

Multiparticle one-loop amplitudes and S-duality in closed superstring theory

Michael B. Green,^a Carlos R. Mafra^a and Oliver Schlotterer^{a,b}

^a*Department of Applied Mathematics and Theoretical Physics,
Wilberforce Road, Cambridge CB3 0WA, U.K.*

^b*Max-Planck-Institut für Gravitationsphysik,
Albert-Einstein-Institut, 14476 Golm, Germany*

E-mail: M.B.Green@damtp.cam.ac.uk, C.R.Mafra@damtp.cam.ac.uk,
oliver.schlotterer@aei.mpg.de

ABSTRACT: Explicit expressions for one-loop five supergraviton scattering amplitudes in both type II superstring theories are determined by making use of the pure spinor formalism. The type IIB amplitude can be expressed in terms of a doubling of ten-dimensional super Yang-Mills tree amplitude, while the type IIA amplitude has additional pieces that cannot be expressed in that manner. We evaluate the coefficients of terms in the analytic part of the low energy expansion of the amplitude, which correspond to a series of terms in an effective action of the schematic form $D^{2k}R^5$ for $0 \leq k \leq 5$ (where R is the Riemann curvature). Comparison with earlier analyses of the tree amplitudes and of the four-particle one-loop amplitude leads to an interesting extension of the action of $SL(2, \mathbb{Z})$ S-duality on the moduli-dependent coefficients in the type IIB theory. We also investigate closed-string five-particle amplitudes that violate conservation of the $U(1)$ R-symmetry charge — processes that are forbidden in supergravity. The coefficients of their low energy expansion are shown to agree with S-duality systematics. A less detailed analysis is also given of the six-point function, resulting in the vanishing of the analytic parts of the R^6 and D^4R^6 interactions in the ten-dimensional effective action, but not in lower dimensions.

KEYWORDS: Superstrings and Heterotic Strings, Superspaces, String Duality

ARXIV EPRINT: [1307.3534](https://arxiv.org/abs/1307.3534)

Contents

1	Introduction and overview	1
1.1	Introduction	1
1.2	Overview of paper and brief summary of results	2
2	Review of background material	8
2.1	The structure of superstring tree amplitudes	8
2.2	Low energy expansion of superstring tree amplitudes	9
2.2.1	Four- and five-particle examples of closed-string tree amplitudes	11
2.3	One-loop amplitudes of the open superstring	11
2.4	Some S-duality constraints on Type IIB amplitudes	12
2.4.1	U(1)-conserving amplitudes	12
2.4.2	U(1)-violating amplitudes	14
3	Type II one-loop amplitudes in pure spinor superspace	15
3.1	The structure of the five-point closed-string correlator	17
3.2	Integration by parts and $K^{(b)}$	18
3.3	Interactions between left and right-movers and $K^{(c)}$	19
3.4	BRST invariance	20
3.5	The five closed-string amplitude in terms of SYM trees	22
3.6	The distinction between type IIA and type IIB in ten dimensions	24
4	Low energy expansion of type IIB one-loop amplitudes	25
4.1	The four-particle amplitude and its world-sheet diagrams	26
4.2	Five-particle world-sheet integrals and their diagrammatic expansion	28
4.3	Order by order expansion of five-point integrals	31
5	Low energy type IIB amplitudes and S-duality	35
5.1	The four-particle one-loop amplitude	35
5.2	The U(1)-conserving five-particle amplitudes	37
5.3	The U(1)-violating five-particle amplitudes	39
5.4	S-duality of U(1)-conserving amplitudes	40
5.5	S-duality of U(1)-violating type IIB amplitudes	43
6	A glimpse of the six-particle amplitude	45
7	Summary and comments	47
A	Comparison of conventions	49

B	Pure spinor superspace calculations	50
B.1	The pentagon numerator equation in superspace	50
B.2	Permutation symmetry of the five-graviton amplitude	51
C	Further 1PI world-sheet diagrams	51
C.1	Weight $w = 5$	52
C.2	Weight $w = 6$	52
D	Five-point integrals to weight $w = 4$	53
E	Six-point integrals to weight $w = 4$	55
E.1	The singular part of six-point integrals	55
E.2	The regular part of six-point integrals	55

1 Introduction and overview

1.1 Introduction

Over the past few years the study of scattering amplitudes in supersymmetric quantum field theory and string theory has led to a stimulating interplay between physical ideas and structures of mathematical interest. These beautiful relationships strongly constrain theories with maximal supersymmetry, which can be viewed as descending from ten-dimensional open or closed-string theories. This has led to the discovery of a number of intriguing properties of both perturbative and non-perturbative aspects of such theories in various backgrounds and in a variety of dimensions. In particular, the relationship between perturbative and non-perturbative properties of string theory embodied in its duality symmetries has stimulated many fruitful research avenues.

The objectives of this paper are twofold. The first is to investigate properties of closed-string one-loop amplitudes in type II superstring theory with more than four external particles, which have not previously been studied systematically. Here we will mainly discuss five-particle amplitudes, with some additional results for six-particle amplitudes. This will make use of the general rules for constructing open-string one-loop amplitudes that were derived in [1] by use of the pure spinor formalism [2, 3], combined with the constraints imposed by BRST invariance. In the type IIB case we will see that the kinematic structure of the five-particle amplitude is similar to that of the tree-level amplitude of [4–6], as well as with the structure of four-particle tree and loop amplitudes. We will also present a detailed analysis of properties of the low energy expansion of these amplitudes.

The second objective is to make use of these perturbative results to extend the understanding of the exact, non-perturbative, structure of terms in the low energy expansion of the amplitude. These are strongly constrained by S-duality, which relates strong and weak coupling. Such higher dimensional interactions have coefficients that are functions of the scalar fields, or moduli. The ten-dimensional type IIB theory provides the simplest

nontrivial example, with duality group $SL(2, \mathbb{Z})$ [7]. Compactification to $D = 10 - d$ dimensions on a d -torus results in a theory with maximal supersymmetry with a rank- $(d+1)$ duality group that is an arithmetic subgroup of the real split version of E_{d+1} [7].¹

Past results concerning $SL(2, \mathbb{Z})$ duality of four-particle scattering have led to expressions for the exact dependence on the moduli of the coefficients of certain BPS interactions that arise at the first three orders in the low energy expansion of the effective action beyond classical supergravity [12–15]. Four-particle processes conserve the $U(1)$ R-charge of type IIB supergravity, but it is not generally conserved in N -particle string theory amplitudes with $N \geq 5$. The modular properties of the coefficients of a number of $U(1)$ -violating effective interactions at the same order as R^4 in the low energy expansion were considered in [16–21]. In the following we will show how this structure matches with our results for the five-particle amplitude and we will find interesting extra structure that is not seen in the four-point function and which might lead to further insights into the exact moduli dependence.

1.2 Overview of paper and brief summary of results

We will devote section 2 to a review of background material that is relevant to the results of the subsequent sections. Section 2.1 will review the construction of closed-superstring tree amplitudes from open-string amplitudes by an extension of the Kawai-Lewellen-Tye (KLT) procedure [22]. The open-string N -point tree amplitudes were constructed in [4, 5] making use of the manifestly supersymmetric pure spinor formalism [2], which packages all processes related by linearised supersymmetry into a single expression. Any colour-ordered open-string tree amplitude is expressed by a simple extension of massless supersymmetric Yang-Mills (YM) amplitudes in the schematic form valid for any multiplicity N

$$\mathcal{A}_{\text{tree}} = F_{\text{tree}}(s_{ij}) A_{YM}. \quad (1.1)$$

In this expression, $\mathcal{A}_{\text{tree}}$ and A_{YM} are $(N-3)!$ component vectors of superstring- and YM tree amplitudes, respectively, referring to a basis of independent colour-ordered amplitudes [23–25]. The quantity $F_{\text{tree}}(s_{ij})$ is a $(N-3)! \times (N-3)!$ matrix that depends on external on-shell momenta k_i through the dimensionless Mandelstam invariants, $s_{ij} = \frac{\alpha'}{4}(k_i + k_j)^2$, in a manner that is determined by generalised Selberg integrals of $N-3$ colour-ordered vertex operators on the boundary of a disk world-sheet. These amplitudes and their expansions in powers of the Mandelstam invariants s_{ij} , or equivalently, their expansions in powers of α' , were studied in detail in [6, 26, 27]. The coefficients of terms in this low energy expansion exhibit a fascinating pattern of kinematic factors involving nontrivial multi-zeta-values (MZVs), which will be reviewed in section 2.2. We will also review the structure of closed-string multiparticle tree-level amplitudes that were also studied in [6] and shown to have the structure

$$\mathcal{M}_{\text{tree}} = A_{YM}^t \mathcal{S}_{\text{tree}}(s_{ij}) \tilde{A}_{YM}. \quad (1.2)$$

¹For $d = 1, \dots, 7$ compact dimensions, these are specific discrete subgroups of the global supergravity groups [8–11], $SL(2)$, $SL(2) \times SL(3)$, $SL(5)$, $Spin(5, 5)$, E_6 , E_7 , E_8 . The two distinct $d = 0$ ($D = 10$) theories are type IIA and IIB, with S-duality groups 1 and $SL(2)$, respectively.

Here, $\mathcal{S}_{\text{tree}}$ is again an $(N-3)! \times (N-3)!$ matrix acting in the space spanned by a basis of colour-ordered YM tree amplitudes A_{YM}, \tilde{A}_{YM} associated with the left- and right-movers, respectively. This matrix is a function of the Mandelstam invariants and is given by an integral over $N-3$ vertex insertion points in a spherical world-sheet. It is again of interest to study the “low energy” expansion of the amplitude, which involves an expansion of $\mathcal{S}_{\text{tree}}(s_{ij})$ in powers of Mandelstam invariants

$$\mathcal{S}_{\text{tree}}(s_{ij}) = S_0 \sum_{\{k_i\} \in (2\mathbb{N}+1)^\times} (MZV_{\{k_i\}}) \times (M_{k_1} M_{k_2} M_{k_3} \dots + \dots), \quad (1.3)$$

The factors M_{k_i} are $(N-3)! \times (N-3)!$ matrices with entries that are homogeneous polynomials in Mandelstam invariants of odd degree k_i which already enter the open-string α' corrections (1.1) along with ζ_{k_i} . The notation $(MZV_{\{k_i\}})$ indicates (linear combinations of) MZV products whose overall weight $w = \sum_i k_i$ matches the order in α' . Non-trivial MZVs (i.e., MZVs that are not polynomial in ordinary Riemann-zeta values) do not occur until order $(\alpha')^{11}$ in this expansion, whereas they first occur at order $(\alpha')^8$ for the open string. The leading low energy term S_0 (i.e., $k_i = 0 \ \forall i$) proportional to $N-3$ powers of Mandelstam invariants is known as the momentum kernel [28–30], and reproduces the supergravity tree. Higher order terms, on the other hand, are proportional to monomials in the M_{k_i} ’s and represent stringy corrections. These matrices in the α' expansion were determined explicitly at multiplicity $N=5$ in [6] and later in [31], and a systematic derivation of their form at general multiplicity N is given in [26, 27]. The precise form of the expansion in (1.3) will be given up to order $(\alpha')^{11}$ in (2.9).

The structure of the open-string one-loop multiparticle amplitudes $\mathcal{A}_{1\text{-loop}}$ was established in [1], as will be reviewed in section 2.3. The construction again made use of the pure spinor formalism and led to amplitudes of the form

$$\mathcal{A}_{1\text{-loop}} = F_{1\text{-loop}}(s_{ij}) A_{YM}, \quad (1.4)$$

where the matrix function of the invariants, $F_{1\text{-loop}}(s_{ij})$, is given in terms of integrals over vertex positions on either boundary of an annular world-sheet. These integrals, which generalise the Selberg integrals of the tree-level term, have not yet been systematically analysed in their general low energy behaviour, even for small values of N . The vector of A_{YM} still refers to the independent tree-level amplitudes in YM field theory.

A major focus of this paper is the construction of N -particle closed-string one-loop amplitudes and, in the type IIB case, how their relationship to tree amplitudes is constrained by S-duality. Since this is a non-perturbative symmetry and therefore constrains the structure of the theory over the whole of moduli space. Up to now almost all the work on closed-string loop amplitudes has been restricted to four-particle ($N=4$) scattering except for [32–34]. Even though this is a particularly special case, it has provided interesting input for analysing the constraints of S-duality, as will be reviewed in section 2.4. There we will describe the manner in which the $\text{SL}(2, \mathbb{Z})$ S-duality group acts on the low energy expansion of the four-particle closed-string scattering amplitude in the ten-dimensional type IIB theory. This requires the terms in the low energy expansion of the four graviton

amplitude to have coefficients that are modular functions of the complex scalar field, Ω . For example, the lowest order correction to the classical supergravity tree amplitude is a term of order $\alpha'^3 R^4$ (where R is the Weyl curvature) and its coefficient is a particular $\text{SL}(2, \mathbb{Z})$ Eisenstein series. The dependence on Ω of the coefficients of the higher derivative terms of order $D^4 R^4$ and $D^6 R^4$ will also be discussed.

The construction of closed-string one-loop amplitudes in the pure spinor formalism is the subject of section 3. Certain technical problems that arise with the composite b ghost are alleviated by imposing BRST invariance as a restriction on the form of the amplitude. Although these amplitudes can again be viewed, in a certain sense, as a doubling of the open-string amplitudes, they incorporate an important new feature. In contrast to tree-level N particle amplitudes and the one-loop $N = 4$ amplitude, the $N > 4$ one-loop amplitudes involve contractions between left and right moving world-sheet fields. This happens both through OPE contractions and through integration by parts and leads to new classes of terms. As a result, with present methods the explicit construction of the loop amplitudes becomes very complicated as N increases and we will limit the discussion in section 3 to the case $N = 5$.

There is a qualitative distinction between the structure of the type IIA and IIB amplitudes. We will find that the type IIB amplitude can once again be expressed in terms of a doubling of the YM tree amplitude in the form

$$N = 5 : \quad \mathcal{M}_{1\text{-loop}} = A_{YM}^t \mathcal{S}_{1\text{-loop}}(s_{ij}) \tilde{A}_{YM}. \quad (1.5)$$

This provides the first nontrivial indication that the polarisation dependence in closed-string loop amplitudes is captured by bilinears of YM trees. It seems plausible that the structure in (1.5) extends to higher numbers of loops and possibly to higher N . Furthermore, since the low-energy limit of closed-string theory reproduces maximal supergravity, these comments should also apply to loop amplitudes in (maximal) supergravity. The structure of (1.5) will be shown to apply not only to five-particle amplitudes that conserve the $\text{U}(1)$ R-symmetry charge of classical supergravity, but also those that do not. The dependence on the charge violation is encoded in the coefficient function, $\mathcal{S}_{1\text{-loop}}(s_{ij})$.

We will also see that the type IIA five-particle amplitude contains extra terms that cannot be expressed in the form (1.5), as will be discussed in section 3.6. These include parity-violating components (terms with a single ϵ tensor), such as the amplitude with a Neveu-Schwarz/Neveu-Schwarz antisymmetric potential and four gravitons, which reproduces the familiar BR^4 interaction. While the type IIA and IIB theories are distinct in $D = 10$ dimensions, they are equivalent upon toroidal compactification to dimension $D = 10 - d$ in which the Yang-Mills theory is non-chiral. Therefore, the compactified version of (1.5) applies to the scattering of massless states in either of these theories.

The expression $\mathcal{S}_{1\text{-loop}}(s_{ij})$ in (1.5) involves integrals of the vertex positions over a toroidal world-sheet of complex structure τ , which is also to be integrated with an $\text{SL}(2, \mathbb{Z})$ -invariant measure. This makes it very difficult to analyse the full amplitude, but a great deal of information about its low energy expansion can be obtained, as we will see in section 4. This is an expansion in powers of the world-sheet scalar Green function and its derivatives. At least to the order we consider in this paper the amplitude can be

separated into a non-analytic piece that contains thresholds and an analytic piece that can be expanded in powers of the Mandelstam invariants. The terms that arise at a given order in α' are world-sheet Feynman diagrams with free propagators joining the external vertex positions. To the order that we will reach in this paper most of these diagrams were evaluated in [35] in studying the low energy expansion of the one-loop four-particle amplitude. Furthermore, following section 5 of [35] it is straightforward to generalise the discussion to compactifications of the amplitude to lower dimensions on tori.

In this paper we will concentrate on the analytic part of the loop amplitude, although the interplay of the analytic and non-analytic parts is significant in determining properties of the amplitude. The analysis in sections 4 and 5 will determine the detailed structure of the expansion of $\mathcal{S}_{1\text{-loop}}$ in a form that can be compared with the tree amplitude structure given in (1.3),

$$S_{1\text{-loop}}(s_{ij}) = S_0 \sum_{\substack{\{k_i\} \in (2\mathbb{N}+1)^\times \\ \{l_i\} \in (\mathbb{N}+6)^\times}} \Xi_{\{k_i, l_i, \dots\}}^{(d)} \times (M_{k_1} M_{k_2} \dots M'_{l_1} M'_{l_2} \dots + \dots), \quad (1.6)$$

where M_{k_i} and M'_{l_i} are 2×2 matrices that depend on the Mandelstam invariants and S_0 again describes the field theory limit of (1.3). A striking feature of this set of matrices is the augmentation of tree-level matrices M_{k_i} of odd degree in s_{ij} by additional matrices M'_{l_i} of (non necessarily odd) degrees ≥ 7 . The quantities $\Xi_{\{k_i, l_i, \dots\}}^{(d)}$ in (1.6) are determined by a sum of world-sheet Feynman diagrams with $w = \sum_i (k_i + l_i) - 3$ propagators and represent the coefficients of terms such as the supersymmetric completion of $(\alpha')^{w+3} D^{2w-2} R^5$ in the low energy effective action. The diagrams that contribute to the $\Xi_{\{k_i, l_i, \dots\}}^{(d)}$'s up to order $w = 6$ in the four-point function were evaluated in [35, 36] and resulted in polynomials in Riemann-zeta values in $D = 9, 10$ space-time dimensions. A slightly extended set of diagrams enters in the calculation of the five-point function, the first novel example showing up at order $(\alpha')^8$. One of the major impediments to obtaining results at higher order in α' is the difficulty in calculating these diagrams. A very interesting issue concerning the structure of the diagrams beyond the order at which they have so far been evaluated is at what order nontrivial MZVs arise (recalling that they arise in the factor $(MZV_{\{k_i\}})$ in (1.3) for $w \geq 11$).

The results in section 4 on the low energy expansion of type IIB loop amplitudes and the parity-conserving components of the type IIA loop amplitudes are restricted to parts of the torus integrals that are analytic in the Mandelstam invariants. Hence, they concern the local part of the one-loop effective action of the type II theories. General features of these results are shown in tables 1 and 3. Table 1 indicates the dimension of the space of kinematic invariants that arise in the expansion of the tree and one-loop graviton amplitudes with $N = 4, 5$ and 6 at each order in α' . The precise form of the rank $2k+4N$ tensors that contract with the derivatives and curvature tensors in terms of the form $D^{2k} R^N$ can in principle be extracted from the amplitudes and the polynomial structure of the matrices M_k, M'_l . Novel techniques to bypass the tedious procedure and to obtain superstring effective actions without excessive diagram bookkeeping are described in [37].

We will find that the pattern of kinematic invariants that arises for $N > 4$ is more elaborate. Generically, curvature interactions at order $(\alpha')^{k+3}$ are organised in sequence

	$n = 0$	1	2	3	4	5	6	
$(\alpha')^{n+3} D^{2n} R^4$	1	0	1	1	1	1	2	tree
	1	0	<u>1</u>	1	<u>1</u>	1	2	1-loop
$(\alpha')^{n+3} D^{2n-2} R^5$	0	0	1	1	1	1	2	tree
	0	0	<u>1</u>	1	<u>2</u>	2	4	1-loop
$(\alpha')^{n+3} D^{2n-4} R^6$	0	0	1	1	1	1	2	tree
	0	0	<u>≤ 1</u>	≤ 2	<u>≤ 4</u>	??	??	1-loop

Table 1. A schematic list of the number of independent kinematic invariants that contribute at each order in the α' expansion of $N \leq 6$ graviton amplitudes. The underlined entries vanish in the $D = 10$ case, but not in lower dimensions. The detailed pattern of contraction of indices between the curvatures and derivatives is encoded in the A_{YM}, \tilde{A}_{YM} and M_{ki}, M'_{li} , where the expressions include the other process related by supersymmetry that conserve the U(1) charge. The inequalities and the ?? in the last row indicate our presently incomplete knowledge of the six-particle amplitude.

boson	q	fermion	q
(anti-)holomorphic axio-dilaton	± 2	dilatino	$\pm 3/2$
(anti-)holomorphic three-form	± 1	gravitino	$\pm 1/2$
graviton and five-form	0		

Table 2. States of the massless type IIB supermultiplet and their R symmetry charges q .

of the form $D^{2k-2l} R^{4+l}$ with $l = 0, 1, 2, \dots, k$, i.e. which start with a four curvature term. However, the five-particle one-loop amplitude generates additional $D^{2k-2} R^5$ interactions at $k \geq 4$ that do not have any four-point ancestor, $D^{2k} R^4$. At tree level, the first example of a sequence of $N \geq 5$ interactions that is not related to an $N = 4$ interaction occurs at weight eleven and has the schematic form $(\alpha')^{11} \{D^{14} R^5, D^{12} R^6, \dots\}$ (which is not shown in table 1 but see [6]). Strikingly, the coefficient of this sequence involves a triple zeta value, $\zeta_{3,3,5}$, that cannot be reduced to a monomial in Riemann zeta values. We will find that the one-loop results contain additional kinematic structures at lower orders 7, 8, 9 in the α' expansion, as indicated in table 1. There we see that there are two distinct contributions at order $(\alpha')^7 D^6 R^5$. One of these belongs to the same family of invariants $(\alpha')^7 D^{8-2l} R^{4+l}$ that arises at tree level and the second is a new contribution that does not match any tree-level result. The number of independent contributions increases further to four at order $(\alpha')^9 D^{10} R^5$ as shown in the last column of table 1. This pattern has important consequences for the implementation of $SL(2, \mathbb{Z})$ duality, as we will see in section 5.4.

The terms in table 1 that are underlined with a wavy line have coefficients that vanish in ten dimensions, but not after toroidal compactification to lower dimensions. The prototype for such terms is the $D^4 R^4$ interaction, which was shown to vanish at one loop in ten dimensions in [36] but not in nine dimensions [35]. More generally, this interaction is nonzero in any dimension less than ten.

There are many other amplitudes describing scattering of other component massless fields that are related to the graviton amplitudes by supersymmetry. Any interaction

	$n = 0$	1	2	3	4	5	6	
$(\alpha')^{n+3} D^{2n} G^2 R^3$	1	0	1	1	1	1	2	tree
	1	0	<u>1</u>	1	<u>2</u>	2	5	1-loop
$(\alpha')^{n+3} D^{2n-2} G^2 R^4$	0	0	1	1	1	1	2	tree
	0	0	<u>≤ 1</u>	≤ 2	<u>≤ 4</u>	??	??	1-loop

Table 3. The degeneracy of kinematic invariants that violate the conservation of U(1) by 2 units. This applies to all the interactions of the same dimension as $D^{2k} G^2 R^3$ and $D^{2k} G^2 R^4$. Once again, the underlined entries vanish in the $D = 10$ case, but not in lower dimensions.

that conserves the U(1) R-symmetry charge q of type IIB supergravity follows the same pattern. Table 2 reviews the q charges of the states in the type IIB supergravity multiplet. Four-particle amplitudes conserve the U(1) charge of the type IIB theory, but this may be violated in five-particle scattering. In fact, as noted in [17], the maximal U(1) charge violation, q , of an N -particle amplitude is

$$q = \pm(2N - 8), \quad (1.7)$$

so when $N = 5$ the only U(1)-violating processes are ones for which $q = \pm 2$. These U(1)-violating five-particle processes are also discussed in sections 3 and 4, leading to results for the degeneracies of the kinematic invariants displayed in table 3 using the example of derivatives acting on $G^2 R^3$ interactions. Here G is the complex combination of Neveu-Schwarz/Neveu-Schwarz (NSNS) and Ramond/Ramond (RR) three-form field strengths that carries unit U(1) charge.² As we will see, the coefficients in the low energy expansion of the the five-particle one-loop amplitude (1.6) in the $q = \pm 2$ sector are related to those in the $q = 0$ sector in a simple manner.

Some implications of these results for S-duality are described in section 2.4 and 5. In particular, we will see that the results concerning the modular functions of the modulus Ω that implement type IIB S-duality in the four-particle amplitude extend to the five-particle and higher point analogues in a simple manner. However, as mentioned above, the explicit one-loop amplitudes for N -particle amplitudes with $N > 4$ have additional pieces that enter at the order $(\alpha')^7 D^8 R^4$ that are not present in the $N = 4$ case. Therefore, there must be modular invariant coefficients for these kinematic factors that begin at one loop and have no tree-level pieces. Interestingly, this is the order at which an understanding of the modular invariance of the four-particle amplitude is extremely limited and there is still much to be understood. We will also consider the implications of S-duality for the U(1)-violating five-particle interactions listed in table 3. In these cases the coefficient functions are expected to be modular forms of nontrivial weight that are related to those of the U(1)-conserving graviton interactions. We will see that certain modular functions described in section 2.4 tie in with the α' expansion in the U(1)-violating sector displayed

²We should emphasise that these $G^2 R^3$ operators are understood to involve two alike combinations of field strengths rather than complex conjugate pairs $G\bar{G}$.

in section 5.3. Furthermore, we offer an explanation for the pattern of $q = \pm 2$ and $q = 0$ coefficients referred to in the previous paragraph.

Some general properties of the six-particle amplitude will be considered in section 6 although our analysis is not yet complete due to technical obstacles that will be described. The extent to which we have analysed the world-sheet integrals gives rise to the information listed in the last column of table 1. At present, we can only give upper bounds on the number of independent $D^{2k}R^6$ operators after classifying the lattice sums appearing in the low energy expansion of the underlying torus integrals. The main six-point result is the vanishing of the R^6 and D^4R^6 interactions in $D = 10$ dimensions (see table 1) as well as the related $U(1)$ -violating operators of the same dimension (see e.g. table 3).

The paper concludes with a brief summary and some comments in section 7.

Several technical issues are left for the appendices. The first appendix A helps to compare some of the present results with the literature. Appendix B contains a derivation of an important pentagon numerator identity using pure spinor superspace in B.1, and a proof that the low energy limit of the five-particle amplitude is totally symmetric in B.2. The diagrams that enter the low energy expansion of the five-particle amplitude are exhibited in detail in appendix C. A third appendix D summarizes the analytic parts of the five-point torus integrals to order $O(\alpha'^7)$. Finally, appendix E is devoted to the key ingredients in the momentum expansion of the six-particle amplitude.

2 Review of background material

2.1 The structure of superstring tree amplitudes

Tree level amplitudes involving any number of massless open-string states have been computed in [4, 5] based on pure spinor cohomology methods [38, 39]. The colour stripped amplitude was found to be

$$\mathcal{A}_{\text{tree}}(1, \sigma(2, 3, \dots, N-2), N-1, N) = \sum_{\pi \in S_{N-3}} A_{YM}^{\pi} F_{\text{tree}}^{\sigma, \pi}(s_{ij}) . \quad (2.1)$$

Remarkably, all the polarisation dependence of the superstring amplitude (2.1) is encoded in the (super-)YM field theory subamplitudes A_{YM}^{π} . The objects $F_{\text{tree}}^{\sigma, \pi}(s_{ij})$ originate in world-sheet integrals over the disk boundary, encode the string theory modifications to the field theory amplitude and can be mathematically classified as generalised Euler or Selberg integrals. Both ingredients on the right-hand side of (2.1) appear in their $(N-3)!$ element basis [25]

$$A_{YM}^{\pi} := A_{YM}(1, \pi(2, 3, \dots, N-2), N-1, N) \quad (2.2)$$

$$F_{\text{tree}}^{\sigma, \pi}(s_{ij}) := 4^{N-3} \int_{z_{\sigma(i)} < z_{\sigma(i+1)}} dz_2 \dots \int dz_{N-2} \prod_{i < j}^{N-1} |z_{ij}|^{4s_{ij}} \\ \times \pi \left\{ \prod_{k=2}^{\lfloor N/2 \rfloor} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \prod_{k=\lfloor N/2 \rfloor + 1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right\} , \quad (2.3)$$

and we have also restricted the string subamplitudes on the left-hand side to their $(N-3)!$ dimensional basis [23, 24]. The conformal Killing group $SL(2, \mathbb{R})$ of the disk topology has been used to fix three world-sheet positions in (2.3) to the values $(z_1, z_{N-1}, z_N) = (0, 1, \infty)$. The S_{N-3} permutation π acts on the labels $2, 3, \dots, N-2$ of the variables³

$$z_{ij} := z_i - z_j, \quad s_{ij} := \frac{\alpha'}{4} (k_i + k_j)^2 \quad (2.4)$$

in the curly bracket whereas $\pi(1, N-1) = (1, N-1)$ are unaffected. The vectors k_i denote external on-shell momenta.

Tree level correlation functions of closed-string states factorise into the left- and right moving correlators, so the open-string result (2.1) yields the following closed-string amplitude

$$\mathcal{M}_{\text{tree}} = \sum_{\sigma, \pi \in S_{N-3}} A_{YM}^\sigma \mathcal{S}_{\text{tree}}^{\sigma, \pi}(s_{ij}) \tilde{A}_{YM}^\pi. \quad (2.5)$$

The integrals over the spherical closed-string world-sheet at genus zero form an $(N-3)! \times (N-3)!$ matrix

$$\begin{aligned} \mathcal{S}_{\text{tree}}^{\sigma, \pi}(s_{ij}) := & \int d^2 z_2 \dots \int d^2 z_{N-2} \prod_{i < j}^{N-1} |z_{ij}|^{2s_{ij}} \sigma \left\{ \prod_{k=2}^{\lfloor N/2 \rfloor} \sum_{m=1}^{k-1} \frac{s_{mk}}{z_{mk}} \prod_{k=\lfloor N/2 \rfloor + 1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{z_{kn}} \right\} \\ & \times \pi \left\{ \prod_{k=2}^{\lfloor N/2 \rfloor} \sum_{m=1}^{k-1} \frac{s_{mk}}{\bar{z}_{mk}} \prod_{k=\lfloor N/2 \rfloor + 1}^{N-2} \sum_{n=k+1}^{N-1} \frac{s_{kn}}{\bar{z}_{kn}} \right\} \end{aligned} \quad (2.6)$$

which describes the string theory extension of the supergravity tree amplitude.

2.2 Low energy expansion of superstring tree amplitudes

The $(\alpha')^w$ order in the low energy expansion of massless superstring tree amplitude involves multiple zeta values (MZVs) [5, 40–43],

$$\zeta_{n_1, \dots, n_r} := \sum_{0 < k_1 < \dots < k_r} \prod_{l=1}^r k_l^{-n_l}, \quad n_l \in \mathbb{N}, \quad n_r \geq 2, \quad (2.7)$$

of weight $w = \sum_{j=1}^r n_j$. The number r of arguments in (2.7) is referred to as the depth of a MZV. The systematics of their appearance was analysed in [6] and extended to the closed-string sector through the KLT relations [22]. Let M_{2k+1} denote the ζ_{2k+1} coefficient⁴ of the matrix (2.3) of open-string α' corrections,⁵

$$M_{2k+1} := F_{\text{tree}}\left(\frac{s_{ij}}{4}\right) \Big|_{\zeta_{2k+1}}, \quad (2.8)$$

³Our definition of s_{ij} incorporates an extra factor of $\frac{1}{4}$ compared to many references such as [4–6]. This facilitates the discussion of closed strings and introduces the unusual factors of 4 into the open string statement (2.3).

⁴The coefficient of primitive zeta values ζ_{2k+1} depend on the \mathbb{Q} basis chosen for MZVs of weight $2k+1$, see e.g. [44] for a minimal depth basis at weights ≤ 22 which was also used in [6] at weights ≤ 16 .

⁵The M_{2k+1} matrices from [6, 26, 27] are written in open string conventions with $s_{ij}^{\text{open}} := \alpha'(k_i + k_j)^2$. The present closed string discussion preserves the functional dependence $M_{2k+1}(s_{ij}^{\text{open}}) \mapsto M_{2k+1}(s_{ij})$ to implement the “rescaling” of α' in the translation between open and closed string results (see in particular [22]).

its entries are degree $2k + 1$ polynomials in the dimensionless Mandelstam variables (2.4) to be determined through the methods of [26, 27]. Moreover, let S_0 denote the field theory limit of the closed-string integral matrix (2.6), then the α' -expansion of (2.6) has the structure

$$\begin{aligned} \mathcal{S}_{\text{tree}} = & S_0 \left(1 + 2\zeta_3 M_3 + 2\zeta_5 M_5 + 2\zeta_3^2 M_3^2 + 2\zeta_7 M_7 + 2\zeta_3\zeta_5 \{M_3, M_5\} + 2\zeta_9 M_9 \right. \\ & + \frac{4}{3} \zeta_3^3 M_3^3 + 2\zeta_5^2 M_5^2 + 2\zeta_3\zeta_7 \{M_3, M_7\} + 2\zeta_{11} M_{11} + \zeta_3^2\zeta_5 \{M_3, \{M_3, M_5\}\} \\ & \left. + 2 \left(\frac{1}{5} \zeta_{3,3,5} - \frac{4}{35} \zeta_3^3 \zeta_5 + \frac{6}{25} \zeta_2^2 \zeta_7 + 9\zeta_2 \zeta_9 \right) [M_3, [M_3, M_5]] + \dots \right) \end{aligned} \quad (2.9)$$

The coefficients in this expansion are products of Riemann zeta values up to weight 11, where the first irreducible MZV arises (whereas in the low energy expansion of the open-string tree (2.1) the first irreducible MZV arises at weight 8). The MZV content of the string corrections can be better understood by lifting the $\zeta_{n_1, n_2, \dots, n_r} \in \mathbb{R}$ to their motivic version $\zeta_{n_1, n_2, \dots, n_r}^{\mathfrak{m}}$ [6, 45]. The latter are endowed with a Hopf algebra structure which can be used to make the basis more transparent: References [6, 45] describe a Hopf algebra isomorphism ϕ which maps motivic MZVs to noncommutative polynomials in cogenerators f_3, f_5, f_7, \dots supplemented by a commutative element $f_2 = \phi(\zeta_2^{\mathfrak{m}})$. The latter turns out to be absent in the motivic version $\mathcal{S}_{\text{tree}}^{\mathfrak{m}}$ of $\mathcal{S}_{\text{tree}}$:

$$\begin{aligned} \mathcal{S}_{\text{tree}}^{\mathfrak{m}} = & S_0 \phi^{-1} \left(1 + 2f_3 M_3 + 2f_5 M_5 + 4f_3^2 M_3^2 + 2f_7 M_7 + 2f_3 \sqcup f_5 \{M_3, M_5\} \right. \\ & + 2f_9 M_9 + 8f_3^3 M_3^3 + 4f_5^2 M_5^2 + 2f_3 \sqcup f_7 \{M_3, M_7\} + 2f_{11} M_{11} \\ & \left. + f_3 \sqcup f_3 \sqcup f_5 \{M_3, \{M_3, M_5\}\} + 2f_5 f_3^2 [M_3, [M_3, M_5]] + \dots \right) \end{aligned} \quad (2.10)$$

$$= S_0 \phi^{-1} \left(\sum_{p=0}^{\infty} \sum_{\substack{i_1, i_2, \dots, i_p \\ \in 2\mathbb{N}+1}} M_{i_1} \dots M_{i_p} \sum_{k=0}^p f_{i_1} \dots f_{i_k} \sqcup f_{i_p} f_{i_{p-1}} \dots f_{i_{k+1}} \right) \quad (2.11)$$

The \sqcup symbol denotes the commutative shuffle product⁶ on the non-commutative words in f_{2k+1} . The simplicity of the ϕ image of $\mathcal{S}_{\text{tree}}^{\mathfrak{m}}$ can firstly be seen at weight 11 where the awkward coefficient of $[M_3, [M_3, M_5]]$ in (2.9) is mapped to $\phi(\frac{1}{5}\zeta_{3,3,5}^{\mathfrak{m}} - \frac{4}{35}(\zeta_2^{\mathfrak{m}})^3\zeta_5^{\mathfrak{m}} + \frac{6}{25}(\zeta_2^{\mathfrak{m}})^2\zeta_7^{\mathfrak{m}} + 9\zeta_2^{\mathfrak{m}}\zeta_9^{\mathfrak{m}}) = f_5 f_3^2$. The absence of the commutative element f_2 in (2.11) reflects the cancellation of ζ_{2k} at low weights of the KLT relations. The projection of the noncommutative words in f_{2k+1} to $\sum_{k=0}^p f_{i_1} f_{i_2} \dots f_{i_k} \sqcup f_{i_p} f_{i_{p-1}} \dots f_{i_{k+1}}$ imposes a selection rule⁷ on the MZVs of depth $r \geq 2$. As one can see from (2.10), depth two MZVs at weight

⁶The shuffle product on non-commutative words in f_{i_j} with $i_j \in 2\mathbb{N} + 1$ is defined by

$$\begin{aligned} f_2^p (f_{i_1} f_{i_2} \dots f_{i_r}) \sqcup f_2^q (f_{i_{r+1}} f_{i_{r+2}} \dots f_{i_{r+s}}) &= f_2^{p+q} \sum_{\sigma \in \Sigma(r,s)} f_{i_{\sigma(1)}} f_{i_{\sigma(2)}} \dots f_{i_{\sigma(r+s)}} \\ \Sigma(r,s) &= \{ \sigma \in S_{r+s} : \sigma^{-1}(1) < \sigma^{-1}(2) < \dots < \sigma^{-1}(r) \text{ and } \sigma^{-1}(r+1) < \dots < \sigma^{-1}(r+s) \} \end{aligned} \quad (2.12)$$

⁷We should stress that the form of (2.11) is independent on the choice of \mathbb{Q} basis for MZVs even though the polynomial structure of matrices M_w at $w \leq 11$ is subject to possible redefinitions by (possibly nested) commutators in $M_{i < w}$. This ambiguity in M_w is compensated by a reshuffling of f_w in the ϕ images at weight w .

$w = 8, 10$ such as $\zeta_{3,5}^m$ and $\zeta_{3,7}^m$ do not enter gravity tree amplitudes, and the first instance $\zeta_{3,3,5}^m$ of a depth $r > 1$ MZV occurs at weight $w = 11$.

2.2.1 Four- and five-particle examples of closed-string tree amplitudes

The S-duality connection between the four- and five-particle closed-string amplitudes at tree level and one loop are the main topic of this work. This motivates us to explicitly spell out the $N = 4, 5$ version of the ingredients in (2.5): Multiplicity $N = 4$ gives rise to scalars

$$N = 4 : \quad A_{YM} = A_{YM}(1, 2, 3, 4), \quad S_0 = \frac{\pi s_{12} s_{14}}{s_{13}}, \quad M_w = -\frac{1}{w}(s_{12}^w + s_{13}^w + s_{14}^w) \quad (2.13)$$

such that all the commutators among M_{2k+1} and therefore all the MZVs of depth ≥ 2 cancel.

The $N = 5$ point amplitude, on the other hand, is built from two component kinematic vectors A_{YM} and 2×2 matrices S_0, M_{2k+1}

$$N = 5 : \quad A_{YM} = \begin{pmatrix} A_{YM}(1, 2, 3, 4, 5) \\ A_{YM}(1, 3, 2, 4, 5) \end{pmatrix}, \quad S_0 = \frac{\pi^2}{s_{14} s_{25} s_{35}} \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix} \quad (2.14)$$

with entries $\sigma_{12} = -s_{12} s_{34} s_{13} s_{24} (s_{45} + s_{51})$ as well as $\sigma_{22} = -s_{13} s_{24} (s_{12} s_{23} s_{45} + \text{cyclic}(12345))$ and $\sigma_{11} = \sigma_{22}|_{2 \leftrightarrow 3}$. The simplest example of the matrices M_{2k+1} reads

$$N = 5 : \quad M_3 = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}, \quad \begin{aligned} m_{11} &= s_3[-s_1(s_1 + 2s_2 + s_3) + s_3 s_4 + s_4^2] + s_1 s_5 (s_1 + s_5) \\ m_{12} &= -s_{13} s_{24} (s_1 + s_2 + s_3 + s_4 + s_5) \end{aligned} \quad (2.15)$$

(with $m_{21} = m_{12}|_{2 \leftrightarrow 3}$ and $m_{22} = m_{11}|_{2 \leftrightarrow 3}$ as well as $s_i := s_{i,i+1}$ subject to $s_5 = s_{15}$). There are several avenues to determine their explicit form in more general cases, either by exploiting the representation of F_{tree} (2.3) in terms of (multiple Gaussian) hypergeometric functions (different techniques were used in [6, 46–48] and [31]), or by use of polylogarithm integration [26], or by use of the Drinfeld associator [27].⁸ The explicit expressions for M_3, M_5, M_7 and M_9 in the $N = 5$ case are contained in the auxiliary files to this paper and to [31], as well as on the website [52], where matrices for higher multiplicity are available.

2.3 One-loop amplitudes of the open superstring

The structure of one loop amplitudes among massless open-string states has been analysed in [1]. Their BRST invariant part unaffected by the hexagon anomaly [53, 54] was found to again boil down to linear combinations of YM tree subamplitudes

$$\mathcal{A}_{1\text{-loop}}(1, \Sigma(2, 3, \dots, N-2, N-1, N)) = \sum_{\pi \in S_{N-3}} A_{YM}^\pi F_{1\text{-loop}}^{\Sigma, \pi}(s_{ij}). \quad (2.16)$$

We introduce a collective notation Σ for both planar and non-planar arrangements of vertex operators along the boundary of the cylinder or Moebius strip world-sheet. It

⁸See [49, 50] for mathematical background to the Drinfeld associator, and [51] for its first connection with superstring amplitudes.

governs the integration range for the vertex operator positions and the modular parameter of the genus one Riemann surface in $F_{1\text{-loop}}^{\Sigma,\pi}(s_{ij})$. We will later on make use of the five-point correlator (3.9) underlying (2.16) whose double copy furnishes a subsector of one loop amplitudes of the closed string.

In contrast to the tree-level results (2.1) and (2.5) which are completely universal with respect to the number of space-time dimensions and supercharges preserved, the structure of one loop amplitudes (2.16) crucially depends on maximal supersymmetry.

2.4 Some S-duality constraints on Type IIB amplitudes

Ten-dimensional closed-string perturbation theory is an expansion around a limit in moduli space in which the type IIA string coupling, $g_A = e^{\varphi_A} \rightarrow 0$, or the type IIB string coupling, $g_B = e^{\varphi_B} \rightarrow 0$, (where $\varphi_{A,B}$ are the dilatons of the type II theories). The complete dependence of the amplitude on the moduli involves the nonperturbative completion of the perturbative amplitude. The only modulus in the type IIA theory is e^{φ_A} and the duality group is trivial. The type IIB theory depends on the complex scalar, $\Omega = \Omega_1 + i\Omega_2 = C^{(0)} + ie^{\varphi_B}$, which transforms in the standard nonlinear manner under $\text{SL}(2, \mathbb{Z})$,

$$\Omega \rightarrow \frac{a\Omega + b}{c\Omega + d}, \quad (2.17)$$

where $a, b, c, d \in \mathbb{Z}$ and $ad - bc = 1$. The fields in the IIB supergravity supermultiplet transform under $\text{SL}(2, \mathbb{Z})$ with weights that correspond to the $\text{U}(1)$ charges shown in table 2, where a field Φ_q of charge q transforms as

$$\Phi_q \rightarrow \left(\frac{c\Omega + d}{c\bar{\Omega} + d} \right)^{q/2} \Phi_q. \quad (2.18)$$

Invariance of the type IIB theory and its effective action under $\text{SL}(2, \mathbb{Z})$ transformations of the scattering states severely constrains the structure of the amplitudes. When combined with the constraints of maximal supersymmetry, $\text{SL}(2, \mathbb{Z})$ invariance determines the precise dependence on the modulus, Ω , of terms of low order in the α' expansion. In other words, it determines the precise non-perturbative behaviour of these low order terms.

2.4.1 $\text{U}(1)$ -conserving amplitudes

In the case of four-particle scattering, the $\text{U}(1)$ charge is conserved and the analytic part of the (both perturbatively and non-perturbatively completed) amplitude can be written in component form as

$$\mathcal{M}(\Omega) = s_{12}^2 s_{13} s_{14} A_{YM}(1, 2, 3, 4) \tilde{A}_{YM}(1, 2, 4, 3) T(s_{ij}; \Omega), \quad (2.19)$$

where the totally symmetric kinematic factor $s_{12}^2 s_{13} s_{14} A_{YM}(1, 2, 3, 4) \tilde{A}_{YM}(1, 2, 4, 3)$ reproduces the standard $t_8 t_8$ tensor [55] in the four graviton component. The symmetry of (2.19) in the external states implies that the low energy expansion of the analytic part of the scalar

function⁹ is a symmetric function of powers of the Mandelstam variables and has the form

$$T(s_{ij}; \Omega) \Big|_{\text{analytic}} = \frac{1}{s_{12}s_{13}s_{14}} \left(\Omega_2^2 + M_3 \Omega_2^{\frac{1}{2}} \mathcal{E}_3 + M_5 \Omega_2^{-\frac{1}{2}} \mathcal{E}_5 + M_3^2 \Omega_2^{-1} \mathcal{E}_{3,3} + M_7 \Omega_2^{-\frac{3}{2}} \mathcal{E}_7 \right. \\ \left. + \{M_3, M_5\} \Omega_2^{-2} \mathcal{E}_{\{3,5\}} + M_3^3 \Omega_2^{-\frac{5}{2}} \mathcal{E}_{3,3,3} + M_9 \Omega_2^{-\frac{5}{2}} \mathcal{E}_9 + \dots \right), \quad (2.20)$$

with four-point string corrections M_w defined by (2.13).

The coefficients $\mathcal{E}_{\dots}(\Omega)$ are $\text{SL}(2, \mathbb{Z})$ -invariant functions in the $D = 10$ type IIB theory (and the explicit powers of Ω_2 are absent after transforming from the string frame to the Einstein frame). More generally, after toroidal compactification to D dimensions on a $(10-D)$ -torus the type IIA and type IIB theories are identified and the \mathcal{E}_{\dots} 's are functions of the moduli space associated with the S-duality group. The first term in the above expansion gives classical tree-level supergravity when substituted into (2.19). The above notation differs somewhat from earlier conventions in the literature. Appendix A gathers a couple of conversion rules to compare the subsequent statements with references such as [35].

The kinematic factor $s_{12}^2 s_{13} s_{14} A_{YM}(1, 2, 3, 4) \tilde{A}_{YM}(1, 2, 4, 3)$ in the amplitude (2.19) is completely determined by supersymmetry but the challenge is to determine the dynamical quantity $T(s_{ij}; \Omega)$. Although its exact form is not known, there are some interesting results concerning the first few terms in its low energy expansion, (2.20). The first term in the expansion beyond the supergravity amplitude is given by terms of order R^4 , which are 1/2-BPS interactions that may be expressed as integrals over 16 Grassmann coordinates, see [56] for an on-shell linearized superspace description of type IIB supergravity. The next terms are those of order $D^4 R^4$, which are 1/4-BPS interactions that may be expressed as integrals over 24 superspace Grassmann coordinates. These have Ω -dependent coefficients [12–14]

$$\mathcal{E}_3(\Omega) = E_{\frac{3}{2}}(\Omega), \quad \mathcal{E}_5(\Omega) = E_{\frac{5}{2}}(\Omega), \quad (2.21)$$

where $E_s(\Omega)$ is an $\text{SL}(2, \mathbb{Z})$ Eisenstein series, which satisfies the Laplace eigenvalue equation

$$\Delta_{\Omega} E_s(\Omega) = s(s-1) E_s(\Omega) \quad (2.22)$$

with respect to the modulus, $\Delta_{\Omega} := \Omega^2 \partial_{\Omega} \partial_{\bar{\Omega}}$. The unique $\text{SL}(2, \mathbb{Z})$ -invariant solution of this equation that is power behaved in the weak coupling limit, $\Omega_2 \rightarrow \infty$, is¹⁰ the non-holomorphic Eisenstein series

$$E_s(\Omega) = \sum_{(m,n) \neq (0,0)} \frac{\Omega_2^s}{|m + n\Omega|^{2s}}. \quad (2.23)$$

with Fourier expansion

$$E_s(\Omega) = \sum_{N \neq 0} \mathcal{F}_N(\Omega_2) e^{2i\pi N \Omega_1}. \quad (2.24)$$

The non-zero modes $\mathcal{F}_{N \neq 0}(\Omega_2)$ contain the effects of D-instantons, with exponentially suppressed asymptotic behaviour at weak coupling ($\Omega_2 \rightarrow \infty$). The zero modes, on the

⁹The interplay of the analytic and non-analytic parts of the amplitude is discussed in [35].

¹⁰The normalisation convention is the one used in [35] appendix A.1.

other hand, are a sum of two power behaved terms $\Omega_2^s = g_s^{-s}$ and $\Omega_2^{1-s} = g_s^{s-1}$ which correspond to particular terms in string perturbation theory:

$$\mathcal{F}_0(\Omega_2) = 2\zeta_{2s} \Omega_2^s + \frac{2\pi^{1/2}\Gamma(s-1/2)}{\Gamma(s)} \zeta_{2s-1} \Omega_2^{1-s}, \quad (2.25)$$

In transforming from the string frame to the Einstein frame, the R^4 interaction is multiplied a factor of $\Omega_2^{-1/2}$ so the two perturbative terms in the coefficient $E_{3/2}(\Omega)$ correspond to a tree-level piece proportional to $\Omega_2^{3/2}$, and a one-loop piece proportional to $\Omega_2^{-1/2}$. Similarly the $D^4 R^4$ interaction picks up a factor of $\Omega_2^{1/2}$ in transforming from string frame to Einstein frame so the two perturbative terms in the coefficient $E_{5/2}(\Omega)$ correspond to a tree-level piece proportional to $\Omega_2^{5/2}$, and a two-loop piece proportional to $\Omega_2^{-3/2}$. These results explain why the low energy expansion of the ten-dimensional one-loop four-particle amplitude has a R^4 term but does not have a $D^4 R^4$ part. Furthermore, the coefficient of the two-loop $D^4 R^4$ interaction predicted by these arguments has been checked by explicit amplitude calculations in [57, 58]. The generalisation of these results to lower dimensional theories with maximal supersymmetry obtained by toroidal compactification [35] involve combinations of Eisenstein series for higher-rank duality groups, which are functions of more moduli [59, 60] (see also [61]). An important general feature is that in dimensions $D < 10$ the $D^4 R^4$ coefficient, \mathcal{E}_5 , does have a one-loop contribution (as is indicated by the wavy underlining of the $D^4 R^4$ one-loop term in table 1).

The coefficient of the 1/8-BPS terms of order $D^6 R^4$ in the low energy expansion, $\mathcal{E}_{3,3}$, is not an Eisenstein series but is expected to be a solution of the inhomogeneous Laplace equation¹¹

$$(\Delta_\Omega - 12) \mathcal{E}_{3,3}(\Omega) = -3[\mathcal{E}_3(\Omega)]^2 \quad (2.26)$$

(which was motivated by M-theory considerations in [15]). The solution to this equation has a constant term (the zero Fourier mode in Ω_1) of the form

$$\int_0^1 d\Omega_1 \mathcal{E}_{3,3}(\Omega) = 2\zeta_3^2 \Omega_2^3 + 4\zeta_2 \zeta_3 \Omega_2 + \frac{24}{5} \zeta_2^2 \Omega_2^{-1} + \frac{4}{9} \zeta_6 \Omega_2^{-3} + O(\exp(-4\pi\Omega_2)), \quad (2.27)$$

which contains four terms power behaved in Ω_2 that correspond to tree-level, one-loop, two-loop and three-loop string theory contributions,¹² together with an infinite sum of D-instanton/anti D-instanton contributions. The ratio of the tree-level and one-loop contributions agrees with the explicit string perturbation theory calculations (and the overall normalisation has been chosen to be consistent with a tree-level amplitude normalised to $1/s_{12}s_{13}s_{14}$).

2.4.2 U(1)-violating amplitudes

Amplitudes with $N > 4$ particles generally do not conserve the U(1) charge, with the violation of the charge bounded by $|q| \leq (2N - 8)$ (see (1.7)). The moduli-dependent coefficients

¹¹Note that the function $\mathcal{E}_{3,3}(\Omega)$ in the present notation is related to the $D^6 R^4$ coefficient $\mathcal{E}_{(0,1)}(\Omega)$ in [35] by $\mathcal{E}_{3,3}(\Omega) = 3\mathcal{E}_{(0,1)}(\Omega)$.

¹²The string frame amplitude is obtained by multiplying $\mathcal{E}_{3,3}$ by Ω_2^{-1} .

of interactions that violate U(1) must transform as modular forms of non-zero holomorphic and anti-holomorphic weights in order to ensure that the complete interaction, including its moduli dependence, is invariant under $\text{SL}(2, \mathbb{Z})$. Recall that the transformation under (2.17) of a modular form $f^{(w, w')}(\Omega)$ (with holomorphic and anti-holomorphic weights (w, w')) is given by $f^{(w, w')}(\Omega) \rightarrow (c\Omega + d)^w (c\bar{\Omega} + d)^{w'} f^{(w, w')}(\Omega)$ so a form with weights $(w, -w)$ transforms with a phase,

$$f^{(w, -w)}(\Omega) \rightarrow f^{(w, -w)} \left(\frac{a\Omega + b}{c\Omega + d} \right) = \left(\frac{c\Omega + d}{c\bar{\Omega} + d} \right)^w f^{(w, -w)}(\Omega), \quad (2.28)$$

which corresponds to a charge $q = 2w$.

A modular form of weight $(w + 1, -w - 1)$, or $q = 2w + 2$, can be obtained from one that transforms with weight $(w, -w)$ by applying a covariant derivative

$$\mathcal{D}f^{(w, -w)} = \Omega_2 \left(i \frac{\partial}{\partial \Omega} + \frac{w}{2} \right) f^{(w, -w)} =: f^{(w+1, -w-1)}, \quad (2.29)$$

while the charge is lowered by the operator $\bar{\mathcal{D}}$ defined by $\bar{\mathcal{D}}f^{(w, -w)} = \Omega_2 (-i\partial/\partial\bar{\Omega} + w/2) f^{(w-1, -w+1)}$.

The coefficients of the leading interactions in the low energy expansion of amplitudes that violate U(1) charge conservation were related by nonlinear supersymmetry to $(\alpha')^3 \mathcal{E}_3(\Omega) R^4$ in [13] and have the form arrived at by arguments based on M-theory duality in [16]. An example is the $q = -2$ five-particle amplitude with two G 's and three gravitons. The lowest-order interaction in the low energy expansion is $(\alpha')^3 G^2 R^3$, which has a coefficient $\mathcal{E}_3^{(-1, 1)}(\Omega)$ that is a $(1, -1)$ -form given by

$$\mathcal{E}_3^{(1, -1)}(\Omega) := \mathcal{D}\mathcal{E}_3(\Omega) = \sum_{(m, n) \neq (0, 0)} \frac{\Omega_2^s}{|m + n\Omega|^{2s}} \left(\frac{m + n\bar{\Omega}}{m + n\Omega} \right). \quad (2.30)$$

The zero mode of this expression is easily obtained by acting with \mathcal{D} on the \mathcal{E}_3 zero mode in (2.25), giving

$$\mathcal{D}\mathcal{F}_0(\Omega_2) = s \zeta_{2s} \Omega_2^s - \frac{\pi^{1/2} \Gamma(s - 1/2)}{\Gamma(s - 1)} \zeta_{2s-1} \Omega_2^{1-s}, \quad (2.31)$$

which again has two perturbative contributions that have different coefficients from those of the R^4 interaction. The explicit tree-level and one-loop calculations in later sections will provide further information concerning U(1)-violating processes. Some aspects of the modular forms associated with higher dimension U(1)-conserving and U(1)-violating interactions have been considered in [18–20].

3 Type II one-loop amplitudes in pure spinor superspace

The prescription for computing N -point superstring amplitudes at one-loop using the minimal pure spinor formalism is given by [3]

$$\mathcal{M}_{1\text{-loop}} \sim \int d^2\tau \langle |(\mu, b) \prod_{P=2}^{10} Z_{B_P} Z_J \prod_{I=1}^{11} Y_{C_I}|^2 V_1 \tilde{V}_1(0) \prod_{j=2}^N \int d^2 z_j U^j \tilde{U}^j(z_j, \bar{z}_j) \rangle. \quad (3.1)$$

The massless closed-string states are represented by double copies of the pure spinor vertex operators V and U for the ten-dimensional $\mathcal{N} = 1$ SYM gauge multiplet [3]. Moreover, μ is the Beltrami differential, τ is the Teichmüller parameter of the genus one Riemann surface, and the angle brackets $\langle \dots \rangle$ denote the path integral discussed in detail in [3]. Finally, Z_{B_P} , Z_J , Y_{C_I} are picture-changing operators and b is the b-ghost whose schematic form is given by [3, 62]

$$\begin{aligned} b = & (\Pi d + N \partial \theta + J \partial \theta) d \delta(N) + (w \partial \lambda + J \partial N + N \partial J + N \partial N) \delta(N) \\ & + (N \Pi + J \Pi + \partial \Pi + d^2)(\Pi \delta(N) + d^2 \delta'(N)) \\ & + (N d + J d)(\partial \theta \delta(N) + d \Pi \delta'(N) + d^3 \delta''(N)) \\ & + (N^2 + J N + J^2)(d \partial \theta \delta'(N) + \Pi^2 \delta'(N) + \Pi d^2 \delta''(N) + d^4 \delta'''(N)) \end{aligned} \quad (3.2)$$

where $\delta'(x) = \frac{\partial}{\partial x} \delta(x)$ and the variables on the right-hand side are conformal fields of the world-sheet theory of the pure spinor formalism [3]. In particular, λ^α is a bosonic ghost subject to the pure spinor constraint $\lambda^\alpha \gamma_{\alpha\beta}^m \lambda^\beta = 0$ when contracting the 16×16 Pauli matrices $\gamma_{\alpha\beta}^m$ of $\text{SO}(1, 9)$.

Using the above prescription to compute amplitudes involving more than four external strings can be rather challenging, mostly due to the complicated nature of the b-ghost and the picture-changing operators. Fortunately there are some shortcuts which can be taken to simplify this task.

As explained in the four-point computation of [3], the systematics of how the d_α zero-modes — 16 Weyl spinor components — are saturated can be exploited to bypass many complicated features of the b-ghost. Only a few terms of (3.2) actually give non-vanishing contributions and Lorentz invariance uniquely fixes the result of their path integral up to an overall constant. For higher-point *open* superstring amplitudes similar arguments were used in [1] to perform the integration over the d_α zero modes while BRST invariance of the resulting pure spinor superspace expressions fixed their relative coefficients.

The four-point amplitude of [3] did not involve any OPEs among the vertices (except through the standard Koba-Nielsen factor) since otherwise the d_α zero modes would not be saturated. But starting at $N = 5$ points there are non-vanishing contributions featuring at most $N - 4$ OPE contractions such as

$$d_\alpha(z_i) \theta^\beta(z_j) \rightarrow \frac{\delta_\alpha^\beta}{z_{ij}}, \quad \Pi_m(z_i) x^n(z_j, \bar{z}_j) \rightarrow -\frac{\delta_m^n}{z_{ij}}, \quad N^{mn}(z_i) \lambda^\alpha(z_j) \rightarrow \frac{(\gamma^{mn})^\alpha{}_\beta \lambda^\beta}{2 z_{ij}}. \quad (3.3)$$

The singularities of the genus one correlator caused by primary fields d_α , Π_m and N^{mn} of conformal weight one enter through the torus Green function¹³

$$X_{ij} := s_{ij} \frac{\partial \ln \chi_{ij}}{\partial z_i}, \quad \ln \chi_{ij} := \ln \left| \frac{\theta_1(z_{ij})}{\theta_1'(0)} \right|^2 - \frac{2\pi}{\tau_2} [\text{Im}(z_{ij})]^2 + C(\tau, \bar{\tau}). \quad (3.4)$$

Its zero modes $C(\tau, \bar{\tau}) := 2 \ln |\sqrt{2\pi} \eta(\tau)|^2$ involving the Dedekind eta function $\eta(\tau)$ drop out of the scattering amplitude and have thus been subtracted in (3.4) for later convenience [35, 36]. The definition of X_{ij} including the Mandelstam variable (2.4) is motivated by integration by parts (see section 3.2 and [1]).

¹³Note that $\langle x(z_i, \bar{z}_i) x(z_j, \bar{z}_j) \rangle = -\frac{\alpha'}{2} \left(\ln \left| \frac{\theta_1(z_{ij})}{\theta_1'(0)} \right|^2 - \frac{2\pi}{\tau_2} [\text{Im}(z_{ij})]^2 \right)$.

The novelty appearing in higher-point computations of *closed* string amplitudes stems from terms involving the contraction between left- and right-movers — either directly using the OPE (involving the volume $\tau_2 := \text{Im}(\tau)$ of the torus)

$$\Pi^m(z)\bar{\Pi}_n(\bar{z}) = \delta_n^m \pi \left(\frac{1}{\tau_2} - \delta^2(z, \bar{z}) \right). \quad (3.5)$$

or indirectly through integration by parts, as will be explained below. In the following, we will refer to the right-hand side of the OPE (3.5) as

$$\Omega := \pi \left(\delta^2(z, \bar{z}) - \frac{1}{\tau_2} \right) = \partial\bar{\partial} \ln \chi \quad (3.6)$$

and drop any reference to the argument z because the $\delta^2(z, \bar{z})$ does not contribute in presence of the Koba-Nielsen factor (3.8).¹⁴

3.1 The structure of the five-point closed-string correlator

The computation of the closed-string five-point correlator can be separated into three parts:

- a) purely holomorphic-square¹⁵ terms multiplying world-sheet functions $X_{ij}\tilde{X}_{kl}$;
- b) holomorphic-square terms proportional to Ω generated from integration by parts;
- c) left/right mixing terms arising from the OPE (3.5).

These contributions will be labelled $K^{(a)}$, $K^{(b)}$ and $K^{(c)}$, respectively. BRST invariance will be used to obtain their relative weights in the final answer,

$$\mathcal{M}_{1\text{-loop}} = 2 \int \frac{d^2\tau}{\tau_2^5} \int d^2z_2 \dots d^2z_5 \mathcal{I}(s_{ij}) (K^{(a)} + K^{(b)} + K^{(c)}). \quad (3.7)$$

The overall normalisation -2 can be obtained by unitarity. In (3.7) we introduced the following shorthand for the Koba-Nielsen factor,

$$\mathcal{I}(s_{ij}) := \left\langle \prod_{i=1}^N e^{ik_i \cdot x(z_i, \bar{z}_i)} \right\rangle = \prod_{i < j}^N e^{-k_i \cdot k_j \langle x(z_i, \bar{z}_i) x(z_j, \bar{z}_j) \rangle} = \prod_{i < j}^N \chi_{ij}^{s_{ij}}, \quad (3.8)$$

with χ_{ij} given by the exponential of (3.4). The contributions to $K^{(a)}$ and $K^{(b)}$ can be obtained from the holomorphic square of the open-string results of [1], where it was shown that the left-moving CFT correlator can be written as¹⁶

$$\left(X_{12} \frac{T_3^i T_4^j T_5^k}{s_{12}} + (2 \leftrightarrow 3, 4, 5) \right) + \left(X_{23} \frac{T_1^i T_{23}^j T_4^k T_5^k}{s_{23}} + (23 \leftrightarrow 24, 25, 34, 35, 45) \right) \quad (3.9)$$

¹⁴This can be seen upon analytic continuation to the kinematic region where $s_{ij} > 0$ [32].

¹⁵Throughout this work, the term “holomorphic” square refers to products of expressions from the left- and right moving sector of the closed string, with z_i and τ dependencies which are complex conjugate to each other.

¹⁶The (i, j, k) notation is explained in [1] and should not be confused with vector indices.

in terms of X_{ij} defined by (3.4). The objects $T_1 := V_1$ and T_{12} denote BRST building blocks which were introduced in [38, 39] to compactly represent tree-level kinematic factors in pure spinor superspace. Their relatives T_3^i and T_{23}^i are specific to one-loop open superstring kinematics [1]. Both $T_{12\dots p}$ and $T_{23\dots p}^i$ can be thought of as the single pole residue of iterated OPEs among the pure spinor vertex operators V and U . The efficiency of these superfields in streamlining amplitude computations stems from their covariant BRST variations [38, 39].

The holomorphic square of (3.9) generates 100 integrals but, as we shall see in the next section, only 37 are independent under integration by parts. In the open-string amplitude these manipulations lead to BRST-closed kinematic factors¹⁷

$$C_{1,23} := \frac{T_{12} T_3^i T_4^j T_5^k}{s_{12}} + \frac{T_{31} T_2^i T_4^j T_5^k}{s_{13}} + \frac{T_1 T_{23}^i T_4^j T_5^k}{s_{23}} \quad (3.10)$$

for the left-moving superfields. In the closed-string case, it will be shown in the next subsection that reducing the integrals to a basis leads to a manifestly BRST-closed piece $K^{(a)}$ composed from the holomorphic square of (3.10) together with a correction $K^{(b)}$ proportional to the function Ω of (3.6) which is not by itself BRST-closed.

3.2 Integration by parts and $K^{(b)}$

In view of the definition (3.8) of the Koba-Nielsen factor, the vanishing of total world-sheet derivatives under the z_i integrals leads to identities among the X_{ij} defined in (3.4) in open-string amplitudes. In the $N = 5$ point context, for example [1],

$$0 = \int dz_2 \partial_2 \mathcal{I}(s_{ij}) = \int dz_2 (X_{21} + X_{23} + X_{24} + X_{25}) \mathcal{I}(s_{ij}). \quad (3.11)$$

For closed-string amplitudes, however, the presence of a right-moving \tilde{X}_{ij} interferes with integration by parts performed on the left-moving variables because

$$\tilde{\partial}_i X_{ij} = s_{ij} \Omega, \quad (3.12)$$

which follows from (3.4) and (3.6). As an example, the following five-point identity

$$0 = \int d^2 z_2 \partial_2 (\mathcal{I}(s_{ij}) \tilde{X}_{2j}) = \int d^2 z_2 ((X_{21} + X_{23} + X_{24} + X_{25}) \tilde{X}_{2j} + s_{2j} \Omega) \mathcal{I}(s_{ij}), \quad (3.13)$$

generates a fake left/right-mixing term proportional to $s_{2j} \Omega$. Whenever there's no chance of confusion the Koba-Nielsen factor and the integration sign will be omitted from now on, so the relations (3.11) and (3.13) will be denoted

$$X_{12} = \sum_{k=3}^5 X_{2k}, \quad X_{12} \tilde{X}_{2j} = \sum_{k=3}^5 X_{2k} \tilde{X}_{2j} + s_{2j} \Omega. \quad (3.14)$$

¹⁷The precise form of the BRST building blocks T_{12}, T_1, T_3^i and T_{23}^i is not needed for the purpose of this work once the BRST invariant (3.10) is expressed in terms of YM trees via (3.34).

In order to express the holomorphic square of the open-string CFT correlator (3.9) in terms of a minimal set of integrals, it will be sufficient to consider four prototype integration by parts identities from which all others follow through relabelling,

$$\begin{aligned}
X_{12} \tilde{X}_{23} &= (X_{23} + X_{24} + X_{25}) \tilde{X}_{23} + s_{23} \Omega \\
X_{12} \tilde{X}_{13} &= (X_{23} + X_{24} + X_{25}) (\tilde{X}_{32} + \tilde{X}_{34} + \tilde{X}_{35}) - s_{23} \Omega \\
X_{12} \tilde{X}_{12} &= (X_{23} + X_{24} + X_{25}) (\tilde{X}_{23} + \tilde{X}_{24} + \tilde{X}_{25}) - 2s_{12} \Omega \\
X_{12} \tilde{X}_{34} &= (X_{23} + X_{24} + X_{25}) \tilde{X}_{34} .
\end{aligned} \tag{3.15}$$

After some algebra one finds that the square of (3.9) yields

$$K^{(a)} = \sum_{\substack{2 \leq i < j \\ 2 \leq k < l}}^5 \left(X_{ij} \tilde{X}_{kl} + \delta_{ik} \delta_{jl} s_{ij} \Omega \right) C_{1,ij} \tilde{C}_{1,kl} \tag{3.16}$$

$$\begin{aligned}
K^{(b)} &= -\Omega \left(\frac{T_{12} T_3^i T_4^j T_5^k \tilde{T}_{12} \tilde{T}_3^i \tilde{T}_4^j \tilde{T}_5^k}{s_{12}} + (2 \leftrightarrow 3, 4, 5) \right. \\
&\quad \left. + \frac{T_1 T_{23}^i T_4^j T_5^k \tilde{T}_1 \tilde{T}_{23}^i \tilde{T}_4^j \tilde{T}_5^k}{s_{23}} + (23 \leftrightarrow 24, 25, 34, 35, 45) \right) ,
\end{aligned} \tag{3.17}$$

i.e. the contributions to a) and b) comprise 37 integrals and vary as follows under the pure spinor BRST operator Q :

$$QK^{(a)} = 0 , \tag{3.18}$$

$$\begin{aligned}
QK^{(b)} &= -\Omega V_1 V_2 T_3^i T_4^j T_5^k [\tilde{T}_{12} \tilde{T}_3^i \tilde{T}_4^j \tilde{T}_5^k - \tilde{V}_1 \tilde{T}_{23}^i \tilde{T}_4^j \tilde{T}_5^k - \tilde{V}_1 \tilde{T}_{24}^i \tilde{T}_3^j \tilde{T}_5^k - \tilde{V}_1 \tilde{T}_{25}^i \tilde{T}_3^j \tilde{T}_4^k] \\
&\quad + (2 \leftrightarrow 3, 4, 5) .
\end{aligned} \tag{3.19}$$

Using the explicit superfield representation of the BRST blocks of [1] it is not difficult to check that both $s_{pq} C_{1,ij} \tilde{C}_{1,kl}$ and $|T_{12} T_3^i T_4^j T_5^k|^2 / s_{12}$ have the same dimension as $k^{10} A_{YM} \tilde{A}_{YM}$, i.e. their five-graviton components have dimensions of R^5 / k^2 .

As will become clear later when considering the α' expansions of the integrals, it is natural to include the term $s_{ij} \Omega$ together with the diagonal contributions $X_{ij} \tilde{X}_{ij}$ when writing (3.16) since the resulting integral over $(X_{ij} \tilde{X}_{ij} + s_{ij} \Omega)$ has no leading low energy contribution or kinematic poles. Therefore only $K^{(b)}$ can contribute to the $\mathcal{O}(\alpha'^3)$ factorisation channels and once it is combined with $K^{(c)}$ to form a BRST-closed quantity, the resulting kinematic factor agrees with the holomorphic square of SYM tree amplitudes in the precise combination dictated by the KLT formula at order $(\alpha')^3$.

3.3 Interactions between left and right-movers and $K^{(c)}$

The contributions discussed in the last subsections are those in which the pure spinor variables in the holomorphic and anti-holomorphic sectors are treated separately, as in the open-string case. However, there are other ways to saturate the sixteen d_α zero-modes in both the left- and the right moving sectors of the pure spinor formalism, which also involve an OPE contraction between the two sectors.

When both the left- and right-moving b-ghosts contribute through terms of the form $d^4\delta'(N)$ one can also saturate all d zero modes if a left-moving $\Pi^m(z_i)$ contracts with a right-moving $\bar{\Pi}^n(\bar{z}_j)$ in the external vertices. Similar arguments as in the open-string calculations of [1] can be used to integrate the d_α zero modes, giving a left-moving contribution of

$$V_1 A_2^m T_3^i T_4^j T_5^k + (2 \leftrightarrow 3, 4, 5), \quad (3.20)$$

with a similar expression for the right-movers. The vector indices m of the A^m superfields of ten dimensional $\mathcal{N} = 1$ SYM are contracted between the two sides. Both A^m and its spinorial field strength W^α (see (3.21) and subsequent equations) enters the (left-moving) integrated vertex operator U .

The b-ghost (3.2) admits other possibilities involving left/right mixing. For example, there can be a contraction between a $\bar{\Pi}^m \bar{d}^2$ from the right-moving b-ghost with a $\Pi^n A_n$ from a left-moving vertex or vice-versa. This can be achieved with a left-moving b contribution of $d^4\delta'(N)$ together with a right-moving \tilde{b} proportional to $\bar{\Pi}^m \bar{d}^2$. The left-moving d_α zero-mode integration gives rise to the same contribution $V_1 A_2^m T_3^i T_4^j T_5^k + (2 \leftrightarrow 3, 4, 5)$ as discussed above, but now four \bar{d}_α are required from the right-moving vertices. Their contribution is given by $\tilde{V}_1 \tilde{W}_{2,3,4,5}^m + (2 \leftrightarrow 3, 4, 5)$, where

$$\begin{aligned} W_{2,3,4,5}^m := \frac{1}{12} [& (\lambda\gamma^n W^2)(\lambda\gamma^p W^3)(W^4\gamma^{mnp}W^5) + (\lambda\gamma^n W^2)(\lambda\gamma^p W^4)(W^3\gamma^{mnp}W^5) \\ & + (\lambda\gamma^n W^2)(\lambda\gamma^p W^5)(W^3\gamma^{mnp}W^4) + (\lambda\gamma^n W^3)(\lambda\gamma^p W^4)(W^2\gamma^{mnp}W^5) \\ & + (\lambda\gamma^n W^3)(\lambda\gamma^p W^5)(W^2\gamma^{mnp}W^4) + (\lambda\gamma^n W^4)(\lambda\gamma^p W^5)(W^2\gamma^{mnp}W^3)]. \end{aligned} \quad (3.21)$$

To see this note that group theory considerations imply that only one vector can be constructed using two pure spinors λ^α and four W^β , therefore (3.21) is the unique such vector which is symmetric in the labels 2, 3, 4 and 5. The normalisation was chosen for later convenience.

Another kind of interaction between left- and right-movers appears when there is a $\Pi^m \bar{\Pi}^n$ contraction between the b-ghosts themselves through the schematic term $(\Pi^m d^2)(\bar{\Pi}^n \bar{d}^2)$. In this case both sides of external vertices contribute with factors of $V_1 W_{2,3,4,5}^m$.

3.4 BRST invariance

After obtaining the different CFT contributions one needs to assemble the parts in order to obtain a BRST-invariant result in pure spinor superspace. Our claim is that the correct amplitude is obtained up to an overall coefficient after this step is completed.

The purely holomorphic square part $K^{(a)}$ in (3.16) is written in terms of $C_{1,ij}$ and therefore manifestly BRST-invariant. The relative coefficient between $K^{(b)}$ in (3.17) and $K^{(c)}$ from section 3.3 is uniquely fixed by demanding that they combine to form a BRST invariant quantity. Since $K^{(b)}$ does not involve contractions between the left- and right-moving fields the only way that the purely left/right mixing terms from $K^{(c)}$ can cancel the BRST variation of $K^{(b)}$ is if $QK^{(c)}$ is also holomorphically factorised.

Noting that $QA^m = k^m V + (\lambda \gamma^m W)$ as well as $QT_2^i T_3^j T_4^k = 0$ and

$$QW_{2,3,4,5}^m = -(\lambda \gamma^m W^2) T_3^i T_4^j T_5^k - (2 \leftrightarrow 3, 4, 5), \quad (3.22)$$

one sees that the combination

$$T_{2,3,4,5}^m := A_2^m T_3^i T_4^j T_5^k + A_3^m T_2^i T_4^j T_5^k + A_4^m T_2^i T_3^j T_5^k + A_5^m T_2^i T_3^j T_4^k + W_{2,3,4,5}^m \quad (3.23)$$

has the property that its BRST variation contains the vector index m only in momenta k^m ,

$$QT_{2,3,4,5}^m = k_2^m V_2 T_3^i T_4^j T_5^k + (2 \leftrightarrow 3, 4, 5). \quad (3.24)$$

This observation justifies the normalisation of (3.21) and suggests that the left/right mixing terms in $K^{(c)}$ are given by

$$K^{(c)} = -\Omega V_1 T_{2,3,4,5}^m \tilde{V}_1 \tilde{T}_{2,3,4,5}^m, \quad (3.25)$$

because

$$QK^{(c)} = \Omega V_1 V_2 T_3^i T_4^j T_5^k k_m^2 \tilde{V}_1 \tilde{T}_{2,3,4,5}^m + (2 \leftrightarrow 3, 4, 5) \quad (3.26)$$

is holomorphically factorised and therefore has a chance of cancelling the BRST variation of $K^{(b)}$. Indeed one can show that

$$K^{(b)} + K^{(c)} = -\Omega C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m \quad (3.27)$$

where¹⁸

$$\begin{aligned} C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m := & V_1 T_{2,3,4,5}^m \tilde{V}_1 \tilde{T}_{2,3,4,5}^m + \left(\frac{T_{12} T_3^i T_4^j T_5^k \tilde{T}_{12} \tilde{T}_3^i \tilde{T}_4^j \tilde{T}_5^k}{s_{12}} + (2 \leftrightarrow 3, 4, 5) \right) \\ & + \left(\frac{T_1 T_{23}^i T_4^j T_5^k \tilde{T}_1 \tilde{T}_{23}^i \tilde{T}_4^j \tilde{T}_5^k}{s_{23}} + (23 \leftrightarrow 24, 25, 34, 35, 45) \right). \end{aligned} \quad (3.28)$$

is BRST-closed and therefore fixes the relative normalisation among the different CFT contributions to the closed-string amplitude in (3.7).

To verify that (3.28) is BRST closed one uses (3.19) and (3.26) to obtain

$$QC_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m = V_1 V_2 T_3^i T_4^j T_5^k \tilde{J}_{1|2|345} + (2 \leftrightarrow 3, 4, 5), \quad (3.29)$$

where the right-moving kinematic factor

$$\tilde{J}_{1|2|345} := \tilde{T}_{21} \tilde{T}_3^i \tilde{T}_4^j \tilde{T}_5^k + \tilde{V}_1 \tilde{T}_{23}^i \tilde{T}_4^j \tilde{T}_5^k + \tilde{V}_1 \tilde{T}_{24}^i \tilde{T}_3^j \tilde{T}_5^k + \tilde{V}_1 \tilde{T}_{25}^i \tilde{T}_3^j \tilde{T}_4^k + \tilde{V}_1 k_2^m \tilde{T}_{2,3,4,5}^m \quad (3.30)$$

is BRST-closed and satisfies [63]

$$\langle \tilde{J}_{1|2|345} \rangle = 0. \quad (3.31)$$

The pure spinor bracket $\langle \dots \rangle$ here denotes the integration over the zero-modes of λ^α and θ^α as all non-zero-modes have been integrated out already (via operator product expansions).

¹⁸The m superscripts in $C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m$ are meant to be symbolic reminders that some of the terms therein involve vector index contractions between left- and right-movers. One cannot view $C_{1,2,3,4,5}^m$ as a separate vector of its own right.

The prescription for zero mode integration has the schematic form $\langle(\lambda^3\theta^5)\rangle = 1$ and is reviewed in [4]. Therefore the vanishing of the left-moving BRST variation of (3.28) follows from the vanishing of the pure spinor zero-mode integration of the superfield combination $\tilde{J}_{1|2|345}$ that builds up on the right-moving sector.

Note that the vanishing of $\langle J_{1|2|345} + (2 \leftrightarrow 3, 4, 5) \rangle$ yields

$$\langle T_{12}T_3^iT_4^iT_5^i + T_{13}T_2^iT_4^iT_5^i + T_{14}T_2^iT_3^iT_5^i + T_{15}T_2^iT_3^iT_4^i + k_m^1 V_1 T_{2,3,4,5}^m \rangle = 0. \quad (3.32)$$

The left-hand side will be shown in appendix B.1 to be BRST-exact.

It is interesting to observe that — up to the left/right mixing terms — (3.30), (3.31) and (3.32) can be viewed as kinematic relations dual to Jacobi identities among certain colour factors¹⁹ (see the BCJ-like identity (7.33) of [1]). The fifth term $\sim k_m^i V_1 T_{2,3,4,5}^m$ of (3.30) and (3.32) ties in with the field theoretic identity (3.9) of [64] connecting numerators of box diagrams with the vectorial part of a pentagon numerator contracting a loop momentum.²⁰ The derivation in [64] rests on demanding kinematic Jacobi identities dual to Lie algebraic colour relations between cubic diagrams [25]. Hence, we are lead to interpret the superspace expression $\langle T_{\dots}T_{\dots}^iT_{\dots}^jT_{\dots}^k \rangle$ as a box numerator (with a massive corner represented by the rank two building block T_{pq} or T_{pq}^i). Likewise, $\langle V_1 T_{2,3,4,5}^m \rangle$ qualifies to represent the loop momentum dependent part of a pentagon numerator. Given the vanishing of (3.32) in the BRST cohomology, we can regard it as a pure spinor superspace derivation of certain kinematic Jacobi relations at loop level.

3.5 The five closed-string amplitude in terms of SYM trees

By assembling the results from the previous subsections, we conclude that the supersymmetric closed-string five-particle amplitude is given by

$$\begin{aligned} \mathcal{M}_{1\text{-loop}} &= 2 \int \frac{d^2\tau}{\tau_2^5} \int d^2z_2 \dots d^2z_5 \mathcal{I}(s_{ij}) \\ &\times \left\{ \sum_{\substack{2 \leq i < j \\ 2 \leq k < l}}^5 (X_{ij}\tilde{X}_{kl} + \delta_{ik}\delta_{jl} s_{ij}\Omega) \langle C_{1,ij}\tilde{C}_{1,kl} \rangle - \Omega \langle C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m \rangle \right\}. \end{aligned} \quad (3.33)$$

The superspace expressions $\langle s_{pq}C_{1,ij}\tilde{C}_{1,kl} \rangle$ and $\langle C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m \rangle$ have the same dimensions as $k^{10}A_{YM}\tilde{A}_{YM}$, which translates into dimensions of R^5/k^2 for their five-graviton components. Since z_i integration over the $(X_{ij}\tilde{X}_{kl} + \delta_{ik}\delta_{jl} s_{ij}\Omega)$ will be shown to have no leading contribution in α' , the leading low energy behaviour of (3.33) is given entirely by

¹⁹Pure spinor superspace expressions of the form $\langle T_{\dots}T_{\dots}^iT_{\dots}^jT_{\dots}^k \rangle$ were argued in [1] to be in correspondence with colour tensors built from structure constants f^{abc} contracted with one symmetrised four-trace $d^{a_1a_2a_3a_4} := \frac{1}{6} \sum_{\sigma \in S_3} \text{Tr}\{t^{a_1}t^{a_{\sigma(2)}}t^{a_{\sigma(3)}}t^{a_{\sigma(4)}}\}$. The trace is taken in the fundamental representation of the gauge group generators t^a . These tensors obey four term Jacobi relations such as $d^{abce}f^{edg} + d^{abde}f^{ecg} + d^{acde}f^{ebg} + d^{bcde}f^{eag} = 0$ which formally resembles the first four terms in (3.32) by identifying $\langle T_{12}T_3^iT_4^iT_5^i \rangle \leftrightarrow f^{a_1a_2b}d^{ba_3a_4a_5}$.

²⁰Note that a representation of the one-loop five-particle amplitude in $\mathcal{N} = 4$ SYM theory was found in [65] which satisfies all kinematic Jacobi identities and where the loop momentum dependence in pentagon numerators vanishes.

$\langle C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m \rangle$. In appendix B.2 we will give a superspace proof that this low-energy limit is totally symmetric in all the labels even though its definition (3.28) superficially treats the first external leg on a different footing.

So far, we have expressed the world-sheet integrand for the five-particle closed-string amplitude in terms of BRST invariant kinematic factors $C_{1,ij}$ and $C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m$ in pure spinor superspace. We shall now translate these superfields into tree-level YM field-theory amplitudes [39].

The open-string BRST invariants $C_{1,ij}$ were found in [1] to match with (permutations of) the $(\alpha')^2 \zeta_2$ corrections to disk amplitudes. According to (2.1), those in turn furnish linear combinations of the two independent YM tree amplitudes $A_{YM}(1, 2, 3, 4, 5)$ and $A_{YM}(1, 3, 2, 4, 5)$,

$$\langle C_{1,23} \rangle = s_{45} (s_{24} A_{YM}(1, 3, 2, 4, 5) - s_{34} A_{YM}(1, 2, 3, 4, 5)) \quad (3.34)$$

with bilinear coefficients in Mandelstam variables [4, 5]. This guarantees that the $\langle C_{1,ij} \tilde{C}_{1,kl} \rangle$ terms in (3.33) arising from $K^{(a)}$ can be expressed in terms of the four independent kinematic factors formed by bilinears $\{A_{YM}(1, 2, 3, 4, 5), A_{YM}(1, 3, 2, 4, 5)\} \times \{\tilde{A}_{YM}(1, 2, 3, 4, 5), \tilde{A}_{YM}(1, 3, 2, 4, 5)\}$.

The additional BRST invariant, $C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m$ in (3.28), involves contractions between vectorial superfields from the left- and right-movers. Hence, the results from the open-string sector is not enough to determine the relation with A_{YM} bilinears. Instead, we shall evaluate individual supercomponents of the last term in (3.33) using the methods of [63] and compare the result with the component representation of a A_{YM} ansatz.

It turns out that the any type IIB component expansion encoded in the pure spinor superspace expression (3.28) can be matched with a bilinear combination of A_{YM} tree amplitudes. The expansion coefficients, however, depend on the overall U(1) charge q of the components of the maximal supergravity multiplet being scattered. Therefore one cannot hope to find a single superspace expression like (3.34) which contains all supercomponents at the same time. In fact, the component evaluation based on [63] shows that (3.28) can be expressed in terms of the tree amplitude (2.5) in the form

$$\langle C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m \rangle = \begin{cases} +1 A_{YM}^t S_0 M_3 \tilde{A}_{YM} : U(1) \text{ conserved, } q = 0. \\ -\frac{1}{3} A_{YM}^t S_0 M_3 \tilde{A}_{YM} : U(1) \text{ violated, } q = \pm 2. \end{cases} \quad (3.35)$$

The specific 2×2 matrices, S_0 and M_3 , for the five-particle amplitude are given in (2.14) and (2.15). The five-graviton and four-graviton-one-dilaton component amplitudes were used to probe the U(1)-conserving and U(1)-violating sectors, respectively, including the relative prefactor $-1/3$ in (3.35). As we will see in section 5.3, the latter agrees with expectations based on type IIB S-duality introduced in section 2.4.2.

After substituting (3.34) and (3.35) into (3.33) we see that the U(1)-conserving and U(1)-violating components of the one-loop five-particle amplitude both have the structure

$$\mathcal{M}_{1\text{-loop}}^q = A_{YM}^t \mathcal{S}_{1\text{-loop}}^q(s_{ij}) \tilde{A}_{YM} \quad (3.36)$$

with the two-component vectors, A_{YM} and \tilde{A}_{YM} , defined by (2.14) encoding all the polarisation dependence. The 2×2 matrix $\mathcal{S}_{1\text{-loop}}^q(s_{ij})$, on the other hand, only depends on

the dimensionless Mandelstam variables through world-sheet integrals and captures the α' dependence. The structure of (3.36) resembles that of the tree amplitude, but it should be stressed that the tree-level amplitude did not involve a contraction between left-moving and right-moving superfields. The BRST invariant (3.35) in the one-loop case with such a contraction can be expanded in a $A_{YM}^t \tilde{A}_{YM}$ basis with coefficients depending on the U(1) charge. Hence, the form of $\mathcal{S}_{1\text{-loop}}^q(s_{ij})$ depends on the U(1) charge-violation of the amplitude. In section 4 we will obtain the low energy expansions of the analytic part of $\mathcal{S}_{1\text{-loop}}^q(s_{ij})$ for both the cases $q = 0$ and $q = \pm 2$ and compare their coefficients with those in the expansion of the five closed-string tree amplitude given in (2.9) (and derived in [6]).

3.6 The distinction between type IIA and type IIB in ten dimensions

The derivation of the five-particle closed-string amplitude (3.33) in pure spinor superspace is equally valid for both the type IIA and type IIB theories. For type IIA the left- and right-movers have opposite space-time chirality while for type IIB these chiralities are the same. As a consequence, when extracting the supercomponent expansions of (3.33) the difference between type IIA and IIB comes entirely from the different sign of the ten dimensional Levi-Civita tensors for the pure spinor correlators of the form $\langle(\lambda^3 \theta^5)\rangle$ given in the appendix of [66]. Since there are only 9 linearly independent vectors for either the left- or right-movers (5 polarisations and 4 momenta), the difference between these two theories is restricted to the kinematic factor which involves at least one contraction between the left- and right-moving superfields. This arises in the first term on the right-hand side of (3.28). More precisely, one can show that²¹ [63]

$$\langle \tilde{V}_1 \tilde{T}_{2,3,4,5}^m \rangle = \dots + \frac{1}{5760} \begin{cases} + \epsilon^m(k^1, k^2, k^3, k^4, \tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4, \tilde{e}^5) & (\text{type IIB}) \\ - \epsilon^m(k^1, k^2, k^3, k^4, \tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4, \tilde{e}^5) & (\text{type IIA}) \end{cases} \quad (3.37)$$

where terms in the ellipsis are identical in type IIB and type IIA theory, and e_m, \tilde{e}_n denote the bosonic polarisations of the left- and right-moving sector. So we see that the ϵ^{\dots} term arising from the right-moving $\tilde{V}_1 \tilde{T}_{2,3,4,5}^m$ flips sign between the two theories. However, the ϵ^{\dots} contribution to the left-moving factor $V_1 T_{2,3,4,5}^m$ in (3.28) has the same sign in type IIA as in type IIB.

As a result, the terms in $\langle V_1 T_{2,3,4,5}^m \tilde{V}_1 \tilde{T}_{2,3,4,5}^m \rangle$ containing the product of two ϵ^{\dots} 's contracted on at least one index have opposite signs in the type IIA and type IIB cases. Such bilinears in ϵ^{\dots} reduce to products of Kronecker δ 's and are not parity-violating. Analogous observations were demonstrated in the somewhat different sigma model calculations in [67, 68]. In the type IIB case the presence of these terms quadratic in ϵ^{\dots} are important in ensuring that the amplitude can be expressed as a bilinear in Yang-Mills tree amplitudes, generalising the structure of the type IIB tree amplitude. By contrast, in the type IIA case, the sign of the ϵ^{\dots} bilinears is different and the amplitude cannot be expressed as a bilinear in Yang-Mills trees.

²¹We are using the following shorthand notation for contractions of the ϵ tensor:

$$\epsilon^m(k^1, k^2, k^3, k^4, \tilde{e}^1, \tilde{e}^2, \tilde{e}^3, \tilde{e}^4, \tilde{e}^5) := k_{n_1}^1 k_{n_2}^2 k_{n_3}^3 k_{n_4}^4 \tilde{e}_{n_5}^1 \tilde{e}_{n_6}^2 \tilde{e}_{n_7}^3 \tilde{e}_{n_8}^4 \tilde{e}_{n_9}^5 \epsilon^{n_1 n_2 \dots n_9 m}$$

The same kind of shorthand is used for the t_8 tensor in (3.38).

Furthermore, in the type IIB case the parity-violating single ϵ^{\dots} term cancels out of (3.28) whereas the type IIA amplitude does contain such a term. An example of such a parity-violating component amplitude is the amplitude with one NSNS-two form and four gravitons,

$$\mathcal{M}_{1\text{-loop}}^{BR^4} = \frac{2}{(360)^2} \int \frac{d^2\tau}{\tau_2^5} \int d^2z_2 \dots d^2z_5 \mathcal{I}(s_{ij}) \Omega$$

$$\times \begin{cases} 0 & \text{(type IIB)} \\ B_{mn}^1 \epsilon^{mn}(k^2, e^2, k^3, e^3, k^4, e^4, k^5, e^5) t_8(k^2, \tilde{e}^2, k^3, \tilde{e}^3, k^4, \tilde{e}^4, k^5, \tilde{e}^5) & \text{(type IIA)} \end{cases} \quad (3.38)$$

where $B_{mn}^1 = e_{[m}^1 \otimes \tilde{e}_{n]}^1$ denotes the two-form polarisation and t_8 is defined in [55]. This reproduces the one-loop calculation in [69]. It again cannot be expressed as a bilinear of ten-dimensional YM tree amplitudes.

Upon compactification to dimensions $D < 10$, however, maximally supersymmetric YM theory is not chiral and there is no distinction between the type IIA and type IIB theories. Therefore, the property of five-particle type IIB amplitudes that the polarisation dependence is contained in $A_{YM}^t \tilde{A}_{YM}$ also applies to the compactified type IIA theory.

Note further that the identity (3.32) motivates the interpretation of the vector $\langle V_1 T_{2,3,4,5}^m \rangle$ as the loop momentum dependent piece of a pentagon numerator in field theory.

4 Low energy expansion of type IIB one-loop amplitudes

In this section we will consider the analytic terms in the low energy expansion of the five-graviton one-loop amplitude (3.36) in powers of the Mandelstam invariants. We will concentrate on the type IIB case since in section 5 we will be interested in studying how our results match with considerations of $SL(2, \mathbb{Z})$ duality. We will begin in section (4.1) by reviewing the corresponding expansion of the four-graviton one-loop amplitude studied in [35]. This involves the expansion of the loop integrand as a power series in the world-sheet Green functions. At order $(\alpha')^{n+3}$ this results in a sum of one-particle irreducible (1PI) vacuum diagrams with n propagators joining the four points, corresponding to the external vertex operators. These diagrams are then integrated over the fundamental domain of the modular parameter τ . Care has to be taken to separate the non-analytic parts of the amplitude that correspond to non-local terms in the effective action, which is an unambiguous procedure, at least at low orders in α' . We will then, in section (4.2), systematically expand the analytic part of the matrix $\mathcal{S}_{1\text{-loop}}^q(s_{ij})$ in the five-particle amplitude (3.36) in terms of the same 1PI diagrams together with a few five-point generalisations. This extends partial results of [32] and paves the way to identification of tree-level matrices M_{2k+1} defined by (2.8) in the momentum expansion of the one-loop amplitude (3.36).

In order to include the possibility of describing amplitudes in compactifications on a d -torus to $D = 10 - d$ dimensions the loop integrand must be multiplied by the standard lattice factor that accounts for Kaluza-Klein charges and winding modes in the loop measure. This means that we should integrate over the world-sheet torus with measure

$$\int d\mu_d(\tau) := \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \Gamma_{d,d}(B, G; \tau), \quad (4.1)$$

where the τ integral is over a fundamental domain \mathcal{F} and

$$\Gamma_{d,d}(B, G; \tau) := \sum_{m^i, n^i \in \mathbb{Z}^d \times \mathbb{Z}^d} \exp \left(-\frac{\pi}{\tau_2} (G_{ij} + B_{ij})(m^i - \tau n^i)(m^j - \bar{\tau} n^j) \right). \quad (4.2)$$

and G_{ij} , B_{ij} ($i, j = 1, \dots, d$) are the metric and antisymmetric potential on the d -torus [70]. These scalar fields parameterise moduli space defined by the coset $\text{SO}(d, d, \mathbb{Z}) \backslash \text{SO}(d, d, \mathbb{R}) / (\text{SO}(d, \mathbb{R}) \times \text{SO}(d, \mathbb{R}))$ associated with T-duality. Such compactifications on flat manifolds preserve all 32 supercharges.

4.1 The four-particle amplitude and its world-sheet diagrams

The four-particle genus-one amplitude involves only one type of world-sheet integral [70] (with Koba-Nielsen factor $\mathcal{I}(s_{ij})$ given by (3.8)),

$$\begin{aligned} I &:= \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 d^2 z_3 d^2 z_4 \mathcal{I}(s_{ij}) \\ &= \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 d^2 z_3 d^2 z_4 \prod_{i < j}^4 \left(\sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!} s_{ij}^{n_{ij}} (\ln \chi_{ij})^{n_{ij}} \right), \end{aligned} \quad (4.3)$$

where the integration measure is defined in (4.1). The terms in the sum in the last factor in the integrand with a given value of $w = \sum_{i < j} n_{ij}$ involve w powers of the propagator and contribute to the terms in the expansion of order $(\alpha')^{w+3}$ relative to the classical supergravity terms. In the following we will represent a propagator $\ln \chi_{ij}$ by a line joining points i and j ,

$$i \bullet \text{---} \bullet j = \ln \chi_{ij}.$$

Terms in the sum in (4.3) of order $(\alpha')^{w+3}$ are therefore represented by diagrams with w propagators, so we will also refer to them as having weight w .

The z_i integrations within any diagram are conveniently performed by Fourier transforming the propagator,²² $\ln \chi_{ij}$ with its zero mode subtracted,²³ defined by (3.4),

$$\ln \chi_{ij} = -\frac{1}{\pi} \sum_{(m,n) \neq (0,0)} \frac{\tau_2}{|m\tau + n|^2} \exp \left(\frac{2\pi i}{\tau_2} (m \text{Re}(z_{ij})\tau_2 - (m\tau_1 + n) \text{Im}(z_{ij})) \right). \quad (4.4)$$

The summation variables m and n are the integer components of the world-sheet momentum conjugate to z_{ij} , which are conserved at each vertex of the diagram. The z_i integration within a diagram results in a modular function of τ expressed as a multiple sum over the internal world-sheet momenta. This has to be integrated over a fundamental τ domain with measure $\mu_d(\tau)$ defined by (4.1). Such diagrams were constructed in [35] up to order

²²Note that the normalization of $\ln \chi_{ij}$ chosen in this work differs by a factor of 4 from [35] and by a factor of -2 from [32]. As a consequence, any w propagator diagram must be rescaled by factors of 4^w and $(-2)^w$ for comparison with [35] and [32], respectively.

²³Momentum conservation allows a simultaneous shift in all the two point functions in (3.8) by a z independent function of $\tau, \bar{\tau}$ without changing the Koba-Nielsen factor [36]. This modification also drops out from derivatives $\partial \ln \chi_{ij}$.

$(\alpha')^9$ and studied in ten space-time dimensions (as well as in the S^1 compactification to nine dimensions).

Note on conventions: Within any diagram there may be multiple propagators joining a particular pair of points, i and j . This will be represented by the symbol

$$i \bullet \text{---} (n) \text{---} \bullet j = (\ln \chi_{ij})^n .$$

An immediate consequence of world-sheet momentum conservation at each vertex is that one-particle reducible diagrams integrate to zero. Thus, any diagram that contains a vertex connected by a single propagator gives a zero contribution,²⁴ such as $\ln \chi_{ij} \ln \chi_{jk}$ in the following figure,

$$\begin{array}{c} j \bullet \\ | \\ i \bullet \end{array} \quad \begin{array}{c} \diagdown \\ \bullet k \end{array} = \ln \chi_{ij} \ln \chi_{jk} .$$

The structure of the low order diagrams is summarised as follows. The diagram with no propagators is simply the $n_{ij} = 0$ term in the sum in (4.3) and will be labelled $D_0 := 1$. The diagram consisting of a single propagator vanishes since the integral of (4.4) is zero at a fixed finite value of τ_2 , so $D_1 = 0$. There is only one non-vanishing contribution at weight $w = 2$, given by a diagram with a double line,

$$\bullet \text{---} (2) \text{---} \bullet =: D_2 .$$

At weight $w = 3$, there are two diagrams of different topologies,

$$\bullet \text{---} (3) \text{---} \bullet =: D_3 , \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} =: D_{111} ,$$

and weight $w = 4$ gives diagrams of four distinct topologies:²⁵

$$\bullet \text{---} (4) \text{---} \bullet =: D_4 , \quad \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ (2) \end{array} =: D_{211} , \quad \begin{array}{c} \bullet \quad \bullet \\ \hline \bullet \quad \bullet \end{array} =: D_{1111} , \quad \begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ (2) \quad (2) \\ | \quad | \\ \bullet \quad \bullet \end{array} =: D_2^2$$

²⁴However, although this is true at fixed τ , there is a nonuniformity in the large- τ_2 limit leading to important threshold singularities that contribute to the non-analytic part of the amplitude [35], which we will not discuss further in this paper.

²⁵The D_2^2 term can either be represented by a disconnected diagram or by an equivalent connected diagram,

$$\begin{array}{c} \bullet \text{---} (2) \text{---} \bullet \\ \bullet \text{---} (2) \text{---} \bullet \end{array} = D_2^2 = \begin{array}{c} \bullet \text{---} (2) \text{---} \bullet \\ \diagup \quad \diagdown \\ \bullet \text{---} (2) \text{---} \bullet \end{array}$$

Using the Fourier expansion (4.4) each propagator is associated with a factor of $-\tau_2/(\pi|m\tau + n|^2)$, where the components m, n of the discrete world-sheet loop momentum are conserved at every vertex. So the modular functions, D_{\dots} , are given by multiple sums over integer loop momenta. For example, the lowest-weight cases are given by

$$\begin{aligned} D_2(\tau) &= \frac{1}{\pi^2} \sum_{(m,n) \neq (0,0)} \frac{\tau_2^2}{|m\tau + n|^4}, & D_{111}(\tau) &= -\frac{1}{\pi^3} \sum_{(m,n) \neq (0,0)} \frac{\tau_2^3}{|m\tau + n|^6} \\ D_3(\tau) &= -\frac{1}{\pi^3} \sum_{\substack{(m_1, n_1), (m_2, n_2) \neq (0,0) \\ (m_1 + m_2, n_1 + n_2) \neq (0,0)}} \frac{\tau_2^3}{|m_1\tau + n_1|^2 |m_2\tau + n_2|^2 |(m_1 + m_2)\tau + (n_1 + n_2)|^2}, \end{aligned} \quad (4.5)$$

with analogous expressions for the weight-four diagrams listed above and the higher-order diagrams that enter into the α' expansion of the four-particle amplitude. In the α' expansion of the four-point one-loop amplitude, the coefficients of any product $M_{l_1} M_{l_2} \dots$ are given by the integral of a particular linear combination of D_{\dots} 's. We will use the notation

$$\Xi_{l_1, l_2, \dots}^{(d)} = \int d\mu_d(\tau) \sum D_{\dots}, \quad (4.6)$$

where the l_1, l_2, \dots subscript refers to the accompanying $M_{l_1} M_{l_2} \dots$. Explicit examples are given in (5.3) and other parts of section 5. In the $D = 10$ case (where $d = 0$) the coefficients $\Xi_{\dots}^{(0)}$ are constants that reduce to multi-zeta values in the cases that we will consider, whereas the quantities $\Xi_{\dots}^{(d)}$ depend on the moduli of the $(10 - d)$ -dimensional theory.

The complete set of such sums that arise up to order $(\alpha')^9$ (as well as certain classes of higher weight diagrams) were analysed in [35] in sufficient detail to determine their contributions to the $D = 10$ (or $d = 0$) four-particle α' expansion. Upon compactification on a circle of radius r to $D = 9$ (i.e., when $d = 1$) the analysis in [35] determined, for each power of α' , the terms that are power-behaved in r in the limit $r^2 \gg 1/\alpha'$. For compactifications to lower dimensions (larger values of d) the coefficients of the terms in the α' expansion depend on the moduli that parameterise the coset $\text{SO}(d, d)/(\text{SO}(d) \times \text{SO}(d))$ as discussed in [59, 60]

The evaluation of these multiple sums rapidly gets unwieldy and we do not have a general procedure for analysing the contribution of higher-weight diagrams. However, it is possible to analyse completely certain infinite subsets of diagrams. For example, the subset of weight- k diagrams that form k -sided polygons are non-holomorphic Eisenstein series',

$$E_k(\tau) = (-\pi)^k D_{\underbrace{11 \dots 1}_k} = \sum_{(m,n) \neq (0,0)} \frac{\tau_2^k}{|m\tau + n|^{2k}}. \quad (4.7)$$

In the following we will extend the diagrammatic discussion to the five-particle amplitude up to order $(\alpha')^9 D^{10} R^5$.

4.2 Five-particle world-sheet integrals and their diagrammatic expansion

We now turn to consider the expansion of the five-particle amplitude. The integrand of the amplitude (3.33) comprises terms proportional to Ω , see (3.6), and terms proportional

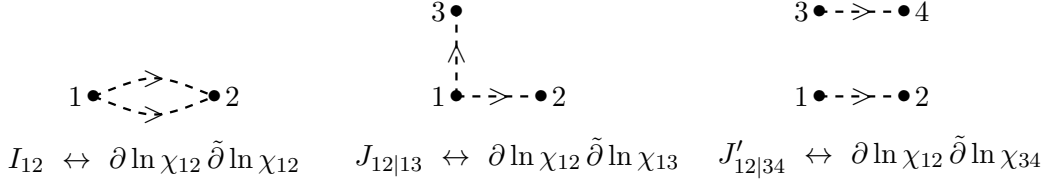


Figure 1. Three different topologies of five-point integrals: Directed dashed lines represent both holomorphic and antiholomorphic derivatives $\partial \ln \chi_{ij}$ and $\tilde{\partial} \ln \chi_{ij}$.

to $X_{ij} \tilde{X}_{kl}$. The latter in turn is proportional to $\partial \log \chi_{ij} \tilde{\partial} \log \chi_{kl}$ by (3.4). As a result, the diagrams that enter into the low energy expansion have the additional feature that they include lines with holomorphic or anti-holomorphic derivatives acting on them. In the following such a line will be represented by

$$1 \bullet \text{---} \text{---} \text{---} \text{---} \bullet 2 = \partial \ln \chi_{12}$$

where the arrow indicates a derivative (and it is not necessary to distinguish holomorphic or anti-holomorphic derivatives at the level of the five-particle amplitude).

The complete five-particle amplitude involves the sum of a number of integrals over the moduli space of the torus: One of these is a totally symmetric integral

$$K := -\frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \mathcal{I}(s_{ij}) \Omega \quad (4.8)$$

and the others comprise 36 integrals involving a pair of holomorphic and antiholomorphic propagator derivatives $X_{ij} \tilde{X}_{pq} \sim \partial \ln \chi_{ij} \tilde{\partial} \ln \chi_{pq}$. Depending on the relative labels $i, j \leftrightarrow p, q$, we have to distinguish three topologies [32] of integrands $\sim X_{ij} \tilde{X}_{pq}$ shown in figure 1. The integral expressions associated with these diagrams are given by

$$I_{rs} := \frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \mathcal{I}(s_{ij}) \partial \ln \chi_{rs} \tilde{\partial} \ln \chi_{rs} \quad (4.9)$$

$$J_{rs|rt} := \frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \mathcal{I}(s_{ij}) \partial \ln \chi_{rs} \tilde{\partial} \ln \chi_{rt} \quad (4.10)$$

$$J'_{rs|tu} := \frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \mathcal{I}(s_{ij}) \partial \ln \chi_{rs} \tilde{\partial} \ln \chi_{tu} . \quad (4.11)$$

Expanding these integrals is a more substantial challenge. Integrals of J and J' type do not involve any poles in s_{ij} , and the power series expansion in α' can be performed at the level of the Koba-Nielsen factor (3.8) in the integrand. In the case of the I_{12} topology, however, we first have to subtract the s_{12}^{-1} pole whose residue is given by the four-point integral (4.3) with momenta $k_1 + k_2, k_3, k_4, k_5$ [32]:

$$I_{12} = \frac{1}{s_{12}} I(k_1 + k_2, k_3, k_4, k_5) + I_{12}^{\text{reg}} \quad (4.12)$$

$$I_{12}^{\text{reg}} = \frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \prod_{i < j}^5 \left(\sum_{n_{ij}=0}^{\infty} \frac{1}{n_{ij}!} s_{ij}^{n_{ij}} (\ln \chi_{ij})^{n_{ij}} \right) \partial \ln \chi_{rs} \tilde{\partial} \ln \chi_{rs} \quad (4.13)$$

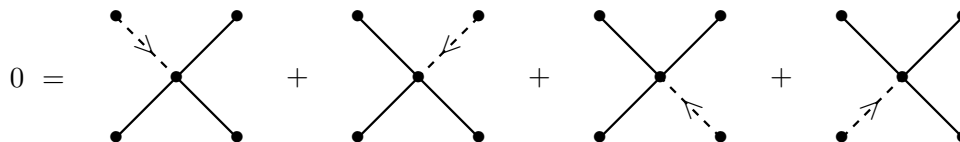


Figure 2. Momentum conservation at vertices of world-sheet diagrams following from z_i integration.

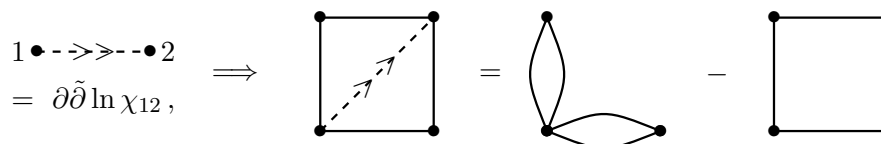


Figure 3. Graphical formulation of the Laplace equation: The first diagram on the right-hand side originates from the τ_2^{-1} factor in $\partial \bar{\partial} \ln \chi_{ij} \sim (\delta^2(z_{ij}) - \tau_2^{-1})$, the second diagram is due to the delta function admixture.

In other words, removal of the pole in (4.12) paves the way towards Taylor expanding the Koba-Nielsen factor. This generalises the treatment of multiparticle disk integrals with kinematic poles in [26]. The low energy behaviour of the regular part I_{12}^{reg} can then be obtained by the same methods as those applied to J and J' . Hence, the residual task is the reduction of integrals of the form

$$\int d^2 z_2 \dots d^2 z_5 \partial \ln \chi_{rs} \bar{\partial} \ln \chi_{tu} \prod_{i < j}^5 (\ln \chi_{ij})^{n_{ij}}, \quad n_{ij} \in \mathbb{N} \quad (4.14)$$

to an appropriate basis of multiple sums.

At low orders in the expansion expressions of the form (4.14), can be reduced to the expressions that define the D_{\dots} 's that arose in expanding the four-particle amplitude (4.3). This reduction involves:

- Integration by parts, or momentum conservation (as illustrated in figure 2). This can be used to express diagrams with non-coincident derivatives, as in the integrals (4.10), (4.11) and (4.13), in terms of those with a laplacian acting on one propagator. However, this fails for some diagrams with more than 5 propagators, as will be discussed later.
- The Laplace equation (3.6) and its diagrammatic formulation shown in figure 3 reduces any diagram with a double derivative on a propagator to a diagram with one less propagator. In diagrammatic language, the action of the laplacian on a propagator, $\partial \bar{\partial} \ln \chi_{ij} \sim (\delta^2(z_{ij}) - \tau_2^{-1})$, amounts to either shrinking or deleting it (with a relative sign), as in figure 3.

For diagrams with less than six propagators this procedure reduces the factor of $\partial \ln \chi_{rs} \bar{\partial} \ln \chi_{tu}$ to a single propagator $\ln \chi_{ij}$ and the weight w of the resulting expression is then related to the integer exponents n_{ij} in (4.14) via $w = 1 + \sum_{i < j} n_{ij}$. Certain

diagrams with $w > 4$ cannot be reduced to the diagrams that arose in the case of the expansion of the four-particle amplitude and led to the expressions D_{\dots} .

4.3 Order by order expansion of five-point integrals

In this section, we investigate low weight examples in detail in order to illustrate the procedure outlined above. The goal is to expand the five-point integrals (4.8) to (4.11). In evaluating K , we may replace Ω by $-\pi/\tau_2$ and proceed along the lines of section 4.1 where its four-particle relative (4.3) is analysed,

$$\begin{aligned}
 K = \int d\mu_d(\tau) & \left(D_0 + \frac{D_2}{2} \sum_{i < j} s_{ij}^2 + D_{111}(s_{12}s_{13}s_{23} + 9\text{perms.}) \right. \\
 & + \frac{D_3}{6} \sum_{i < j} s_{ij}^3 + \frac{D_4}{24} \sum_{i < j} s_{ij}^4 + \frac{D_2^2}{4}(s_{12}^2 s_{13}^2 + s_{12}^2 s_{34}^2 + 43\text{perms.}) \\
 & \left. + \frac{D_{211}}{2}(s_{12}s_{13}s_{23}s_{123} + 9\text{perms.}) + D_{1111}(s_{12}s_{23}s_{34}s_{41} + 14\text{perms.}) + \dots \right)
 \end{aligned} \tag{4.15}$$

Expanding the remaining integrals (4.9) to (4.11) is more complicated since they have an overall factor of two propagators with a derivative acting on each of them. The following enumerates the contributions that arise at each weight in the expansion.

Weight 1. The zeroth order term in the expansion has $n_{ij} = 0$ in (4.14) and so involves the product of two propagators with a derivative acting on each of them. At fixed τ_2 this integrates to zero as one can also see from the absence of a non-vanishing lattice sum D_1 . This not only ties in with the vanishing of the linearised $D^2 R^4$ contribution to the graviton effective action, which vanishes by integration by parts, but also implies the vanishing of its nonlinear contribution to the five-graviton amplitude, as well as the vanishing of the R^5 contribution to the five-graviton amplitude (as pointed out in [32]²⁶). However, as noted earlier, the fact that the propagator $\log \chi_{ij}$ (in (4.4)) is proportional to τ_2 does lead to non-zero contributions to the non-analytic part of amplitudes from the large τ_2 limit. This is responsible for the threshold terms of the symbolic form $(s \ln s + \dots) R^4$ and $(\ln s + \dots) R^5$ [35].

Weight 2. In this case $\sum_{i < j} n_{ij} = 1$ in (4.14) and the diagrams have three propagators. The following diagrams contribute to I_{12}^{reg} and $J_{12|13}$ and are evaluated by integration by parts and use of the Laplace operator on one propagator.

$$\begin{aligned}
 \text{Diagram 1} &= -\frac{1}{2} \text{Diagram 2} = \frac{1}{2} D_2 \\
 \text{Diagram 3} &= - \text{Diagram 4} = -D_2
 \end{aligned}$$

For completeness, the following is a diagrammatic representation of the Laplace equation identities that are used in deriving the above expressions:

²⁶The contributions of $D^2 R^4$ and R^5 to the five-particle amplitude in $D = 4$ dimensions have also been argued to vanish by virtue of supersymmetry in theories with 32 supercharges [71–73].

$$\begin{aligned}
 \text{Triangle with circled 2 and dashed bottom edges} &= \times - \text{Triangle with circled 2 and solid bottom edges} = -D_2 \\
 \text{Triangle with dashed bottom edge and solid other edges} &= \text{Vertical oval} - \text{Triangle with solid edges} = +D_2
 \end{aligned}$$

Propagators contracted to a point through the delta function in (3.6) do not contribute in this case, which reflects the regularisation $\ln \chi_{ij}(0, \bar{0}) \rightarrow 0$.

Diagrams that arise in the expansion of $J'_{12|34}$ at this order vanish and hence $J'_{12|34}|_{w=2} = 0$. The above examples exhaust the nonvanishing $w = 2$ contributions of all the five-point integrals, resulting in

$$I_{12}^{\text{reg}} \Big|_{w=2} = \frac{1}{2} \int d\mu_d(\tau) D_2 s_{12}, \quad J_{12|13} \Big|_{w=2} = - \int d\mu_d(\tau) D_2 s_{23}. \quad (4.16)$$

Weight 3. The contributions of this order are terms with $\sum_{i<j} n_{ij} = 2$ in (4.14) that correspond to diagrams with four propagators. The following diagrams arise and can again be reduced to standard form by integration by parts and use of the Laplace equation on one propagator.

$$\begin{aligned}
 \text{Triangle with circled 2 and dashed bottom edges} &= \frac{1}{3} D_3 & \text{Triangle with solid top edge and dashed bottom edges} &= D_{111} + \frac{1}{2} D_3 \\
 \text{Triangle with dashed bottom edge and solid other edges} &= -\frac{1}{2} D_3 & \text{Triangle with dashed top edge and solid bottom edges} &= -D_3 \\
 \text{Rectangle with dashed bottom and left edges, solid top and right edges} &= -D_{111} & \text{Rectangle with dashed top and bottom edges, solid left and right edges} &= -D_{111}
 \end{aligned}$$

The Laplace equation identities that are used in determining these expressions (following integration by parts) are the following.

$$\begin{aligned}
 \text{Triangle with circled 3 and dashed bottom edges} &= -D_3, & \text{Rectangle with solid top and bottom edges, dashed left and right edges} &= D_{111} \\
 \text{Triangle with dashed bottom edge and solid other edges} &= -D_{111}, & \text{Triangle with dashed top edge and solid bottom edges} &= D_3
 \end{aligned}$$

We conclude that the contributions of all diagrams at this order reduce to rational multiples of D_3 and D_{111} . Using the above results, the expansion of the integrals (4.10), (4.11)

and (4.13) to weight $w = 3$ is

$$\begin{aligned}
J_{12|13} &= - \int d\mu_d(\tau) \left(D_2 s_{23} + D_{111}(s_{24}s_{34} + s_{25}s_{35}) + \frac{D_3}{2} s_{23}(s_{12} + s_{13} + s_{23}) + \dots \right) \\
J'_{12|34} &= \int d\mu_d(\tau) D_{111}(s_{14}s_{23} - s_{13}s_{24}) + \dots \\
I_{12}^{\text{reg}} &= \int d\mu_d(\tau) \left(\frac{D_2}{2} s_{12} + D_{111}(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25}) \right. \\
&\quad \left. + \frac{D_3}{6} (s_{12}^2 + 3(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25})) + \dots \right)
\end{aligned} \tag{4.17}$$

which reproduce the expansion results in [32].

A feature that is worth highlighting is that in reconstructing the expansion of the five-particle amplitude (3.33), the left-right symmetric expression $X_{ij}\tilde{X}_{ij}$ is always accompanied by $s_{ij}\Omega$. As a consequence, integrals of I_{ij} type (4.9) only appear in combination with the K integral (4.8) and important cancellations occur (e.g. the complete D_0 coefficient drops out). Therefore, the expansion of the piece of the amplitude that contains I_{ij} is given, up to weight three, by

$$\begin{aligned}
\frac{1}{\pi} \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_5 \mathcal{I}(s_{ij}) [X_{12}\tilde{X}_{12} + s_{12}\Omega] &= s_{12} \int d\mu_d(\tau) \\
&\left(D_2 (s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25}) + \frac{D_3}{2} (s_{12}s_{13}s_{23} + s_{12}s_{14}s_{24} + s_{12}s_{15}s_{25}) \right. \\
&\quad + D_{111} (s_{13}s_{24}s_{34} + s_{13}s_{25}s_{35} + s_{14}s_{23}s_{34} + s_{14}s_{25}s_{45} + s_{15}s_{23}s_{35} + s_{15}s_{24}s_{45}) \\
&\quad \left. + \frac{D_3}{2} (s_{13}^2 s_{23} + s_{13} s_{23}^2 + s_{14} s_{24}^2 + s_{14}^2 s_{24} + s_{15} s_{25}^2 + s_{15}^2 s_{25}) + \dots \right). \tag{4.18}
\end{aligned}$$

Weight 4. The diagrammatic framework is easily applied to higher weight contributions. Terms with $\sum_{i<j} n_{ij} = 3$ in (4.14) give rise to diagrams with five propagators, two of which have derivatives acting on them. If we ignore the derivatives, there are eight distinct five-propagator diagrams, which are listed in appendix C.1. These diagrams (without arrows) will be directly relevant for the $w = 5$ case as described below. However, attaching derivatives to two lines, which can be done in many ways, and using the integration by parts procedure and Laplace equation, reduces the diagrams to the standard weight-four D_{\dots} that were encountered in the case of the four-particle amplitude. As an example, consider attaching two arrows to the diagram that is equal to D'_{1111} in appendix C.1, which has a particularly convoluted topology. The weight-four contributions that result are the following

$$\begin{aligned}
 & \text{Diagram 1} = - \text{Diagram 2} = -D_{211} \\
 & \text{Diagram 3} = - \text{Diagram 4} - \text{Diagram 5} = \frac{1}{2} (D_{1111} - D_2^2) \\
 & \text{Diagram 6} = - \text{Diagram 7} - \text{Diagram 8} = -D_{211} + \frac{1}{2} (D_2^2 - D_{1111})
 \end{aligned}$$

These expressions make use of the following identities involving the action of the Laplace operator on a propagator

$$\begin{aligned}
 & \text{Diagram 9} = \text{Diagram 10} - \text{Diagram 11} = D_2^2 - D_{1111} \\
 & \text{Diagram 12} = \text{Diagram 13} - \text{Diagram 14} = D_{211}
 \end{aligned}$$

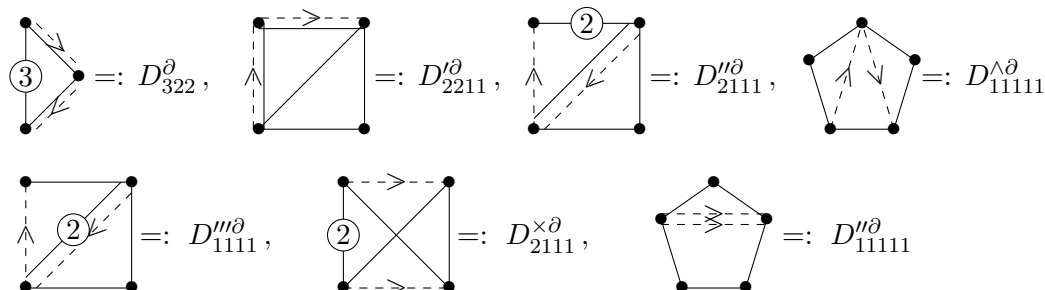
We see that any diagram of this topology appearing in the expansions of I_{12}^{reg} , $J_{12|13}$ and $J'_{12|34}$ can be reduced to rational combinations of $(D_{1111} - D_2^2)$ and D_{211} . The analysis of the remaining topologies involving five propagators (such as $D_5, D_2 D_3, \dots$ in the terminology of appendix C.1) is simpler and we will not display it here. The complete expansion of the five-point integrals up to $w = 4$ is fully displayed in appendix D.

Weights 5 and 6. Starting at weight five, not all the derivatives in the $\partial \ln \chi_{ij} \tilde{\partial} \ln \chi_{pq}$ integrand can be eliminated by this method. For $w = 5$ (terms in (4.14) with $\sum_{i < j} n_{ij} = 4$) the diagrams have six propagators, two of which have derivatives. The derivatives can be eliminated from most types of diagrams as shown in appendix C.2, although some of these were not encountered in the expansion of the four-particle amplitude because they involve more than four vertices. This leave two types of diagrams from which the derivatives cannot be eliminated by the above procedure. The precise definition of these diagrams is ambiguous since they are only defined up to terms arising from integration by parts, but they can be chosen to be the ones shown below

$$\begin{aligned}
 & \text{Diagram 15} =: D_{222}^{\partial} \\
 & \text{Diagram 16} =: D_{1111}^{\partial\partial}
 \end{aligned}$$

Together with the eight diagrams shown in appendix C.1, the diagrams of this weight span a ten-dimensional vector space of lattice sums.

A similar analysis of the $w = 6$ contributions (diagrams with seven propagators, two of which have derivatives) leads to the 20 diagrams from which derivatives have been eliminated that are listed in appendix C.2, and leaves the following seven undetermined diagrams from which the derivatives cannot be eliminated by integration by parts.



So we see that the diagrams of weight 6 span a 27-dimensional vector space of lattice sums. Again there are ambiguities in the definition of these diagrams since they are only defined modulo integration by parts, corresponding to redefinitions of the basis.

5 Low energy type IIB amplitudes and S-duality

In this section, we will compare our results for the low energy expansion of four- and five-particle type IIB amplitudes up to order $(\alpha')^9$. This is of particular interest in the type IIB theory, in which the constraints of $SL(2, \mathbb{Z})$ S-duality are expected to relate different orders in perturbation theory in an interesting manner. Since the amplitudes at tree level and one loop both take the form $A_{YM}^t \mathcal{S}_{\text{tree}} \tilde{A}_{YM}$ and $A_{YM}^t \mathcal{S}_{1\text{-loop}} \tilde{A}_{YM}$, we can restrict our discussion to the quantities \mathcal{S}_{\dots} . It turns out that several combinations of multiple sums D_{\dots} arise in the α' expansions of $\mathcal{S}_{1\text{-loop}}$ in both the $N = 4$ and $N = 5$ cases. It is also striking that the set of tree-level kinematic invariants at any order in α' (encoded in the M matrices) is reproduced in the one-loop amplitudes, although extra invariants also appear at one loop in the $N = 5$ case starting at order $(\alpha')^7 D^6 R^5$. In the case of U(1)-conserving amplitudes we will see that the ratio of the coefficients of the tree-level invariants and the coefficients of the same invariants at one-loop is identical in the $N = 4$ and $N = 5$ cases. This has suggestive implications for the pattern of non-perturbative $SL(2, \mathbb{Z})$ S-duality of the type IIB amplitudes that extends the well-established $N = 4$ pattern.

We will also determine the α' expansion of the amplitudes that violate the U(1) charge by $q = \pm 2$ units (which arises when $N \geq 5$) up to order $(\alpha')^9$. We will see that the pattern of rational coefficients of such terms extends the systematics of S-duality at order $(\alpha')^3$ described in section 2.4 in a compelling manner.

5.1 The four-particle one-loop amplitude

In order to make contact between four-particle and five-particle closed-string amplitudes at tree-level and one-loop, we summarise the four-particle one-loop results of [35]. In our present convention, the four-point amplitude

$$\mathcal{M}_{1\text{-loop}} = 2\pi I \cdot A_{YM} S_0 M_3 \tilde{A}_{YM} \quad (5.1)$$

proportional to the integral I defined by (4.3) allows to read off

$$\mathcal{S}_{1\text{-loop}} = S_0 \left(\Xi_3^{(d)} M_3 + \Xi_5^{(d)} M_5 + \Xi_{3,3}^{(d)} M_3^2 + \Xi_7^{(d)} M_7 + \Xi_{\{3,5\}}^{(d)} \{M_3, M_5\} \right. \\ \left. + \Xi_9^{(d)} M_9 + \Xi_{3,3,3}^{(d)} M_3^3 + \dots \right), \quad (5.2)$$

where S_0 and M_w are scalar functions of Mandelstam invariants defined in (2.13). The accompanying coefficients, $\Xi_{\dots}^{(d)}$, are combinations of the integrated multiple sums, $\int d\mu_d(\tau) D_{\dots}(\tau)$, and their dependence on the moduli of the $(10-d)$ -dimensional theory (which has been suppressed) enters through the measure factor $d\mu_d(\tau)$. The precise combinations in (5.2) are

$$\begin{aligned} \Xi_3^{(d)} &:= 2\pi \int d\mu_d(\tau) D_0 \\ \Xi_5^{(d)} &:= 4\pi \int d\mu_d(\tau) D_2 \\ \Xi_{3,3}^{(d)} &:= -2\pi \int d\mu_d(\tau) (4D_{111} + D_3) \\ \Xi_7^{(d)} &:= \frac{\pi}{3} \int d\mu_d(\tau) (D_4 + 9D_2^2 + 6D_{1111}) \\ \Xi_{\{3,5\}}^{(d)} &:= -\frac{\pi}{12} \int d\mu_d(\tau) (D_5 + 16D_{311} - 12D_{221} + 12D_{2111} - 24D'_{1111} + 14D_3 D_2 + 48D_{111} D_2) \\ \Xi_9^{(d)} &:= \frac{\pi}{90} \int d\mu_d(\tau) (D_6 - 90D_{2211} + 90D_2^3 + 120D_{3111} - 10D_3^2 + 45D_4 D_2) \\ \Xi_{3,3,3}^{(d)} &:= \frac{\pi}{540} \int d\mu_d(\tau) (7D_6 + 540(D_{411} - 2D_{321} + D_{222}) + 2610D_{2211} - 450D_2^3 - 1320D_{3111} \\ &\quad + 2160D_3 D_{111} + 470D_3^2 - 225D_4 D_2 - 6480D'_{2111} + 3240D''_{1111} + 1080D_{1111}^\times). \end{aligned} \quad (5.3)$$

The diagrams associated with multiple sums with up to four propagators were displayed in section 4.1, whereas the more complicated weight-five and six D_{\dots} entering $\Xi_{\{3,5\}}^{(d)}$, $\Xi_9^{(d)}$ and $\Xi_{3,3,3}^{(d)}$ are reviewed in appendix C.

The expressions for $\Xi_w^{(d)}$ were analysed explicitly in [35] in $D = 10$ and $D = 9$ dimensions, after compactification on a circle of radius r . For example, the $\Xi_w^{(0)}$'s that arise up to weight four in $D = 10$, where the measure $d\mu_0(\tau)$ trivialises, are given in terms of

$$\begin{aligned} \int d\mu_0(\tau) D_0 &\approx \frac{\pi}{3}, & \int d\mu_0(\tau) D_2 &\approx 0 \\ \int d\mu_0(\tau) D_3 &\approx -\frac{\pi\zeta_3}{3}, & \int d\mu_0(\tau) D_{111} &\approx 0 \\ \int d\mu_0(\tau) D_4 &\approx 0, & \int d\mu_0(\tau) D_{211} &\approx 0 \\ \int d\mu_0(\tau) D_{1111} &\approx 0, & \int d\mu_0(\tau) D_2^2 &\approx 0. \end{aligned} \quad (5.4)$$

The \approx sign indicates that the integrations actually diverge at the large τ_2 boundary of the fundamental domain, but they are regulated by taking into account the non-analytic

4pt M_w 's	Interaction	Tree	1-loop generic d	$d = 0$
1	R	1	0	0
M_3	R^4	$2\zeta_3$	$\Xi_3^{(d)}$	$4\zeta_2$
M_5	$D^4 R^4$	$2\zeta_5$	$\Xi_5^{(d)}$	0
M_3^2	$D^6 R^4$	$2\zeta_3^2$	$\Xi_{3,3}^{(d)}$	$4\zeta_2\zeta_3$
M_7	$D^8 R^4$	$2\zeta_7$	$\Xi_7^{(d)}$	0
$\{M_3, M_5\}$	$D^{10} R^4$	$2\zeta_3\zeta_5$	$\Xi_{\{3,5\}}^{(d)}$	$\frac{97}{90}\zeta_2\zeta_5$
M_9	$D^{12} R^4$	$2\zeta_9$	$\Xi_9^{(d)}$	$\frac{16}{15}\zeta_2\zeta_3^2$
M_3^3	$D^{12} R^4$	$\frac{4}{3}\zeta_3^3$	$\Xi_{3,3,3}^{(d)}$	$\frac{151}{90}\zeta_2\zeta_3^2$

Table 4. Summary of tree-level and one-loop contributions to terms in the effective action of the form $D^{2k}R^4$, which can be extracted from the four-graviton components of the α' expansion of the superamplitudes shown in (2.9) and (5.2). The M_9, M_3^3 degeneracy at order $(\alpha')^9$ gives rise to two distinct $D^{12}R^4$ operators which differ in the tensor contractions of the derivatives and curvature.

thresholds that arise from the large- τ_2 limit. This was discussed in detail in [35, 36] and is an indication of the important interplay between the analytic and non-analytic parts of the amplitude. We refer the reader to [35] for details of the integrals that arise at weights five and six.

Table 4 summarises the M_r matrices that correspond to the $D^{2k}R^4$ interactions in the expansion of the $N = 4$ tree-level and one-loop amplitudes up to order $(\alpha')^9$ that are given in (2.9) and (5.2), together with the corresponding tree-level and one-loop $D = 10$ coefficients.

Notably, the values of the coefficients in table 4 up to order M_3^2 match with the tree-level and one-loop contributions contained in the modular invariant coefficient functions, \mathcal{E}_3 , \mathcal{E}_5 and $\mathcal{E}_{3,3}$. As reviewed in section 2.4 [12–14], these accompany the interactions R^4 , $D^4 R^4$ and $D^6 R^4$ in the $D = 10$ type IIB effective action. In particular, it follows from (5.4) that the $D = 10$ coefficients, $\Xi_5^{(0)}$ and $\Xi_7^{(0)}$, vanish. As a consequence, the interactions of order $D^4 R^4$ and $D^8 R^4$, as well as their supersymmetric partners, do not receive one-loop contributions to their analytic parts in ten dimensions.²⁷ Nevertheless, generic compactifications to $D < 10$ lead to non-vanishing $\Xi_5^{(d)}$ and $\Xi_7^{(d)}$. This is denoted in tables 1 and 3 by the wavy underlining of those terms that are zero only in $D = 10$ dimensions.

5.2 The U(1)-conserving five-particle amplitudes

We have generalised the procedure used for the four-particle amplitude to expand the five-particle integrals (4.9)–(4.11) and (4.15) to all weights $w \leq 6$, using integration by parts identities (3.13) as a cross-check. In U(1)-conserving five-curvature components, this leads

²⁷However, as discussed in [35] there is a non-analytic threshold term of the schematic form $(\alpha')^7 s^4 \ln(\alpha' s) R^4$ that is related to a logarithmic divergence in $\int d\mu_0(\tau) D_2^2$.

to the following α' expansion for the matrix $\mathcal{S}_{1\text{-loop}}$ in the five point amplitude (3.36),

$$\begin{aligned} \mathcal{S}_{1\text{-loop}}^{q=0} = S_0 \Big(& \Xi_3^{(d)} M_3 + \Xi_5^{(d)} M_5 + \Xi_{3,3}^{(d)} M_3^2 + \Xi_7^{(d)} M_7 + \Xi_{7'}^{(d)} M_7' + \Xi_{\{3,5\}}^{(d)} \{M_3, M_5\} \\ & + \Xi_{8'}^{(d)} M_8' + \Xi_9^{(d)} M_9 + \Xi_{3,3,3}^{(d)} M_3^3 + \Xi_{9'}^{(d)} M_9' + \Xi_{9''}^{(d)} M_9'' + \dots \Big), \end{aligned} \quad (5.5)$$

In the present $N = 5$ case, the quantities S_0 , M_k and M_k' are 2×2 matrices. The unprimed matrices M_k are defined by (2.8) in a tree-level context whereas M_k' do not appear at genus zero. Their entries are degree k polynomials in Mandelstam invariants (and, by (2.4), in α') whose explicit form can be obtained from an extra Mathematica file accompanying the arXiv submission of this paper. Strikingly, the combinations of terms that make up the coefficients $\Xi_k^{(d)}$ of the matrices M_k in (5.5) are exactly the same as those in the four-particle amplitude, (5.2) and (5.3). However, the following additional combinations of multiple sums arise

$$\begin{aligned} \Xi_{7'}^{(d)} &:= -10\pi \int d\mu_d(\tau) \left(\frac{1}{96} D_4 + \frac{19}{32} D_2^2 - D_{211} - \frac{99}{80} D_{1111} \right) \\ \Xi_{8'}^{(d)} &:= \frac{24\pi}{5} \int d\mu_d(\tau) \left(\frac{7}{96} D_5 + \frac{1}{6} D_{311} - \frac{1}{8} D_{221} + \frac{1}{8} D_{2111} + \frac{1}{4} D'_{1111} \right. \\ &\quad \left. + \frac{13}{48} D_3 D_2 + D_{11111} - \frac{1}{4} D_{222}^\partial - D_{1111}^{\prime\prime\partial} \right) \\ \Xi_{9'}^{(d)} &:= -2\pi \int d\mu_d(\tau) \left(-\frac{667}{12960} D_6 - 4D_{1111} D_2 - 3D_{111}^2 + \frac{2}{3} D_{11,11,11} + D_{21111} - 5D_{211} D_2 \right. \\ &\quad - \frac{1}{2} D_{222} + \frac{55}{144} D_{2211} + \frac{191}{144} D_2^3 - \frac{235}{108} D_{3111} - \frac{7}{6} D_{321} + \frac{7}{6} D_3 D_{111} - \frac{1}{6} D_{411} + \frac{257}{288} D_4 D_2 \\ &\quad + \frac{667}{1296} D_3^2 - 2D'_{11111} + D'_{2111} + D''_{1111} - D_{1111}^\times + 2D_{1111}^{\prime\prime\partial} + \frac{1}{6} D_{322}^\partial + 4D_{2211}^{\prime\partial} \\ &\quad \left. - 6D_{2111}^{\prime\prime\partial} + 4D_{1111}^{\prime\prime\prime\partial} - 2D_{2111}^{\times\partial} - 2D_{1111}^{\wedge\partial} \right) \\ \Xi_{9''}^{(d)} &:= -2\pi \int d\mu_d(\tau) \left(\frac{1}{10368} D_6 - D_{1111} D_2 - \frac{1}{2} D_{111}^2 + \frac{1}{3} D_{11,11,11} + D_{21111} - \frac{1}{4} D_{211} D_2 \right. \\ &\quad + \frac{355}{576} D_{2211} - \frac{1}{8} D_{222} + \frac{101}{576} D_2^3 - \frac{211}{432} D_{3111} - \frac{1}{2} D_{321} + \frac{1}{2} D_3 D_{111} + \frac{1291}{5184} D_3^2 + \frac{3}{8} D_{411} \\ &\quad \left. + \frac{341}{1152} D_4 D_2 - 2D'_{11111} - 2D'_{2111} + D''_{1111} - 2D_{2111}^{\prime\prime\partial} + D_{1111}^{\prime\prime\prime\partial} - D_{2111}^{\times\partial} - 2D_{1111}^{\wedge\partial} \right) \end{aligned} \quad (5.6)$$

The terms in (5.5) are associated with the contributions to the low energy expansion of the five-particle amplitude due to (supersymmetrised) combinations of $D^{2w} R^4$ and $D^{2w-2} R^5$ interactions (with w denoting the weight of the accompanying multiple sum at order $(\alpha')^{w+3}$). It is worth noting the following points:

- the five-particle matrices M_k arise in both the tree-level and one-loop amplitudes, (2.9) and (5.5), respectively;
- the specific linear combinations $\Xi_{\dots}^{(d)}$ of one-loop multiple sums that arose in the four-particle case (5.3) also contribute to the five-particle one-loop amplitude.

We therefore conclude that the $D^{2w} R^4$ and $D^{2w-2} R^5$ interactions associated with the $w \leq 3$ results in (5.5) involve exactly the same tree-level and one-loop coefficients. For

$4 \leq w \leq 6$ one of the kinematic invariants in $D^{2w-2}R^5$ (the one involving only unprimed M_k matrices) has the same coefficient as $D^{2w}R^4$. Such perturbative results provide useful input for constraining the exact form of the modular invariant coefficient functions \mathcal{E}_{\dots} in the type IIB theory, generalising the arguments in section 2.4. After discussing analogous results concerning U(1)-violating five-particle amplitudes in the following section, we will see, in section 5.4, how these results may be interpreted in terms of $\text{SL}(2, \mathbb{Z})$ duality of the type IIB amplitude.

A class of $D^{2w-2}R^5$ interactions which does not occur at tree level is signalled by novel matrices M'_7, M'_8, M'_9 and M''_9 in (5.5), which share a lot of algebraic properties with the M_k . For example, they again preserve the BCJ and KK relations [25] among the A_{YM} they act on. But they do not appear in the tree amplitude (2.9). In fact, these matrices were identified in the research leading to [26] as the unique deformations of the constituents M_7, P_8, M_9 of disk amplitudes that are consistent with their factorisation properties, cyclicity and monodromy relations [23, 24].

A small comment: There is an apparent mismatch between the new five-particle kinematic invariants associated with the matrices M'_7, M'_8, M'_9 and M''_9 and the classification of candidate counterterms in four dimensional $\mathcal{N} = 8$ supergravity given in [73]. In particular, this reference rules out an independent D^6R^5 operator based on four dimensional spinor helicity methods. The fact that we have found a term of the form $A_{YM}^t S_0 M'_7 \tilde{A}_{YM}$ is, however, compatible with the analysis of [73] since this term vanishes in the phase space of four dimensional on-shell kinematics whereas in dimensions $D \geq 5$ the interaction is non-vanishing. We are grateful to the authors of [73] for helpful email correspondence which clarified this issue.

5.3 The U(1)-violating five-particle amplitudes

Also U(1)-violating five-particle closed-string amplitudes, such as the amplitude with two G 's and three gravitons, can be expressed in terms of A_{YM} bilinears. According to (3.35), their coefficients encoded in the matrix $\mathcal{S}_{1\text{-loop}}^q$ are different for R-symmetry charges $q = 0$ and $q = \pm 2$. In the latter case, we arrive at the following low energy expansion

$$\begin{aligned} \mathcal{S}_{1\text{-loop}}^{q=\pm 2} = S_0 & \left(-\frac{1}{3} \Xi_3^{(d)} M_3 + \frac{1}{5} \Xi_5^{(d)} M_5 + \frac{1}{3} \Xi_{3,3}^{(d)} M_3^2 + \frac{3}{7} \Xi_7^{(d)} M_7 + \hat{\Xi}_{7'}^{(d)} M'_7 + \frac{1}{2} \Xi_{\{3,5\}}^{(d)} \{M_3, M_5\} \right. \\ & \left. + \hat{\Xi}_{8'}^{(d)} M'_8 + \frac{5}{9} \Xi_9^{(d)} M_9 + \frac{5}{9} \Xi_{3,3,3}^{(d)} M_3^3 + \hat{\Xi}_{9'}^{(d)} M'_9 + \hat{\Xi}_{9''}^{(d)} M''_9 + \hat{\Xi}_{9'''}^{(d)} M'''_9 + \dots \right) \end{aligned} \quad (5.7)$$

Apart from the terms with coefficients $\Xi_{\dots}^{(d)}$, which arose in the U(1)-conserving amplitudes (5.2) and (5.5), novel linear combinations, $\hat{\Xi}_{\dots}^{(d)}$, of multiple sums appear (which are different from $\Xi_{7'}^{(d)}, \dots, \Xi_{9'''}^{(d)}$ of the $q = 0$ case in (5.6)), which accompany the M'_k, M''_k and

M_9''' matrices,

$$\begin{aligned}
 \hat{\Xi}_{7'}^{(d)} &:= \pi \int d\mu_d(\tau) \left(\frac{7}{48} D_4 - \frac{251}{16} D_2^2 + 18 D_{211} + \frac{207}{8} D_{1111} \right) \\
 \hat{\Xi}_{8'}^{(d)} &:= \pi \int d\mu_d(\tau) \left(\frac{107}{576} D_5 - \frac{479}{180} D_{311} + \frac{767}{240} D_{221} - \frac{767}{240} D_{2111} + \frac{817}{120} D'_{1111} + \frac{313}{30} D_{11111} \right. \\
 &\quad \left. + \frac{6}{5} D_{111} D_2 - \frac{4067}{1440} D_3 D_2 - \frac{5}{6} D''_{1111} - \frac{5}{24} D''_{222} \right) \\
 \hat{\Xi}_{9'}^{(d)} &:= -2\pi \int d\mu_d(\tau) \left(-\frac{4847}{116640} D_6 - 3 D_{111}^2 + \frac{2}{3} D_{11,11,11} + 3 D_{21111} - 10 D_{211} D_2 \right. \\
 &\quad + \frac{2579}{1296} D_{2211} - \frac{1}{6} D_{222} + \frac{7195}{1296} D_2^3 - \frac{6143}{972} D_{3111} - \frac{3}{2} D_{321} + \frac{7}{6} D_3 D_{111} + \frac{2903}{11664} D_3^2 \\
 &\quad - \frac{1}{2} D_{411} + \frac{2605}{2592} D_4 D_2 - 6 D'_{11111} + 7 D'_{2111} - D''_{1111} - \frac{7}{3} D_{1111}^\times + 2 D''_{11111} \\
 &\quad \left. + \frac{1}{6} D_{322}^\partial + 4 D_{2211}^\partial - 6 D_{2111}'' + 4 D_{1111}''' - 2 D_{2111}^{\times\partial} - 2 D_{11111}^{\wedge\partial} \right) \quad (5.8) \\
 \hat{\Xi}_{9''}^{(d)} &:= -2\pi \int d\mu_d(\tau) \left(\frac{6073}{466560} D_6 + 3 D_{1111} D_2 + \frac{3}{2} D_{111}^2 + \frac{1}{3} D_{11,11,11} + 3 D_{21111} - \frac{9}{4} D_{211} D_2 \right. \\
 &\quad - \frac{7}{24} D_{222} + \frac{2351}{5184} D_{2211} + \frac{8665}{5184} D_2^3 - \frac{8831}{3888} D_{3111} - \frac{1}{6} D_3 D_{111} - \frac{2185}{46656} D_3^2 - 6 D'_{11111} \\
 &\quad + \frac{457}{10368} D_4 D_2 - \frac{1}{6} D_{321} + \frac{5}{24} D_{411} + 4 D'_{2111} - D''_{1111} - \frac{4}{3} D_{1111}^\times - 2 D_{2111}'' + D_{1111}''' \\
 &\quad \left. - D_{2111}^{\times\partial} - 2 D_{11111}^{\wedge\partial} \right) \\
 \hat{\Xi}_{9'''}^{(d)} &:= -2\pi \int d\mu_d(\tau) \left(-\frac{7}{270} D_6 - 12 D_{1111} D_2 + 4 D_{11,11,11} - \frac{11}{3} D_{2211} - \frac{1}{3} D_2^3 + \frac{44}{9} D_{3111} \right. \\
 &\quad \left. + \frac{7}{27} D_3^2 + \frac{5}{6} D_4 D_2 \right) .
 \end{aligned}$$

The coefficients of terms in the expression (5.7) have a striking pattern when compared with the coefficients of terms in the U(1)-conserving sector. In particular, the coefficient along with (products of) unprimed matrices M_k that arise at order $(\alpha')^{w+3}$ has a factor

$$\frac{w-1}{w+3} \quad (5.9)$$

in (5.7) relative to the corresponding coefficient in (5.5). This simple pattern appears to fit in well with considerations based on $\text{SL}(2, \mathbb{Z})$, the $D=10$ type IIB S-duality group, as we will argue at the end of the following section.

5.4 S-duality of U(1)-conserving amplitudes

Section 2.4 reviewed the manner in which the moduli dependence of perturbative contributions to the terms in the low energy expansion of the ten-dimensional type IIB four-particle amplitude fit into duality-invariant functions, $\mathcal{E}_{\dots}(\Omega)$. In particular, we presented explicit expressions for the low dimension terms, $\mathcal{E}_3 R^4$, $\mathcal{E}_5 D^4 R^4$ and $\mathcal{E}_{3,3} D^6 R^4$. Here we would like to see to what extent combining information from the $N=5$ loop amplitudes with previous results on tree amplitudes and $N=4$ loop amplitudes may extend our understanding of the non-perturbative structure of amplitudes in the type IIB theory.

In order to characterise the N -particle effective action, we need some notation that distinguishes the distinct invariants that are contained in interactions such as $D^{2w}R^4$ and $D^{2w-2}R^5$ when $w > 3$. We will therefore introduce the following shorthand notation:

- Terms in the order $(\alpha')^{w+3}$ N -graviton amplitude are generated by effective interactions of the form $D^{2w-2\ell}R^{4+\ell}$ with $\ell = 0, 1, \dots, w$. The family of interactions which gives rise to a product of matrices $M_{p_1}M_{p_2}\dots$ of order $(\alpha')^{w+3}$ in the amplitude (with $w = \sum_j p_j - 3$) is denoted by $(\bigoplus_{\ell=0}^w D^{2w-2\ell}R^{4+\ell})_{p_1,p_2,\dots}$ with subscripts p_1, p_2, \dots referring to the matrices.
- More generally, $(D^{2w}R^4 \oplus \dots)_{[p_1,p_2]\dots}$ or $(D^{2w}R^4 \oplus \dots)_{\{p_1,p_2\}\dots}$ indicate kinematic structures associated with (anti-)commutators of matrices $[M_{p_1}, M_{p_2}], \dots$ or $\{M_{p_1}, M_{p_2}\}, \dots$, respectively.
- A quantity such as $(D^{2w-2}R^5 \oplus \dots)_{p'_1,p'_2,\dots}$ is associated with primed matrices, $M'_{p_1}M'_{p_2}\dots$.

The results for U(1)-conserving five-particle amplitudes described above suggest that the same $\mathcal{E}\dots$ defined at the four-point level by (2.20) accompany the combinations $(D^{2w}R^4 \oplus D^{2w-2}R^5)_{p_1,\dots}$ of appropriate dimensions, at least to the orders $2p + 3q \leq 6$ investigated here. Our incomplete knowledge of one-loop $N \geq 6$ point amplitudes makes the analogous $D^{2w-4}R^6, D^{2w-6}R^7, \dots$ inaccessible to the present analysis.

The absence of M'_7, M'_8, M'_9 and M''_9 in tree-level amplitudes implies that they arise with new SL(2, \mathbb{Z})-invariant coefficients to be denoted by $\mathcal{E}_{w'}$ and $\mathcal{E}_{9''}$ in the following. These must contain perturbative terms that begin at one loop, and their one-loop contribution must have a coefficient that is given by the integrated multiple sums, $\Xi_{w'}^{(d)}$ and $\Xi_{9''}^{(d)}$, in (5.6).

Since the matrices M'_7, M'_8, M'_9 and M''_9 do not contribute to collinear limits,²⁸ $\Xi_{7'}^{(d)}, \Xi_{8'}^{(d)}, \Xi_{9'}^{(d)}$ and $\Xi_{9''}^{(d)}$ are associated with local five-field operators of dimension $(D^6R^5)_{7'}$, $(D^8R^5)_{8'}$, $(D^{10}R^5)_{9'}$ and $(D^{10}R^5)_{9''}$, respectively.

In the following we will restrict our discussion to the ten-dimensional case with S-duality group SL(2, \mathbb{Z}) although this should extend to the higher-rank groups associated with compactification on a d -torus. The low-energy expansion of the exact SL(2, \mathbb{Z})-invariant amplitudes for $N \leq 5$ must include the perturbative terms corresponding to the tree-level and analytic parts of the one-loop amplitudes described above. This suggests that these terms should be incorporated into an SL(2, \mathbb{Z})-invariant effective action in

²⁸The absence of poles in open string-like expressions $M'_w \tilde{A}_{YM}$ implies locality of $A_{YM}^t S_0 M'_w \tilde{A}_{YM}$ for the following reason: The latter object is totally symmetric (thanks to the field theory monodromy relations preserved by M'_w) but it can only have the poles of A_{YM}^t . Since some of the pole channels (such as s_{15}^{-1}, s_{24}^{-1} and s_{34}^{-1} in the five-particle case) are absent in $A_{YM}^t S_0$, the other ones which superficially occur in A_{YM}^t or S_0 must also be cancelled due to the total symmetry of $A_{YM}^t S_0 M'_w \tilde{A}_{YM}$.

M_w 's	Interactions	Tree	1-loop	Coefficient
1	R	1	0	1
M_3	$(R^4)_3$	$2\zeta_3$	$\Xi_3^{(0)}$	\mathcal{E}_3
M_5	$(D^4 R^4 \oplus D^2 R^5)_5$	$2\zeta_5$	$\Xi_5^{(0)}$	\mathcal{E}_5
M_3^2	$(D^6 R^4 \oplus D^4 R^5)_{3,3}$	$2\zeta_3^2$	$\Xi_{3,3}^{(0)}$	$\mathcal{E}_{3,3}$
M_7	$(D^8 R^4 \oplus D^6 R^5)_7$	$2\zeta_7$	$\Xi_7^{(0)}$	\mathcal{E}_7
M_7'	$(D^6 R^5)_{7'}$	0	$\Xi_{7'}^{(0)}$	$\mathcal{E}_{7'}$
$\{M_3, M_5\}$	$(D^{10} R^4 \oplus D^8 R^5)_{\{3,5\}}$	$2\zeta_3 \zeta_5$	$\Xi_{\{3,5\}}^{(0)}$	$\mathcal{E}_{\{3,5\}}$
M_8'	$(D^8 R^5)_{8'}$	0	$\Xi_{8'}^{(0)}$	$\mathcal{E}_{8'}$
M_9	$(D^{12} R^4 \oplus D^{10} R^5)_9$	$2\zeta_9$	$\Xi_9^{(0)}$	\mathcal{E}_9
M_3^3	$(D^{12} R^4 \oplus D^{10} R^5)_{3,3,3}$	$\frac{4}{3}\zeta_3^3$	$\Xi_{3,3,3}^{(0)}$	$\mathcal{E}_{3,3,3}$
M_9'	$(D^{10} R^5)_{9'}$	0	$\Xi_{9'}^{(0)}$	$\mathcal{E}_{9'}$
M_9''	$(D^{10} R^5)_{9''}$	0	$\Xi_{9''}^{(0)}$	$\mathcal{E}_{9''}$

Table 5. Tree-level and one-loop contributions to U(1)-conserving terms in the $D = 10$ low energy effective action of the form $D^{2k} R^5$, which can be extracted from the five-graviton components of the superamplitudes expanded in (2.9) and (5.5).

Einstein frame of the schematic form,

$$\begin{aligned}
 S_{\text{eff}}^{q=0} \Big|_{\text{local}} = & \int d^{10}x \sqrt{-g} \Big(R + \mathcal{E}_3(R^4)_3 + \mathcal{E}_5(D^4 R^4 + D^2 R^5)_5 + \mathcal{E}_{3,3}(D^6 R^4 + D^4 R^5)_{3,3} \\
 & + \mathcal{E}_7(D^8 R^4 + D^6 R^5)_7 + \mathcal{E}_{7'}(D^6 R^5)_{7'} + \mathcal{E}_{\{3,5\}}(D^{10} R^4 + D^8 R^5)_{\{3,5\}} + \mathcal{E}_{8'}(D^8 R^5)_{8'} \\
 & + \mathcal{E}_9(D^{12} R^4 + D^{10} R^5)_9 + \mathcal{E}_{3,3,3}(D^{12} R^4 + D^{10} R^5)_{3,3,3} \\
 & + \mathcal{E}_{9'}(D^{10} R^5)_{9'} + \mathcal{E}_{9''}(D^{10} R^5)_{9''} + \mathcal{O}(\alpha'^{10}) \Big), \tag{5.10}
 \end{aligned}$$

with g denoting the determinant of the space-time metric. The coefficients $\mathcal{E}_{\dots}(\Omega)$ and $\mathcal{E}_{w'}(\Omega)$ and $\mathcal{E}_{9''}(\Omega)$ are $\text{SL}(2, \mathbb{Z})$ -invariant functions that have perturbative expansions that contain the required tree-level MZV 's and one-loop $\Xi_{\dots}^{(0)}$ contributions. There is strong evidence that the first three of these functions, $\mathcal{E}_3(\Omega)$, $\mathcal{E}_5(\Omega)$ and $\mathcal{E}_{3,3}(\Omega)$ have the form reviewed in section 2.4, which was based on the $N = 4$ case. The purpose of (5.10) is to indicate the general pattern of modular functions that are expected to arise, based on the tree-level and one-loop results.²⁹ Although we do not know the form of the modular functions \mathcal{E}_{\dots} (that arose in the $N = 4$ case) and the new ones, $\mathcal{E}_{w'}(\Omega)$ and $\mathcal{E}_{9''}(\Omega)$, we now know that they must contain the MZV s coefficients of tree-level powers of Ω_2 (2.9) and the lattice sums in the one-loop powers of Ω_2 (5.5).

For clarity, the coefficients of the α' expansion of the U(1)-conserving 4-particle and 5-particle tree-level and one-loop amplitudes in (5.10) are summarised in table 5. The second column of the table indicates the combinations of $D^{2w} R^4$ and $D^{2w-2} R^5$ interactions that

²⁹Of course, the full amplitude also includes the host of terms that are non-analytic in the Mandelstam invariants that we have been ignoring, which should appear as complicated non-local terms in the effective action.

have the same coefficients.³⁰ The precise structure of the kinematic invariants associated with the interactions is defined by the polynomials of M_k associated with them, which are listed in the first column. The tree and one-loop coefficients are listed in the third and fourth columns, respectively. The last column of the table indicates how the coefficients of these perturbative terms fit into $SL(2, \mathbb{Z})$ -invariant functions, \mathcal{E}_{\dots} .

It is notable that the ratio of one-loop to tree-level contributions to the $D^4 R^4$ and $D^2 R^5$ interactions are identical, which fits in with the fact that they are in the same supermultiplet. They should therefore both be associated with the same modular function, \mathcal{E}_5 . Similarly, the ratio of tree-level and one-loop contributions to $D^6 R^4$ and $D^4 R^5$ are identical and they should be contained in the same modular invariant function, $\mathcal{E}_{3,3}$. More generally, interactions of 5-particle interactions of the form $D^{2w-2} R^5$ with $w > 3$ have more kinematic invariants than the corresponding 4-particle interaction, $D^{2w} R^4$. However, as shown in the second column of the table, for each w there is at least one five-point invariant that pairs with a corresponding 4-particle invariant and these particular invariants are presumably related by supersymmetry. In that case they should also be associated with a single modular function \mathcal{E}_{\dots} . The five-particle invariants listed in column 2 that are not paired with corresponding 4-particle interactions are ones that do not have tree-level contributions. They must therefore be contained in distinct modular functions $\mathcal{E}_{w'}$ and $\mathcal{E}_{9''}$ in which the one-loop term is the leading perturbative contribution.

5.5 S-duality of U(1)-violating type IIB amplitudes

Analogous considerations apply to the terms in the α' expansion of the U(1)-violating amplitude discussed in section 5.3 where the one-loop coefficients were presented in (5.7). Properties of the one-loop coefficients of U(1)-violating five-particle interactions of the form $D^{2w} G^2 R^3$ together with their tree-level counterparts are listed in table 6.³¹ Again these are particular examples of the complete set of interactions that are related by supersymmetry. In this case the degeneracy of kinematic invariants of a given weight grows faster than in the U(1)-conserving case. For example, there are 5 invariants at order $D^{12} G^2 R^3$, which accounts for the coefficient 5 in the entry in the second row and last column of table 3. In the last column of table 6 we have speculated as to how these perturbative terms might be incorporated into $(1, -1)$ modular forms, analogous to $E_{3/2}^{(1, -1)}(\Omega)$ defined in (2.30) in the context of the terms of order $G^2 R^3$. In particular, for those interactions that are partners of corresponding U(1)-conserving four-particle interactions (i.e., that have the same structure of the unprimed M_k matrices) it is tempting to make the ansatz that the $(1, -1)$ modular form is the one obtained by applying a covariant derivative (2.29) to the corresponding modular function, \mathcal{E}_{\dots} , so it has the form $\mathcal{D}\mathcal{E}_{\dots}$.

³⁰These are particular components of the complete set of interactions that result from the pure spinor construction, which naturally produces a supersymmetric expression that contains all interactions that are related by supersymmetry.

³¹Similar to the U(1) preserving cases (see early section 5.4), we use the notation $(D^{2w} G^2 R^3)_{p_1, p_2, \dots}$ and $(D^{2w} G^2 R^3)_{p'_1, p'_2, \dots}$ to specify interactions related to $M_{p_1} M_{p_2} \dots$ and $M'_{p_1} M'_{p_2} \dots$ terms in the amplitude (5.7).

M_w 's	Interactions	Tree	1-loop	$(1, -1)$ modular form
1	\times	0	0	0
M_3	$(G^2 R^3)_3$	$2\zeta_3$	$-\frac{1}{3}\Xi_3^{(0)}$	\mathcal{DE}_3
M_5	$(D^4 G^2 R^3)_5$	$2\zeta_5$	$\frac{1}{5}\Xi_5^{(0)}$	\mathcal{DE}_5
M_3^2	$(D^6 G^2 R^3)_{3,3}$	$2\zeta_3^2$	$\frac{1}{3}\Xi_{3,3}^{(0)}$	$\mathcal{DE}_{3,3}$
M_7	$(D^8 G^2 R^3)_7$	$2\zeta_7$	$\frac{3}{7}\Xi_7^{(0)}$	\mathcal{DE}_7
M_7'	$(D^8 G^2 R^3)_{7'}$	0	$\hat{\Xi}_{7'}^{(0)}$	$\mathcal{E}_{7'}^+$
$\{M_3, M_5\}$	$(D^{10} G^2 R^3)_{\{3,5\}}$	$2\zeta_3\zeta_5$	$\frac{1}{2}\Xi_{\{3,5\}}^{(0)}$	$\mathcal{DE}_{\{3,5\}}$
M_8'	$(D^{10} G^2 R^3)_{8'}$	0	$\hat{\Xi}_{8'}^{(0)}$	$\mathcal{E}_{8'}^+$
M_9	$(D^{12} G^2 R^3)_9$	$2\zeta_9$	$\frac{5}{9}\Xi_9^{(0)}$	\mathcal{DE}_9
M_3^3	$(D^{12} G^2 R^3)_{3,3,3}$	$\frac{4}{3}\zeta_3^3$	$\frac{5}{9}\Xi_{3,3,3}^{(0)}$	$\mathcal{DE}_{3,3,3}$
M_9'	$(D^{12} G^2 R^3)_{9'}$	0	$\hat{\Xi}_{9'}^{(0)}$	$\mathcal{E}_{9'}^+$
M_9''	$(D^{12} G^2 R^3)_{9''}$	0	$\hat{\Xi}_{9''}^{(0)}$	$\mathcal{E}_{9''}^+$
M_9'''	$(D^{12} G^2 R^3)_{9'''}$	0	$\hat{\Xi}_{9'''}^{(0)}$	$\mathcal{E}_{9'''}^+$

Table 6. Tree-level and one-loop contributions to the coefficients of interactions of the form $D^{2k} G^2 R^3$, which can be extracted from the α' expansion of the U(1)-violating five-particle amplitudes given in (2.9) and (5.7).

This ansatz relates the terms that are power-behaved in Ω_2 that contribute to the constant terms (the zero modes with respect to Ω_1) of the coefficient functions in the expansion of the U(1)-violating ($q = -2$) amplitude to those of the U(1)-conserving ($q = 0$) amplitude, as follows. The zero Fourier mode of a coefficient \mathcal{E}_{\dots} of a U(1)-conserving interaction $D^{2w} R^4$, with $w = 2p + 3q$, has power behaved terms given by

$$\mathcal{F}_0^w =: a_w \Omega_2^{(w+3)/2} + b_w \Omega_2^{(w-1)/2} + O(\Omega_2^{(w-5)/2}). \quad (5.11)$$

The zero mode of the corresponding $(-1, 1)$ form, \mathcal{DE}_{\dots} , is given by

$$\begin{aligned} \mathcal{DF}_0^w &= a_w \frac{3+w}{2} \Omega_2^{(3+w)/2} + b_w \frac{w-1}{2} \Omega_2^{(w-1)/2} \\ &=: A_w \Omega_2^{(3+w)/2} + B_w \Omega_2^{(w-1)/2} + O(\Omega_2^{(w-5)/2}). \end{aligned} \quad (5.12)$$

We see that the ratio of tree-level and one-loop amplitudes for the $q = -2$ process is related to that of the $q = 0$ processes by

$$\frac{B_w/A_w}{b_w/a_w} = \frac{w-1}{w+3}. \quad (5.13)$$

This provides a natural explanation for the result (5.9) that we observed in the five-particle amplitudes (5.5) and (5.7). In fact, this also accounts for the factor of $-\frac{1}{3}$ in the kinematic identity (3.35).

The kinematic structures that arise at one loop but have no tree-level partners are associated with matrices M_7' , M_8' , M_9' , M_9'' and M_9''' listed in column 1 of table 6. Their

coefficients are $(1, -1)$ forms, denoted by $\mathcal{E}_{w'}^+$, $\mathcal{E}_{w''}^+$ and $\mathcal{E}_{w'''}^+$ must have zero modes that contain the multiple sums (5.8), but they do not have any obvious connection with the $\mathcal{E}_{w'}$ and $\mathcal{E}_{g''}$ coefficients of the expansion of the $U(1)$ -conserving amplitude that arise in (5.10).

Finally, we note that with the above ansatz, we can write a schematic form for the local part of the $U(1)$ -violating effective action for the low energy expansion of the five-particle amplitude up to order $(\alpha')^9$ in the form

$$\begin{aligned}
 S_{\text{eff}}^{q=2} \Big|_{\text{local}} = & \int d^{10}x \sqrt{-g} \Big(\mathcal{D}\mathcal{E}_3(G^2 R^3)_3 + \mathcal{D}\mathcal{E}_5(D^4 G^2 R^3)_5 + \mathcal{D}\mathcal{E}_{3,3}(D^6 G^2 R^3)_{3,3} \\
 & + \mathcal{D}\mathcal{E}_7(D^8 G^2 R^3)_7 + \mathcal{E}_{7'}^+(D^8 G^2 R^3)_{7'} + \mathcal{D}\mathcal{E}_{\{3,5\}}(D^{10} G^2 R^3)_{\{3,5\}} + \mathcal{E}_{8'}^+(D^{10} G^2 R^3)_{8'} \\
 & + \mathcal{D}\mathcal{E}_9(D^{12} G^2 R^3)_9 + \mathcal{D}\mathcal{E}_{3,3,3}(D^{12} G^2 R^3)_{3,3,3} + \mathcal{E}_{9'}^+(D^{12} G^2 R^3)_{9'} \\
 & + \mathcal{E}_{9''}^+(D^{12} G^2 R^3)_{9''} + \mathcal{E}_{9'''}^+(D^{12} G^2 R^3)_{9'''} + \mathcal{O}(\alpha'^{10}) \Big). \quad (5.14)
 \end{aligned}$$

6 A glimpse of the six-particle amplitude

The CFT calculations of section 3 and the set of integrals appearing in the five-particle amplitude (3.33) provide a convenient starting point for developing the structure of the integrand at higher multiplicity. According to [1], the N point open-string correlator at one loop is built from $N - 4$ powers of propagator derivatives X_{ij} as defined in (3.4), and the closed-string integrand augments its holomorphic square by interactions between left- and right-movers. Sections 3.2 and 3.3 give two examples in which such interactions stem either from integration by parts or from the OPE involving $\Pi^m \bar{\Pi}^n$ fields from opposite sectors as in (3.5). Both cases introduce a factor of Ω given in (3.6) into the integrand instead of a product $X_{ij} \tilde{X}_{kl}$ due to separate OPEs within the left- and right moving sectors.

In the prescription (3.1) for the six closed-string correlator on the torus, zero mode saturation is compatible with up to two contractions between left- and right-movers. As a consequence, the six-point integrand involves three classes of z dependencies:

$$\begin{aligned}
 \mathcal{M}_{1\text{-loop}}^6 \sim & \int d\mu_d(\tau) \tau_2^{-3} \int d^2 z_2 \dots d^2 z_6 \mathcal{I}(s_{ij}) \\
 & \times \left\{ X_{ij} X_{kl} \tilde{X}_{pq} \tilde{X}_{rs} K_{ij,kl,pq,rs} + \Omega X_{ij} \tilde{X}_{pq} K_{ij,pq}^\Omega + \Omega^2 K^{\Omega^2} \right\}, \quad (6.1)
 \end{aligned}$$

where the kinematic factor $K_{ij,kl,pq,rs}$ has the dimension of R^6/k^8 (where k is a momentum), $K_{ij,pq}^\Omega$ has the dimension of R^6/k^6 and K^{Ω^2} has the dimension of R^6/k^4 . A careful evaluation of these kinematic factors in superspace is left for future work [74]. In this case the presence of anomalies (associated with BRST non-invariant kinematic factors) in the open-string six-particle loop amplitude leads to delicate issues in building the non-anomalous closed-string amplitude. However, as we will argue in the following, we can still extract general statements on the six-particle low energy effective action from the classification (6.1) of contributing world-sheet integrals, even without precise knowledge of the kinematic factors. The analytic α' dependence of (6.1) is sufficient to derive the selection rules summarised in table 1.

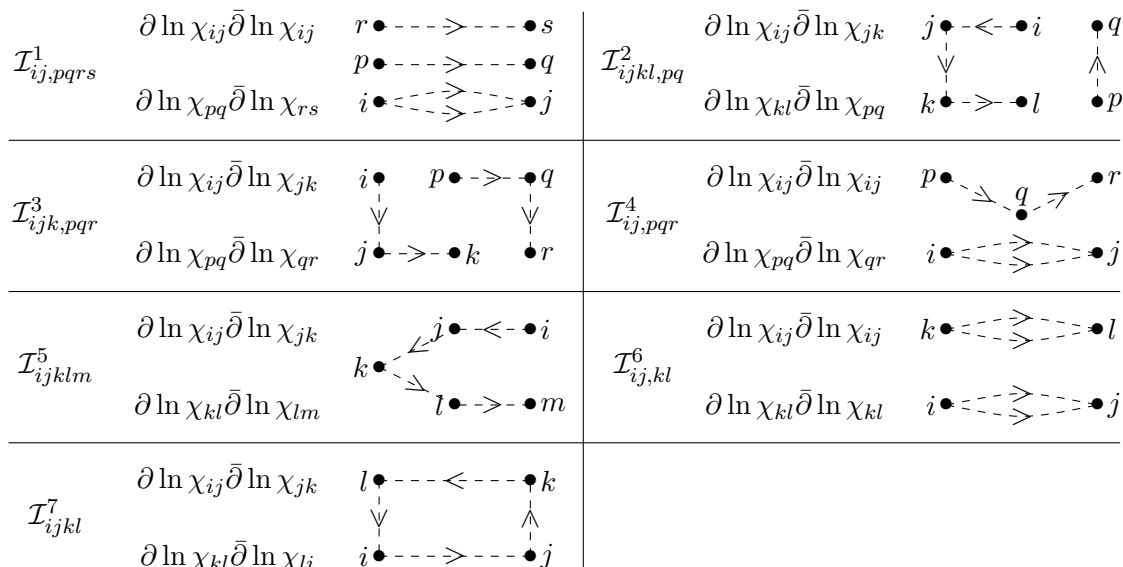


Figure 4. Possible topologies of six-particle integrals with four $\ln \chi$ derivatives where both the $\partial \ln \chi_{ij}$'s and the $\tilde{\partial} \ln \chi_{ij}$'s carry four different labels. The integration measure for the above expressions can be found in the first line of (6.1). A distinction between ∂ and $\tilde{\partial}$ is not needed since dashed lines with alike derivatives never end on the same vertex

First of all, the momentum expansion of the Ω^2 and $\Omega X_{ij} \tilde{X}_{pq}$ integrals can be almost literally³² inferred from the results in (4.15) and appendix D on five-particle integrals over Ω and $X_{ij} \tilde{X}_{pq}$, respectively. Up to weight $w = 4$, they are shown to introduce no extra multiple sums beyond $D_0, D_2, D_{111}, D_3, D_{1111}, D_{211}, D_2^2$ and D_4 that are known from the four-particle amplitude (see section 4.1). The only potential source of new multiple sums are the integrands of schematic form $X_{ij} X_{kl} \tilde{X}_{pq} \tilde{X}_{rs}$ due to the product of left- and right moving correlators.

The six-point open-string correlator computed in [1] is characterised by two topologies of BRST invariants and accompanying X_{ij} bilinears: The final expression for its integrand comprises 20 permutations of $X_{23}(X_{24} + X_{34})$ and 15 permutations of $X_{23}X_{45}$ with respect to integrated labels 2, 3, 4, 5, 6. Integration by parts can be used to eliminate the former topology via $X_{23}(X_{24} + X_{34}) = X_{23}(X_{41} + X_{45} + X_{46})$, possibly at the expense of introducing Ω admixtures from the right-movers, see (3.14). Hence, it is sufficient to expand elementary integrals $X_{23}X_{45}\tilde{X}_{pq}\tilde{X}_{rs}$ (where all the labels $\{p, q, r, s\}$ are pairwise distinct) to understand the low energy behaviour of the six closed-string integrals. Within this topology, there are seven inequivalent ways of arranging the right moving labels $\{p, q, r, s\}$ relative to the left moving ones $\{2, 3, 4, 5\}$, which can be conveniently visualised through the diagrams in figure 4 (which generalise the five-particle specific figure 1).

The integrals $\mathcal{I}_{ij,pqrs}^1, \mathcal{I}_{ij,pqr}^4$ and $\mathcal{I}_{ij,pq}^7$ defined in figure 4 have kinematic poles with residues given by lower point amplitudes along the lines of (4.12) and (4.13) (see ap-

³²Starting from weight $w = 5$, the possibility of having six vertex diagrams introduces extra terms into the six-point version of $\Omega X_{ij} \tilde{X}_{va}$ integrals which cannot appear in the five-particle setting.

pendix E.1 for details). The intrinsic six-particle information stems from regular parts \mathcal{I}^{reg} where the Koba-Nielsen factor is Taylor expanded before performing the z_i integration. Using the diagrammatic methods introduced in sections 4.2 and 4.3, we have expanded the seven inequivalent \mathcal{I}^{reg} functions up to weight four, the results are displayed in appendix E.2.

As shown in appendix E.2, the 1PI diagrams $D_0, D_2, D_{111}, D_3, D_{1111}, D_{211}, D_2^2$ and D_4 are sufficient to express all the regular parts of the integrals in figure 4 up to weight $w \leq 4$. The fact that there are no further diagrams, apart from those that contributed to the four- and five-particle amplitudes that contribute to the six-particle function up to this weight imposes upper bounds 1, 2 and 4 on the number of independent $R^6, D^2 R^6$ and $D^4 R^6$ interactions, respectively, see table 1. In particular, in view of the vanishing integrated multiple sums at weight $w = 2, 4$ in $D = 10$ — see (5.4) — the R^6 and $D^4 R^6$ interactions must be absent in ten space-time dimensions. If the kinematic factors along with D_{111}, D_3 as well as D_{1111}, D_{211}, D_2^2 and D_4 satisfy linear relations as they do for the five-particle closed-string amplitudes (5.5) and (5.7), then there could be less such operators, and the numbers could possibly match with the five-field operators in table 1 with the same mass dimension. Settling these questions is one motivation for detailed evaluation of N -point amplitudes in superspace with $N > 5$ [74].

7 Summary and comments

In this paper we have investigated one-loop amplitudes for the scattering of five (and to some extent, six) massless superparticles in type IIA and type IIB closed superstring theories. The world-sheet integrand (3.33) was computed in pure spinor superspace and expressed in terms of a minimal set of functions. The structure of the type IIB amplitude shown in (1.5),

$$\mathcal{M}_{1\text{-loop}} = A_{YM}^t \mathcal{S}_{1\text{-loop}}(s_{ij}) \tilde{A}_{YM}, \quad (7.1)$$

has the form of a bilinear in YM tree amplitudes contracted with a matrix function of the Mandelstam invariants, just as for the closed-string tree amplitudes sketched in (1.2). The type IIA five-particle amplitude has extra pieces that do not have this form, as discussed in section 3.6. One of these is a parity-conserving piece that arises from the product of two ϵ tensors contracted on at least one index. The second is a parity-violating component that was also constructed in section 3.6 and includes the familiar BR^4 component interaction.

The one-loop expression, $\mathcal{S}_{1\text{-loop}}(s_{ij})$, is defined in terms of integrals over a world-sheet torus. Their contributions to local terms in the low energy effective action involving four or five powers of the (supersymmetrised) curvatures were obtained up to order $(\alpha')^9$ and compared with analogous tree-level expressions. In fact, all of the kinematic invariants that appear in the α' expansion of the tree amplitude up to order $(\alpha')^9$ were found to reappear in the one-loop N -particle amplitude with $N = 4$ and $N = 5$.

Whereas at tree-level the coefficients of the terms in the α' expansion are (rationals multiplying) MZVs, the coefficients, $\Xi_w^{(d)}$, of the α' expansion of the one-loop amplitudes are rational combinations of integrated multiple sums $D_{\dots}(\tau)$. We systematically reduced

the weight $w \leq 6$ multiple sums in the five-particle amplitude to a tentative basis. This has the flavour of a higher-genus generalisation of the \mathbb{Q} basis reduction of MZVs.

Some implications for S-duality ($\text{SL}(2, \mathbb{Z})$ covariance) of the $D = 10$ type IIB theory were considered in section 5. One important fact that emerged from the five-particle amplitude is that for every tree-level interaction of the form $D^{2w-2}R^5$ associated with a particular combination of M_i matrices, there is a corresponding one-loop counterpart. Moreover, the ratio of the coefficients of these five-curvature terms is precisely the same as the ratio of tree-level to one-loop coefficients of the $D^{2w}R^4$ interaction that was extracted from the four-particle amplitude. This strongly suggests that these terms arise in the combination $\mathcal{E}_{l_1, l_2, \dots}(\Omega)(D^{2w}R^4 \oplus D^{2w-2}R^5)$, with $\sum_i l_i = w + 3$ in the effective action, where $\mathcal{E}_{l_1, l_2, \dots}(\Omega)$ is a common modular invariant coefficient. Furthermore, the presence of new kinematic invariants at one-loop that are absent at tree-level requires modular invariant coefficients that have no genus-zero contributions. This concerns order $(\alpha')^7 D^6 R^5$ and higher.

We also considered tree-level and one-loop features of the modular forms associated with U(1)-violating interactions, such as $D^{2w}G^2R^3$. In particular, the coefficients in the U(1)-violating and U(1)-conserving processes shown in (5.7) and (5.5) fit perfectly with the expected pattern, as described in (5.11)–(5.13).

In order to get a better understanding of the S-duality systematics of type IIB amplitudes it would be interesting to extend our knowledge of kinematic factors and one-loop world-sheet integrals to higher multiplicity and higher orders in α' . For example, we have not yet checked whether the one-loop coefficients of the tree-level matrices M_{2k+1} (and products thereof) are truly independent of the number N of external legs for $N > 5$ i.e. whether the full families $\{D^{2k-2l}R^{N+l}, l = 0, 1, \dots, k\}$ of interactions are really accompanied by the same modular function in the exact type IIB effective action. Multiparticle amplitudes also allow for a broader range of U(1) charges and therefore provide further information for pinning down the modular forms of non-zero weight in the effective action. Higher powers in α' are expected to enter via new matrices $M'_{w>9}$ but also via products of various lower order matrices present in (5.5). It would be interesting to explore the matrix multiplicative patterns at orders beyond $(\alpha')^9$.

On the other hand, even up to weight $w \leq 6$ the coefficients (5.3), (5.6) and (5.8) in our expressions point to hidden systematics underlying the multiple sums $D \dots$. Their unwieldy rational prefactors are reminiscent of the α' expansion of the tree amplitude [6] where the MZVs have not yet been mapped to the more transparent alphabet of noncommutative generators f_i [45]. Our results suggest that there is a more natural language to describe and arrange the multiple sums, and one might speculate whether further hidden structures become visible after performing the modular τ integrals. Finally, in view of the all orders result in α' obtained for open-string trees from the Drinfeld associator [27] (extending the work of [51]), it would be desirable to develop a unified description of α' corrections to higher genus string amplitudes. In the same way as the Drinfeld associator encodes the universal monodromy of the Knizhnik-Zamolodchikov equation governing correlators on the disk and sphere, one might envision a reduction of loop amplitudes to universal monodromies associated with the Riemann surface in question.

Finally, it is important to stress that we have concentrated entirely on analytic con-

tributions to the low energy expansion of the amplitude, but it is important to develop a better understanding of the interplay between these terms and the non-analytic contributions (discussed to a limited extent in [59]), which are crucial in understanding the nonlocal structure of the quantum effective action.

Acknowledgments

We are very grateful to Francis Brown and Don Zagier, as well as to Herbert Gangl and the other organisers of the Newton Institute workshop on “Grothendieck-Teichmueller Groups, Deformation and Operads”, for many mathematical insights. OS is grateful to Johannes Brödel and Stephan Stieberger for collaboration on related topics. The research leading to these results has received funding from the European Research Council under the European Community’s Seventh Framework Programme (FP7/2007-2013) / ERC grant agreement no. [247252].

A Comparison of conventions

The notation for the $N = 4$ case in section 2.4 differs somewhat from earlier conventions in the literature. This change in convention is necessary in order to uniformly describe higher multiplicities with $N \geq 4$. We will here review the correspondence between these different conventions.

In [35] the expansion of the analytic part of the amplitude was written in the form (in Einstein frame)

$$\begin{aligned} T(s_{ij}; \Omega) \Big|_{\text{analytic}} &= \sum_{p=0, q=-1}^{\infty} \mathcal{E}_{(p,q)}(\Omega) \sigma_2^p \sigma_3^q \\ &= \frac{3}{\sigma_3} + \mathcal{E}_{(0,0)} + \mathcal{E}_{(1,0)} \sigma_2 + \mathcal{E}_{(0,1)} \sigma_3 + \mathcal{O}(\alpha'^7), \end{aligned} \quad (\text{A.1})$$

where the dependence on the Mandelstam variables $s := s_{12} = s_{34}$, $t := s_{14} = s_{23}$ and $u := s_{13} = s_{24}$ (with $s + t + u = 0$), is contained in powers of $\sigma_2 = (s^2 + t^2 + u^2)$ and $\sigma_3 = (s^3 + t^3 + u^3)$. In order to make contact with the notation in (2.20) note that [36]

$$M_w = -\frac{s^w + t^w + u^w}{w} = -\sum_{2p+3q=w} \frac{(p+q-1)!}{p!q!} \left(\frac{\sigma_2}{2}\right)^p \left(\frac{\sigma_3}{3}\right)^q. \quad (\text{A.2})$$

The duality-invariant coefficients $\mathcal{E}_{(p,q)}(\Omega)$ in (A.1) are linear combinations of those $\mathcal{E}_{l_1, l_2, \dots}$ in (2.20) with $\sum_i l_i = 2p + 3q + 3$. The tree-level supergravity amplitude is given by the $p = 0, q = -1$ term with coefficient $\mathcal{E}_{(0,-1)}(\Omega) = 3$.

The representation (A.1) of the four-particle amplitude makes use of the fact that any symmetric polynomial of the four-point Mandelstam invariants can be written as a product of the form $\sigma_2^p \sigma_3^q$. The kinematic combinations involving four fields are unique below order $(\alpha')^9 D^{12} R^4$. At that order the expansions of both the tree-level and one-loop four-point amplitudes, the superpositions of a $(\sigma_2)^3$ term and $(\sigma_3)^2$ term [35] ties in with the two-fold degeneracy in the MZV content ζ_3^3 and ζ_9 of the tree amplitude at order $(\alpha')^9$, as indicated

in the last column of table 1. At higher orders, however, the $\sigma_2^p \sigma_3^q$ monomials might not capture the full variety of M_l products. For example, at weight $(\alpha')^{10} D^{14} R^4$ there are two elements, M_5^2 and $\{M_3, M_7\}$, whereas only the element $\sigma_2^2 \sigma_3^2$ exists at that weight in the $\sigma_2^p \sigma_3^q$ basis.³³ This shortcoming of the $\sigma_2^p \sigma_3^q$ basis as well as the straightforward $N \geq 5$ point generalizability of the M_l 's motivate us to only refer to the basis formed by products of M_l 's in this paper.

B Pure spinor superspace calculations

This appendix gives a more detailed discussion of certain pure spinor superspace identities which were not proved in the main text.

B.1 The pentagon numerator equation in superspace

In this subsection we provide a pure spinor superspace derivation for the field theory relation between box numerators and loop momentum dependent parts of pentagon numerators, see (3.9) of [64]. For this purpose, we demonstrate that the left-hand side of (3.32) is BRST-exact. As a starting point, consider $0 = \langle Q(A_m^1 T_{2,3,4,5}^m) \rangle$. Using (3.24) it is easy to show that

$$0 = \langle Q(A_m^1 T_{2,3,4,5}^m) \rangle = \langle (\lambda \gamma^m W^1) T_{2,3,4,5}^m + k_m^1 V^1 T_{2,3,4,5}^m + (k^2 \cdot A^1) V^2 T_3^i T_4^j T_5^k + (k^3 \cdot A^1) V^3 T_2^i T_4^j T_5^k + (k^4 \cdot A^1) V^4 T_2^i T_3^j T_5^k + (k^5 \cdot A^1) V^5 T_2^i T_3^j T_4^k \rangle. \quad (\text{B.1})$$

Using the tree-level building blocks L_{1j} [38, 39] one rewrites

$$(k^j \cdot A^1) V^j = -L_{1j} - A_m^j (\lambda \gamma^m W^1), \quad j = 2, 3, 4, 5 \quad (\text{B.2})$$

to obtain, after a few trivial cancellations, that

$$0 = \langle Q(A_m^1 T_{2,3,4,5}^m) \rangle = \langle (\lambda \gamma^m W^1) W_{2,3,4,5}^m + k_m^1 V^1 T_{2,3,4,5}^m - L_{12} T_3^i T_4^j T_5^k - L_{13} T_2^i T_4^j T_5^k - L_{14} T_2^i T_3^j T_5^k - L_{15} T_2^i T_3^j T_4^k \rangle. \quad (\text{B.3})$$

The bosonic component of the term $\langle (\lambda \gamma^m W^1) W_{2,3,4,5}^m \rangle$ was shown in [66] to be proportional to $\epsilon_{10} F^5$ so it vanishes identically using the momentum phase space of five particles. Therefore,

$$\begin{aligned} & \langle T_{12} T_3^i T_4^i T_5^i + T_{13} T_2^i T_4^i T_5^i + T_{14} T_2^i T_3^i T_5^i + T_{15} T_2^i T_3^i T_4^i + k_m^1 V_1 T_{2,3,4,5}^m \rangle \\ &= \langle Q \left[A_m^1 T_{2,3,4,5}^m - \frac{1}{2} \left[(A^1 \cdot A^2) T_3^i T_4^j T_5^k + (2 \leftrightarrow 3, 4, 5) \right] \right] \rangle \end{aligned} \quad (\text{B.4})$$

where we used $L_{1j} = -T_{1j} - \frac{1}{2} Q(A^1 \cdot A^j)$, finishing the proof that (3.32) is BRST-exact.

³³In the $N = 4$ case where $M_l \in \mathbb{R}$ the element $\{M_3, M_7\}$ is equivalent to $2M_3 M_7$, but for higher N it is important that it is a symmetrised product of $(N-3)! \times (N-3)!$ -dimensional matrices.

B.2 Permutation symmetry of the five-graviton amplitude

This subsection is devoted to a superspace proof that the expression (3.28) for the leading low energy contribution of the five-graviton amplitude (3.33) is totally symmetric even though label 1 associated with the unintegrated vertex appears to enter on special footing. For this purpose, we rewrite

$$\begin{aligned}
 C_{1,2,3,4,5}^m \tilde{C}_{1,2,3,4,5}^m - C_{2,1,3,4,5}^m \tilde{C}_{2,1,3,4,5}^m &= (V_1 T_2^i \tilde{V}_1 \tilde{T}_2^i - V_2 T_1^i \tilde{V}_2 \tilde{T}_1^i) \left(\frac{T_{34}^j T_5^k \tilde{T}_{34}^j \tilde{T}_5^k}{s_{34}} + (3 \leftrightarrow 4, 5) \right) \\
 &+ (V_1 T_{2,3,4,5}^m - V_2 T_{1,3,4,5}^m) \tilde{V}_1 \tilde{T}_{2,3,4,5}^m + V_2 T_{1,3,4,5}^m (\tilde{V}_1 \tilde{T}_{2,3,4,5}^m - \tilde{V}_2 \tilde{T}_{1,3,4,5}^m) \\
 &+ \left(\frac{|T_{13} T_2^i T_4^j T_5^k|^2 - |V_2 T_{13}^i T_4^j T_5^k|^2}{s_{13}} - \frac{|T_{23} T_1^i T_4^j T_5^k|^2 - |V_1 T_{23}^i T_4^j T_5^k|^2}{s_{23}} + (3 \leftrightarrow 4, 5) \right)
 \end{aligned} \quad (B.5)$$

(where e.g. $|T_{13} T_2^i T_4^j T_5^k|^2 := T_{13} T_2^i T_4^j T_5^k \tilde{T}_{13} \tilde{T}_2^i \tilde{T}_4^j \tilde{T}_5^k$) and insert identities

$$V_1 T_2^i \tilde{V}_1 \tilde{T}_2^i - V_2 T_1^i \tilde{V}_2 \tilde{T}_1^i = \frac{Q T_{12}^i \tilde{V}_1 \tilde{T}_2^i + V_2 T_1^i Q \tilde{T}_{12}^i}{s_{12}} \quad (B.6)$$

$$\langle V_1 T_{2,3,4,5}^m - V_2 T_{1,3,4,5}^m \rangle = \frac{k_{12}^m \langle T_{21} T_3^i T_4^j T_5^k \rangle + (k_3^m \langle V_3 T_{21}^i T_4^j T_5^k \rangle + (3 \leftrightarrow 4, 5))}{s_{12}} \quad (B.7)$$

$$\langle k_1^m V_{1,2,3,4,5}^m \rangle = -\langle T_{12} T_3^i T_4^j T_5^k \rangle + (2 \leftrightarrow 3, 4, 5) \quad (B.8)$$

$$\langle k_2^m V_{1,2,3,4,5}^m \rangle = \langle T_{12} T_3^i T_4^j T_5^k \rangle - \langle V_1 T_{23}^i T_4^j T_5^k \rangle + (3 \leftrightarrow 4, 5) \quad (B.9)$$

as well as

$$\begin{aligned}
 \frac{1}{s_{13}} \langle |T_{13} T_2^i T_4^j T_5^k|^2 - |V_2 T_{13}^i T_4^j T_5^k|^2 \rangle &= \frac{1}{s_{12}} \langle (V_3 T_{12}^i - T_{12} T_3^i) T_4^j T_5^k \tilde{T}_{13} \tilde{T}_2^i \tilde{T}_4^j \tilde{T}_5^k \rangle \\
 &+ \frac{1}{s_{12}} \langle V_2 T_{13}^i T_4^j T_5^k (\tilde{V}_3 \tilde{T}_{12}^i - \tilde{T}_{12} \tilde{T}_3^i) \tilde{T}_4^j \tilde{T}_5^k \rangle.
 \end{aligned} \quad (B.10)$$

After making repeated use of these manipulations, all the terms on the right-hand side of (B.5) are proportional to s_{12}^{-1} and cancel pairwise.

C Further 1PI world-sheet diagrams

Starting at weight $w = 5$, it is essential to have a systematic classification scheme for the large number of world-sheet 1PI diagrams. A first criterion is the distinction between indecomposable and possibly disconnected diagrams (such as D_2^2 , see section 4.1). Secondly, we can characterise diagrams according to the number of vertices that have propagators ending on them. Note that the weight- w diagram involving the greatest number of vertices is the w -gon. This includes the pentagon at $w = 5$, which does not contribute to the four-particle amplitude, and the hexagon at $w = 6$, which does not contribute to the four- or five-particle amplitude.

The following table 7 summarises the number of inequivalent diagrams at $w \leq 8$ (both the overall number and the number of disconnected representatives). Their graphical representation for $w = 5$ and $w = 6$ will be given in the subsequent subsections.

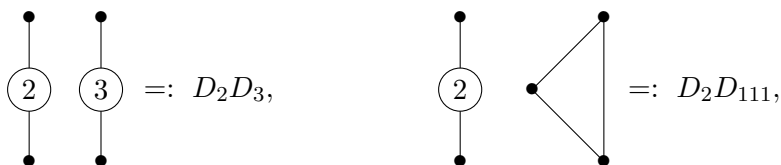
weight w	0	1	2	3	4	5	6	7	8
overall # diagrams	1	0	1	2	4	8	20	42	109
# indecomposables	1	0	1	2	3	6	13	28	71

Table 7. The number of inequivalent 1PI world-sheet diagrams and their indecomposable representatives at weights $w \leq 8$.

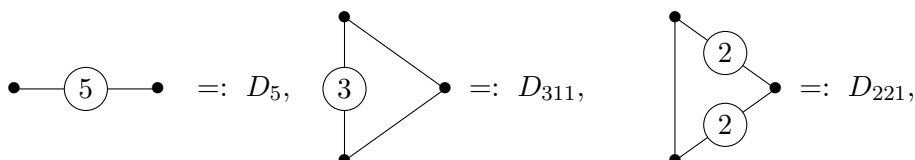
C.1 Weight $w = 5$

There are the following 8 diagrams (6 of them indecomposable):

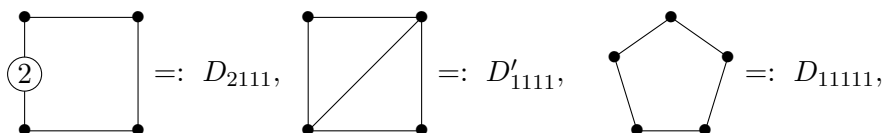
- 2 disconnected diagrams



- 3 diagrams with two or three vertices



- 3 diagrams with four or five vertices



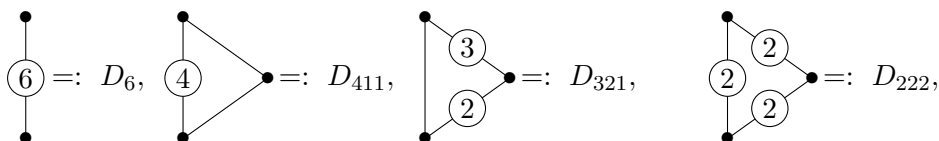
C.2 Weight $w = 6$

There are the following 20 diagrams (13 of them indecomposable):

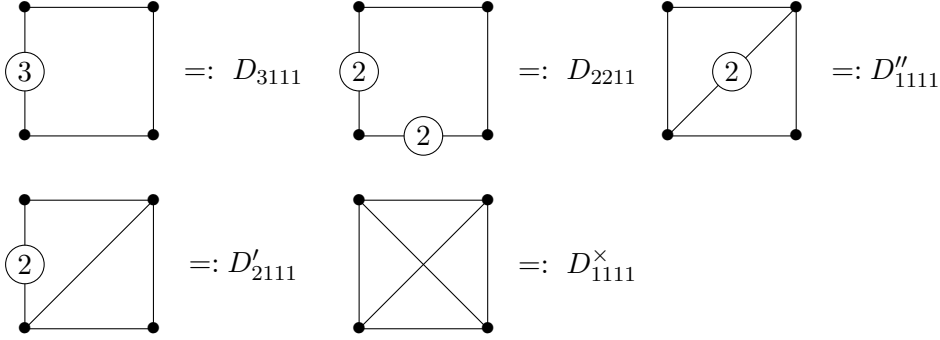
- 7 disconnected diagrams

- 1 with 2+2+2 partition of the six propagators: D_2^3
- 3 with 4+2 partition of the six propagators: $D_4D_2, D_{211}D_2, D_{1111}D_2$
- 3 with 3+3 partition of the six propagators: $D_3^2, D_3D_{111}, D_{111}^2$

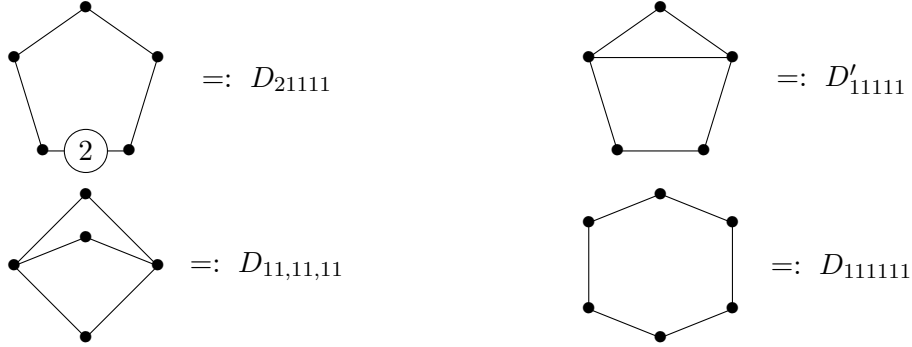
- 4 diagrams with two or three vertices



- 5 diagrams with four vertices



- 4 diagrams with five or six vertices



Note that polygons with neighbouring propagators interchanged are indistinguishable under the translation invariant torus integration measure:

$$\begin{aligned}
 & \text{Diagram: Square with circles '2' on left and bottom edges} = \text{Diagram: Square with circles '2' on left and right edges} \\
 &= \frac{\tau_2^6}{\pi^6} \sum_{(m,n) \neq (0,0)} \frac{1}{|m\tau + n|^4} \sum_{(p,q),(r,s) \neq (0,0)} \\
 & \quad \times |p\tau + q|^{-2} |(p+m)\tau + (n+q)|^{-2} \\
 & \quad \times |r\tau + s|^{-2} |(r+m)\tau + (n+s)|^{-2}
 \end{aligned}$$

D Five-point integrals to weight $w = 4$

The analytic string corrections to the five-particle closed-string amplitude at one loop were given up to order $(\alpha')^9$ in (5.5) and (5.7). These results are based on the expansion of the five-point integrals (4.15) as well as (4.9) to (4.11) to sixth subleading order. The latter were explicitly expanded to weight $w \leq 3$ in (4.17), and this appendix extends these results to weight $w = 4$. Two inequivalent five-point integrals are free of massless poles by

themselves:

$$\begin{aligned}
 J_{12|13} = & - \int d\mu_d(\tau) \left(D_2 s_{23} + D_{111}(s_{24}s_{34} + s_{25}s_{35}) + \frac{D_3}{2}s_{23}(s_{12} + s_{13} + s_{23}) \right. \\
 & + \frac{D_4}{6}s_{23}\left(s_{23}^2 + \frac{3}{2}s_{23}(s_{12} + s_{13}) + s_{12}^2 + s_{13}^2 + \frac{3}{2}s_{12}s_{13}\right) \\
 & - \frac{D_2^2}{4}\left(s_{23}^2(s_{12} + s_{13}) + s_{12}s_{13}s_{23} - 2s_{23}(s_{14}^2 + s_{15}^2 + s_{24}^2 + s_{25}^2 + s_{34}^2 + s_{35}^2 + s_{45}^2) \right. \\
 & \left. - 2s_{23}(s_{14}(s_{24} + s_{34}) + s_{15}(s_{25} + s_{35})) + 2(s_{14}s_{24}s_{34} + s_{15}s_{25}s_{35}) \right) \\
 & + \frac{D_{211}}{2}\left(2s_{23}(s_{24}s_{34} + s_{25}s_{35}) + s_{24}s_{34}(s_{24} + s_{34}) + s_{25}s_{35}(s_{25} + s_{35}) \right. \\
 & \left. + (s_{12} + s_{13})(s_{24}s_{34} + s_{25}s_{35}) + 2(s_{14}s_{24}s_{34} + s_{15}s_{25}s_{35}) \right) \\
 & - \frac{D_{1111}}{2}\left(s_{23}(s_{14}s_{24} + s_{14}s_{34} + s_{15}s_{25} + s_{15}s_{35}) - s_{14}s_{24}s_{34} - s_{15}s_{25}s_{35} \right. \\
 & \left. - 2s_{45}(s_{24}s_{35} + s_{25}s_{34}) \right) + \dots \Big) \quad (D.1)
 \end{aligned}$$

$$\begin{aligned}
 J'_{12|34} = & \int d\mu_d(\tau) \left(D_{111}(s_{14}s_{23} - s_{13}s_{24}) + \frac{D_2^2}{2}\left(s_{13}s_{24}(s_{14} + s_{23}) - s_{14}s_{23}(s_{13} + s_{24}) \right) \right. \\
 & + \frac{D_{211}}{2}\left(s_{14}s_{23}(s_{14} + s_{23}) - s_{13}s_{24}(s_{13} + s_{24}) + (s_{12} + s_{34})(s_{14}s_{23} - s_{13}s_{24}) \right. \\
 & \left. + 2s_{14}s_{23}(s_{13} + s_{24}) - 2s_{13}s_{24}(s_{14} + s_{23}) \right) \\
 & + \frac{D_{1111}}{2}\left(s_{14}s_{23}(s_{13} + s_{24}) - s_{13}s_{24}(s_{14} + s_{23}) + 2(s_{14}s_{25}s_{35} + s_{23}s_{15}s_{45}) \right. \\
 & \left. - 2(s_{13}s_{25}s_{45} + s_{24}s_{15}s_{35}) \right) + \dots \Big) \quad (D.2)
 \end{aligned}$$

For the third type of holomorphically factored five-point integral (4.9), the residue of its massless pole is determined by the four-point integral (4.12). The residual task is to expand its regular part defined by (4.13):

$$\begin{aligned}
 I_{12}^{\text{reg}} = & \int d\mu_d(\tau) \left(\frac{D_2}{2}s_{12} + D_{111}(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25}) \right. \\
 & + \frac{D_3}{6}\left(s_{12}^2 + 3(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25})\right) \\
 & + \frac{D_4}{24}\left(s_{12}^3 + 4s_{12}(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25}) \right. \\
 & \left. + 6(s_{13}^2s_{23} + s_{13}s_{23}^2 + s_{14}^2s_{24} + s_{14}s_{24}^2 + s_{15}^2s_{25} + s_{15}s_{25}^2) \right) \\
 & + \frac{D_2^2}{4}\left(s_{12}(s_{13}^2 + s_{14}^2 + s_{15}^2 + s_{23}^2 + s_{24}^2 + s_{25}^2 + s_{34}^2 + s_{35}^2 + s_{45}^2) \right. \\
 & \left. - (s_{13}^2s_{23} + s_{13}s_{23}^2 + s_{14}^2s_{24} + s_{14}s_{24}^2 + s_{15}^2s_{25} + s_{15}s_{25}^2) \right) \\
 & + \frac{D_{211}}{2}\left(s_{12}(s_{13}s_{23} + s_{14}s_{24} + s_{15}s_{25}) + s_{34}(s_{13}s_{24} + s_{14}s_{23}) + s_{35}(s_{13}s_{25} + s_{15}s_{23}) \right. \\
 & \left. + s_{45}(s_{14}s_{25} + s_{15}s_{24}) + s_{13}^2s_{23} + s_{13}s_{23}^2 + s_{14}^2s_{24} + s_{14}s_{24}^2 + s_{15}^2s_{25} + s_{15}s_{25}^2 \right) \\
 & + D_{1111}\left(s_{34}(s_{13}s_{24} + s_{14}s_{23}) + s_{35}(s_{13}s_{25} + s_{15}s_{23}) + s_{45}(s_{14}s_{25} + s_{15}s_{24}) \right) + \dots \Big) \quad (D.3)
 \end{aligned}$$

The weight $w = 5, 6$ analogues can be obtained from the auxiliary file included in the arXiv submission.

E Six-point integrals to weight $w = 4$

In this appendix, we provide the analytic part of the α' expansion to weight $w = 4$ for a set of six-point world-sheet integrals appearing in the six closed-string amplitude at one-loop. As argued in section 6, these results are sufficient to infer weights $w \leq 4$ for any other world-sheet integral in the six-particle amplitude (6.1).

E.1 The singular part of six-point integrals

Among the seven topologies of six-point integrals with four propagator derivatives shown in figure 4, the integrals $\mathcal{I}_{ij,pqrs}^1$, $\mathcal{I}_{ij,pqr}^4$ and $\mathcal{I}_{ij,pq}^7$ have kinematic poles in s_{ij} due to the integration region $z_i \rightarrow z_j$ where $\partial \ln \chi_{ij} \partial \ln \chi_{ij}$ products in the integrand diverge as $\sim \frac{1}{|z_{ij}|^2}$. The single pole residues can be expressed through five-point integrals $J_{qp|qr}$, $J'_{pq|rs}$ and I_{pq}^{reg} given by (4.10), (4.11) and (4.13), promoted to functions of five momenta $(k_p, k_q, k_r, k_s, k_t)$.³⁴ Let $\mathcal{I}^{i,\text{reg}}$ denote the regular part of the integrals after subtracting off the poles, then

$$\mathcal{I}_{ij,pqrs}^1 = \mathcal{I}_{ij,pqrs}^{1,\text{reg}} + \frac{1}{s_{ij}} J'_{pq|rs}(k_p, k_q, k_r, k_s, k_i + k_j) \quad (\text{E.1})$$

$$\mathcal{I}_{ij,pqr}^4 = \mathcal{I}_{ij,pqr}^{4,\text{reg}} - \frac{1}{s_{ij}} J_{qp|qr}(k_q, k_p, k_r, k_i + k_j, k_s) \quad (\text{E.2})$$

$$\begin{aligned} \mathcal{I}_{ij,pq}^6 &= \mathcal{I}_{ij,pq}^{6,\text{reg}} + \frac{1}{s_{ij}} I_{pq}^{\text{reg}}(k_p, k_q, k_i + k_j, k_r, k_s) + \frac{1}{s_{pq}} I_{ij}^{\text{reg}}(k_i, k_j, k_p + k_q, k_r, k_s) \\ &+ \frac{1}{s_{ij}s_{pq}} I(k_i + k_j, k_p + k_q, k_r, k_s). \end{aligned} \quad (\text{E.3})$$

The remaining \mathcal{I}^i at $i = 2, 3, 5, 7$ do not have poles and coincide with their regular parts $\mathcal{I}^{i,\text{reg}}$.

E.2 The regular part of six-point integrals

In the following, we display the diagrammatic expansion of the six-point integrals defined in figure 4. Dashed lines represent derivatives $\partial \ln \chi$ or $\tilde{\partial} \ln \chi$, and a distinction between ∂ and $\tilde{\partial}$ is not needed since dashed lines with alike derivatives do not touch the same vertex

³⁴Up to $(\alpha')^3$ order, for instance, we have

$$\begin{aligned} J'_{pq|rs}(k_p, k_q, k_r, k_s, k_i + k_j) &= \int d\mu_d(\tau) D_{111}(s_{ps}s_{qr} - s_{pr}s_{qs}) \\ J_{qp|qr}(k_q, k_p, k_r, k_i + k_j, k_s) &= - \int d\mu_d(\tau) \left(D_2 s_{pr} + D_{111}((s_{pi} + s_{pj})(s_{ri} + s_{rj}) + s_{ps}s_{rs}) + \frac{D_3}{2} s_{pr}s_{pqr} \right) \\ I_{pq}^{\text{reg}}(k_p, k_q, k_i + k_j, k_r, k_s) &= \int d\mu_d(\tau) \left(\frac{D_2}{2} s_{pq} + \frac{D_3}{6} s_{pq}^2 \right. \\ &\quad \left. + \left(D_{111} + \frac{D_3}{2} \right) ((s_{pi} + s_{pj})(s_{qi} + s_{qj}) + s_{pr}s_{qr} + s_{ps}s_{qs}) \right) \end{aligned}$$

for the integrals under consideration. The ellipses refers to contributions of higher weight $w \geq 5$.

$$\begin{aligned}
 \mathcal{I}_{ij,pqrs}^{1,\text{reg}} &= \dots \\
 \mathcal{I}_{ijkl,pq}^{2,\text{reg}} &= \int d\mu_d(\tau) D_{1111}(s_{iq}s_{lp} - s_{ip}s_{lq}) + \dots \\
 \mathcal{I}_{ijk,pqr}^{3,\text{reg}} &= \int d\mu_d(\tau) \left(D_2^2 s_{ik}s_{pr} + D_{1111}(s_{ip}s_{kr} + s_{ir}s_{kp}) + \dots \right) \\
 \mathcal{I}_{ij,pqr}^{4,\text{reg}} &= \int d\mu_d(\tau) \left(\frac{D_2^2}{2} s_{ij}s_{pr} + \left(D_{1111} + \frac{D_{211}}{2} \right) (s_{ip}s_{jr} + s_{ir}s_{jp}) + \dots \right) \\
 \mathcal{I}_{ijklm}^{5,\text{reg}} &= \int d\mu_d(\tau) \left(D_{111}s_{im} + \frac{D_{211}}{2} s_{im}^2 + D_{1111} \sum_{p \neq i,j,k,l,m} s_{ip}s_{mp} + D_{211}s_{im}(s_{ik} + s_{jl} + s_{km}) \right. \\
 &\quad + \frac{D_{211}}{2} s_{im}(s_{ij} + s_{jk} + s_{kl} + s_{lm}) + \frac{(D_2^2 - D_{1111})}{2} (s_{ik}s_{jm} + s_{il}s_{km}) \\
 &\quad + D_2^2 s_{ik}s_{km} + \left(\frac{(D_{1111} - D_2^2)}{2} + D_{211} \right) (s_{im}(s_{jm} + s_{il}) - s_{il}s_{jm}) + \dots \Big) \\
 \mathcal{I}_{ij,kl}^{6,\text{reg}} &= \int d\mu_d(\tau) \left(\frac{D_2^2}{4} s_{ij}s_{kl} + \left((D_{1111} + D_{211}) + \frac{D_4 - D_2^2}{4} \right) (s_{ik}s_{jl} + s_{il}s_{jk}) + \dots \right) \\
 \mathcal{I}_{ijkl}^{7,\text{reg}} &= \int d\mu_d(\tau) \left(D_2 + \frac{D_3}{2} (s_{ij} + s_{jk} + s_{kl} + s_{il} + 2s_{ik} + 2s_{jl}) + \frac{D_2^2}{2} \sum_{p < q \neq i,j,k,l} s_{pq}^2 \right. \\
 &\quad + \frac{D_2^2}{2} \sum_{p \neq i,j,k,l} (s_{ip}^2 + s_{jp}^2 + s_{kp}^2 + s_{lp}^2) + \frac{D_4 - D_2^2}{2} (s_{ij} + s_{jk} + s_{kl} + s_{il})(s_{ik} + s_{jl}) \\
 &\quad + \frac{D_4}{6} (s_{ij}^2 + s_{jk}^2 + s_{kl}^2 + s_{il}^2 + 3s_{ik}^2 + 3s_{jl}^2) + \frac{D_4 - 2D_2^2 + D_{1111}}{2} s_{ik}s_{jl} \\
 &\quad + \frac{D_4 - D_2^2}{4} (s_{ij}s_{kl} + s_{il}s_{jk} + s_{ij}s_{jk} + s_{jk}s_{kl} + s_{kl}s_{il} + s_{ij}s_{il}) \\
 &\quad + \frac{D_2^2 - D_{1111}}{2} \sum_{p \neq i,j,k,l} (s_{ip}s_{jp} + s_{jp}s_{kp} + s_{kp}s_{lp} + s_{ip}s_{lp}) \\
 &\quad \left. + D_{211} \sum_{p \neq i,j,k,l} (s_{ip}s_{kp} + s_{jp}s_{lp}) + \dots \right)
 \end{aligned}$$

We emphasise that all the above terms are spanned by the same set of $w \leq 4$ 1PI diagrams $D_0, D_2, D_{111}, D_3, D_{1111}, D_{211}, D_2^2$ and D_4 appearing in the four-particle amplitude. This justifies the upper bounds on the six-point one-loop effective action in table 1 and 3, also see the discussion in section 6.

References

- [1] C.R. Mafra and O. Schlotterer, *The Structure of n-Point One-Loop Open Superstring Amplitudes*, [arXiv:1203.6215](#) [[INSPIRE](#)].
- [2] N. Berkovits, *Super Poincaré covariant quantization of the superstring*, [JHEP 04 \(2000\) 018](#) [[hep-th/0001035](#)] [[INSPIRE](#)].

- [3] N. Berkovits, *Multiloop amplitudes and vanishing theorems using the pure spinor formalism for the superstring*, *JHEP* **09** (2004) 047 [[hep-th/0406055](#)] [[INSPIRE](#)].
- [4] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude I. Pure Spinor Computation*, *Nucl. Phys. B* **873** (2013) 419 [[arXiv:1106.2645](#)] [[INSPIRE](#)].
- [5] C.R. Mafra, O. Schlotterer and S. Stieberger, *Complete N-Point Superstring Disk Amplitude II. Amplitude and Hypergeometric Function Structure*, *Nucl. Phys. B* **873** (2013) 461 [[arXiv:1106.2646](#)] [[INSPIRE](#)].
- [6] O. Schlotterer and S. Stieberger, *Motivic Multiple Zeta Values and Superstring Amplitudes*, [arXiv:1205.1516](#) [[INSPIRE](#)].
- [7] C. Hull and P. Townsend, *Unity of superstring dualities*, *Nucl. Phys. B* **438** (1995) 109 [[hep-th/9410167](#)] [[INSPIRE](#)].
- [8] E. Cremmer and B. Julia, *The $N = 8$ Supergravity Theory. 1. The Lagrangian*, *Phys. Lett. B* **80** (1978) 48 [[INSPIRE](#)].
- [9] E. Cremmer and B. Julia, *The $SO(8)$ Supergravity*, *Nucl. Phys. B* **159** (1979) 141 [[INSPIRE](#)].
- [10] B. Julia, *Group Disintegrations*, in *Supergravity and Superspace*, S.W. Hawking and M. Rocek, C.U.P. Cambridge (1981).
- [11] B. Julia, *Kac-Moody symmetry of gravitation and supergravity theories*, Proc. AMS-SIAM Summer Seminar on Applications of Group Theory in Physics and Mathematical Physics, Chicago (1982), Vertex Operators in Mathematics and Physics, *Mathematical Sciences Research Institute Publications* **3** (1985) 393.
- [12] M.B. Green and M. Gutperle, *Effects of D instantons*, *Nucl. Phys. B* **498** (1997) 195 [[hep-th/9701093](#)] [[INSPIRE](#)].
- [13] M.B. Green and S. Sethi, *Supersymmetry constraints on type IIB supergravity*, *Phys. Rev. D* **59** (1999) 046006 [[hep-th/9808061](#)] [[INSPIRE](#)].
- [14] A. Sinha, *The $\hat{G}^4\lambda^{16}$ term in IIB supergravity*, *JHEP* **08** (2002) 017 [[hep-th/0207070](#)] [[INSPIRE](#)].
- [15] M.B. Green and P. Vanhove, *Duality and higher derivative terms in M-theory*, *JHEP* **01** (2006) 093 [[hep-th/0510027](#)] [[INSPIRE](#)].
- [16] M.B. Green, M. Gutperle and H.-h. Kwon, *Sixteen fermion and related terms in M-theory on T^2* , *Phys. Lett. B* **421** (1998) 149 [[hep-th/9710151](#)] [[INSPIRE](#)].
- [17] M.B. Green, *Interconnections between type-II superstrings, M-theory and $N = 4$ supersymmetric Yang-Mills*, [hep-th/9903124](#) [[INSPIRE](#)].
- [18] N. Berkovits and C. Vafa, *Type IIB $R^4 H^{(4g-4)}$ conjectures*, *Nucl. Phys. B* **533** (1998) 181 [[hep-th/9803145](#)] [[INSPIRE](#)].
- [19] A. Basu and S. Sethi, *Recursion Relations from Space-time Supersymmetry*, *JHEP* **09** (2008) 081 [[arXiv:0808.1250](#)] [[INSPIRE](#)].
- [20] R.H. Boels, *Maximal R-symmetry violating amplitudes in type IIB superstring theory*, *Phys. Rev. Lett.* **109** (2012) 081602 [[arXiv:1204.4208](#)] [[INSPIRE](#)].
- [21] A. Basu, *The structure of the R^8 term in type IIB string theory*, [arXiv:1306.2501](#) [[INSPIRE](#)].
- [22] H. Kawai, D. Lewellen and S. Tye, *A Relation Between Tree Amplitudes of Closed and Open Strings*, *Nucl. Phys. B* **269** (1986) 1 [[INSPIRE](#)].

- [23] N. Bjerrum-Bohr, P.H. Damgaard and P. Vanhove, *Minimal Basis for Gauge Theory Amplitudes*, *Phys. Rev. Lett.* **103** (2009) 161602 [[arXiv:0907.1425](#)] [[INSPIRE](#)].
- [24] S. Stieberger, *Open & Closed vs. Pure Open String Disk Amplitudes*, [arXiv:0907.2211](#) [[INSPIRE](#)].
- [25] Z. Bern, J. Carrasco and H. Johansson, *New Relations for Gauge-Theory Amplitudes*, *Phys. Rev. D* **78** (2008) 085011 [[arXiv:0805.3993](#)] [[INSPIRE](#)].
- [26] J. Broedel, O. Schlotterer and S. Stieberger, *Polylogarithms, Multiple Zeta Values and Superstring Amplitudes*, *Fortsch. Phys.* **61** (2013) 812 [[arXiv:1304.7267](#)] [[INSPIRE](#)].
- [27] J. Broedel, O. Schlotterer, S. Stieberger and T. Terasoma, *All order α' -expansion of superstring trees from the Drinfeld associator*, [arXiv:1304.7304](#) [[INSPIRE](#)].
- [28] N. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *Gravity and Yang-Mills Amplitude Relations*, *Phys. Rev. D* **82** (2010) 107702 [[arXiv:1005.4367](#)] [[INSPIRE](#)].
- [29] N. Bjerrum-Bohr, P.H. Damgaard, B. Feng and T. Sondergaard, *Proof of Gravity and Yang-Mills Amplitude Relations*, *JHEP* **09** (2010) 067 [[arXiv:1007.3111](#)] [[INSPIRE](#)].
- [30] N. Bjerrum-Bohr, P.H. Damgaard, T. Sondergaard and P. Vanhove, *The Momentum Kernel of Gauge and Gravity Theories*, *JHEP* **01** (2011) 001 [[arXiv:1010.3933](#)] [[INSPIRE](#)].
- [31] R.H. Boels, *On the field theory expansion of superstring five point amplitudes*, *Nucl. Phys. B* **876** (2013) 215 [[arXiv:1304.7918](#)] [[INSPIRE](#)].
- [32] D.M. Richards, *The One-Loop Five-Graviton Amplitude and the Effective Action*, *JHEP* **10** (2008) 042 [[arXiv:0807.2421](#)] [[INSPIRE](#)].
- [33] N. Bjerrum-Bohr and P. Vanhove, *Explicit Cancellation of Triangles in One-loop Gravity Amplitudes*, *JHEP* **04** (2008) 065 [[arXiv:0802.0868](#)] [[INSPIRE](#)].
- [34] N. Bjerrum-Bohr and P. Vanhove, *Absence of Triangles in Maximal Supergravity Amplitudes*, *JHEP* **10** (2008) 006 [[arXiv:0805.3682](#)] [[INSPIRE](#)].
- [35] M.B. Green, J.G. Russo and P. Vanhove, *Low energy expansion of the four-particle genus-one amplitude in type-II superstring theory*, *JHEP* **02** (2008) 020 [[arXiv:0801.0322](#)] [[INSPIRE](#)].
- [36] M.B. Green and P. Vanhove, *The Low-energy expansion of the one loop type-II superstring amplitude*, *Phys. Rev. D* **61** (2000) 104011 [[hep-th/9910056](#)] [[INSPIRE](#)].
- [37] L.A. Barreiro and R. Medina, *Revisiting the S-matrix approach to the open superstring low energy effective lagrangian*, *JHEP* **10** (2012) 108 [[arXiv:1208.6066](#)] [[INSPIRE](#)].
- [38] C.R. Mafra, *Towards Field Theory Amplitudes From the Cohomology of Pure Spinor Superspace*, *JHEP* **11** (2010) 096 [[arXiv:1007.3639](#)] [[INSPIRE](#)].
- [39] C.R. Mafra, O. Schlotterer, S. Stieberger and D. Tsimpis, *A recursive method for SYM n -point tree amplitudes*, *Phys. Rev. D* **83** (2011) 126012 [[arXiv:1012.3981](#)] [[INSPIRE](#)].
- [40] S. Stieberger, *Constraints on Tree-Level Higher Order Gravitational Couplings in Superstring Theory*, *Phys. Rev. Lett.* **106** (2011) 111601 [[arXiv:0910.0180](#)] [[INSPIRE](#)].
- [41] F.C. Brown, *Multiple zeta values and periods of moduli spaces $\mathcal{M}_{0,n}$* , *Annales Sci. Ecole Norm. Sup.* **42** (2009) 371 [[math/0606419](#)] [[INSPIRE](#)].
- [42] F. Brown, *Périodes des espaces des modules $\mathcal{M}_{0,n}$ et valeurs zêtas multiples*, *C.R. Acad. Sci. Paris Ser. I* **342** (2006) 949.
- [43] T. Terasoma, *Selberg Integrals and Multiple Zeta Values*, *Comp. Math.* **133** (2002) 1.

- [44] J. Blumlein, D. Broadhurst and J. Vermaseren, *The Multiple Zeta Value Data Mine*, *Comput. Phys. Commun.* **181** (2010) 582 [[arXiv:0907.2557](#)] [[INSPIRE](#)].
- [45] F. Brown, *On the decomposition of motivic multiple zeta values*, [arXiv:1102.1310](#) [[INSPIRE](#)].
- [46] D. Oprisa and S. Stieberger, *Six gluon open superstring disk amplitude, multiple hypergeometric series and Euler-Zagier sums*, [hep-th/0509042](#) [[INSPIRE](#)].
- [47] S. Stieberger and T.R. Taylor, *Amplitude for N-Gluon Superstring Scattering*, *Phys. Rev. Lett.* **97** (2006) 211601 [[hep-th/0607184](#)] [[INSPIRE](#)].
- [48] S. Stieberger and T.R. Taylor, *Multi-Gluon Scattering in Open Superstring Theory*, *Phys. Rev. D* **74** (2006) 126007 [[hep-th/0609175](#)] [[INSPIRE](#)].
- [49] V.G. Drinfeld, *Quasi Hopf algebras*, *Leningrad Math. J.* **1** (1989) 1419.
- [50] V.G. Drinfeld, *On quasitriangular quasi-Hopf algebras and a group that is closely connected with $Gal(\bar{\mathbb{Q}}/\mathbb{Q})$* , *Leningrad Math. J.* **2** (1991) 829.
- [51] J. Drummond and É. Ragoucy, *Superstring amplitudes and the associator*, *JHEP* **08** (2013) 135 [[arXiv:1301.0794](#)] [[INSPIRE](#)].
- [52] J. Broedel, O. Schlotterer and S. Stieberger, *α' -expansion of open superstring amplitudes*, <http://mzv.mpp.mpg.de>.
- [53] M.B. Green and J.H. Schwarz, *Anomaly Cancellation in Supersymmetric $D = 10$ Gauge Theory and Superstring Theory*, *Phys. Lett. B* **149** (1984) 117 [[INSPIRE](#)].
- [54] M.B. Green and J.H. Schwarz, *The Hexagon Gauge Anomaly in Type I Superstring Theory*, *Nucl. Phys. B* **255** (1985) 93 [[INSPIRE](#)].
- [55] M.B. Green, J.H. Schwarz and E. Witten, *Superstring Theory. Vol. 2: Loop Amplitudes, Anomalies And Phenomenology*, Cambridge University Press, Cambridge U.K. (1987).
- [56] P.S. Howe and P.C. West, *The Complete $N = 2$, $D = 10$ Supergravity*, *Nucl. Phys. B* **238** (1984) 181 [[INSPIRE](#)].
- [57] E. D'Hoker, M. Gutperle and D. Phong, *Two-loop superstrings and S-duality*, *Nucl. Phys. B* **722** (2005) 81 [[hep-th/0503180](#)] [[INSPIRE](#)].
- [58] H. Gomez and C.R. Mafra, *The Overall Coefficient of the Two-loop Superstring Amplitude Using Pure Spinors*, *JHEP* **05** (2010) 017 [[arXiv:1003.0678](#)] [[INSPIRE](#)].
- [59] M.B. Green, J.G. Russo and P. Vanhove, *Automorphic properties of low energy string amplitudes in various dimensions*, *Phys. Rev. D* **81** (2010) 086008 [[arXiv:1001.2535](#)] [[INSPIRE](#)].
- [60] M.B. Green, S.D. Miller, J.G. Russo and P. Vanhove, *Eisenstein series for higher-rank groups and string theory amplitudes*, *Commun. Num. Theor. Phys.* **4** (2010) 551 [[arXiv:1004.0163](#)] [[INSPIRE](#)].
- [61] B. Pioline, *R^4 couplings and automorphic unipotent representations*, *JHEP* **03** (2010) 116 [[arXiv:1001.3647](#)] [[INSPIRE](#)].
- [62] I. Oda and M. Tonin, *Y-formalism in pure spinor quantization of superstrings*, *Nucl. Phys. B* **727** (2005) 176 [[hep-th/0505277](#)] [[INSPIRE](#)].
- [63] C.R. Mafra, *PSS: A FORM Program to Evaluate Pure Spinor Superspace Expressions*, [arXiv:1007.4999](#) [[INSPIRE](#)].

- [64] N.E.J. Bjerrum-Bohr, T. Dennen, R. Monteiro and D. O’Connell, *Integrand Oxidation and One-Loop Colour-Dual Numerators in $N = 4$ Gauge Theory*, *JHEP* **07** (2013) 092 [[arXiv:1303.2913](#)] [[INSPIRE](#)].
- [65] J.J. Carrasco and H. Johansson, *Five-Point Amplitudes in $N = 4$ super-Yang-Mills Theory and $N = 8$ Supergravity*, *Phys. Rev. D* **85** (2012) 025006 [[arXiv:1106.4711](#)] [[INSPIRE](#)].
- [66] N. Berkovits and C.R. Mafra, *Some Superstring Amplitude Computations with the Non-Minimal Pure Spinor Formalism*, *JHEP* **11** (2006) 079 [[hep-th/0607187](#)] [[INSPIRE](#)].
- [67] M.T. Grisaru, A. van de Ven and D. Zanon, *Four Loop β -function for the $N = 1$ and $N = 2$ Supersymmetric Nonlinear σ -model in Two-Dimensions*, *Phys. Lett. B* **173** (1986) 423 [[INSPIRE](#)].
- [68] M.T. Grisaru and D. Zanon, *σ Model Superstring Corrections to the Einstein-hilbert Action*, *Phys. Lett. B* **177** (1986) 347 [[INSPIRE](#)].
- [69] C. Vafa and E. Witten, *A One loop test of string duality*, *Nucl. Phys. B* **447** (1995) 261 [[hep-th/9505053](#)] [[INSPIRE](#)].
- [70] M.B. Green, J.H. Schwarz and L. Brink, *$N=4$ Yang-Mills and $N = 8$ Supergravity as Limits of String Theories*, *Nucl. Phys. B* **198** (1982) 474 [[INSPIRE](#)].
- [71] J. Drummond, P. Heslop, P. Howe and S. Kerstan, *Integral invariants in $N = 4$ SYM and the effective action for coincident D-branes*, *JHEP* **08** (2003) 016 [[hep-th/0305202](#)] [[INSPIRE](#)].
- [72] R. Kallosh, *On UV Finiteness of the Four Loop $N = 8$ Supergravity*, *JHEP* **09** (2009) 116 [[arXiv:0906.3495](#)] [[INSPIRE](#)].
- [73] H. Elvang, D.Z. Freedman and M. Kiermaier, *A simple approach to counterterms in $N = 8$ supergravity*, *JHEP* **11** (2010) 016 [[arXiv:1003.5018](#)] [[INSPIRE](#)].
- [74] M.B. Green, C.R. Mafra and O. Schlotterer, work in progress.