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Criticality in conserved dynamical systems: Experimental observation vs. exact properties

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Conserved dynamical systems are generally considered to be critical. We study a class of critical routing models, equivalent to random maps, which can be solved rigorously in the thermodynamic limit. The information flow is conserved for these routing models and governed by cyclic attractors. We consider two classes of information flow, Markovian routing without memory and vertex routing involving a one-step routing memory. Investigating the respective cycle length distributions for complete graphs, we find log corrections to power-law scaling for the mean cycle length, as a function of the number of vertices, and a sub-polynomial growth for the overall number of cycles. When observing experimentally a real-world dynamical system one normally samples stochastically its phase space. The number and the length of the attractors are then weighted by the size of their respective basins of attraction. This situation is equivalent, for theory studies, to "on the fly" generation of the dynamical transition probabilities. For the case of vertex routing models, we find in this case power law scaling for the weighted average length of attractors, for both conserved routing models. These results show that the critical dynamical systems are generically not scale-invariant but may show power-law scaling when sampled stochastically. It is hence important to distinguish between intrinsic properties of a critical dynamical system and its behavior that one would observe when randomly probing its phase space. © 2013 American Institute of Physics. [http://dx.doi.org/10.1063/1.4773003]

Power law scaling is observed in many real-world phenomena, like neural avalanches in the brain. In statistical physics, all critical systems, at the point of a second-order phase transition, show power law scaling. Power law scaling is hence commonly attributed to criticality, but it is an open question to which extend this relation is satisfied for complex dynamical systems. There is, in addition, a difference between the distribution an observer may be able to sample and the exact properties of the underlying dynamical system. An observer will sample in general the number and the size of attractors as weighted by size of their respective basins of attraction. Here, we investigate the critical models for information routing and show that the number and the length of attractors does not obey power law scaling, while, on the other hand, an external observer, sampling the weighted distribution, would find power law scaling. Hence when drawing conclusions from experimentally observed power law scaling one needs to take into account the implicitly employed sampling procedures.

I. INTRODUCTION

The propagation of perturbations is a central notion in dynamical system theory. One speaks of a frozen state when a perturbation tends to die out, on the average, during the course of time evolution and of a chaotic state when perturbations tend to spread out.[1,2] A given class of dynamical systems may change from frozen to chaotic behavior as a function of parameters, being critical right at the transition point.

At criticality, information is on the average conserved,[3] as one can regard a perturbation of a state as the information about the persistence of small differences. A well studied example of a critical dynamical system is the Kauffman net with connectivity $K = 2$, an example of a random Boolean network.[4–6] In statistical mechanics, critical systems are generically scale invariant,[7] and it has been widely assumed that this statement would also hold for critical dynamical systems. Indeed, numerical simulations seemed to support scaling in critical Boolean networks, notably a $\sqrt{N}$ scaling for the number of attractors as a function of the number of vertices N had been proposed.[4,5]

An important clarification then came with the exact proof that the number of attractors actually grows faster than any power of N, and that the results of the numerical simulations suffered from systematic undersampling of phase space.[8] It could be shown, on the other side, that the number of frozen and the number of relevant nodes in a large class of critical Boolean networks obey power law scaling.[9] The situation is then that certain properties of critical dynamical systems, at least for the case of random Boolean networks, obey power law scaling while others do not. It is hence important to investigate the possible occurrence of scaling in different classes of dynamical systems.

We study a class of dynamical systems describing the transport of conserved quantities on network structures that is quantities which cannot be multiplied or separated into

smaller parts during the transport between network nodes. We denote such a process a routing process, since only one node is active at each time step, the one containing the transmitted quantity. A routing process can be seen alternatively as the transport of perturbations between network elements and as such represents a critical process because the perturbation neither spreads out through the entire network nor does it die out. A routing process initiated from a given network node will eventually follow a limiting cycle, thus the total number of nodes affected by the perturbation will be a finite fraction of the whole network. Hence, a routing process satisfies the conditions needed for it to be considered as a critical dynamical process.[10]

Transport on networks, such as the spreading of rumors[11] and diseases[12] in social networks or the flow of capital in financial networks[13] has been studied intensively, indeed transport constitutes a basic process in biology quite in general,[14] as well as in sociology and technical applications. In many cases, the quantity transported is not conserved, e.g., when considering the spreading of rumors in social networks. Routing processes, investigated here, model the transport of a conserved quantity, like conserved information packages. Information packages are sent from node to node and are routed at every vertex, as illustrated in Fig. 1. A routing process eventually ends up in one of the cyclic attractors, the members of the attractors benefiting hence from a continuous flow of information arriving from the respective basins of attraction. We have shown previously that the geometric arrangement of the attractors on the network gives rise in the thermodynamic limit to a non-trivial distribution for the information centrality, which measures the number of attractors intersecting at a given vertex.[15]

We present here the solution for two types of routing models, Markovian routing in the absence of a routing memory and vertex routing in the presence of an one-step memory. The solutions are asymptotically exact in the ther-

modynamic limit $N \to \infty$, they can be evaluated for large networks containing thousands to millions of sites. We present results for the scaling behavior of the overall number of attractors and for the mean of the cycle length distribution. We find that the number of cycles increases as $\log(N)$ and that the mean cycle length scales like $\sqrt{N}/\log(N)$ and $N/\log(N)$, respectively, for the model without and with routing memory.

We also derive rigorous results for the case of stochastic sampling of phase space, which yields a cycle length distribution weighted by the size of the respective basins of attraction. This kind of "on the fly" sampling is generically equivalent to an experimental observation of a real-world dynamical system. We find power law scaling for on-the-fly sampling, logarithmic corrections are absent. We conclude that real-world investigations of scaling in complex dynamical systems, like the brain, need to be interpreted carefully.

II. MODELS

The two classes of models we consider differ with respect to the absence/presence of a routing memory. The phase space volume Ω is, respectively, linear and quadratic in the number of vertices N.

- For the Markovian model, the selection of the next active vertex is independent of the previous state.[16] At every point in time only one vertex is active, the vertex with the information package. The phase space is hence identical with the collection of vertices; $\Omega = N$;
- For the vertex routing model, the phase space is given by the collection of directed links; $\Omega = N(N-1)$. At every point in time one directed link is active, the link currently transporting the information package, compare Fig. 1.

In both setups the routing of information packages is realized through static routing tables. For every incoming edge the routing table specifies an allowed outgoing edge. A vertex k will transmit an information package, which was received from a vertex j, to a specific neighboring vertex i. The vertex routing table $\hat{T}$ corresponds to a tensor of binary elements $T_{ikj} = (\hat{T})_{ikj} \in \{0,1\}$,

$$T_{ikj} = \begin{cases} 0 & \text{no routing allowed} \\ 1 & \text{routing from} \end{cases} \quad \vec{e}_{jk} \text{ to } \vec{e}_{ki} , \quad (1)$$

where $\vec{e}_{jk}$ denotes a directed edge from vertex j to vertex k. An example of a routing table for a four-site network is presented in Fig. 1. In Fig. 1(a), allowed routing paths are color coded and mapped to a four-site network. The complete phase space of this network is obtained by representing each edge (Fig. 1(b)) as a node in an iterated graph which is shown in Fig. 1(c). Here, each node corresponds to a same colored and numbered edge shown in Fig. 1(b). In Fig. 1(d), we show again a single realization of routing tables, but now in the iterated phase space graph. The edges of the phase space graph shown correspond to allowed routing directions, that is, to non-zero entries of the routing table $\hat{T}$.

We consider here critical models, *viz.*, models where the number of information packages is conserved. When the
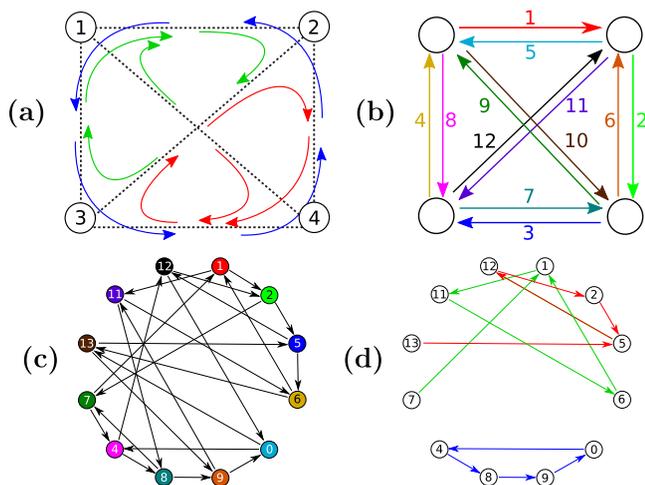

FIG. 1. Vertex routing dynamics for a $N=4$ complete graph (a) A realization of the routing tables. Routing through the first vertex follows $T_{312} = T_{213} = T_{214} = 1$, with all other T_{i1j} vanishing. There are three cyclic attractors, namely (123), (243), and (1342). (b) Enumeration of all $N(N-1)=12$ directed edges, the phase-space elements. (c) The corresponding phase-space graph. (d) The same realization of the routing table as in (a), now in terms of the phase-space graph.

information is received along edge $\vec{e}_{jk}$, it can hence be transmitted along only one outgoing edge $\vec{e}_{ki}$,

$$\sum_i T_{ikj} = 1, \qquad \sum_{ij} T_{ikj} = z_k, \qquad (2)$$

the non-zero entries of the routing table are drawn randomly. Here, z_k is the degree of vertex k, which is $N - 1$ for fully connected networks considered here. For the Markovian model, the routing table T_{ikj} is independent of j, that is, routing depends only on the node which received the information package and not on the direction along the information was received.

III. CYCLE LENGTH DISTRIBUTION

The dynamics consists of random walks through configuration space, as illustrated in Fig. 2. One can hence adapt the considerations,[2] used for solving the Kauffman network for large connectivity $K \to \infty$, in order to solve the vertex routing model analytically. In addition to the previously derived expression for cycle length distribution in the case of the Markovian model,[15] we present here the solution of the vertex routing model.

The general expression for the average number of cycles $\langle C_L \rangle$ of length L is given by

$$\langle C_L \rangle_N(r) = \frac{N(N-1)^r}{L(N-1)^{r+1}} \; q_r(t = L - 1), \qquad (3)$$

where $r = 0$ for the Markovian model and $r = 1$ for the vertex routing model. Here, the factor $1/L$ cancels overcounting of a cycle of length L, while the factor $N(N-1)^r$ is the number of phase space elements, that is, the number of possible starting elements. The factor $1/(N-1)^{r+1}$ gives the probability to close the cycle exactly at the starting phase space element. For the Markovian model the probability to close the cycle at the starting node is inversely proportional to the number of neighbors, whereas in the vertex routing model this probability is inversely proportional to the squared number of neighbors as the initial edge has to be matched for closing the path (see Fig. 2). The $q_r(t = L - 1)$ is the probability that a path containing L nodes is still open. At a time step $t = 0, 1, \ldots$, we have already visited t nodes. Thus, a probability that the next node in the sequence was already visited is $t/(N-1)$. For the trajectory to enter a cycle, the routing has

to retrace the existing path. The probability for this to happen is $1/(N-1)^r$. The relative probability of closing the path at next time step is then $\rho_r(t) = t/(N-1)^{r+1}$.

The probability of still having an open path after $t + 1$ steps is

$$q_r(t+1) = q_r(t)(1 - \rho_r(t)). \qquad (4)$$

Expanding the equation till the term $q_r(1) = 1$ and substituting the expression for relative probability one obtains

$$q_r(t) = \frac{((N-1)^{r+1} - 1)!}{(N-1)^{(r+1)(t-1)}((N-1)^{r+1} - t)!} . \qquad (5)$$

Substituting Eq. (5) in Eq. (3) for the Markovian model, given by $r = 0$, one finds

$$\langle C_L \rangle_m(N) = \frac{N!}{L(N-1)^L(N-L)!} \qquad (6)$$

for the average number of cycles of length L. For the vertex routing model, given by $r = 1$, the average number of cycles is

$$\langle C_L \rangle_v(N) = \frac{N((N-1)^2)!}{L(N-1)^{2L-1}((N-1)^2 + 1 - L)!} . \qquad (7)$$

Note that for the Markovian model the cycle length L falls within a range $\{2, N\}$, while $L \in \{2, (N-1)^2 + 1\}$ for the vertex routing model.

Relation (7) is an approximation to the average number of cycles as it does not take into account corrections for self intersecting paths. These corrections drop, however, as $1/N$ and can be neglected in the thermodynamic limit. Furthermore, the graph of the phase space elements (see Fig. 1(c)) is not fully connected and thus not Hamiltonian for arbitrary network size N, which means that cycle visiting every element of the phase space do in general not exist. Formulas (6) and (7) are based on a mapping to random maps and can be generalized to the case of routing on NK networks.

The probability of observing a cycle of length L is obtained by dividing the average number of cycles of length L from Eqs. (6) and (7) by the total number of cycles in a single realization of the routing table which is given as

$$\langle n \rangle_{v,m} = \sum_L \langle C_L \rangle_{v,m} .$$

We denote with

$$\rho_{m,v}(L,N), \qquad \sum_L \rho_{m,v}(L,N) = 1$$

the normalized cycle length distributions for the Markovian (m) and for the vertex routing model (v), Note that substituting N by $(N-1)^2 + 1$ in Eq. (6) one obtains for large N the approximate scaling relation
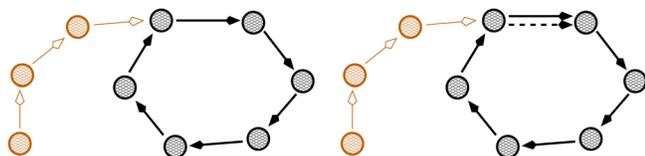

FIG. 2. Random walks through configuration space for the Markovian model (left) and for the vertex routing model (right). In order to find an attractor independent of the size of their basins of attraction (light color) one needs to close the path at the respective starting points. The probability to find a given attractor is, on the other side, proportional to the size of its basin of attraction for stochastic 'on the fly' sampling of phase space.

$$\langle C_L \rangle_v (N) \sim \langle C_L \rangle_m ((N-1)^2 + 1) \qquad (8)$$

between the number of cycles of the vertex routing and the Markovian model, $\langle C_L \rangle_v$ and $\langle C_L \rangle_m$.

IV. RESULTS

The analytic expressions (6) and (7) for the number of attractors are valid for quenched dynamics,[2] viz., for fixed routing tables. One can, in addition, evaluate the number of cycles obtained when randomly sampling phase space, which corresponds to generating the routing tables on the fly. The corresponding results will be discussed in Sec. IV B.

A. Quenched dynamics

Evaluating numerically the number of cycles (6) and (7) we find, see inset of Fig. 3, that the total number of attractors

$$\langle n \rangle_{v,m} = \sum_L \langle C_L \rangle_{v,m} \qquad (9)$$

grows logarithmically, as a function of phase space volume Ω. This result is consistent with a direct evaluation of the number of attractors for random maps.[17] The total number of cycles hence grows slower than any polynomial of the number of vertices N, in contrast to critical Kauffman models, where it grows faster than any power of N.[8]

The normalized cycle length distributions $\rho_{v,m}(L) = \langle C_L \rangle_{v,m} / \langle n \rangle_{v,m}$ thus scale as $1/\log(\Omega)$, due to the divisor $\langle n \rangle_{v,m}$. The rescaled distributions $\log(\Omega)\rho_{v,m}(L)$ approach the thermodynamic limit rapidly, compare Fig. 3. For small cycle lengths L, the limiting functional form of the rescaled distributions is $2/L$, while for large $L \to L_{max}$ it falls off as $(1 - L/L_{max})^{(L-L_{max}-1/2)} e^{-L}$. The limiting behavior of

$\log(\Omega)\rho_{v,m}(L)$ is identical for both models, due to the inter-model scaling relation (8).

The total cycle length, viz., the combined length of all cyclic attractors present for a given system size N, is on the average

$$\langle T \rangle_{v,m} = \sum_L L \langle C_L \rangle_{v,m} . \qquad (10)$$

The total cycle length follows a polynomial growth as the function of phase space volume Ω (see the inset of Fig. 3). This algebraic dependence of the total cycle length can be obtained analytically by generalizing the analysis[17] for the $N \to \infty$ limiting behavior of the mean cycle length (9) to $\langle T \rangle_{v,m}$.

The determination of the scaling behavior is somewhat more subtle for the mean cycle length (see Fig. 4).

$$\langle L \rangle_{v,m} = \frac{\langle T \rangle_{v,m}}{\langle n \rangle_{v,m}} = \sum_L L \rho_{v,m}(L). \qquad (11)$$

We find that the functional dependence on the phase space volume is best reproduced by $a + b\sqrt{\Omega}/\log(\Omega) + c/\log(\Omega)$, where a, b, c are free parameters. This assumption perfectly fits mean cycle length, whereas assuming a power law $a\prime + b\prime\Omega^{c\prime}$ leads to a worse fit of the mean cycle length for the case of quenched dynamics; the opposite will hold in the case of stochastic sampling of phase space. This dependence is obtained by keeping the fastest growing terms of mean cycle length as $\Omega \to \infty$. Note that a, and, respectively, $a\prime$, are finite size corrections not obtainable when evaluating analytically the scaling of Eqs. (9) and (10) separately. Interestingly, log-corrections to power law scaling have been studied also in sandpile models at the upper critical dimension[20] and in epidemic percolation.[21] An overview of the obtained scaling relations is given in Table I, where
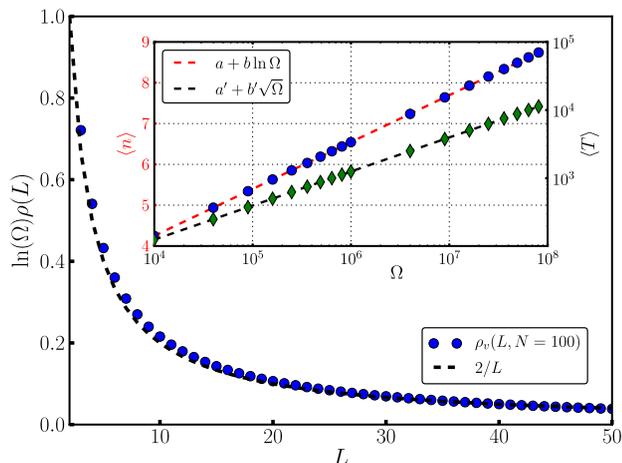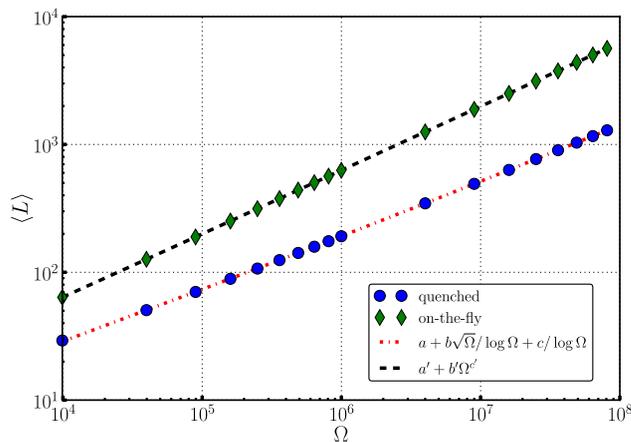

FIG. 3. The cycle length distributions $\rho_v(L)$, rescaled by $\log(\Omega)$, for the vertex routing model. The dashed line, $2/L$, represents the large-N and small-L limiting behavior. In the inset two quantities are plotted as a function of the phase space volume Ω. The average number of cycles $\langle n \rangle$ (see Eq. (9), filled blue circles, log-linear plot) and the expected total cycle length $\langle T \rangle$ (see Eq. (10), green filled diamonds, log-log plot). Also included are fits using $a + b \ln \Omega$ (red dashed line), with $a = -0.345(3)$ and $b = 0.4988(2)$, and using $a\prime + b\prime\sqrt{\Omega}$ (black dashed line) with $a\prime = -0.3311(5)$ and $b\prime = 1.25331 \pm 2 \cdot 10^{-7}$. The coefficient of determination is $R^2 = 1.0$ in both cases, within the numerical precision.

FIG. 4. Log-log plot, as a function of the phase space volume Ω, of the mean cycle lengths $\langle L \rangle_{v,\tilde{v}}$, see Eq. (11), for the vertex routing with quenched dynamics ($\langle L \rangle_v$, blue circles) and the vertex routing with on the fly dynamics ($\langle L \rangle \tilde{v}$, green diamonds). The dotted and dashed lines are fits using $a + b\sqrt{\Omega}/\log(\Omega) + c/\log(\Omega)$ and $a\prime + b\prime\Omega^{c\prime}$, respectively, with $a = 8.1(8)$, $b = 2.6035(9)$, $c = -69(9)$, and $a\prime = 1.3319(3)$, $b\prime = 0.627 \pm 2 \cdot 10^{-6}$, $c\prime = 0.5 \pm 9 \cdot 10^{-8}$. The coefficient of determination is $R^2 = 1.0$ in both cases, within the numerical precision.

TABLE I. Scaling relations, as a function of the number of vertices N, for the number of cycles and for the mean of the cycle length distribution, respectively, for vertex routing (v) and the Markovian (m) model. The routing table distribution is either quenched (exact result) or generated on the fly, as it corresponds to a stochastic sampling of phase space. Only relative quantities can be evaluated for on the fly dynamics.

		Quenched	on the fly
(v)	Number of cycles	$\log(N)$	—
	mean cycle length	$N/\log(N)$	N
(m)	Number of cycles	$\log(N)$	—
	mean cycle length	$\sqrt{N}/\log(N)$	$\sqrt{N}$

"quenched dynamics" denotes the results for quenched distributions of routing tables (exact result). Note that in Figs. 3 and 4 we present only the data for the vertex routing model as it completely overlaps for large phase spaces Ω, due to the scaling (8), with the results for the Markovian model.

B. Stochastic sampling of phase space

In addition to working with predetermined (quenched) vertex routing tables one can generate dynamics "on the fly" without explicitly creating initially routing tables for all vertices of the network. For this kind of dynamics, which correspond to a stochastic sampling of phase space, a routing for a given vertex is selected only when the trajectory visits this vertex. A cyclic attractor is then found when one state of the phase space (edge or node) is visited more then once. The probability to find a cycle is hence weighted by the size of its basin of attraction.

The probability of observing a closed cycle of length L in a randomly generated path of length t after a total number of t routing steps is

$$p(L|\ t) = \frac{\Theta(t-L)\Theta(L-2)}{t-1}, \qquad (12)$$

where $\Theta(x)$ is the Heaviside step function with $\Theta(0) = 1$. The joint probability distribution $P(L,\ t)$ is given as $P(L,t) = p(L|\ t)p_t$, where $p_t = q_t\rho_t$ is the probability of closing a cycle at the next time step $t+1$. Then, the probability of generating a cycle of length L becomes simply the sum over all possible path lengths, with the maximum path length $t_{\max} = N$ for the Markovian routing and $(N-1)^2 + 1$ for routing with memory. Thus, the probability to find an L-cycle is

$$\tilde{\rho}_v(L,N) = \sum_{t=L}^{L_{max}} \frac{((N-1)^2)!}{(N-1)^{2t}((N-1)^2 + 1 - t)!},$$

where we denoted with $\tilde{\rho}_v(L,N)$ the weighted cycle length distribution for the vertex routing model, viz., the cycle length distribution for on-the-fly dynamics. An analogous relation holds for the Markovian model. By generalizing the scaling relation (8), one finds $\tilde{\rho}_v(L,N) = \tilde{\rho}_m(L, (N-1)^2 + 1)$ and consequently $\langle L \rangle \tilde{v}(N) = \langle L \rangle \tilde{m}((N-1)^2 + 1)$, where $\tilde{\rho}_m$ denotes the weighted cycle length distributions for the Markovian model.

Fitting the data, as shown in Fig. 4 for the vertex routing model, with and without log-corrections, we find evidence for a scaling $\sim N$ and $\sim\sqrt{N}$ for the mean cycle lengths of the vertex routing and the Markovian model, respectively, with on-the-fly dynamics. Note that the overall number of cycles cannot be obtained when routing on the fly, only relative quantities can be evaluated.

V. DISCUSSION

For Boolean networks, the phase space volume Ω is 2^N and hence grows exponentially with the number of vertices N. The fact,[8] that the number of attractors grows faster than any power of N could in principle be related to the exponential growth of the phase space volume. Our results, however, show that the critical properties of the Kauffman networks for connectivity $Z = 2$ and of the vertex routing models considered here are not related. The scaling $\sim\log(\Omega)$ valid for vertex routing models would imply a polynomial scaling with the system size

$$\log(\Omega) \sim N, \qquad \Omega = 2^N$$

for critical Kauffman nets, which are, however, not observed.[8] Our results hence indicate that scaling in critical dynamical systems may generically be non-universal, depending on the details of the microscopic dynamics.

We also note that other properties of critical dynamical systems, like the scaling of the number of frozen or relevant nodes for critical Boolean networks,[9] may show highly non-trivial behavior. For the case of vertex routing models, one may define a measure of centrality, information centrality, determined by the number of attractors intersecting a given vertex, which scales to a non-trivial limiting distribution in the thermodynamic limit.[15]

Our results may also be seen in the context of the surge in interests in modelling[18,19] and in experimentally investigating[22,23] the spontaneous neural dynamics of the brain. The observation of power law scaling relations[24] has been interpreted as evidence of a critical self-organized neural state.[25] The power law scaling in neural activity was observed in spite of strong sub-sampling of neural avalanches resulting from small number of electrodes relative to total number of neurons within the cortex. Priesemann and colleges[26] have recently demonstrated that sub-sampling of critical avalanches results in the loss of power law scaling. This suggests that the power law scaling of neural avalanches observed in various experiments in spite of sub-sampling, might have different origins.

Our results suggest, to some extent, that there is no universal relation in dynamical systems theory between criticality and power law scaling and that scaling is generically dependent on the observation modus. The unbiased statistics of a certain property, like the number of attractors or avalanches, may differ from a statistics obtained via stochastic sampling ($\rho_{v,m}(L)$ and $\tilde{\rho}_{v,m}(L)$ in our case). The later will in general be dependent on the size of the respective basins of attraction of the dynamical process considered, viz., of a cycle or an avalanche. For the case of the vertex routing

models studied here we found logarithmic corrections to power law scaling for the unbiased, quenched statistics and pure power law scaling for stochastic on the fly sampling. We conclude that experimental observations of real-world systems, when investigating scaling, need to be interpreted carefully.

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