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ON EMBEDDINGS IN CYCLES

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Abstract

We prove exact results on dilations in cycles for important parallel computer interconnection networks as complete trees, hypercubes and 2- and 3-dimensional meshes. Moreover we show that trees, X-trees, n-dimensional meshes, pyramides and trees of meshes have the same dilations both in the path and the cycle.

1 Introduction

A lot of work has been done in recent years in the study of the properties of interconnection networks for parallel computer systems [15]. An important feature of an interconnection network is its ability to efficiently simulate programs written for other architectures. Such simulation problem can be mathematically formulated as a graph embedding. Informally, the graph embedding problem is to label the vertices of a "guest" graph (interconnection network) G by distinct vertices of a "host" graph (network) H. The quality of the embedding corresponding to the efficiency of the simulation (time delay of the communication among processors) is the maximum distance in H between adjacent vertices in G. The minimum maximum distance over all embeddings is called the dilation of G in H and denoted by $\mathrm{dil}(G, H)$. To know the dilation of a graph G in a graph H can also help in such important tasks as the minimization of wire lengths in VLSI layout [2] or the representation of a data structure by another data structure [17].

In this paper we shall investigate embeddings of well-known interconnection networks in cycles. The importance of this task was pointed out by Chung [4]. We note that to decide whether for a given graph G and a given number k, $\mathrm{dil}(G, C_n) < k$ holds is NP-complete (C_n denotes the cycle of n vertices). To our knowledge the only known exact result for the dilation of graphs in cycles is $\mathrm{dil}(C_n \times C_n, C_{n^2}) = n$ [14]. We prove several exact results for the dilation of well-known interconnection networks in cycles, namely: $\mathrm{dil}(T_{t,r}, C_{(t^{r-1}-1)/(r-1)}) =$

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$\lceil t(t^{r-1}-1)/(2(r-1)(t-1))\rceil$, for complete r-level t-ary trees, $\mathrm{dil}(Q_n, C_{2^n}) = \sum_{k=0}^{n-1} \binom{k}{\lfloor \frac{k}{2} \rfloor}$, for n-dimensional hypercubes, $\mathrm{dil}(P_n \times P_n \times P_n, C_{n^3}) = \lfloor 3n^2/4 + n/2 \rfloor$, for 3-dimensional meshes (where P_n is an n-vertex path), $\mathrm{dil}(P_m \times P_n, C_{mn}) = \mathrm{dil}(C_m \times P_n, C_{mn}) = \mathrm{dil}(C_m \times C_n, C_{mn}) = \min\{m, n\}$, for 2-dimensional ordinary, cylindrical and toroidal meshes, respectively. Thus we solve three remaining open problems of the type "$\mathrm{dil}(X \times Y, Z) = ?$", where X, Y and Z are paths or cycles. The result on $\mathrm{dil}(P_m \times P_n, C_m n)$ was independently achieved by Fujita and Hsu [8] and Leighton [13]. The previously known results are: $\mathrm{dil}(P_m \times P_n, P_{mn}) = \min\{m, n\}$ [5, 7], $\mathrm{dil}(C_m \times P_n, P_{mn}) = \min\{m, 2n\}$ [6] and $\mathrm{dil}(C_m \times C_n, P_{mn}) = 2\min\{m, n\}$, if $m \neq n$, otherwise $\mathrm{dil}(C_n \times C_n) = 2n - 1$ [16].

The proofs of the above stated results are based on the following technique. One can easily observe that, for any graph G, $\mathrm{dil}(G, P_n)/2 \leq \mathrm{dil}(G, C_n) \leq \mathrm{dil}(G, P_n)$. We find a suficient condition for a graph G which assures the equality $\mathrm{dil}(G, C_n) = \mathrm{dil}(G, P_n)$. We prove that trees, X-trees, meshes, hypercubes, pyramides and tree of meshes satisfy the condition. Using known optimal dilations of some of these networks (complete trees [4], hypercubes [10] and 2- and 3-dimensional meshes [5, 7]) in paths we get the exact results on their dilations in cycles.

Our paper is organized as follows. Section 2 contains some basic definitions. The sufficient condition assuring $\mathrm{dil}(G, C_n) = \mathrm{dil}(G, P_n)$ is formulated and proved in Section 3. In Section 4 we apply the condition to obtain optimal embeddings of above mentioned graphs in cycles. Finally, in Section 5 we discuss some possibilities for further investigation in this area.

2 Preliminaries

Let $G = (V_G, E_G), H = (V_H, E_H)$ be two graphs. An embedding of G into H is an injective mapping $\phi : V_G \to V_H$. Let $\mathrm{dist}_H(x, y)$ denote the distance of vertices x, y in H. An important measure of efficiency of the embedding is the so called dilation of ϕ:

$$\mathrm{dil}_\phi(G, H) = \max_{(u,v) \in E_G} \{\mathrm{dist}_H(\phi(u), \phi(v))\}.$$

The dilation of G in H is defined as

$$\mathrm{dil}(G, H) = \min_\phi \{\mathrm{dil}_\phi(G, H)\}.$$

Let P_n (C_n) denote an n-vertex path (cycle), respectively. Let $V_{P_n} = V_{C_n} = \{0, 1, ..., n-1\}$ and $E_{P_n} = \{ij : \; |i - j| = 1, \; i, j \in V_{P_n}\}$, $E_{C_n} = \{ij : \; (i - j) \bmod n = 1, \; i, j \in V_{C_n}\}$.

Let $G_1 = (V_1, E_1), G_2 = (V_2, E_2)$ be graphs. Then $G_1 \times G_2$ denotes the graph with the set of vertices $V_1 \times V_2$ and in which $(i, j), (r, s)$ are adjacent iff either $i = r$ and $js \in E_2$ or $j = s$ and $ir \in E_1$. Further, $G_1 \cup G_2$ denotes the graph $(V_1 \cup V_2, E_1 \cup E_2)$.

Let X and Y be two cycles which intersect in a path P of length at least 1. The sum of the cycles X, Y is the cycle Z obtained from the graph $X \cup Y$ by deleting edges and inner vertices of P. For more cycles the sum is defined inductively. For precise definition see e.g. [9] (Chapter 4). A set of cycles in a graph G is called basic if any cycle in G either belongs to the basic set or can be expressed as a sum of cycles of the basic set. Trivially, the set of all cycles is a basic set. For completeness assume that trees have emty basic sets.

Shrinking denotes the graph operation that deletes an edge xy in a graph G and identifies x and y. The set of neighbours of the new vertex is the union of sets of neighbours of x and y in G.

3 A sufficient condition

First, we mention two obvious but useful facts.

Lemma 3.1
1. If P and P' are paths then $\mathrm{dil}(G,P) = \mathrm{dil}(G,P')$.
2. If G is a subgraph of G' and H' is a subgraph of H then $\mathrm{dil}(G,H) \leq \mathrm{dil}(G',H')$.

Theorem 3.1 *Suppose that the set of all cycles in G of length $\leq n/\mathrm{dil}(G,P_n)$ is a basic set. Then*

$$\mathrm{dil}(G,C_n) = \mathrm{dil}(G,P_n).$$

Proof: Idea of Proof. Assume that the dilations are different. Then there are two possibilities. Either there exists a cycle in G which is "stretched" around C_n or G is "wrapped" around C_n. The first case implies that there exists a basic cycle which is stretched around C_n. This is impossible due to the assumptions. The second case implies an embedding of G into a path with dilation not greater than $\mathrm{dil}(G,C_n)$.

Lemma 3.1 implies $\mathrm{dil}(G,C_n) \leq \mathrm{dil}(G,P_n)$. Suppose that $\mathrm{dil}(G,C_n) < \mathrm{dil}(G,P_n)$. Let $\phi : V_G \longrightarrow V_{C_n}$ be the optimal embedding i.e.

$$\min\{|\phi(x) - \phi(y)|, n - |\phi(x) - \phi(y)|\} \leq \mathrm{dil}(G,C_n)$$

for arbitrary adjacent vertices $x,y \in V_G$. Let $D \subset V_G \times V_G$ such that $(x,y) \in D$ iff $xy \in E_G$. Define a function $\alpha : D \longrightarrow \{-1,0,1\}$ as follows:

$$\alpha(x,y) = \begin{cases} 0 & \text{if} \quad |\phi(x) - \phi(y)| \leq \mathrm{dil}(G,C_n) \\ 1 & \text{if} \quad \phi(x) - \phi(y) \geq n - \mathrm{dil}(G,C_n) \\ -1 & \text{if} \quad \phi(x) - \phi(y) \leq \mathrm{dil}(G,C_n) - n. \end{cases}$$

Note that $\alpha(x,y) + \alpha(y,x) = 0$.

Lemma 3.2 *For any cycle $x_0x_1x_2...x_{l-1}$ in G it holds*

$$\sum_{i=0}^{l-1} \alpha(x_i, x_{i+1}) = 0, \qquad (1)$$

where $x_l \equiv x_0$.

Proof: First, let us suppose that the cycle belongs to the basic set.
Define a mapping $\psi : D \longrightarrow$ {set of directed paths in C_n} such that $\psi(x,y)$ is the shortest path starting in $\phi(x)$ and ending in $\phi(y)$. Let $|\psi(x,y)|$ be the length of the path. Consider the set $\{\psi(x_i,x_{i+1}) | \, i = 0,1,2,...,l-1\}$. We assert that the set of paths does not cover the whole cycle C_n. Indeed, we have

$$\sum_{i=0}^{l-1} |\psi(x_i,x_{i+1})| \leq l\mathrm{dil}(G,C_n) < l\mathrm{dil}(G,P_n) \leq n.$$

Hence the set of the paths covers a path in C_n with endvertices $\phi(u)$ and $\phi(v)$. Let us proceed along the directed paths from the set $\{\psi(x_i,x_{i+1}) | \, i = 0,1,2,...,l-1\}$ starting in $\phi(u)$. Clearly, the edge between 0 and $n-1$ is traversed by the paths the same times in the clockwise and counterclockwise direction. This implies that the number of 1's and -1's in (1) is the same.

If the cycle does not belong to the basic set we express the cycle as a sum of basic cycles and proceed by induction on the legth m of the expression.

Assume $m = 2$. Let the cycle $x_0x_1...x_{l-1}$ be a sum of two basic cycles $x_0x_1...x_iy_1...y_p$, where $y_p \equiv x_0$, and $x_0y_{p-1}...y_1x_i...x_{l-1}$. Then

$$\sum_{i=0}^{l-1}\alpha(x_i,x_{i+1}) = \alpha(x_0,x_1)+...+\alpha(x_{i-1},x_i)+\alpha(x_i,y_1)+...+\alpha(y_{p-1},x_0)+$$

$$+\alpha(x_0,y_{p-1})+...+\alpha(y_1,x_i)+\alpha(x_i,x_{i+1})+...+\alpha(x_{l-1},x_0) = 0+0 = 0.$$

Assume that the claim is true for all cycles which are expressable as a sum of $\leq m-1$ basic cycles. Consider any cycle formed by adding m basic cycles. Sum up the first $m-1$ basic cycles. The resulting cycle satisfies the inductive assumption. Hence we have two cycles whose sum gives the original cycle and for which the equation (1) holds. The rest of Proof is the same as in case $m = 2$. □

Let us define a function $\beta : V_G \to \mathbf{Z}$ as follows: Fix a vertex $x_0 \in V_G$ and set $\beta(x_0) = 0$. Set $\beta(x) = \alpha(x_o,x_1)+\alpha(x_1,x_2)+...+\alpha(x_p,x)$, where $x_0x_1x_2,...x_px$ is an arbitrary path joining x_0 and x in G. We prove that the function β is well defined. Let $x_0y_1...y_qx$ be another path joining x_0 and x. Assume that the paths have no common inner vertices. If $\beta(x)$ defined through the two paths receives two different values, i.e.

$$\alpha(x_0,x_1)+...+\alpha(x_p,x) \neq \alpha(x_0,y_1)+...+\alpha(y_q,x)$$

then

$$\alpha(x_0,x_1)+...+\alpha(x_p,x)+\alpha(x,y_q)+...+\alpha(y_1,x_0) \neq 0,$$

which contradicts to Lemma 3.2. Assume that u is the first common inner vertex of the paths, starting from x_0. Using the same argument as above we prove that $\beta(u)$ is unique. Then we omit the subpaths between x_0 and u and repeat the above procedure.

Note that if $xy \in E_G$ then

$$\beta(y) = \beta(x)+\alpha(x,y).$$

Finally, define a function $\phi^* : V_G \to \mathbf{Z}$ as follows:

$$\phi^*(x) = \phi(x)+n\beta(x).$$

We verify that ϕ^* is injective. Suppose that $\phi^*(x) = \phi^*(y)$. It implies $\phi(x)-\phi(y) = n(\beta(y)-\beta(x))$. If $\beta(x) = \beta(y)$ then $\phi(x) = \phi(y)$. Hence $x = y$. If $\beta(x) \neq \beta(y)$ then $|\phi(x)-\phi(y)| \geq n$, which contradicts to the fact that both $\phi(x)$ and $\phi(y)$ lie in the interval $[0, n-1]$.

Now let $xy \in E_G$.

If $\alpha(x,y) = 0$ then

$|\phi^*(y)-\phi^*(x)| = |\phi(y)-\phi(x)+n\alpha(x,y)| = |\phi(x)-\phi(y)| \leq \mathrm{dil}(G,C_n)$.

If $\alpha(x,y) = \pm 1$ then

$|\phi^*(y)-\phi^*(x)| = |\phi(y)-\phi(x)+n\alpha(x,y)| = |\phi(y)-\phi(x)\pm n| \leq \mathrm{dil}(G,C_n)$.

Thus we have constructed an embedding ϕ^* of G in a path P such that

$$\mathrm{dil}_{\phi^*}(G,P) \leq \mathrm{dil}(G,C_n).$$

But Lemma 3.1 implies $\mathrm{dil}(G,P_n) = \mathrm{dil}(G,P)$, a contradiction. □

4 Applications

In this section we show that many graphs corresponding to fundamental parallel architectures (trees, X-trees, hypercubes, meshes, pyramides and tree of meshes) satisfy the condition of

Theorem 3.1, i.e. their embeddings in the path and the cycle have the same dilation. It enables to obtain exact dilations in cycles for complete trees, hypercubes, and 2- and 3-dimensional meshes.

We start with three lemmas. As the first two are obvious, we omit proofs.

Lemma 4.1 *If G_1, G_2 and G_3 are graphs then* $\mathrm{dil}(G_1, G_3) \leq \mathrm{dil}(G_1, G_2) \cdot \mathrm{dil}(G_2, G_3)$.

Lemma 4.2 *Let G and G' be two graphs such that G' is obtained from G by shrinking an edge. If all cycles of length $\leq l$ in G form a basic set of G then all cycles of length $\leq l$ in G' form a basic set of G'.*

Lemma 4.3 *Let all cycles of length $\leq l_1$ ($\leq l_2$) in a graph G_1 (G_2) form a basic set of G_1 (G_2). Then all cycles of length $\leq \max\{4, l_1, l_2\}$ in the graph $G_1 \times G_2$ form a basic set of $G_1 \times G_2$.*

Proof: Recall the definition of the product of graphs. We say that vertices of $G_1 \times G_2$ belong to a horizontal (vertical) level if the vertices have the same first (second) entry. Note that the vertices of $G_1 \times G_2$ can be divided into $|V_{G_1}|$ ($|V_{G_2}|$) horizontal (vertical) levels. Consider any cycle in $G_1 \times G_2$. The vertices of the cycle belong to some number of horizontal and vertical levels. We proceed by induction on the number h of horizontal levels in which the cycle vertices lie.

The case $h = 1$ is trivial. Let $h \geq 2$ and suppose that all cycles having vertices in $\leq h-1$ horizontal levels are expressable as a sum of cycles of length $\leq \max\{4, l_1, l_2\}$. Consider a horizontal level L that contains a vertex of the original cycle. We proceed by induction on the number m of the cycle vertices lying in L.

Assume $m = 1$. Let x_2 be the cycle vertex lying in L. Let x_1 and x_3 be its neighbours in the cycle. Note that x_1, x_2 and x_3 lie in the same vertical level. Clearly, in the vertical level there must exist a path between x_1 and x_3 not containig x_2. Proceed along the path from x_1 to x_3. Let $x \neq x_1$ be the first cycle vertex on the path. If $x = x_3$ then the path divides the original cycle into two cycles. One of them contains vertices from $\leq h-1$ horizontal levels, the second one lies in one vertical level. Otherwise the path between x_1 and x divides the original cycle into two cycles. One of them contains vertices from $\leq h-1$ levels. Replace x by x_1 and repeat the above step for the second cycle until $x = x_3$.

Let $m \geq 2$. Assume that all cycles having $\leq m-1$ vertices in the horizontal level can be expressed as a sum of cycles having vertices in $\leq h$ levels. First suppose that there are at least two adjacent cycle vertices lying in L. Let x_1, x_2, x_3 be a subpath of the cycle such that $x_2, x_3 \in L$ and $x_1 \notin L$. Let x be the unique vertex lying in the same level as x_1 and adjacent to both x_1 and x_3. By inserting the edges x_1x and x_3x into the cycle we express the original cycle as a sum of two or three cycles (depending on whether x is a cycle vertex or not). One of them is of length 4, the second and third one (if exist) contain $\leq m-1$ vertices from L.

Suppose that there are no edges between the cycle vertices lying in L. Choose two arbitrary cycle vertices u and v lying in L such that the shortest path between u and v does not contain cycle vertices, except for u, v. The path divides the original cycle in two cycles, say C and C'. We show that both C and C' are expressable as a sum of cycles of length 4 and cycles having $\leq m-1$ vertices in L. W.l.o.g. consider the cycle C. Let $uu_1u_2...u_qv$ be the shortest path between u and v. Note that this path lies in L. Let $u_0 \neq u_1$ be the cycle neighbour of u. Let x be the unique vertex lying in the same level as u_0 and adjacent to both u_0 and u_1. By inserting the edges u_0x and u_1x the cycle C is expressed as a sum of two or three cycles (depending on whether x is a vertex of C or not). One of them is of length 4, next one (if exists, i.e. $x \in C$) contains $\leq m-1$ vertices in L. We repeat the above procedure with the remaining cycle and the subpath $u_1u_2...u_qv$. After a finite number of steps we obtain a resulting cycle which contains $\leq m-1$ vertices in L. □

Proposition 4.1 *For any n-vertex tree T it holds*

$$\text{dil}(T, C_n) = \text{dil}(T, P_n).$$

Proof: The sufficient condition is satisfied trivially because trees are cycle-free. □

Theorem 4.1 $\text{dil}(T_{t,r}, C_{(t^{r-1}-1)/(r-1)}) = \lceil t(t^{r-1}-1)/(2(r-1)(t-1)) \rceil$.

Proof: The result follows from Proposition 4.1 and the result of [4] on $\text{dil}(T_{t,r}, P_{(t^{r-1}-1)/(r-1)})$. □

The X-tree of height k (see [15]), denoted by $X(k)$, is the graph with the vertex set $V_{X(k)} = \{1, 2, 3, ..., 2^{k+1}-1\}$ and the edge set
$E_{X(k)} = \{(i, 2i), (i, 2i+1) : i = 1, 2, 3, ..., 2^k - 1\} \cup \bigcup_{j=1}^{k} \{(i, i+1) : 2^j \le i \le 2^{j+1} - 2\}$.

Proposition 4.2 *If $k \ge 5$ then* $\text{dil}(X(k), C_{2^{k+1}-1}) = \text{dil}(X(k), P_{2^{k+1}-1})$.

Proof: $X(k)$ can be obtained from the mesh $P_{k+1} \times P_{2^k}$ by shrinking edges. According to Lemmas 4.2 and 4.3 all cycles of length ≤ 4 create a basic set of $X(k)$. The embedding of the complete binary tree of height k described in [4] implies that $\text{dil}(X(k), P_{2^{k+1}-1}) \le 2\lceil 2^k - 1/k \rceil$. The sufficient condition is satisfied because $8\lceil (2^k-1)/k \rceil \le 2^{k+1} - 1$ for $k > 4$. □

The n-dimensional hypercube graph is usually defined by means of the graph product: $Q_1 = P_2$, $Q_n = Q_{n-1} \times P_2$, for $n > 2$.

Theorem 4.2 *If $n \ge 11$ then*

$$\text{dil}(Q_n, C_{2^n}) = \sum_{k=0}^{n-1} \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

Proof: Using Lemma 4.3 one can easily prove by induction on n that all cycles of length 4 in Q_n form a basic set. For $n \ge 11$ it holds

$$4\text{dil}(Q_n, P_{2^n}) = 4\sum_{k=0}^{n-1} \binom{k}{\lfloor \frac{k}{2} \rfloor} \le 4\binom{n}{\lfloor \frac{n}{2} \rfloor} \le 2^n.$$

The first inequality can be proved by induction. The second one follows from the Stirling approximation for factorials. The sufficient condition is satisfied. The rest of Proof follows from the result of Harper [10]. □

Let $m_1 \le m_2 \le ... \le m_r$, $n_1 \le n_2 \le ... \le n_s$ and $N = \prod_{i=1}^{r} m_i \prod_{j=1}^{s} n_j$. Let $G = \prod_{i=1}^{r} P_{m_i} \times \prod_{j=1}^{s} C_{n_j}$ denote an $(r+s)$-dimensional cylindrical mesh.

Proposition 4.3 *If $4 \le n_s \le m_r$ then*

$$\text{dil}(G, C_N) = \text{dil}(G, P_N).$$

Proof: By repeated application of Lemma 4.3 we prove that all cycles of length $\le n_s$ in G form a basic set. The general result from [3] on the dilation of the product of graphs in paths implies: $\text{dil}(G, P_N) \le N \min\{1/m_r, 2/n_s\}$. Hence

$$n_s \text{dil}(G, P_N) \le n_s N \min\{1/m_r, 2/n_s\} \le N.$$

The sufficient condition is satisfied. □

Theorem 4.3
1. $\text{dil}(P_m \times P_n, C_{mn}) = \min\{m, n\}$, *for* $\max\{m, n\} > 3$.
2. $\text{dil}(P_n \times P_n \times P_n, C_{n^3}) = \lfloor 3n^2/4 + n/2 \rfloor$, *for* $n > 3$.

Proof: The results follow from Proposition 4.3 and [5, 7] (see Introduction). □

The first result in Theorem 4.3 implies:

Corollary 4.1 *If* $\max\{m,n\} > 3$ *then*

$$\mathrm{dil}(C_m \times C_n, C_{mn}) = \mathrm{dil}(C_m \times P_n, C_{mn}) = \min\{m,n\}.$$

Proof: The lower bound follows from Theorem 4.3 and Lemma 3.1. The upper bound can be easily obtained by placing the "rows" of $C_m \times C_n$ consecutively on C_{mn}, provided that $m \geq n$. □

The first result in Corollary 4.1 is an extension of the result from [14] for arbitrary toroidal meshes. The first result in Theorem 4.3 and results in Corollary 4.1 complete the investigation of problems of the type "$\mathrm{dil}(X \times Y, Z) =$?", where X, Y and Z are paths or cycles.

Let $M_n = P_{2^n} \times P_{2^n}$. Define $P(n)$ to be the pyramide graph with
$V_{P(n)} = \{(i,x,y) : (x,y) \in V_{M_i}, i = 0,1,2,...,n\}$,
$E_{P(n)} = \{((i,x,y),(i,u,v)) : (x,y)(u,v) \in E_{M_i}, i = 1,2,3,...,n\} \cup$
$\cup \{((i,x,y),(i-1,\lfloor x/2 \rfloor, \lfloor y/2 \rfloor)) : (x,y) \in V_{M_i}, i = 1,2,3,...,n\}$.
Intuitively, $P(n)$ contains M_0 through M_n, with each vertex in M_i having four offsprings in M_{i+1}. $V_{P(n)}$ can be separated into levels; specifically $(i,x,y) \in V_{P(n)}$ is on level i. Note that $|V_{P(n)}| = (4^{n+1}-1)/3$.

Proposition 4.4 *For* $n > 45$ *it holds*

$$\mathrm{dil}(P(n), C_{(4^{n+1}-1)/3}) = \mathrm{dil}(P(n), P_{(4^{n+1}-1)/3}).$$

Proof: All cycles of length ≤ 4 form a basic set as $P(n)$ can be obtained from the mesh $P_{n+1} \times P_{2^n} \times P_{2^n}$ by shrinking edges. Put $G_1 = P(n), G_2 = Q_{2n+1}$ and $G_3 = P_{2^{2n+1}}$ in Lemma 4.1. We obtain

$$\mathrm{dil}(P(n), P_{2^{2n+1}}) \leq \mathrm{dil}(P(n), Q_{2n+1})\mathrm{dil}(Q_{2n+1}, P_{2^{2n+1}}).$$

The first term on the right hand side is equal to 2 [11], the second one is bounded from above by $\binom{2n+1}{n}$ according to Theorem 4.2. Hence, for $n > 45$, we have

$$4\mathrm{dil}(P(n), P_{(4^{n+1}-1)/3}) \leq 4\mathrm{dil}(P(n), P_{2^{2n+1}}) \leq 8\binom{2n+1}{n} \leq \frac{4^{n+1}-1}{3}.$$

The sufficient condition is satisfied. □

The tree of meshes TM_n is defined as follows [2]: The root of a complete binary tree is replaced by an $n \times n$ mesh, their sons are replaced by $n/2 \times n$ meshes, their sons are replaced by $n/2 \times n/2$ meshes and so on until the leaves are replaced by 1×1 meshes. Each right edge of the binary tree is replaced by edges which connect the rightmost column of vertices of the mesh corresponding to the father to the topmost row of vertices in the mesh corresponding to the right son. Similar replacement are made for left edges of the binary tree. The number of vertices of TM_n is given by $|V_{TM_n}| = n^2(2\log n + 1)$.

Proposition 4.5 *For* $n \geq 16$ *it holds*

$$\mathrm{dil}(TM_n, C_{n^2(2\log n+1)}) = \mathrm{dil}(TM_n, P_{n^2(2\log n+1)}).$$

Proof: It is evident that all cycles of length 4 form a basic set. Now we construct an embedding $\phi : TM_n \rightarrow P_{n^2(2\log n+1)}$. Let us divide the path $P_{n^2(\log n+1)}$ into $2\log n + 1$ subpaths of length $n^2 - 1$. Embed somehow the mesh $n \times n$ into the first subpath, the meshes of type $n/2 \times n$ into

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