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Nonlinear Kaluza-Klein theory for dual fields

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We present nonlinear uplift *Ansätze* for all the bosonic degrees of freedom and dual fields in the S^7 reduction of $D = 11$ supergravity to maximal SO(8) gauged supergravity and test them for the $\mathrm{SO}(7)^{\pm}$ invariant solutions. In particular, we complete the known *Ansätze* for the internal components of the metric and four-form flux by constructing a nonlinear *Ansatz* for the internal components of the dual seven-form flux. Furthermore, we provide *Ansätze* for the complete set of 56 vector fields, which are given by more general structures than those available from standard Kaluza-Klein theory. The novel features encountered here have no conventional geometric interpretation and provide a new perspective on Kaluza-Klein theory. We study the recently found set of generalized vielbein postulates and, for the S^7 compactification, we show that they reduce to the $\mathrm{E}_{7(7)}$ Cartan equation of maximal SO(8) gauged supergravity in four dimensions. The significance of this framework for a higher-dimensional understanding of the embedding tensor and other gauged maximal supergravities is briefly discussed.

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I. INTRODUCTION

In this article, we continue the investigation of the generalized geometry underlying maximal supergravity on the basis of the $\mathrm{SO}(1,3) \times \mathrm{SU}(8)$ invariant reformulation [1] of $D = 11$ supergravity, which has been further developed in very recent work [2–4]. Our main concern here will be the question of whether the consistency results of standard Kaluza-Klein theory can be extended to the "nongeometrical" structures that arise in this context, and in particular from the vector and form fields *and their duals*. Focusing on $D = 11$ supergravity [5], and more specifically on its S^7 compactification [6,7] to maximal gauged SO(8) supergravity in four dimensions [8], is amply justified by the fact that this theory presents by far the richest structure of all Kaluza-Klein models studied so far, but of course, we expect our results to be relevant also in a more general context.

The question of whether a four-dimensional theory can be obtained by dimensional reduction from higher dimensions, and the question of whether a given compactification of a higher-dimensional theory can be associated with a *consistent truncation* is clearly an important and pertinent one. Consistency here by definition is taken to mean that any full solution of the lower-dimensional theory should admit an uplift to a full solution of the nonlinear higher-dimensional field equations. However, establishing such a relation and its consistency is far from obvious in all but the most trivial examples. In particular, due to the generic emergence of non-Abelian gauge theories in Kaluza-Klein compactifications, we have to deal with *gauged* supergravities. The most efficient framework for understanding these theories (in any dimension) is the embedding tensor formalism [9–12]. Therefore, any general scheme that aims to address the issue of the higher-dimensional origin of four-dimensional theories should provide a higher-dimensional perspective on the embedding tensor. Furthermore, given a consistent truncation, yet another challenging task is to give explicit uplift *Ansätze* for all relevant fields, something that standard Kaluza-Klein theory cannot give for fields other than the graviton and the vector fields.

In Ref. [1], a 4 + 7 splitting of $D = 11$ supergravity is considered with an appropriate decomposition of all 11-dimensional fields with respect to this splitting, while retaining full on-shell equivalence to the original theory. This reformulation has manifest local SU(8) invariance, and emphasizes and generalizes the structures that would appear upon a toroidal reduction of the theory to four dimensions [13,14]. The construction of Ref. [1] relies on an analysis of the supersymmetry transformations of the redefined fields, and a crucial object that emerges from the supersymmetry transformation of the graviphoton is an SU(8) tensor, the generalized vielbein. The graviphoton gives rise to vector fields upon reduction. However, in the reduced theory these are complemented by other vector fields. In particular, the three-form potential also contributes to the vector degrees of freedom. The supersymmetry transformation of these vectors in the $D = 11$ theory gives rise to yet another generalized vielbein [2]. The observation made in Ref. [4] is that by considering dualization of 11-dimensional fields, a full set of 56 vectors is obtained whose supersymmetry transformations give rise naturally to an $\mathrm{E}_{7(7)}$ vielbein in 11 dimensions.

The emergence [4] of $\mathrm{E}_{7(7)}$ structures in $D = 11$ supergravity gives a new perspective on the extent to which duality symmetries play a role in the full unreduced $D = 11$ theory and the necessity to transcend usual notions

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of geometry.[1] However, the framework that is developed in Ref. [4]—based on the SU(8) invariant reformulation [1] of $D = 11$ supergravity and extending the recent results of Ref. [2]—is also the most natural setting in which to understand the 11-dimensional origins of four-dimensional gauged theories. The construction in Ref. [4] highlights structures in 11 dimensions that are manifest in the reduced theory, enabling one to address questions concerning uplift *Ansätze* and the appearance of particular gaugings in four dimensions from a reduction point of view.

In this paper, we use the S^7 reduction to maximal gauged supergravity to illustrate the effectiveness of the framework presented in Ref. [4]. In particular, we extend the Kaluza-Klein *Ansätze* for the graviphoton [16–18] and the vector associated with the three-form potential [2] to vectors associated with dual fields, revealing novel features. These *Ansätze* allow us to derive a nonlinear *Ansatz* for the internal components of the six-form potential complementing the nonlinear *Ansätze* for the internal components of the metric [19] and the three-form potential [2]. A remarkable feature of this analysis is the existence of a nonlinear *Ansatz* not only for a *dual field*, but for *all* fields.

To illustrate the novelty of the present construction, let us first recall some well-known facts from standard Kaluza-Klein theory [16–18]. Starting from the higher-dimensional vielbein $E_M{}^A(z) \equiv E_M{}^A(x, y)$, where the higher-dimensional coordinates $\{z^M\}$ are split into four-dimensional coordinates $\{x^\mu\}$ and internal coordinates $\{y^m\}$, respectively, one proceeds from the *Ansatz*

$$E_M{}^A(x, y) = \begin{pmatrix} \Delta^{-1/2} e'_\mu{}^\alpha & B_\mu{}^m e_m{}^a \\ 0 & e_m{}^a \end{pmatrix}, \tag{1}$$

where $e_m{}^a(x, y)$ is the vielbein associated with the internal manifold on which the higher-dimensional theory is assumed to be compactified, $\Delta \equiv \det e_m{}^a$, and $e'_\mu{}^\alpha$ is the Weyl rescaled vierbein of the compactified theory; the triangular form of $E_M{}^A$ is arrived at by making partial use of the local Lorentz symmetry of the higher-dimensional theory. A consistent truncation for the spin-2 field (graviton) is achieved simply by setting

$$e'_\mu{}^\alpha(x, y) \equiv e'_\mu{}^\alpha(x),$$

that is, by dropping all dependence on the internal coordinates. Likewise, for the vectors $B_\mu{}^m(x, y)$, the exact consistent *Ansatz* has been known for a very long time [16–18]; it reads

$$B_\mu{}^m(x, y) = K^{m\mathcal{I}}(y) A_\mu^{\mathcal{I}}(x), \tag{2}$$

where the index $\mathcal{I}$ labels the Killing vectors $K^{m\mathcal{I}}(y)$ on the internal manifold. It is a key result of Kaluza-Klein theory that the non-Abelian gauge interactions of the compactified theory then originate from the commutator of two Killing vector fields

$$[K^{m\mathcal{I}}\partial_m, K^{n\mathcal{J}}\partial_n] = f^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}} K^{p\mathcal{K}}\partial_p, \tag{3}$$

where $f^{\mathcal{I}\mathcal{J}}{}_{\mathcal{K}}$ are the structure constants of the isometry group of the internal manifold. In this way the gauge group of the compactified theory is completely explained in geometric terms. While this has been well understood for many decades, the main difficulty in establishing the full consistency of the Kaluza-Klein reduction resides in the scalar sector, in particular involving the search for consistent *Ansätze* for the internal vielbein $e_m{}^a(x, y)$ and other fields of a tensorial nature under internal symmetries. The main focus of the present work, then, is to develop a similar theory for "nongeometrical" vector fields and matter fields, and in particular, for those fields arising from dual fields in higher dimensions, for which no readily applicable formulas are available from general Kaluza-Klein theory—hence the need for a "generalized geometry."

The structure of the paper is as follows: In Sec. II, we review the main results of Ref. [4]. We briefly discuss how the consideration of dual fields in 11 dimensions can be used to construct 56 vectors, the supersymmetry transformations of which give rise to a set of generalized vielbeine that are parametrized by the "internal" components of the 11-dimensional metric, three-form potential and its dual six-form. The generalized vielbeine can be viewed as components of an $E_{7(7)}$ matrix and the supersymmetry transformations of the bosonic fields can be cast into a form that mirrors the analogous supersymmetry transformations in four dimensions. Furthermore, the generalized vielbein postulates [1,4] are summarized in Sec. II C.

Specializing to the S^7 reduction of $D = 11$ supergravity to maximal SO(8) gauged supergravity in four dimensions in Sec. III, we use the results summarized in Sec. II to derive nonlinear *Ansätze* for all bosonic degrees of freedom. In particular, extending the result in Ref. [2], we give Kaluza-Klein *Ansätze* for all 56 vector fields. These *Ansätze* include not only Killing vector fields on S^7, but also tensors and the potential for the volume form on S^7, which is not globally defined. Comparing the 11- and four-dimensional supersymmetry transformations of the vectors, along the lines of Refs. [2,19], allows us to express the generalized vielbeine in terms of the four-dimensional scalars. These relations are then used to find a nonlinear *Ansatz* for the internal components of the six-form dual potential. The set of *Ansätze* for the vectors and the internal components of the metric, three-form potential and its dual comprise a set of uplift formulas for all bosonic degrees of freedom. In section V, we test the nonlinear *Ansatz* for the six-form field by explicitly checking that the *Ansatz* reproduces the internal component of the six-form potential of the $SO(7)^\pm$ invariant solutions of $D = 11$ supergravity [20,21] from the scalar expectation values of the $SO(7)^\pm$ invariant stationary points [22] of maximal SO(8) gauged

[1]For alternative approaches to generalized geometry and a list of recent references with bibliographies, see Ref. [15].

supergravity. The possibility to perform such explicit checks of all formulas against various nontrivial compactifications is a feature that distinguishes our formalism from other approaches to generalized geometry.

From a four-dimensional point of view, the 56 vector fields include the full set of electric and magnetic vectors that can be gauged. In Sec. IV, we show that the generalized vielbein postulates determine exactly which of the vector fields are gauged in the S^7 compactification. In particular, we show that upon inserting the *Ansätze* relevant for the S^7 compactification given in Sec. III, the magnetic vector fields, which come from the reduction of the three-form potential, drop out of the expressions. Moreover, the generalized vielbein postulates reduce to the $E_{7(7)}$ Cartan equation with an SO(8) gauge covariant derivative. The SO(8) gauge fields are solely electric and arise from the graviphoton and the six-form potential. More generally, for any compactification, the generalized vielbein postulates reduce to the four-dimensional Cartan equation with the appropriate gauge covariant derivative. Therefore, the generalized vielbein postulates provide an understanding of how the gauge vectors are selected from the 56 vector fields available. This goes some way towards establishing the origin of the embedding tensor in the higher-dimensional $D = 11$ theory.

We conclude in Sec. VI with a brief, general discussion of other compactifications that could lead to more general gaugings in four dimensions. Of particular interest are examples of compactifications where both electric and magnetic vectors are gauged. We also discuss the possibility of using our framework to provide an 11-dimensional perspective on the recently discovered continuous family of SO(8) gauged supergravities [23].

In summary, the key results and issues raised in this paper are as follows:

(i) Kaluza-Klein theory is developed for "nongeometric" vector fields.
(ii) Consistent nonlinear *Ansätze* are obtained for all fields, including dual fields.
(iii) All formulas can be tested against nontrivial compactifications of $D = 11$ supergravity and the associated stationary points of $N = 8$ supergravity.
(iv) With 56 "electric" and "magnetic" vectors present in all $D = 11$ relations, one can now study the higher-dimensional origins of the embedding tensor.
(v) Preliminary evidence is presented that the ω-deformed SO(8) gaugings of Ref. [23] correspond to ω-deformations of $D = 11$ supergravity.

The conventions and index notations used in this paper are as in Refs. [1,4].

II. DUAL FIELDS AND $E_{7(7)}$ IN $D = 11$ SUPERGRAVITY

In Ref. [4], the two generalized vielbeine previously known from the literature [1,2] are completed to an $E_{7(7)}$ matrix in 11 dimensions by constructing two further generalized vielbeine that are intimately related to the dualization of the metric and the three-form potential in 11 dimensions. While the significance of this construction from an 11-dimensional point of view is clear in that it establishes the role of the $E_{7(7)}$ duality group in the full $D = 11$ theory, its importance from a practical point of view in relating the four-dimensional maximal supergravity to $D = 11$ supergravity is what will be addressed in this paper. In particular, the results of Ref. [4] give uplift *Ansätze* for *all* bosonic degrees of freedom including a nonlinear *Ansatz* for the six-form potential dual to the three-form potential. We will illustrate this in great detail for the S^7 compactification of $D = 11$ supergravity in the following sections, but from the generality of our results it should be clear that our construction furnishes similar information for other compactifications of $D = 11$ supergravity. Consequently, we will first summarize the general results in this section, without reference to any specific compactification.

A. Dual vector fields and generalized vielbeine

Working in the context of the SU(8) invariant reformulation of $D = 11$ supergravity, one identifies certain SU(8) objects starting from an analysis of the fermionic sector [1].[2] In the bosonic sector, the most prominent of these objects are the so-called *generalized vielbeine*, which can be regarded as components of an $E_{7(7)}$ matrix in 11 dimensions. The generalized vielbeine appear when one considers the supersymmetry transformation of those components of the elfbein, three-form potential and their dual fields which in a proper reduction to four dimensions would give rise to vector fields. However, it is important to keep in mind the main feature of the present analysis (and of Ref. [1]), namely that *we retain the full coordinate dependence on all 11 coordinates throughout*. Therefore, we will not be dealing with a dimensional reduction in the strict sense of the word, but rather a 4 + 7 split and a subsequent reformulation of the theory. Furthermore, the reformulated theory is on-shell equivalent to the original $D = 11$ supergravity of Ref. [5] *at all stages of the construction*.

Let us therefore first consider the spin-1 sector of the theory. In the direct dimensional reduction of $D = 11$ supergravity to four dimensions there appear only 28 vector fields, namely [14]

$$B_\mu{}^m \quad \text{and} \quad B_{\mu mn} = A_{\mu mn} - B_\mu{}^p A_{pmn}. \tag{4}$$

The first seven of these are just the standard Kaluza-Klein vector fields in the decomposition of the elfbein displayed in Eq. (1), while the second set of vectors originates from the three-form field A_{MNP}. As explained in our previous

[2] See also Sec. 3.1 of Ref. [4] for a brief description of the SU(8) invariant reformulation.

work [4], this set of vector fields is complemented by another set of vectors related to the dual fields in 11 dimensions, viz.[3]

$$B_{\mu m_1\ldots m_5} = A_{\mu m_1\ldots m_5} - B_\mu{}^p A_{p m_1\ldots m_5} - \frac{\sqrt{2}}{4}(A_{\mu[m_1 m_2} - B_\mu{}^p A_{p[m_1 m_2})A_{m_3 m_4 m_5]}, \qquad (5)$$

$$\begin{aligned} B_{\mu m_1\ldots m_7,n} = {} & A_{\mu m_1\ldots m_7,n} + (3\tilde{c}-1) \\ & \times (A_{\mu[m_1\ldots m_5} - B_\mu{}^p A_{p[m_1\ldots m_5})A_{m_6 m_7]n} \\ & + \tilde{c}A_{[m_1\ldots m_6}(A_{|\mu|m_7]n} - B_\mu{}^p A_{|p|m_7]n}) \\ & + \frac{\sqrt{2}}{12}(A_{\mu[m_1 m_2} - B_\mu{}^p A_{p[m_1 m_2})A_{m_3 m_4 m_5}A_{m_6 m_7]n}. \end{aligned} \qquad (6)$$

The vector components in Eq. (5) thus originate from the six-form $A_{(6)}$ which is the dual potential associated with the three-form potential $A_{(3)}$, whence it is clear that these fields are simultaneously defined only on shell, as explained in Ref. [4]. The vector fields in Eq. (6) are related to the dual gravity field in 11 dimensions, and are defined only up to a real constant $\tilde{c}$. While the precise relation of $A_{\mu m_1\ldots m_7,n}$ to the $D = 11$ fields is not known, the indeterminacy encoded in the parameter $\tilde{c}$ can be traced back to the fact that dual gravity does not give rise to scalar degrees of freedom in the $4 + 7$ split. We also note that the nonlinear modifications in these equations which involve the Kaluza-Klein vectors $B_\mu{}^m$ can be understood geometrically via the conversion of curved to flat indices, whereas the remaining nonlinear modifications are required by the consistency of the supersymmetry variations, but have no direct explanation in terms of 11-dimensional geometry.[4]

Thus, in all we have identified 56 such vector fields in 11 dimensions, starting from the fields of $D = 11$ supergravity and their duals. These make up part of the bosonic sector of our reformulation of $D = 11$ supergravity in the framework of the "generalized geometry" introduced in Ref. [4]. However, not all of these vector fields will correspond to independent propagating vectors in a given compactification of the $D = 11$ theory. In particular, for compactifications related to $N = 8$ supergravity and deformations thereof, we know that there can be at most 28 propagating spin-1 degrees of freedom. This is most easily seen in the T^7 reduction of Ref. [14], where the seven "electric" vectors from $B_\mu{}^m$ (corresponding to the seven Killing vectors on T^7) combine with 21 "magnetic" vectors from $B_{\mu mn}$ to give 28 Abelian vector fields. The other 28 vectors correspond to their four-dimensional duals such that the 11-dimensional duality relations reduce to the "twisted self-duality constraint" of Ref. [14] in the reduction to four dimensions. For nontrivial compactifications of the theory, the situation is, however, much more complicated because of the appearance of *non-Abelian* gauge interactions, for which the usual (Abelian) dualization of vector fields does not work.

A judicious analysis of the supersymmetry transformations of these 56 vector fields [1,2,4] leads to the generalized vielbeine. For the vector fields in Eq. (4), the latter can be directly obtained from the $D = 11$ theory, while the variation of $B_{\mu m_1\ldots m_5}$ in Eq. (5) is determined from the variation of $A_{(6)}$ [4]. The supersymmetry transformation of $B_{\mu m_1\ldots m_7,n}$ is also given in Ref. [4], but it cannot be obtained from the $D = 11$ theory. It can, however, be obtained by imposing consistency with the supersymmetry variations of the other vector fields. Somewhat lengthy computations show that [4]

$$\delta B_\mu{}^m = \frac{\sqrt{2}}{8} e^m_{AB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.}, \qquad (7)$$

$$\delta B_{\mu mn} = \frac{\sqrt{2}}{8} e_{mnAB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.}, \quad (8)$$

$$\delta B_{\mu m_1\ldots m_5} = \frac{\sqrt{2}}{8} e_{m_1\ldots m_5 AB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.}, \qquad (9)$$

$$\delta B_{\mu m_1\ldots m_7,n} = \frac{\sqrt{2}}{8} e_{m_1\ldots m_7,nAB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.}, \qquad (10)$$

where φ^A_μ and χ^{ABC} are the (chiral) fermions in the SU(8) invariant reformulation, and where $\gamma'_\mu = e'_\mu{}^\alpha\gamma_\alpha$ with the Weyl rescaled vierbein $e'_\mu{}^\alpha$ from Eq. (1). The scalar coefficients in front of the fermionic bilinears make up the generalized vielbeine, and are given by [1,2,4]

$$e^m_{AB} = i\Delta^{-1/2}\Gamma^m_{AB}, \qquad (11)$$

$$e_{mnAB} = -\frac{\sqrt{2}}{12} i\Delta^{-1/2}\Big(\Gamma_{mnAB} + 6\sqrt{2}A_{mnp}\Gamma^p_{AB}\Big), \qquad (12)$$

[3]Note the slight change in notation here compared with that used in Ref. [4]. In particular, here, we reserve the notation $B_{\mu m_1\ldots m_5}$ for the vector whose supersymmetry transformation gives rise to the generalized vielbein $e_{m_1\ldots m_5 AB}$. A similar change of notation is made for the fourth vector. In addition, the arbitrary constant $\tilde{c}$ here is related to c in Ref. [4] by $\tilde{c} = 5!c/\sqrt{2}$.

[4]A geometrical explanation might, however, follow from E_{10} or E_{11}, where the "vielbein" comprises not only the gravitational, but also the three-form and six-form fields.

$$e_{m_1\ldots m_5AB} = \frac{1}{6!\sqrt{2}} i\Delta^{-1/2}\Big[\Gamma_{m_1\ldots m_5AB} + 60\sqrt{2}A_{[m_1m_2m_3}\Gamma_{m_4m_5]AB} - 6!\sqrt{2}\Big(A_{pm_1\ldots m_5} - \frac{\sqrt{2}}{4}A_{p[m_1m_2}A_{m_3m_4m_5]}\Big)\Gamma^p_{AB}\Big], \tag{13}$$

$$\begin{aligned} e_{m_1\ldots m_7,nAB} = {} & -\frac{2}{9!} i\Delta^{-1/2}\Big[(\Gamma_{m_1\ldots m_7}\Gamma_n)_{AB} + 126\sqrt{2}A_{n[m_1m_2}\Gamma_{m_3\ldots m_7]AB} \\ & + 3\sqrt{2}\times 7!\Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{4}A_{n[m_1m_2}A_{m_3m_4m_5}\Big)\Gamma_{m_6m_7]AB} + \frac{9!}{2}\Big(A_{n[m_1\ldots m_5} + \frac{\sqrt{2}}{12}A_{n[m_1m_2}A_{m_3m_4m_5}\Big)A_{m_6m_7]p}\Gamma^p{}_{AB}\Big]. \end{aligned} \tag{14}$$

These objects carry SU(8) indices $A, B, \ldots$ and are to be regarded as SU(8) tensors in a specific gauge, as explained in Ref. [1]. As also shown there, one thereby enlarges the original tangent space symmetry from SO(7) to a local SU(8) symmetry that acts on the chiral fermions. Observe that the new vielbein components, $e_{m_1\ldots m_5AB}$ and $e_{m_1\ldots m_7,nAB}$, which originate from the variations of the dual vectors (5) and (6), themselves depend on the dual six-form field $A_{m_1\ldots m_6}$, and hence again are only defined on shell.

B. Emergence of $E_{7(7)}$ structure

One can make the relation of the above expressions to $E_{7(7)}$ more explicit by combining the matrix blocks into a single 56-bein *in 11 dimensions* [4],

$$\mathcal{V}(z) = (\mathcal{V}^{\mathtt{MN}}{}_{AB}(z), \mathcal{V}_{\mathtt{MN}AB}(z)), \tag{15}$$

by means of the identifications

$$\begin{aligned} \mathcal{V}^{m8}{}_{AB} &= \frac{\sqrt{2}}{8}e^m_{AB}, \qquad \mathcal{V}_{mnAB} = -\frac{3}{2}e_{mnAB}, \\ \mathcal{V}^{mn}{}_{AB} &= -\frac{3}{2}\Delta\epsilon^{mnp_1\ldots p_5}e_{p_1\ldots p_5AB}, \\ \mathcal{V}_{m8AB} &= \frac{9\sqrt{2}}{2}\Delta\epsilon^{n_1\ldots n_7}e_{n_1\ldots n_7,mAB}. \end{aligned} \tag{16}$$

This $E_{7(7)}$ vielbein is equivalent (see Ref. [4]) to the one considered in Ref. [24] in the context of another proposal to realize an exceptional geometry. Note that complex conjugation acts by raising or lowering the SU(8) indices, viz.

$$(\mathcal{V}^{\mathtt{MN}}{}_{AB})^* = \mathcal{V}^{\mathtt{MN}AB}, \qquad (\mathcal{V}_{\mathtt{MN}}{}^{AB})^* = \mathcal{V}_{\mathtt{MN}AB}, \tag{17}$$

but leaves the position of the $SL(8,\mathbb{R})$ indices $\mathtt{M}$, $\mathtt{N}$ unaffected. The 56-bein $\mathcal{V}(z)$ as defined above is a coset element of $E_{7(7)}/SU(8)$ written in terms of the decomposition of the **56** of $E_{7(7)}$ under initially its $SL(8,\mathbb{R})$ and then its $GL(7,\mathbb{R})$ subgroups:

$$\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}} \to \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \bar{\mathbf{7}}. \tag{18}$$

The world indices $m, n, \ldots$ labeling the seven-dimensional directions and originally transforming under seven-dimensional diffeomorphisms thus become associated with the $GL(7,\mathbb{R})$ subgroup of $E_{7(7)}$. In contrast to Ref. [4], we have adjusted the normalization of the matrix blocks in such a way that $\mathcal{V}(z)$, as defined in Eq. (16), satisfies the following identity:

$$\mathcal{V}_{\mathtt{MN}AB}\mathcal{V}^{\mathtt{MN}CD} - \mathcal{V}^{\mathtt{MN}}{}_{AB}\mathcal{V}_{\mathtt{MN}}{}^{CD} = i\delta^{CD}_{AB}. \tag{19}$$

This is simply the statement that the inverse of an $E_{7(7)}$ matrix is related to its complex conjugate; more specifically, Eq. (19) is a necessary condition expressing the fact that any $E_{7(7)}$ matrix automatically belongs to $Sp(56,\mathbb{R})$ [8,14]. The direct verification of Eq. (19) by substitution of Eqs. (11)–(14) into Eq. (16) is straightforward.

The vectors can be arranged into a similar object of the form $(\mathcal{B}_\mu{}^{\mathtt{MN}}, \mathcal{B}_{\mu\mathtt{MN}})$, such that

$$\begin{aligned} \mathcal{B}_\mu{}^{m8} &= -\frac{1}{2}B_\mu{}^m, \qquad \mathcal{B}_{\mu mn} = 3\sqrt{2}B_{\mu mn}, \\ \mathcal{B}_\mu{}^{mn} &= 3\sqrt{2}\Delta\epsilon^{mnp_1\ldots p_5}B_{\mu p_1\ldots p_5}, \\ \mathcal{B}_{\mu m8} &= -18\Delta\epsilon^{n_1\ldots n_7}B_{\mu n_1\ldots n_7,m}. \end{aligned} \tag{20}$$

Thereby, the 28 + 28 vectors are combined into a **56** of $E_{7(7)}$. In this language, the supersymmetry transformations in Eqs. (7)–(10) can be compactly written as

$$\begin{aligned} \delta\mathcal{B}_\mu{}^{\mathtt{MN}} &= -\frac{1}{2}\mathcal{V}^{\mathtt{MN}}{}_{AB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.}, \\ \delta\mathcal{B}_{\mu\mathtt{MN}} &= -\frac{1}{2}\mathcal{V}_{\mathtt{MN}AB}\Big[2\sqrt{2}\bar{\varepsilon}^A\varphi^B_\mu + \bar{\varepsilon}_C\gamma'_\mu\chi^{ABC}\Big] + \text{H.c.} \end{aligned} \tag{21}$$

In Ref. [4], we have also shown that the matrix blocks making up the 56-bein [Eq. (15)] transform uniformly under local supersymmetry

$$\begin{aligned} \delta\mathcal{V}^{\mathtt{MN}}{}_{AB}(z) &= -\sqrt{2}\Sigma_{ABCD}\mathcal{V}^{\mathtt{MN}CD}(z), \\ \delta\mathcal{V}_{\mathtt{MN}AB}(z) &= -\sqrt{2}\Sigma_{ABCD}\mathcal{V}_{\mathtt{MN}}{}^{CD}(z) \end{aligned} \tag{22}$$

with the complex self-dual tensor

$$\Sigma_{ABCD} \equiv \bar{\varepsilon}_{[A}\chi_{BCD]} + \frac{1}{24}\epsilon_{ABCDEFGH}\bar{\varepsilon}^E\chi^{FGH}, \tag{23}$$

where we have discarded a local SU(8) rotation that also acts uniformly on all components. In the forms of Eqs. (21) and (22), the supersymmetry variations of compactified maximal supergravity can be read off directly from the $D = 11$ formulas.

C. Generalized vielbein postulates

The generalized vielbeine satisfy certain differential constraints derived in Refs. [1,4]. These constraints are generalizations of the vielbein postulate in Riemannian geometry which establishes the relation between affine and spin connections. For our subsequent analysis, here we only need their components along the four space-time directions, which are [1,4]

$$\mathcal{D}_\mu e^m_{AB} + \frac{1}{2}\partial_n B_\mu{}^n e^m_{AB} + \partial_n B_\mu{}^m e^n_{AB} + \mathcal{Q}^C_{\mu[A} e^m_{B]C} + \mathcal{P}_{\mu ABCD} e^{mCD} = 0, \tag{24}$$

$$\mathcal{D}_\mu e_{mnAB} + \frac{1}{2}\partial_p B_\mu{}^p e_{mnAB} + 2\partial_{[m} B_{|\mu|}{}^p e_{n]pAB} + 3\partial_{[m} B_{|\mu|np]} e^p_{AB} + \mathcal{Q}^C_{\mu[A} e_{mnB]C} + \mathcal{P}_{\mu ABCD} e_{mn}{}^{CD} = 0, \tag{25}$$

$$\mathcal{D}_\mu e_{m_1\ldots m_5 AB} + \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\ldots m_5 AB} - 5\partial_{[m_1} B_{|\mu|}{}^p e_{m_2\ldots m_5]pAB} + \frac{3}{\sqrt{2}}\partial_{[m_1} B_{|\mu|m_2 m_3} e_{m_4 m_5]AB} - 6\partial_{[m_1} B_{|\mu|m_2\ldots m_5 p]} e^p_{AB} + \mathcal{Q}^C_{\mu[A} e_{m_1\ldots m_5 B]C} + \mathcal{P}_{\mu ABCD} e_{m_1\ldots m_5}{}^{CD} = 0, \tag{26}$$

$$\mathcal{D}_\mu e_{m_1\ldots m_7,nAB} - \frac{1}{2}\partial_p B_\mu{}^p e_{m_1\ldots m_7,nAB} - \partial_n B_\mu{}^p e_{m_1\ldots m_7,pAB} + 5\partial_{[m_1} B_{|\mu|m_2 m_3} e_{m_4\ldots m_7]nAB} - 2\partial_{[m_1} B_{|\mu|m_2\ldots m_6} e_{m_7]nAB} + \mathcal{Q}^C_{\mu[A} e_{m_1\ldots m_7,nB]C} + \mathcal{P}_{\mu ABCD} e_{m_1\ldots m_7,n}{}^{CD} = 0, \tag{27}$$

where[5]

$$\mathcal{D}_\mu \equiv \partial_\mu - B_\mu{}^m \partial_m. \tag{28}$$

In comparison with Eq. (88) of Ref. [4], the generalized vielbein postulate for $e_{m_1\ldots m_7,nAB}$ has been simplified by using the Schouten identity over eight indices. The connection coefficients Q_μ and P_μ appearing in these equations are valued in the $E_{7(7)}$ Lie algebra, and are related to the connections and four-form field strengths of $D = 11$ supergravity as follows [1]:

$$\mathcal{Q}^A_{\mu B} = -\frac{1}{2}[e^m{}_a \partial_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^{ab}_{AB} - \frac{\sqrt{2}}{12}\Delta^{-1/2} e'_\mu{}^\alpha (F_{\alpha abc}\Gamma^{abc}_{AB} - \eta_{\alpha\beta\gamma\delta}F^{\beta\gamma\delta a}\Gamma_{aAB}), \tag{29}$$

$$\mathcal{P}_{\mu ABCD} = \frac{3}{4}[e^m{}_a \partial_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^a_{[AB}\Gamma^b_{CD]} - \frac{\sqrt{2}}{8}\Delta^{-1/2} e'_\mu{}^\alpha F_{abc\alpha}\Gamma^a_{[AB}\Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{48}\Delta^{-1/2} e'_{\mu\alpha}\eta^{\alpha\beta\gamma\delta}F_{\alpha\beta\gamma\delta}\Gamma_{b[AB}\Gamma^{ab}_{CD]}. \tag{30}$$

Below, we will use them in a slightly modified form, again adapted to the S^7 compactification.

III. NONLINEAR *ANSÄTZE* FOR MAXIMAL SUPERGRAVITY ON S^7

We will now illustrate the usefulness of the results in the foregoing section by specializing to the S^7 compactification of $D = 11$ supergravity [6,7], where our formalism furnishes numerous new insights and results, most notably with regard to the dual fields of $D = 11$ supergravity. To this aim, we will present a detailed analysis of the nonlinear *Ansätze* and the generalized vielbein postulate in the context of the S^7 compactification.

A. Compactification on S^7

To begin with, and for the reader's convenience, here we collect some relevant (and well-known) formulas, see e.g. Refs. [3,7], related to the seven-sphere. Denoting the S^7 background covariant derivative by $\mathring{D}_m$, we recall that S^7 admits eight Killing spinors $\eta^I(y)$ obeying

$$\left(\mathring{D}_m + \frac{1}{2} i m_7 \mathring{e}_m{}^a \Gamma_a\right)\eta^I = 0, \tag{31}$$

where $I, J, \ldots = 1, \ldots, 8$, and $\mathring{e}_m{}^a$ is the siebenbein on the round S^7. Written out in components, the Killing spinors η^I_A are orthonormal matrices; that is,

$$\eta^A_I \eta^J_A = \delta^J_I, \qquad \eta^A_I \eta^I_B = \delta^A_B. \tag{32}$$

The 28 Killing vectors $K^{mIJ}(y)$ and their derivatives $K_{mn}{}^{IJ}(y)$ can then be represented as bilinears in terms of Killing spinors, viz.

$$K^{mIJ} = i\mathring{e}^{ma}\bar{\eta}^I\Gamma_a\eta^J, \qquad K_{mn}{}^{IJ} = \mathring{e}_m{}^a \mathring{e}_n{}^b \bar{\eta}^I \Gamma_{ab}\eta^J; \tag{33}$$

clearly,

$$\mathring{D}_n K^{IJ}_m = m_7 K^{IJ}_{mn} \Rightarrow \mathring{D}_m K^{IJ}_n + \mathring{D}_n K^{IJ}_m = 0. \tag{34}$$

Hence, $\mathring{D}_n K^{nIJ} = 0$. Observe the different "canonical" positions of the world indices on K^m and K_{mn}, and the fact that both of these are always and by definition related to the "flat" objects by means of the S^7 background siebenbein and its inverse. The vector fields $K^{IJ} \equiv K^{mIJ}\mathring{D}_m$ generate the SO(8) isometry group of the seven-sphere via the Lie bracket

$$[K^{IJ}, K^{KL}] = -8m_7 \delta^{[I|[K} K^{L]|J]}, \tag{35}$$

[5]Below, we will work with a slightly modified operator $\mathcal{D}_\mu$ adapted to the S^7 compactification, cf. Eq. (59).

or equivalently

$$K^{nIJ}\mathring{D}_n K^{mKL} - K^{nKL}\mathring{D}_n K^{mIJ} = -8m_7 \delta^{[I|[K}K^{mL]|J]}. \tag{36}$$

However, in standard Kaluza-Klein geometry, there is no corresponding interpretation for the tensor fields $K_{mn}{}^{IJ}$ [nor for Eq. (47) below].

B. The nonlinear *Ansätze*

The nonlinear *Ansätze* for maximal supergravity are obtained by comparing the supersymmetry transformations in Eq. (21) with the analogous supersymmetry transformations of the vectors in $D = 4$ maximal supergravity [8,12]:

$$\delta A_\mu{}^{IJ} = -\frac{1}{2}(u_{ij}{}^{IJ} + v_{ijIJ})\Big[2\sqrt{2}\bar{\varepsilon}^i \varphi^j_\mu + \bar{\varepsilon}_k \gamma'_\mu \chi^{ijk}\Big] + \text{H.c.}, \tag{37}$$

$$\delta A_{\mu IJ} = -\frac{1}{2}i(u_{ij}{}^{IJ} - v_{ijIJ})\Big[2\sqrt{2}\bar{\varepsilon}^i \varphi^j_\mu + \bar{\varepsilon}_k \gamma'_\mu \chi^{ijk}\Big] + \text{H.c.} \tag{38}$$

We will refer to the 28 + 28 vector fields $A_\mu{}^{IJ}$ and $A_{\mu IJ}$ as "electric" and "magnetic" vectors, respectively.

In order to relate the $D = 4$ vectors to the vector fields identified in the previous section, we now need to choose appropriate Kaluza-Klein *Ansätze* for all vectors in Eqs. (4)–(6). For the Kaluza-Klein vector $B_\mu{}^m$, this *Ansatz* is well known, as we explained in the Introduction; choosing appropriate normalizations, we have

$$B_\mu{}^m(x, y) = -\frac{\sqrt{2}}{4}K^{mIJ}(y)A_\mu{}^{IJ}(x), \tag{39}$$

where K^{mIJ} are the 28 Killing vectors defined in Eq. (33). However, for the remaining vector fields, and in particular for those arising from the dual fields in higher dimensions, there are no such *Ansätze* available from general Kaluza-Klein theory, and therefore we have to proceed in a different manner. In fact, for the "nongeometrical" vector fields $B_{\mu mn}(x, y)$, the appropriate *Ansatz* was already found in Ref. [2]; it reads[6]

$$B_{\mu mn}(x, y) = \frac{1}{24}K_{mn}{}^{IJ}(y)A_{\mu IJ}(x). \tag{40}$$

We stress that the "canonical" position of the world indices as defined in Ref. [2] is in accord with the position of indices on $B_\mu{}^m$ and $B_{\mu mn}$. We emphasize again that, unlike for the standard Kaluza-Klein vector, there is *a priori* no geometric argument to fix the *Ansatz* for $B_{\mu mn}$. Furthermore, the normalization had to be determined in Ref. [3] by comparison with the $D = 4$ theory.

Adopting the Kaluza-Klein *Ansätze* (39) and (40) for $B_\mu{}^m$ and $B_{\mu mn}$, respectively, and comparing the 11-dimensional supersymmetry transformations (7) and (8) with the respective four-dimensional transformations (37) and (38) gives a relation between the generalized vielbeine e^m_{AB} and e_{mnAB} and the four-dimensional scalars $u_{ij}{}^{IJ}$ and v_{ijIJ}. More precisely, the identification between 11-dimensional and 4-dimensional SU(8) indices is made by means of the orthonormal Killing spinors on the round sphere η^i_A, which convert "curved" SU(8) indices $A, B, \ldots$ (appropriate to the $D = 11$ theory) into "flat" SU(8) indices $i, j, k, \ldots$ (appropriate to maximal $D = 4$ gauged supergravity) and vice versa in the terminology of Ref. [25]. Hence,

$$\begin{aligned} X_{ijk\ldots} &= \eta^A_i \eta^B_j \eta^C_k \ldots X_{ABC\ldots} \Leftrightarrow \\ X_{ABC\ldots} &= \eta^i_A \eta^j_B \eta^k_C \ldots X_{ijk\ldots} \end{aligned} \tag{41}$$

for any SU(8) tensor by orthonormality of the Killing spinors [Eq. (32)]. Accordingly, we define

$$\begin{aligned} e^m_{ij}(x, y) &\equiv e^m_{AB}(x, y)\eta_i{}^A(y)\eta_j{}^B(y), \\ e_{mnij}(x, y) &\equiv e_{mnAB}(x, y)\eta_i{}^A(y)\eta_j{}^B(y). \end{aligned} \tag{42}$$

The nonlinear *Ansätze* derived in previous work are then given by

$$e^m_{ij}(x, y) = K^{mIJ}(y)[u_{ij}{}^{IJ} + v_{ijIJ}](x), \tag{43}$$

$$e_{mnij}(x, y) = -\frac{\sqrt{2}}{12}iK_{mn}{}^{IJ}(y)[u_{ij}{}^{IJ} - v_{ijIJ}](x). \tag{44}$$

The *Ansätze* for the remaining vectors originating from the dual $D = 11$ fields are more tricky and, in fact, can only be arrived at by imposing consistency of the relevant supersymmetry variations. More specifically, our analysis implies the identifications

$$B_{\mu m_1\ldots m_5}(x, y) = -\frac{1}{4\cdot 6!}(K_{m_1\ldots m_5}{}^{IJ} - 6\cdot 6!\mathring{\zeta}_{m_1\ldots m_5 p}K^{pIJ})(y)A_\mu{}^{IJ}(x), \tag{45}$$

$$B_{\mu m_1\ldots m_7,n}(x, y) = -\frac{1}{9!\sqrt{2}}(\mathring{\eta}_{m_1\ldots m_7}K_n{}^{IJ} + 6\cdot 7!\mathring{\zeta}_{[m_1\ldots m_6}K_{m_7]n}{}^{IJ})(y)A_{\mu IJ}(x), \tag{46}$$

where

$$K_{m_1\ldots m_5}{}^{IJ} = i\mathring{e}_{m_1}{}^{a_1}\ldots\mathring{e}_{m_5}{}^{a_5}\bar{\eta}^I\Gamma_{a_1\ldots a_5}\eta^J \tag{47}$$

[6] Note that the difference between the coefficient in the *Ansatz* for $B_{\mu mn}$ here and in Ref. [2] is due to differing conventions for $F = dA$. In Ref. [2], this constant was fixed based on the tests of the nonlinear flux *Ansatz* in Ref. [3] where $F = 4\partial A$. However, here, as in much of the related literature, we use the convention $F = 4!\partial A$.

is again a bilinear in the Killing spinors. The quantity $\mathring{\zeta}_{m_1\ldots m_6}(y)$ is defined such that

$$7!\partial_{[m_1}\mathring{\zeta}_{m_2\ldots m_7]} = m_7\,\mathring{\eta}_{m_1\ldots m_7}, \tag{48}$$

and is thus to be regarded as a potential for the volume form on the round seven-sphere.

While the first terms on the right-hand side of the *Ansätze* (45) and (46) are not completely unexpected, the crucial new feature in comparison with formulas (39) and (40) is the presence of the nonglobally defined field $\mathring{\zeta}_{m_1\ldots m_6}$. One way of understanding its presence in the above *Ansätze* is to observe that the components of the field strength $F_{(4)}$ along the four dimensions $F_{\mu\nu\rho\sigma}$ is nonzero and, for maximally symmetric solutions, is proportional to the volume form in four dimensions [6]. Equivalently, this nonzero component of the three-form potential $A_{\mu\nu\rho}$ can be viewed as a nonzero component of its six-form dual along the seven-dimensional directions [4], namely

$$A_{m_1\ldots m_6} \sim \mathring{\zeta}_{m_1\ldots m_6}. \tag{49}$$

Thus, even for the S^7 solution [7] where the scalar expectation values are essentially trivial, one would have to obtain such a nonzero value for $A_{m_1\ldots m_6}$ from a nonlinear *Ansatz*. Indeed, the coefficients in *Ansätze* (45) and (46) have been fixed by requiring consistency with the S^7 solution. In fact, the "vacuum expectation value" of the six-form field will be nonvanishing for *any* nontrivial compactification (that is, other than the T^7 reduction of Ref. [14]) of $D = 11$ supergravity.

As before, comparing the four-dimensional supersymmetry transformations (37) and (38) with their higher-dimensional analogues (9) and (10) gives

$$e_{m_1\ldots m_5 ij}(x,y) = \frac{1}{6!\sqrt{2}}(K_{m_1\ldots m_5}{}^{IJ} - 6\cdot 6!\,\mathring{\zeta}_{m_1\ldots m_5 p}K^{pIJ})(y) \times [u_{ij}{}^{IJ} + v_{ijIJ}](x), \tag{50}$$

$$e_{m_1\ldots m_7,nij}(x,y) = \frac{2}{9!}i(\mathring{\eta}_{m_1\ldots m_7}K_n{}^{IJ} + 6\cdot 7!\,\mathring{\zeta}_{[m_1\ldots m_6}K_{m_7]n}{}^{IJ})(y) \times [u_{ij}{}^{IJ} - v_{ijIJ}](x), \tag{51}$$

where we have again converted "curved" to "flat" SU(8) indices by means of the relations in Eq. (41).

The fact that we now have two expressions for the generalized vielbeine, one in terms of 11-dimensional fields [Eqs. (11)–(14)], and one in terms of four-dimensional scalars [Eqs. (43), (44), (50), and (51)], allows us to derive nonlinear *Ansätze* for internal fields. The nonlinear *Ansatz* for the metric [19] is found by considering the expression

$$e^m{}_{AB}e^{nAB}, \tag{52}$$

which gives [19]

$$\Delta^{-1}g^{mn}(x,y) = \frac{1}{8}K^{mIJ}K^{nKL}(y) \times [(u^{ij}{}_{IJ} + v^{ijIJ})(u_{ij}{}^{KL} + v_{ijKL})](x). \tag{53}$$

Similarly, the nonlinear flux *Ansatz* [2] is found by considering the expression

$$e_{mn}{}^{AB}e^p{}_{AB}, \tag{54}$$

which gives[7]

$$A_{mnp}(x,y) = -\frac{\sqrt{2}}{96}i\Delta g_{pq}(x,y)K_{mn}{}^{IJ}K^{qKL}(y)[(u^{ij}{}_{IJ} - v^{ijIJ}) \times (u_{ij}{}^{KL} + v_{ijKL})](x), \tag{55}$$

where one uses the metric *Ansatz* (53) to compute Δg_{pq} in the equation above. Now, considering

$$e_{m_1\ldots m_5 AB}e^{nAB} \tag{56}$$

gives a nonlinear *Ansatz* for the internal six-form components. It is simple to show that by equating the contraction of the two vielbeine using definitions (11) and (13) on one side and definitions (43) and (50) on the other gives

$$A_{nm_1\ldots m_5} - \frac{\sqrt{2}}{4}A_{n[m_1m_2}A_{m_3m_4m_5]} = -\frac{\sqrt{2}}{16\cdot 6!}\Delta g_{np}(K_{m_1\ldots m_5}{}^{IJ} - 6\cdot 6!\,\mathring{\zeta}_{m_1\ldots m_5 q}K^{qIJ}) \times K^{pKL}[(u_{ij}{}^{IJ} + v_{ijIJ})(u^{ij}{}_{KL} + v^{ijKL})], \tag{57}$$

where the internal metric and three-form potential components are derived using *Ansätze* (53) and (55). We note that the complete antisymmetry of $A_{m_1\ldots m_6}$ in all six indices is not manifest from this expression; our explicit tests for the SO(7)$^\pm$ invariant solutions (see Sec. V) show, however, that this consistency requirement is met in nontrivial examples.

An *Ansatz* for the six-form potential can also be obtained by considering the fourth generalized vielbeine $e_{m_1\ldots m_7,nAB}$. The relation of this *Ansatz* to *Ansatz* (57) above will not be obvious, but clearly the two *Ansätze* must be equivalent. This completes the set of uplift *Ansätze* for all bosonic degrees of freedom from maximal gauged supergravity to 11 dimensions.

IV. GENERALIZED VIELBEIN POSTULATES AND THE S^7 COMPACTIFICATION

The generalized vielbein postulate for e^m_{AB} plays an important role in establishing the consistency of the S^7 reduction of $D = 11$ supergravity [25,26]. In particular, in

[7] See footnote 6 for an explanation of the extra factor of 6 between this expression and the nonlinear flux *Ansatz* in Refs. [2,3].

Ref. [25] it is shown that upon the S^7 compactification, the $d = 4$ generalized vielbein postulate reduces to the $E_{7(7)}$ Cartan equation of gauged maximal supergravity, to wit

$$\mathcal{V}^{-1}(x)(\partial_\mu - gA_\mu^{IJ}(x)X^{IJ})\mathcal{V}(x) = \mathcal{Q}_\mu(x) + P_\mu(x), \quad (58)$$

where X^{IJ} generate the compact SO(8) subgroup inside $SL(8,\mathbb{R}) \subset E_{7(7)}$, and g is the gauge coupling constant. In this section, then, we explore the generalized vielbein postulates in a more general context than Refs. [25,26] by investigating the full set of relations [Eqs. (24)–(27)] for the extra vielbein components [Eqs. (11)–(14)], hence taking into account the full set of 56 vector fields identified in Sec. II A. The presence of both electric *and* magnetic vectors in these relations indicates that our construction should eventually allow one to derive more general gaugings of $N = 8$ supergravity from compactification, and thereby to understand how the embedding tensor emerges from the $D = 11$ theory upon different compactifications. However, here we will concentrate on the S^7 compactification, as this case already by itself provides a wealth of new insights, in particular concerning the role of dual vector fields in non-Abelian gaugings. We will briefly return to the more general case in the final section, postponing a detailed discussion to later work.

One issue that we will specifically address and resolve in this section is the following: The fact that not all of the 56 vector fields can correspond to independent propagating degrees of freedom, and the generic emergence of *non-Abelian* gauge interactions for nontrivial compactifications (for which the standard Abelian dualization linking electric and magnetic vectors no longer works), immediately raises the question of how the theory can dispose of the unwanted vectors and thereby ensure the consistency of compactified theory. Here we will establish the consistency of the equations for the S^7 compactification by explicitly showing how the magnetic vector fields drop out of the generalized vielbein postulates, leaving only electric gaugings.

For the S^7 compactification, the reduction *Ansatz* of the vector fields and the generalized vielbeine are given by Eqs. (39), (40), (45), (46), (43), (44), (50), and (51), respectively. To adapt to the S^7 compactification, we introduce a minor modification by replacing Eq. (28) with

$$\mathcal{D}_\mu = \partial_\mu - B_\mu{}^m \mathring{D}_m, \quad (59)$$

and the connections in Eqs. (29) and (30) with

$$\mathcal{Q}^A_{\mu B} = -\frac{1}{2}[e^m{}_a \mathring{D}_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^{ab}_{AB} - \frac{\sqrt{2}}{12}\Delta^{-1/2} e'_\mu{}^\alpha (F_{\alpha abc}\Gamma^{abc}_{AB} - \eta_{\alpha\beta\gamma\delta}F^{\beta\gamma\delta a}\Gamma_{aAB}), \quad (60)$$

$$\mathcal{P}_{\mu ABCD} = \frac{3}{4}[e^m{}_a \mathring{D}_m B_\mu{}^n e_{nb} - (e^p{}_a \mathcal{D}_\mu e_{pb})]\Gamma^a_{[AB}\Gamma^b_{CD]} - \frac{\sqrt{2}}{8}\Delta^{-1/2} e'_\mu{}^\alpha F_{abc\alpha}\Gamma^a_{[AB}\Gamma^{bc}_{CD]} - \frac{\sqrt{2}}{48}\Delta^{-1/2} e'_{\mu\alpha}\eta^{\alpha\beta\gamma\delta}F_{a\beta\gamma\delta}\Gamma_{b[AB}\Gamma^{ab}_{CD]}; \quad (61)$$

that is, replacing the partial derivative ∂_m with the S^7 background covariant derivative $\mathring{D}_m$ everywhere. Likewise, this replacement is to be made everywhere in the vielbein postulate [Eqs. (24)–(27)].

As before [cf. Eq. (41)], we have to convert SU(8) indices in order to relate the connection coefficients above to their four-dimensional counterparts for the S^7 compactification. This change of basis is covariant for all fields, with the exception of [25]

$$\mathcal{Q}^i_{\mu j} = \eta^i_A \eta^B_j \left(\mathcal{Q}^A_{\mu B} - i\frac{\sqrt{2}}{4} m_7 A^{KL}_\mu K^{nKL} \mathring{e}_n{}^a \Gamma_{aAB}\right). \quad (62)$$

Using the Killing spinor equation and the equation above,

$$\begin{aligned}
&\mathcal{D}_\mu e^m_{AB} + \mathcal{Q}^C_{\mu[A} e^m_{B]C} \\
&= (\partial_\mu - B_\mu{}^m \mathring{D}_m)[\eta^i_A(y)\eta^j_B(y) e^m_{ij}(x,y)] + \mathcal{Q}^C_{\mu[A} e^m_{B]C}, \\
&= \eta^i_A \eta^j_B \mathcal{D}_\mu e^m_{ij} + i m_7 B_\mu{}^m \mathring{e}_m{}^a \Gamma^a{}_{C[A} e^m_{B]C} + \mathcal{Q}^C_{\mu[A} e^m_{B]C}, \\
&= \eta^i_A \eta^j_B (\mathcal{D}_\mu e^m_{ij} + \mathcal{Q}^k_{\mu[i} e^m_{j]k}).
\end{aligned} \quad (63)$$

Analogous relations hold for the other generalized vielbeine:

$$\mathcal{D}_\mu e_{mnAB} + \mathcal{Q}^C_{\mu[A} e_{mnB]C} = \eta^i_A \eta^j_B (\mathcal{D}_\mu e_{mnij} + \mathcal{Q}^k_{\mu[i} e_{mnj]k}), \quad (64)$$

$$\mathcal{D}_\mu e_{m_1\ldots m_5 AB} + \mathcal{Q}^C_{\mu[A} e_{m_1\ldots m_5 B]C} = \eta^i_A \eta^j_B (\mathcal{D}_\mu e_{m_1\ldots m_5 ij} + \mathcal{Q}^k_{\mu[i} e_{m_1\ldots m_5 j]k}), \quad (65)$$

$$\mathcal{D}_\mu e_{m_1\ldots m_7, nAB} + \mathcal{Q}^C_{\mu[A} e_{m_1\ldots m_7, nB]C} = \eta^i_A \eta^j_B (\mathcal{D}_\mu e_{m_1\ldots m_7, nij} + \mathcal{Q}^k_{\mu[i} e_{m_1\ldots m_7, nj]k}). \quad (66)$$

The noncovariant term in Eq. (62) thus ensures that we can freely convert between "curved" and "flat" SU(8) indices in all relations.

Let us first consider the generalized vielbein postulate for e^m_{AB}, which is already analyzed in Ref. [25]. The supersymmetry transformation of the graviphoton $B_\mu{}^m$ gives rise to the generalized vielbein e^m_{AB} [Eq. (7)]. In Kaluza-Klein theory, the exact *Ansatz* relating the graviphoton to the four-dimensional vector field is given by the the Killing vectors of the internal space [Eq. (39)]. As we have already mentioned, this *Ansatz*, via Eqs. (37) and (7), also furnishes an *Ansatz* (43) for the generalized vielbein e^m_{AB}. The emergence of the SO(8) covariantization of the four-dimensional derivative is then easily seen to be a consequence, in accordance with general Kaluza-Klein theory, of the appearance of the commutator of two Killing vector fields in

$$(\partial_\mu - B_\mu{}^n \mathring{D}_n) e^m_{ij} + \mathring{D}_n B_\mu{}^m e^n_{ij} \tag{67}$$

and the fact that $\mathring{D}_m B_\mu{}^m = 0$ for any Killing vector. More precisely, plugging these *Ansätze* into the generalized vielbein postulate (24) and using Eq. (63), the latter reduces to

$$\begin{aligned}\partial_\mu e^m_{ij} - &\frac{\sqrt{2}}{8}\mathring{D}_n K^{nIJ} A_\mu{}^{IJ} e^m_{ij} \\ &+ \frac{\sqrt{2}}{4}(K^{nIJ}\mathring{D}_n K^{mKL} - K^{nKL}\mathring{D}_n K^{mIJ}) \\ &\quad \times A_\mu{}^{IJ}[u_{ij}{}^{KL} + v_{ijKL}] + \mathcal{Q}^k_{\mu[i} e^m_{j]k} + \mathcal{P}_{\mu ijkl} e^{mkl} = 0.\end{aligned} \tag{68}$$

Using Eq. (36), the first three terms in the generalized vielbein postulate (68) reduce to

$$K^{mIJ}(\delta^I_K \partial_\mu - 2\sqrt{2} A_\mu{}^{IK})[u_{ij}{}^{KJ} + v_{ijKJ}], \tag{69}$$

which is a contraction of the SO(8) gauge covariant derivative on the scalar fields. Denoting

$$w^+{}_{ij}{}^{IJ} = u_{ij}{}^{IJ} + v_{ijIJ}, \qquad w^-{}_{ij}{}^{IJ} = i[u_{ij}{}^{IJ} - v_{ijIJ}], \tag{70}$$

and with $w^{+ij}{}_{IJ}$ and $w^{-ij}{}_{IJ}$ as their complex conjugates, respectively, the generalized vielbein postulate gives

$$K^{mIJ}(D^{\mathrm{SO}(8)}_\mu w^+{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^+{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{+kl}{}_{IJ}) = 0, \tag{71}$$

where the SO(8) gauge covariant derivative is defined as

$$D^{\mathrm{SO}(8)}_\mu w^\pm{}_{ij}{}^{IJ} = \partial_\mu w^\pm{}_{ij}{}^{IJ} - 2\sqrt{2} m_7 A_\mu{}^{K[I} w^\pm{}_{ij}{}^{J]K}. \tag{72}$$

Thence, we identify $A_\mu{}^{IJ}$ as the SO(8) gauge fields and $\sqrt{2}m_7$ as the SO(8) gauge coupling. Thus, the generalized vielbein postulate reduces to a particular component of the $E_{7(7)}$ Cartan equation with SO(8) covariant derivatives, as claimed above.

While this part of the argument was already given in Ref. [25], the SO(8) covariantization on the other components of the generalized vielbein cannot be traced back to geometrical arguments of this type. In Ref. [25], it is argued that Eq. (71) in fact implies the $E_{7(7)}$ Cartan equation with SO(8) covariant derivatives. However, here, by considering *all* of the generalized vielbein postulates, we can see that this equation follows directly upon compactification on S^7. In other words, the rest of the generalized vielbein postulates give rise to the "missing" components of the Cartan equation in Eq. (58). We will show this in turn for each of the generalized vielbein postulates [Eqs. (25)–(27)].

The generalized vielbein postulate for e_{mnAB} [Eq. (25)] becomes, after conversion to "flat" SU(8) indices using Eq. (64) and again using $\mathring{D}_m B_\mu{}^m = 0$,

$$\begin{aligned}\mathcal{D}_\mu e_{mnij} &+ 2\mathring{D}_{[m} B_{|\mu|}{}^p e_{n]pij} + 3\mathring{D}_{[m} B_{|\mu|np]} e^p_{ij} \\ &+ \mathcal{Q}^k_{\mu[i} e_{mnj]k} + \mathcal{P}_{\mu ijkl} e_{mn}{}^{kl} = 0.\end{aligned} \tag{73}$$

The new feature here is the presence of the "magnetic" vectors $B_{\mu mn}$, which according to Eq. (40) could in principle lead to the gauging of magnetic vector fields in the four-dimensional theory. However, note that the relations between e_{mnAB} [Eq. (44)] and $B_{\mu mn}$ [Eq. (40)] and the four-dimensional fields are not made with respect to Killing vectors but via the tensor K^{IJ}_{mn}, which from the Killing spinor equation satisfies

$$\mathring{D}_p K_{mn}{}^{IJ} = 2m_7 \mathring{g}_{p[m} K_{n]}{}^{IJ}. \tag{74}$$

This immediately implies that

$$\mathring{D}_{[m} B_{|\mu|np]} = 0. \tag{75}$$

Hence, *the magnetic vector fields drop out of relation* (25) in the S^7 reduction, thus ensuring that effectively only the 28 electric vectors appear in the four-dimensional theory with their non-Abelian interactions, while the magnetic vectors all decouple. Using Eqs. (34) and (74), the generalized vielbein postulate then further simplifies to

$$\begin{aligned}\partial_\mu e_{mnij} &+ \frac{1}{12} m_7 A_\mu{}^{IJ} w^-{}_{ij}{}^{KL}(K^p{}_{[m}{}^{IJ} K_{n]p}{}^{KL} - K_{[m}{}^{IJ} K_{n]}{}^{KL}) \\ &+ \mathcal{Q}^k_{\mu[i} e_{mnj]k} + \mathcal{P}_{\mu ijkl} e_{mn}{}^{kl} = 0,\end{aligned} \tag{76}$$

which, using Eq. (B6), gives another component of the $E_{7(7)}$ Cartan equation:

$$K_{mn}{}^{IJ}(D^{\mathrm{SO}(8)}_\mu w^-{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^-{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{-kl}{}_{IJ}) = 0. \tag{77}$$

Next we consider the third equation, (26), which becomes, using $\mathring{D}_m B_\mu{}^m = 0$ and Eqs. (65) and (75),

$$\begin{aligned}\partial_\mu e_{m_1\ldots m_5 ij} &- B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_5 ij} - 5\mathring{D}_{[m_1} B_{|\mu|}{}^p e_{m_2\ldots m_5]pij} \\ &- 6\mathring{D}_{[m_1} B_{|\mu|m_2\ldots m_5 p]} e^p_{ij} + \mathcal{Q}^k_{\mu[i} e_{m_1\ldots m_5 j]k} \\ &+ \mathcal{P}_{\mu ijkl} e_{m_1\ldots m_5}{}^{kl} = 0.\end{aligned} \tag{78}$$

In this case, the reduction *Ansätze* (45) and (50) not only contain tensors, rather than Killing vectors, but they also contain the potential for the volume form on the round seven-sphere $\mathring{\zeta}$, which is not globally defined. As we shall see below, these terms are not only crucial for obtaining the correct nonlinear flux *Ansätze*, but also equally crucial in the reduction of the generalized vielbein postulate.

Inserting the reduction *Ansätze* for the generalized vielbeine and the vector fields and using Eqs. (B3), (34), and (74), we obtain

$$B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_5 ij} = \frac{1}{4\cdot 6!} A_\mu{}^{IJ} w^+{}_{ij}{}^{KL} K^{pIJ}(m_7 \mathring{\eta}_{m_1\ldots m_5 pq} K^{qKL} + 6\cdot 6! \mathring{D}_p \mathring{\zeta}_{m_1\ldots m_5 q} K^{qKL} + 6\cdot 6! m_7 \mathring{\zeta}_{m_1\ldots m_5 q} K^q{}_p{}^{KL}), \tag{79}$$

$$\mathring{D}_{[m_1} B_{|\mu| m_2\ldots m_5 p]} e^p_{ij} = -\frac{1}{4\cdot 6!} A_\mu{}^{IJ} w^+{}_{ij}{}^{KL} K^{pKL}(m_7 \mathring{\eta}_{m_1\ldots m_5 pq} K^{qIJ} - 6\cdot 6! \mathring{D}_{[m_1} \mathring{\zeta}_{m_2\ldots m_5 p]q} K^{qIJ} - 6\cdot 6! m_7 \mathring{\zeta}_{q[m_1\ldots m_5} K^q{}_{p]}{}^{IJ}). \tag{80}$$

Hence,

$$\begin{aligned} B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_5 ij} + 6\mathring{D}_{[m_1} B_{|\mu| m_2\ldots m_5 p]} e^p_{ij} &= \frac{1}{4\cdot 6!} A_\mu{}^{IJ} w^+{}_{ij}{}^{KL}(7 m_7 \mathring{\eta}_{m_1\ldots m_5 pq} K^{pIJ} K^{qKL} + 6\cdot 7! \mathring{D}_{[p} \mathring{\zeta}_{m_1\ldots m_5 q]} K^{pIJ} K^{qKL} \\ &\quad + 6\cdot 6! m_7 \mathring{\zeta}_{m_1\ldots m_5 q} K^{pIJ} K^q{}_p{}^{KL} + 36\cdot 6! m_7 \mathring{\zeta}_{q[m_1\ldots m_5} K^q{}_{p]}{}^{IJ} K^{pKL}), \\ &= \frac{1}{4\cdot 6!} m_7 A_\mu{}^{IJ} w^+{}_{ij}{}^{KL}(\mathring{\eta}_{m_1\ldots m_5 pq} K^{pIJ} K^{qKL} + 48\cdot 6! \mathring{\zeta}_{m_1\ldots m_5 q} \delta^{IK} K^{qJL} \\ &\quad + 30\cdot 6! \mathring{\zeta}_{pq[m_1\ldots m_4} K^q{}_{m_5]}{}^{IJ} K^{pKL}), \end{aligned} \tag{81}$$

where in the second equality above we have used Eqs. (48) and (B5). A straightforward substitution of the *Ansätze* (39) and (50) also gives

$$5\mathring{D}_{[m_1} B_{|\mu|}{}^p e_{m_2\ldots m_5] pij} = \frac{5 m_7}{8\cdot 6!} A_\mu{}^{IJ} w^+{}_{ij}{}^{KL} K^p{}_{[m_1}{}^{IJ}(\mathring{\eta}_{m_2\ldots m_5] pqr} K^{qrKL} + 12\cdot 6! \mathring{\zeta}_{m_2\ldots m_5] pq} K^{qKL}). \tag{82}$$

Now, using the identity

$$5K^p{}_{[m_1}{}^{IJ} \mathring{\eta}_{m_2\ldots m_5] pqr} K^{qrKL} = 2K^p{}_r{}^{IJ} \mathring{\eta}_{m_1\ldots m_5 pq} K^{qrKL} \tag{83}$$

and Eq. (B6), the first terms on the right-hand sides of Eqs. (81) and (82) precisely combine to give

$$\frac{1}{6!} m_7 A_\mu{}^{IJ} w^+{}_{ij}{}^{KL} \mathring{\eta}_{m_1\ldots m_5 pq} \delta^{IK} K^{pqJL} = -\frac{2}{6!} m_7 A_\mu{}^{IJ} w^+{}_{ij}{}^{KL} \delta^{IK} K_{m_1\ldots m_5}{}^{JL}. \tag{84}$$

Moreover, the third term on the right-hand side of Eq. (81) cancels the second term on the right-hand side of Eq. (82). Therefore, in all, Eq. (78) simplifies to

$$(K_{m_1\ldots m_5}{}^{IJ} - 6\cdot 6! \mathring{\zeta}_{m_1\ldots m_5 p} K^{pIJ})[D_\mu^{\mathrm{SO}(8)} w^+{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^+{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{+kl}{}_{IJ}] = 0. \tag{85}$$

Using Eqs. (71) and (B3), the above equation implies

$$K^{mnIJ}(D_\mu^{\mathrm{SO}(8)} w^+{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^+{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{+kl}{}_{IJ}) = 0. \tag{86}$$

Finally, we consider the generalized vielbein postulate for $e_{m_1\ldots m_7,n}$, which, using the same equations as before, simplifies to

$$\partial_\mu e_{m_1\ldots m_7,nij} - B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_7,nij} - \partial_n B_\mu{}^p e_{m_1\ldots m_7,pij} - 2\partial_{[m_1} B_{|\mu| m_2\ldots m_6} e_{m_7]nij} + \mathcal{Q}^k_{\mu[i} e_{m_1\ldots m_7,nj]k} + \mathcal{P}_{\mu ijkl} e_{m_1\ldots m_7,n}{}^{kl} = 0. \tag{87}$$

A similar calculation to the one outlined above for $e_{m_1\ldots m_5 AB}$ gives

$$B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_7,nij} = -\frac{\sqrt{2}}{2\cdot 9!} A_\mu{}^{IJ} w^-{}_{ij}{}^{KL}(5 m_7 \mathring{\eta}_{m_1\ldots m_7} K^{pIJ} K^{KL}_{pn} + 36\cdot 7! \mathring{D}_{[m_1|} \mathring{\zeta}_{p|m_2\ldots m_6} K^{pIJ} K^{KL}_{m_7]n} + 6\cdot 7! m_7 \mathring{\zeta}_{[m_1\ldots m_6}(K_{m_7]}{}^{IJ} K^{KL}_n - K_{m_7]}{}^{KL} K^{IJ}_n)), \tag{88}$$

$$\partial_n B_\mu{}^p e_{m_1\ldots m_7,pij} = -\frac{\sqrt{2}}{2\cdot 9!} m_7 A_\mu{}^{IJ} w^-{}_{ij}{}^{KL} K^p{}_n{}^{IJ}(\mathring{\eta}_{m_1\ldots m_7} K_p{}^{KL} + 6\cdot 7! \mathring{\zeta}_{[m_1\ldots m_6} K_{m_7]p}{}^{KL}), \tag{89}$$

$$2\partial_{[m_1} B_{|\mu| m_2\ldots m_6} e_{m_7]nij} = \frac{\sqrt{2}}{2\cdot 9!} A_\mu{}^{IJ} w^-{}_{ij}{}^{KL}(6 m_7 \mathring{\eta}_{m_1\ldots m_7} K^{pIJ} K^{KL}_{pn} + 36\cdot 7! \mathring{D}_{[m_1|} \mathring{\zeta}_{p|m_2\ldots m_6} K^{pIJ} K^{KL}_{m_7]n} + 6\cdot 7! m_7 \mathring{\zeta}_{[m_1\ldots m_6} K_{m_7]p}{}^{IJ} K^p{}_n{}^{KL}), \tag{90}$$

where we have used Eqs. (34), (48), (74), and (B3) and antisymmetrizations over eight indices, which vanish, to simplify the expressions above. It is now simple to verify, using Eqs. (B5) and (B6), that

$$B_\mu{}^p \mathring{D}_p e_{m_1\ldots m_7,nij} + \partial_n B_\mu{}^p e_{m_1\ldots m_7,pij} + 2\partial_{[m_1} B_{|\mu|m_2\ldots m_6} e_{m_7]nij} = \frac{4\sqrt{2}}{9!} m_7 A_\mu{}^{KI} w^-{}_{ij}{}^{JK} (\mathring{\eta}_{m_1\ldots m_7} K_n^{IJ} + 6\cdot 7!\, \mathring{\zeta}_{[m_1\ldots m_6} K_{m_7]n}{}^{IJ}). \tag{91}$$

Hence, Eq. (87) reduces to

$$K_m{}^{IJ}(D_\mu^{\mathrm{SO}(8)} w^-{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^-{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{-kl}{}_{IJ}) = 0, \tag{92}$$

where we have used Eq. (77) to eliminate the expression proportional to $\mathring{\zeta}$.

To sum up, in the reduction of Eq. (24), the SO(8) gauge covariant derivative arose from the geometrical properties of Killing vectors on S^7. However, for the other generalized vielbeine, the emergence of the SO(8) gauge covariant derivative is not so direct. Indeed, the reduction of the other generalized vielbein postulates is particularly novel given that the postulates (25)–(27) contain fields ($B_{\mu mn}$ and $B_{\mu m_1\ldots m_5}$) for which the identification with the four-dimensional vector fields is not made with the S^7 Killing vectors, but with more general structures on the seven-sphere. We stress again that in the derivation of the last two equations, (86) and (92), the $\mathring{\zeta}$ terms in the *Ansätze* are crucial for obtaining the SO(8) gauge covariant terms. Therefore, the SO(8) gauge covariant derivatives emerge, not in spite of but *because of* these more general structures.

The results that we have obtained from the reduction of the generalized vielbein postulates, Eqs. (71), (77), (86), and (92), can be summarized as

$$K^{aIJ}(D_\mu^{\mathrm{SO}(8)} w^\pm{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^\pm{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{\pm kl}{}_{IJ}) = 0, \tag{93}$$

$$K^{abIJ}(D_\mu^{\mathrm{SO}(8)} w^\pm{}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} w^\pm{}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} w^{\pm kl}{}_{IJ}) = 0. \tag{94}$$

Since K^{aIJ} and K^{abIJ} form a basis of antisymmetric 28×28 matrices, these equations are equivalent to

$$D_\mu^{\mathrm{SO}(8)} \mathcal{V}_{ij}{}^{IJ} + \mathcal{Q}^k_{\mu[i} \mathcal{V}_{j]k}{}^{IJ} + \mathcal{P}_{\mu ijkl} \mathcal{V}^{kl}{}_{IJ} = 0, \tag{95}$$

where $\mathcal{V}(x)$ is the $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$ coset element parametrized by the scalar fields. In Ref. [25] this equation was argued, somewhat indirectly, to hold solely on the basis of the generalized vielbein postulate for e^m_{AB} [Eq. (24)]. Here we see that it naturally follows from the full set of generalized vielbein postulates.

In summary, we find that, in the case of the S^7 compactification, both $B_\mu{}^m$ and $B_{\mu m_1\ldots m_5}$ contribute to the electric vector fields, while the magnetic vector fields drop out of the expressions. Indeed, from Eqs. (39) and (45) we see that this is natural, because $B_\mu{}^m$ and $B_{\mu m_1\ldots m_5}$ project onto different SO(8) components of the electric vector field $A_\mu{}^{IJ}$.

V. A FIRST TEST OF THE NONLINEAR SIX-FORM *ANSATZ*

In this section, we check the consistency of the relations derived in Sec. III, in particular the nonlinear *Ansatz* for the dual six-form using the relatively simple, yet nontrivial $\mathrm{SO}(7)^\pm$ invariant solutions of gauged supergravity [22] for which the higher-dimensional solutions are known [20,21]. For the convenience of the reader, we give a brief description of these solutions from both a four- and a higher-dimensional perspective in Appendix A. The nonlinear metric and flux *Ansätze*, Eqs. (53) and (55), respectively, have been subjected to some very nontrivial tests, which they have passed with remarkable success, most recently in Ref. [3] (where references to earlier work can also be found). In particular, these *Ansätze* correctly reproduce the $\mathrm{SO}(7)^\pm$ invariant solutions [3,19,26]. Therefore, let us consider the nonlinear *Ansatz*, Eq. (57), for the dual six-form potential of the $\mathrm{SO}(7)^\pm$ invariant solutions.

Using the following $\mathrm{E}_{7(7)}$ properties satisfied by $u^{ij}{}_{KL}$ and v^{ijKL} [8]

$$u^{ij}{}_{IJ} u_{ij}{}^{KL} - v_{ijIJ} v^{ijKL} = \delta^{KL}_{IJ}, \tag{96}$$

$$u^{ij}{}_{IJ} v_{ijKL} - v_{ijIJ} u^{ij}{}_{KL} = 0, \tag{97}$$

one can show that

$$(u_{ij}{}^{IJ} + v_{ijIJ})(u^{ij}{}_{KL} + v^{ijKL}) = \delta^{IJ}_{KL} + 2v_{IJMN} v^{KLMN} + 2\,\mathrm{Re}(u_{MN}{}^{IJ} v^{MNKL}). \tag{98}$$

Now, using the form of $u^{ij}{}_{KL}$ and v^{ijKL} for the $\mathrm{SO}(7)^\pm$ invariant solutions, given in Appendix A, and the following identifies satisfied by $C_\pm^{IJKL}$ [20,27],

$$C_+^{IJMN} C_+^{MNKL} = 12\delta^{IJ}_{KL} + 4C_+^{IJKL}, \tag{99}$$

$$C_-^{IJMN} C_-^{MNKL} = 12\delta^{IJ}_{KL} - 4C_-^{IJKL}, \tag{100}$$

the above equation reduces to

$$(u_{ij}{}^{IJ} + v_{ijIJ})(u^{ij}{}_{KL} + v^{ijKL}) = (c^3 + \epsilon s^3)\delta^{IJ}_{KL} + \frac{1}{2}\epsilon\, cs(c+s) C_+^{IJKL} - \frac{1}{2}(1-\epsilon)\, cs^2 C_-^{IJKL}, \tag{101}$$

where

$$\epsilon = \begin{cases} 1 & \mathrm{SO}(7)^+ \\ 0 & \mathrm{SO}(7)^- \end{cases}. \tag{102}$$

Dualizing $\Gamma_{m_1\ldots m_5}$ and using the identities satisfied by the contraction of $C_\pm^{IJKL}$ with K^{mIJ} and K^{mnIJ} (see Ref. [3]) gives

$$
\begin{aligned}
A_{nm_1\ldots m_5} &- \frac{\sqrt{2}}{4} A_{n[m_1m_2}A_{m_3m_4m_5]} \\
&= \frac{\sqrt{2}}{4\cdot 6!}\Delta g_{np}\Big\{12\cdot 6!(c^3+\epsilon s^3)\mathring{\zeta}_{m_1\ldots m_5}{}^{p} \\
&\quad - \frac{cs(c+s)}{3}\epsilon[\mathring{\eta}_{m_1\ldots m_5q}{}^{p}\xi^q + 6\cdot 6!\mathring{\zeta}_{m_1\ldots m_5q}((3+\xi)\mathring{g}^{pq} \\
&\quad - (21+\xi)\hat{\xi}^m\hat{\xi}^n)] - (1-\epsilon)cs^2\mathring{\eta}_{m_1\ldots m_5rs}\mathring{S}^{prs}\Big\}.
\end{aligned} \tag{103}
$$

To compare this result, obtained from the uplift formula, with the results directly obtained by solving the $D = 11$ field equations, let us first consider the $\mathrm{SO}(7)^+$ invariant solution. Using

$$
cs(c+s) = \gamma^{1/2}/5, \qquad c^3+s^3 = 2\gamma^{1/2}/5 \tag{104}
$$

and noting that $A_{mnp} = 0$ for this solution, Eq. (103) reduces to

$$
A_{m_1\ldots m_6} = \frac{\sqrt{2}}{2\cdot 6!}\frac{1}{9-\xi}\mathring{\eta}_{m_1\ldots m_6p}\xi^p - 3\sqrt{2}\mathring{\zeta}_{m_1\ldots m_6}, \tag{105}
$$

where we have used the form of the metric g_{mn} given in Eq. (A5). Now, taking the exterior derivative of this equation gives

$$
7!\mathring{D}_{[m_1}A_{m_2\ldots m_7]} = -\frac{180\sqrt{2}}{(9-\xi)^2}m_7\mathring{\eta}_{m_1\ldots m_7}, \tag{106}
$$

which agrees precisely with the expression found by dualizing the Freund-Rubin field strength for the value of γ set by the nonlinear *Ansätze*, see Eq. (A9).

Next, consider the $\mathrm{SO}(7)^-$ invariant solution. Using the expressions for A_{mnp} and g_{mn} in Appendix A, it is simple to show that Eq. (103) reduces to

$$
A_{m_1\ldots m_6} = -3\sqrt{2}\mathring{\zeta}_{m_1\ldots m_6} + \frac{\sqrt{2}}{16\cdot 6!}\gamma^{-1/3}\mathring{S}^{pq}{}_{[m_1}\mathring{\eta}_{m_2\ldots m_6]pq}. \tag{107}
$$

However, the second term on the right-hand side vanishes by the Schouten identity and the tracelessness of the torsion. Hence,

$$
A_{m_1\ldots m_6} = -3\sqrt{2}\mathring{\zeta}_{m_1\ldots m_6}, \tag{108}
$$

which agrees with the expression for $A_{m_1\ldots m_6}$ given in Eq. (A19) for the particular value of γ set by the *Ansätze*.

VI. OUTLOOK: MAGNETIC VECTORS AND THE EMBEDDING TENSOR

Having established the full consistency of all equations for the S^7 compactification, we now return to the most remarkable feature of the vielbein postulate equations (24)–(27), namely the fact that they simultaneously involve the Kaluza-Klein vectors, the "nongeometric vectors" coming from the three-form field, and the $D = 11$ dual vector fields. In principle, it is therefore clear that both the "electric" vector fields, coming from the reduction of $B_\mu{}^m$ and $B_{\mu m_1\ldots m_5}$, and the "magnetic" vector fields, coming from the reduction of $B_{\mu mn}$, can be gauged. This is a feature that our construction shares with the embedding tensor formalism as applied to gaugings of maximal supergravity in four dimensions [10–12]. There as well, one initially works with the full set of 56 electric and magnetic vector fields, replacing the Cartan equations (58) with the more general *Ansatz*

$$
\begin{aligned}
\mathcal{V}^{-1}(x)[\partial_\mu &- gA_\mu^{IJ}(x)\Theta_{IJ\mathcal{A}}\mathcal{Y}^{\mathcal{A}} \\
&- gA_{\mu IJ}(x)\Theta^{IJ}{}_{\mathcal{A}}\mathcal{Y}^{\mathcal{A}}]\mathcal{V}(x) = Q_\mu(x) + P_\mu(x),
\end{aligned} \tag{109}
$$

where $\mathcal{Y}^{\mathcal{A}}$ ($\mathcal{A} = 1,\ldots,133$) are the generators of $\mathrm{E}_{7(7)}$, $(\Theta_{IJ\mathcal{A}}, \Theta^{IJ}{}_{\mathcal{A}})$ is the embedding tensor, and A_μ^{IJ} and $A_{\mu IJ}$ are the electric and magnetic vectors introduced in Eqs. (37) and (38), respectively. The embedding tensor thus transforms in the product $\mathbf{56}\otimes\mathbf{133}$, but a consistent gauging with 28 propagating gauge fields exists only when Θ restricts to the $\mathbf{912}$ representation of $\mathrm{E}_{7(7)}$ in this product [10–12] (and in addition satisfies a quadratic identity). The choice of embedding tensor not only determines the gauge group, but also decides which 28 vector fields out of the initial 56 vectors become propagating non-Abelian vectors. Consequently, studying the vielbein equations (24)–(27) in parallel with Eq. (109) should thus enable one to understand the embedding tensor and its relation to any particular compactification directly from the 11-dimensional perspective. Although we will leave the full exploration of these possibilities to future work, we conclude with some comments.

In the S^7 compactification considered in Sec. IV, the SO(8) gauge covariant derivative term comes from the following terms in the generalized vielbein postulates:

$$
\partial_m B_\mu{}^n \quad \text{and} \quad \partial_{[m_1}B_{|\mu|m_2\ldots m_6]}. \tag{110}
$$

In particular, terms of the forms

$$
\partial_m B_\mu{}^m \quad \text{and} \quad \partial_{[m}B_{|\mu|np]} \tag{111}
$$

do not contribute. The first expression above vanishes because the Kaluza-Klein *Ansatz* for the graviphoton is given by S^7 Killing vectors, which are divergence-free, while the second expression vanishes because of the form of the Kaluza-Klein *Ansatz* for $B_{\mu mn}$, Eq. (40), and properties of S^7 Killing spinors.

A natural question to ask is whether one can find examples of compactifications where the expressions (111), that vanish for the S^7 reduction, contribute to the four-dimensional Cartan equation (109). For example, while the first expression vanishes for Killing vectors, it is nonzero for conformal Killing vectors, which also form a simple Lie algebra. An interesting question is whether one can carry out more general reductions of this type.

Furthermore, a particularly interesting class of gaugings to investigate in this context are the Scherk-Schwarz compactifications [28] and twisted seven-torus flux compactifications [29–33], which lead to various gaugings in four dimensions (see Ref. [12] for a review of known gaugings in four dimensions). While the original Scherk-Schwarz reductions on flat groups are known to lead to electric gaugings [34], flux compactifications provide examples where both electric and magnetic vector fields contribute to the gauging [31].

A study of the generalized vielbein postulates may also shed light on the higher-dimensional origins of the recently discovered continuous family of inequivalent maximal SO(8) gauged supergravities [23]. While the original SO(8) gauged supergravity [8] in the $\mathrm{SL}(8,\mathbb{R})$ symplectic frame only contains electric gaugings, there is a deformation that allows both electric and magnetic gaugings in the aforementioned symplectic frame. For a given range of the angle of rotation between gaugings of electric and magnetic vector fields, the theory is inequivalent to the original theory. While $D = 11$ supergravity apparently cannot explain the existence of these new supergravities, with the framework presented here it is possible to investigate whether $D = 11$ supergravity admits an analogous deformation that rotates $B_\mu{}^{\mathrm{MN}}$ and $B_{\mu\mathrm{MN}}$, defined in Eq. (20), into each other and that would be implemented by a rotation on the 56-bein [Eq. (15)] in complete analogy with the $D = 4$ theory [2]. The S^7 compactification of these putative theories would then give rise to the magnetic gaugings in the deformed theories found in Ref. [23].

APPENDIX A: SO(7)$^\pm$ INVARIANT SOLUTIONS

In this appendix, we summarize the SO(7)$^\pm$ invariant stationary points of maximal gauged supergravity [22] and their respective 11-dimensional counterpart solutions [20,21]. The nonlinear metric and flux *Ansätze* have been confirmed for these solutions in Refs. [3,19]. Much of the necessary information regarding these solutions is explained in Ref. [3], and in particular its Appendix A. Therefore, for brevity, we refer the reader there for the definitions of the relevant structures and content ourselves here with a list of the most important properties of these solutions that will be relevant for the calculations in Sec. V.

The scalar profile for the SO(7)$^+$ invariant stationary point is given by [19,27]

$$u^{IJ}{}_{KL} = p^3\delta^{IJ}_{KL} + \frac{1}{2}pq^2C_+^{IJKL}, \tag{A1}$$

$$v^{IJKL} = q^3\delta^{IJ}_{KL} + \frac{1}{2}p^2qC_+^{IJKL}, \tag{A2}$$

where constants p and q are such that [22]

$$c^2 = (p^2+q^2)^2 = \frac{1}{2}\Big(3/\sqrt{5}+1\Big), \tag{A3}$$

$$s^2 = (2pq)^2 = \frac{1}{2}\Big(3/\sqrt{5}-1\Big). \tag{A4}$$

The 11-dimensional solution is of the form [20]

$$g_{MN} = \gamma^{7/18}\,30^{-2/3}(9-\xi)^{2/3} \times\Big(\mathring{\eta}_{\mu\nu}, \frac{\gamma^{-1/2}}{9-\xi}[30\,\mathring{g}_{mn} - (21+\xi)\hat{\xi}_m\hat{\xi}_n]\Big), \tag{A5}$$

$$F_{MNPQ} = \Big(\frac{\sqrt{6}}{3}im_7\gamma^{5/6}\mathring{\eta}_{\mu\nu\rho\sigma}, 0\Big), \tag{A6}$$

where γ is an arbitrary constant, which takes the value

$$\gamma = 5^{3/2} \tag{A7}$$

when the solution is constructed via the nonlinear *Ansätze* [3]. Note that the determinant of the siebenbein

$$\Delta = \det(e_m{}^a) = \sqrt{\det(g_{mn})} = \gamma^{-7/18}\,30^{2/3}(9-\xi)^{-2/3}. \tag{A8}$$

In addition, due to the existence of the Freund-Rubin term, the dual potential $A_{(6)}$ is nonzero and of the form

$$7!D_{[M_1}A_{M_2\ldots M_7]} = \begin{cases} -180\sqrt{10}m_7\gamma^{-1/3}(9-\xi)^{-2}\mathring{\eta}_{m_1\ldots m_7} & [m_1\ldots m_7] \\ 0 & \text{otherwise} \end{cases}. \tag{A9}$$

The scalar profile of the SO(7)$^-$ invariant stationary point of maximal supergravity is of the form [19,27]

$$u^{IJ}{}_{KL} = p^3\delta^{IJ}_{KL} - \frac{1}{2}pq^2C_-^{IJKL}, \tag{A10}$$

$$v^{IJKL} = iq^3\delta^{IJ}_{KL} - \frac{1}{2}ip^2qC_-^{IJKL}, \tag{A11}$$

where constants the c and s, related to p and q as above, take the values [22]

$$c^2 = \frac{5}{4}, \qquad s^2 = \frac{1}{4}. \tag{A12}$$

The 11-dimensional solution is of the form [21]

$$g_{MN} = \gamma^{7/18}(\mathring{\eta}_{\mu\nu}, \gamma^{-1/2}\mathring{g}_{mn}), \tag{A13}$$

$$F_{MNPQ} = \Big(2\sqrt{2}im_7\gamma^{5/6}\mathring{\eta}_{\mu\nu\rho\sigma}, \frac{\sqrt{2}}{6}m_7\gamma^{-1/6}\mathring{\eta}_{mnpqrst}\mathring{S}^{rst}\Big), \tag{A14}$$

and in particular,

$$A_{MNP} = \begin{cases} 2\sqrt{2}i\gamma^{5/6}\mathring{\zeta}_{\mu\nu\rho} & [\mu\nu\rho] \\ \frac{\sqrt{2}}{4!}\gamma^{-1/6}\mathring{S}_{mnp} & [mnp] \\ 0 & \text{otherwise} \end{cases}, \tag{A15}$$

where $\mathring{\zeta}_{\mu\nu\rho}$ is the potential for the Freund-Rubin field strength

$$4!\partial_{[\mu}\mathring{\zeta}_{\nu\rho\sigma]} = m_7\,\mathring{\eta}_{\mu\nu\rho\sigma}. \tag{A16}$$

As before, γ is an arbitrary constant that is fixed by the nonlinear *Ansätze* to take the value [3]

$$\gamma^{1/3} = 5/4. \tag{A17}$$

Furthermore,

$$\Delta = \det(e_m{}^a) = \sqrt{\det(g_{mn})} = \gamma^{-7/18}. \tag{A18}$$

The six-form potential for this solution is of the form [4]

$$A_{M_1\ldots M_6} = \begin{cases} \frac{\sqrt{2}}{12}i\gamma^{2/3}\mathring{\zeta}_{\mu\nu\rho}\mathring{S}_{mnp} & [\mu\nu\rho mnp] \\ -\frac{15\sqrt{2}}{4}\gamma^{-1/3}\mathring{\zeta}_{m_1\ldots m_6} & [m_1\ldots m_6] \\ 0 & \text{otherwise} \end{cases}, \tag{A19}$$

where $\mathring{\zeta}_{m_1\ldots m_6}$ is defined in Eq. (48).

APPENDIX B: USEFUL IDENTITIES

We list some useful identities satisfied by seven-dimensional Γ matrices. These identities already appear in Refs. [1,14].

$$\Gamma^{a_1\ldots a_7} = -i\epsilon^{a_1\ldots a_7}, \tag{B1}$$

$$\Gamma^{a_1\ldots a_6} = -i\epsilon^{a_1\ldots a_6 b}\Gamma^b, \tag{B2}$$

$$\Gamma^{a_1\ldots a_5} = \frac{i}{2}\epsilon^{a_1\ldots a_5 bc}\Gamma^{bc}, \tag{B3}$$

$$\Gamma^{a_1\ldots a_4} = \frac{i}{3!}\epsilon^{a_1\ldots a_4 bcd}\Gamma^{bcd}, \tag{B4}$$

$$\Gamma^{ab}_{AB}\Gamma^{b}_{CD} - \Gamma^{b}_{AB}\Gamma^{ab}_{CD} = 8\delta_{[C|[A}\Gamma^{a}_{B]|D]}, \tag{B5}$$

$$\Gamma^{c[a}_{AB}\Gamma^{b]c}_{CD} + \Gamma^{[a}_{AB}\Gamma^{b]}_{CD} = 4\delta_{[C|[A}\Gamma^{ab}_{B]|D]}. \tag{B6}$$

These identities are exactly identities (A.1), (A.6), and (A.7) in Appendix A of Ref. [1].

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