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A local version of Bando's theorem on the real-analyticity of solutions to the Ricci flow

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Abstract

It is a theorem of Bando that if $g(t)$ is a solution to the Ricci flow on a compact manifold M, then $(M, g(t))$ is real-analytic for each $t > 0$. In this note, we extend his result to smooth solutions on open domains $U \subset M$.

1. *Introduction*

Suppose that U is an open subset of $M = M^n$. We consider a smooth solution $g(t)$ to the Ricci flow

$$\frac{\partial}{\partial t} g = -2\,\mathrm{Rc}(g) \tag{1.1}$$

on $U \times [0, T]$. The purpose of this note is to establish the following result.

Theorem 1.1. *For $0 < t \leqslant T$, $(U, g(t))$ is a real-analytic manifold.*

More precisely, about each point $p \in U$ there is a neighborhood on which the expression of $g(t)$ in geodesic normal coordinates is real-analytic. This is a result of Bando [1] when M is compact, and his argument extends, essentially without change, to the case when $(M, g(t))$ is complete and of uniformly bounded curvature. Our aim is to eliminate the global assumptions on the metric, and verify that, as is typical of parabolic equations, instantaneous analyticity in the spatial variables is a purely local phenomenon.

From Theorem 1.1 and a classical monodromy-type argument (cf. [7, Corollary 6.4]), it is then automatic to obtain the following qualitative unique-continuation results for complete solutions (of possibly unbounded curvature).

Corollary 1.2. *Suppose that M is connected and simply connected, and $g(t)$, $\tilde{g}(t)$ are complete solutions to the Ricci flow on $M \times (a, b)$. Let $t_0 \in (a, b)$.*

(1) *If $g(\cdot, t_0) = \tilde{g}(\cdot, t_0)$ on a connected open set $U \subset M$, then there exists a diffeomorphism $\phi : M \to M$ such that $g(t_0) = \phi^* \tilde{g}(t_0)$.*

(2) *Any local isometry, $\tilde{\phi} : (U, g|_U(t_0)) \to (V, g|_V(t_0))$, between connected open sets $U, V \subset M$ can be uniquely extended to a global isometry $\phi : M \to M$.*

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Since any local Ricci soliton may be transformed into a local (self-similar) solution to Ricci flow, Theorem 1.1 also provides a new proof of the real-analyticity of Ricci solitons, a fact that can be proved, much as for Einstein metrics (cf. [**4**, **6**]), by the use of harmonic coordinates.

COROLLARY 1.3. *Suppose that (M, g, X, λ) is a Ricci soliton, that is,*

$$\mathrm{Rc}(g) + \mathcal{L}_X g + \lambda g = 0$$

for some smooth vector field X and scalar λ. Then (M, g) is a real-analytic manifold.

Note that since $\Delta X = -\mathrm{Rc}(X)$ on a Ricci soliton, it follows that the representation of the vector field X in geodesic normal coordinates will also be analytic.

Theorem 1.1 will be a consequence of the following estimate. Here and below, R denotes the Riemann curvature tensor, $T_* \doteqdot \max\{T, 1\}$, and $\Omega(p, r)$ and $\Omega(p, r, T)$ denote, respectively, $B_{g(0)}(p, r)$ and $\Omega(p, r) \times [0, T]$.

THEOREM 1.4. *Suppose that $g(x, t)$ is a smooth solution to the Ricci flow on the open set $U \subset M^n$ for $t \in [0, T]$. Let $p \in U$ and $\rho > 0$ such that $\overline{\Omega(p, 3\rho)}$ is compactly contained in U and define $M_0 \doteqdot \sup_{\Omega(p,3\rho,T)} |R|_{g(t)}$. Then there exist positive constants C, N, and τ depending only on n, ρ, T_* and M_0 such that, for all $m \in \mathbb{N} \cup \{0\}$,*

$$t^{m/2} |\nabla^m R|_{g(t)}(x, t) \leqslant C N^{m/2} (m + 1)! \tag{1.2}$$

on $\Omega(p, \rho, \tau)$.

The estimates (1.2) are variants of the well-known local estimates of Shi [**10**] and Hamilton [**5**] (see also [**8**, **9**]), which likewise take the form

$$t^m |\nabla^m R|^2 \leqslant C(n, m, \rho, M_0, T).$$

The only new content is an explicit accounting of the dependency of the constants $C(n, m, \rho, M_0, T)$ on the order m, a dependency that can be safely ignored in many applications and, that is, consequently, rather obscure in the variants of the estimates of which we are aware. In fact, it is an interesting question whether, for example, the constants generated inductively in Shi's argument are of sufficiently slow growth in m to ensure the real-analyticity. The application of the heat operator to the quantity $|\nabla^m R|^2$, the subsequent commutation of the Laplacian with the m-fold covariant derivative, and the m-fold differentiation of the reaction terms on the right-hand side of $(\partial/\partial t - \Delta)R = R * R$ together generate a number of lower-order terms that grow with m. In order to control these terms, we found it easiest to use a localized modification of Bando's original quantity

$$\varphi = \sum_{k=0}^{m} \frac{t^k}{((k+1)!)^2} |\nabla^k R|^2,$$

whose evolution equation can be arranged, as in the global case, to produce a comparison with the solution of an appropriate ordinary differential equation.

2. *Proof of the local estimates*

For the remainder of this paper, we will work in the setting of the statement of Theorem 1.4. We first describe our cut-off function.

LEMMA 2.1. *Under the assumptions of Theorem* 1.4, *there exists a cut-off function* $\eta : U \to [0,1]$ *that is compactly supported in* $\Omega(p,2\rho)$, *satisfies* $\eta \equiv 1$ *on* $\Omega(p,\rho)$, *and whose derivatives satisfy*

$$|\nabla\eta|^2_{g(t)} - \eta\Delta_{g(t)}\eta \leqslant C_0\eta \tag{2.1}$$

on $U \times [0,T]$ *for some constant* $C_0 = C_0(n,\rho,T_*,M_0)$.

Proof. For the proof, see, for example, [3, Lemma 14.4]. The key is that the local curvature bound implies the uniform equivalence of the metrics $g(t)$ on $\bar{\Omega}(p,3\rho,T)$ and the local first derivative estimate of either Shi [10] or Hamilton [5] supplies a bound of the form

$$\sup_{\bar{\Omega}(p,2\rho,T)} t|\nabla R|^2_{g(t)}(x,t) \leqslant C(n,\rho,T_*,M_0),$$

and these, together, are sufficient to control the $g(t)$-gradient and Laplacian of the cut-off function. □

Although η is constructed by composition with a Riemannian distance function, namely, that of the initial metric $g(0)$, we may, as usual, on account of Calabi's trick [2], regard the resulting function as smooth for the purpose of applying the maximum principle. (Alternatively, at the outset of what follows, we may simply decrease ρ if necessary to ensure that $3\rho < \mathrm{inj}_p(g(0))$.)

Now we introduce the principal quantity in our estimate, a simple modification of that introduced by Bando [1]. We define

$$A_k \doteqdot \left(\frac{t}{N}\right)^{k/2} \frac{|\nabla^k R|}{(k+1)!}, \quad \phi_k \doteqdot \eta^{k+1} A_k^2$$

for $k = 0,1,2,\ldots$ and

$$B_k \doteqdot \left(\frac{t}{N}\right)^{(k-1)/2} \frac{|\nabla^k R|}{k!}, \quad \psi_k \doteqdot \eta^k B_k^2$$

for $k = 1,2,3,\ldots$. We then compute

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi_k = \eta^{k+1}\left(\frac{\partial}{\partial t} - \Delta\right)A_k^2 + A_k^2\left(\frac{\partial}{\partial t} - \Delta\right)\eta^{k+1} - 2\langle\nabla\eta^{k+1}, \nabla A_k^2\rangle. \tag{2.2}$$

We will split our computations into two cases, depending as $k \geqslant 1$ or $k = 0$. We consider first the case $k \geqslant 1$. As in [1], the quantity A_k^2 can be seen to satisfy

$$\left(\frac{\partial}{\partial t} - \Delta\right)A_k^2 \leqslant -2B_{k+1}^2 + \frac{k}{N(k+1)^2}B_k^2 + C_1 M_0 A_k^2 + \frac{C_1 t}{N}S_k, \tag{2.3}$$

where $C_1 = C_1(n)$ is independent of k and

$$S_k \doteqdot \sum_{i=0}^{k-1} \frac{A_i B_{k-i} B_k}{i+2}.$$

Using (2.3) and $0 \leqslant \eta \leqslant 1$, the first term of (2.2) thus satisfies

$$\eta^{k+1}\left(\frac{\partial}{\partial t} - \Delta\right)A_k^2 \leqslant -2\psi_{k+1} + \frac{1}{N}\psi_k + C_1 M_0\phi_k + \frac{C_1 t}{N}\theta_k$$

for $t \leqslant T$, where

$$\theta_k \doteqdot \eta^{k+1/2} S_k = \sum_{i=0}^{k-1} \frac{\phi_i^{1/2}\psi_{k-i}^{1/2}\psi_k^{1/2}}{i+2}.$$

Then, since

$$\begin{aligned}-\Delta\eta^{k+1} &= -k(k+1)\eta^{k-1}|\nabla\eta|^2 - (k+1)\eta^k\Delta\eta\\ &\leqslant (k+1)C_0\eta^k,\end{aligned}$$

and $(k+1)A_k = \sqrt{t/N}B_k$, the second term of (2.2) satisfies

$$A_k^2\left(\frac{\partial}{\partial t} - \Delta\right)\eta^{k+1} \leqslant (k+1)C_0\eta^k A_k^2 \leqslant \frac{C_0 T}{N(k+1)}\psi_k. \tag{2.4}$$

Finally, on $\operatorname{supp}\eta$, the last term of (2.2) may be estimated by

$$\begin{aligned}-2\langle\nabla\eta^{k+1}, \nabla A_k^2\rangle &\leqslant 4\frac{(k+1)}{((k+1)!)^2}\left(\frac{t}{N}\right)^k \eta^k|\nabla\eta||\nabla^{k+1}R||\nabla^k R|\\ &\leqslant 4\frac{|\nabla\eta|}{\eta^{1/2}}\sqrt{\frac{t}{N}}\left(t^{k/2}\eta^{(k+1)/2}\frac{|\nabla^{k+1}R|}{(k+1)!}\right)\left(t^{(k-1)/2}\eta^{k/2}\frac{|\nabla^k R|}{k!}\right)\\ &\leqslant \psi_{k+1} + \frac{4C_0 t}{N}\psi_k.\end{aligned} \tag{2.5}$$

Taken together, (2.3)–(2.5) imply that, for $k \geqslant 1$,

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi_k \leqslant -\psi_{k+1} + \frac{C_2}{N}\psi_k + C_1 M_0\phi_k + \frac{C_1 t}{N}\theta_k, \tag{2.6}$$

where $C_2 = C_2(n, M_0, \rho, T_*)$. In the case $k = 0$, we may estimate $\phi_0 = \eta|R|^2$ in the same way, obtaining

$$\left(\frac{\partial}{\partial t} - \Delta\right)\phi_0 \leqslant -\psi_1 + C_1 M_0\phi_0 + C_3 \tag{2.7}$$

for some constant $C_3 = C_3(n, M_0, \rho, T_*)$.

Now we define

$$\Phi_m \doteqdot \sum_{k=0}^m \phi_m, \quad \Psi_m \doteqdot \sum_{k=1}^m \psi_k, \quad \Theta_m \doteqdot \sum_{k=1}^m \theta_k.$$

Provided we choose $N = N(n, M_0, \rho, T_*)$ suitably large ($N > 2 \times \max(C_1, C_2)$ is sufficient), equations (2.6) and (2.7) combine to produce the estimate

$$\left(\frac{\partial}{\partial t} - \Delta\right)\Phi_m \leqslant -\frac{1}{2}(\Psi_m - t\Theta_m) + C_4(\Phi_m + 1) \tag{2.8}$$

for $C_4 = C_1 M_0 + C_3$. Before we apply the maximum principle, it remains only to estimate Θ_m, and this may be done just as for the corresponding quantity in [**1**] (see also [**3**, Chapter 13.2]); we reproduce the estimate here for completeness:

$$\begin{aligned}\Theta_m^2 &= \left(\sum_{k=1}^m\sum_{i=0}^{k-1}\frac{1}{i+2}\phi_i^{1/2}\psi_{k-i}^{1/2}\psi_k^{1/2}\right)^2\\ &\leqslant \sum_{k=1}^m\left(\sum_{i=0}^{k-1}\frac{1}{i+2}\phi_i^{1/2}\psi_{k-i}^{1/2}\right)^2\sum_{k=1}^m\psi_k\end{aligned}$$

$$\leqslant \sum_{k=1}^{m} \left\{ \left(\sum_{i=0}^{k-1} \frac{1}{(i+2)^2} \right) \left(\sum_{i=0}^{k-1} \phi_i \psi_{k-i} \right) \right\} \Psi_m$$
$$\leqslant \Phi_m \Psi_m^2,$$

where we have used that $\sum_{i=0}^{\infty} 1/(i+2)^2 < 1$. So $\Theta_m \leqslant \Phi_m^{1/2} \Psi_m$, and returning to (2.8), we have

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Phi_m \leqslant -\frac{1}{2} \Psi_m (1 - t\Phi_m^{1/2}) + C_4(\Phi_m + 1). \tag{2.9}$$

For the time being, let τ_m denote

$$\tau_m = \sup\{a \in [0, T] \mid t\Phi_m^2(x, t) \leqslant 1 \text{ for all } (x, t) \in U \times [0, a]\}.$$

We will soon show that there exists a constant $\tau = \tau(n, M_0, \rho, T_*) > 0$ for which $\tau_m \geqslant \tau$ for all m, but for now, simply note that, owing to the compact support of each $\Phi_m(\cdot, t)$ in $\Omega(p, 2\rho)$, we at least have $\tau_m > 0$ for all m.

Let

$$F(t) = (M_0^2 + 1) \exp(C_4 t) - 1,$$

so that F solves $F' = C_4(F + 1)$ with $F(0) = M_0^2$. The function $\Upsilon_m \doteqdot \Phi_m - F$ then satisfies $\Upsilon_m \leqslant 0$ on the parabolic boundary of $\Omega(p, 2\rho, T)$ and

$$\left(\frac{\partial}{\partial t} - \Delta \right) \Upsilon_m \leqslant 0$$

on $\Omega(p, 2\rho, \tau_m)$. Thus, on $\Omega(p, 2\rho, \tau_m)$ we have, by the maximum principle,

$$\Phi_m(x, t) \leqslant F(t) \leqslant (M_0^2 + 1) \exp(C_4 T) \doteqdot C(n, \rho, T_*, M_0). \tag{2.10}$$

But it is clear now that if τ is the lesser of T and $C^{-1/2}$, then $t^2 \Phi_m(x, t) \leqslant 1$ for $t \leqslant \tau$. So we have $\tau_m \geqslant \tau$ for any m. From (2.10), it follows in particular that, for all $m \geqslant 0$ and $(x, t) \in \Omega(p, \rho, \tau)$, we have

$$t^m |\nabla^m R|^2(x, t) = t^m \eta^{m+1}(x) |\nabla^m R|^2(x, t) \leqslant C N^m ((m+1)!)^2,$$

which is the estimate (1.2). Hence, $g(x, t)$ is real-analytic (in geodesic coordinates) at (p, t) for any $0 < t \leqslant \tau$. Iterating this argument proves the same for any $t \in (0, T]$, and it follows that $(U, g(t))$ is real-analytic for any $0 < t \leqslant T$.

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