Commuting Simplicity and Closure Constraints for 4D Spin Foam Models

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Abstract

Spin Foam Models are supposed to be discretised path integrals for quantum gravity constructed from the Plebanski-Holst action. The reason for there being several models currently under consideration is that no consensus has been reached for how to implement the simplicity constraints.

Indeed, none of these models strictly follows from the original path integral with commuting B fields, rather, by some non standard manipulations one always ends up with non commuting B fields and the simplicity constraints become in fact anomalous which is the source for there being several inequivalent strategies to circumvent the associated problems.

In this article, we construct a new Euclidian Spin Foam Model which is constructed by standard methods from the Plebanski-Holst path integral with commuting B fields discretised on a 4D simplicial complex. The resulting model differs from the current ones in several aspects, one of them being that the closure constraint needs special care. Only when dropping the closure constraint by hand and only in the large spin limit can the vertex amplitudes of this model be related to those of the FK_γ Model but even then the face and edge amplitude differ.

Interestingly, a non-commutative deformation of the B^{IJ} variables leads from our new model to the Barrett-Crane Model in the case of $\gamma = \infty$.

Contents

1 Introduction

Loop Quantum Gravity (LQG) is an attempt to make a background independent, non-perturbative quantization of 4-dimensional General Relativity (GR) – for reviews, see [1, 2, 3]. It is inspired by the formulation of GR as a dynamical theory of connections [4]. Starting from this formulation, the kinematics of LQG is well-studied and results in a successful kinematical framework (see the corresponding chapters in the books [1]), which is also unique in a certain sense [5]. However, the framework of the dynamics in LQG is still largely open so far. There are two main approaches to the dynamics of LQG, they are (1) the Operator formalism of LQG, which follows the spirit of Dirac quantization of constrained dynamical system, and performs a canonical quantization of GR [6, 7]; (2) the Path integral formulation of LQG, which is currently understood in terms of the Spin-foam Models (SFMs) [3, 10, 11, 12, 13]. The relation between these two approaches is well-understood in the case of 3-dimensional gravity [14], while for 4-dimensional gravity, the situation is much more complicated and there are some attempts [15] for relating these two approaches.

The present article is concerned with the following issue in the framework of spin-foam models. The current spin-foam models are mostly inspired by the 4-dimensional Plebanski formulation of GR [16] (Plebanski-Holst formulation by including the Barbero-Immirzi parameter γ), whose action reads

$$S_{\mathrm{PH}}[A,B,\varphi]:=\int\left(B+\frac{1}{\gamma}*B\right)^{IJ}\wedge F_{IJ}+\frac{1}{4}\int \mathrm{d}^4x\ \varphi^{\alpha\beta\gamma\delta}B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta}\epsilon_{IJKL} \tag{1.1}$$

where B is a so(4)-valued 2-form field, $F:=\mathrm{d}A+A\wedge A$ is the curvature of the so(4)-connection field A and $\varphi^{\alpha\beta\gamma\delta}=\varphi^{[\alpha\beta][\gamma\delta]}$ is a densitized tensor, symmetrized under interchanging $[\alpha\beta]$ and $[\gamma\delta]$, and traceless $\epsilon_{\alpha\beta\gamma\delta}\varphi^{\alpha\beta\gamma\delta}=0$. For the illustrative purposes of this article, we consider only Euclidean GR in the present article, however, the lessons learnt will extend also to the Lorentzian theory. One can show that the equations of motion implied by the Plebanski-Holst action are equivalent to the Einstein equations of GR. Moreover, if we consider formally the following path integral partition function of the Plebanski-Holst action and perform the integral of $\varphi^{\alpha\beta\gamma\delta}$

$$Z:=\int[DA\ DB\ D\varphi]\ e^{iS_{\mathrm{PH}}[A,B,\varphi]}=\int[DA\ DB]\ \delta\left(\epsilon_{IJJKL}B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta}-\mathcal{V}\epsilon_{\alpha\beta\gamma\delta}/4!\right)\ e^{i\int\left(B+\frac{1}{\gamma}*B\right)^{IJ}\wedge F_{IJ}} \tag{1.2}$$

we obtain the partition function of BF theory [17] whose paths are, however, constrained by 20 *Simplicity Constraint* equations

$$\epsilon_{IJJKL}B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta}-\frac{1}{4!}\mathcal{V}\epsilon_{\alpha\beta\gamma\delta} \tag{1.3}$$

The point of this formulation is of course that the path integral of BF theory has been formulated as a concrete spin-foam model (subject to the divergence issue, see the corresponding chapters in [1]) and thus the idea is to rely on those results and to implement the simplicity constraints properly into the partition function of BF theory. We remark that even for Euclidian gravity, the partition function (1.2) is unlikely to be derived from the canonical formulation because of the presence of second class constraints which affect the choice of the measure in (1.2), see the first and third reference in [15] for a detailed discussion. Since in current spin foam models the proper choice of measure is also regarded as a nontrivial problem and as we want to draw attention to a different issue for the current spin foam models, we also will not deal with the measure issue in this article and leave this for future research.

The partition function of BF theory, after discretization on a 4-dimensional simplicial complex $\mathcal{K}$ and its dual complex $\mathcal{K}^*$, can be expressed as a sum over certain spin-foam amplitudes. Here a spin-foam amplitude is obtained by (1) assigning an SO(4) unitary irreducible representation to each triangle f of $\mathcal{K}$ (we label the representation by a pair (j^+_f,j^-_f) for each triangle); (2) assigning a 4-valent SO(4) intertwiner to each tetrahedron t of $\mathcal{K}$ (we label the intertwiner by a pair (i^+_t,i^-_t) for each tetrahedron). Then the partition function of BF theory can be written as

$$Z_{\mathrm{BF}}(\mathcal{K})=\sum_{\{j^\pm_f\}_f}\sum_{\{i^\pm_t\}_t}\prod_f\dim(j^+_f)\dim(j^-_f)\prod_\sigma\{15j\}_{\mathrm{SO}(4)}\left(j^\pm_f,i^\pm_t\right) \tag{1.4}$$

where the 15j-symbol is the 4-simplex/vertex amplitude corresponding to the 4-simplex σ. The partition function $Z_{\rm BF}$ turns out to be formally independent of the triangulation $\mathcal{K}$. Clearly, as shown explicitly in Eq.(1.2), in order to obtain the partition function for quantum gravity as a sum of spin-foam amplitudes, one has to impose the simplicity constraint in the BF theory measure. When doing that, the resulting partition function is no longer triangulation independent[1] and thus one should in fact consider all possible discretizations and not only simplicial ones. This is also necessary in order to make contact with the canonical LQG Hilbert space which contains all possible graphs and not only 4-valent ones. This has been recently emphasised in [8, 9] and the current spin foam models already have been generalised in that respect. We believe our model also to be generalisable but will not deal with this aspect in the present work as this would draw attention away from our main point.

Essentially, the very method of imposing the simplicity constraint *defines* the corresponding candidate spin-foam model for quantum gravity which why its proper implementation deserves so much attention. Currently the three most studied spin-foam models for quantum gravity (in Plebanski or Plebanski-Holst formulation) are the Barrett-Crane Model [10], the EPRL Model [11], and FK$_\gamma$ Model [12]. These three, a priori, different models are defined by three different ways to impose simplicity constraint on the measure of the BF partition function $Z_{\rm BF}$. We will review these different methods of imposing the simplicity constraint briefly in what follows.

First of all, in the context of the discretized path integral, the simplicity constraint also takes a discretized expression. For each triangle f we define an so(4) Lie algebra element B_f which corresponds to the integral of the two form B over the triangle f. Then in terms of the B_f for each 4-simplex σ the discretised simplicity constraints read

$$\epsilon_{IJKL}B^{IJ}_fB^{KL}_{f'}=0,\qquad f,f' \text{ belong to the same tetrahedron } t \tag{1.5}$$

$$\epsilon_{IJKL}B^{IJ}_{f_1}B^{KL}_{f_1'}=\epsilon_{IJKL}B^{IJ}_{f_2}B^{KL}_{f_2'},\qquad f_i,f_i' \text{ belong to the two different tetrahedrons in } \sigma \tag{1.6}$$

The Barrett-Crane Model, the EPRL Model, and the FK$_\gamma$ Model all explicitly impose the first type of simplicity constraint Eq.(1.5), called tetrahedron constraint, in some way to the spin-foam partition function of BF theory. On the other hand, all of them replace the second type of simplicity constraint, called 4-simplex constraint Eq.(1.6) by the so called *Closure Constraint*

$$\sum_{f\subset t}B^{IJ}_f=0\qquad \text{for each tetrahedron } t. \tag{1.7}$$

It is not difficult to see that the closure constraints together with the tetrahedron constraints imply the 4-simplex constraints but not vice versa. Thus, sprictly speaking, imposing the closure constraint constrains the BF measure more than the classical theory would precribe. It is unknown and also beyond the scope of the present paper whether this replacement is harmless or is in conflict with the classical theory. In this paper, as we are merely interested in comparing the standard way of imposing the simplicity constraints (commuting B fields) with the non standard methods defining the BC, EPRL and FK models (non commuting B fields), we proceed as in those other spin foam models and also replace the 4-simplex constraint by the closure constraint. To distinguish these two different types of constraints, in what follows we use the terminology "simplicity constraint" for Eq.(1.5) and "closure constraint" for Eq. (1.7). Notice that the BC Model, EPRL Model, and FK$_\gamma$ Model argue that the closure constraint is "automatically" implemented in their spin-foam amplitude. We will come back to this argument in a moment. Because of that argument, in none of these models the closure constraint is further analysed. The proper implemementation of the simplicty and closure constraints is one of the most active research areas in the spin foam model community and there are many issues that yet have to be understood [18].

For both the Barrett-Crane Model and EPRL Model, the strategy for imposing the simplicity constraint is the following: In order to take advantage of the knowledge of BF spin-foam Model, one formally takes the delta distribution

[1] As it should not be because GR is not a TQFT in the classical level. Triangulation independence is understood as a feature in the quantization of classical TQFT, which should not be expected in the quantization of gravity.

on the B variables out of the integral over B by a standard trick known from ordinary quantum field theories: One (formally) just has to replace B by $\delta/\delta F$ because the integrand of the B integral is of the form $\exp(iF \cdot B)$. Due to the discretization upon which F is replaced by a holonomy around a face of the dual triangulation and B by an integral over a triangle of the triangulation, $\delta/\delta F$ can be rewritten in terms of the right invariant vector fields X on the copy of $SO(4)$ corresponding to the given holonomy with holonomy dependent coefficients. One now argues that these coefficients can be replaced by their chromatic evaluation (setting the holonomy equal to unity) because the integration over B leads to $\delta(F)$ enforcing the measure on the space of connections to be supported on flat ones. Clearly, this argument is not obviously water tight because $\delta(\delta^2/\delta F^2) \cdot \delta(F)$ may not be supported at $F = 0$. In fact it should not be if we are interested in gravity rather than BF theory. See the chapter on spin-foams in the second reference of [1] for more details. In any case, this way of proceeding now leads to replacing the commutative derivations $\delta/\delta F$ by the non commutative right invariant vector fields X.

An alternative argument that has been given is the following: The kinetical boundary Hilbert space of the spin foam path integral should be the canonical LQG Hilbert space (restricted to the 4-valent boundary graph of the given simplicial triangulation) and here the B field would be quantised as $\delta/\delta A$ where A is the underlying connection. On functions of holonomies this again becomes a right invariant vector field labelled by the triangles dual (in the 3D sense) to the corresponding boundary edges which in turn correspond to the faces of the dual triangulation dual (in the 4D sense) to those triangles. The physical boundary Hilbert space should therefore be the kernel of that quantised boundary simplicity constraints. In order to write the corresponding spin foam model, one has to define the projector on that physical Hilbert space. To do this properly, one should canonically quantize Plebanski – Holst gravity, identify all the first and second class constraints and define the projector via Dirac bracket and group averaging which then leads to a spin foam path integral. How complicated this becomes if one really performs all the necessary steps is outlined in [15]. However, this is not what is done in [11]. The first observation is that since the spin foam path integral naturally involves SO(4), the kinematical boundary Hilbert space is naturally also in terms of SO(4) spin network functions. One now studies the restrictions that the simplicity constraints impose on the spins and intertwiners of the boundary SO(4) Hilbert space spin network functions. The detailed structure of these restrictions suggests a natural one to one map with spin network states in the canonical SU(2) Hilbert space. Finally, using locality arguments, one conjectures that these restrictions should not only hold on the boundary but also in the bulk of the BF SO(4) spin foam model. See [35] for a particularly simple and clear exposition of this procedure. It has recently been criticised in [18] on the ground that the BF symplectic structure and the LQG symplectic structure have wrongly been identified in the afore mentioned identification map.

In any case, whether or not the map is the correct correpondence, the simplicity constraints were again quantised as non commuting (anomalous) constraints. If one understands the kernel in the strong operator topology then one obtains the BC model, if one understands it in the weak operator topology (Gupta – Bleuler procedure) one obtains the EPRL model. Because of the anomaly, imposing the constraint operators strongly apparently makes the Barrett-Crane Model lose some important information about non-degenerate quantum geometry [19]. Imposing the constraints weakly is less restrictive and thus may lead to a better behaved model. More in detail, first of all the quadratic expression of the simplicity constraint Eq.(1.5) is replaced by a linearized expression. It is given by asking that for each tetrahedron t, there exists a unit vector u_t^I, such that

$$* B_f^{IJ} u_{t,I} = 0 \tag{1.8}$$

The equivalence of the linearized simplicity constraint Eq.(1.8) with original simplicity constraint Eq.(1.5) will be reviewed in Section 3 (in the gravitational sector of the solution). In the original construction of EPRL spin-foam model in [11], the unit vector u_t^I is gauge fixed to be $\delta^{0,I}$, and a "Master constraint" $M_f := \sum_j C_f^j C_f^j$ is defined (to replace the cross-diagonal part of the simplicity constraint Eq.(1.5)), where $C_f^j := *B_f^{0j}$ from Eq.(1.8). The corresponding "Master constraint operator" is defined by replacing B_f^{IJ} by right invariant derivatives. This Master constraint solves the problem of non-commutativity/anomaly of the quantum simplicity to a certain extent, because a single Master constraint replaces all the cross-diagonal components of Eq.(1.5). Moreover the diagonal part of Eq.(1.5) and this Master constraint operator restrict the Hilbert space spanned by the 4-valent SO(4) spin-networks to its subspace, which can be identified with 4-valent SU(2) spin-networks and thus can be imbedded into the kinematical Hilbert space of LQG. For each of these SU(2) spin-networks, the SU(2) unitary irreducible representations labelled by $k \in \frac{1}{2}\mathbb{N}$ has the following relation with the original SO(4) representations on all the boundary edges dual to the boundary triangles

$$j^{\pm} = \frac{|1 \pm \gamma|}{2} k \tag{1.9}$$

Here the Barbero-Immirzi parameter γ can only take discrete values, i.e.

$$\begin{aligned} \text{If } |\gamma| > 1: \qquad & \gamma = \frac{j_f^+ + j_f^-}{j_f^+ - j_f^-} \\ \text{If } |\gamma| < 1: \qquad & \gamma = \frac{j_f^+ - j_f^-}{j_f^+ + j_f^-} \end{aligned} \tag{1.10}$$

More importantly, the recent results in [20, 9] show that the boundary Hilbert space used in the EPRL Model solves the linear version of simplicity constraint Eq.(1.8) (and the closure constraint Eq.(1.7)) weakly, i.e. the matrix elements (with respect to the boundary SO(4) Hilbert space) of the constraint operators vanish on the space of solutions

$$\left\langle f, \hat{C} f' \right\rangle = 0, \qquad \text{for all } f, f' \text{ in the Hilbert space of solutions.} \tag{1.11}$$

in contrast to the strong implementation of the constraints in the Barrett-Crane Model. Finally the (Euclidean) EPRL spin-foam partition function is expressed by

$$Z_{\text{EPRL}}(\mathcal{K}) = \sum_{\{k_f\}_f} \sum_{\{i_t\}_t} \prod_f \dim(k_f) \prod_\sigma \sum_{i_t^\pm} \{15j\}_{\text{SO}(4)} \left(j_f^\pm, i_t^\pm\right) \prod_{\overrightarrow{(\sigma,t)}} f^{i_t}_{i_t^+, i_t^-} \left(j_f^\pm, k_f\right) \tag{1.12}$$

where for each spin-foam amplitude, an SU(2) unitary irreducible representation k_f is assigned to each triangle f, satisfying the relation Eq.(1.9), and an SU(2) 4-valent intertwiner i_t is assigned to each tetrahedron t. Here

$$\sum_{i_t^\pm} \{15j\}_{\text{SO}(4)} \left(j_f^\pm, i_t^\pm\right) \prod_{\overrightarrow{(\sigma,t)}} f^{i_t}_{i_t^+, i_t^-} \left(j_f^\pm, k_f\right) \tag{1.13}$$

is the 4-simplex/vertex amplitude for the EPRL Model, where $f^{i_t}_{i_t^+, i_t^-}$ are a fusion coefficients defined in [11].

The FK$_\gamma$ Model follows a different strategy to impose the simplicity constraint, namely by using the coherent states for SU(2) group [21, 22]. Given a unitary irreducible representation space V^j of SU(2), the coherent state is defined by

$$|j, n\rangle := n\,|j, j\rangle = \sum_{m=-j}^{j} |j, m\rangle\, \pi^j_{mj}(n) \qquad n \in \text{SU}(2) \tag{1.14}$$

We then immediately have the resolution of identity on V^j

$$1_j = \dim(j) \int_{SU(2)} \mathrm{d}n \;\; |j, n\rangle \langle j, n| \tag{1.15}$$

This coherent state has a certain geometrical interpretation, which can be seen by computing the expectation value of the su(2) generator (σ_i are Pauli matrices)

$$\langle j, n|\, \hat{X}\, |j, n\rangle = \langle j, n|\, \hat{J}^i\, |j, n\rangle\, \sigma_i = j n \sigma_3 n^{-1} \tag{1.16}$$

If we identify the Lie algebra su(2) with $\mathbb{R}^3$, we can see that the coherent state $|j,n\rangle$ describes a vector in $\mathbb{R}^3$ with length j, its direction is determined by the action of n on a unit reference vector (the direction of σ_3). From the expression $n\sigma_3 n^{-1}$ we see that n can be parameterized by the coset $\mathrm{SU}(2)/\mathrm{U}(1)=S^2$. In addition, the integral in the resolution of identity is essentially over $\mathrm{SU}(2)/\mathrm{U}(1)=S^2$. It is not hard to show that the (Euclidean) BF partition function can be expressed in terms of the coherent states (we write (g^+,g^-) for each SO(4) element, (j^+,j^-) for an SO(4) unitary irreducible representation)

$$\begin{aligned} Z_{\mathrm{BF}}(\mathcal{K}) &= \sum_{\{j_f^\pm\}_f}\prod_f \dim(j_f^+)\dim(j_f^-)\int\prod_{(\sigma,t)}\mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \\ &\prod_{(t,f)}\dim(j_f^+)\dim(j_f^-)\int \mathrm{d}n^+_{tf}\mathrm{d}n^-_{tf}\prod_{(\sigma f)}\left\langle j_f^+,n_{tf}^+\right| g^+_{t\sigma}g^+_{\sigma t}\left|j_f^+,n^+_{t'f}\right\rangle\left\langle j_f^-,n_{tf}^-\right| g^-_{t\sigma}g^-_{\sigma t}\left|j_f^-,n^-_{t'f}\right\rangle \end{aligned} \quad (1.17)$$

where $(g^+_{\sigma t},g^-_{sigt})$ is a SO(4) holonomy along the edge from the center of 4-simplex σ to the center of tetrahedron t. Then the strategy of imposing simplicity constraint in FK$_\gamma$ Model is to use the interpretation (1.16) of the coherent state labels $j_f^\pm n_{tf}^\pm\tau_3(n_{tf}^\pm)^{-1}$ as the self-dual/anti-self-dual part $X_{tf}^\pm$ of the so(4) variable B_{tf} associated with a triangle f seen from a tetrahedron t. (More precisely, we know that the previously defined B_f can be decomposed into self-dual and anti-self-dual part $X_f^\pm$. The *interpretations* of $j_f^\pm n_{tf}^\pm\tau_3(n_{tf}^\pm)^{-1}$, $X_{tf}^\pm$ are considered as the parallel transport of $X_f^\pm$ from the center of triangle f to the center of tetrahedron t, i.e. $X_{tf}^\pm=g_{tf}^\pm X_f^\pm g_{ft}^\pm$, where $g_{tf}^\pm$ is the holonomy along the edge from the center of triangle f to the center of tetrahedron t). That is, the simplicity constraint is imposed on the coherent state labels, which results in the following restrictions:

$$\frac{j^+}{j^-}=\left|\frac{\gamma+1}{\gamma-1}\right|,\quad \text{and}\quad \begin{cases}(n^+_{tf},n^-_{tf})=(n_{tf}h_{\phi_{tf}},u_t n_{tf}h^{-1}_{\phi_{tf}}), & \text{for } -1<\gamma<1;\\ (n^+_{tf},n^-_{tf})=(n_{tf}h_{\phi_{tf}},u_t n_{tf}h^{-1}_{\phi_{tf}}\epsilon), & \text{for } \gamma<-1 \text{ or } \gamma>1.\end{cases} \quad (1.18)$$

where u_t is some normal to t, $h_{\phi_{tf}}$ takes values in the U(1) subgroup of SU(2) generated by σ_3 and $\epsilon=i\sigma_2$. In more detail, the proposal is then to simply replace in (1.17) $n_{tf}^\pm$ by these expressions and the Haar measure $dn^+_{tf}\,dn^-_{tf}$ by the Haar measure $dn_{tf}\;du_t\;dh_{\phi_{tf}}$. We emphasize that this is an interesting but non standard procedure: while the identification of the coherent state labels $j_f^\pm,n_{tf}^\pm$ with the so(4) variables B_{tf} is certainly well motivated, the resulting expression does not arise by integrating out the B fields in the presence of the delta distributions enforcing the simplicit constraints. Rather, in (1.17) the B fields have already been integrated out. To restrict measure and integrand by hand afterwards according to (1.18) is not obviously equivalent with the standard procedure of solving the $\delta-$distributions. One would hope that the resulting procedures coincide in the semiclassical or the "large-j" limit [23]. Indeed, the "large-j" limit result in Section 4 will support this expectation. Finally the spin-foam partition function of FK$_\gamma$ Model coincides (at least up to a slight change of edge amplitude) with EPRL partition function when the Barbero-Immirzi parameter $-1<\gamma<1$. However when $\gamma<-1$ or $\gamma>1$, FK$_\gamma$ partition function is rather different from the EPRL partition function. Here we only show explicitly the 4-simplex/vertex amplitude of FK$_\gamma$ model when $\gamma<-1$ or $\gamma>1$

$$\sum_{i_t^\pm}\{15j\}_{\mathrm{SO}(4)}\left(j_f^\pm,i_t^\pm\right)\prod_{\overrightarrow{(\sigma,t)}}f^{i_t}_{i_t^+,i_t^-}\left(j_f^\pm,k_{tf}\right) \quad (1.19)$$

Here although the relation between $j_f^\pm$

$$\frac{j^+}{j^-}=\left|\frac{\gamma+1}{\gamma-1}\right| \quad (1.20)$$

is the same as in EPRL Model, in FK$_\gamma$ model for $\gamma<-1$ or $\gamma>1$, there are some additional degrees of freedom associated with the label k_{tf}, which are the values of spins from the coupling of j_f^+ and j_f^-, i.e. k_{tf} could take values in $|j_f^+ - j_f^-|, \cdots, j_f^+ + j_f^-$. The final partition function is obtained by summing over j_f^-, i_t, and k_{tf} with some measure factors (see [12] for details).

In the previous three paragraphs, we briefly revisited the main strategies of imposing simplicity constraint in Barrett-Crane, EPRL and FK_γ Models. We have seen that these in general different spin-foam models came from two different ways of imposing simplicity constraint, i.e. Barrett-Crane and EPRL Model quantize the simplicity constraint as operators and imposed them (strongly or weakly) on the boundary spin-networks, while FK_γ Model imposes the constraint on the coherent state labels. However, as we have reviewed, none of the three models is derived from the original path integral formula Eq.(1.2) of the Plebanski action (or the discretized version of the path integral) without using some non standard methods. Therefore a natural question arises:
Is any of those three spin-foam models consistent with the path integral formula Eq.(1.2) and its discretized version? This question is non trivial because in all three types of models one deals with non commutative B fields and simplicity constraints as operators on some Hilbert space while the original path integral is in terms of commutative c-number variables so that anomalies cannot arise. Because of this issue, it is interesting to investigate what kind of spin-foam model we will obtain, if we start from the (discretization of) the path integral formula Eq.(1.2) with commutative B^{IJ} variables. It is also interesting to find some possible bridges linking the (discretization of) the path integral formula Eq.(1.2) with commutative B^{IJ} variables to the existing spin-foam models using non-commutative B^{IJ} variables.

In this article, we consider the discretization of the path integral formula Eq.(1.2), which will be Eq.(2.1). As announced in [36], in contrast to the Barrett-Crane, EPRL, and FK_γ Models, we always consider the variables B^{IJ} as commutative c-numbers. The simplicity constraint (and closure constraint) is (are) imposed by the c-number delta functions inserted in the path integral formula, which one gets by integrating over the Lagrange multiplier and which constrain the path integral measure. In our concrete analysis in Section 4, the most important difference between our derivation and the derivation in any of Barrett- Crane, EPRL, and FK_γ Models is the following: in any of Barrett-Crane, EPRL, and FK_γ Models, one always imposes the respective version of the simplicity constraint constraint on the BF spin-foam partition function Eq.(1.4) or (1.17) *after* integration over B^{IJ}. This feature is essentially the reason why it is difficult to find a relation between the simplicity constraint imposed in any of Barrett-Crane, EPRL, and FK_γ Models and the simplicity constraint in the path integral formula Eq.(1.2). By contrast, our derivation in Section 4 will *not* start from the spin-foam partition function of BF theory, but instead we impose the delta function of the simplicity constraint (and closure constraint) before the integration over B^{IJ}, and we will see that solving these constraints gives rise to a non trivial modification of the path integral measure. There were early works analyzing the simplicity constraint toward this direction, see e.g. [26].

As also announced in [36], regarding the B^{IJ} variables as commutative c-numbers also makes the treatment of closure constraint different. We know that the closure constraint Eq.(1.7) is necessary in order that the full set of simplicity constraint Eq.(1.5) and (1.6) is satisfied. In Barrett-Crane Model the closure constraint is argued to be automatically satisfied by the SO(4) gauge invariance of the vertex amplitude. However, as shown in [36], this is only true *after* performing the Haar measure integrals which essentially project everything on the gauge invariant sector. It is clear that the closure constraint must be imposed *before* performing the integral over the connections. In the EPRL Model, the argument is improved in that both simplicity constraint and closure constraint vanishes *weakly* on the EPRL boundary Hilbert space [20]. Moreover, in [24], it is shown that in both EPRL and FK_γ Model, the closure constraint can be implemented in terms of geometric quantization and by the commutativity of the quantization and phase space reduction [25]. As defined, an additional closure constraint would be redundant for both EPRL and FK_γ Model, since they are already on the constraint surface of closure constraint (if one interprets the coherent state labels to be the B^{IJ} variables), although the original definitions of both models didn't impose closure constraint explicitly. We feel that this is again due to the fact that the Haar integrals have already been performed. In our analysis we find that the implementation of closure constraint gives non-trivial restrictions on the measure.

In order to understand what happens when one ignores the clsoure constraint and to follow more closely the procedure followed by existing spin foam models, in section 4, we first consider a simplified partition function $Z_{\text{Simplified}}(\mathcal{K})$ in which the delta functions of closure constraint is dropped (as it is discussed in [26]), and derive an expression of $Z_{\text{Simplified}}(\mathcal{K})$ as a sum of all possible spin-foam amplitudes (constrained only by the simplicity constraints). Then we also compute the true partition function $Z(\mathcal{K})$ with the closure constraint implemented. When we compare $Z_{\text{Simplified}}(\mathcal{K})$ with the true partition function $Z(\mathcal{K})$, we find the closure constraint non-trivially affect the spin-foam expression of partition function. But all the spin-foams (transition channels) admitted in the simplified partition function $Z_{\text{Simplified}}(\mathcal{K})$ still contribute to the full partition function $Z(\mathcal{K})$ (with some changes for the triangle/face amplitude and tetrahedron/edge amplitude).

Another key feature of our derivation is a different discretization of the BF action. Here we first break the faces dual to the triangles into wedges (see FIG.1) and then write the discretized BF action in terms of the holonomies along the boundary of the wedges. Here, as usual, a wedge in the dual face f is determined by a dual vertex or original 4-simplex σ and thus denoted by (σ, f). Its boundary consists of four segments defined as follows: The original (piecewise linear) 4-simplex has a barycentre $\hat{\sigma}$ which is the dual vertex. The dual edges connect these barycentres. A pair of dual edges e, e' adjacent to the same dual vertex defines a face. Conversely, given a face and a dual vertex which is one of the corners of the face, we obtain two dual edges. These are dual to two tetrahedra t, t' of the original complex. The boundary of the wedge (σ, f) is now given by $(\hat{\sigma}, \hat{e}) \circ (\hat{e}, \hat{f}) \circ (\hat{f}, \hat{e}') \circ (\hat{e}', \hat{\sigma})$ where the hat denotes the respective barycentres. In an unfortunate abuse of notation which exploits the duality one also writes this as $(\sigma, t) \circ (t, f) \circ (f, t') \circ (t', \sigma)$. Using this notation we have (cf. FIG.1)

$$
\begin{aligned}
\int_M \left[B + \frac{1}{\gamma} * B\right]^{IJ} \wedge F_{IJ} \;&=\; \int_M \left(1 + \frac{1}{\gamma}\right) \operatorname{tr}\left(X^+ \wedge F^+\right) + \int_M \left(1 - \frac{1}{\gamma}\right) \operatorname{tr}\left(X^- \wedge F^-\right) \\
=\;& \sum_f \left(1 + \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^+ F_f^+\right) + \sum_f \left(1 - \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^- F_f^-\right) \\
=\;& \sum_{(\sigma,f)} \left(1 + \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^+ F^+_{(\sigma,f)}\right) + \sum_{(\sigma,f)} \left(1 - \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^- F^-_{(\sigma,f)}\right) \\
\simeq\;& \sum_{(\sigma,f)} \left(1 + \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^+ g^+_{ft} g^+_{t\sigma} g^+_{\sigma t'} g^+_{t'f}\right) + \sum_{(\sigma,f)} \left(1 - \frac{1}{\gamma}\right) \operatorname{tr}\left(X_f^- g^-_{ft} g^-_{t\sigma} g^-_{\sigma t'} g^-_{t'f}\right)
\end{aligned}
\tag{1.21}
$$

where $F_{(\sigma,f)}$ is the curvature 2-form integrated on the wedge determined by (σ, f) and t, t' respectively are the afore mentioned unique tetrahedra (or dual edges). This starting point results in the following structures in the resulting spin-foam model $Z_{\text{Simplified}}(\mathcal{K})$ (these structures turn out to be similar to the structure proposed in [26]):

- In contrast to the existing spin-foam models, where the SO(4) representations (j_f^+, j_f^-) were labeling the faces f, the new spin-foam model derived in Section 4 have SO(4) representations $(j^+_{\sigma f}, j^-_{\sigma f})$ labeling the wedges, i.e. a dual face f having n vertices (corners) in general has n different pairs $(j^+_{\sigma f}, j^-_{\sigma f})$, one for each wedge determined by the vertex dual to σ. However in the large-j limit, the triangle/ face amplitude is concentrated on SO(4) representations $j^\pm_{\sigma f} = j^\pm_{\sigma' f}$) for any vertices σ, σ' of the same face f.

- Two neighboring wedges (σ, f) and (σ', f) of a face f share a segment (t, f) (c.f. FIG.1) whose end points are the center of the face f and the center of the edge dual to the tetrahedron $t = \sigma \cap \sigma'$. For each segment (t, f) there is an SU(2) representation k_{tf} "mediating" the SO(4) representations on the two neighboring wedges, $(j^+_{\sigma f}, j^-_{\sigma f})$ and $(j^+_{\sigma' f}, j^-_{\sigma' f})$, in the sense that k_{tf} has to lie in the range of the joint Clebsh - Gordan decomposition of $j^+_{\sigma f} \otimes j^-_{\sigma f}$ and $j^+_{\sigma' f} \otimes j^-_{\sigma' f}$ (c.f. FIG.4), thus

$$
k_{tf} \in \left\{|j^+_{\sigma f} - j^-_{\sigma f}|, \cdots, j^+_{\sigma f} + j^-_{\sigma f}\right\} \cap \left\{|j^+_{\sigma' f} - j^-_{\sigma' f}|, \cdots, j^+_{\sigma' f} + j^-_{\sigma' f}\right\}.
\tag{1.22}
$$

Note that the idea for implementing c-number simplicity constraint strongly in the spin-foam model is not new, and has been employed in [26]. Some calculations, e.g. solving the simplicity constraint, toward $Z_{\text{Simplified}}(\mathcal{K})$ is similar to the derivation in [26] (especially in the first reference in [26]). However the discrete action Eq.(1.21) here is different from the one used in [26]. The action here turns out to be important to understand the non-commutative deformation and the relation to Barrett-Crane Model in Appendix A, which is one of the key points in this paper.

An interesting result from the analysis here is the relations between the new spin-foam model derived here and the existing spin-foam models e.g. Barrett-Crane, EPRL, and FK_γ Models. From the analysis in Section 4, we find that, firstly, in the large-j and large-area limit the spin-foams in our new model $Z_{\text{Simplified}}(\mathcal{K})$ reduces to the spin-foams in FK_γ Model (with identical 4-simplex/ vertex amplitude but different tetrahedron/edge and triangle/face amplitudes) at least for $|\gamma| > 1$. Secondly, in Appendix A, we study the non-commutative deformation of the partition function Eq.(2.1), in order to study how the non-commutative nature of the B^{IJ} variables in the existing spin-foam models emerges in our commutative context. The non-commutative deformation we employ here comes from a generalized Fourier transformation on the compact group [29] (the deformed partition function will be denote by $Z_\star(\mathcal{K})$). With this deformation, we find that the closure constraint really becomes redundant when we set the deformation parameter $a = \ell_p^2$, while the redundancy is hard to be shown with a general deformation parameter. With the setting of the deformation parameter $a = \ell_p^2$, we show that the non-commutative deformation of our new spin-foam model leads to Barrett-Crane model when the Barbero-Immirzi parameter $\gamma = \infty$. This result explains how the non-commutative nature of the B^{IJ} variables in Barrett-Crane model relates to the commutative context of our new spin- foam model in Section 4, and also explains to some extent the reason why in the Barrett-Crane model the closure constraint is redundant (such an explanation also appears in the first reference of [30] from the group field theory perspective). On the other hand, the relation with EPRL Model and FK_γ ($|\gamma| < 1$) is still veiled. What we know is that the allowed spin-foams (transition channels) in EPRL Model form a subset of those allowed in our new spin-foam model (with the same 4-simplex/vertex amplitude but different different tetrahedron/edge and triangle/face amplitudes) and this fact also holds for FK_γ Model for any γ. All above relations between various spin-foam models are summarized in the following diagram, where the sets $\{Z_{\dots}\}$ are the collections of spin-foams (transition channels) which respectively contribute their partition functions $Z_{\dots}(\mathcal{K})$:

$$\begin{array}{ccccccc} \{Z_{\text{EPRL}}\}, \{Z_{\text{FK}_\gamma}\} & \overset{\text{inclusion}}{\subset} & \{Z_{\text{Simplified}}\} & \overset{\text{inclusion}}{\subset} & \{Z\} & \overset{\text{noncomm. deform.}}{\rightsquigarrow} & \{Z_{\text{BC}}\} \\ & & \downarrow \scriptstyle{\text{large-}j\text{, large area, } |\gamma|>1} & & & & \\ & & \{Z_{\text{FK}_{|\gamma|>1}}\} & & & & \end{array}$$

where $\subset$ means the inclusion in terms of contributing spin-foam amplitudes. We will discuss the details in Section 4.2.

2 Starting Point of the New Model

2.1 The Partition Function

In the last section we reviewed the approaches of simplicity constraint and closure constraint in the existing spin-foam models, and summarized the approach and main results of the present article. In this section, we present the detailed construction and analysis of our new spin-foam model. We take a simplicial complex $\mathcal{K}$ of the 4-dimensional manifold M[2], where we denote the simplices by σ, the tetrahedra by t and the triangles by f. And we take the following

[2]in most of the discussions of the present paper, the manifold M is assumed to be without boundary, then the partition function $Z(\mathcal{K})$ is a number associated to the triangulation. But the discussion can be easily generalized to the case with a boundary.

discretized partition function as the staring point for constructing the spin-foam model[3]:

$$
\begin{aligned}
Z(\mathcal{K}) \quad := \quad & \int \prod_f \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \prod_{(\sigma,t)} \mathrm{d}g_{\sigma t}^+ \mathrm{d}g_{\sigma t}^- \prod_{(t,f)} \mathrm{d}g_{tf}^+ \mathrm{d}g_{tf}^- \prod_{t;f,f'\subset t} \delta\left(X_{tf}^+ \cdot X_{tf'}^+ - X_{tf}^- \cdot X_{tf'}^-\right) \prod_t \delta\Big(\sum_{f\subset t} X_{tf}^+\Big) \\
& \times \prod_{(\sigma,f)} e^{i(1+\frac{1}{\gamma})\mathrm{tr}\left(X_f^+ g_{ft}^+ g_{t\sigma}^+ g_{\sigma t'}^+ g_{t'f}^+\right)} \prod_{(\sigma,f)} e^{i(1-\frac{1}{\gamma})\mathrm{tr}\left(X_f^- g_{ft}^- g_{t\sigma}^- g_{\sigma t'}^- g_{t'f}^-\right)}
\end{aligned} \tag{2.1}
$$

We explain the meaning of the variables appearing in the above definition:

- $X_f^+, X_f^- \in \mathfrak{su}(2)$ are respectively the self-dual and anti-self-dual part of the $\mathfrak{so}(4)$ flux variable B_f^{IJ}, which is the $\mathfrak{so}(4)$-valued 2-form field $B_{\alpha\beta}^{IJ}$ smeared on the triangle dual to f while

$$X_{tf}^\pm := g_{tf}^\pm X_f^\pm g_{ft}^\pm. \tag{2.2}$$

  So given two tetrahedra t, t' sharing a face f, the relation between X_{tf} and $X_{t'f}$ is thus

$$X_{t'f}^\pm := g_{t't}^\pm X_{tf}^\pm g_{tt'}^\pm \tag{2.3}$$

  where $g_{t't}^\pm = g_{t'f}^\pm g_{ft}^\pm$ and $g_{t't}^\pm = (g_{tt'}^\pm)^{-1}$. Such a "parallel-transportation condition" for $X_{tf}^\pm$ means that each triangle f associates a unique pair $X_f^\pm$, which ensures the right number of degrees of freedom as a discretization of Plebanski-Holst gravity. $X_{tf}^\pm$ are the auxiliary variables which are useful in the following derivation.

- $\mathrm{d}g$ is the Haar measure on SU(2). $g_{\sigma t}^+, g_{\sigma t}^- \in \mathrm{SU}(2)$ is the self- dual and anti-self-dual part of the SO(4) holonomy along the half edge $\overrightarrow{(\sigma, t)}$ outgoing from the vertex σ while g_{tf}^+, g_{tf}^- are respectively the self-dual and anti-self-dual part of the SO(4) holonomy along the segments $\overrightarrow{(t, f)}$ (see FIG.1).

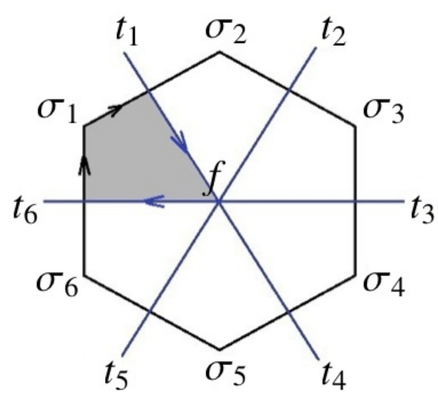

Figure 1: A face dual to the triangle f. The vertices of the face are dual to the 4-simplices σ. Each edge of the face is dual to a tetrahedron t. The fundamental region (in gray) in the face determined by a 4-simplex and two tetrahedra is called a wedge. Each tetrahedron is shared by two 4-simplices. A triangle is shared by n simplices, where n is the number of vertices of the dual face.

- The delta function $\delta\left(X_{tf}^+ \cdot X_{tf'}^+ - X_{tf}^- \cdot X_{tf'}^-\right)$ imposes the simplicity constraint for each tetrahedron:

$$\epsilon_{IJKL} B_{tf}^{IJ} B_{tf'}^{KL} = 0 \qquad f, f' \text{ belonging to the same tetrahedron} \tag{2.4}$$

  while the delta function $\delta\Big(\sum_{f\subset t} X_{tf}^+\Big)$ imposes the self-dual closure constraint for each tetrahedron. Note that there is no closure constraint for X_f^- because the closure of X_f^- is implied by the self-dual closure constraint

[3] Such a spin-foam partition function can be undertood as a sum over the histories of SO(4) spin-networks, as we will see in the following discussion.

and the simplicity constraint as we will demonstrate shortly. So including it would be equivalent to multiplying the partition function with a divergent constant which drops out in expectation values. In addition, the closure constraint and simplicity constraint Eq.(1.5) imply the 4-simplex constraints ($i,j,k,l \in \{1,2,3,4,5\}$):

$$\epsilon_{IJKL}B^{IJ}_{\sigma f_{ij}}B^{KL}_{\sigma f'_{kl}} = \epsilon_{IJKL}B^{IJ}_{\sigma f_{ik}}B^{KL}_{\sigma f'_{lj}} = \epsilon_{IJKL}B^{IJ}_{\sigma f_{il}}B^{KL}_{\sigma f'_{jk}}$$
$$f_{ij}\text{face dual to the triangle } t_i \cap t_j, \text{ where } t_i \text{ are the 5 tetrahedra of } \sigma \tag{2.5}$$

Here $X^{\pm}_{\sigma f} = g^{\pm}_{\sigma t}X^{\pm}_{tf}g^{\pm}_{t\sigma}$ and $B_{\sigma f} = X^{+}_{\sigma f} + X^{-}_{\sigma f}$. In the continuum limit of Eqs.(2.4) and (2.5), in which the holonomies can be replaced by the group unit, we recover the Plebanski simplicity constraints (20 equations):

$$\epsilon_{IJKL}B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta} = \mathcal{V}\epsilon_{\alpha\beta\gamma\delta}/4! \tag{2.6}$$

where $\mathcal{V} := \epsilon^{\alpha\beta\gamma\delta}\epsilon_{IJKL}B^{IJ}_{\alpha\beta}B^{KL}_{\gamma\delta}$ is the 4-dimensional volume element. Note that there are essentially 20 constraint equations while the trace part of Eq.(2.6) is an identity. The solutions of the simplicity constraints is well- known: given a non-degenerate co-tetrad e^I_α, there are five sectors of solutions of the simplicity constraints [3]

$$\begin{aligned} I\pm: &\quad B^{IJ} = \pm e^I \wedge e^J \\ II\pm: &\quad B^{IJ} = \pm\frac{1}{2}\epsilon^{IJ}{}_{KL}e^K \wedge e^L \\ \text{Deg}: &\quad B^+ = B^- \end{aligned} \tag{2.7}$$

where $B^{\pm}$ are the self-dual and anti-self-dual parts of B^{IJ}.

- The exponentials in $\prod_{(\sigma,f)} e^{i(1+\frac{1}{\gamma})\mathrm{tr}\left(X^+_f g^+_{ft}g^+_{t\sigma}g^+_{\sigma t'}g^+_{t'f}\right)} \prod_{(\sigma,f)} e^{i(1-\frac{1}{\gamma})\mathrm{tr}\left(X^-_f g^-_{ft}g^-_{t\sigma}g^-_{\sigma t'}g^-_{t'f}\right)}$ come from the exponential of the BF action, discretized in terms of wedge holonomies $g^{\pm}_{\sigma t}g^{\pm}_{tt'}g^{\pm}_{t'\sigma}$. In more detail,

$$\begin{aligned} \int_M \left[B + \frac{1}{\gamma} * B\right]^{IJ} \wedge F_{IJ} &= \int_M \left(1+\frac{1}{\gamma}\right)\mathrm{tr}\left(X^+ \wedge F^+\right) + \int_M \left(1-\frac{1}{\gamma}\right)\mathrm{tr}\left(X^- \wedge F^-\right) \\ = \sum_f \left(1+\frac{1}{\gamma}\right)\mathrm{tr}\left(X^+_f F^+_f\right) &+ \sum_f \left(1-\frac{1}{\gamma}\right)\mathrm{tr}\left(X^-_f F^-_f\right) \\ = \sum_{(\sigma,f)} \left(1+\frac{1}{\gamma}\right)\mathrm{tr}\left(X^+_f F^+_{(\sigma,f)}\right) &+ \sum_{(\sigma,f)} \left(1-\frac{1}{\gamma}\right)\mathrm{tr}\left(X^-_f F^-_{(\sigma,f)}\right) \\ \simeq \sum_{(\sigma,f)} \left(1+\frac{1}{\gamma}\right)\mathrm{tr}\left(X^+_f g^+_{ft}g^+_{t\sigma}g^+_{\sigma t'}g^+_{t'f}\right) &+ \sum_{(\sigma,f)} \left(1-\frac{1}{\gamma}\right)\mathrm{tr}\left(X^-_f g^-_{ft}g^-_{t\sigma}g^-_{\sigma t'}g^-_{t'f}\right) \end{aligned} \tag{2.8}$$

where $F_{(\sigma,f)}$ is the curvature 2-form integrated on the wedge determined by (σ, f)

- Finally we note that under the SO(4) gauge transformations:

$$g^{\pm}_{tf} \mapsto h^{\pm}_t g^{\pm}_{tf}(h^{\pm}_f)^{-1} \quad g^{\pm}_{\sigma t} \mapsto h^{\pm}_\sigma g^{\pm}_{\sigma t}(h^{\pm}_t)^{-1} \quad X^{\pm}_f \mapsto h^{\pm}_f X^{\pm}_f (h^{\pm}_f)^{-1} \quad X^{\pm}_{tf} \mapsto h^{\pm}_t X^{\pm}_{tf}(h^{\pm}_t)^{-1} \tag{2.9}$$

where $h : \Sigma \to SO(4);\ x \mapsto h(x)$ denotes a gauge transformation and $h_\sigma := h(\hat\sigma),\ h_t := h(\hat t),\ h_f(\hat f)$ with $\hat\sigma$ the barycenter of σ etc.

Hence the traces of the exponentials

$$\mathrm{tr}\left(X^{\pm}_f g^{\pm}_{ft}g^{\pm}_{t\sigma}g^{\pm}_{\sigma t'}g^{\pm}_{t'f}\right) \tag{2.10}$$

and the simplicity constraint

$$X^+_{tf} \cdot X^+_{tf'} - X^-_{tf} \cdot X^-_{tf'} \tag{2.11}$$

are invariant quantities while the closure constraint transforms covariantly

$$\sum_{f\subset t} X^+_{tf} \mapsto h_t \left(\sum_{f\subset t} X^+_{tf} \right) h_t^{-1}. \tag{2.12}$$

The desire to maintain gauge (co)invariance of action and constraints in the discretisation motivated to introduce the quantities $X^\pm_{\sigma f}$ and $X^\pm_{\tau f}$ which in the continuum limit reduce to $X^\pm_f$ to leading order in the discretisation regulator.

- One may wonder why we do not include δ functions enforcing the closure constraint for the "minus" sector. As we will see, the measure is supported on configurations satisfying $X^-_{tf} - u_t X^+_{tf} u_t^{-1}$ for some $u_t \in$SU(2). Thus

$$\sum_{f\subset t} X^-_{tf} = -u_t [\sum_{f\subset t} X^+_{tf}] u_t^{-1} \tag{2.13}$$

is already implied by the "Plus" sector. So we could include it but that would result in an infinite constant $\delta(0)$ which drops out in correlators. We assume to have done this already.

Remark:
It appears awkward, that here are more holonomies than B fields, suggesting a mismatch in the number of B and A degrees of freedom in contrast to the classical theory. Here we remark that the natural definition of the dual of a triangle really is the gluing of wedges (see e.g. the second reference of [1] in the notation used here and references therein). The boundary ∂f is naturally a composition of the half edges $[\hat{t}, \hat{\sigma}]$ where the hat denotes the barycentre of tetrahedron and 4-simplex respectively. Thus, if we would discretize the action using the holonomy around the ∂f rather than around the wedges, the discretized action only would depend on the edges $e = [\hat{\sigma}\sigma \hat{\cap} \sigma'] \cap [\sigma \hat{\cap} \sigma', \hat{\sigma}']$ and the properties of the Haar measure ensure that the inegrals over $g^\pm_{\sigma f}, g^\pm_{tf}$ reduce to the integrals over $g^\pm_e$. Thus, what we are doing here is to approximate $\mathrm{tr}(B_f \cdot g_f)$ by $\sum_{\hat{\sigma}\in f} \mathrm{tr}(B_f \cdot g_{f,\sigma})$ where $g_{f,\sigma} = g_{ft}\; g_{t\sigma}\; g_{\sigma t'}\; g_{t'f}$ is the corresponding wedge holonomy *after* having introduced the redundant variables $g_{t\sigma},\; g_{tf}$. We are aware that this presents a further modification of the model but it should be a mild one because both discretised actions have the same continuum limit. In fact we will see that in the semiclassical (large-j) limit the representations on the wedges essentially coincide so that effectively only the face holonomies are of relevance. It is certainly possible to define the commutative B field model without this step, however, it is very helpful to do so as it facilitates the solution to otherwise cumbersome bookkeeping problems. We leave the definition of the model without a priori introduction of wedges for future work.

2.2 Expansion of The Exponentials

For the preparation of the integration of the holonomies $g^\pm_{\sigma t}$ and $g^\pm_{tf}$, we would like to expand the factors $e^{i(1\pm\frac{1}{\gamma})\mathrm{tr}\left(X^\pm_f g^\pm_{ft} g^\pm_{t\sigma} g^\pm_{\sigma t'} g^\pm_{t'f}\right)}$ in terms of the SU(2) unitary irreducible representation matrix elements $\pi^j_{mn}(g)$. So we define the matrix $K^j_{mn}(Y)$, $Y \in \mathfrak{su}(2)$, such that

$$e^{i\mathrm{tr}(Yg)} = \sum_{j,m,n} K^j_{mn}(Y) \pi^j_{mn}(g) \tag{2.14}$$

while the expression of $K^j_{mn}(Y)$ can be obtained by

$$\frac{1}{\dim(j)} K^j_{mn}(Y) = \int \mathrm{d}g\; e^{i\mathrm{tr}(Yg)} \overline{\pi^j_{mn}(g)} = \int \mathrm{d}g\; e^{i\mathrm{tr}(Yg)} \pi^j_{nm}(g^{-1}) \tag{2.15}$$

Since $iY \equiv i\vec{y}\cdot\vec{\tau} = \vec{y}\cdot\vec{\sigma}$ (σ_j are Pauli matrices, $\tau_j = -i\sigma_j$), we have the following relation

$$iY = \frac{|\vec{y}|}{i} i\hat{y}\cdot\vec{\sigma} = \frac{|\vec{y}|}{i} e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}} \tag{2.16}$$

Therefore

$$\frac{1}{\dim(j)} K^j_{mn}(Y) = \int \mathrm{d}g \; e^{\frac{|\vec{y}|}{i}\mathrm{tr}\left(g e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}}\right)} \pi^j_{nm}(g^{-1}) = \int \mathrm{d}g \; e^{\frac{|\vec{y}|}{i}\mathrm{tr}(g)} \pi^j_{nm}(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}} g^{-1}) \tag{2.17}$$

where in the last step we made a translation $g \to g e^{-i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}}$. Moreover we can expand the function $e^{\frac{|\vec{y}|}{i}\mathrm{tr}(g)}$ by the SU(2) characters

$$e^{-i|\vec{y}|\mathrm{tr}(g)} = \sum_{k\in\mathbb{N}/2} \beta_k(|\vec{y}|)\chi_k(g) \tag{2.18}$$

Then

$$\begin{aligned}
\frac{1}{\dim(j)} K^j_{mn}(Y) &= \sum_{k\in\mathbb{N}/2} \beta_k(|\vec{y}|) \sum_l \pi^j_{nl}(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}}) \int \mathrm{d}g \; \pi^j_{lm}(g^{-1})\chi_k(g) \\
&= \sum_{k\in\mathbb{N}/2} \beta_k(|\vec{y}|) \sum_l \pi^j_{nl}(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}}) \int \mathrm{d}g \; \overline{\pi^j_{ml}(g)}\chi_k(g) \\
&= \sum_{k\in\mathbb{N}/2} \beta_k(|\vec{y}|) \sum_l \pi^j_{nl}(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}}) \frac{1}{\dim(j)} \delta_{jk}\delta_{ml} \\
&= \frac{1}{\dim(j)} \beta_j(|\vec{y}|) \pi^j_{nm}(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}})
\end{aligned} \tag{2.19}$$

Then plugging this result back into Eq.(2.14) yields

$$e^{i\mathrm{tr}(Yg)} = \sum_j \beta_j(|\vec{y}|)\; \mathrm{tr}_j\left(e^{i\frac{\pi}{2}\hat{y}\cdot\vec{\sigma}} g\right) = \sum_j \beta_j(|\vec{y}|)\; \mathrm{tr}_j\left(i\hat{y}\cdot\vec{\sigma} g\right) \tag{2.20}$$

by using this identity, we have ($X^\pm \equiv \vec{X}^\pm \cdot \vec{\tau} = \vec{X}^\pm \cdot (-i\vec{\sigma})$)

$$e^{i(1\pm\frac{1}{\gamma})\mathrm{tr}\left(X^\pm_f g^\pm_{ft} g^\pm_{t\sigma} g^\pm_{\sigma t'} g^\pm_{t'f}\right)} = \sum_{j^\pm_{\sigma f}} \beta_{j^\pm_{\sigma f}}\left(\left|1 \pm \frac{1}{\gamma}\right| \left|\vec{X}^\pm_f\right|\right)\; \mathrm{tr}_{j^\pm_{\sigma f}}\left(i\hat{X}^\pm_f \cdot \vec{\sigma} g^\pm_{ft} g^\pm_{t\sigma} g^\pm_{\sigma t'} g^\pm_{t'f}\right) \tag{2.21}$$

Inserting this result into the expression of the partition function, we obtain

$$\begin{aligned}
Z(\mathcal{K}) \;=\; & \int_{II\pm} \prod_f \mathrm{d}^3X^+_f \mathrm{d}^3X^-_f \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}g^+_{tf}\mathrm{d}g^-_{tf} \prod_{t;f,f'\subset t} \delta\left(X^+_{tf}\cdot X^+_{tf'} - X^-_{tf}\cdot X^-_{tf'}\right) \prod_t \delta\Big(\sum_{f\subset t} X^+_{tf}\Big) \\
& \times \sum_{\{j^+_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right| \left|\vec{X}^+_f\right|\right) \mathrm{tr}_{j^+_{\sigma f}}\left(i\hat{X}^+_f \cdot \vec{\sigma} g^+_{ft} g^+_{t\sigma} g^+_{\sigma t'} g^+_{t'f}\right) \\
& \times \sum_{\{j^-_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right| \left|\vec{X}^-_f\right|\right) \mathrm{tr}_{j^-_{\sigma f}}\left(i\hat{X}^-_f \cdot \vec{\sigma} g^-_{ft} g^-_{t\sigma} g^-_{\sigma t'} g^-_{t'f}\right).
\end{aligned} \tag{2.22}$$

3 Implementation of Simplicity Constraint

3.1 Linearizing the Simplicity Constraint

In order to implement the simplicity constraints via the delta functions $\delta\left(X^+_f \cdot X^+_{f'} - X^-_f \cdot X^-_{f'}\right)$ for each tetrahedron it proves convenient to pass from this quadratic expression to an integral of linear expressions directly at the level of measures (in the gravitational sector II±). In this subsection we are dealing with a single tetrahedron t, thus we ignore the t label of $X^\pm_{tf}$.

Consider the four flux variables $X^\pm_f$ ($f = 1, \cdots, 4$) associated with a tetrahedron t. Define the symmetric matrix $l^\pm_{ff'} := X^\pm_f \cdot X^\pm_{f'},\ 1 \le f, f' \le 4$. Then $l^\pm_{ff'}$ determines the $X^\pm_f$ up to an O(3) matrix O. Denote by L the range of the map $\{X^\pm_f\}_{f=1}^f \mapsto \{l^\pm_{ff'}\}_{1\le f\le f'\le 4}$ (as a subset of $\mathbb{R}^{10}$, L is constrained in particular by the Cauchy–Schwarz inequality). Then we can define a map $Y:\ O(3)\otimes L \to \mathbb{R}^{12},\ (g,l) \mapsto (gX_1(l), gX_2(l), gX_3(l), gX_4(l))$ where $X_f(l)$ is any solution of $l_{ff'} = X_f \cdot X_{f'}$.

In the following result we drop the $\pm$ for convenience.

Lemma 3.1. *We have* $\det((l_{ff'}) = 0$. *Given* $F:\ \mathbb{R}^{12} \to \mathbb{R}$ *define* $\tilde{F}:\ O(3)\times L \to \mathbb{R}$ *by* $\tilde{F} := F \circ Y$. *Then*

$$\int_{\mathbb{R}^{12}} \prod_{f=1}^{4} \mathrm{d}^3 X_f\ F = \int_{O(3)} \mathrm{d}g \int_{\mathbb{R}\times L} d^{10}l\ \delta\left(\det((l_{ff'}))\right)\ \tilde{F} \tag{3.1}$$

where dg *is the SU(2) Haar measure (up to normalisation) and* $\tilde{F}$ *is trivially extended off the surface* $\det(l) = 0$.

Proof: Up to measure zero sets, X_1, X_2, X_3 will be linearly independent and define a 3 metric $l_{ab} = X_a \cdot X_b$. Accordingly (since X_4 is a linear combination of X_1, X_2, X_3)

$$X_4 = l^{ab}(X_b \cdot X_4) X_a = l^{ab}\ l_{b4}\ X_a \tag{3.2}$$

is a linear combination of these vectors and $l^{ac} l_{cb} = \delta^a_b$. We obtain the constraint

$$l_{44} = X_4 \cdot X_4 = l^{ab} l_{4a} l_{4b} \tag{3.3}$$

among the $l_{ff'}$. On the other hand

$$\det(l_{ff'}) = \det \begin{pmatrix} l_{44} & l_{4b} \\ l_{a4} & l_{ab} \end{pmatrix} = l_{44} \det(l_{ab} - l_{4a} l_{4b}/l_{44}) = l_{44}^{-2}\ [\det(X^i_a)]^2\ \det(l_{44}\delta_{ij} - l_{4i} l_{4j}) \tag{3.4}$$

with $l_{4i} = X^a_i l_{4a}$ and X^a_i is the inverse of X^i_a. The computation of the remaining determinant is elementary and yields

$$\det(l_{ff'}) = \det(l_{ab})[l_{44} - l^{ab} l_{4a} l_{4b}] \tag{3.5}$$

which is proportional to the constraint Eq.(3.3), hence $\det(l_{ff'}) = 0$.

In order to write an integral over $X_1, .., X_4$ in terms of the independent coordinates $l_{ab}, l_{4a}, \vec{\alpha}$ where α parametrises the rotation g, we must compute the Jacobian

$$J = |\det(\frac{\partial(X_1, X_2, X_3, X_4)}{\partial(l_{ab}, \vec{\alpha}, l_{4a})})| \tag{3.6}$$

Since only X_4 depends on l_{4a} this immediately simplifies to

$$J = \frac{1}{\sqrt{\det(l_{ab})}}\ |\det(\frac{\partial(X_1, X_2, X_3)}{\partial(l_{ab}, \vec{\alpha})})| \tag{3.7}$$

To compute the remaining determinant we choose for instance the following parametrisation

$$\begin{aligned} X_1 &= \frac{l_{13}}{\sqrt{l_{33}}} b_3 + \sqrt{l_{11} - l_{13}^2/l_{33}}(\cos(\gamma+\chi) b_1 + \sin(\gamma+\chi) b_2) \\ X_2 &= \frac{l_{23}}{\sqrt{l_{33}}} b_3 + \sqrt{l_{22} - l_{23}^2/l_{33}}(\cos(\chi) b_1 + \sin(\chi) b_2) \\ X_3 &= \sqrt{l_{33}} b_3 \end{aligned} \tag{3.8}$$

with the Euler angles $\vec{\alpha} = (\phi, \theta, \chi),\ \phi, \chi \in [0, 2\pi],\ \theta \in [0, \pi]$ and the orthonormal right oriented basis

$$b_3 = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)),\ b_1 = b_{3,\theta},\ b_2 = b_{3,\phi}/\sin(\theta) \tag{3.9}$$

together with

$$\cos(\gamma) = \frac{l_{12}l_{33} - l_{13}l_{23}}{\sqrt{(l_{11}l_{33} - l_{13}^2)(l_{22}l_{33} - l_{23}^2)}}, \quad \sin(\gamma) = \frac{\sqrt{\det(l_{ab})l_{33}}}{\sqrt{(l_{11}l_{33} - l_{13}^2)(l_{22}l_{33} - l_{23}^2)}} \tag{3.10}$$

This defines the map Y above and the reader may check that the relations $X_a \cdot X_b = l_{ab}$ are satisfied for any $\vec{\alpha}$. The computation of the Jacobian is much simplified by noticing that the matrix

$$\frac{\partial(X_3, X_2, X_1)}{\partial(l_{33}, \theta, \phi, \chi, l_{22}, l_{23}, l_{11}, l_{12}, l_{13})} \tag{3.11}$$

consists of 3x3 blocks and is upper block trigonal with non singular matrices as diagonal block entries. Accordingly its determinant is the product of the determinants of the diagonal block matrices and yields after a short comutation the value $\sin(\theta)/(8\sqrt{\det(l_{ab})})$. Due to the absolute value the Jacobian is thus given by

$$J = \frac{\sin(\theta)}{8\det(l_{ab})} \tag{3.12}$$

It is not difficult to check that for the Euler angle parametrisation we have up to a normalisation constant the following expression for the Haar measure

$$dg = d\chi \; d\phi \; d\theta \;\; \sin(\theta)/8 \tag{3.13}$$

We can therefore finish the proof by

$$\begin{aligned}
&\int_{\mathbb{R}^{12}} d^3X_1 \; d^3X_2 \; d^3X_3 \; d^3X_4 \; F \\
&= \int_{O(3)} dg \int_B \prod_{a\leq b\leq 3} dl_{ab} \prod_{a=1}^{3} dl_{4a} \; \frac{\tilde{F}}{\det((l_{ab}))} \\
&= \int_{O(3)} dg \int_{B\times\mathbb{R}} \prod_{a\leq b\leq 3} dl_{ab} \prod_{f=1}^{4} dl_{4f} \; \frac{\tilde{F}}{\det((l_{ab}))} \; \delta(l_{44} - l^{ab}l_{4a}l_{4b}) \\
&= \int_{O(3)} dg \int_{B\times\mathbb{R}} \prod_{f\leq f'\leq 4} dl_{ff'} \; \delta(\det(l_{ff'})) \; \tilde{F}
\end{aligned} \tag{3.14}$$

□

As usual in path integrals we will not worry about normalisation constants as they drop out in correlators. The preceding lemma is crucial for establishing the following result.

Lemma 3.2. *For each tetrahedron t ($u_t \in SO(3)^4$). u_t can be viewed as the parametrization of the normal for the tetrahedron t (see Eq.(3.18)).*

$$\prod_{f,f'=1}^{4} \delta\left(X_f^+ \cdot X_{f'}^+ - X_f^- \cdot X_{f'}^-\right) = \delta\left(\det(X_f^+ \cdot X_{f'}^+)\right) \int \mathrm{d}u_t \prod_{f=1}^{4} \delta\left(X_f^+ + u_t X_f^- u_t^{-1}\right) \tag{3.15}$$

in the solution sector II± of the simplicity constraint [12].

Proof: Essentially we need to prove that for all continuous function $f(X_f^+, X_f^-)$ $f = 1, \cdots, 4$ vanishing in the topological solution sector I± of the simplicity constraint

$$\begin{aligned}
&\int \prod_{f=1}^{4} \mathrm{d}^3X_f^+ \mathrm{d}^3X_f^- \prod_{f,f'=1}^{4} \delta\left(X_f^+ \cdot X_{f'}^+ - X_f^- \cdot X_{f'}^-\right) f\left(X_f^+, X_f^-\right) \\
&= \int \prod_{f=1}^{4} \mathrm{d}^3X_f^+ \mathrm{d}^3X_f^- \delta\left(\det(X_f^+ \cdot X_{f'}^+)\right) \int \mathrm{d}u_t \prod_{f=1}^{4} \delta\left(X_f^+ + u_t X_f^- u_t^{-1}\right) f\left(X_f^+, X_f^-\right)
\end{aligned} \tag{3.16}$$

[4] SO(3) is considered as the upper hemisphere of SU(2), while their Haar measure is different by a factor of 2.

From the left hand side, by using Lemma 3.1, we transform the coordinates from $X_f^\pm$ to $u_t^\pm$ and $l_{ff'}^\pm$, constrained by $\det(l_{ff'}^\pm) = 0$

$$
\begin{aligned}
&\int \prod_{f=1}^{4} \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \prod_{f,f'=1}^{4} \delta\left(X_f^+ \cdot X_{f'}^+ - X_f^- \cdot X_{f'}^-\right) f\left(X_f^+, X_f^-\right) \\
= &\int \mathrm{d}u_t^+ \mathrm{d}u_t^- \prod_{f,f'=1}^{4} \mathrm{d}l_{ff'}^+ \mathrm{d}l_{ff'}^- \delta\left(\det(l_{ff'}^+)\right) \delta\left(\det(l_{ff'}^-)\right) \prod_{f,f'=1}^{4} \delta\left(l_{ff'}^+ - l_{ff'}^-\right) f\left(l_{ff'}^+, l_{ff'}^-, u_t^+, u_t^-\right) \\
= &\int \mathrm{d}u_t^+ \mathrm{d}u_t^- \prod_{f,f'=1}^{4} \mathrm{d}l_{ff'}^+ \delta\left(\det(l_{ff'}^+)\right) \delta\left(\det(l_{ff'}^+)\right) f\left(l_{ff'}^+, l_{ff'}^+, u_t^+, u_t^-\right) \\
= &\int \prod_{f=1}^{4} \mathrm{d}^3 X_f^+ \int \mathrm{d}u_t^- \delta\left(\det(X_f^+ \cdot X_{f'}^+)\right) f\left(X_f^+, -u_t^- X_f^+ (u_t^-)^{-1}\right) \\
= &\int \prod_{f=1}^{4} \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \delta\left(\det(X_f^+ \cdot X_{f'}^+)\right) \int \mathrm{d}u_t \prod_{f=1}^{4} \delta\left(X_f^+ + u_t X_f^- u_t^{-1}\right) f\left(X_f^+, X_f^-\right)
\end{aligned}
\tag{3.17}
$$

where we restrict ourself in the gravitational sector $II\pm$. □

Notice that strictly speaking we should be using the Haar measure $\mathrm{d}u_t^\pm$ on O(3) rather than SO(3) which is just the sum of two Haar measures on SO(3) twisted by a reflection so that we actually get an integral over $SO(3)$ of a sum of δ distributions $\delta(X - u_t X^+ u_t^{-1}) + \delta(X + u_t X^+ u_t^{-1})$ with $u_t \in SO(3)$. This is expected because the simplicity constraints do not select either of the two sectors (gravitational and topological). As usual in spin-foam models, we consider a restriction of the model to the purely gravitational sector in the above lemma.

Here we note that the singular factor $\delta(\det(X_{tf}^+ \cdot X_{tf'}^+))$ is essentially a $\delta(0)$ and can be divided out by an appropriate Faddeev-Popov procedure [12]. And the linearized simplicity constraint $\delta\left(X_f^+ + u_t X_f^- u_t^{-1}\right)$ has clear geometrical interpretation that for each tetrahedron t, there exists a unit 4-vector $n_t = (n_t^1, n_t^2, n_t^3, n_t^4)$ corresponding to the SU(2) element

$$
u_t = \begin{pmatrix} n_t^1 + i n_t^2 & n_t^3 + i n_t^4 \\ -(n_t^3 - i n_t^4) & n_t^1 - i n_t^2 \end{pmatrix} \tag{3.18}
$$

such that $*B_f^{IJ} n_{t,I} = 0$.

Thus the constrained measure of the flux variables in Eq.(2.22) is written as (we denote $g_t = u_t$ in what follows)

$$
\prod_f \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \prod_t \int \mathrm{d}u_t \prod_{f\subset t} \delta\left(X_{tf}^+ + u_t X_{tf}^- u_t^{-1}\right) \prod_t \delta\Big(\sum_{f\subset t} X_{tf}^+\Big) \tag{3.19}
$$

Note that the measure $\mathrm{d}^3 X_f^\pm$ can be considered as the measure $\mathrm{d}^3 X_{tf}^\pm$ constrained by the parallel-transportation condition $\delta\left(X_{tf}^\pm - g_{tf}^\pm X_f^\pm (g_{tf}^\pm)^{-1}\right)$.

In particular we see, that it is possible to justify the passing between the quadratic simplicity constraints employed by the BC model and the lineraised simplicity constraints of the EPRL and FK models respectively, at the level of measures in terms of the commuting B variables.

3.2 Imposing the Simplicity Constraint

In what follows we make the ad hoc restriction to the gravitational sector as mentioned at the end of the previous subsection.

Performing a polar decomposition of the variables $X_f^{\pm}$ and $X_{tf}^{\pm}$, we introduce the new variables $\rho_f^{\pm} \in \mathbb{R}^+$ and $N_f^{\pm} \in \mathrm{SU}(2)$

$$X_f^{\pm} = \rho_f^{\pm} N_f^{\pm} \tau_3 (N_f^{\pm})^{-1} \qquad X_{tf}^{\pm} = \rho_f^{\pm} N_{tf}^{\pm} \tau_3 (N_{tf}^{\pm})^{-1} \qquad N_{tf}^{\pm} = g_{tf}^{\pm} N_f^{\pm} \tag{3.20}$$

where $\rho_f^{\pm} = ||X_f^{\pm}||$, $\hat{X}_f^{\pm} \cdot \vec{\tau} = N_f^{\pm} \tau_3 (N_f^{\pm})^{-1}$ and the same for $X_{tf}^{\pm}$. Note that given $X^{\pm} \in \mathfrak{su}(2)$, $N^{\pm} \in \mathrm{SU}(2)$ is determined up to a U(1) rotation $h_\phi \in \mathrm{U}(1)$, which leaves τ_3 invariant.

$$h_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{3.21}$$

The associated equivalence relation is called the Hopf fibration of $SU(2) = S^3$ as a $U(1)$ bundle over the coset space $SU(2)/U(1) \cong S^2$. It is convenient for given unit vector $\vec{n}(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)$ to fix the representative $N = ie^j(\theta,\phi)\sigma_j$ with the unit vector $\vec{e}(\theta,\phi) = (\sin(2\theta)\cos(\phi), \sin(2\theta)\sin(\phi), \cos(2\theta)$ parametrising a point on S^2.

The linearized simplicity constraint $X_{tf}^+ = -u_t X_{tf}^- u_t^{-1}$ implies that there exists a $h_{\phi_{tf}} \in \mathrm{U}(1)$ for each pair of (t, f) such that

$$\rho_f^+ = \rho_f^- = \rho_f \qquad \text{and} \qquad \left(N_{tf}^+, N_{tf}^-\right) = \left(N_{tf} h_{\phi_{tf}}, u_t N_{tf} h_{\phi_{tf}}^{-1} \epsilon\right) \tag{3.22}$$

where the diagonal U(1) invariance is absorbed into the definition of N_{tf}, we only take care of the anti-diagonal one by introducing ϕ_{tf}, and

$$\epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{3.23}$$

We now reexpress the constrained measure in terms of the new variables $\rho_f^{\pm}$ and $N_f^{\pm}$. The Lebesgue measure $\mathrm{d}^3 X$ can be expressed in the spherical coordinates (when one integrates any function f of X independent of the U(1) part)

$$\int f \mathrm{d}^3 X = \int f \rho^2 \mathrm{d}\rho \mathrm{d}^2 \Omega = \int f \rho^2 \, d\rho \, \mathrm{d}N \tag{3.24}$$

where $\mathrm{d}^2\Omega$ is the round measure on S^2 and $\mathrm{d}N$ is the Haar measure on SU(2).

Lemma 3.3. *For any continuous function $f(N^+, N^-)$ on SU(2)×SU(2), up to overall constant factor ($\delta_{S^2}(\cdots)$ is the delta function on S^2)*

$$\begin{aligned} &\int_{SU(2)\times SU(2)} \mathrm{d}N^+ \mathrm{d}N^- \; \delta_{S^2}\left(N^- \tau_3 (N^-)^{-1} + uN^+ \tau_3 (N^+)^{-1} u^{-1}\right) \; f(N^+, N^-) \\ = &\int_{SU(2)\times SU(2)} \mathrm{d}N^+ \mathrm{d}N^- \int_{SU(2)} \mathrm{d}N \int_0^{2\pi} \mathrm{d}\phi \; \delta_{SU(2)}\left(N^+, N h_\phi\right) \; \delta_{SU(2)}\left(N^-, uNh_\phi^{-1}\epsilon\right) \; f(N^+, N^-) \end{aligned} \tag{3.25}$$

which gives

$$\delta_{S^2}\left(N^- \tau_3 (N^-)^{-1} + uN^+ \tau_3 (N^+)^{-1} u^{-1}\right) = \int_0^{2\pi} \mathrm{d}\phi \; \delta_{SU(2)}\left(N^-, uN^+ h_{2\phi}^{-1} \epsilon\right) \tag{3.26}$$

Proof: On the right hand side of Eq.(3.25),

$$\begin{aligned} &\int_{\mathrm{SU}(2)\times \mathrm{SU}(2)} \mathrm{d}N^+ \mathrm{d}N^- \int_{\mathrm{SU}(2)} \mathrm{d}N \int_0^{2\pi} \mathrm{d}\phi \; \delta_{\mathrm{SU}(2)}\left(N^+, N h_\phi\right) \; \delta_{\mathrm{SU}(2)}\left(N^-, uNh_\phi^{-1}\epsilon\right) \; f(N^+, N^-) \\ = &\int_{\mathrm{SU}(2)\times \mathrm{SU}(2)} \mathrm{d}N^+ \mathrm{d}N^- \int_0^{2\pi} \mathrm{d}\phi \; \delta_{\mathrm{SU}(2)}\left(N^-, uN^+ h_{2\phi}^{-1}\epsilon\right) \; f(N^+, N^-) \\ = &\int_{\mathrm{SU}(2)} \mathrm{d}N^+ \int_0^{2\pi} \mathrm{d}\phi \; f\left(N^+, uN^+ h_{2\phi}^{-1}\epsilon\right) \end{aligned} \tag{3.27}$$

On the left hand side, we can express the Haar measure $\mathrm{d}N^-$ in terms of Euler angles

$$\int_{\mathrm{SU}(2)} \mathrm{d}N^- \ldots = \frac{1}{16\pi^2}\int_0^{2\pi} \mathrm{d}\phi_2 \int_0^{\pi} \mathrm{d}\theta\ \sin\theta \int_0^{4\pi} \mathrm{d}\phi_1 \ldots \tag{3.28}$$

And the delta function $\delta_{S^2}\left(N^-\tau_3(N^-)^{-1} + uN^+\tau_3(N^+)^{-1}u^{-1}\right)$ is the delta function on S^2, which is coordinatized by $\theta \in [0,\pi]$ and $\phi_2 \in [0,2\pi]$. By explicit computation

$$\begin{aligned} &\int_0^{2\pi} \mathrm{d}\phi_2 \int_0^{\pi} \mathrm{d}\theta\ \sin\theta\ \delta_{S^2}\left(N^-\tau_3(N^-)^{-1} + uN^+\tau_3(N^+)^{-1}u^{-1}\right)\ f(N^+,N^-) \\ =\ & f\left(N^+, uN^-h_{\phi_1}^{-1}\epsilon\right) \end{aligned} \tag{3.29}$$

Therefore the left hand side of Eq.(3.25) reduces to

$$\int_{\mathrm{SU}(2)} \mathrm{d}N^+ \int_0^{4\pi} \mathrm{d}\phi_1\ f\left(N^+, uN^-h_{\phi_1}^{-1}\epsilon\right). \tag{3.30}$$

which is identical to the right hand side Eq.(3.27).

□

Using this we rewrite the constrained measure up to an unimportant overall constant as

$$\begin{aligned} &\prod_f \mathrm{d}^3X_f^+\mathrm{d}^3X_f^- \prod_t \int \mathrm{d}u_t \prod_{f\subset t}\delta\left(X_{tf}^+ + u_tX_{tf}^-u_t^{-1}\right)\prod_t\delta\Big(\sum_{f\subset t}X_{tf}^+\Big) \\ =\ &\prod_f \mathrm{d}\rho_f^+\mathrm{d}N_f^+\left(\rho_f^+\right)^2\mathrm{d}\rho_f^-\mathrm{d}N_f^-\prod_t\int\mathrm{d}u_t\prod_{f\subset t}\delta\left(\rho_f^+-\rho_f^-\right)\int_0^{2\pi}\mathrm{d}\phi_{tf}\delta\left(N_{tf}^-, u_tN_{tf}^+h_{2\phi_{tf}}^{-1}\epsilon\right) \\ &\times\prod_t\delta\Big(\sum_{f\subset t}\rho_f^+N_{tf}^+\tau_3(N_{tf}^+)^{-1}\Big) \end{aligned} \tag{3.31}$$

We insert this result into the partition function

$$\begin{aligned} Z(\mathcal{K}) =\ &\int\prod_{(\sigma,t)}\mathrm{d}g_{\sigma t}^+\mathrm{d}g_{\sigma t}^-\prod_{(t,f)}\mathrm{d}g_{tf}^+\mathrm{d}g_{tf}^-\prod_f\mathrm{d}\rho_f^+\mathrm{d}N_f^+\left(\rho_f^+\right)^2\mathrm{d}\rho_f^-\mathrm{d}N_f^-\prod_t\mathrm{d}u_t\prod_{f\subset t}\delta\left(\rho_f^+-\rho_f^-\right) \\ &\times\int_0^{2\pi}\mathrm{d}\phi_{tf}\delta\left(N_{tf}^-, u_tN_{tf}^+h_{2\phi_{tf}}^{-1}\epsilon\right)\prod_t\delta\Big(\sum_{f\subset t}\rho_f^+N_{tf}^+\tau_3(N_{tf}^+)^{-1}\Big) \\ &\times\sum_{\{j_{\sigma f}^+\}}\prod_{(\sigma,f)}\beta_{j_{\sigma f}^+}\left(\left|1+\frac{1}{\gamma}\right|\rho_f^+\right)\mathrm{tr}_{j_{\sigma f}^+}\left(iN_f^+\sigma_3\left(N_f^+\right)^{-1}g_{ft}^+g_{t\sigma}^+g_{\sigma t'}^+g_{t'f}^+\right) \\ &\times\sum_{\{j_{\sigma f}^-\}}\prod_{(\sigma,f)}\beta_{j_{\sigma f}^-}\left(\left|1-\frac{1}{\gamma}\right|\rho_f^-\right)\mathrm{tr}_{j_{\sigma f}^-}\left(iN_f^-\sigma_3\left(N_f^-\right)^{-1}g_{ft}^-g_{t\sigma}^-g_{\sigma t'}^-g_{t'f}^-\right) \end{aligned} \tag{3.32}$$

Performing a translation of the Haar measure $\mathrm{d}g_{tf}^\pm$

$$\mathrm{d}g_{tf}^\pm \mapsto \mathrm{d}\left(g_{tf}^\pm N_f^\pm\right) = \mathrm{d}N_{tf}^\pm \tag{3.33}$$

(notice that $\mathrm{d}N_{tf}^\pm$ and $\mathrm{d}N_f^\pm$ are Haar measures on SU(2)) we see that the integrand depends on $N_{tf}^\pm$ only so that the integrals over $\mathrm{d}N_f$ are trivial and give unity (upon proper normalisation). The partition function therefore reduces to

$$\begin{aligned} Z(\mathcal{K}) &= \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}N^+_{tf}\mathrm{d}N^-_{tf} \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_t \mathrm{d}u_t \prod_{f\subset t} \\ &\times \int_0^{2\pi} \mathrm{d}\phi_{tf}\,\delta\left(N^-_{tf}, u_t N^+_{tf} h^{-1}_{2\phi_{tf}}\epsilon\right) \prod_t \delta\Big(\sum_{f\subset t} \rho_f N^+_{tf}\tau_3 (N^+_{tf})^{-1}\Big) \\ &\times \sum_{\{j^+_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^+_{\sigma f}}\left(iN^+_{t'f}\sigma_3\left(N^+_{tf}\right)^{-1} g^+_{t\sigma} g^+_{\sigma t'}\right) \\ &\times \sum_{\{j^-_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^-_{\sigma f}}\left(iN^-_{t'f}\sigma_3\left(N^-_{tf}\right)^{-1} g^-_{t\sigma} g^-_{\sigma t'}\right) \end{aligned} \tag{3.34}$$

where we also performed the integral over ρ^-_f.

Next we perform the integral over $\mathrm{d}N^-_{tf}$ to solve the simplicity constraint (implementing Eq.(3.22))

$$\begin{aligned} Z(\mathcal{K}) &= \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}N_{tf} \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_t \mathrm{d}u_t \prod_{(t,f)} \mathrm{d}\phi_{tf} \prod_t \delta\Big(\sum_{f\subset t} \rho_f N_{tf}\tau_3 N^{-1}_{tf}\Big) \\ &\times \sum_{\{j^+_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^+_{\sigma f}}\left(iN_{t'f}h_{\phi_{t'f}}\sigma_3 h^{-1}_{\phi_{tf}}\left(N_{tf}\right)^{-1} g^+_{t\sigma} g^+_{\sigma t'}\right) \\ &\times \sum_{\{j^-_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^-_{\sigma f}}\left(iu_{t'}N_{t'f}h^{-1}_{\phi_{t'f}}\epsilon\sigma_3\epsilon^{-1}h_{\phi_{tf}}N^{-1}_{tf}u^{-1}_t g^-_{t\sigma} g^-_{\sigma t'}\right) \end{aligned} \tag{3.35}$$

where we also have performed the translation $N^+_{tf} \to N^+_{tf}h_{\phi_{tf}}$. Performing the translation $g^-_{\sigma t} \to g^-_{\sigma t}u^{-1}_t$, the integrand no longer depends on u_t and the u_t integral gives unity, leaving us with

$$\begin{aligned} Z(\mathcal{K}) &= \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}N_{tf} \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)} \mathrm{d}\phi_{tf} \prod_t \delta\Big(\sum_{f\subset t} \rho_f N_{tf}\tau_3 N^{-1}_{tf}\Big) \\ &\times \sum_{\{j^+_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^+_{\sigma f}}\left(iN_{t'f}h_{\phi_{t'f}}\sigma_3 h^{-1}_{\phi_{tf}}\left(N_{tf}\right)^{-1} g^+_{t\sigma} g^+_{\sigma t'}\right) \\ &\times \sum_{\{j^-_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \mathrm{tr}_{j^-_{\sigma f}}\left(iN_{t'f}h^{-1}_{\phi_{t'f}}\epsilon\sigma_3\epsilon^{-1}h_{\phi_{tf}}N^{-1}_{tf} g^-_{t\sigma} g^-_{\sigma t'}\right) \end{aligned} \tag{3.36}$$

Recall that for any SL(2,$\mathbb{C}$) matrix g

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{3.37}$$

the representation matrix element $\pi^j_{mn}(g)$ reads

$$\pi^j_{mn}(g) = \sum_l \frac{\sqrt{(j+m)!\,(j-m)!\,(j+n)!\,(j-n)!}}{(j+n-l)!\,(m-n+l)!\,(j-m-l)!\,l!}\; a^{j+n-l}\; b^{m-n+l}\; c^l\; d^{j-m-l} \tag{3.38}$$

Applying this to $i\sigma_3$

$$i\sigma_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{3.39}$$

yields

$$\begin{aligned} \pi^j_{mn}(i\sigma_3) &= \sum_l \frac{\sqrt{(j+m)!\,(j-m)!\,(j+n)!\,(j-n)!}}{(j+n-l)!\,(m-n+l)!\,(j-m-l)!\,l!}\; i^{j+n-l}\; 0^{m-n+l}\; 0^l\; (-i)^{j-m-l} \\ &= i^{j+m}\,(-i)^{j-m}\,\delta_{mn} \end{aligned} \tag{3.40}$$

Likewise for

$$h_\phi = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \tag{3.41}$$

we obtain

$$\begin{aligned}\pi^j_{mn}(h_\phi) &= \sum_l \frac{\sqrt{(j+m)!\,(j-m)!\,(j+n)!\,(j-n)!}}{(j+n-l)!\,(m-n+l)!\,(j-m-l)!\,l!}\,(e^{i\phi})^{j+n-l}\,0^{m-n+l}\,0^l\,(e^{-i\phi})^{j-m-l} \\ &= (e^{i\phi})^{j+m}(e^{-i\phi})^{j-m}\delta_{mn} = (e^{2im\phi})\delta_{mn}\end{aligned} \tag{3.42}$$

We conclude (summing over repeated indices),

$$\begin{aligned}&\mathrm{tr}_{j^+_{\sigma f}}\left(iN_{t'f}h_{\phi_{t'f}}\sigma_3 h^{-1}_{\phi_{tf}}\,(N_{tf})^{-1}\,g^+_{t\sigma}g^+_{\sigma t'}\right) \\ =\ & i^{2b_{\sigma f}}e^{2ib_{\sigma f}(\phi_{t'f}-\phi_{tf})}\pi^{j^+_{\sigma f}}_{a_{\sigma f}b_{\sigma f}}\,(N_{t'f})\,\pi^{j^+_{\sigma f}}_{b_{\sigma f}c_{\sigma f}}(N^{-1}_{tf})\pi^{j^+_{\sigma f}}_{c_{\sigma f}a_{\sigma f}}\left(g^+_{t\sigma}g^+_{\sigma t'}\right)\end{aligned} \tag{3.43}$$

Since $i\epsilon\sigma_3\epsilon^{-1} = -i\sigma_3$ we have similarly for the anti-self-dual part

$$\begin{aligned}\pi^j_{mn}(-i\sigma_3) &= \sum_l \frac{\sqrt{(j+m)!\,(j-m)!\,(j+n)!\,(j-n)!}}{(j+n-l)!\,(m-n+l)!\,(j-m-l)!\,l!}\,(-i)^{j+n-l}\,0^{m-n+l}\,0^l\,i^{j-m-l} \\ &= (-i)^{j+m}\,i^{j-m}\,\delta_{mn}\end{aligned} \tag{3.44}$$

thus

$$\begin{aligned}&\mathrm{tr}_{j^-_{\sigma f}}\left(iN_{t'f}h^{-1}_{\phi_{t'f}}\epsilon\sigma_3\epsilon^{-1}h_{\phi_{tf}}N^{-1}_{tf}g^-_{t\sigma}g^-_{\sigma t'}\right) \\ =\ & (-i)^{2b_{\sigma f}}e^{-2ib_{\sigma f}(\phi_{t'f}-\phi_{tf})}\pi^{j^-_{\sigma f}}_{a_{\sigma f}b_{\sigma f}}\,(N_{t'f})\,\pi^{j^-_{\sigma f}}_{b_{\sigma f}c_{\sigma f}}(N^{-1}_{tf})\pi^{j^-_{\sigma f}}_{c_{\sigma f}a_{\sigma f}}\left(g^-_{t\sigma}g^-_{\sigma t'}\right)\end{aligned} \tag{3.45}$$

We insert these formulae into the partition function Eq.(3.36)

$$\begin{aligned}Z(\mathcal{K}) &= \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t}\prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)}\mathrm{d}N_{tf}\prod_{(t,f)}\mathrm{d}\phi_{tf}\prod_t \delta\Big(\sum_{f\subset t}\rho_f N_{tf}\tau_3 N^{-1}_{tf}\Big) \\ &\times \sum_{\{j^\pm_{\sigma f}\}}\prod_{(\sigma,f)}\beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\ \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \\ &\times \sum_{a,b,c} i^{2b^+_{\sigma f}}\ e^{2ib^+_{\sigma f}(\phi_{t'f}-\phi_{tf})}\pi^{j^+_{\sigma f}}_{a^+_{\sigma f}b^+_{\sigma f}}\,(N_{t'f})\,\pi^{j^+_{\sigma f}}_{b^+_{\sigma f}c^+_{\sigma f}}(N^{-1}_{tf})\pi^{j^+_{\sigma f}}_{c^+_{\sigma f}a^+_{\sigma f}}\left(g^+_{t\sigma}g^+_{\sigma t'}\right) \\ &\times(-i)^{2b^-_{\sigma f}}\ e^{-2ib^-_{\sigma f}(\phi_{t'f}-\phi_{tf})}\pi^{j^-_{\sigma f}}_{a^-_{\sigma f}b^-_{\sigma f}}\,(N_{t'f})\,\pi^{j^-_{\sigma f}}_{b^-_{\sigma f}c^-_{\sigma f}}(N^{-1}_{tf})\pi^{j^-_{\sigma f}}_{c^-_{\sigma f}a^-_{\sigma f}}\left(g^-_{t\sigma}g^-_{\sigma t'}\right).\end{aligned} \tag{3.46}$$

and perform the integrals over $\mathrm{d}\phi_{tf}$ which enforce $b^-_{\sigma f} = b^+_{\sigma f} \equiv b_{\sigma f}$, and restrict the range of the sum over $b_{\sigma f}$ to the set $\{-j^+_{\sigma f},\cdots,j^+_{\sigma f}\}\cap\{-j^-_{\sigma f},\cdots,j^-_{\sigma f}\}$. Accordingly,

$$\begin{aligned}Z(\mathcal{K}) &= \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t}\prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)}\mathrm{d}N_{tf}\prod_t \delta\Big(\sum_{f\subset t}\rho_f N_{tf}\tau_3 N^{-1}_{tf}\Big) \\ &\times \sum_{\{j^\pm_{\sigma f}\}}\prod_{(\sigma,f)}\beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\ \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \\ &\times \sum_{a,b,c}\pi^{j^+_{\sigma f}}_{a^+_{\sigma f}b_{\sigma f}}\,(N_{t'f})\,\pi^{j^+_{\sigma f}}_{b_{\sigma f}c^+_{\sigma f}}(N^{-1}_{tf})\pi^{j^+_{\sigma f}}_{c^+_{\sigma f}a^+_{\sigma f}}\left(g^+_{t\sigma}g^+_{\sigma t'}\right) \\ &\times\pi^{j^-_{\sigma f}}_{a^-_{\sigma f}b_{\sigma f}}\,(N_{t'f})\,\pi^{j^-_{\sigma f}}_{b_{\sigma f}c^-_{\sigma f}}(N^{-1}_{tf})\pi^{j^-_{\sigma f}}_{c^-_{\sigma f}a^-_{\sigma f}}\left(g^-_{t\sigma}g^-_{\sigma t'}\right).\end{aligned} \tag{3.47}$$

3.3 Topological/Gravitational Sector Duality, γ-Duality

Before performing further computations, in this subsection we consider the topological sector I$\pm$ of the simplicity constraint. Because we consider the model with finite Barbero-Immirzi parameter, the sector I$\pm$ is actually also gravitational here in the following sense: By definition, $\mathrm{tr}(F \wedge *(e \wedge e))$ is the Palatini (gravitational) term while $\mathrm{tr}(F \wedge (e \wedge e))$ is the topological term. Since we are considering the Plebanski – Holst Lagrangian $\mathrm{tr}(F \wedge (B + \frac{1}{\gamma} * B))$, inserting the gravitational solution $B = \pm * (e \wedge e)$ yields (due to $*^2$ =id in Euclidian signature) the Palatini – Holst Lagrangian with Immirzi parameter γ, that is, $\pm\mathrm{tr}(F \wedge (*(e \wedge e) + \frac{1}{\gamma} e \wedge e))$ while inserting the topological solution $B = \pm e \wedge e$ yields Palatini – Holst Lagrangian with Immirzi parameter $1/\gamma$, that is $\pm\frac{1}{\gamma}\mathrm{tr}(F \wedge (*(e \wedge e) + \gamma e \wedge e))$. rescaled by $1/\gamma$. If we change coordinates from $X_f^\pm$ to $\pm X_f^\pm/\gamma$ in the partition function Z_γ (2.1) then we obtain the relation

$$Z_\gamma(\mathcal{K}) = \gamma^{6F-21T}\, Z_{1/\gamma}(\mathcal{K}) \tag{3.48}$$

where F, T respectively denote the number of triangles and tetrahedra respectively in $\mathcal{K}$ (the powers arise from the Lebesgue measure and the δ functions respectively). The appearing power of γ drops out in correlators, hence up to the rescaling of the n-point functions of involving $X_f^\pm$, Z_γ, $Z_{1/\gamma}$ yield the same correlators. It follows that the model (2.1) is a mixture of gravitational and topological sectors as it should be.

This is before restriction to either the gravitational or topological sector respectively and the manipulations (dropping infinite constants) that followed. For comparison, the partition function for the topological (I) and gravitational sector with Immirzi parameter γ respectively read (before expanding the exponentials)

$$\begin{aligned} Z_\gamma^{I/II}(\mathcal{K}) &= \int [\prod_f d^3X_f^+\, d^3X_f^-]\, [\prod_{(\sigma,t)} dg_{\sigma t}^+\, dg_{\sigma t}^-]\, [\prod_{(t,f)} dg_{tf}^+\, dg_{tf}^-] \\ &\times [\prod_t \delta(\sum_{f\subset t} X_{tf}^+)] \int [\prod_t du_t][\prod_{(t,f)} \delta(X_{tf}^+ \mp u_t X_{tf}^- u_t^{-1})] \\ &\times \exp(i[1+\gamma^{-1}] \sum_{(\sigma,f)} \mathrm{Tr}(X_f^+ w_{\sigma f}^+) + i[1-\gamma^{-1}] \sum_{(\sigma,f)} \mathrm{Tr}(X_f^- w_{\sigma f}^-)) \end{aligned} \tag{3.49}$$

The only difference is the sign in the δ distribution enforcing the linearised simplicity constraint. Now change variables $X_f^\pm \to \pm X_f^\pm/\gamma$ in the model I (this induces also $X_{tf}^\pm \to \pm X_{tf}^\pm/\gamma$). This switches the sign of the simplicity constraint to that of the model II, maps the $1/\gamma$ in the exponent to γ and rescales the Lebesgue measure and the δ distributions according to

$$Z_\gamma^I(\mathcal{K}) = \gamma^{6F-5T} Z_{1/\gamma}^{II}(\mathcal{K}) \tag{3.50}$$

The power of γ again drops out in correlators and thus up to γ powers coming from n-point functions, "topological" correlators with respect to γ are essentially the same as "gravitational" correlators with respect to $1/\gamma$. We coin this relation between the two sectors "γ duality". We will therefore not discuss model I any further in this article.

4 The Spin-foam Model

4.1 A Simplified Model without Closure Constraint

In this subsection we discuss a simplified model by removing the closure constraint in the partition function $Z(\mathcal{K})$ by hand as it is also done in existing spin foam models. We do this just for a better comparison between our model and those models as far as the modifications are concerned that result from commuting rather than non commuting B fields. The discussion of the full model and the additional modifications that come from a proper treatment of the closure constraint will follow in the subsequent subsection.

The simplified partition function reads (from Eq.(3.47))

$$Z_{\text{Simplified}}(\mathcal{K}) = \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)} \mathrm{d}N_{tf} \sum_{\{j^\pm_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\ \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right)$$

$$\sum_{a,b,c}\left[\pi^{j^+_{\sigma f}}_{a^+_{\sigma f}b_{\sigma f}}(N_{t'f})\,\pi^{j^+_{\sigma f}}_{b_{\sigma f}c^+_{\sigma f}}(N^{-1}_{tf})\pi^{j^+_{\sigma f}}_{c^+_{\sigma f}a^+_{\sigma f}}\left(g^+_{t\sigma}g^+_{\sigma t'}\right)\right]\left[\pi^{j^-_{\sigma f}}_{a^-_{\sigma f}b_{\sigma f}}(N_{t'f})\,\pi^{j^-_{\sigma f}}_{b_{\sigma f}c^-_{\sigma f}}(N^{-1}_{tf})\pi^{j^-_{\sigma f}}_{c^-_{\sigma f}a^-_{\sigma f}}\left(g^-_{t\sigma}g^-_{\sigma t'}\right)\right] \tag{4.1}$$

In order to explore the structure of the spin-foam amplitude (e.g. vertex amplitude) for this partition function, we use the following recoupling relation ($N \in \mathrm{SU}(2)$):

$$\begin{aligned} & \pi^{j^+}_{a^+b^+}(N)\,\pi^{j^-}_{a^-b^-}(N) \\ = & \ \langle j^+, b^+| \otimes \langle j^-, b^-|U(N)|j^+, a^+\rangle \otimes |j^-, a^-\rangle \\ = & \sum_{k=|j^+-j^-|}^{j^++j^-} \left\langle k, a^+ + a^-|j^+, a^+; j^-, a^-\right\rangle \left\langle j^+, b^+; j^-, b^-|k, b^+ + b^-\right\rangle \pi^k_{a^++a^-,b^++b^-}(N) \end{aligned} \tag{4.2}$$

We denote by $c(k,j^\pm)^{a^+a^-}_\alpha = c(k,j^\pm)^\alpha_{a^+a^-}$ the Clebsch-Gordan coefficients $\langle k,\alpha|j^+,a^+;j^-,a^-\rangle$, which are real and vanish unless $\alpha = a^+ + a^-$. Thus (summing repeated indices)

$$\begin{aligned} \pi^{j^+}_{a^+b^+}(N)\,\pi^{j^-}_{a^-b^-}(N) &= \sum_{k=|j^+-j^-|}^{j^++j^-} c(k,j^\pm)^\alpha_{a^+a^-}c(k,j^\pm)^\beta_{b^+b^-}\pi^k_{\alpha\beta}(N) \\ \pi^k_{\alpha\beta}(N) &= c(k,j^\pm)^{a^+a^-}_\alpha c(k,j^\pm)^{b^+b^-}_\beta \pi^{j^+}_{a^+b^+}(N)\,\pi^{j^-}_{a^-b^-}(N) \\ & \quad (k \in \{|j^+-j^-|,\cdots,j^++j^-\}) \end{aligned} \tag{4.3}$$

By using this recoupling relation we find

$$\begin{aligned} & \pi^{j^+}_{a^+b}(N_{t'f})\,\pi^{j^-}_{a^-b}(N_{t'f})\,\pi^{j^+}_{bc^+}(N^{-1}_{tf})\pi^{j^-}_{bc^-}(N^{-1}_{tf}) \\ = & \sum_{k,k'=|j^+-j^-|}^{j^++j^-} c(k,j^\pm)^\alpha_{a^+a^-}\ \pi^k_{\alpha\beta}(N_{t'f})\ c(k,j^\pm)^\beta_{b,b}\ c(k',j^\pm)^{\alpha'}_{b,b}\ \pi^{k'}_{\alpha'\beta'}(N^{-1}_{tf})\ c(k',j^\pm)^{\beta'}_{c^+c^-} \end{aligned} \tag{4.4}$$

where β and α' are fixed to be $2b$. We note that k and k' are restricted to be greater than or equal to $|2b|$ which we take care of by defining $c(k,j^\pm)^{2b}_{b,b}$ to be zero when $k < |2b|$. Inserting this result back into the partition function $Z_{\text{Simplified}}(\mathcal{K})$ results in

$$\begin{aligned} & Z_{\text{Simplified}}(\mathcal{K}) \\ = & \int \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)} \mathrm{d}N_{tf} \sum_{\{j^\pm_{\sigma f}\}} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\ \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \sum_{k_{\sigma f},k'_{\sigma f}=|j^+_{\sigma f}-j^-_{\sigma f}|}^{j^+_{\sigma f}+j^-_{\sigma f}} \\ & \prod_{(\sigma,f)} \sum_{b_{\sigma f}} \pi^{k_{\sigma f}}_{\alpha_{\sigma f},2b_{\sigma f}}(N_{t'f})\left[c(k_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}\ c(k'_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}\right]\pi^{k'_{\sigma f}}_{2b_{\sigma f},\beta'_{\sigma f}}(N^{-1}_{tf}) \\ & \left[c(k'_{\sigma f},j^\pm)^{\beta'_{\sigma f}}_{c^+_{\sigma f}c^-_{\sigma f}}\ \pi^{j^+_{\sigma f}}_{c^+_{\sigma f}a^+_{\sigma f}}\left(g^+_{t\sigma}g^+_{\sigma t'}\right)\pi^{j^-_{\sigma f}}_{c^-_{\sigma f}a^-_{\sigma f}}\left(g^-_{t\sigma}g^-_{\sigma t'}\right)c(k_{\sigma f},j^\pm_{\sigma f})^{\alpha_{\sigma f}}_{a^+_{\sigma f}a^-_{\sigma f}}\right] \end{aligned} \tag{4.5}$$

Now we focus on a vertex v dual to a 4-simplex σ. We fix the orientation of each dual half edge $\overrightarrow{(\sigma,t)}(=\overrightarrow{(v,\mu)}$ in the notation of FIG. 3) to be outgoing from the vertex and integrate the SU(2) holonomies $g^\pm_{\sigma t}$. The integration of $g^\pm_{\sigma t}$ leads to a result that depends on the orientations of the wedges bounded by $\overrightarrow{(\sigma,t)}$. We say a wedge w bounded by $\overrightarrow{(\sigma,t)}$ is incoming to the edge $\overrightarrow{(\sigma,t)}$, if the orientation along its boundary agrees with $\overrightarrow{(\sigma,t)}$, otherwise we call it to be

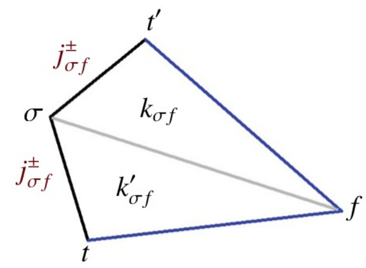

Figure 2: A wedge (σ, f) with a SO(4) representation $(j^+_{\sigma f}, j^-_{\sigma f})$ and two SU(2) representations $k_{\sigma f}$ and $k'_{\sigma f}$, where $k_{\sigma f}, k'_{\sigma f} \in \left\{|j^+_{\sigma f} - j^-_{\sigma f}|, \cdots, j^+_{\sigma f} + j^-_{\sigma f}\right\} \cap \left\{|j^+_{\sigma' f} - j^-_{\sigma' f}|, \cdots, j^+_{\sigma' f} + j^-_{\sigma' f}\right\}$.

outgoing from $\overrightarrow{(\sigma,t)}$. The integrations of $g^{\pm}_{\sigma t}$ in Eq.(4.5)

$$\int \mathrm{d}g_{\sigma t} \bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} \pi^{j_w}\left(g_{\sigma t}\right) \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} \pi^{j_w}\left(g^{-1}_{\sigma t}\right) \tag{4.6}$$

equals a projection operator $\mathfrak{P}_{\sigma t}$ for each dual half edge $\overrightarrow{(\sigma,t)}$

$$\begin{aligned} \mathfrak{P}_{\sigma t} \quad : \quad & \left[\bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} V_{j^+_w} \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} V^*_{j^+_w}\right] \bigotimes \left[\bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} V_{j^-_w} \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} V^*_{j^-_w}\right] \\ \rightarrow \quad & \mathrm{Inv}\left(\bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} V_{j^+_w} \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} V^*_{j^+_w}\right) \bigotimes \mathrm{Inv}\left(\bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} V_{j^-_w} \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} V^*_{j^-_w}\right) \\ \mathfrak{P}_{\sigma t} \quad := \quad & \left[\sum_{i^+_{\sigma t}} C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right) \otimes C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)^\dagger\right] \otimes \left[\sum_{i^-_{\sigma t}} C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right) \otimes C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)^\dagger\right] \end{aligned} \tag{4.7}$$

where V_j is the representation space for SU(2) unitary irreducible representation, and we keep in mind that each pair (σ, f) determines a wedge w, and $C^4_{i^{\pm}_{\sigma t}}\left(j^{\pm}_{\sigma f}\right)$ are the 4-valent SU(2) intertwiners forming an orthonormal basis in

$$\mathrm{Inv}\left(\bigotimes_{w \text{ incoming } \overrightarrow{(\sigma,t)}} V_{j^{\pm}_w} \bigotimes_{w \text{ outgoing } \overrightarrow{(\sigma,t)}} V^*_{j^{\pm}_w}\right) \tag{4.8}$$

Thus the result of the integrations of $g^{\pm}_{\sigma t}$ in Eq.(4.5) is a product of the projection operators $\mathfrak{P}_{\sigma t}$ for all the dual half edges $\overrightarrow{(\sigma,t)}$. According to the index structure appearing in Eq.(4.5), we find that in each $\mathfrak{P}_{\sigma t}$ the adjoint intertwiners $C^4_{i^{\pm}_{\sigma t}}\left(j^{\pm}_{\sigma f}\right)^\dagger$ are combined with the indices $a^{\pm}_{\sigma f}, c^{\pm}_{\sigma f}$, where $a^{\pm}_{\sigma f}$ are for the incoming wedges and $c^{\pm}_{\sigma f}$ are for the outgoing wedges. However the intertwiners $C^4_{i^{\pm}_{\sigma t}}\left(j^{\pm}_{\sigma f}\right)$ for each half edge $\overrightarrow{(\sigma,t)}$ are contracted with other half edge intertwiners of $\overrightarrow{(\sigma,t')}$ at the vertex dual to σ. Summing over the indices $a^{\pm}_{\sigma f}, c^{\pm}_{\sigma f}$, the integrations of $g^{\pm}_{\sigma t}$ in Eq.(4.5) result in a product of

$$\left[\sum_{i^+_{\sigma t}} C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)_{\cdots} \overline{C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)}_{\{a^+_{\sigma f}\},\{c^+_{\sigma f'}\}}\right] \cdot \left[\sum_{i^-_{\sigma t}} C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)_{\cdots} \overline{C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)}_{\{a^-_{\sigma f}\},\{c^-_{\sigma f'}\}}\right] \tag{4.9}$$

for all half edges $\overrightarrow{(\sigma,t)}$, where $\cdots$ are the indices contracted with other half edge intertwiners of $\overrightarrow{(\sigma,t')}$ at the vertex dual to σ. According to the structure of Eq.(4.5), we assign the intertwiners

$$\left[C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)_{\cdots}\right] \cdot \left[C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)_{\cdots}\right] \tag{4.10}$$

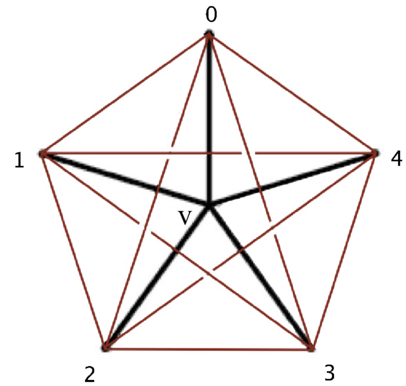

Figure 3: v is the vertex dual to the 4-simplex σ. $\mu = 0, 1, \cdots, 4$ are the labels for the tetrahedra t_μ forming the boundary of σ. The edges (v, μ) $\mu = 0, \cdots, 4$ are the edges dual to the tetrahedron t_μ. The face determined by v, μ, ν $\mu, \nu = 0, \cdots, 4$ is the wedge determined by σ, t_μ, t_ν.

to the beginning point of $\overrightarrow{(\sigma, t)}$, while we assign the adjoint intertwiners

$$\left[\overline{C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)}_{\{a^+_{\sigma f}\},\{c^+_{\sigma f'}\}}\right] \cdot \left[\overline{C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)}_{\{a^-_{\sigma f}\},\{c^-_{\sigma f'}\}}\right] \tag{4.11}$$

to the end point of $\overrightarrow{(\sigma, t)}$.

The contractions of the half edge intertwiners Eq.(4.10) at each vertex dual to σ gives a SO(4) $15j$-symbol

$$\left\{15j\right\}_{\mathrm{SO}(4)}(j^\pm_{\sigma f}, i^\pm_{\sigma t}) := \mathrm{tr}\left[\bigotimes_{\overrightarrow{(\sigma,t)}} C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)\right]\ \mathrm{tr}\left[\bigotimes_{\overrightarrow{(\sigma,t)}} C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)\right] \tag{4.12}$$

to each 4-simplex σ (to each vertex dual to σ).

On the other hand, each of the adjoint intertwiners Eq.(4.11) at the end point of $\overrightarrow{(\sigma, t)}$ is contracted with the Clebsch-Gordan coefficients $c(k, j^\pm)^\alpha_{a^+a^-}$ and $\overline{c(k', j^\pm)^{\beta'}_{c^+c^-}}$. Thus we obtain a 4-valent SU(2) intertwiner $\mathcal{I}^4_{i^\pm_{\sigma t}}(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f})$ at the end point of each half edge $\overrightarrow{(\sigma, t)}$ (summing repeated indices)

$$\begin{aligned} \mathcal{I}^4_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right)_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}} \quad := \quad & \left[\overline{C^4_{i^+_{\sigma t}}\left(j^+_{\sigma f}\right)}_{\{a^+_{\sigma f}\},\{c^+_{\sigma f'}\}}\right] \cdot \left[\overline{C^4_{i^-_{\sigma t}}\left(j^-_{\sigma f}\right)}_{\{a^-_{\sigma f}\},\{c^-_{\sigma f'}\}}\right] \\ & \times \prod_{(\sigma,f)\ \text{incoming}} c(k_{\sigma f}, j^\pm_{\sigma f})^{\alpha_{\sigma f}}_{a^+_{\sigma f} a^-_{\sigma f}} \prod_{(\sigma,f')\ \text{outgoing}} \overline{c(k'_{\sigma f'}, j^\pm_{\sigma f'})^{\beta'_{\sigma f'}}_{c^+_{\sigma f'} c^-_{\sigma f'}}} \end{aligned} \tag{4.13}$$

If we choose an orthonormal basis in the space of 4-valent SU(2) intertwiners (labeled by $l_{\sigma t}$)

$$C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right) \in \mathrm{Inv}\left(\bigotimes_{f\ \text{incoming}\ \overrightarrow{(\sigma,t)}} V_{k_{\sigma f}} \bigotimes_{f'\ \text{outgoing}\ \overrightarrow{(\sigma,t)}} V^*_{k'_{\sigma f'}}\right) \tag{4.14}$$

we may expand $\mathcal{I}^4_{i^\pm_{\sigma t}}(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f})$ in terms of this basis, explicitly

$$\begin{aligned}
&\mathcal{I}^4_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right)_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}} \\
=& \sum_{l_{\sigma t}} \left[\mathcal{I}^4_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right)_{\{\rho_{\sigma f}\},\{\rho'_{\sigma f'}\}} \; C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)^{\{\rho_{\sigma f}\},\{\rho'_{\sigma f'}\}}\right] \overline{C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)}_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}} \\
\equiv& \sum_{l_{\sigma t}} f^{l_{\sigma t}}_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right) \; \overline{C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)}_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}}
\end{aligned} \tag{4.15}$$

Insert these findings into the partition function $Z_{\text{Simplified}}(\mathcal{K})$ yields

$$\begin{aligned}
&Z_{\text{Simplified}}(\mathcal{K}) \\
=& \sum_{\{j^\pm_{\sigma f}\}} \sum_{k_{\sigma f}, k'_{\sigma f}=|j^+_{\sigma f}-j^-_{\sigma f}|}^{j^+_{\sigma f}+j^-_{\sigma f}} \sum_{\{l_{\sigma t}\}} \int \prod_f \mathrm{d}\rho_f\, (\rho_f)^2 \prod_{(t,f)} \mathrm{d}N_{tf} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right) \; \beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right) \\
& \sum_{\{i^\pm_{\sigma t}\}} \prod_\sigma \left\{15j\right\}_{\mathrm{SO}(4)}\left(j^\pm_{\sigma f}, i^\pm_{\sigma t}\right) \prod_{\overrightarrow{(\sigma,t)}} f^{l_{\sigma t}}_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right) \; \overline{C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)}_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}} \\
& \prod_{(\sigma,f)} \pi^{k_{\sigma f}}_{\alpha_{\sigma f}, 2b_{\sigma f}}(N_{t'f}) \left[c(k_{\sigma f}, j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f} b_{\sigma f}} \; c(k'_{\sigma f}, j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f} b_{\sigma f}}\right] \pi^{k'_{\sigma f}}_{2b_{\sigma f}, \beta'_{\sigma f}}(N^{-1}_{tf})
\end{aligned} \tag{4.16}$$

from which we we read the vertex amplitude for each vertex dual to a 4-simplex σ

$$A_\sigma\left(j^\pm_{\sigma f}; k_{\sigma f}, k'_{\sigma f}; l_{\sigma t}\right) := \sum_{\{i^\pm_{\sigma t}\}} \left\{15j\right\}_{\mathrm{SO}(4)}\left(j^\pm_{\sigma f}, i^\pm_{\sigma t}\right) \prod_{\overrightarrow{(\sigma,t)}} f^{l_{\sigma t}}_{i^\pm_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f}; j^\pm_{\sigma f}\right) \tag{4.17}$$

Next, we consider the integrations of $\mathrm{d}N_{tf}$. Since the closure constraint is removed, the integrals over $\mathrm{d}N_{tf}$ can be done immediately. Consider a tetrahedron t_i is shared by two 4-simplex σ_i, σ_{i+1} (c.f. FIG.1), the integral of $\mathrm{d}N_{t_i,f}$ is essentially

$$\int \mathrm{d}N_{t_i f}\; \pi^{k_{\sigma_i f}}_{\alpha_{\sigma_i f}\beta_{\sigma_i f}}(N_{t_i f})\; \pi^{k'_{\sigma_{i+1} f}}_{\alpha'_{\sigma_{i+1} f}\beta'_{\sigma_{i+1} f}}(N^{-1}_{t_i f}) = \frac{1}{\dim(k_{\sigma_i f})}\delta^{k_{\sigma_i f} k'_{\sigma_{i+1} f}}\delta_{\alpha_{\sigma_i f}\beta'_{\sigma_{i+1} f}}\delta_{\beta_{\sigma_i f}\alpha'_{\sigma_{i+1} f}} \tag{4.18}$$

There are three consequences from these integrals:

1. The SU(2) representations $k_{\sigma_i f}$ is identified with $k'_{\sigma_{i+1} f}$, thus we label

$$k_{\sigma_i f} = k'_{\sigma_{i+1} f} \equiv k_{t_i f} \tag{4.19}$$

   where t_i is the tetrahedron shared by the 4-simplices σ_i, σ_{i+1} (see FIG.4).

2. For the SU(2) intertwiners on the half edges $\overrightarrow{(\sigma_i, t_i)}$ and $\overrightarrow{(\sigma_{i+1}, t_i)}$,

$$\begin{aligned}
& \overline{C^4_{l_{\sigma_i t_i}}\left(k_{t_i f}, k_{t_i f'}\right)}_{\{\beta'_{\sigma_i f}\},\{\alpha_{\sigma_i f'}\}} \overline{C^4_{l_{\sigma_{i+1} t_i}}\left(k_{t_i f'}, k_{t_i f}\right)}_{\{\beta'_{\sigma_{i+1} f'}\},\{\alpha_{\sigma_{i+1} f}\}} \\
& \prod_{f \text{ incoming } \overrightarrow{(\sigma_i,\sigma_{i+1})}} \delta_{\alpha_{\sigma_i f}\beta'_{\sigma_{i+1} f}} \prod_{f' \text{ outgoing } \overrightarrow{(\sigma_i,\sigma_{i+1})}} \delta_{\alpha_{\sigma_i f'}\beta'_{\sigma_{i+1} f'}} \\
=& \; \delta_{l_{\sigma_i t_i},\, l^\dagger_{\sigma_{i+1} t_i}}
\end{aligned} \tag{4.20}$$

   which identify the half edge intertwiners into full edge intertwiners

$$l_{\sigma_i t_i} = l^\dagger_{\sigma_{i+1} t_i} \equiv l_{\overrightarrow{(\sigma_i,\sigma_{i+1})}} \equiv l_{e_i} \tag{4.21}$$

   where $e_i := \overrightarrow{(\sigma_i, \sigma_{i+1})}$ is the edge dual to the tetrahedron t_i.

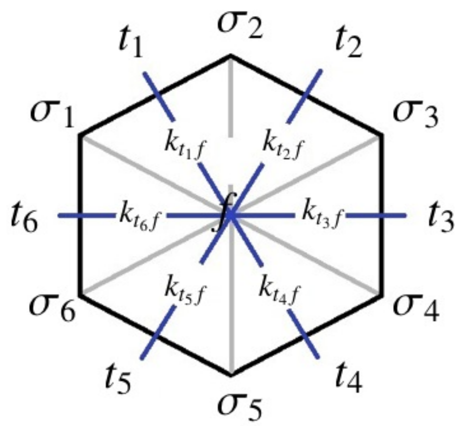

Figure 4: SU(2) representations k_{tf} assigned to each pair of (t, f).

3. For each face dual to f, we have a factor

$$\prod_{i=1}^{|\sigma|_f}\left[c(k_{t_if},j^{\pm}_{\sigma_if})^{2b_{\sigma_if}}_{b_{\sigma_if}b_{\sigma_if}}\ c(k_{t_{i-1}f},j^{\pm}_{\sigma_if})^{2b_{\sigma_if}}_{b_{\sigma_if}b_{\sigma_if}}\right]\prod_{i=1}^{|\sigma|_f}\delta_{2b_{\sigma_if},2b_{\sigma_{i+1}f}}$$
$$= \sum_{b_f}\prod_{i=1}^{|\sigma|_f}c(k_{t_if},j^{\pm}_{\sigma_if})^{2b_f}_{b_fb_f}\ c(k_{t_if},j^{\pm}_{\sigma_{i+1}f})^{2b_f}_{b_fb_f} \tag{4.22}$$

where the indices $b_{\sigma f}$ are identified for the different wedges belonging to the same dual face and the range of b_f is

$$\bigcap_{i=1}^{|\sigma|_f}\left[\{-j^+_{\sigma_if},\cdots,j^+_{\sigma_if}\}\cap\{-j^-_{\sigma_if},\cdots,j^-_{\sigma_if}\}\right] \tag{4.23}$$

and $|\sigma|_f$ is the number of vertices around a face dual to f.

Finally we consider the integrals of $\mathrm{d}\rho_f$. We define a triangle/face amplitude

$$A_f\left(j^{\pm}_{\sigma f},k_{tf}\right) := \sum_{b_f}\prod_{i=1}^{|\sigma|_f}c(k_{t_if},j^{\pm}_{\sigma_if})^{2b_f}_{b_fb_f}\ c(k_{t_if},j^{\pm}_{\sigma_{i+1}f})^{2b_f}_{b_fb_f}$$
$$\times\int_0^\infty \mathrm{d}\rho_f(\rho_f)^2\prod_{i=1}^{|\sigma|_f}\left[\beta_{j^+_{\sigma_if}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\ \beta_{j^-_{\sigma_if}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right)\right] \tag{4.24}$$

By Eq.(2.18), we can directly compute the expression of the function β_j

$$\beta_j(r)=\int \mathrm{d}g\ e^{-ir\mathrm{tr}(g)}\chi_j(g)=i^{-2j}(2j+1)\frac{J_{2j+1}(2r)}{r} \tag{4.25}$$

where $J_n(x)$ is the Bessel function of the first kind. The proof of this relation uses the recurrence relation:

$$J_{2j+2}(2r)+J_{2j}(2r)=(2j+1)\frac{J_{2j+1}(2r)}{r} \tag{4.26}$$

Let's consider the integrand of the integration over areas ρ_f (or considering an integral in large area regime). In the uniform limit of $j\to\infty,\rho\to\infty$ (or $r\to\infty$), the asymptotic behavior of the function β_j is (see e.g. [37], uniform limit can be made by the scaling $j\to\lambda j, r\to\lambda r$ and sending $\lambda\to\infty$)

$$\text{Large-}(j,r):\quad \beta_j(r)\sim i^{-2j}\frac{2j+1}{r}\delta(2r-2j) \tag{4.27}$$

It follows that in the uniform limit $j^{\pm}_{\sigma f} \to \infty, \rho_f \to \infty$, the asymptotic behaviour of the Bessel functions constrains the SO(4) representations on the wedges by

$$j^{\pm}_{\sigma f} = j^{\pm}_{\sigma' f} = j^{\pm}_{f} \tag{4.28}$$

and also impose the well-known constraint on the self-dual and anti-self-dual representations

$$\left|1 - \frac{1}{\gamma}\right| j^{+}_{f} = \left|1 + \frac{1}{\gamma}\right| j^{-}_{f} \tag{4.29}$$

which gives the quantization condition for the Barbero-Immirzi parameter

$$\begin{aligned} &\text{If } |\gamma| > 1: \qquad \gamma = \frac{j^{+}_{f} + j^{-}_{f}}{j^{+}_{f} - j^{-}_{f}} \\ &\text{If } |\gamma| < 1: \qquad \gamma = \frac{j^{+}_{f} - j^{-}_{f}}{j^{+}_{f} + j^{-}_{f}} \end{aligned} \tag{4.30}$$

While it is nice to see that we obtain certain points of contact with the EPRL and FK models respectively, one should keep in mind that these constraints hold only in the sense of large-j. In general, the representations which do not satisfy Eqs.(4.28) and (4.29) still have nontrivial contributions to the spin-foam amplitude and it is not clear whether these “non EPRL/FK configurations” have large or low measure.

Let us summarize the structure of the partition function $Z_{\text{Simplified}}(\mathcal{K})$ in terms of the 4-simplex/vertex amplitude, tetrahedron/edge amplitude and triangle/face amplitude

$$Z_{\text{Simplified}}(\mathcal{K}) = \sum_{\{j^{\pm}_{\sigma f}\}} \sum_{\{k_{tf}\}} \sum_{\{l_e\}} \prod_{f} A_f\left(j^{\pm}_{\sigma f}, k_{tf}\right) \prod_{t} A_t(k_{tf}) \prod_{\sigma} A_\sigma\left(j^{\pm}_{\sigma f}, k_{tf}, l_e\right) \tag{4.31}$$

where k_{tf} is constrained by the condition that for a tetrahedron t shared by both 4-simplices σ, σ' we have

$$k_{tf} \in \left\{|j^{+}_{\sigma f} - j^{-}_{\sigma f}|, \cdots, j^{+}_{\sigma f} + j^{-}_{\sigma f}\right\} \cap \left\{|j^{+}_{\sigma' f} - j^{-}_{\sigma' f}|, \cdots, j^{+}_{\sigma' f} + j^{-}_{\sigma' f}\right\} \tag{4.32}$$

and the 4-simplex/vertex amplitudes, tetrahedron/edge amplitudes and triangle/face amplitudes are respectively identified as

$$\begin{aligned} A_\sigma\left(j^{\pm}_{\sigma f}, k_{tf}, l_e\right) \quad &:= \quad \sum_{\{i^{\pm}_{\sigma t}\}} \left\{15j\right\}_{\mathrm{SO}(4)} \left(j^{\pm}_{\sigma f}, i^{\pm}_{\sigma t}\right) \prod_{\overrightarrow{(\sigma,t)}} f^{l_e}_{i^{\pm}_{\sigma t}}\left(k_{tf}; j^{\pm}_{\sigma f}\right) \\ A_t(k_{tf}) \quad &:= \quad \prod_{f \subset t} \frac{1}{\dim(k_{tf})} \\ A_f\left(j^{\pm}_{\sigma f}, k_{tf}\right) \quad &= \quad \sum_{b_f} \prod_{i=1}^{|\sigma|_f} c(k_{t_i f}, j^{\pm}_{\sigma_i f})^{2b_f}_{b_f b_f}\ c(k_{t_i f}, j^{\pm}_{\sigma_{i+1} f})^{2b_f}_{b_f b_f} \\ &\quad \times \int_0^\infty \mathrm{d}\rho_f (\rho_f)^2 \prod_{i=1}^{|\sigma|_f} \left[\beta_{j^{+}_{\sigma_i f}}\left(\left|1 + \frac{1}{\gamma}\right| \rho_f\right)\ \beta_{j^{-}_{\sigma_i f}}\left(\left|1 - \frac{1}{\gamma}\right| \rho_f\right)\right] \end{aligned} \tag{4.33}$$

When we take the uniform limit: $j^{\pm}_{\sigma f}, \rho_f \to \infty$ for the integrand, by the previous discussion, we obtain the constraints:

$$j^{\pm}_{\sigma f} = j^{\pm}_{\sigma' f} = j^{\pm}_{f} \quad \text{and} \quad \left|1 - \frac{1}{\gamma}\right| j^{+}_{f} = \left|1 + \frac{1}{\gamma}\right| j^{-}_{f} \tag{4.34}$$

Thus the spins $j^{\pm}_{\sigma f}$ for different wedges are identical on the same face dual to f, and j^{+}_{f} and j^{-}_{f} satisfies the “γ-simple” relation in this limit. Then the vertex amplitude reduces to

$$A_\sigma \quad \sim \quad \sum_{\{i^{\pm}_{t}\}} \left\{15j\right\}_{\mathrm{SO}(4)} \left(j^{\pm}_{f}, i^{\pm}_{t}\right) \prod_{\overrightarrow{(\sigma,t)}} f^{l_e}_{i^{\pm}_{t}}\left(k_{tf}; j^{\pm}_{f}\right) \tag{4.35}$$

where j_f^+ and j_f^- subject the relation in Eq.(4.40). We notice that in this limit Eq.(4.35) is nothing but the vertex amplitude of the FK$_\gamma$ Model (when $|\gamma|>1$) [12]. And in the large-j limit the integral over area ρ_f in the large area regime can be approximated by a discrete sum over j_f^- or j_f^+ in the path integral Eq.(2.1). In the usual context of spinfoam formulation, the large-j limit is understood as a semiclassical limit in a certain sense [27, 28].

4.2 On the Implementation of Closure Constraint

In this subsection we properly keep the closure constraint in the partition function:

$$
\begin{aligned}
Z(\mathcal{K}) \;=\; & \sum_{\{j^\pm_{\sigma f}\}} \sum_{k_{\sigma f},k'_{\sigma f}=|j^+_{\sigma f}-j^-_{\sigma f}|}^{j^+_{\sigma f}+j^-_{\sigma f}} \sum_{\{l_{\sigma t}\}} \int \prod_f \mathrm{d}\rho_f\,(\rho_f)^2 \prod_{(t,f)} \mathrm{d}N_{tf} \prod_{(\sigma,f)} \beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\,\beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right)\\
& \sum_{\{i^\pm_{\sigma t}\}} \prod_\sigma \mathcal{A}_\sigma\left(j^\pm_{\sigma f};k_{\sigma f},k'_{\sigma f};l_{\sigma t}\right) \prod_{\overrightarrow{(\sigma,t)}} \overline{C^4_{l_{\sigma t}}\left(k_{\sigma f},k'_{\sigma f'}\right)}_{\{\beta'_{\sigma f}\},\{\alpha_{\sigma f'}\}} \prod_t \delta\Big(\sum_{f\subset t}\rho_f N_{tf}\tau_3 N_{tf}^{-1}\Big)\\
& \prod_{(\sigma,f)} \pi^{k_{\sigma f}}_{\alpha_{\sigma f},2b_{\sigma f}}(N_{t'f})\left[c(k_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}\;c(k'_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}\right]\pi^{k'_{\sigma f}}_{2b_{\sigma f},\beta'_{\sigma f}}(N^{-1}_{tf})
\end{aligned}
\tag{4.36}
$$

Here we can also extract the vertex/4-simplex amplitude $\mathcal{A}_\sigma$, the edge/tetrahedron amplitude $\mathcal{A}_t$, and the face/triangle amplitude $\mathcal{A}_f$

$$
\begin{aligned}
\mathcal{A}_\sigma\left(j^\pm_{\sigma f};k_{\sigma f},k'_{\sigma f};l_{\sigma t}\right) \;&:=\; \sum_{\{i^\pm_{\sigma t}\}} \Big\{15j\Big\}_{\mathrm{SO}(4)}\left(j^\pm_{\sigma f},i^\pm_{\sigma t}\right) \prod_{\overrightarrow{(\sigma,t)}} f^{l_{\sigma t}}_{i^\pm_{\sigma t}}\left(k_{\sigma f},k'_{\sigma f};j^\pm_{\sigma f}\right)\\
\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big) \;&:=\; \int\prod_{f\subset t}\mathrm{d}N_{tf}\;\delta\Big(\sum_{f\subset t}\rho_f N_{tf}\tau_3 N^{-1}_{tf}\Big)\\
&\quad \left[\overline{C^4_{l_{\sigma t}}\left(k_{\sigma f},k'_{\sigma f'}\right)}_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}\right]\left[\overline{C^4_{l_{\sigma' t}}\left(k_{\sigma' f},k'_{\sigma' f'}\right)}_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}\right]\\
&\quad \prod_{f\ \text{outgoing}} \pi^{k'_{\sigma f}}_{2b_{\sigma f},\alpha_{\sigma f}}(N^{-1}_{tf})\;\pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_{\sigma' f}}(N_{tf}) \prod_{f'\ \text{incoming}} \pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{\sigma f'}}(N_{tf'})\pi^{k'_{\sigma' f'}}_{2b_{\sigma' f'},\beta_{\sigma' f'}}(N^{-1}_{tf'})\\
\mathcal{A}_f\left(\rho_f;j^\pm_{\sigma f};k_{\sigma f},k'_{\sigma f};b_{\sigma f}\right) \;&:=\; (\rho_f)^2\prod_{(\sigma,f)}\beta_{j^+_{\sigma f}}\left(\left|1+\frac{1}{\gamma}\right|\rho_f\right)\;\beta_{j^-_{\sigma f}}\left(\left|1-\frac{1}{\gamma}\right|\rho_f\right)\\
&\quad \prod_{(\sigma,f)} c(k_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}\;c(k'_{\sigma f},j^\pm_{\sigma f})^{2b_{\sigma f}}_{b_{\sigma f}b_{\sigma f}}
\end{aligned}
\tag{4.37}
$$

Then the partition function can be written in terms of these amplitudes as:

$$
\begin{aligned}
Z(\mathcal{K}) \;=\; & \sum_{\{j^\pm_{\sigma f}\}}\sum_{\{k_{\sigma f},k'_{\sigma f}\}}\sum_{\{l_{\sigma t}\}}\sum_{\{b_{\sigma f}\}}\int\prod_f\mathrm{d}\rho_f\times\\
& \mathcal{A}_f\left(\rho_f;j^\pm_{\sigma f};k_{\sigma f},k'_{\sigma f};b_{\sigma f}\right)\;\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)\;\mathcal{A}_\sigma\left(j^\pm_{\sigma f};k_{\sigma f},k'_{\sigma f};l_{\sigma t}\right)
\end{aligned}
\tag{4.38}
$$

Note that in the large-(j,ρ) limit,

$$
\text{Large-}(j,\rho):\quad \beta_{j^\pm_{\sigma f}}\left(\left|1\pm\frac{1}{\gamma}\right|\rho_f\right)\sim i^{-2j^\pm_{\sigma f}}\frac{2j^\pm_{\sigma f}+1}{\left|1\pm\frac{1}{\gamma}\right|\rho_f}\;\delta\left(2\left|1\pm\frac{1}{\gamma}\right|\rho_f-2j^\pm_{\sigma f}\right)
\tag{4.39}
$$

In this limit, the integral of ρ_f in the large area regime is completely constrained by the delta functions. Thus, as in the previous section, the delta functions impose the constraints:

$$
j^\pm_{\sigma f}=j^\pm_{\sigma' f}=j^\pm_f \qquad\text{and}\qquad \left|1-\frac{1}{\gamma}\right|j^+_f=\left|1+\frac{1}{\gamma}\right|j^-_f
\tag{4.40}
$$

This shows that in the large-(j,ρ) limit the spins $j^\pm_{\sigma f}$ for different wedges are identical on the same face dual to f, and j^+_f and j^-_f satisfy the "γ-simple" relation in this limit. However, since at the current stage the constraint

$$k_{\sigma_i f}=k'_{\sigma_{i+1} f}\equiv k_{t_i f} \tag{4.41}$$

are not obviously imposed by the integral of N_{tf} (because of the present of closure constraint in $\mathcal{A}_t$), the vertex amplitude $\mathcal{A}_\sigma$, even in the large-j limit, does not approximate the FK_γ vertex amplitude in general.

To explore the structure of this amplitude, we consider the integral of N_{tf} in the expression of $\mathcal{A}_t$, for a tetrahedron t shared by σ,σ'

$$\begin{aligned}&\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)\\ :=\ &\int\prod_{f\subset t}\mathrm{d}N_{tf}\ \delta\Big(\sum_{f\subset t}\rho_f N_{tf}\tau_3 N_{tf}^{-1}\Big)\ \left[\overline{C^4_{l_{\sigma t}}\Big(k_{\sigma f},k'_{\sigma f'}\Big)}_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}\right]\left[\overline{C^4_{l_{\sigma' t}}\Big(k_{\sigma' f},k'_{\sigma' f'}\Big)}_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}\right]\\ &\prod_{f\ \text{outgoing}}\pi^{k'_{\sigma f}}_{2b_{\sigma f},\alpha_{\sigma f}}(N_{tf}^{-1})\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_{\sigma' f}}(N_{tf})\prod_{f'\ \text{incoming}}\pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{\sigma f'}}(N_{tf'})\pi^{k'_{\sigma' f'}}_{2b_{\sigma' f'},\beta_{\sigma' f'}}(N_{tf'}^{-1})\end{aligned} \tag{4.42}$$

Recall that $N_{tf}=g_{tf}N_f$ ($N_{tf}=N^+_{tf}=g^+_{tf}N^+_f$) while the integrand of the partition function only depends on the combination $N_f\tau_3N_f^{-1}$ (recall that the integrand depends on $X^\pm_f$), i.e. the integrand is invariant under $N_f\mapsto N_f h_\phi$ where $h_\phi\in\mathrm{U}(1)$ thus it only depends on $\mathrm{SU}(2)/\mathrm{U}(1)$. Let us parameterize N_f in terms of the spherical coordinates: In terms of the complex coordinates $(z_f,\bar{z}_f)$ on the unit sphere we have

$$N(z_f)\ =\ \frac{1}{\sqrt{1+|z_f|^2}}\begin{pmatrix}1 & z_f\\ -\bar{z}_f & 1\end{pmatrix} \tag{4.43}$$

where the complex coordinates $z,\bar{z}$ are defined by the stereographic projection, and the unit vector $\vec{\Omega}$ on S^2 is expressed in terms of the complex coordinates

$$\vec{\Omega}(z)\ =\ -i\left(-\frac{z+\bar{z}}{1+|z|^2}\ \sigma_1+\frac{1}{i}\frac{z-\bar{z}}{1+|z|^2}\ \sigma_2+\frac{1-|z|^2}{1+|z|^2}\ \sigma_3\right)=N(z)\tau_3N(z)^{-1} \tag{4.44}$$

Under the action of SU(2) group

$$gN(z)=N(z^g)\begin{pmatrix}\frac{\bar{a}-\bar{b}z}{|a-b\bar{z}|} & 0\\ 0 & \frac{a-b\bar{z}}{|a-b\bar{z}|}\end{pmatrix}^{-1}\qquad\text{where}\qquad\begin{pmatrix}\frac{\bar{a}-\bar{b}z}{|a-b\bar{z}|} & 0\\ 0 & \frac{a-b\bar{z}}{|a-b\bar{z}|}\end{pmatrix}\in\mathrm{U}(1) \tag{4.45}$$

where

$$z^g=\frac{az+b}{\bar{a}-\bar{b}z}\qquad\text{with}\qquad g=\begin{pmatrix}a & b\\ -\bar{b} & \bar{a}\end{pmatrix} \tag{4.46}$$

Therefore

$$N_{tf}=g_{tf}N(z_f)=N(z_{tf})h^{-1}_{\phi_{tf}}\qquad h_{\phi_{tf}}\in\mathrm{U}(1) \tag{4.47}$$

where $z_{tf}=z_f^{g_{tf}}$. Note that the above decomposition may also be understood by writing the SU(2) matrix in terms of Euler coordinates, i.e. $u=u(\phi_2)\ u(\theta)\ u(\phi_1)$ for all $u\in\mathrm{SU}(2)$

$$u(\phi_1)=\pm\begin{pmatrix}e^{i\phi_1/2} & 0\\ 0 & e^{i\phi_1/2}\end{pmatrix}\qquad u(\theta)=\pm\begin{pmatrix}\cos\frac{\theta}{2} & i\sin\frac{\theta}{2}\\ i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}\qquad u(\phi_2)=\pm\begin{pmatrix}e^{i\phi_2/2} & 0\\ 0 & e^{i\phi_2/2}\end{pmatrix} \tag{4.48}$$

where $u(\phi_1),u(\phi_2)\in\mathrm{U}(1)$, and $0\le\phi_1,\phi_2\le 2\pi$, $0\le\theta\le\pi$, while the SU(2) Haar measure can also be written as

$$\mathrm{d}g=\frac{1}{16\pi^2}\sin\theta\mathrm{d}\phi_1\mathrm{d}\theta\mathrm{d}\phi_2 \tag{4.49}$$

Hence the integral Eq.(4.42) can be written as

$$\begin{aligned}
&\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)\\
=&\int\prod_{f\subset t}\mathrm{d}^2\Omega_{tf}\mathrm{d}\phi_{tf}\ \delta\Big(\sum_{f\subset t}\rho_f\vec{\Omega}_{tf}\Big)\ \Big[\overline{C^4_{l_{\sigma t}}\Big(k_{\sigma f},k'_{\sigma f'}\Big)}_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}\Big]\Big[\overline{C^4_{l_{\sigma' t}}\Big(k_{\sigma' f},k'_{\sigma' f'}\Big)}_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}\Big]\\
&\prod_{f\ \text{outgoing}}\ \pi^{k'_{\sigma f}}_{2b_{\sigma f},\alpha_{\sigma f}}\Big(h_{\phi_{tf}}N(z_{tf})^{-1}\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_{\sigma' f}}\Big(N(z_{tf})h^{-1}_{\phi_{tf}}\Big)\\
&\prod_{f'\ \text{incoming}}\ \pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{\sigma f'}}\Big(N(z_{tf'})h^{-1}_{\phi_{tf'}}\Big)\pi^{k'_{\sigma' f'}}_{2b_{\sigma' f'},\beta_{\sigma' f'}}\Big(h_{\phi_{tf'}}N(z_{tf'})^{-1}\Big)
\end{aligned}\tag{4.50}$$

where $\mathrm{d}\Omega_{tf}$ is the standard spherical measure on $S^2=\mathrm{SU}(2)/\mathrm{U}(1)$. Since $\pi^j_{mn}(h_\phi)=(e^{2im\phi})\delta_{mn}$ it follows

$$\begin{aligned}
&\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)\\
=&\int\prod_{f\subset t}\mathrm{d}^2\Omega_{tf}\mathrm{d}\phi_{tf}\ \delta\Big(\sum_{f\subset t}\rho_f\vec{\Omega}_{tf}\Big)\ \Big[\overline{C^4_{l_{\sigma t}}\Big(k_{\sigma f},k'_{\sigma f'}\Big)}_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}\Big]\Big[\overline{C^4_{l_{\sigma' t}}\Big(k_{\sigma' f},k'_{\sigma' f'}\Big)}_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}\Big]\\
&\prod_{f\ \text{outgoing}}\ e^{4i(b_{\sigma f}-b_{\sigma' f})\phi_{tf}}\ \pi^{k'_{\sigma f}}_{2b_{\sigma f},\alpha_{\sigma f}}\Big(N(z_{tf})^{-1}\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_{\sigma' f}}\Big(N(z_{tf})\Big)\\
&\prod_{f'\ \text{incoming}}\ e^{4i(b_{\sigma' f'}-b_{\sigma f'})\phi_{tf'}}\pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{\sigma f'}}\Big(N(z_{tf'})\Big)\pi^{k'_{\sigma' f'}}_{2b_{\sigma' f'},\beta_{\sigma' f'}}\Big(N(z_{tf'})^{-1}\Big)
\end{aligned}\tag{4.51}$$

The integrals $\int_0^{2\pi}\mathrm{d}\phi_{tf}$ impose the constraint that $b_{\sigma f}=b_{\sigma' f}\equiv b_f$ for all $f\subset t$, hence

$$\begin{aligned}
&\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)\\
=&\int\prod_{f\subset t}\mathrm{d}^2\Omega_{tf}\ \delta\Big(\sum_{f\subset t}\rho_f\vec{\Omega}_{tf}\Big)\ \Big[\overline{C^4_{l_{\sigma t}}\Big(k_{\sigma f},k'_{\sigma f'}\Big)}_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}\Big]\Big[\overline{C^4_{l_{\sigma' t}}\Big(k_{\sigma' f},k'_{\sigma' f'}\Big)}_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}\Big]\\
&\prod_{f\ \text{outgoing}}\pi^{k'_{\sigma f}}_{2b_f,\alpha_{\sigma f}}\Big(N(z_{tf})^{-1}\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_f}\Big(N(z_{tf})\Big)\prod_{f'\ \text{incoming}}\pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{f'}}\Big(N(z_{tf'})\Big)\ \pi^{k'_{\sigma' f'}}_{2b_{f'},\beta_{\sigma' f'}}\Big(N(z_{tf'})^{-1}\Big)
\end{aligned}\tag{4.52}$$

Moreover for the outgoing dual face f, we have the relation

$$\begin{aligned}
&\pi^{k'_{\sigma f}}_{2b_f,\alpha_{\sigma f}}\Big(N^{-1}\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_f}\Big(N\Big)\ =\ \pi^{k'_{\sigma f}}_{2b_f,\alpha_{\sigma f}}\Big(\epsilon N^T\epsilon^{-1}\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_f}\Big(N\Big)\\
=&\ (-1)^{2k'_{\sigma f}-2b_f-\alpha_{\sigma f}}\pi^{k'_{\sigma f}}_{-\alpha_{\sigma f},-2b_f}\Big(N\Big)\ \pi^{k_{\sigma' f}}_{\beta_{\sigma' f},2b_f}\Big(N\Big)\\
=&\ (-1)^{2k'_{\sigma f}-2b_f-\alpha_{\sigma f}}\sum_{l'_{tf}=|k'_{\sigma f}-k_{\sigma' f}|}^{k'_{\sigma f}+k_{\sigma' f}}c\left(l'_{tf};k'_{\sigma f},k_{\sigma' f}\right)^{\rho'_{tf}}_{-\alpha_{\sigma f},\beta_{\sigma' f}}\ c\big(l'_{tf};k'_{\sigma f},k_{\sigma' f}\big)^{0}_{-2b_f,2b_f}\pi^{l'_{tf}}_{\rho'_{tf},0}\big(N\big)
\end{aligned}\tag{4.53}$$

while for the incoming dual face f' we have similarly

$$\begin{aligned}
&\pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{f'}}\Big(N\Big)\ \pi^{k'_{\sigma' f'}}_{2b_{f'},\beta_{\sigma' f'}}\Big(N^{-1}\Big)\ =\ \pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{f'}}\Big(N\Big)\ \pi^{k'_{\sigma' f'}}_{2b_{f'},\beta_{\sigma' f'}}\Big(\epsilon N^T\epsilon^{-1}\Big)\\
=&\ (-1)^{2k'_{\sigma' f'}-2b_{f'}-\beta_{\sigma' f'}}\pi^{k_{\sigma f'}}_{\alpha_{\sigma f'},2b_{f'}}\Big(N\Big)\ \pi^{k'_{\sigma' f'}}_{-\beta_{\sigma' f'},-2b_{f'}}\Big(N\Big)\\
=&\ (-1)^{2k'_{\sigma' f'}-2b_{f'}-\beta_{\sigma' f'}}\sum_{l_{tf'}=|k_{\sigma f'}-k'_{\sigma' f'}|}^{k_{\sigma f'}+k'_{\sigma' f'}}c\left(l_{tf'};k_{\sigma f'},k'_{\sigma' f'}\right)^{\rho_{tf'}}_{\alpha_{\sigma f'},-\beta_{\sigma' f'}}\ c\big(l_{tf'};k_{\sigma f'},k'_{\sigma' f'}\big)^{0}_{2b_{f'},-2b_{f'}}\pi^{l_{tf'}}_{\rho_{tf'},0}\big(N\big)
\end{aligned}\tag{4.54}$$

Thus the integral reduces to

$$\mathcal{A}_t\Big(\rho_f;k_{\sigma f},k'_{\sigma f};l_{\sigma t};b_{\sigma f}\Big)=\int\prod_{f\subset t}\mathrm{d}^2\Omega_{tf}\mathrm{d}\phi_{tf}\ \delta\Big(\sum_{f\subset t}\rho_f\vec{\Omega}_{tf}\Big)\ \Theta_t\left(l_{\sigma t},l_{\sigma' t},k'_{\sigma f},k_{\sigma' f},k_{\sigma f'},k'_{\sigma' f'},b_f;z_{tf}\right)\tag{4.55}$$

with the integrand (t is the tetrahedron shared by σ, σ')

$$\Theta_t\left(l_{\sigma t}, l_{\sigma' t}, k'_{\sigma f}, k_{\sigma' f}, k_{\sigma f'}, k'_{\sigma' f'}, b_f; z_{tf}\right) :=$$

$$\prod_{f \text{ outgoing}} (-1)^{2k'_{\sigma f}-2b_f-\alpha_{\sigma f}} \sum_{l'_{tf}=|k'_{\sigma f}-k_{\sigma' f}|}^{k'_{\sigma f}+k_{\sigma' f}} c\left(l'_{tf}; k'_{\sigma f}, k_{\sigma' f}\right)^{\rho'_{tf}}_{-\alpha_{\sigma f}, \beta_{\sigma' f}} c\big(l'_{tf}; k'_{\sigma f}, k_{\sigma' f}\big)^{0}_{-2b_f, 2b_f} \pi^{l'_{tf}}_{\rho'_{tf},0}\big(N(z_{tf})\big)$$

$$\prod_{f' \text{ incoming}} (-1)^{2k'_{\sigma' f'}-2b_{f'}-\beta_{\sigma' f'}} \sum_{l_{tf'}=|k_{\sigma f'}-k'_{\sigma' f'}|}^{k_{\sigma f'}+k'_{\sigma' f'}} c\left(l_{tf'}; k_{\sigma f'}, k'_{\sigma' f'}\right)^{\rho_{tf'}}_{\alpha_{\sigma f'}, -\beta_{\sigma' f'}} c\big(l_{tf'}; k_{\sigma f'}, k'_{\sigma' f'}\big)^{0}_{2b_{f'}, -2b_{f'}} \pi^{l_{tf'}}_{\rho_{tf'},0}\big(N(z_{tf'})\big)$$

$$\left[\overline{C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}}\right] \left[\overline{C^4_{l_{\sigma' t}}\left(k_{\sigma' f}, k'_{\sigma' f'}\right)_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}}\right] \tag{4.56}$$

The complete integration of Eq.(4.55) turns out to be rather involved, thus is left as a future research. What one can say is the following: From the expression Eq.(4.56), it is not hard to see that the set of amplitudes contributing to the simplified model $Z_{\text{Simplified}}(\mathcal{K})$ are part of those of the full model $Z(\mathcal{K})$. To see this, notice that the simplest nontrivial contribution of the integral in $\mathcal{A}_t$ comes from the term with $l_{tf'} = l'_{tf} = 0$. With the constraints $l_{tf'} = l'_{tf} = 0$ (and thus $\rho'_{tf} = \rho_{tf'} = 0$), we obtain the same set of constraints as it was in the previous subsection for the simplified model $Z_{\text{Simplified}}(\mathcal{K})$.

$$k'_{\sigma f} = k_{\sigma' f} \equiv k_{tf} \quad k_{\sigma f'} = k'_{\sigma' f'} \equiv k_{tf'} \quad \alpha_{\sigma f'} = \beta_{\sigma' f'} \quad \alpha_{\sigma f} = \beta_{\sigma' f} \tag{4.57}$$

Since

$$c\,(0; k, k')^{0}_{\alpha,\beta} = \delta_{k,k'}\delta_{\alpha,-\beta}\frac{(-1)^{k-\alpha}}{\sqrt{\dim(k)}} \tag{4.58}$$

we obtain, by extracting the term with $l_{tf'} = l'_{tf} = 0$ and dropping the contribution from the other terms

$$\begin{aligned}
&\Theta_t\left(l_{\sigma t}, l_{\sigma' t}, k'_{\sigma f}, k_{\sigma' f}, k_{\sigma f'}, k'_{\sigma' f'}, b_f; z_{tf}\right)\\
\to\; &\prod_{f \text{ outgoing}} (-1)^{2k'_{\sigma f}-2b_f-\alpha_{\sigma f}} \delta_{k'_{\sigma f}, k_{\sigma' f}} \delta_{\alpha_{\sigma f}, \beta_{\sigma' f}} \frac{(-1)^{k'_{\sigma f}+\alpha_{\sigma f}}(-1)^{k'_{\sigma f}+2b_f}}{\dim(k'_{\sigma f})}\\
&\prod_{f' \text{ incoming}} (-1)^{2k'_{\sigma' f'}-2b_{f'}-\beta_{\sigma' f'}} \delta_{k_{\sigma f'}, k'_{\sigma' f'}} \delta_{\alpha_{\sigma f'}, \beta_{\sigma' f'}} \frac{(-1)^{k_{\sigma f'}-\beta_{\sigma' f'}}(-1)^{k_{\sigma f'}-2b_{f'}}}{\dim(k_{\sigma f'})}\\
&\left[\overline{C^4_{l_{\sigma t}}\left(k_{\sigma f}, k'_{\sigma f'}\right)_{\{\alpha_{\sigma f}\},\{\alpha_{\sigma f'}\}}}\right] \left[\overline{C^4_{l_{\sigma' t}}\left(k_{\sigma' f}, k'_{\sigma' f'}\right)_{\{\beta_{\sigma' f'}\},\{\beta_{\sigma' f}\}}}\right]\\
=\; &\prod_{f\subset t} \frac{1}{\dim(k_{tf})} \delta_{k'_{\sigma f}, k_{\sigma' f}} \delta_{k_{\sigma f'}, k'_{\sigma' f'}} \delta_{l_{\sigma t}, l^{\dagger}_{\sigma' t}}
\end{aligned} \tag{4.59}$$

For this subset of amplitude the edge/tetrahedron amplitude reduces to

$$\begin{aligned}
\mathcal{A}_t\Big(\rho_f; k_{\sigma f}, k'_{\sigma f}; l_{\sigma t}; b_{\sigma f}\Big) \;&\to\; \prod_{f\subset t} \frac{1}{\dim(k_{tf})} \delta_{k'_{\sigma f}, k_{\sigma' f}} \delta_{k_{\sigma f'}, k'_{\sigma' f'}} \delta_{l_{\sigma t}, l^{\dagger}_{\sigma' t}} \int \prod_{f\subset t} \mathrm{d}^2\Omega_{tf}\mathrm{d}\phi_{tf}\; \delta\Big(\sum_{f\subset t} \rho_f \vec{\Omega}_{tf}\Big)\\
&\equiv\; \mathcal{A}'_t\Big(\rho_f; k_{\sigma f}, k'_{\sigma f}; l_{\sigma t}; b_{\sigma f}\Big)
\end{aligned} \tag{4.60}$$

Then we can define a spin-foam model by pick out a subset of amplitudes in the full partition function $Z(\mathcal{K})$:

$$\begin{aligned}
Z'(\mathcal{K}) \;=\; &\sum_{\{j^{\pm}_{\sigma f}\}} \sum_{\{k_{\sigma f}, k'_{\sigma f}\}} \sum_{\{l_{\sigma t}\}} \sum_{\{b_{\sigma f}\}} \int \prod_f \mathrm{d}\rho_f \;\times\\
&\mathcal{A}_f\left(\rho_f; j^{\pm}_{\sigma f}; k_{\sigma f}, k'_{\sigma f}; b_{\sigma f}\right)\; \mathcal{A}'_t\Big(\rho_f; k_{\sigma f}, k'_{\sigma f}; l_{\sigma t}; b_{\sigma f}\Big)\; \mathcal{A}_\sigma\left(j^{\pm}_{\sigma f}; k_{\sigma f}, k'_{\sigma f}; l_{\sigma t}\right)
\end{aligned} \tag{4.61}$$

The amplitudes in $Z'(\mathcal{K})$ are contributions with the closure constraint implemented, however unfortunately they may not exhaust all the contributions.

In Eq.(4.60) the Kronecker deltas $\delta_{k'_{\sigma f},k_{\sigma' f}}\delta_{k_{\sigma f'},k'_{\sigma' f'}}\delta_{l_{\sigma t},l^\dagger_{\sigma' t}}$ imply that there is an one-to-one correspondence between the transition channels in the simplified model $Z_{\text{Simplified}}(\mathcal{K})$ and the transition channels in the model $Z'(\mathcal{K})$, which form a subset of the transition channels in $Z(\mathcal{K})$. Consider the sets $\{Z_{\text{Simplified}}\}$ and $\{Z\}$ respectively, which are the collections of spin-foams that contribute to their respective partition functions $Z_{\text{Simplified}}(\mathcal{K})$ and $Z(\mathcal{K})$. Our above analysis then reveals

$$\{Z_{\text{Simplified}}\} \subset \{Z\} \tag{4.62}$$

At this point this is all we can say about the relation between the models with the closure constraint in place or not. The additional weights and contributions in the full model may severely change the correlators (physical inner product) and it is by no means obvious that the simplified model is a good approximation.

As a final remark, the above inclusion is in terms of spin-foam amplitude, in the sense that we write the partition functions as a sum of amplitude over possible spins and intertwiners. Moreover such an inclusion is natural from the path integral point of view. We consider a simple example: Consider a function $f(x,y)$ on $\mathbb{R}^2$ which has a Fourier transform $\tilde{f}(k,q)$ and that we have a "closure constraint" $y=0$. Then the Z integral (with closure) corresponds to (dropping factors of 2π)

$$\begin{aligned} Z &= \int dxdy\delta(y,0)f(x,y) \;=\; \int dxdy\delta(y,0)\int dkdq\tilde{f}(k,q)\exp(i(kx+qy)) \\ &= \int dx\int dkdq\tilde{f}(k,q)\exp(ikx) \;=\; \int dkdq\tilde{f}(k,q)\delta(k,0) \\ &= \int dq\tilde{f}(0,q) \end{aligned} \tag{4.63}$$

On the other hand the Z_{Simplied} integral without closure is

$$\begin{aligned} Z_{\text{Simplied}} &= \int dxdyf(x,y) \;=\; \int dxdy\int dkdq\tilde{f}(k,q)\exp(i(kx+qy)) \;=\; \int dkdq\tilde{f}(k,q)\delta(k,0)\delta(q,0) \\ &= \tilde{f}(0,0) \end{aligned} \tag{4.64}$$

Hence the Z amplitudes are more in Fourier space (k,p) corresponding to spin-foam representation, and less in real space (x,y).

5 Outlook

In section 4 we first carried out the analysis for the simplified partition function without closure constraint and obtained the spin-foam model $Z_{\text{Simplified}}(\mathcal{K})$, then we discussed the complete partition function $Z(\mathcal{K})$ with closure constraint implemented, however we did not compute yet explicitly the full set of possible spin-foam amplitudes. We were only able to show that all the spin-foam amplituded contributing to $Z_{\text{Simplified}}(\mathcal{K})$ are contained in those contributing to the full model $Z(\mathcal{K})$. Therefore, in addition to present spin foam models, our commutative B field model variable sums over additional amplitudes having non- trivial contributions to the partition function $Z(\mathcal{K})$. While we have shown that in the large-j limit the 4-simplex/vertex amplitude of $Z_{\text{Simplified}}(\mathcal{K})$ can be related to the 4-simplex/ vertex amplitude of FK$_\gamma$ Model ($|\gamma|>1$), for the full model $Z(\mathcal{K})$, even in the large-j limit, there exist additional, non-trivial spin-foam amplitudes. It would be important to further specify those unknown spin-foams contributing to $\{Z\}$ but not to $\{Z_{\text{Simplified}}\}$, at least for their large-j asymptotics.

Unfortunately, the relation between our new model and EPRL model is almost untouched in the present article. Although we have seen that all the EPRL spin-foams (with possibly different triangle/face and tetrahedron/edge amplitudes) are included in $\{Z_{\text{Simplified}}\}$ (thus in $\{Z\}$), it seems to us that, however, they are not quite special among the spin-foam amplitudes contributing $Z_{\text{Simplified}}(\mathcal{K})$ or $Z(\mathcal{K})$. We expected that the relation between our model $Z(\mathcal{K})$ and EPRL Model could be realized by the non-commutative deformation, like in the case of Barrett-Crane Model. The reason for our expectation was that (1) both models are defined via the non-commutative operator constraint technique, and (2) when the Barbero-Immirzi parameter $\gamma \to \infty$, EPRL Model reduces to Barrett-Crane Model. However it turns out that our expectation is difficult to realize, since the non-commutative deformation via the group Fourier transformation hardly works for the case of finite γ. It seems to us that if our model $Z(\mathcal{K})$ and the EPRL Model could be related via any non-commutative deformation, we should rather choose a different deformation scheme.

The present article starts from a purely path-integral/spin-foam point of view. If we also consider the relation between the path integral and canonical quantization, then the partition function Eq. (2.1) should probably be modified. It is pointed in [15] that a quantum gravity path integral formula consistent with canonical physical inner product should not only be an naive path integral Eq.(1.2) of Plebanski-Holst action, but also include a suitable local measure factor in the path integral formula. The local measure factor is a product of a certain power of spacetime volume elements and a certain power of spatial volume elements at all the spacetime points. The implementation of such local measure factor in the partition function will modify both the 4-simplex/ vertex and tetrahedron/edge amplitudes. A detailed analysis of this issue will be postponed to future research.

It is interesting to look for relations with other new approaches on the implementation of simplicity constraint in spinfoam models or GFTs. In the appendix, we show that a non-commutative deformation of the above model, as a noncommutative simplicial path integral, relates to the GFT model defined in [30]. One may also compare the approach here with the "holomorphic simplicity constraint" in [38], where the new version of simplicity constraints using spinor/twistor variables are commutative. However this approach closely relates to the operator-constraint approach reviewed in the introduction. The commutative holomorphic simplicity constraints come from the noncommutative algebra of flux variables. It may also be interesting to see the relation with quantum Regge calculus. As far as we have shown, the spinfoam model constructed here comes from a path integral of simplicial Plebanski-Holst action, where the discretization procedure is different from Regge calculus (in 1st or 2nd order formulations). So the resulting spinfoam model doesn't coincide with the quantum Regge calculus in general. But it is possible that they may be related in certain limit. Such a possibility should be studied in the future.

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A Noncommutative Deformation and Barrett-Crane Model

A.1 Noncommutative Deformation

In order to further investigate the question, in which sense the closure constraint is redundant when working with non commutative B fields (as is common practice in existent spin foam models), in this section, we explore a non-commutative deformation of our starting point, the partition function $Z(\mathcal{K})$ in Eq.(2.1). The non-commutative deformation we will employ here comes from a generalized Fourier transformation defined on a compact group [29].

The deformation replaces the normal c-number product in the expression of $Z(\mathcal{K})$ by a non-commutative "$\star$-product" (we will briefly review the definition below). Interestingly, this non-commutative deformation establishes a relation between the new spin- foam model $Z(\mathcal{K})$ we analyzed in the previous section and the Barrett-Crane spin-foam model [10]. In some sense it relates the recent approach of using noncommutative product in the simplicial path integral representation of the Group Field Theory (GFT) [30].

First of all, we recall the partition function $Z(\mathcal{K})$ in the commutative context (after the linearization of simplicity constraint):

$$\begin{aligned} Z(\mathcal{K}) \;\; &:= \;\; \int_{II\pm} \prod_{f} \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \prod_{(\sigma,t)} \mathrm{d}g_{\sigma t}^+ \mathrm{d}g_{\sigma t}^- \prod_{(t,f)} \mathrm{d}g_{tf}^+ \mathrm{d}g_{tf}^- \prod_t \mathrm{d}u_t \prod_{t,f} \delta\left(X_{tf}^- + \beta u_t X_{tf}^+ u_t^{-1}\right) \prod_t \delta\Big(\sum_{f\subset t} X_{tf}^+\Big) \\ &\times \prod_{(\sigma,f)} e^{i\mathrm{tr}\left(X_f^+ g_{ft}^+ g_{t\sigma}^+ g_{\sigma t'}^+ g_{t'f}^+\right)} \prod_{(\sigma,f)} e^{i\mathrm{tr}\left(X_f^- g_{ft}^- g_{t\sigma}^- g_{\sigma t'}^- g_{t'f}^-\right)} \end{aligned} \tag{A.1}$$

where $\beta = \frac{1-1/\gamma}{1+1/\gamma}$ and for the convenience of the following analysis, we have made a change of variables

$$X_f^\pm \mapsto \left(1 \pm \frac{1}{\gamma}\right)^{-1} X_f^\pm. \tag{A.2}$$

and dropped a constant γ dependent factor. Here we assume that our structure group is SO(3)$\times$SO(3) instead of SO(4)$\simeq$SU(2)$\times$SU(2)/$\mathbb{Z}_2$. The reason for this replacement is to be compatible with the group fourier transformation, which will be seen shortly.

We now replace (by hand) the commutative c-number product in Eq.(2.1) by the non- commutative $\star$-product on $\mathfrak{su}(2) \simeq \mathbb{R}^3$ defined in [29], that is

$$e^{\frac{i}{2a}\mathrm{tr}(X|g_1|)} \star e^{\frac{i}{2a}\mathrm{tr}(X|g_2|)} := e^{\frac{i}{2a}\mathrm{tr}(X|g_1 g_2|)} \tag{A.3}$$

where a is the deformation parameter, $X = X^j \tau_j$ and $\tau_j = -i\sigma_j$ with σ_j the Pauli matrices $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, $g \in \mathrm{SU}(2)$ represented by a 2×2 matrix and $|g| = \mathrm{sgn}(\mathrm{tr}g)g$ so that $|-g| = |g|$. We can write $g \in \mathrm{SU}(2)$ as

$$g = P_0 + ia\vec{P}\cdot\vec{\sigma}, \qquad P_0^2 + a^2||\vec{P}||^2 = 1 \tag{A.4}$$

Thus $|g|$ is the projection of g on the upper "hemisphere" of SU(2) with $P_0 \geq 0$. Therefore the "plane wave" in Eq.(A.3) can be written

$$e_g(X) := e^{\frac{i}{2a}\mathrm{tr}(X|g|)} = e^{i\vec{P}\cdot\vec{X}\mathrm{sgn}(\mathrm{tr}g)} \tag{A.5}$$

depends on SO(3) only (its character expansio depnds on integral representations only because it is an even function under reflection $g \to -g$). With these "plane waves" we can define an invertible "Group Fourier Transformation" from the functions $f(g)$ on SO(3) ($f(g) = f(-g)$ for $g \in$ SU (2)) to the functions $\tilde{f}(X)$ on the Lie algebra $\mathfrak{su}(2)$

$$\begin{aligned} \tilde{f}(X) \;\; &= \;\; \int \mathrm{d}g \; f(g)\; e_g(X) \\ f(g) \;\; &= \;\; \frac{1}{8\pi a^3}\int \mathrm{d}^3X\; \tilde{f}(X)\star e_{g^{-1}}(X) \;\; = \;\; \frac{\sqrt{1-a^2|\vec{P}(g)|^2}}{8\pi a^3}\int \mathrm{d}^3X\; \tilde{f}(X)\; e_{g^{-1}}(X) \end{aligned} \tag{A.6}$$

Given two functions $\tilde{f}_1(X)$ and $\tilde{f}_2(X)$ in the image of the group Fourier transformation, their $\star$-product is defined as

$$\tilde{f}_1(X)\star\tilde{f}_2(X) = \int \mathrm{d}g_1 \mathrm{d}g_2\; f_1(g_1)\; f_2(g_2)\; e_{g_1}(X)\star e_{g_2}(X) \tag{A.7}$$

and when the deformation parameter turns to $a \to 0$, the $\star$-product reproduces the normal commutative product (if we keep $P_0, \vec{P}$ fixed, see (A.4)).

We also have two identities for delta functions

$$\delta_{\mathrm{SO}(3)}(g) = \frac{1}{8\pi a^3}\int \mathrm{d}^3X\ e_g(X)$$
$$\delta_X(X') = \int \mathrm{d}g\ e_{g^{-1}}(X)\ e_g(X') \tag{A.8}$$

where the second delta function is the Dirac distribution in the noncommutative sense, that is

$$\int \mathrm{d}^3X'\ \left(\delta_X \star f\right)(X') = \int \mathrm{d}^3X'\ \left(f\star\delta_X\right)(X') = f(X) \tag{A.9}$$

With the above definitions, we can make a noncommutative deformation of the integrand in Eq.(A.1). In the following we fix the deformation parameter to

$$a = \ell_p^2 = 1 \tag{A.10}$$

The reason for this choice is that only in this case the closure constraint turns out to be redundant and can be removed from Eq.(A.1), which is necessary in order to derive the Barrett-Crane model. We will show this immediately in the next paragraph. On the other hand, fixing $a = \ell_p^2$ makes it impossible to study the commutative limit $a \to 0$ of the non commutative model $Z_a(\mathcal{K})$ which we denote by and thus we cannot compare with the commutative model $Z(\mathcal{K})$.

We first define the noncommutative deformation of

$$\prod_t \delta\Big(\sum_{f\subset t} X^+_{tf}\Big)\prod_{(\sigma,f)} e^{\mathrm{itr}\left(X^+_f g^+_{ft} g^+_{t\sigma} g^+_{\sigma t'} g^+_{t'f}\right)} = \prod_t \delta\Big(\sum_{f\subset t} g^+_{tf} X^+_f g^+_{ft}\Big)\prod_{(\sigma,f)} e^{\mathrm{itr}\left(X^+_f g^+_{ft} g^+_{t\sigma} g^+_{\sigma t'} g^+_{t'f}\right)} \tag{A.11}$$

Given a face dual to the triangle f with n vertices dual to the 4-simplices $\sigma_1, \cdots, \sigma_n$ (cf. FIG.1), we define the quantity

$$\mathcal{G}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, h_t) := \Big[e_{g^+_{ft_n} h_{t_n} g^+_{t_n f}} \star e_{g^+_{ft_n} g^+_{t_n\sigma_1} g^+_{\sigma_1 t_1} g^+_{t_1 f}} \star e_{g^+_{ft_1} h_{t_1} g^+_{t_1 f}} \star e_{g^+_{ft_1} g^+_{t_1\sigma_2} g^+_{\sigma_2 t_2} g^+_{t_2 f}} \star$$
$$\star \cdots \star e_{g^+_{ft_{n-1}} h_{t_{n-1}} g^+_{t_{n-1} f}} \star e_{g^+_{ft_{n-1}} g^+_{t_{n-1}\sigma_n} g^+_{\sigma_n t_n} g^+_{t_n f}}\Big](X^+_f) \tag{A.12}$$

A possible noncommutative deformation of Eq.(A.11) is

$$\int \prod_t \mathrm{d}h_t \prod_f \mathcal{G}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, h_t) \tag{A.13}$$

because the noncommutative Dirac distribution for the closure constraint is

$$\delta\Big(\sum_{f\subset t} g^+_{tf} X^+_f g^+_{ft}\Big) = \int \mathrm{d}h_t \prod_{f\subset t} e_{g^+_{ft} h_t g^+_{tf}}(X^+_f). \tag{A.14}$$

It is here where the choice $a = \ell_P^2$ was important because we have implicitly set $\ell_P^2 = 1$ in the exponential so far (it comes from the fact that the flux field has dimension cm^2 and the Plebanski action is multiplied by $1/\kappa$ where $\kappa\hbar = \ell_P^2$) so restoring it we can combine the ordinary product of exponentials into star products only if the deformation parameter is given by $a = \ell_P^2$ [5].

However, since

$$\mathcal{G}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, h_t) = \Big[e_{g^+_{ft_n} h_{t_n} g^+_{t_n\sigma_1} g^+_{\sigma_1 t_1} g^+_{t_1 f}} \star e_{g^+_{ft_1} h_{t_1} g^+_{t_1\sigma_2} g^+_{\sigma_2 t_2} g^+_{t_2 f}} \star \cdots \star e_{g^+_{ft_{n-1}} h_{t_{n-1}} g^+_{t_{n-1}\sigma_n} g^+_{\sigma_n t_n} g^+_{t_n f}}\Big](X^+_f) \tag{A.15}$$

we can absorb h_t into $g_{t\sigma}$ by a change of variables

$$g^+_{t_i\sigma_{i+1}} \mapsto h^{-1}_{t_i} g^+_{t_i\sigma_{i+1}} \tag{A.16}$$

[5] the point here is that one should make the exponential of action to look like a "plane-wave" Eq.(A.5) of group Fourier transformation. However it is ℓ_P^{-2} in front of the action but not a^{-1} (The plane-wave in Eq.(A.14) is with a^{-1} not ℓ_P^2). So we have to set $a = \ell_p^2$ to resolve the mismatch, in order to remove the closure condition from the ($\star$-deformed) path integral.

while $\mathrm{d}g^+_{\sigma t}$ does not change since it is Haar measure. Therefore finally the integral of h_t gives unity, which shows the redundancy of the closure constraint for this particular non commutative deformation!

Next we consider the simplicity constraint

$$\delta(X^-_{tf} + \beta u_t X^+_{tf} u_t^{-1}) = \delta(g^-_{tf} X^-_f g^-_{ft} + \beta u_t g^+_{tf} X^+_f g^+_{ft} u_t^{-1}) \tag{A.17}$$

whose noncommutative version is

$$\begin{aligned}\delta(g^-_{tf} X^-_f g^-_{ft} + \beta u_t g^+_{tf} X^+_f g^+_{ft} u_t^{-1}) &= \int \mathrm{d}v_{tf}\ e_{v_{tf}}(g^-_{tf} X^-_f g^-_{ft} + \beta u_t g^+_{tf} X^+_f g^+_{ft} u_t^{-1}) \\ &= \int \mathrm{d}v_{tf}\ e_{v_{tf}}(g^-_{tf} X^-_f g^-_{ft})\ e_{v_{tf}}(\beta u_t g^+_{tf} X^+_f g^+_{ft} u_t^{-1}) \\ &= \int \mathrm{d}v_{tf}\ e_{g^-_{ft} v_{tf} g^-_{tf}}(X^-_f)\ e_{g^+_{ft} u_t^{-1} v_{tf} u_t g^+_{tf}}(\beta X^+_f)\end{aligned} \tag{A.18}$$

For the above factor related to β in the above integrand, we can write

$$e^\beta_g(X) := e_g(\beta X) \tag{A.19}$$

Thus for each face dual to the triangle f with n vertices dual to the 4-simplices $\sigma_1, \cdots, \sigma_n$, we define

$$\begin{aligned}\mathcal{F}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, h_t, u_t, v_{tf}, \beta) := &\Big[e_{g^+_{ft_n} h_{t_n} g^+_{t_n f}} \star e_{g^+_{ft_n} g^+_{t_n\sigma_1} g^+_{\sigma_1 t_1} g^+_{t_1 f}} \star e^\beta_{g^+_{ft_1} u_{t_1}^{-1} v_{t_1 f} u_{t_1} g^+_{t_1 f}} \star \\ &\star e_{g^+_{ft_1} h_{t_1} g^+_{t_1 f}} \star e_{g^+_{ft_1} g^+_{t_1\sigma_2} g^+_{\sigma_2 t_2} g^+_{t_2 f}} \star e^\beta_{g^+_{ft_2} u_{t_2}^{-1} v_{t_2 f} u_{t_2} g^+_{t_2 f}} \star \\ &\star \cdots \star \\ &\star e_{g^+_{ft_{n-1}} h_{t_{n-1}} g^+_{t_{n-1} f}} \star e_{g^+_{ft_{n-1}} g^+_{t_{n-1}\sigma_n} g^+_{\sigma_n t_n} g^+_{t_n f}} \star e^\beta_{g^+_{ft_n} u_{t_n}^{-1} v_{t_n f} u_{t_n} g^+_{t_n f}}\Big](X^+_f)\end{aligned} \tag{A.20}$$

and

$$\begin{aligned}\mathcal{F}^-_f(X^-_f, g^-_{\sigma t}, g^-_{tf}, v_{tf}) := &\Big[e_{g^-_{ft_n} g^-_{t_n\sigma_1} g^-_{\sigma_1 t_1} g^-_{t_1 f}} \star e_{g^-_{ft_1} v_{t_1 f} g^-_{t_1 f}} \star e_{g^-_{ft_1} g^-_{t_1\sigma_2} g^-_{\sigma_2 t_2} g^-_{t_2 f}} \star e_{g^-_{ft_2} v_{t_2 f} g^-_{t_2 f}} \star \\ &\star \cdots \star e_{g^-_{ft_{n-1}} g^-_{t_{n-1}\sigma_n} g^-_{\sigma_n t_n} g^-_{t_n f}} \star e_{g^-_{ft_n} v_{t_n f} g^-_{t_n f}}\Big](X^-_f)\end{aligned} \tag{A.21}$$

Then the deformed partition function is defined by

$$\begin{aligned}Z_\star(\mathcal{K}) := &\int \prod_f \mathrm{d}^3X^+_f \mathrm{d}^3X^-_f \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}g^+_{tf}\mathrm{d}g^-_{tf} \prod_t \mathrm{d}h_t \mathrm{d}u_t \prod_{(t,f)} \mathrm{d}v_{tf} \\ &\prod_f \mathcal{F}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, h_t, u_t, v_{tf}, \beta)\ \mathcal{F}^-_f(X^-_f, g^-_{\sigma t}, g^-_{tf}, v_{tf})\end{aligned} \tag{A.22}$$

which is the noncommutative deformation of Eq.(A.1). However since we have shown the redundancy of the closure constraint in $Z_\star(\mathcal{K})$, we can equivalently write

$$\begin{aligned}Z_\star(\mathcal{K}) := &\int \prod_f \mathrm{d}^3X^+_f \mathrm{d}^3X^-_f \prod_{(\sigma,t)} \mathrm{d}g^+_{\sigma t}\mathrm{d}g^-_{\sigma t} \prod_{(t,f)} \mathrm{d}g^+_{tf}\mathrm{d}g^-_{tf} \prod_t \mathrm{d}u_t \prod_{(t,f)} \mathrm{d}v_{tf} \\ &\prod_f \mathcal{F}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, u_t, v_{tf}, \beta)\ \mathcal{F}^-_f(X^-_f, g^-_{\sigma t}, g^-_{tf}, v_{tf})\end{aligned} \tag{A.23}$$

where $\mathcal{F}^+_f$ is replaced by

$$\begin{aligned}\mathcal{F}^+_f(X^+_f, g^+_{\sigma t}, g^+_{tf}, u_t, v_{tf}, \beta) := &\Big[e_{g^+_{ft_n} g^+_{t_n\sigma_1} g^+_{\sigma_1 t_1} g^+_{t_1 f}} \star e^\beta_{g^+_{ft_1} u_{t_1}^{-1} v_{t_1 f} u_{t_1} g^+_{t_1 f}} \star \\ &\star e_{g^+_{ft_1} g^+_{t_1\sigma_2} g^+_{\sigma_2 t_2} g^+_{t_2 f}} \star e^\beta_{g^+_{ft_2} u_{t_2}^{-1} v_{t_2 f} u_{t_2} g^+_{t_2 f}} \star \\ &\star \cdots \star \\ &\star e_{g^+_{ft_{n-1}} g^+_{t_{n-1}\sigma_n} g^+_{\sigma_n t_n} g^+_{t_n f}} \star e^\beta_{g^+_{ft_n} u_{t_n}^{-1} v_{t_n f} u_{t_n} g^+_{t_n f}}\Big](X^+_f).\end{aligned} \tag{A.24}$$

A.2 $\gamma = \infty$ and Barrett-Crane Model

The computation with general β is difficult, because it involves the $\star$-product between two different types of plane waves e_g and e_g^β, which is even not well-defined in general (since they could consider having different deformation parameter). Therefore here we only consider the simplified case that $\gamma = \infty$. Then $\beta = 1$ and in this case we can directly compute $\mathcal{F}_f^+$ to be

$$\begin{aligned} & \mathcal{F}_f^+(X_f^+, g_{\sigma t}^+, g_{tf}^+, u_t, v_{tf}, \beta = 1) \\ = \; & e_{g_{ft_n}^+ g_{t_n\sigma_1}^+ g_{\sigma_1 t_1}^+ u_{t_1}^{-1} v_{t_1 f} u_{t_1} g_{t_1\sigma_2}^+ g_{\sigma_2 t_2}^+ u_{t_2}^{-1} v_{t_2 f} u_{t_2} \cdots g_{t_{n-1}\sigma_n}^+ g_{\sigma_n t_n}^+ u_{t_n}^{-1} v_{t_n f} u_{t_n} g_{t_n f}^+}(X_f^+) \\ = \; & e_{g_{t_n\sigma_1}^+ g_{\sigma_1 t_1}^+ u_{t_1}^{-1} v_{t_1 f} u_{t_1} g_{t_1\sigma_2}^+ g_{\sigma_2 t_2}^+ u_{t_2}^{-1} v_{t_2 f} u_{t_2} \cdots g_{t_{n-1}\sigma_n}^+ g_{\sigma_n t_n}^+ u_{t_n}^{-1} v_{t_n f} u_{t_n}}(g_{t_n f}^+ X_f^+ g_{ft_n}^+) \end{aligned} \tag{A.25}$$

Similarly for the anti-self-dual part

$$\begin{aligned} & \mathcal{F}_f^-(X_f^-, g_{\sigma t}^-, g_{tf}^-, v_{tf}) \\ = \; & e_{g_{ft_n}^- g_{t_n\sigma_1}^- g_{\sigma_1 t_1}^- v_{t_1 f} g_{t_1\sigma_2}^- g_{\sigma_2 t_2}^- v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^- g_{\sigma_n t_n}^- v_{t_n f} g_{t_n f}^-}(X_f^-) \\ = \; & e_{g_{t_n\sigma_1}^- g_{\sigma_1 t_1}^- v_{t_1 f} g_{t_1\sigma_2}^- g_{\sigma_2 t_2}^- v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^- g_{\sigma_n t_n}^- v_{t_n f}}(g_{t_n f}^- X_f^- g_{ft_n}^-) \end{aligned} \tag{A.26}$$

We define the following changes of the variables

$$g_{\sigma_i t_i}^+ \mapsto g_{\sigma_i t_i}^+ u_{t_i} \quad g_{t_i\sigma_{i+1}}^+ \mapsto u_{t_i}^{-1} g_{t_i\sigma_{i+1}}^+ \quad X_f^+ \mapsto g_{ft_n}^+ u_{t_n}^{-1} X_f^+ u_{t_n} g_{t_n f}^+ \quad X_f^- \mapsto g_{ft_n}^- X_f^- g_{t_n f}^-, \tag{A.27}$$

where for each face dual to f, a unique $t_n(f)$ is chosen as the base point of the dual face. Thus the partition function can be written as

$$\begin{aligned} Z_\star(\mathcal{K}) \; := \; & \int \prod_f \mathrm{d}^3 X_f^+ \mathrm{d}^3 X_f^- \prod_{(\sigma,t)} \mathrm{d}g_{\sigma t}^+ \mathrm{d}g_{\sigma t}^- \prod_{(t,f)} \mathrm{d}v_{tf} \\ & \prod_f e_{g_{t_n\sigma_1}^+ g_{\sigma_1 t_1}^+ v_{t_1 f} g_{t_1\sigma_2}^+ g_{\sigma_2 t_2}^+ v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^+ g_{\sigma_n t_n}^+ v_{t_n f}}(X_f^+) \\ & \prod_f e_{g_{t_n\sigma_1}^- g_{\sigma_1 t_1}^- v_{t_1 f} g_{t_1\sigma_2}^- g_{\sigma_2 t_2}^- v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^- g_{\sigma_n t_n}^- v_{t_n f}}(X_f^-) \end{aligned} \tag{A.28}$$

We perform the integrals over X_f^+, X_f^- and obtain

$$\begin{aligned} Z_\star(\mathcal{K}) \; := \; & \int \prod_{(\sigma,t)} \mathrm{d}g_{\sigma t}^+ \mathrm{d}g_{\sigma t}^- \prod_{(t,f)} \mathrm{d}v_{tf} \\ & \prod_f \delta\left(g_{t_n\sigma_1}^+ g_{\sigma_1 t_1}^+ v_{t_1 f} g_{t_1\sigma_2}^+ g_{\sigma_2 t_2}^+ v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^+ g_{\sigma_n t_n}^+ v_{t_n f}\right) \\ & \prod_f \delta\left(g_{t_n\sigma_1}^- g_{\sigma_1 t_1}^- v_{t_1 f} g_{t_1\sigma_2}^- g_{\sigma_2 t_2}^- v_{t_2 f} \cdots g_{t_{n-1}\sigma_n}^- g_{\sigma_n t_n}^- v_{t_n f}\right) \end{aligned} \tag{A.29}$$

which gives Barrett-Crane vertex amplitude [10]. This result is consistent with the work done by colleagues [30, 26]. On a given triangulation, the GFT model constructed by Baratin and Oriti in [30] reproduce the $\star$-deformed simplicial path integral considered in this appendix, which is the $\star$-deformation of the c-number simplicial path integral Eq.(2.1). Thus the spinfoam model constructed in the main part of the paper may be viewed as the commutative limit of the model in [30] as the triangulation is fixed.

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