{pIJ}\Gamma^{IJ}_{AB} e_{mnB} - \frac{1}{2}\Gamma^{IJ}_{AB}\mathcal{P}_p{}^B e_{mnIJ} = 0. \tag{5.5}$$

Note the striking resemblance of these equations to their $\mathrm{E}_{7(7)}$ counterparts, equations (4.22) and (4.5). In particular, note the presence of the vector field $B_{\mu mn}$, the supersymmetry transformation of which gives $e_{mn\mathcal{A}}$ in equation (5.4) and the non-gauge invariant "connection" term, analogous to connection (4.8), in equation (5.5).

The construction of the $\mathrm{E}_{6(6)}$ matrix from the eleven-dimensional fields is more straightforward and only requires consideration of the eleven-dimensional metric, 3-form field and its 6-form dual, because the dual gravity field does not give rise to any physical degrees of freedom. In the $5 + 6$ split, the components of the eleven-dimensional fields that give rise to vector and scalar degrees of freedom under reduction to five dimensions are

$$B_\mu{}^m\,,\; B_{\mu mn}\,,\; B_{\mu\nu m}\,,\; g_{mn}\,,\; A_{mnp}\,,\; B_{\mu\nu\rho}, \tag{5.6}$$

[16] See for example [34], where the $\mathrm{E}_{8(8)}$ matrix is found by group theoretic means.

[17] Our apologies to the reader for the multiple different uses of these letters.

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where now $\mu, \nu, \rho = 0, \ldots 4$ and $m, n, p = 5, \ldots, 10$. Note that in 5-dimensions, 3-forms are dual to scalars. Therefore, in total there are 42 scalars coming from g_{mn}, A_{mnp} and $B_{\mu\nu\rho}$ that parametrise the $\mathrm{E}_6/\mathrm{USp}(8)$ coset.

The $\mathrm{E}_{6(6)}$ matrix in eleven dimensions can be constructed from the generalised vielbeine that arise from the supersymmetry transformations of $B_\mu{}^m, B_{\mu mn}, B_{\mu m_1\ldots m_5}$ in a USp(8) invariant reformulation of $D = 11$ supergravity along the lines of [3, 6]. The $\mathrm{E}_{6(6)}$ matrix is parametrised by g_{mn}, A_{mnp} and the dual 6-form $A_{m_1\ldots m_6}$. We stress once more that the construction of the $\mathrm{E}_{6(6)}$ does not involve the dual gravity field and only depends on fields that are well-understood in eleven dimensions. The $\mathrm{E}_{6(6)}$ matrix thus constructed should be equivalent to the $\mathrm{E}_{6(6)}$ matrix constructed in [33] by group theory.

6 Concluding remarks

In this paper, we have established new structures in $D = 11$ supergravity, which demonstrate most clearly and explicitly the extent to which the duality group $\mathrm{E}_{7(7)}$ plays a role in eleven dimensions. At the heart of the formalism that we develop, which is on-shell equivalent to the Cremmer-Julia-Scherk theory [1], are the generalised vielbeine that are packaged into a 56-bein $\mathcal{V}(z)$. The 56-bein contains all eleven-dimensional degrees of freedom that reduce to scalar degrees of freedom in four dimensions and is determined in terms of fields that are fully understood in $D = 11$ supergravity, namely components of the elfbein, 3-form and dual 6-form potentials. Furthermore, we show that the components of the 56-bein transform, under the supersymmetry transformations of $D = 11$ supergravity, according to equations (3.16), (3.17), (3.24), (3.33), which are analogues of the supersymmetry transformation of the scalars in the four-dimensional ungauged theory [5].

The first three components of the 56-bein, (3.12), (3.15) and (3.23), are found by considering supersymmetry transformations (3.13) [3], (3.14) [2] and (3.22). The final component, (3.34) is *uniquely* determined by requiring that it satisfies the supersymmetry transformation (3.33). Therefore, we obtain the $\mathrm{E}_{7(7)}$ valued matrix. Note that this final component does not contain any new degrees of freedom. This is expected as the first three components already contain all the degrees of freedom that are associated with the scalars in the reduced theory. In this sense the last component of the 56-bein is somewhat auxiliary.

A natural completion of the would-be vectors would be to identify a field of the form $\mathcal{B}_{\mu m_1\ldots m_7, n}$ whose supersymmetry transformation leads to the final component, equation (3.35). The structure of this field, and how it emerges, suggests a clear link with dual gravity. While we do not understand the relation of this field to the $D = 11$ supergravity fields, we can deduce that such a field must satisfy a supersymmetry transformation of the form given in (3.37). In any case this field plays no role in the derivation of the 56-bein, non-linear uplift ansätze nor the generalised vielbein postulates. Nevertheless, with this field, the 56-bein is related to a set of $\mathcal{B}$ that give rise to the 56 vector degrees of freedom in the reduction to four dimensions. Simply stated, the 56-bein $\mathcal{V}$ is given by the supersymmetry transformation of $\mathcal{B}$:

$$\delta\mathcal{B} \sim \mathcal{V}(\text{fermions}).$$

The generalised vielbeine satisfy differential constraints, which we refer to as generalised vielbein postulates (see section 3.3). The generalised vielbein postulates reveal a remarkably rich structure, some of which we point out. However, a deeper analysis of these equations may reveal interesting insights into the unification of gravitational and matter degrees of freedom ("generalised geometry") and the precise role of $E_{7(7)}$ within this context.

Apart from more fundamental issues related to generalised geometry, the formalism that we develop in this paper and, in particular the generalised vielbein postulates (4.19), (4.22)–(4.24), provide an appropriate setting for understanding issues related to four dimensional gauged theories. In particular, there is a direct relation between these generalised vielbein postulates and the embedding tensor of maximal gauged supergravities in four dimensions. This is not so surprising given that the embedding tensor enters an $E_{7(7)}$ Cartan equation that includes derivatives of the 56-bein and the gauge vectors — this is precisely the form of the generalised vielbein postulates. Furthermore, it is not so surprising given the old results of ref. [9]. In ref. [9], it is shown how *one* of the generalised vielbein postulates, (4.19) (the only one known up to now), can be used to derive the embedding tensor of maximal SO(8) gauged supergravity. Now that we have all generalised vielbein postulates with derivatives a more direct analysis of the embedding tensor from an eleven dimensional perspective is possible.

Indeed, in [12], we rederive the embedding tensor of maximal SO(8) gauged supergravity using the generalised vielbein postulates. As expected, this derivation is much simpler than that undertaken in ref. [9]. Furthermore, we give explicit uplift ansaetze, including for dual fields, for a fully constructive uplifting of the solutions of SO(8) gauged maximal supergravity to eleven dimensions and demonstrate their non-triviality and validity for some non-trivial examples. This is the first time that non-trivial solutions of maximal gauged supergravity have been uplifted to eleven dimensional solutions in a fully constructive approach without recourse to numerical methods.

In a forthcoming paper [72], we demonstrate explicitly how our methods provide a complete understanding of how the embedding tensor emerges for Scherk-Schwarz compactification with background flux.

One of our main motivations in this program is a possible higher dimensional understanding of the new deformed SO(8) gauged supergravities [4]. We hope to address this exciting prospect in the future.

Acknowledgments

We are grateful to Axel Kleinschmidt and Bernard de Wit for stimulating discussions, and to Bernard Schutz for inspiration [73]. We are also grateful to O. Hohm and H. Samtleben for alerting us to ref. [74] where four-dimensional Einstein theory is reformulated in terms of a constrained $(3 + 3)$-dimensional theory with manifest Ehlers symmetry.

A Conventions

We use the following conventions:

$$
\begin{aligned}
A_{[a_1\ldots a_p]} &= \frac{1}{p!}\left(A_{a_1\ldots a_p} + (p!-1)\text{ terms}\right),\\
(dA)_{a_1\ldots a_{p+1}} &= (p+1)!\partial_{[a_1}A_{a_2\ldots a_{p+1}]},\\
(\star A)_{a_1\ldots a_{d-p}} &= \frac{i}{p!}\epsilon_{a_1\ldots a_{d-p}b_1\ldots b_p}A^{b_1\ldots b_p}.
\end{aligned}
$$

B Supersymmetry transformation identities

Below we list some equations that prove to be useful in deriving the supersymmetry transformations of the generalised vielbeine.

$$
\delta\left(i\Delta^{-1/2}\Gamma^p{}_{AB}\right) = -\sqrt{2}\,\Sigma_{ABCD}\left(i\Delta^{-1/2}\Gamma^p{}_{CD}\right) - 2\Lambda^C{}_{[A}\left(i\Delta^{-1/2}\Gamma^p{}_{B]C}\right), \tag{B.1}
$$

$$
\begin{aligned}
\delta\left(i\Delta^{-1/2}\Gamma_{mnAB}\right) = &-\sqrt{2}\,\Sigma_{ABCD}\left(-i\Delta^{-1/2}\Gamma_{mnCD}\right) - 2\Lambda^C{}_{[A}\left(i\Delta^{-1/2}\Gamma_{mnB]C}\right)\\
&+\frac{3}{2}\,i\Delta^{-1/2}\,\bar{\varepsilon}\Gamma_{[mn}\Psi_{p]}\,\Gamma^p{}_{AB},
\end{aligned} \tag{B.2}
$$

$$
\begin{aligned}
\delta\left(i\Delta^{-1/2}\Gamma_{m_1\ldots m_5AB}\right) = &-\sqrt{2}\,\Sigma_{ABCD}\left(i\Delta^{-1/2}\Gamma_{m_1\ldots m_5CD}\right) - 2\Lambda^C{}_{[A}\left(i\Delta^{-1/2}\Gamma_{m_1\ldots m_5B]C}\right)\\
&+15\,i\Delta^{-1/2}\,\bar{\varepsilon}\Gamma_{[m_1m_2}\Psi_{m_3}\Gamma_{m_4m_5]AB} + 3\,i\Delta^{-1/2}\,\bar{\varepsilon}\gamma_5\Gamma_{[m_1\ldots m_5}\Psi_{p]}\,\Gamma^p{}_{AB},
\end{aligned} \tag{B.3}
$$

$$
\begin{aligned}
\delta\left(\Delta^{-1/2}\epsilon_{m_1\ldots m_7}\Gamma_{nAB}\right) = &-\sqrt{2}\,\Sigma_{ABCD}\left(-\Delta^{-1/2}\epsilon_{m_1\ldots m_7}\Gamma_{nCD}\right) - 2\Lambda^C{}_{[A}\left(\Delta^{-1/2}\epsilon_{m_1\ldots m_7}\Gamma_{nB]C}\right)\\
&-\Delta^{-1/2}\epsilon_{m_1\ldots m_7}\left(\frac{3}{4}\bar{\varepsilon}\Gamma_{[pq}\Psi_{n]}\Gamma^{pq}_{AB} + \frac{1}{2}\bar{\varepsilon}\gamma_5\Gamma^{pq}\Psi_p\,\Gamma_{qnAB}\right).
\end{aligned} \tag{B.4}
$$

The first equation in the list above is the precisely the supersymmetry transformation of the generalised vielbein e^m_{AB} found in [3]. The second equation [2] is used to derive the supersymmetry transformation of the generalised vielbein e_{mnAB}.

C Six-form potential of the Englert solution

In this appendix, we demonstrate that even for a very simple eleven-dimensional solution with non-vanishing flux, the Englert solution [46], the six-form potential is non-zero and contains non-vanishing components mixing space-time and internal components (thus vitiating one of the basic assumptions made in several recent approaches to generalised geometry). This leads us to expect that the six-form potential will in general acquire a non-trivial form for all solutions with non-vanishing flux, that is, all solutions other than the torus compactification.

The Englert solution satisfies the Freund-Rubin ansatz [75] and preserves an $SO(7)^-$ subgroup of $SO(8)$. More explicitly, the solution is of the form

$$
\begin{aligned}
g_{MN} &= \gamma^{7/18}\left(\mathring{\eta}_{\mu\nu},\ \gamma^{-1/2}\mathring{g}_{mn}\right),\\
F_{MNPQ} &= \left(2\sqrt{2}i\,m_7\,\gamma^{5/6}\mathring{\eta}_{\mu\nu\rho\sigma},\ \frac{\sqrt{2}}{6}m_7\,\gamma^{-1/6}\,\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}\right),
\end{aligned} \tag{C.1}
$$

where $\mathring{\eta}_{\mu\nu}$ is the anti-de Sitter metric, $\mathring{g}_{mn}$ is the round metric on the seven-sphere with inverse radius m_7 and all quantities with four-dimensional (seven-dimensional) indices are tensors with respect to $\mathring{\eta}_{\mu\nu}$ ($\mathring{g}_{mn}$). γ is an arbitrary positive constant, which takes the value $\gamma^{1/3} = 5/4$ when the solution is constructed via the non-linear flux ansatz [11]. Furthermore, the torsion tensor $\mathring{S}_{mnp}$ satisfies the relation

$$\mathring{D}_m S_{npq} = \frac{1}{6} m_7 \mathring{\eta}_{mnpqrst} \mathring{S}^{rst}. \tag{C.2}$$

From equation (2.13), the six-form potential is given by the following equation

$$7! D_{[M_1} A_{M_2 \dots M_7]} = \frac{i}{4!} \eta_{M_1 \dots M_{11}} F^{M_8 \dots M_{11}} - \frac{7! \sqrt{2}}{4!\, 2} A_{[M_1 \dots M_3} F_{M_4 \dots M_7]}. \tag{C.3}$$

Note that

$$\eta_{\mu\nu\rho\sigma m_1 \dots m_7} = \gamma^{7/18} \mathring{\eta}_{\mu\nu\rho\sigma} \mathring{\eta}_{m_1 \dots m_7}, \tag{C.4}$$

while

$$F^{MNPQ} = \left(2\sqrt{2} i\, m_7\, \gamma^{-13/18} \mathring{\eta}^{\mu\nu\rho\sigma},\ \frac{\sqrt{2}}{6} m_7\, \gamma^{5/18}\, \mathring{\eta}^{mnpqrst} \mathring{S}_{rst} \right), \tag{C.5}$$

where the indices on F_{MNPQ} are raised using the eleven-dimensional (inverse) metric g^{MN}. Clearly, the right hand side of equation (C.3) is only non-zero for $[M_1 \dots M_7]$ equal to $[m_1 \dots m_7]$, $[\mu\nu\rho\sigma mnp]$ or $[\mu\nu\rho m_1 \dots m_4]$. Thus,

$$7!\ D_{[M_1} A_{M_2 \dots M_7]} = \begin{cases} -\frac{15\sqrt{2}}{4} m_7\, \gamma^{-1/3} \mathring{\eta}_{m_1 \dots m_7} & [m_1 \dots m_7] \\ \frac{\sqrt{2}}{2} i\, m_7\, \gamma^{2/3} \mathring{\eta}_{\mu\nu\rho\sigma} \mathring{S}_{mnp} & [\mu\nu\rho\sigma mnp] \\ -2\sqrt{2} i\, m_7\, \gamma^{2/3} \mathring{\zeta}_{\mu\nu\rho} \mathring{\eta}_{m_1 \dots m_7} \mathring{S}^{m_5 \dots m_7} & [\mu\nu\rho m_1 \dots m_4] \\ 0 & \text{otherwise} \end{cases}, \tag{C.6}$$

where $\mathring{\zeta}_{\mu\nu\rho}$ is the potential for the Freund-Rubin field strength and is only defined locally

$$4!\, \partial_{[\mu} \mathring{\zeta}_{\nu\rho\sigma]} = m_7\, \mathring{\eta}_{\mu\nu\rho\sigma}. \tag{C.7}$$

Hence,

$$A_{M_1 \dots M_6} = \begin{cases} \frac{\sqrt{2}}{12} i\, \gamma^{2/3} \mathring{\zeta}_{\mu\nu\rho} \mathring{S}_{mnp} & [\mu\nu\rho mnp] \\ -\frac{15\sqrt{2}}{4} \gamma^{-1/3} \mathring{\zeta}_{m_1 \dots m_6} & [m_1 \dots m_6] \\ 0 & \text{otherwise} \end{cases}, \tag{C.8}$$

where $\mathring{\zeta}_{m_1 \dots m_6}$ is such that

$$7!\, \partial_{[m_1} \mathring{\zeta}_{m_2 \dots m_7]} = m_7\, \mathring{\eta}_{m_1 \dots m_7}. \tag{C.9}$$

As anticipated, $A_{M_1 \dots M_6}$ has non-vanishing components with *both* space-time and internal indices.

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