

The Universal Phase Space of AdS₃ Gravity

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Abstract: We describe what can be called the “universal” phase space of AdS₃ gravity, in which the moduli spaces of globally hyperbolic AdS spacetimes with compact spatial sections, as well as the moduli spaces of multi-black-hole spacetimes are realized as submanifolds. The universal phase space is parametrized by two copies of the universal Teichmüller space $\mathcal{T}(1)$ and is obtained from the correspondence between maximal surfaces in AdS₃ and quasisymmetric homeomorphisms of the unit circle. We also relate our parametrization to the Chern-Simons formulation of 2+1 gravity and, infinitesimally, to the holographic (Fefferman-Graham) description. In particular, we obtain a relation between the generators of quasiconformal deformations in each $\mathcal{T}(1)$ sector and the chiral Brown-Henneaux vector fields. We also relate the charges arising in the holographic description (such as the mass and angular momentum of an AdS₃ spacetime) to the periods of the quadratic differentials arising via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$. Our construction also yields a symplectic map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ generalizing the well-known Mess map in the compact spatial surface setting.

1. Introduction

Since the discoveries by Brown and Henneaux [1] that the group of symmetries of an asymptotically AdS₃ spacetime is a centrally extended conformal group in two dimensions, and then by Banados, Teitelboim and Zanelli [2] that black holes can exist in such spacetimes, the subject of negative cosmological constant gravity in 2+1 dimensions continues to fascinate researchers. The result [1] is now considered to be a precursor of the AdS/CFT correspondence of string theory [3], and the value of the central charge determined in [1] is an essential ingredient of the conformal field theoretic explanation [4] of the microscopic origin of the black hole entropy.

The AdS₃/CFT₂ story is reasonably well-understood in the string theory setting of 3-dimensional gravity coupled to a large number of fields of string (and extra dimensional) origin. At the same time, the question of whether there is a CFT dual to pure

AdS₃ gravity remains open, see [5] and [6] for the most recent (unsuccessful) attempts in this direction. In particular, the attempt [6] to construct the genus one would-be CFT partition function by summing over the modular images of the partition function of pure AdS leads to discouraging conclusions. It thus appears that pure AdS₃ gravity either does not have enough “states” to account for the BH entropy microscopically, or that the known such states cannot be consistently put together into some CFT structure.

The current lack of understanding of pure AdS₃ gravity quantum mechanically is particularly surprising given the fact that, in a sense, the theory is trivial since pure gravity in 2+1 dimensions does not have any propagating degrees of freedom. In the setting of compact spatial sections the phase space of 2+1 gravity (i.e. the space of constant curvature metrics in a $\mathbb{R} \times \Sigma$, with Σ a genus $g > 1$ Riemann surface) is easy to describe (for all values of the cosmological constant). The constant mean curvature foliation of such a spacetime is particularly useful for this purpose. One finds, see [7] and also [8] for a more recent description, that the phase space is the cotangent bundle over the Teichmüller space of the spatial slice (for any value of the cosmological constant). The zero cosmological constant result [7] also follows quite straightforwardly from the Chern-Simons (CS) description given in [9]. In the setting of AdS₃ manifolds with compact spatial slices, there is yet another description of the same phase space, first discovered by Mess [10]. This is given by two copies of the Teichmüller space of the spatial slice, or, equivalently, by two hyperbolic metrics on the spatial slice Riemann surface. The Mess description is related to the Chern-Simons description of AdS₃ gravity in terms of two copies of $SL(2, \mathbb{R})$ CS theory.

It appears sensible to tackle the problem of quantum gravity as a problem of quantization of the arising classical phase space. One could argue that this approach is unlikely to succeed in 3+1 and higher dimensions, where the phase spaces that arise this way are infinite dimensional (because of the existence of local excitations — gravitational waves). However, in the setting of 2+1 gravity, at least in the setting of spacetimes with compact spatial slices one deals with a finite-dimensional dynamical system and the problem of quantum gravity seems to reduce to a problem from quantum mechanics. In spite of this being a tractable problem, the immediate worry with this approach is that the Hilbert space of quantum states one can obtain by quantizing such a finite-dimensional phase space would not be sufficiently large to account for the black hole entropy.

At the same time, in the context of black holes one should consider non-compact spatial slices. The classical phase space that should arise in this context is somewhat less understood. On one hand, we now know that there is not just the simple BTZ BH [2], but also a much more involved zoo of multi-black hole (MBH) spacetimes first described in [11]. A rather general description of such MBH’s using causal diamonds at their conformal infinity is given in [12]. As a by-product of a construction in [13] using earthquakes, another description of MBH geometries is also available. There are also descriptions of MBH spacetimes in the physics literature, see [14] and [15]. These descriptions show that, like in the compact spatial slice setting with its Mess parametrization, the geometry of multi-black holes continues to be parametrized by two hyperbolic metrics on their spatial slice (or, equivalently, by the cotangent bundle of the corresponding Teichmüller space). The main difference with the compact setting is that the spatial slices are now Riemann surfaces with a geodesic boundary (or with hyperbolic ends attached), and there are now additional moduli, namely the sizes of the boundary components. These new length parameters, two for each boundary component (because there are two hyperbolic metrics involved in the parametrization) determine the geometrical characteristics of the corresponding black hole horizon, such as its length and angular velocity. An explicit formula of this sort can be found in e.g. [13], see formula (1) of the first (arxiv)

version of this paper. All in all, there is a reasonable understanding of the geometry of the multi-black-hole spacetimes, as well as an efficient parametrization of these spacetimes by two copies of the Teichmüller space of Riemann surfaces with boundaries. It thus might seem that the phase space of non-compact spatial slices AdS₃ gravity is as finite dimensional as in the compact setting.

It is however clear that a geometrical description of the multi-black hole spacetime is just half of the story. Indeed, one of the most exciting aspects of 2+1 gravity in an asymptotically AdS setting is the fact [1] that the diffeomorphisms that are asymptotically non-trivial should no longer be interpreted as gauge. Indeed, they map one asymptotically AdS spacetime into a non-equivalent one. Thus, asymptotic symmetries applied e.g. to the AdS₃ create an infinitely large class of asymptotically AdS₃ spacetimes described by Brown and Henneaux [1]. The phase space consisting of all such deformations of a reference spacetime is then infinite dimensional and the problem of its quantization therefore becomes much more non-trivial than in the compact spatial slice setting. It could be that the CFT dual of pure 2+1 gravity can be discovered by quantizing this phase space. And indeed, the states obtained from the AdS “vacuum” by an action of the Virasoro generators is what was summed over in [6] in the authors’ attempt to build the genus one pure gravity partition function.

We can now formulate the main objective of this paper. Our main aim is to give a description of the phase space of AdS₃ gravity that is equally applicable to both compact and non-compact spatial section spacetimes. At the same time, we would like our description to include the Brown-Henneaux asymptotic deformations. As we shall see, there is a “universal” way of doing so, where one constructs what can be called the universal phase space, in which all the moduli spaces of fixed spatial topology are realized as submanifolds. We achieve this in the same way as in the context of the universal Teichmüller space, where the fixed topology Teichmüller spaces are realized as (complex) submanifolds of the universal Teichmüller space. The construction of this paper can then be seen as a generalization of the description of [10] to the setting of the universal Teichmüller space.

We want to emphasize that we do not consider here the quantum theory that would arise by quantizing the classical phase space of 2+1 gravity. This is left to future studies. Rather, our main aim here is to describe the phase space in as explicit terms as possible, thus setting the stage for its quantization. We shall see that the universal phase space is extremely non-trivial, and is parametrized by two copies $\mathcal{T}(1) \times \mathcal{T}(1)$ of the universal Teichmüller space $\mathcal{T}(1)$, or, equivalently, the cotangent bundle $T^*\mathcal{T}(1)$ over $\mathcal{T}(1)$. This generalizes Mess’s description [10] of the compact spatial slice setting, where there is similarly two equivalent parametrizations of the moduli space of AdS₃ spacetimes.

Let us briefly indicate how the universal Teichmüller space comes about. In one possible definition of the latter, this is the space of quasisymmetric homeomorphisms of the unit circle (such homeomorphisms are boundary values of quasiconformal maps from the unit disc to itself). Thus, in very general terms, the universal phase space of AdS₃ gravity is parametrized by two functions on the circle. To obtain the moduli space of spacetimes of fixed spatial topology, e.g. that of fixed topological type multi-black holes, one imposes the condition that the functions in question be invariant under a suitable discrete subgroup of the group of Möbius transformations. In the case of multi-black hole spacetimes this produces a moduli space that is still infinite-dimensional and that includes the Brown-Henneaux “excitations” in all asymptotic regions. The cardinality is that of a pair of functions for each asymptotic region. While the freedom of prescribing two functions on the circle could be anticipated already from the Fefferman-Graham type description, see below, one novelty of our construction is that the phase space includes

all possible multi-black-hole spacetimes. Another novelty of the constructions of this paper is a precise characterization of which functions on the circle are relevant in the context of AdS₃ gravity. Indeed, our description in terms of two points in the universal Teichmüller space shows that these are associated to quasimetric maps of the unit circle which is larger than the class of smooth maps.

As we have already mentioned, the fact that in the non-compact spatial slice setting the phase space becomes an infinite dimensional space of certain (pairs of) functions on \mathbb{S}^1 can be expected already from the AdS/CFT perspective. Indeed, we know that a possible description of an asymptotically AdS spacetime is in terms of an expansion of the spacetime metric in a neighbourhood of the conformal boundary, see e.g. [16, 17] for such expansions in the AdS₃ context. For any asymptotically AdS₃ spacetime one can find the so called Fefferman-Graham coordinates in a neighbourhood of (a component of) the conformal boundary where the bulk metric takes the form

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{\rho^2}(g_{(0)} + \rho^2 g_{(2)} + \rho^4 g_{(4)}).$$

Here $g_{(0)}$ is a representative of the conformal class on the conformal boundary and

$$g_{(2)} = \frac{1}{2}(R_{(0)}g_{(0)} + T), \quad g_{(4)} = \frac{1}{4}g_{(2)}g_{(0)}^{-1}g_{(2)}$$

with $R_{(0)}$ the Ricci scalar of $g_{(0)}$ and T the quasilocal stress-tensor [18, 19]. Note that for fixed $g_{(0)}$ the only freedom in specifying the space time metric are the components of T .

For a flat boundary metric (which is always achievable by choosing ρ appropriately), the most general quasilocal stress tensor can be written

$$T = adt^2 + 2btdt\theta + ad\theta^2,$$

with a and b given by sum and difference of two arbitrary chiral functions

$$a(t, \theta) = a_+(t + \theta) + a_-(t - \theta), \quad b(t, \theta) = a_+(t + \theta) - a_-(t - \theta),$$

and the spacetime metric becomes

$$ds^2 = \frac{d\rho^2}{\rho^2} + \frac{1}{4\rho^2}(-dt^2 + d\theta^2) + \frac{1}{2}(adt^2 + 2btdt\theta + ad\theta^2) + \frac{\rho^2}{4}(a^2 - b^2)(-dt^2 + d\theta^2). \quad (1)$$

Note that the possibility of writing down the Fefferman-Graham type expansion in a closed form (with a finite number of terms) is peculiar to 2+1 dimensions, and is due to the absence of any local degrees of freedom in this theory. It thus becomes clear that asymptotically AdS₃ spacetimes are parametrized by certain pairs of functions of $t \pm \theta$. More explicitly, the work of Brown and Henneaux [1] shows that the group of asymptotic symmetries of asymptotically AdS spacetimes is given by two copies of $\text{Diff}_+(\mathbb{S}^1)$. The phase space of asymptotically AdS₃ spacetimes could then be described as the quotient space of this group modulo the AdS₃ isometry group, giving overall two copies of $\text{Diff}_+(\mathbb{S}^1)/\text{SL}(2, \mathbb{R})$, see e.g. Sect. 2.2 of [6].

The above description is, however, not entirely satisfactory. Indeed, the Fefferman-Graham coordinate ρ extends only over a portion of the spacetime near its conformal

boundary. Thus, only very little control over what happens inside the spacetime is available. In particular, it is not possible to know whether a spacetime (1) contains any non-trivial topology. It is also very hard to characterize those choices of a_{\pm} that lead to non-singular spacetimes. For all these reasons the description (1), while indicating that there is some infinite-dimensionality to be expected, is not a satisfactory description of the phase space of asymptotically AdS₃ gravity.

As we will show in this paper, a description of the phase space that overcomes these difficulties is possible by using embedded maximal surfaces. Indeed, one particularly powerful description of the compact spatial slice situation is based on maximal surfaces, see [8]. It can be shown that each AdS₃ spacetime with a compact spatial slice (such spacetimes were referred to as globally hyperbolic maximal compact (GHMC) AdS in [8]) contains a unique maximal surface. The first and second fundamental forms induced on such a surface then become the configurational and momentum variables. It can be shown that the free data are those of a conformal structure and a certain quadratic differential on the maximal surface, and these together parametrize a point in the cotangent bundle of the Teichmüller space of the Cauchy surface. The data on the maximal surface can in turn be used to produce two hyperbolic metrics via a generalized Gauss map, and thus two points in the Teichmüller space, and this way one obtains an explicit realization of the Mess parametrization [10].

The present work more or less generalizes the above compact case description to the non-compact setting. Thus, similar to the construction described in [8], we shall present two parametrizations of the phase space. One of them works with conformal structures and quadratic differentials on the disc, and thus provides an analogue of the cotangent bundle description. The other works with two conformal structures on the disc, and is the analogue of the two copies of the Teichmüller space description. The relation between both parametrizations is obtained explicitly from the harmonic decomposition of quasi-conformal minimal Lagrangian diffeomorphisms of the disc, and generalizes the Mess map to a symplectomorphism $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$. Our main mathematical result is a description of this highly non-trivial map, and a proof of the fact that it is bijective. We also give a “physicist’s” proof that this map is a symplectomorphism making use of the Chern-Simons description of 2+1 gravity.

We would like to emphasize that our description of the phase space of AdS₃ gravity by two copies of the universal Teichmüller includes and in a sense supercedes the description by two copies of $\text{Diff}_+(\mathbb{S}^1)/\text{SL}(2, \mathbb{R})$ that follows from the Brown-Henneaux work, see e.g. [6], Sect. 2.2, for a particularly clear account of the Brown-Henneaux parametrization of the phase space. Indeed, as we shall see below, in one possible description $\mathcal{T}(1)$ is realized as the space $\text{QS}(\mathbb{S}^1)/\text{SL}(2, \mathbb{R})$ of (normalized) quasisymmetric homeomorphism of \mathbb{S}^1 , and this contains $\text{Diff}_+(\mathbb{S}^1)/\text{SL}(2, \mathbb{R})$ as a submanifold. However, more important than the extension from the space of smooth to that of quasisymmetric maps, our construction also describes how the bulk moduli (non-trivial topology) can be encoded in the phase space.

The constructions in this paper builds on and extends those in [20] and [21]. Thus, there is not much new mathematics in this work. Rather, we take some results obtained by mathematicians (with different aims), and use them to describe the phase space of AdS₃ gravity. In particular, our description is based on the result in [20] that proved the existence and uniqueness of maximal surfaces in AdS₃ with a given boundary curve. This boundary curve is, in turn, parametrized by a single quasisymmetric homeomorphism on the circle, so there is a one-to-one correspondence between quasisymmetric homeomorphisms of the circle (i.e. points in the universal Teichmüller space) and maximal surfaces in AdS₃. We use these, as well as some results from the universal Teichmüller

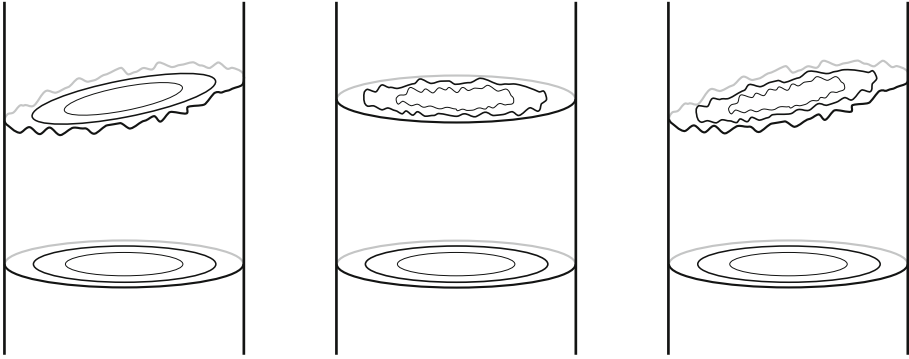


Fig. 1. Two deformation directions. One (left figure) corresponds to deforming the curve along which the maximal surface intersects the boundary. The other (middle figure) corresponds to deforming the complex structure on the maximal surface, or, geometrically, to deforming the constant “radial” coordinate foliations of the surface. A general point in the phase space deforms both the curve at infinity as well as the constant radial coordinate foliation of the maximal surface (right figure)

literature, to describe the phase space, seen as the space of all AdS_3 spacetimes, in terms of deformations of the domain of dependence of a totally geodesic spacelike surface in AdS_3 .

One non-obvious point of our construction, which is also where we depart from the works cited above, is the existence of two independent directions in the phase space. Thus, the work [20] makes it clear that a single point in $\mathcal{T}(1)$ gives rise to a maximal surface in AdS_3 , which then comes equipped with its first and second fundamental forms (induced by the embedding in AdS_3). Thus, it can appear that a single quasisymmetric homeomorphism on the circle (single point in $\mathcal{T}(1)$) is sufficient to specify all of the “initial” data necessary for the maximal surface description. The question is then where does the second phase space direction, i.e. a second point in $\mathcal{T}(1)$, come from. As we shall describe in more details below, this other direction comes from the possibility of an additional quasiconformal deformation on the maximal surface. Being a diffeomorphism it does not change the first and second fundamental forms on the maximal surface, but being asymptotically non-trivial it gives rise to deformations that has to be considered as non-gauge. And we shall verify that the two types of deformations — the geometric ones corresponding to changing the curve along which the maximal surface intersects the boundary, and the non-geometric one corresponding to just performing an asymptotically non-trivial diffeomorphism on the maximal surface — are canonically conjugate to each other in the symplectic structure induced by the gravitational action. Thus, both are equally important as far as AdS_3 gravity is concerned. The two phase space directions — those deforming the curve along which the maximal surface intersects the cylinder at infinity and those deforming the complex coordinate on the maximal surface — are graphically depicted in Fig. 1.

Another, more geometrical description of our phase space was suggested to us by Jean-Marc Schlenker after the first version of this paper was put on the arxiv [22]. Thus, the phase space can be described as the space of quasiconformal maximal embeddings of the unit disc Δ into AdS_3 , considered up to a natural in this context identification: two quasi-conformal maps $u, v : \Delta \rightarrow \text{AdS}_3$ are considered equivalent if their composition $u^{-1} \circ v$ is the identity at infinity.

As in the compact setting, the parametrization by two copies of the (universal) Teichmüller space is related to the Chern-Simons formulation of 2+1 gravity

introduced in [9]. Indeed, we recall that, for AdS spacetimes, the Einstein-Hilbert action can be written as two copies of $SL(2, \mathbb{R})$ Chern-Simons action. Every AdS metrics can, therefore, be described by an associated pair of flat $SL(2, \mathbb{R})$ connections. Then a simple explicit computation shows that, in our parametrization, each copy of $\mathcal{T}(1)$ corresponds to one of these connections. In fact, this relation to the CS formulation is the easiest way to understand why the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ is a symplectomorphism, see below for a further discussion of this point.

The outline of the present paper is as follows. In Sect. 2 we give a brief review of the compact case. Sect. 3 deals with the maximal surfaces in AdS₃. The construction of the phase space of globally hyperbolic AdS spacetimes is described in Sect. 4. We present the generalized Mess map in Sect. 5. A relation to the holographic description is worked out in Sects. 6, 7, 8. We finish with a discussion. For those not familiar with (universal) Teichmüller theory we present a quick overview in the Appendix.

2. Compact Spatial Topology

In this section we consider globally hyperbolic AdS₃ spacetimes $M = \mathbb{R} \times \Sigma$ whose Cauchy surface is a genus $g \geq 2$ Riemann surface Σ . In the Hamiltonian formulation of general relativity one starts by foliating spacetime by spacelike hypersurfaces. The spacetime metric is then described in terms of the first and second fundamental forms (I, II) of the initial Cauchy surface Σ . The first and second fundamental forms have to satisfy certain relations, known as the Gauss-Codazzi equations (or as the Hamiltonian and momentum constraints in the GR community). In [7], the phase space of flat 2+1 gravity was shown to be parametrized by the cotangent bundle over Teichmüller space of the initial surface by choosing a foliation by constant mean curvature surfaces. In fact, in isothermal coordinates on Σ , associated with its conformal structure, we can write

$$I = e^{2\varphi}|dz|^2, \quad II = \frac{1}{2}(qdz^2 + \bar{q}d\bar{z}^2) + He^{2\varphi}|dz|^2,$$

where H is the mean curvature of Σ . Then, Codazzi equation imposes holomorphicity of the quadratic differential qdz^2 defined by the traceless part of II and Gauss equation becomes an equation for the “Liouville” field φ . This equation becomes particularly simple on a maximal surface $H = 0$ and reads:

$$4\partial_z\partial_{\bar{z}}\varphi = e^{2\varphi} - e^{-2\varphi}|q|^2. \quad (2)$$

Note that this is the equation relevant for the AdS₃ setting. Similar equations (with some sign changes) hold also in the positive or zero scalar curvature settings. The important fact is that the existence and uniqueness of the solution of the above Gauss equation holds, implying the existence and uniqueness of a maximal surface in any globally hyperbolic AdS₃ spacetime (here with compact spatial slices). Given this existence and uniqueness result, the pair (I, II) is completely determined by a conformal structure z and a holomorphic quadratic differential qdz^2 , thus a point in $T^*\mathcal{T}(\Sigma)$. This gives an efficient explicit description of the spacetime geometry as parametrized by the data on the maximal surface. Indeed, the 3-metric can be written in the equidistant coordinates to the maximal surface as

$$ds^2 = -d\tau^2 + \cos^2 \tau I + 2 \sin \tau \cos \tau II + \sin^2 \tau II I^{-1} II. \quad (3)$$

A direct computation then shows the gravitational symplectic structure agrees with the canonical cotangent bundle one, see [8] for more details. We thus have a description of the phase space of AdS₃ gravity in the compact spatial slice setting as $T^*\mathcal{T}(\Sigma)$.

In [10] Mess obtained another parametrization of the same spacetimes by two copies of Teichmüller space $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. His construction can be understood as follows. In the projective model, AdS₃ can be seen as the image of the quadric $X = \{x \in \mathbb{R}^{2,2}; \langle x, x \rangle = -1\}$, with its induced metric, under the projection $\pi : X \rightarrow \mathbb{R}P^3$,

$$\text{AdS}_3 = \pi(X) = \{[x] \in \mathbb{R}P^3; \langle x, x \rangle < 0\}.$$

A point on AdS₃ is then in correspondence with a line in $\mathbb{R}^{2,2}$ passing through a point in the quadric X and the origin. The boundary of AdS₃, the projective quadric

$$\partial\text{AdS}_3 = \{[x] \in \mathbb{R}P^3; \langle x, x \rangle = 0\},$$

is known to be foliated by two families of projective lines \mathcal{L}_+ and \mathcal{L}_- (corresponding to the left and right null geodesics). Since each line of one family intersects a line of the other family a single time, this provides an identification of ∂AdS_3 with the torus $\mathbb{S}^1 \times \mathbb{S}^1$.

Now, a line in \mathcal{L}_+ (resp. \mathcal{L}_-) meets the boundary of any spacelike surface at a single point so one can use these “left” and “right” families to define maps between the boundaries of any pair of spacelike surfaces. Let us first require these spacelike surfaces to be totally geodesic. Given a pair of such totally geodesic spacelike surfaces P_0 and P , let $\pi_+, \pi_- : \partial P \rightarrow \partial P_0$ be the “left” and “right” maps of their boundaries. These then uniquely extend to isometries $\Phi_+^P, \Phi_-^P : P \rightarrow P_0$ of AdS₃ sending P to P_0 . Now, taking an arbitrary spacelike surface S , one can associate to S a pair of diffeomorphisms $\Phi_+, \Phi_- : S \rightarrow P_0$ by

$$\Phi_+(x) = \Phi_+^{P(x)}(x), \quad \Phi_-(x) = \Phi_-^{P(x)}(x),$$

where $P(x)$ is the totally geodesic spacelike surface tangent to S at x . Taken modulo an overall isometry in AdS₃, this construction is independent of the choice of P_0 .

Given an GHMC AdS spacetime M , let Σ be some smooth embedded spacelike surface and consider its lift S into the universal cover of M , which is AdS₃. Taking the pull-back of the hyperbolic metric on P_0 by Φ_+ and Φ_- defines two hyperbolic metrics on S , which in turn descend to hyperbolic metrics I_+ and I_- on Σ , thus defining a point in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$. This construction can be applied to any spacelike surface Σ in M and it can be shown that the point in $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ one gets is independent of this choice. When Σ is maximal (or at least of constant mean curvature), the data on Σ also gives rise to a point in $T^*\mathcal{T}(\Sigma)$, as we reviewed above, so this defines what can be referred to as the Mess map,

$$\text{Mess} : T^*\mathcal{T}(\Sigma) \rightarrow \mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma), \tag{4}$$

taking (I, II) on the maximal surface into (I_+, I_-) . It can be shown that this map is a bijection, see [8], so there are two equivalent descriptions of the moduli space of globally hyperbolic AdS₃ spacetimes with spatial sections of fixed topology. One is given by $T^*\mathcal{T}(\Sigma)$, the other by $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$.

By a calculation, also available in [8], the Mess map can be explicitly described as follows

$$I_{\pm} = I(E \pm JI^{-1}II \cdot, E \pm JI^{-1}II \cdot),$$

where E is the identity operator on $T\Sigma$ and J is the complex structure associated with the conformal structure of Σ . These two metrics on Σ are hyperbolic provided I, II satisfy the Gauss and Codazzi equations. We note that each of the maps Φ_{\pm} is harmonic (with their Hopf differentials adding to zero) so the maps Φ_{\pm} can be referred to as the generalized Gauss map, see e.g. [21]. This terminology is legitimate given the resemblance of the construction of the metrics I_{\pm} with the famous Gauss map between the data on a constant mean curvature surface in $\mathbb{R}^{2,1}$ and hyperbolic metrics.

The work [8] also described an explicit inverse of the Mess map, by providing a map between a pair I_{\pm} of hyperbolic metrics on Σ and the first and second fundamental forms of the *maximal* surface in the spacetime M that corresponds to I_{\pm} . This map uses the existence and uniqueness of a minimal Lagrangian diffeomorphism between a surface Σ equipped with the “left” and “right” hyperbolic metrics I_{\pm} . We shall denote this map by $F : (\Sigma, I_+) \rightarrow (\Sigma, I_-)$. It is an area preserving diffeomorphism (hence the term Lagrangian) whose graph is minimal in the product $(\Sigma \times \Sigma, I_+ \times I_-)$ (hence the term minimal). As is reviewed in [8], the existence of a minimal Lagrangian F is equivalent to the existence of an operator $b : T\Sigma \rightarrow T\Sigma$ satisfying

1. $\det b = 1$;
2. b is self-adjoint with respect to I_+ ;
3. $d^{\nabla^+} b = 0$, where ∇^+ is the Levi-Civita connection of I_+ ;
4. $F^*I_- = I_+(b \cdot, b \cdot)$.

In terms of b one can construct a metric and a symmetric bilinear form on Σ

$$I = \frac{1}{4}I_+(E + b \cdot, E + b \cdot), \quad II = -IJ(E + b)^{-1}(E - b) \quad (5)$$

which satisfy the Gauss-Codazzi equations. Thus, the problem of constructing the inverse map reduces to the problem of determining the operator b . Once this is known, the first and second fundamental form I, II obtained by the above formulas are those of the *maximal* surface in the spacetime corresponding to the pair I_{\pm} . An explicit expression for the spacetime metric is then given by (3) providing an efficient parametrization of the space of globally hyperbolic AdS₃ spacetime (with compact spatial slices) by two copies of the Teichmüller space. This description of the inverse of the Mess map admits a direct generalization to the non-compact setting of interest to us, and will play an important role in the next sections.

We also note that the gravitational symplectic structure, evaluated in the parametrization $T^*\mathcal{T}(\Sigma)$, is just the canonical cotangent bundle symplectic structure, see [8] for a simple calculation that demonstrates this. It can also be verified that the map Mess is a symplectomorphism with respect to the natural symplectic structure on $\mathcal{T}(\Sigma) \times \mathcal{T}(\Sigma)$ given by two copies of the Weil-Petersson symplectic structure. The easiest way to see this is to use the Chern-Simons description in which the left and right metrics of the Mess parametrization encode the monodromies of the left and right $SL(2, \mathbb{R})$ connections on Σ . Since the CS formulation provides an equivalent description of AdS₃ gravity, and the symplectic structure of $SL(2, \mathbb{R})$ CS theory reduces to the Weil-Petersson symplectic structure on $\mathcal{T}(\Sigma)$, the Mess map must be symplectic. We shall see this explicitly (at the origin of both spaces) in Sect. 9.

3. Maximal Surfaces in AdS₃ and Universal Teichmüller Space

As a preparation for our consideration of the non-compact setting, we start by reviewing the relation between the universal Teichmüller space and maximal surfaces in AdS₃

described in [20]. We shall also present some known facts relating maximal surfaces in AdS_3 , harmonic maps and minimal Lagrangian maps between the unit disc, see [21].

We start by reviewing some details about the universal Teichmüller space. We follow most conventions of [24], and the reader is advised to consult these references for more details. In general terms, the universal Teichmüller space is the quasiconformal deformation space of the unit disc Δ and can be realized as the space of certain equivalence classes of bounded Beltrami coefficients on Δ by solving the Beltrami equation. More concretely, let

$$L^\infty(\Delta)_1 = \left\{ \mu : \Delta \rightarrow \mathbb{C}; |\mu|_\infty = \sup_\Delta |\mu(w)| < 1 \right\},$$

be the unit ball in the space of bounded Beltrami coefficients in the unit disc Δ . Given $\mu, \nu \in L^\infty(\Delta)_1$ one solves the Beltrami equation

$$\tilde{\mu} = \partial_{\bar{w}} z / \partial_w z$$

in $\hat{\mathbb{C}}$ with coefficients extended by reflection

$$\tilde{\mu}(w) = \begin{cases} \overline{\mu(1/\bar{w})} w^2 / \bar{w}^2, & w \in \hat{\mathbb{C}} \setminus \Delta, \\ \mu(w), & w \in \Delta, \end{cases}$$

similarly for ν . The equivalence relation between μ, ν is then given if the corresponding solutions, normalized to fix $-1, -i$ and 1 , agree on the conformal boundary \mathbb{S}^1 ,

$$z_\mu|_{\mathbb{S}^1} = z_\nu|_{\mathbb{S}^1}. \tag{6}$$

Since the boundary values of quasiconformal diffeomorphisms of the disc are quasymmetric homeomorphisms of the unit circle, we have an identification between the universal Teichmüller space $\mathcal{T}(1)$ and the space $\text{QS}(\mathbb{S}^1)/\text{SL}(2, \mathbb{R})$ of Möbius normalized quasisymmetric homeomorphisms of \mathbb{S}^1 . For purposes of this section it will be sufficient to think about $\mathcal{T}(1)$ as such space.

This is the so-called A-model of the universal Teichmüller space. The complex structure in $\mathcal{T}(1)$ is not easily described in this model, but become much more explicit in another description, the so-called B-model, which works with solutions of the Beltrami equation where the Beltrami coefficient is extended to the outside of the unit disc as $\mu = 0$. Some more facts about the two models and their relation are described in Sect. 7.

We note that the universal Teichmüller space has a (formal) symplectic manifold structure generalizing the Weil-Petersson symplectic structure of the closed topology Teichmüller spaces. The symplectic structure is described in some detail in the Appendix, but we refer the reader to a more complete exposition in [24].

Before we explain a relation between $\mathcal{T}(1)$ and the maximal surfaces in AdS_3 , we would like to describe how the fixed topology Teichmüller spaces $\mathcal{T}(\Sigma)$ can be realized as submanifolds in $\mathcal{T}(1)$, thus justifying the nomenclature used in this theory. This is achieved by restricting the Beltrami coefficients introduced above to have fixed periodicity properties with respect to some “base point” Fuchsian group. Thus, let Γ be a fixed Fuchsian group such that the quotient Δ/Γ is a Riemann surface of the required topology. We then define a space of Beltrami coefficients for Γ ,

$$L^\infty(\Delta, \Gamma)_1 = \left\{ \mu \in L^\infty(\Delta)_1 : \mu \circ \gamma \frac{\bar{\gamma}'}{\gamma'} = \mu, \forall \gamma \in \Gamma \right\},$$

and the corresponding Teichmüller space of $\Sigma = \Delta / \Gamma$ is obtained as

$$\mathcal{T}(\Sigma) = \mathcal{T}(\Gamma) = L^\infty(\Delta, \Gamma)_1 / \sim,$$

where the equivalence relation is the same as the one introduced above, see (6). It can be shown that this is just the space of all fixed topology Fuchsian groups, which arise as

$$\Gamma_\mu = z_\mu \circ \Gamma \circ z_\mu^{-1}.$$

Thus, we have given a description of the Teichmüller space $\mathcal{T}(\Sigma)$ as the quasiconformal deformation space of Γ . This can be seen as a ball of radius one in the space of (Γ -periodic) Beltrami coefficients centred at the preferred surface Δ / Γ . Note that the group Γ can be chosen to be rather arbitrary here. One possible choice is that Δ / Γ is a compact surface of given genus. However, a choice where Δ / Γ is an infinite area surface with hyperbolic ends is also possible. This latter choice is the one relevant for the description of multi-black holes.

We would also like to note, without going into much details, that it is possible to generalize the discussion to include the case of non-orientable spatial topology. Fuchsian groups should then be replaced by the so called non-euclidean crystallographic (NEC) groups, discrete groups of isometries of Δ including orientation reversing elements. The invariance property of Beltrami coefficients under these additional elements should then be $\mu \circ \gamma(\overline{\gamma'/\gamma'}) = \bar{\mu}$. Then every Klein surface Σ has an orientable complex double cover Σ^c and the Teichmüller space of Σ embeds as an open submanifold of $\mathcal{T}(\Sigma^c)$. We refer the reader to [25] for an exposition on Kleinian surfaces. Thus, if desired, the universal construction of this paper also includes the ‘‘geon’’ spacetimes studied in e.g. [23].

We now describe a relation between points in $\mathcal{T}(1)$ and maximal surfaces in AdS₃, established in [20]. The key idea here is that, given a quasisymmetric map $\mathbb{S}^1 \rightarrow \mathbb{S}^1$, its graph in $\mathbb{S}^1 \times \mathbb{S}^1$ can be viewed as a spacelike curve on the conformal boundary ∂AdS_3 . Indeed, as we recalled in the previous section, ∂AdS_3 is ruled by two families of left and right null geodesics, and is therefore (a 2-to-1 cover of) $\mathbb{S}^1 \times \mathbb{S}^1$. Now, given a spacelike curve on ∂AdS_3 , it is shown there is a unique maximal surface in AdS₃ intersecting the conformal boundary along this curve. The existence part here is quite general and makes only very weak assumptions about the curve at infinity. It is in the proof of uniqueness that quasisymmetric boundary curves become relevant. Thus, [20] introduces the notion of a width $w(\Gamma)$ of the boundary curve Γ which is the supremum of the (time) distance between the upper and lower boundaries of its convex hull $C(\Gamma)$ in AdS₃. In one hand, this provides a new characterization of quasisymmetric maps in terms of AdS₃ geometry. It is shown that for any boundary curve the width is at most $\pi/2$, being strictly less than $\pi/2$ if and only if it is the graph of a quasisymmetric map $\mathbb{S}^1 \rightarrow \mathbb{S}^1$. On the other hand it gives sufficient conditions for the uniqueness result. First, the condition $w(\Gamma) < \pi/2$ is shown to imply that the corresponding maximal surface has sectional curvature bounded above by a negative constant. Then, based on convexity properties, it is shown that this maximal surface with uniformly negative sectional curvature is unique among complete maximal graphs with the given boundary curve and bounded sectional curvature.

The existence and uniqueness of maximal surfaces in AdS₃ corresponding to points in $\mathcal{T}(1)$, i.e. quasisymmetric maps, can then be seen equivalent to the existence and uniqueness of quasiconformal minimal Lagrangian extensions of quasisymmetric homeomorphisms of \mathbb{S}^1 to the interior of the disc. This last point is essentially the same construction as occurs in the compact setting, see the previous section, where a maximal surface in AdS₃ gives rise to a minimal Lagrangian diffeomorphism between

the Riemann surfaces (Σ, I_+) and (Σ, I_-) . The same construction now extends to the non-compact setting, as was used in [21], and allowed [20] to prove the existence and uniqueness of minimal Lagrangian extensions of quasisymmetric maps.

4. The Generalized Mess Parametrization of the AdS₃ Phase Space

In this section we describe the universal phase space of AdS spacetimes as parametrized by two copies of the universal Teichmüller space. This is essentially a generalization of the inverse of the Mess map, as described towards the end of Sect. 2. The moduli space of fixed spatial topology AdS₃ manifolds is then obtainable from the universal phase space by restricting the Beltrami coefficients to be invariant under appropriate topology discrete subgroups of $SL(2, \mathbb{R})$. In Sect. 6 we shall see how the “non-geometrical” asymptotic degrees of freedom of Brown-Henneaux are encoded in this phase space.

Thus, let us take two points in $\mathcal{T}(1)$, which, we remind the reader, can be thought of as two (normalized to fix $-1, 1, -i$) quasisymmetric homeomorphisms of the circle. Let us denote these homeomorphisms by $f_{\pm} : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. As will become clear below, it will be useful to think about f_{\pm} as the boundary values of two quasiconformal maps z_{\pm} from a “base point” unit disc representing a preferred base point in $\mathcal{T}(1)$. We call the complex coordinate on the base disc w , see Fig. 2, and interpret the maps z_{\pm} as deformations of this disc defining complex coordinates, which we also call z_{\pm} , on the target discs Δ_{\pm} . Each of the discs Δ_{\pm} has its standard hyperbolic metric which we denote

$$I_{\pm} = \frac{4|dz_{\pm}|^2}{(1 - |z_{\pm}|^2)^2}.$$

Note that I_{\pm} , although both hyperbolic, are to be considered as representatives of inequivalent conformal structures of the disc. Although I_{\pm} are in fact isometric, the isometry mapping I_+ to I_- acts nontrivially at infinity changing the equivalence class. The choice of a base point unit disc thus becomes quite helpful in avoiding confusion. In fact, when

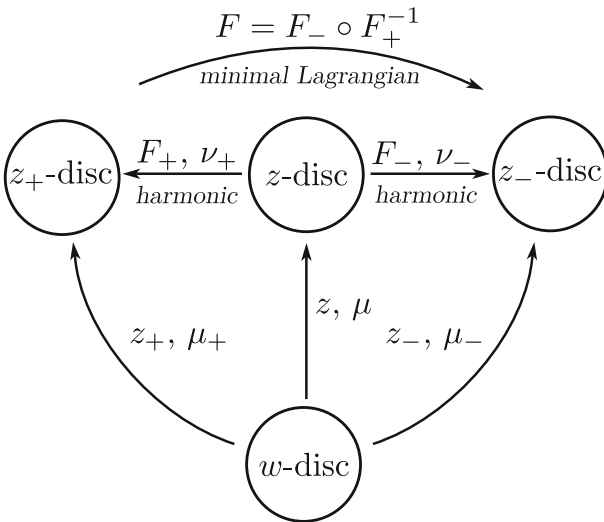


Fig. 2. The diagram of maps

pulled back to the base disc the metrics I_{\pm} explicitly involve the Beltrami coefficients associated with points in $\mathcal{T}(1)$ they represent

$$I_{\pm} = \frac{4|\partial_w z_{\pm}|^2}{(1 - |z_{\pm}(w)|^2)^2} |dw + \mu_{\pm} d\bar{w}|^2,$$

where $\mu_{\pm} = \partial_{\bar{w}} z_{\pm} / \partial_w z_{\pm}$. It is in this sense that we use the hyperbolic metrics I_{\pm} on Δ as representatives of points in $\mathcal{T}(1)$.

A word is in order about which quasiconformal extensions z_{\pm} of quasisymmetric boundary maps f_{\pm} are considered. Indeed, there are many quasiconformal maps z_{\pm} in the same equivalence class in the sense of universal Teichmüller theory, i.e. having the same restrictions f_{\pm} to the circle. We shall see below that for our purposes the composition $z_- \circ z_+^{-1}$ will need to satisfy certain property which makes it unique, given the boundary values. Apart from this restriction, the extensions z_{\pm} are arbitrary, and for our construction of the phase space it will not matter which specific extension is chosen.

Let us now consider $f = f_- \circ f_+^{-1}$ obtained by composing the homeomorphisms f_{\pm} . This is also a quasisymmetric homeomorphism of \mathbb{S}^1 , and according to [20] there is a unique maximal surface (with uniformly negative sectional curvature) in AdS₃ whose boundary curve is the graph of f . Let $F : \Delta_+ \rightarrow \Delta_-$ denote the minimal Lagrangian extension of f obtained in [20]. We now fix the arbitrariness in z_{\pm} (to some extent) by requiring their composition to agree with this minimal Lagrangian diffeomorphism

$$F = z_- \circ z_+^{-1}.$$

Note that this is a condition on the maps z_{\pm} rather than on F , the latter being completely fixed by the boundary quasisymmetric maps f_{\pm} .

As in the compact case, the knowledge of F and the hyperbolic metric on e.g. the disc Δ_+ are enough to reconstruct both the first and second fundamental forms on the maximal surface. Formula (5) gives their expressions evaluated at the z_+ -disc. This description, however, completely hides one direction in the phase space. To obtain the complete description with the dependence on both (f_+, f_-) explicit, we take the pull-back to the base disc. The nature of the two deformation directions in the phase space then becomes clear. Their “difference” $f = f_- \circ f_+^{-1}$ defines a maximal surface in AdS₃, with its induced first and second fundamental forms and the corresponding isothermal coordinate. We interpret this as a “geometric” direction as it completely describes the geometry of the maximal surface, and of its Cauchy development. The other direction, corresponding to the particular way f decomposes into f_{\pm} , determines another quasiconformal deformation of the maximal surface (or a choice of the complex coordinate on the maximal surface possibly different from the isothermal coordinate z there). This measures how distant our spacetime is from the preferred base point in the phase space and can be interpreted as a “non-geometric” deformation determining a constant radius foliation of the maximal surface.

To write down the first and second fundamental forms on the maximal surface in terms of the complex coordinate measuring how distant our spacetime w is on the base disc, we need to describe the operator $b : T\Delta_+ \rightarrow T\Delta_+$ associated with the minimal Lagrangian diffeomorphism F . Then, the inverse Mess map (5) directly gives the first and second fundamental forms of the corresponding maximal surface. It is possible to write b explicitly as

$$b = |\partial F| \left(\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+ + \partial_{\bar{z}_+} d\bar{z}_+ \right),$$

where $\lambda = \partial_{\bar{z}_+} F / \partial_{z_+} F$ is the Beltrami coefficient of F and

$$|\partial F| = \frac{1 - |z_+|^2}{1 - |F|^2} |\partial_{z_+} F|$$

is the holomorphic energy density of F . When F is a minimal Lagrangian this operator satisfies the conditions (1-4) in the definition of the inverse of the Mess map and we can construct the fundamental forms of the maximal surface as in the compact case

$$I = \frac{|\partial F| (|\partial F| + 1)}{(1 - |z_+|^2)^2} \left(2|dz_+|^2 + \bar{\lambda} dz_+^2 + \lambda d\bar{z}_+^2 \right).$$

One then computes the almost-complex structure of I and the operator $(E+b)^{-1}(E-b)$ directly

$$J = i|\partial F| \left(\partial_{z_+} dz_+ + \lambda \partial_{z_+} d\bar{z}_+ - \bar{\lambda} \partial_{\bar{z}_+} dz_+ - \partial_{\bar{z}_+} d\bar{z}_+ \right)$$

and

$$(E+b)^{-1}(E-b) = -\frac{|\partial F|}{|\partial F| + 1} \left(\lambda \partial_{z_+} d\bar{z}_+ + \bar{\lambda} \partial_{\bar{z}_+} dz_+ \right).$$

Therefore the second fundamental form II is given by

$$II = -i \frac{|\partial F|}{(1 - |z_+|^2)^2} (\bar{\lambda} dz_+^2 - \lambda d\bar{z}_+^2).$$

These determine the first and second fundamental forms on the maximal surface, written in terms of the coordinate z_+ on the disc Δ_+ .

We now pull-back (I, II) to the base w -disc using the map z_+ . The transformation of the Beltrami coefficient λ and the holomorphic energy density $|\partial F|$ are easily obtained computing derivatives of $z_- = F \circ z_+$,

$$\lambda \circ z_+ \frac{\partial \bar{w} \bar{z}_+}{\partial w z_+} = \frac{\mu_- - \mu_+}{1 - \mu_- \bar{\mu}_+}, \quad |\partial F| \circ z_+ |\partial_w z_+| = |\partial z_-| \frac{|1 - \bar{\mu}_+ \mu_-|}{1 - |\mu_+|^2}.$$

Using the area preserving condition for F , we then get the following expressions

$$I = \frac{1}{(1 - |w|^2)^2} \left(\frac{1}{2} \frac{|\partial z_+|^2}{1 - |\mu_-|^2} + \frac{1}{2} \frac{|\partial z_-|^2}{1 - |\mu_+|^2} + \frac{|\partial z_+| |\partial z_-|}{|1 - \mu_- \bar{\mu}_+|} \right) \left(2(1 - |\mu_-|^2 |\mu_+|^2) |dw|^2 \right. \\ \left. + (\bar{\mu}_+(1 - |\mu_-|^2) + \bar{\mu}_-(1 - |\mu_+|^2)) dw^2 + (\mu_+(1 - |\mu_-|^2) + \mu_-(1 - |\mu_+|^2)) d\bar{w}^2 \right) \quad (7)$$

and

$$II = \frac{i}{(1 - |w|^2)^2} \frac{|\partial z_+| |\partial z_-|}{|1 - \mu_- \bar{\mu}_+|} \left(2(\mu_+ \bar{\mu}_- - \bar{\mu}_+ \mu_-) |dw|^2 + (\bar{\mu}_+(1 + |\mu_-|^2) \right. \\ \left. - \bar{\mu}_-(1 + |\mu_+|^2)) dw^2 - (\mu_+(1 + |\mu_-|^2) - \mu_-(1 + |\mu_+|^2)) d\bar{w}^2 \right). \quad (8)$$

Here

$$|\partial z_{\pm}| = \frac{1 - |w|^2}{1 - |z_{\pm}|^2} |\partial_w z_{\pm}|,$$

are the holomorphic energy densities of z_{\pm} . The pair (I, II) satisfies the Gauss-Codazzi equations, and can be used to construct the spacetime metric via (3).

We close our discussion of the parametrization of AdS₃ spacetimes by two copies of $\mathcal{T}(1)$ with a more detailed discussion of the ambiguity that entered into the above construction. Indeed, recall that the maps z_{\pm} are arbitrary quasiconformal extensions of the boundary quasisymmetric maps f_{\pm} , with the condition that $z_{-} \circ z_{+}^{-1}$ is the minimal Lagrangian diffeomorphism extending $f_{-} \circ f_{+}^{-1}$. One can now see that nothing depends on the remaining extension ambiguity. Indeed, choosing two different extensions for f_{\pm} , say \tilde{z}_{\pm} , which are in the same universal Teichmüller class as z_{\pm} and still satisfy $\tilde{z}_{-} \circ \tilde{z}_{+}^{-1} = F$, we obtain two pairs (I, II) and (\tilde{I}, \tilde{II}) , as well as the corresponding spacetime metrics. It is however clear that the corresponding spacetimes can be mapped into one another by the (purely spatial) diffeomorphism $\tilde{z}_{+} \circ z_{+}^{-1}$, which is clearly asymptotically trivial since their restriction to the boundary is given by

$$\tilde{z}_{+} \circ z_{+}^{-1} = f_{+} \circ f_{+}^{-1} = \text{Id}.$$

Therefore these spacetimes should be considered equivalent and nothing in the above construction depends on which particular extension of f_{\pm} are chosen, provided the minimal Lagrangian condition on $z_{-} \circ z_{+}^{-1}$ holds.

5. The Generalized Mess Map

The aim of this section is to describe an analogue of the cotangent bundle parametrization of the universal phase space. To this end, we first verify the existence of a special decomposition of minimal Lagrangian diffeomorphisms in terms of harmonic maps with opposite Hopf differentials. The corresponding coordinate on the source disc of these maps is then shown to agree with the isothermal coordinate on the maximal surface and the Hopf differentials are just (i times) plus or minus the quadratic differential parametrizing the cotangent direction. These facts are not new and can be found in [21]. We give them here for completeness. We then describe what can be called a generalized Mess map from $T^*\mathcal{T}(1)$ to two copies of $\mathcal{T}(1)$. This arises precisely in the same way as in the compact setting, see (4), the only non-trivial point being the existence and uniqueness of a solution to the Gauss equation (2), which follows from results of [26]. In Sect. 9 we give a physicist’s proof that this map is a (formal) symplectomorphism with respect to the natural symplectic structures on $T^*\mathcal{T}(1)$ and $\mathcal{T}(1) \times \mathcal{T}(1)$. The arising description of a symplectomorphism between these spaces is a new result, as far as we are aware.

Any minimal Lagrangian diffeomorphism F of the unit disc can be uniquely written as a composition $F = F_{-} \circ F_{+}^{-1}$ of two harmonic maps F_{\pm} whose Hopf differentials add up to zero, see e.g. Lemma 2.1 of [21]. As we shall now see, it is this decomposition that leads to the particularly simple expressions for the data on the maximal surface. Let us refer to the coordinate on the source disc of F_{\pm} by z , see Fig. 2. The Hopf differentials are then given by

$$\text{Hopf}(F_{\pm}) = \frac{4\partial_z F_{\pm} \partial_z \bar{F}_{\pm}}{(1 - |F_{\pm}|^2)^2} dz^2.$$

Using the associated Beltrami differentials

$$v_{\pm} = \partial_z F_{\pm} / \partial_z F_{\pm},$$

we can write these Hopf differentials as

$$\text{Hopf}(F_{\pm}) = \frac{4|\partial F_{\pm}|^2 \bar{v}_{\pm}}{(1 - |z|^2)^2} dz^2,$$

where

$$|\partial F_{\pm}| = \frac{1 - |z|^2}{1 - |F_{\pm}|^2} |\partial_z F_{\pm}|$$

are the corresponding holomorphic energy densities. The Hopf differentials are required to add up to zero, which gives

$$v_+ = -\frac{|\partial F_-|^2}{|\partial F_+|^2} v_-.$$

Then the area preserving condition for F then reduces to

$$\frac{|\partial F_-|^4}{|\partial F_+|^4} |v_-|^2 + \frac{|\partial F_-|^2}{|\partial F_+|^2} (1 - |v_-|^2) - 1 = 0,$$

and this implies

$$\frac{|\partial F_-|^2}{|\partial F_+|^2} = 1,$$

in particular, $v_+ = -v_- = v$. The fundamental forms (7), (8) of the maximal surface thus become

$$I = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2} |dz|^2, \quad II = \frac{i}{2} \left(\text{Hopf}(F_+) - \overline{\text{Hopf}(F_+)} \right). \tag{9}$$

Note that the functions F_{\pm} are not holomorphic or anti-holomorphic in z , and so the metric I is not hyperbolic, despite its seeming resemblance to $4|dF_+|^2/(1 - |F_+|^2)^2$. The formulas (9) are also contained in Proposition 3.1 of [21].

It is clear that what we have obtained is just the cotangent bundle description with the conformal factor and the holomorphic quadratic differential given by

$$e^{2\varphi} = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2}, \quad q dz^2 = i \text{Hopf}(F_+).$$

We note that the Gauss-Codazzi equations for φ and q follow directly from the harmonicity of F_+ ,

$$\partial_z \partial_{\bar{z}} F_+ + \frac{2\bar{F}_+}{1 - |F_+|^2} \partial_z F_+ \partial_{\bar{z}} F_+ = 0.$$

We have thus seen that the Gauss maps composing the generalized Mess map (4), from the data on the maximal surface to the hyperbolic discs Δ_{\pm} , are harmonic. This allowed for a simple description (9) of the map $\mathcal{T}(1) \times \mathcal{T}(1) \rightarrow T^*\mathcal{T}(1)$. This fact also allows for a description of the map in the opposite direction. Thus, given a point

$(\mu, qdz^2) \in T^*\mathcal{T}(1)$, one can solve for two harmonic maps F_{\pm} with prescribed Hopf differentials

$$i\text{Hopf}(F_{\pm}) = \pm q.$$

The existence and uniqueness of harmonic maps with prescribed Hopf differentials was given in [26] by proving that there exists a unique solution of the Gauss equation (2) such that the right- and left-hand-sides are non-negative and such that $e^{2\varphi}|dz|^2$ is a complete metric. This then allows to construct the harmonic maps explicitly via the Mess map (4). For more details of this construction the reader can consult [26]. We note that, although the treatment in this reference is carried for CMC surfaces in the Minkowski space $\mathbb{R}^{2,1}$, it needs very little adaptation to the present situation.

Finally, writing $\nu_{\pm} = \pm\nu$ for the corresponding Beltrami coefficients, it is now just a matter of using the group structure of $\mathcal{T}(1)$ to get the Beltrami coefficients of the maps from the base disc:

$$\mu_{\pm} = \frac{\mu \pm \nu \circ z(\partial_{\bar{w}}\bar{z}/\partial_w z)}{1 \pm \bar{\mu}\nu \circ z(\partial_{\bar{w}}\bar{z}/\partial_w z)}, \quad \nu = i \frac{(1 - |z|^2)^2}{4|\partial F_{\pm}|^2} \bar{q}.$$

This gives an explicit description of the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ and a proof it is a bijection.

6. Relation to the Fefferman-Graham Description: The Infinitesimal Case

In this section we relate the description (3) of spacetimes as evolving data (7), (8) to the Fefferman-Graham type description of asymptotically AdS₃ spacetimes presented in the Introduction. Here we treat the infinitesimal case only, relating the objects appearing in our parametrization to the chiral functions a_{\pm} determining the Fefferman-Graham metric (1). We shall accomplish this by computing the generators of both quasiconformal and asymptotic deformations of AdS₃, which we take to be the reference spacetime in both descriptions.

We start with the infinitesimal version of the spacetime metric in terms of initial data on the maximal surface. This is obtained by taking infinitesimal Beltrami coefficients $\delta\mu_{\pm}$ as representatives of the pair of quasisymmetric homeomorphisms $f_{\pm} \in \mathcal{T}(1)$. This can be interpreted in terms of infinitesimal quasiconformal deformations of the preferred base disc in $\mathcal{T}(1)$ in the direction of $\delta\mu_{\pm}$, now thought of as tangent vectors to $\mathcal{T}(1)$ at the origin. These deformations can be written

$$z_{\pm} = w + \delta z_{\pm} + \dots,$$

where the first variations δz_{\pm} are solutions of the first variations of the Beltrami equation

$$\partial_{\bar{w}}\delta z_{\pm} = \delta\mu_{\pm}. \tag{10}$$

We refer the reader to the Appendix for more details on the tangent space to $\mathcal{T}(1)$.

It is then easy to obtain the corresponding infinitesimal versions of the data (7) and (8)

$$I = \frac{4|dw|^2}{(1 - |w|^2)^2} + \frac{2}{(1 - |w|^2)^2} \left[(\delta\bar{\mu}_+ + \delta\bar{\mu}_-)dw^2 + (\delta\mu_+ + \delta\mu_-)d\bar{w}^2 \right],$$

$$II = \frac{i}{(1 - |w|^2)^2} \left[(\delta\bar{\mu}_+ - \delta\bar{\mu}_-)dw^2 - (\delta\mu_+ - \delta\mu_-)d\bar{w}^2 \right],$$

and the infinitesimal metric

$$ds^2 = ds_{\text{AdS}_3}^2 + \frac{2 \cos^2 \tau}{(1 - |w|^2)^2} \left[(\delta \bar{\mu}_+ + \delta \bar{\mu}_-) dw^2 + (\delta \mu_+ + \delta \mu_-) d\bar{w}^2 \right] + \frac{2i \sin \tau \cos \tau}{(1 - |w|^2)^2} \left[(\delta \bar{\mu}_+ - \delta \bar{\mu}_-) dw^2 - (\delta \mu_+ - \delta \mu_-) d\bar{w}^2 \right], \quad (11)$$

where

$$ds_{\text{AdS}_3}^2 = -d\tau^2 + \frac{4 \cos^2 \tau |dw|^2}{(1 - |w|^2)^2} \quad (12)$$

is the AdS₃ metric in equidistant coordinates.

Another fact we shall need for our computation is an identity from [27] saying that the infinitesimal quasiconformal maps $w + \delta z_{\pm}$ are area preserving:

$$\text{Re} \frac{\partial}{\partial w} \frac{\delta z}{(1 - |w|^2)^2} = 0.$$

Equivalently, by expanding and multiplying by $(1 - |w|^2)$, we have

$$2 \frac{\bar{w} \delta z + w \delta \bar{z}}{(1 - |w|^2)} + \partial_w \delta z + \partial_{\bar{w}} \delta \bar{z} = 0. \quad (13)$$

Now, to compare the metric (11) arising in the maximal surface description with that in the Fefferman-Graham setting, see below, we could just apply the same coordinate transformation that puts the AdS metrics (12) into the Fefferman-Graham form to the infinitesimal part of the metric in (11). The arising metric could then be expected to be of the Fefferman-Graham type, and one could read off the quantities a , b in terms of the Beltrami coefficients $\delta \mu_{\pm}$. However, this direct approach for performing the computation seems to be too difficult to carry out, and we proceed in a different way.

Thus, let us consider a general vector field

$$\xi = \xi^\tau \partial_\tau + \xi^w \partial_w + \xi^{\bar{w}} \partial_{\bar{w}}$$

written in the coordinates relevant for the maximal surface description. We would like to describe the vector fields whose action on the AdS₃ metric (12) gives us the infinitesimal metric (11). We will then be able to compare (asymptotically) such vector field with the Brown-Henneaux vector field generating the infinitesimal asymptotically AdS metric, see below, and relate the defining parameters in the Brown-Henneaux vector fields with the Beltrami coefficients $\delta \mu_{\pm}$. This will finally lead to a relation between the parameters in the metrics (11) and (1).

Let's start computing the Lie derivative of $ds_{\text{AdS}_3}^2$ in the direction of ξ ,

$$\begin{aligned} & -2\partial_\tau \xi^\tau d\tau^2 + \left(\frac{4 \cos^2 \tau \partial_\tau \xi^{\bar{w}}}{(1 - |w|^2)^2} - 2\partial_w \xi^\tau \right) d\tau dw + \left(\frac{4 \cos^2 \tau \partial_\tau \xi^w}{(1 - |w|^2)^2} - 2\partial_{\bar{w}} \xi^\tau \right) d\tau d\bar{w} \\ & + \frac{4 \cos^2 \tau \partial_w \xi^{\bar{w}} dw^2}{(1 - |w|^2)^2} + \frac{4 \cos^2 \tau \partial_{\bar{w}} \xi^w d\bar{w}^2}{(1 - |w|^2)^2} \\ & + \frac{4 \cos^2 \tau}{(1 - |w|^2)^2} \left(2 \frac{\bar{w} \xi^w + w \xi^{\bar{w}}}{(1 - |w|^2)} + \partial_w \xi^w + \partial_{\bar{w}} \xi^{\bar{w}} - 2 \tan \tau \xi^\tau \right) |dw|^2. \end{aligned}$$

We now equate this tensor with the infinitesimal part of the metric (11), which leads to the following set of equations:

$$\begin{aligned}\partial_\tau \xi^\tau &= 0, & \frac{4 \cos^2 \tau \partial_\tau \xi^{\bar{w}}}{(1 - |w|^2)^2} - 2\partial_w \xi^\tau &= 0, \\ 2 \frac{\bar{w} \xi^w + w \xi^{\bar{w}}}{(1 - |w|^2)} + \partial_w \xi^w + \partial_{\bar{w}} \xi^{\bar{w}} - 2 \tan \tau \xi^\tau &= 0, \\ 2\partial_{\bar{w}} \xi^w &= (1 - i \tan \tau) \delta \mu_+ + (1 + i \tan \tau) \delta \mu_-.\end{aligned}$$

In view of (10), the last equation is clearly satisfied by

$$\xi^w = \frac{1}{2}(1 - i \tan \tau) \delta z_+ + \frac{1}{2}(1 + i \tan \tau) \delta z_- = \frac{1}{2}(\delta z_+ + \delta z_-) + \frac{1}{2i} \tan \tau (\delta z_+ - \delta z_-),$$

and the third equation becomes

$$\begin{aligned}2 \tan \tau \xi^\tau &= \frac{\bar{w}(\delta z_+ + \delta z_-) + w(\delta \bar{z}_+ + \delta \bar{z}_-)}{(1 - |w|^2)} + \frac{1}{2} \partial_w (\delta z_+ + \delta z_-) + \frac{1}{2} \partial_{\bar{w}} (\delta \bar{z}_+ + \delta \bar{z}_-) \\ &+ \tan \tau \left(\frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-) - w(\delta \bar{z}_+ - \delta \bar{z}_-)}{(1 - |w|^2)} + \frac{1}{2i} \partial_w (\delta z_+ - \delta z_-) - \frac{1}{2i} \partial_{\bar{w}} (\delta \bar{z}_+ - \delta \bar{z}_-) \right).\end{aligned}$$

Using the identity (13) for each δz_\pm we have

$$\begin{aligned}\xi^\tau &= \frac{1}{2i} \frac{\bar{w}(\delta z_+ - \delta z_-) - w(\delta \bar{z}_+ - \delta \bar{z}_-)}{(1 - |w|^2)} + \frac{1}{4i} \partial_w (\delta z_+ - \delta z_-) - \frac{1}{4i} \partial_{\bar{w}} (\delta \bar{z}_+ - \delta \bar{z}_-) \\ &= \frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-)}{(1 - |w|^2)} + \frac{1}{2i} \partial_w (\delta z_+ - \delta z_-).\end{aligned}$$

Therefore, the generator of the infinitesimal metric (11) has components

$$\begin{aligned}\xi_T^\tau &= \frac{1}{i} \frac{\bar{w}(\delta z_+ - \delta z_-)}{(1 - |w|^2)} + \frac{1}{2i} \partial_w (\delta z_+ - \delta z_-), \\ \xi_T^w &= \frac{1}{2}(\delta z_+ + \delta z_-) + \frac{1}{2i} \tan \tau (\delta z_+ - \delta z_-).\end{aligned}$$

This gives us an expression for what can be interpreted as a Brown-Henneaux vector field in the maximal surface description.

Let us now compare this to the holographic description that was sketched in the Introduction. We find it convenient to work with a radial coordinate $\chi = \log(1/\rho)$. The metric (1) can then be written as

$$\begin{aligned}ds^2 &= \frac{e^{2\chi}}{4} (-dt^2 + d\theta^2) + d\chi^2 + \frac{1}{2} (adt^2 + 2bdt d\theta + ad\theta^2) \\ &+ \frac{e^{-2\chi}}{4} (a^2 - b^2) (-dt^2 + d\theta^2).\end{aligned}\tag{14}$$

The Brown-Henneaux vector fields [1], generators of the group of asymptotic symmetries, are parametrized by two functions on the boundary

$$\begin{aligned}\xi_{BH}^\chi &= -\frac{1}{2} (\partial_+ \xi_+ + \partial_- \xi_-) + O(e^{-4\chi}), \\ \xi_{BH}^t &= \frac{1}{2} (\xi_+ + \xi_-) + e^{-2\chi} (\partial_+^2 \xi_+ + \partial_-^2 \xi_-) + O(e^{-4\chi}), \\ \xi_{BH}^\theta &= \frac{1}{2} (\xi_+ - \xi_-) - e^{-2\chi} (\partial_+^2 \xi_+ - \partial_-^2 \xi_-) + O(e^{-4\chi}),\end{aligned}\tag{15}$$

with $\xi_\pm = \xi_\pm(t \pm \theta)$.

Then, we consider an infinitesimal version of the metric (14) given by

$$ds^2 = ds_{\text{AdS}_3}^2 + \frac{1}{2}(\delta a dt^2 + 2\delta b dt d\theta + \delta a d\theta^2),$$

where the infinitesimal part is obtained by applying the Lie derivative with respect to the Brown-Henneaux vector field to the AdS₃ metric

$$ds_{\text{AdS}_3}^2 = -\cosh^2 \chi dt^2 + d\chi^2 + \sinh^2 \chi d\theta^2,$$

corresponding to $b = 0$, $a = -1$ in (14). The arising relation between the perturbations δa , δb and the functions ξ_{\pm} parametrizing the vector field (15) is then given by:

$$\begin{aligned} \delta a &= -\frac{1}{2}(\partial_+ \xi_+ + \partial_+^3 \xi_+) - \frac{1}{2}(\partial_- \xi_- + \partial_-^3 \xi_-), \\ \delta b &= -\frac{1}{2}(\partial_+ \xi_+ + \partial_+^3 \xi_+) + \frac{1}{2}(\partial_- \xi_- + \partial_-^3 \xi_-). \end{aligned} \quad (16)$$

We now relate the two descriptions using the fact that they represent different coordinates in the same spacetime. Note that this is indeed possible in a neighbourhood of the boundary curve of the maximal surface where both the Fefferman-Graham and the coordinates of (3) are defined. Thus, let us compute the components of the Brown-Henneaux vector fields (15) in the coordinates used in (12). The coordinate transformation relating the AdS₃ metric in the form (14) and its equidistant coordinates description is

$$\tan t = \frac{1 - |w|^2}{1 + |w|^2} \tan \tau, \quad \sinh \chi = \frac{2|w|}{1 - |w|^2} \cos \tau, \quad \theta = \arg w.$$

We can therefore obtain the Brown-Henneaux vector field (15) in equidistant coordinates:

$$\begin{aligned} \xi_{BH}^{\tau} &= \frac{1 + |w|^2}{1 - |w|^2} \xi_{BH}^t + \frac{2|w| \sin \tau}{[(1 - |w|^2)^2 + 4|w|^2 \cos^2 \tau]^{1/2}} \xi_{BH}^{\chi}, \\ \xi_{BH}^w &= w \tan \tau \xi_{BH}^t - \frac{1}{2} \frac{w}{|w|} \frac{(1 - |w|^4) \sec \tau}{[(1 - |w|^2)^2 + 4|w|^2 \cos^2 \tau]^{1/2}} \xi_{BH}^{\chi} + iw \xi_{BH}^{\theta}. \end{aligned}$$

We need only the asymptotic behaviour of these components, evaluated on the $\tau = 0$ surface. Thus, we get, asymptotically,

$$\xi_{BH}^{\tau} = \frac{\xi_+ + \xi_-}{1 - |w|^2} + \dots, \quad \xi_{BH}^w = \frac{iw}{2} (\xi_+ - \xi_-) + \dots,$$

where the dots stand for the subleading components.

On the other hand, the maximal surface vector fields (evaluated on the $\tau = 0$ surface) are, asymptotically,

$$\xi_T^{\tau} = \frac{1}{i} \bar{w} \frac{\delta z_+ - \delta z_-}{1 - |w|^2} + \dots, \quad \xi_T^w = \frac{1}{2} (\delta z_+ + \delta z_-),$$

and, equating

$$\xi_{BH}^{\tau} = \xi_T^{\tau}, \quad \xi_{BH}^w = \xi_T^w,$$

we immediately read off the sought relations:

$$\delta z_{\pm} \Big|_{|w|=1} = \pm i e^{i\theta} \xi_{\pm}. \quad (17)$$

7. The Fefferman-Graham Stress Tensor and the Bers Embedding

We now develop an interpretation of the obtained relation (17) between the maximal surface and the holographic descriptions. We remain at the infinitesimal level, leaving the question of explicitly relating the general metric from evolving data (7) and (8) and the Fefferman-Graham metric (14) to future studies.

So far, we have seen how the two functions ξ_{\pm} parametrizing the Brown-Henneaux vector fields are given in terms of the boundary values of the (infinitesimal) quasiconformal maps δz_{\pm} . However, a more interesting question is that of a relation between the holographic stress-energy tensor components — functions a, b in (14) — and these quasiconformal maps. In this section we shall see that this relation is that between the holomorphic quadratic differentials arising via the so-called Bers embedding of $\mathcal{T}(1)$ and the quasiconformal maps parametrizing $\mathcal{T}(1)$. In other words, we shall see that the stress-energy tensor components of the holographic description are nothing but the components of the quadratic differentials that arise via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$.

Let us start by reminding the reader some facts about the Bers embedding. We are necessarily brief here, and for more details the reader can consult a very accessible exposition in [28]. The Bers embedding arises via the so-called B-model of the universal Teichmüller space. So far we have been solving the Beltrami equation

$$\partial_{\bar{w}}z = \mu\partial_wz$$

for a quasiconformal map z_{μ} starting from a (bounded) Beltrami coefficient μ on the unit disc Δ , and then extending μ symmetrically to its complement Δ^* . This gives rise to a quasiconformal map z_{μ} , depending real-analytically on μ , that leaves the unit disc invariant (and thus also the unit circle and the complement of the disc), and reduces to a quasisymmetric map on the unit circle. This gave rise to the A-model of the universal Teichmüller space.

The B-model is obtained by taking the same bounded Beltrami coefficients on Δ , and then extending them to be zero on Δ^* . The corresponding quasiconformal map z^{μ} is now conformal on Δ^* , it is in fact biholomorphic onto its image, and depends on μ complex-analytically. The universal Teichmüller space in the B-model is defined as the space of equivalence classes of Beltrami coefficients (or quasiconformal maps), with two coefficients considered equivalent if the corresponding solutions agree outside the unit disc. Thus $\mathcal{T}(1)$ in the B-model is parametrized by univalent holomorphic functions on Δ^* and one may use the holomorphicity of these maps to obtain an embedding of $\mathcal{T}(1)$ in the space of holomorphic quadratic differentials on the complement of Δ via Schwarzian derivative.

At the infinitesimal level, there exists an explicit relation between the two models exhibited in [28]. Thus, the tangent vectors to $\mathcal{T}(1)$ in the A-model can be described as functions $u : \mathbb{S}^1 \rightarrow \mathbb{R}$ on the circle that are related to the so-called Zygmund class functions on the real line, see [28] and references therein for more details. What is important for us here is that the functions u are defined from the boundary values of the corresponding infinitesimal quasiconformal maps δz via:

$$u(e^{i\theta}) = \frac{\delta z(e^{i\theta})}{ie^{i\theta}}. \tag{18}$$

The functions arising as tangent vectors to $\mathcal{T}(1)$ are those for which

$$F(x) = \frac{1}{2}(x^2 + 1)u\left(\frac{x - i}{x + i}\right)$$

are of the Zygmund class, see [28]. The functions u can be expanded into a Fourier series:

$$u(e^{i\theta}) = \sum_{k=-\infty}^{\infty} u_k e^{ik\theta},$$

where $u_{-k} = \bar{u}_k$, as required by the fact $u(e^{i\theta})$ is a real function.

The tangent space to the B-model universal Teichmüller space is described as follows. In this model the solution z^μ of the Beltrami equation is of the form $z^\mu(w) = w + \delta z(w)$, with the function δz now being holomorphic on Δ^* . It thus admits an expansion

$$w + \delta z(w) = w \left(1 + \frac{c_2}{w^2} + \frac{c_3}{w^3} + \dots \right) \quad \text{in } |w| > 1, \tag{19}$$

where a Möbius transformation is used to remove the $1/w$ term in the brackets and to set the first term to unity. The corresponding holomorphic quadratic differential, obtained as the (infinitesimal) Schwarzian derivative $\delta_w^3 \delta z$ of (19), can also be expanded as

$$h(w) = \frac{1}{w^4} \left(h_0 + \frac{h_1}{w} + \frac{h_2}{w^2} + \dots \right) \quad \text{in } |w| > 1, \tag{20}$$

where the coefficients h_k are related to those in (19) via

$$h_{k-2} = c_k(k - k^3), \quad k \geq 2. \tag{21}$$

Finally, the relation between the A- and B-model Fourier coefficients u_k and c_k is given by, see [28]

$$c_k = i\bar{u}_k. \tag{22}$$

For the later purposes, we now note that in all the discussions of the B-model above we could have replaced Δ and Δ^* . In fact, this is the choice made in some of the references, see e.g. [24]. In this case one works with bounded Beltrami differentials on Δ^* , solves the Beltrami equation continuing $\mu = 0$ inside, and gets a univalent holomorphic function on Δ , whose Schwarzian derivative produces a holomorphic quadratic differential on the unit disc. We could have as well worked with these models for the universal Teichmüller space. In fact, as we shall see below, it will be natural to work with holomorphic functions on Δ^* for one copy of $\mathcal{T}(1)$ and with holomorphic functions on Δ for the other. The analogues of (19) and (20) in this realization of $\mathcal{T}(1)$ are then given by

$$w + \delta \hat{z}(w) = w(1 + \hat{c}_2 w^2 + \hat{c}_3 w^3 + \dots), \quad |w| < 1,$$

and

$$\hat{h}(w) = \hat{h}_0 + \hat{h}_1 w + \hat{h}_2 w^2 + \dots \quad |w| < 1, \tag{23}$$

where we have denoted the quantities arising in this realization of the B-model by letters with an extra hat. We also note that there is an extra minus as compared to (21) in the relation between the coefficients in this realization

$$\hat{h}_{k-2} = -\hat{c}_k(k - k^3).$$

Also, note that we can always map a holomorphic function inside the disc to an anti-holomorphic function outside by $w \rightarrow 1/\bar{w}$. By applying this to the quadratic differential (23) we get a new anti-holomorphic quadratic differential $h(\bar{w})$ outside of the disc by complex conjugating

$$\overline{h(\bar{w})} = \hat{h} \left(\frac{1}{w} \right) \frac{1}{w^4} = \frac{1}{w^4} \left(\hat{h}_0 + \frac{\hat{h}_1}{w} + \frac{\hat{h}_2}{w^2} + \dots \right) \quad \text{in } |w| > 1, \quad (24)$$

which gives the same expression as in (20), but with the change $w \rightarrow \bar{w}$. We shall use this realization in terms of anti-holomorphic functions on Δ^* for the second copy of $\mathcal{T}(1)$, and the “usual” realization in terms of holomorphic functions on Δ^* for the first copy of the universal Teichmüller.

Another relation we shall need is that between the c -coefficients in the realization of the B-model in terms of Beltrami coefficients on Δ^* (we have denoted these coefficients by \hat{c}_k above), and the Fourier coefficients u_k of the Zygmund functions u of the A-model. Formula (22) gives such relation for one realization of the B-model, and we need to derive a similar relation for the other. The derivation is a straightforward adaptation of the Proof I in [28]. One finds

$$\hat{c}_k = -\frac{1}{\pi} \int_{\Delta^*} \frac{\mu(w)}{w^{k+2}} dx dy. \quad (25)$$

In relating it to the A-model Fourier coefficients we use the fact that the A-model Beltrami coefficient on Δ^* is obtained by reflection. Thus, the integral in (25) can be taken over the unit disc with the reflected Beltrami given by

$$\mu \left(\frac{1}{\bar{w}} \right) = \overline{\mu(w)} \frac{w^2}{\bar{w}^2}.$$

Substituting this to the integral, and taking into account the change of integration measure $dx dy \rightarrow -dx dy/w^2 \bar{w}^2$ we get

$$\hat{c}_k = \frac{1}{\pi} \int_{\Delta} \overline{\mu(w)} \bar{w}^{k-2} dx dy.$$

The hatted c -coefficients are thus related to the unhatted ones by complex conjugation

$$\hat{c}_k = \bar{c}_k.$$

We can now continue to think about both points in $\mathcal{T}(1) \times \mathcal{T}(1)$ as being parametrized by Beltrami coefficients inside Δ . This gives two A-model quasiconformal maps in Δ whose boundary values produce two Zygmund functions u . Expanding these into Fourier modes we get two sets of coefficients u_k , which we shall later denote by u_k^\pm . Now our convention is that the B-model for the first copy of $\mathcal{T}(1)$ is obtained by setting to zero the Beltrami on the complement of Δ , thus producing holomorphic functions on Δ^* , and the holomorphic quadratic differentials as in (20). The B-model for the second copy of $\mathcal{T}(1)$ will be realized by first reflecting the Beltrami coefficient, then setting it to zero inside Δ , thus producing a holomorphic quadratic differential on Δ , which in turn can be interpreted as an anti-holomorphic quadratic differential on Δ^* , as in (24).

Collecting all the relations above we can write the following relations between the u - and h -coefficients of the two copies of $\mathcal{T}(1)$:

$$h_{k-2}^\pm = \pm i \bar{u}_k^\pm (k - k^3), \quad (26)$$

where we have now differentiated between the two copies of $\mathcal{T}(1)$ by assigning plus and minus labels. We also used the following notation in (24) relating the hatted h coefficients and the coefficients of the anti-holographic quadratic differential:

$$\hat{h}_{k-2}^- = \bar{h}_{k-2}^-.$$

Now, to prepare for the relation between the Bers embedding quadratic differentials and the holomorphic stress-energy tensor that we will derive, let us rewrite the stress tensor

$$T = adt^2 + 2bdtd\theta + ad\theta^2$$

of the Fefferman-Graham metric in a more suggestive way. To this end, we will analytically continue the t coordinate to the imaginary values. Thus, let us continue all the functions appearing in T via

$$t = \frac{1}{2i} \log |w|^2, \quad \theta = \frac{1}{2i} \log \frac{w}{\bar{w}}, \tag{27}$$

so that the new (imaginary) time coordinate runs between $-i\infty$ and $i\infty$ while w runs over the complex plane with the unit circle $|w| = 1$ corresponding to $t = 0$. With this choice we have

$$t + \theta = \frac{1}{i} \log w, \quad t - \theta = \frac{1}{i} \log \bar{w},$$

so that functions of $e^{i(t\pm\theta)}$ become holomorphic (anti-holomorphic) functions on the complex plane. In particular, the functions a_{\pm} whose sum and difference give a, b would seem to become a holomorphic and anti-holomorphic function on the complex plane. However, there is no (bounded) holomorphic function on the whole complex plane apart from a constant. Thus, we need to be very careful when designing the analytic continuation.

Let us expand a_{\pm} into Fourier modes. When restricted to $t = 0$ these are periodic functions of θ , and so the Fourier expansion is possible. We have

$$a_{\pm}(t \pm \theta) = \sum_{k=-\infty}^{\infty} a_k^{\pm} e^{ik(t\pm\theta)}.$$

As before, we have $\bar{a}_k^{\pm} = a_{-k}^{\pm}$ so that these are real functions. It is clear that we cannot continue a_{\pm} as holomorphic or anti-holomorphic functions into the whole complex plane. What is possible is to take what can be called the chiral part of a_{\pm} , containing only, say, the negative frequency modes, and continue this part only. Thus, let us introduce

$$\tilde{a}_{\pm}(t \pm \theta) = \sum_{k=-\infty}^{-2} a_k^{\pm} e^{ik(t\pm\theta)}$$

for the chiral parts. Here we have used the fact that $|k| \geq 2$ in these expansions, which will become manifest below. We now continue the chiral parts via (27) to the complement of the unit disc to get:

$$\tilde{a}_+(w) = \frac{a_{-2}^+}{w^2} + \frac{a_{-3}^+}{w^3} + \dots, \quad \tilde{a}_-(\bar{w}) = \frac{a_{-2}^-}{\bar{w}^2} + \frac{a_{-3}^-}{\bar{w}^3} + \dots, \tag{28}$$

which are, respectively, holomorphic and anti-holomorphic functions on Δ^* . We can now analytically continue the chiral part \tilde{T} of the stress tensor (23), which is the tensor T with functions a_{\pm} replaced by their chiral parts. A simple computation gives:

$$\tilde{T} = -\frac{\tilde{a}_+(w)}{w^2}dw^2 - \frac{\tilde{a}_-(\bar{w})}{\bar{w}^2}d\bar{w}^2.$$

We may now relate the chiral parts \tilde{a}_{\pm} of the stress-energy tensor to the quadratic differentials arising via the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$. To this end, let us first obtain the relation between the Fourier coefficients a_k^{\pm} and those of the Fourier expansions of the functions ξ_{\pm} appearing in the Brown-Henneaux vector fields. From (16) we have

$$-2a_{\pm} = \partial_{\pm}\xi_{\pm} + \partial_{\pm}^3\xi_{\pm}.$$

Thus, if we expand

$$\xi_{\pm}(t \pm \theta) = \sum_{k=-\infty}^{\infty} \xi_k^{\pm} e^{ik(t \pm \theta)},$$

with $\xi_{-k}^{\pm} = \bar{\xi}_k^{\pm}$, we get the following relation between the Fourier coefficients:

$$-2a_k^{\pm} = i\xi_k^{\pm}(k - k^3).$$

We now come back to the problem of relating the maximal surface and holographic descriptions. We have seen that the relation between the Brown-Henneaux functions ξ_{\pm} and the boundary values of the quasiconformal maps δz_{\pm} is given by (17). In view of (18), this can then be rewritten as

$$\xi_{\pm}(\pm\theta) = \pm u^{\pm}(e^{i\theta}),$$

where on the left-hand-side the functions ξ_{\pm} are restricted to the circle $t = 0$. This now implies the following relation between the Fourier coefficients of the ξ -functions and the A-model functions u_{\pm} ,

$$\xi_k^{\pm} = \pm u_k^{\pm},$$

with $u_{-k}^{\pm} = \bar{u}_k^{\pm}$. We can therefore write $-2a_k^{\pm} = \pm i u_k^{\pm}(k - k^3)$ or

$$2a_{-k}^{\pm} = \pm i \bar{u}_k^{\pm}(k - k^3).$$

Comparing this with (26) we see that

$$h_{k-2}^{\pm} = 2a_{-k}^{\pm}, \quad k \geq 2.$$

This gives us the desired relation between the coefficients in the expansions (28) of the chiral parts of the stress-energy tensor and those of the Bers embedding quadratic differentials (20), (24), and implies

$$\frac{2\tilde{a}_+(w)}{w^2} = h^+(w), \quad \frac{2\tilde{a}_-(\bar{w})}{\bar{w}^2} = h^-(\bar{w}). \tag{29}$$

Finally, the analytic continuation of the chiral part \tilde{T} of the stress-energy tensor is equal to (minus half) the sum of two quadratic differentials arising via the Bers embedding:

$$\tilde{T} = -\frac{1}{2}h^+(w)dw^2 - \frac{1}{2}h^-(\bar{w})d\bar{w}^2, \tag{30}$$

which is our final result for the (infinitesimal) relation between the maximal surface and the holographic descriptions.

8. Charges

Having the relation (30) we now wish to derive an expression for the asymptotic charges for a general spacetime in terms of the maximal surface parametrization. Similar to the previous section, we shall do so at the infinitesimal level. However, the answer that we obtain admits an obvious generalization to the finite case. We shall see that the charges are given simply by the (real parts of) the periods of the Bers embedding quadratic differentials.

We start by computing the charges of an asymptotically AdS spacetime in the holographic Fefferman-Graham description, see [19]. Here one starts with the Einstein-Hilbert action complemented with the York-Gibbons-Hawking boundary term and a volume renormalization counterterm

$$S = \frac{1}{2} \int_M d^3x \sqrt{-g} (R - 2\Lambda) + \int_{\partial M} d^2x \sqrt{-\gamma} (\Theta - 1).$$

The quasi-local stress-energy tensor is then obtained from the variation of the action with respect to the boundary metric

$$T_{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \frac{\delta S}{\delta \gamma^{\mu\nu}}.$$

One gets

$$T_{\mu\nu} = -\left(\Theta_{\mu\nu} - \Theta^\rho{}_\rho \gamma_{\mu\nu} + \gamma_{\mu\nu}\right),$$

where γ is the boundary metric and Θ the boundary extrinsic curvature.

Let us now consider a spacetime asymptotically described by a Fefferman-Graham type metric (14). Let M_{χ_0} denote the portion of the spacetime manifold where $\chi < \chi_0$. The metric induced on $\chi = \chi_0$ surface is

$$\gamma = \frac{1}{4} e^{2\chi_0} (-dt^2 + d\theta^2) + \frac{1}{2} (adt^2 + 2bdt d\theta + ad\theta^2) + \frac{1}{4} e^{-2\chi_0} (a^2 - b^2) (-dt^2 + d\theta^2).$$

The extrinsic curvature of the $\chi = \chi_0$ surface is given by

$$\Theta_{\mu\nu} = -\mathcal{L}_n \gamma_{\mu\nu}|_{\chi_0} = -\nabla_\mu n_\nu|_{\chi_0} = -\frac{1}{2} \partial_\chi g_{\mu\nu}|_{\chi_0},$$

where $n = -\partial_\chi$ is the unit normal vector field to the boundary ∂M_{χ_0} . Therefore

$$\Theta = \frac{1}{4} \left[e^{2\chi_0} - e^{-2\chi_0} (a^2 - b^2) \right] (-dt^2 + d\theta^2)$$

and, taking the limit $\chi_0 \rightarrow \infty$, we get

$$T =adt^2 + 2bdt d\theta + ad\theta^2.$$

Then for each asymptotic Killing vector field ξ we have a conserved charge given by

$$Q_\xi = \lim_{\chi_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial \Sigma_{\chi_0}} d\theta \sqrt{|\sigma|} u^\mu \xi^\nu T_{\mu\nu}.$$

Here we need to consider a spacelike slice $\Sigma = \{t = 0\}$ and compute its unit normal timelike vector field

$$u = \frac{2}{e^{2\chi} - e^{-2\chi}(a^2 - b^2)} \times \left[\sqrt{e^{2\chi} + 2a + e^{-2\chi}(a^2 - b^2)} \partial_t - \frac{2b}{\sqrt{e^{2\chi} + 2a + e^{-2\chi}(a^2 - b^2)}} \partial_\theta \right].$$

We also consider $\Sigma_{x_0} = M_{x_0} \cap \Sigma$. The induced metric on its boundary $\partial\Sigma_{x_0}$ is

$$\sigma = \frac{1}{4} \left[e^{2x_0} + 2a + e^{-2x_0}(a^2 - b^2) \right] d\theta^2.$$

Mass is the conserved charge associated with time translation ($\xi = \partial_t$)

$$M = \lim_{x_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial\Sigma_{x_0}} d\theta \sqrt{|\sigma|} (u^0 T_{00} + u^2 T_{20}) = \frac{1}{2\pi} \int_{\partial\Sigma} d\theta a(\theta)$$

and angular momentum the one associated with rotation ($\xi = -\partial_\theta$)

$$J = \lim_{x_0 \rightarrow \infty} \frac{1}{2\pi} \int_{\partial\Sigma_{x_0}} d\theta \sqrt{|\sigma|} (u^0 T_{02} + u^2 T_{22}) = \frac{1}{2\pi} \int_{\partial\Sigma} d\theta b(\theta).$$

Taking into account the relations

$$a(t, \theta) = a_+(t + \theta) + a_-(t - \theta), \quad b(t, \theta) = a_+(t + \theta) - a_-(t - \theta)$$

to the chiral functions a_\pm we can rewrite the above formulas for the charges compactly as

$$\frac{1}{2}(M \pm J) = \frac{1}{2\pi} \oint_{\partial\Sigma} a_\pm.$$

Note that, even prior to any relation to the maximal surface description, these can be expressed as the real parts of the periods of the holomorphic quadratic differentials arising via the analytic continuation of the chiral parts of a_\pm , see the previous section.

We now relate this to the maximal surface description. From (29) we know that the chiral parts of the functions a_\pm are basically the (analytic continuations of the) quadratic differentials h_\pm arising from the Bers embedding. The full functions a_\pm on the circle can be obtained by taking their chiral parts and adding the complex conjugate. Thus, we have $2\text{Re}(\tilde{a}^\pm)|_{|w|=1} = a^\pm(\theta)$ and therefore

$$\frac{1}{2}(M + J) = \frac{1}{2\pi} \text{Re} \oint_{|w|=1} w^2 h^+, \quad \frac{1}{2}(M - J) = \frac{1}{2\pi} \text{Re} \oint_{|w|=1} \bar{w}^2 h^-, \quad (31)$$

which are just the (real parts of the) periods of the Bers embedding quadratic differentials h^\pm . The formulas in terms of the Bers embedding quadratic differentials are of course only valid at the infinitesimal level, where we have a relation between the functions a_\pm of the holographic description and the data on the maximal surface. However, as we noted above, the same formulas are valid even in the finite case if one understands that h_\pm are the (multiples of the) analytic continuations of the chiral parts of a_\pm , see (29). It is then natural to conjecture that the analytic continuations of the chiral parts of a_\pm continue to be related to the Bers embedding quadratic differentials in the same

way as they do in the infinitesimal case, and that (31) gives a general formula for the charges in terms of the maximal surface data. We leave an attempt at demonstrating this finite case relation to future work. We also note that in this infinitesimal case the charges (31) are actually zero, for there is no $1/w^2$, $1/\bar{w}^2$ terms in the expansion of the infinitesimal quadratic differentials h^\pm , see (20), (24). So, the first order variation of the charges, computed at the origin corresponding to the AdS_3 is zero. It is clear however that considering non-trivial spatial topologies, obtained as the quotients of AdS_3 by some discrete groups of isometries, will render non-trivial periods for the (anti-)holomorphic quadratic differentials and therefore non-trivial charges in each asymptotic region.

9. The Phase Space Symplectic Structure

We turn to the description of the gravitational symplectic structure on the universal phase space. First, we must warn the reader that, due to noncompactness of Δ , this is only a formal symplectic structure. In fact, already in the universal Teichmüller space context the universal Weil-Petersson symplectic structure (in fact the Weil-Petersson hermitian metric) diverges for certain tangent directions. A solution for this problem was introduced in [24] where a new topology is introduced in $\mathcal{T}(1)$ which makes it into a Hermitian manifold with a well defined Weil-Petersson hermitian metric in each tangent space. We shall not reproduce their arguments here and refer the reader to [24] for more information. We note however that the results of [24] directly extend to the universal phase space and, therefore, the formal results obtained below can be readily made more rigorous.

We first compute the symplectic structure in the cotangent bundle description, and then translate it into the generalized Mess description $\mathcal{T}(1) \times \mathcal{T}(1)$ using the Mess map. We shall see that the pull-back of the canonical cotangent bundle symplectic structure on $T^*\mathcal{T}(1)$ to $\mathcal{T}(1) \times \mathcal{T}(1)$ coincides with the difference of Weil-Petersson symplectic structures coming from each copy of $\mathcal{T}(1)$. Thus, the generalized Mess map is symplectic. We do computations by comparing the symplectic structures at the origin of both spaces. The result at an arbitrary point should then follow using the group structure of the universal Teichmüller space $\mathcal{T}(1)$, but we shall not attempt to demonstrate this in the present paper. Instead, the next section shows the Chern-Simons $SL(2, R)$ connections are respectively parametrized by μ_\pm , which then implies the gravitational symplectic structure should indeed coincide with the difference of Weil-Petersson symplectic structures in each sector of the theory.

From the Hamiltonian formulation of general relativity, one knows that the pre-symplectic 1-form is given by

$$\Theta = \frac{1}{2} \int_{\Delta} (\mathbb{I}, \delta I)_I da_I = \frac{1}{2} \int_{\Delta} \text{tr}(I^{-1} \mathbb{I} I^{-1} \delta I) da_I.$$

When working in the maximal surface with its first and second fundamental forms given by

$$I = e^{2\varphi} |dz|^2, \quad \mathbb{I} = \frac{1}{2} (qdz^2 + \bar{q}d\bar{z}^2),$$

we have for the first variation of I :

$$\delta I = e^{2\varphi} (\delta \bar{\mu} dw^2 + \delta \mu d\bar{w}^2 + (2\delta\varphi + \partial_w \delta z + \partial_{\bar{w}} \delta \bar{z}) |dw|^2).$$

Here μ is the Beltrami differential describing variations of the conformal structure of the maximal surface. The pre-symplectic 1-form is therefore

$$\Theta = \int_{\Delta} d^2w (q \delta\mu + \bar{q} \delta\bar{\mu}).$$

We see that the holomorphic quadratic differential determining the second fundamental form is canonically conjugated to the variable μ parametrizing the conformal structure of the maximal surface. We note that it is the same computation that is valid in the context of compact spatial sections of AdS₃ manifolds and in our context of asymptotically AdS₃ spacetimes. Taking the variation of the pre-symplectic 1-form we get

$$\Omega = \int_{\Delta} d^2w (\delta q \wedge \delta\mu + \delta\bar{q} \wedge \delta\bar{\mu}),$$

which shows that the symplectic structure induced by the Einstein-Hilbert functional is just the canonical cotangent bundle symplectic structure on $T^*\mathcal{T}(1)$.

Now, using the Mess map we can write the variations of μ and q at the origin (corresponding to AdS₃) in terms of those of μ_{\pm} ,

$$\delta\mu = \frac{1}{2}(\delta\mu_+ + \delta\mu_-), \quad \delta q = \frac{4i}{(1-|w|^2)^2} \delta\bar{v} = \frac{2i}{(1-|w|^2)^2} (\delta\bar{\mu}_+ - \delta\bar{\mu}_-).$$

The gravitational symplectic form, evaluated at the origin of the phase space, therefore becomes

$$\Omega = \frac{1}{2i} \int_{\Delta} \frac{4d^2w}{(1-|w|^2)^2} (\delta\mu_+ \wedge \delta\bar{\mu}_+ - \delta\mu_- \wedge \delta\bar{\mu}_-), \quad (32)$$

which is just a copy of the Weil-Petersson symplectic form in each $\mathcal{T}(1)$. This shows the generalized Mess map $T^*\mathcal{T}(1) \rightarrow \mathcal{T}(1) \times \mathcal{T}(1)$ (at the origin of both spaces) is symplectic. It should be possible to extend this to an arbitrary point by using the group structure of the universal Teichmüller space, see the Appendix, but we shall not attempt this here.

10. Chern-Simons Connections

Finally, in this section, we present a relation between the $\mathcal{T}(1) \times \mathcal{T}(1)$ parametrization of the phase space and Chern-Simons formulation of 2+1 general relativity [9]. We remind the reader that in the first order formalism the variables one works with are a frame field e and a spin connection ω . For negative cosmological constant, these may be combined into a $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ connection $A = (A^+, A^-)$ over spacetime $M = \mathbb{R} \times \Delta$,

$$A_{\mu}^{\pm} = (\omega_{\mu}^a \pm e_{\mu}^a) T_a,$$

where T_a are the generators of $\text{SL}(2, \mathbb{R})$. Here we choose to work with $\text{SU}(1, 1)$ generators

$$T_0 = \frac{i}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad T_1 = \frac{1}{2} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad T_2 = \frac{1}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

so that we have

$$\text{tr}(T_a T_b) = \frac{1}{2} \eta_{ab}, \quad [T_a, T_b] = \epsilon_{ab}{}^c T_c.$$

Written in terms of A^+ and A^- , the Einstein-Hilbert action becomes the difference of two Chern-Simons actions

$$S_{EH}[A^+, A^-] = S_{CS}[A^+] - S_{CS}[A^-].$$

In this sense we say that 2+1 GR is equivalent to Chern-Simons theory. Note, however, that the phase space of Chern-Simons theory, which is the space of all solutions of the equations of motion, is much bigger than that of GR. Basically, some connections define non-invertible frames e so that singular metrics are also included. The gauge group of CS theory also includes some transformations (large gauge transformations) that cannot be considered as gauge from the point of GR. In spite of this, the CS point of view on AdS₃ gravity is very convenient, because it gives the simplest way to understand how the Mess-type description by two copies of the Teichmüller space can be possible.

The CS formulation thus shows there exists a pair of flat $SL(2, \mathbb{R})$ connections associated with any AdS metric. To relate our phase space construction to CS theory we compute the flat $SL(2, \mathbb{R})$ connections associated with the AdS metric parametrized by (f_+, f_-) . Again, it is convenient to start working at the maximal surface. The 3-metric can then be written

$$ds^2 = -d\tau^2 + \cos^2 \tau e^{2\varphi} |dz|^2 + \sin \tau \cos \tau (q dz^2 + \bar{q} d\bar{z}^2) + \sin^2 \tau e^{-2\varphi} |q|^2 |dz|^2,$$

and it is a simple computation to find the associated flat $SL(2, \mathbb{R})$ connections

$$A_z^\pm = \frac{1}{2} \begin{bmatrix} \partial_z \varphi & \mp e^\varphi \\ i e^{-\varphi} q & -\partial_z \varphi \end{bmatrix}, \quad A_{\bar{z}}^\pm = \frac{1}{2} \begin{bmatrix} -\partial_{\bar{z}} \varphi & \mp i e^{-\varphi} \bar{q} \\ \mp e^\varphi & \partial_{\bar{z}} \varphi \end{bmatrix}.$$

Here we have eliminated the τ dependence using a gauge transformation. Recalling that the Liouville field and the holomorphic quadratic differential can be written, in terms of F_\pm , as

$$e^{2\varphi} = \frac{4|\partial F_+|^2}{(1 - |z|^2)^2}, \quad q dz^2 = i \text{Hopf}(F_+),$$

we need just another gauge transformation $a_\pm \rightarrow g^{-1} a_\pm g + g^{-1} dg$, with

$$g = \begin{bmatrix} (\partial_z F_\pm / |\partial_z F_\pm|)^{-1/2} & 0 \\ 0 & (\partial_z F_\pm / |\partial_z F_\pm|)^{1/2} \end{bmatrix},$$

to see the connections decouple. A pull-back to the base disc then gives us

$$A_w^\pm = \frac{1}{(1 - |z_\pm|^2)} \begin{bmatrix} \frac{1}{2}(\bar{z}_\pm \partial_w z_\pm - z_\pm \partial_w \bar{z}_\pm) & \mp \partial_w z_\pm \\ \mp \partial_w \bar{z}_\pm & -\frac{1}{2}(\bar{z}_\pm \partial_w z_\pm - z_\pm \partial_w \bar{z}_\pm) \end{bmatrix},$$

$$A_{\bar{w}}^\pm = \frac{1}{(1 - |z_\pm|^2)} \begin{bmatrix} -\frac{1}{2}(\bar{z}_\pm \partial_{\bar{w}} z_\pm - z_\pm \partial_{\bar{w}} \bar{z}_\pm) & \mp \partial_{\bar{w}} \bar{z}_\pm \\ \mp \partial_{\bar{w}} z_\pm & \frac{1}{2}(\bar{z}_\pm \partial_{\bar{w}} z_\pm - z_\pm \partial_{\bar{w}} \bar{z}_\pm) \end{bmatrix},$$

and we see that each copy of $\mathcal{T}(1)$ parametrizes one of the CS sectors, as expected.

We may then compute the Chern-Simons pre-symplectic structure in this parametrization. Let's work with a single $SL(2, \mathbb{R})$ Chern-Simons theory on $\mathbb{R} \times \Delta$ for the

moment. We again compute the symplectic structure at the base point in $\mathcal{T}(1)$. Then, a flat $SL(2, \mathbb{R})$ connection is simply given by

$$A = \frac{1}{(1 - |w|^2)} \begin{bmatrix} \frac{1}{2}(\bar{w}dw - w d\bar{w}) & -dw \\ -d\bar{w} & -\frac{1}{2}(\bar{w}dw - w d\bar{w}) \end{bmatrix}.$$

Its variation in the direction of a tangent vector $\delta\mu \in T_{[0]}\mathcal{T}(1)$ is then easily obtained

$$\delta A_w = \frac{1}{2} \begin{bmatrix} -\frac{1}{2}\partial_w(\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}) & -\frac{\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}}{(1-|w|^2)} \\ -\frac{2\delta\bar{\mu}}{(1-|w|^2)} & \frac{1}{2}\partial_w(\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}) \end{bmatrix},$$

$$\delta A_{\bar{w}} = \frac{1}{2} \begin{bmatrix} -\frac{1}{2}\partial_{\bar{w}}(\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}) & -\frac{2\delta\mu}{(1-|w|^2)} \\ \frac{\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}}{(1-|w|^2)} & \frac{1}{2}\partial_{\bar{w}}(\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}) \end{bmatrix}.$$

Here we made use of identity (13) as well as

$$\partial_w\delta\mu = -\frac{2\bar{w}\delta\mu}{(1 - |w|^2)},$$

which follow directly from the representation of Harmonic Beltrami coefficients in terms of holomorphic quadratic differentials.

The pre-symplectic structure is then obtained by restriction from the natural symplectic structure on the space of all connections,

$$\Omega_{CS} = \int_S \text{tr}(\delta A \wedge \delta A) = i \int_S d^2w \text{tr}(\delta A_w \wedge \delta A_{\bar{w}}).$$

Up to a boundary term, this is simply the Weil-Petersson symplectic structure

$$\Omega_{CS} = i \int_S d^2w \frac{\delta\bar{\mu} \wedge \delta\mu}{(1 - |w|^2)^2} + \frac{1}{2} \int_{\partial S} (\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w}) \wedge d(\partial_w\delta w - \partial_{\bar{w}}\delta\bar{w})$$

in agreement with the results of the previous section.

11. Discussion

In this paper we described an explicit parametrization of a large class of AdS₃ manifolds by two copies of the universal Teichmüller space $\mathcal{T}(1)$. Our construction proceeds by first determining the first and second fundamental forms on the maximal surface that corresponds to a given point in $\mathcal{T}(1) \times \mathcal{T}(1)$, and then evolving this initial data using (3) to get the spacetime metric. We note that only half of the data in $\mathcal{T}(1) \times \mathcal{T}(1)$ is needed to get the maximal surface geometric initial data. The other half of the phase space coordinates determines a complex structure on the maximal surface, which may or not coincide with the complex structure of the isothermal complex coordinate on this surface. This non-geometric half of the initial data can be interpreted as determining how the maximal surface is foliated by $|w| = \text{const}$ curves while the other, geometric half, determines the curve along which the maximal surface intersects the boundary at infinity, see Fig. 1. We have also seen that an equally good description of the same class

of spacetimes is provided by $T^*\mathcal{T}(1)$, and that the generalized Mess map between the two descriptions is a symplectomorphism.

We have then studied the relation between the maximal surface description of AdS_3 spacetimes given in this paper and the more standard holographic description by the Fefferman-Graham expansion of the spacetime metrics (14). We have only been able to give an infinitesimal relation between two such metrics that are close to the standard metric on AdS_3 . However, the interpretation of such relation is a natural one. Namely, we interpret the phase space of AdS_3 spacetimes as a deformation space of a given fixed reference spacetime. On one hand, in the usual holographic description we deform each asymptotic region of the reference spacetime with the group of asymptotic symmetries generated by nontrivial Brown-Henneaux vector fields. On the other, with the maximal surface parametrization, we consider quasiconformal deformations of the associate pair of hyperbolic surfaces obtained via the generalized Mess map. These are generated by harmonic Beltrami coefficients and the relation (17) is simply identifying these generators with the Brown-Henneaux vector fields. It is thus expected that such relation between generators can be extended to finite transformations thus identifying, to a certain extent, the asymptotic and quasiconformal deformation spaces. Note that this cannot be a one-to-one identification since, although enough to describe all possible asymptotically AdS_3 metrics in a neighbourhood of conformal infinity, the Brown-Henneaux vector field does not contain the bulk moduli, that is, they do not fix the internal spacetime topology. This is the main advantage of the new parametrization proposed in this work as the maximal surface data also provides such moduli.

Even without a general finite relation between the descriptions, we were able to obtain an expression (31) for the charges (mass and angular momentum) of a spacetime in terms of data on the maximal surface. This expression admits an immediate generalization to the finite case, where the charges would simply be given by the real parts of the periods of the holomorphic quadratic differentials arising from the Bers embedding of $\mathcal{T}(1) \times \mathcal{T}(1)$. It would be very interesting to see that this is indeed the case for a general metric from our family. We leave this to future work. We have also shown that our description in terms of $\mathcal{T}(1) \times \mathcal{T}(1)$ is natural in terms of the Chern-Simons description of AdS_3 gravity, in that the two Chern-Simons connections corresponding to our AdS_3 metrics decouple with each being parametrized by a single copy of $\mathcal{T}(1)$.

The natural question that arises is what our constructions can add to the debate as to the microscopic origin of the entropy of 2+1 dimensional black holes. Here we can only give some speculations on this issue. As we have already mentioned in the Introduction, it seems sensible to approach the problem of quantum gravity in 2+1 dimensions as the problem of quantization of the moduli space of 2+1 dimensional constant curvature manifolds. In the context of the negative cosmological constant all fixed spatial topology moduli spaces are realized as submanifolds of the universal moduli space described in the present work. The universal space therefore includes all possible multi black holes, together with the Brown-Henneaux excitations in each of their asymptotic regions. It also includes all compact spatial slice spacetimes (in this case one should simply take the initial data to be invariant under a Fuchsian group of a compact surface), but these spacetimes are unlikely to be relevant to the problem of BH entropy. One can then reformulate the question of computing the BH entropy as that of computing the partition function over all possible multi-black-hole spacetimes with fixed mass and angular momentum of one of the asymptotic regions. The entropy could then be extracted from this ‘‘canonical’’ partition function by the standard thermodynamic formulas. Our (infinitesimal case) expression (31) for the charges is then the first step in this direction.

It would be very interesting if it were possible to reformulate the partition function computation as that in the context of some conformal field theory. In this respect we note that the Gauss-Codazzi equations that arise on the maximal surface in AdS₃ are integrable, and are those of the so-called \mathfrak{sl}_2 affine Toda system. It thus could be that the conformal field theory associated to the \mathfrak{sl}_2 affine Toda is the CFT relevant for the quantum description of AdS₃ gravity. We note that this CFT would naturally live on the maximal surface, not on the asymptotic boundary. But we have seen that the analytic continuation (to the imaginary time) of the functions on the AdS₃ boundary cylinder has a natural interpretation in terms of data on the maximal surface. Thus, it appears that the Euclidean signature CFT on the spatial slice can, when analytically continued, be relevant for the AdS/CFT type description of 2+1 dimensional quantum gravity. Whether any of these speculations have a chance to come out true only future works on the subject can tell.

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12. Appendix A: Universal Teichmüller Theory

We give a very basic introduction to the theory of universal Teichmüller space trying to keep the work as self-contained as possible. Our presentation follows closely the presentations of [24, 28–30].

For a compact Riemann surface Σ the Teichmüller space $\mathcal{T}(\Sigma)$ is defined as the space of conformal structures on Σ modulo (small) diffeomorphisms in the connected component of the identity. This definition is usually described in an equivalent manner in terms of the possible hyperbolic structures on Σ . However, in generalizing the construction of Teichmüller space for noncompact Riemann surfaces with hyperbolic ends, it is important to keep track of the relation between the structures on the interior of the surface and its conformal boundary. The good definition of $\mathcal{T}(\Sigma)$ is then given in terms of quasiconformal deformations of the conformal structure of Σ . More concretely, Teichmüller space is then defined as the space of Beltrami coefficients on Σ up to equivalence relation describing when two Beltrami coefficients define the same conformal structure.

To define the universal Teichmüller space, let $\Delta = \{w \in \hat{\mathbb{C}}; |z| < 1\}$ and $\Delta^* = \{w \in \hat{\mathbb{C}}; |z| > 1\}$ be the unit disc and its exterior in the Riemann sphere $\hat{\mathbb{C}}$ and let

$$L^\infty(\Delta)_1 = \left\{ \mu : \Delta \rightarrow \mathbb{C}; |\mu|_\infty = \sup_{\Delta} |\mu(w)| < 1 \right\},$$

be the unit ball in the space of bounded Beltrami differentials on Δ . We define $\mathcal{T}(1)$ as the space of equivalence classes of such bounded Beltrami coefficients on Δ ,

$$\mathcal{T}(1) = L^\infty(\Delta)_1 / \sim,$$

the equivalence relation being defined as follows.

Model A. Given two bounded Beltrami coefficients $\mu, \nu \in L^\infty(\Delta)_1$ one solves Beltrami equations in \mathbb{C} with coefficients extended to Δ^* by reflection

$$\tilde{\mu}(w) = \begin{cases} \overline{\mu(1/\bar{w})}w^2/\bar{w}^2, & w \in \Delta^*, \\ \mu(w), & w \in \Delta, \end{cases}$$

similarly for ν . Then μ, ν are taken to be equivalent if the corresponding solutions, normalized to fix $-1, -i$ and 1 , agree in \mathbb{S}^1 ,

$$z_\mu|_{\mathbb{S}^1} = z_\nu|_{\mathbb{S}^1}.$$

Model B. Equivalently, one can define the equivalence relation by solving the Beltrami equations in \mathbb{C} with Beltrami coefficients given by

$$\tilde{\mu}(w) = \begin{cases} 0, & w \in \Delta^*, \\ \mu(w), & w \in \Delta, \end{cases}$$

similarly for ν . Now, μ, ν are considered equivalent if the corresponding solutions, normalized to have a simple pole of residue 1 at ∞ and to satisfy $z(w) - w \rightarrow 0$ for $w \rightarrow \infty$, agree on Δ^* ,

$$z^\mu|_{\Delta^*} = z^\nu|_{\Delta^*}.$$

The equivalence relations above describe the conformal equivalence classes among the quasiconformal deformations of the conformal structure on Δ . One can, therefore, describe the universal Teichmüller space either as the space of (normalized) quasiasymmetric homeomorphisms on \mathbb{S}^1 or the space of (normalized) univalent functions on Δ^* .

Model B allows an important realization of the Teichmüller space as an embedded subspace of holomorphic quadratic differentials on Δ^* ,

$$A_\infty(\Delta^*) = \{h : \Delta^* \rightarrow \mathbb{C} \text{ holomorphic; } |h(w)(1 - |w|^2)|_\infty < \infty\}.$$

This is the so-called Bers embedding of $\mathcal{T}(1)$ and is obtained via the Schwarzian derivative of $z^\mu|_{\Delta^*}$,

$$\{z^\mu|_{\Delta^*}, w\} = \frac{\partial_w \partial_w \partial_w z^\mu}{\partial_w z^\mu} - \frac{3}{2} \left(\frac{\partial_w \partial_w z^\mu}{\partial_w z^\mu} \right)^2.$$

This defines, in particular, a structure of the complex Banach manifold on $\mathcal{T}(1)$, compatible with the one coming from $L^\infty(\Delta)_1$ via projection.

The space of bounded Beltrami coefficients $L^\infty(\Delta)_1$ also carries a group structure induced by the composition of quasiconformal maps. The group multiplication is defined as $\lambda = \nu * \mu$ iff the following relation is satisfied:

$$(\nu \circ z_\mu) = \frac{\lambda - \mu}{1 - \lambda \bar{\mu}} \frac{\partial_w z_\mu}{\partial_w \bar{z}_\mu}.$$

More explicitly, λ is the Beltrami coefficient of $z_\lambda = z_\nu \circ z_\mu$ and is given by

$$\lambda = \frac{\mu + \nu \circ z_\mu \frac{\partial_w \bar{z}_\mu}{\partial_w z_\mu}}{1 + \bar{\mu} \nu \circ z_\mu \frac{\partial_w \bar{z}_\mu}{\partial_w z_\mu}}.$$

Such group structure also descends to $\mathcal{T}(1)$.

Tangent space. Let's denote by $\Phi : L^\infty(\Delta)_1 \rightarrow \mathcal{T}(1)$ the quotient map sending each bounded Beltrami coefficient μ on Δ into its equivalence class $[\mu] \in \mathcal{T}(1)$. Then, the derivative map $D_\mu \Phi : L^\infty(\Delta) \rightarrow T_{[\mu]}\mathcal{T}(1)$ identifies the tangent space to universal Teichmüller space $T_{[\mu]}\mathcal{T}(1)$ with the quotient space $L^\infty(z_\mu(\Delta))/N(z_\mu(\Delta))$ of the space of Beltrami coefficients on $z_\mu(\Delta)$ by its subspace $N(\Delta_\mu)$ of infinitesimally trivial coefficients, the kernel of $D_\mu \Phi$.

Let's set $\mu = 0$ and consider the tangent space at the base point of $\mathcal{T}(1)$. We denote by $\delta\mu$ an element of $L^\infty(\Delta)$ thought of as a tangent vector at the origin. The infinitesimal version of the Beltrami equation is then given by

$$\partial_{\bar{w}} f = t \delta\mu \partial_w f,$$

and its infinitesimal solutions can be written

$$f_{t\delta\mu}(w) = w + t\delta z + O(t^2), \quad \partial_{\bar{w}} \delta z = \delta\mu.$$

Thus, a tangent vector $\delta\mu \in L^\infty(\Delta)$ defines a one parameter family of quasiconformal transformations from the A model procedure. We shall say that $\delta\mu$ is infinitesimally trivial if, to first order in t , the restriction to \mathbb{S}^1 of this family is given just by the identity transformation, that is, if the corresponding variation $\delta z|_{\mathbb{S}^1}$ vanishes identically. This, of course, means that $\delta\mu$ does not change the conformal structure on Δ and, therefore, that it is in the kernel $N(\Delta)$ of the derivative map $D_0 \Phi : L^\infty(\Delta) \rightarrow T_{[0]}\mathcal{T}(1)$.

The infinitesimally trivial condition can be given different characterizations. It can be shown, see [29], that $\delta\mu \in N(\Delta)$ is equivalent to

$$\int_{\Delta} d^2 w h \delta\mu = 0,$$

for any holomorphic quadratic differential $h \in A_\infty(\Delta)$. Here, we define the space of holomorphic quadratic differentials as

$$A_\infty(\Delta) = \left\{ h : \Delta \rightarrow \mathbb{C} \text{ holomorphic; } |h(w)(1 - |w|^2)^2|_\infty < \infty \right\},$$

and we may write

$$N(\Delta) = \left\{ \delta\mu \in L^\infty(\Delta); \int_{\Delta} d^2 w h \delta\mu = 0, \forall h \in A_\infty(\Delta) \right\}.$$

The map $D_0 \Phi : L^\infty(\Delta) \rightarrow T_{[0]}\mathcal{T}(1)$ thus identifies the tangent space $T_{[0]}\mathcal{T}(1)$ with the space of harmonic Beltrami differentials on Δ ,

$$\Omega^{-1,1}(\Delta) = \left\{ \delta\mu = -\frac{(1 - |w|^2)^2}{2} \overline{h(w)}; h \in A_\infty(\Delta) \right\},$$

since the space $L^\infty(\Delta)$ can be decomposed as

$$L^\infty(\Delta) = N(\Delta) \oplus \Omega^{-1,1}(\Delta).$$

One may also think of the family $f_{t\delta\mu}$ as the one-parameter flow of the vector field $\delta z \partial_w$. Its restriction to \mathbb{S}^1 is

$$\delta z \partial_w|_{\mathbb{S}^1} = u(e^{i\theta}) \partial_\theta$$

with

$$u = \frac{\delta z(e^{i\theta})}{ie^{i\theta}} = \sum_{k \neq -1,0,1} u_k e^{ik\theta}.$$

This is an element of the so-called Zygmund class on \mathbb{S}^1 , $\Lambda(\mathbb{S}^1)$, defined by

$$\Lambda(\mathbb{S}^1) = \left\{ u : \mathbb{S}^1 \rightarrow \mathbb{R} \text{ continuous such that } A_u \in \Lambda(\mathbb{R}) \right\},$$

where $A_u(x) = \frac{1}{2}(x^2 + 1)u\left(\frac{x-i}{x+i}\right)$ and

$$\Lambda(\mathbb{R}) = \{A : \mathbb{R} \rightarrow \mathbb{R} \text{ continuous such that } |A(x+t) + A(x-t) - 2A(x)| \leq \kappa|t|, \kappa > 0\}.$$

Note that the coefficients u_{-1}, u_0, u_1 were dropped due to the normalization condition. Consequently, u belongs to the quotient $\Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1)$ and the construction above provide an identification between $T_{[0]}\mathcal{T}(1)$ and the Möbius normalized Zygmund class on \mathbb{S}^1 .

The tangent space to the B-model universal Teichmüller space is obtained similarly by considering the infinitesimal solutions of Beltrami equation. Now, the one-parameter family $f^{t\mu}$ is of the form

$$f^{t\mu}(w) = w + t\delta z(w) + O(t^2),$$

with the function δz being holomorphic on Δ^* . It thus admits an expansion in Δ^* ,

$$f^{t\mu}(w) = w \left(1 + \frac{tc_2}{w^2} + \frac{tc_3}{w^3} + \dots \right).$$

The associated Bers embedding holomorphic quadratic differential can also be expanded in Δ^* as

$$h(w) = \frac{1}{w^4} \left(h_0 + \frac{h_1}{w} + \frac{h_2}{w^2} + \dots \right),$$

where the coefficients h_k are related to those of δz via

$$h_{k-2} = c_k(k - k^3), \quad k \geq 2.$$

The relation between the coefficients in the A and B models now becomes quite simple

$$u_k = i\bar{c}_k = i \frac{\bar{h}_{k-2}}{(k - k^3)}, \quad k \geq 2,$$

see [28] for a proof. This could in fact be expected from the identification of $T_{[0]}\mathcal{T}(1)$ with the space of the harmonic Beltrami coefficient $\Omega^{-1,1}(\Delta)$, in which the infinitesimal Beltrami coefficient is given in terms of a dual holomorphic quadratic differential $q \in A_\infty(\Delta)$. Writing

$$q(w) = \sum_{k \geq 0} q_k w^k$$

for the Laurent expansion of q , one can explicitly find the A model δz by integration. For $w \in \Delta$ we get

$$\delta z(w) = \frac{1}{2} \sum_{k \geq 2} \frac{\bar{q}_{k-2} \bar{w}^{k-1}}{(k-k^3)} \left[k(k+1) - 2(k^2-1)|w|^2 + k(k-1)|w|^4 \right] + F(w),$$

where F is some holomorphic function on Δ , and, by reflection symmetry

$$\delta z(w) = -w^2 \overline{\delta w(1/\bar{w})},$$

for $w \in \Delta^*$

$$\delta z(w) = -\frac{1}{2} \sum_{k \geq 2} \frac{q_{k-2} \bar{w}^{-k-1}}{(k-k^3)} \left[k(k+1)|w|^4 - 2(k^2-1)|w|^2 + k(k-1) \right] - w^2 \overline{F(1/\bar{w})}.$$

We may then write F as

$$F(w) = \sum_{k \geq 0} v_k w^k$$

and restrict δz to \mathbb{S}^1 to get

$$\begin{aligned} \delta z(e^{i\theta}) &= \sum_{k \geq 2} \frac{\bar{q}_{k-2} e^{-(k-1)i\theta}}{(k-k^3)} + v_0 + v_1 e^{i\theta} + v_2 e^{2i\theta} + \sum_{k \geq 2} v_{k+1} e^{(k+1)i\theta} \\ &= -\sum_{k \geq 2} \frac{q_{k-2} e^{(k+1)i\theta}}{(k-k^3)} - \bar{v}_0 e^{2i\theta} - \bar{v}_1 e^{i\theta} - \bar{v}_2 - \sum_{k \geq 2} \bar{v}_{k+1} e^{-(k-1)i\theta} \end{aligned}$$

so $v_0 = -\bar{v}_2, v_1 = -\bar{v}_1, v_{k+1} = iu_k$ for $k \geq 2$. Thus

$$\begin{aligned} \delta z(w) &= \frac{1}{2} \sum_{k \geq 2} \frac{\bar{q}_{k-2} \bar{w}^{k-1}}{(k-k^3)} \left[k(k+1) - 2(k^2-1)|w|^2 + k(k-1)|w|^4 \right] \\ &\quad + v_0 + v_1 w + v_2 w^2 - \sum_{k \geq 2} \frac{q_{k-2} w^{k+1}}{(k-k^3)}, \quad w \in \Delta. \end{aligned}$$

Note that, because of the normalization condition imposing δz to vanish at $-1, -i$ and 1 , the coefficients v_0, v_1, v_2 are completely determined by the $u_k, k \geq 2$. In fact, the Möbius group is realized exactly as

$$\text{Möb}(\mathbb{S}^1) \approx \left\{ u(e^{i\theta}) = \bar{u}_1 e^{-i\theta} + u_0 + u_1 e^{i\theta} \right\}$$

and, therefore, those coefficients are gauge. From now on we will drop the coefficients, understanding that they acquire the necessary values to make δz vanish at $-1, -i, 1$.

We can now read off the Fourier coefficients of the Zygmund function u . With our choice of coefficients for q as above, u is simply given by

$$u(e^{i\theta}) = \sum_{k \neq -1, 0, 1} u_k e^{ik\theta},$$

with

$$u_k = i \frac{q_{k-2}}{(k - k^3)}, \quad u_{-k} = \bar{u}_k.$$

In particular, the dual quadratic differential to $\delta\mu$ relates to the Bers embedding quadratic differential by the simple reflection rule,

$$q(w) \in A_\infty(\Delta) \mapsto h(w) = \overline{q(1/\bar{w})} \frac{1}{w^4} \in A_\infty(\Delta^*).$$

Weil-Petersson hermitian metric. The almost complex structure at the origin of $\mathcal{T}(1)$ is most clear from the model B point of view in which $J : T_{[0]}\mathcal{T}(1) \rightarrow T_{[0]}\mathcal{T}(1)$ is just

$$Jh = ih.$$

By the isomorphism above described we have the almost complex structure

$$Ju = i \sum_{k \neq -1, 0, 1} \text{sgn}(k) u_k e^{in\theta}$$

on the space of normalized Zygmund class functions.

The Weil-Petersson hermitian metric on Teichmüller space of Riemann surfaces can be easily generalized to a (formal) hermitian metric on universal Teichmüller space. Explicitly, given $\delta\mu, \delta\nu \in T_{[0]}\mathcal{T}(1)$ we define

$$\langle \delta\mu, \delta\nu \rangle_{\text{WP}} = \int_{\Delta} \frac{4d^2w}{(1 - |w|^2)^2} \delta\mu(w) \delta\bar{\nu}(w).$$

Bers embedding then gives

$$\langle h, q \rangle_{\text{WP}} = \int_{\Delta} d^2w (1 - |w|^2)^2 h(w) \bar{q}(w),$$

for $h, q \in A_\infty(\Delta^*)$, and the isomorphism above $\Lambda(\mathbb{S}^1)/\text{Möb}(\mathbb{S}^1) \rightarrow A_\infty(\Delta^*)$,

$$\langle u, v \rangle_{\text{WP}} = \sum_{k, l \geq 2} k(k^2 - 1)l(l^2 - 1)u_k \bar{v}_l \int_{\Delta} d^2w (1 - |w|^2)^2 w^{k-2} \bar{w}^{l-2}.$$

Using

$$\int_{\Delta} d^2w (1 - |w|^2)^2 w^{k-2} \bar{w}^{l-2} = \int_{\Delta} dr d\theta (1 - r^2)^2 r^{k+l-3} e^{i(k-l)\theta} = \frac{2\pi}{k(k^2 - k)} \delta_{kl},$$

we get the Weil-Petersson symplectic structure in terms of the coefficients of the Zygmund functions,

$$\langle u, v \rangle_{\text{WP}} = 2\pi \sum_{k \geq 2} k(k^2 - 1)u_k \bar{v}_l.$$

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