Frame dragging and Eulerian frames in General Relativity

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The physical interpretation of cold dark matter perturbations is clarified by associating Bertschinger's Poisson gauge with a Eulerian/observer's frame of reference. We obtain such an association by using a Lagrangian approach to relativistic cosmological structure formation. Explicitly, we begin with the second-order solution of the Einstein equations in a synchronous/comoving coordinate system—which defines the Lagrangian frame, and transform it to a Poissonian coordinate system. The generating vector of this coordinate/gauge transformation is found to be the relativistic displacement field. The metric perturbations in the Poissonian coordinate system contain known results from standard/Eulerian Newtonian perturbation theory, but contain also purely relativistic corrections. On sub-horizon scales these relativistic corrections are dominated by the Newtonian bulk part. These corrections however set up non-linear constraints for the density and for the velocity which become important on scales close to the horizon. Furthermore, we report the occurence of a transverse component in the displacement field, and find that it induces a non-linear frame dragging as seen in the observer's frame, which is sub-dominant at late-times and sub-horizon scales. Finally, we find two other gauges which can be associated with a Eulerian frame. We argue that the Poisson gauge is to be preferred because it comes with the simplest physical interpretation.

I. INTRODUCTION

It is believed that the large-scale structure (LSS) of the Universe is the result of gravitational instability. The governing evolution equations are provided by general relativity (GR), although the simpler Newtonian theory yields reasonable estimates at most scales of interest. Exact analytic solutions—for generic initial conditions and without any symmetry, in both GR and Newton theory are not possible, so one has to use either numerical approaches (Newtonian N-body simulations) or analytical approximations (cosmological perturbation theory [1–6]; CPT). In CPT the equations of motion of the cold dark matter (CDM) particles are usually solved within the irrotational-fluid-dust approximation, which restricts the validity of the approach to sufficiently large scales. The (additional) use of the Newtonian approximation, on the other hand, is assumed to be valid only on interaction scales well below the causality bound. To study the evolution of perturbations close to the causality bound, a relativistic treatment becomes mandatory.

We should seek for a relativistic treatment accompanied with a direct correspondence to the Newtonian solutions. Only such a treatment is capable to deliver straightforward physical interpretations, since one can parametrise the relativistic corrections as deviations from the Newtonian bulk part. A close correspondence becomes increasingly important especially when studying "gauge-dependent" (i.e., frame-dependent) quantities as we shall do in the following.

Lagrangian perturbation theory (LPT) is a promising avenue of the gravitational instability, mostly since it is an intrinsically non-linear approach to non-linear structure formation, but also as it is required to set up initial conditions for N-body simulations. Additionally, the Lagrangian representation comes with a simple physical interpretation as one follows simply the trajectories of fluid particles. The only dynamical quantity in Newtonian LPT is the displacement field $\boldsymbol{F}$, which parametrises the gravitationally induced deviation of the fluid particle from its initial Lagrangian position $\boldsymbol{q}$. The Newtonian coordinate transformation to the Eulerian coordinate $\boldsymbol{x}$ is

$$\boldsymbol{x}(t, \boldsymbol{q}) = \boldsymbol{q} + \boldsymbol{F}(t, \boldsymbol{q}). \tag{1}$$

Newtonian LPT has inspired hundreds of works since Refs. [7–10]. Explicit solutions up to third order in LPT were derived in Refs. [11, 12]. The fourth-order scheme in LPT has been derived in Ref. [13], and a general recursion relation in LPT has been reported in Ref. [14]. Important improvements about LPT related to convergence issues were recently given in Refs. [15, 16].

Significant efforts have been made to obtain a general relativistic generalisation of LPT, see e.g. [17–20]. Recently, we obtained a relativistic generalisation of LPT [21, 22] from a somewhat different perspective than the aforementioned references; we identified LPT in terms of a coordinate transformation of a perturbed synchronous metric—resulting from a relativistic gradient expansion, to a Eulerian/Newtonian coordinate system. This perspective offers a unique interpretation of gauge transformations in GR, a perspective we shall further develop in the following. Furthermore, by including not only scalar perturbations but also vector and tensor perturbations, we generalise the findings of [21, 22].

Identifying and interpreting relativistic effects within cosmological structure formation are the two key objectives we shall study in this *paper*. We specifically focus on relativistic effects of the density and the velocity fields. To understand this *paper* it is very helpful to recall that

the density and velocity are Eulerian fields and not Lagrangian fields. The density and velocity field are frame dependent, and only the Eulerian density and the Eulerian velocity—evaluated at the Eulerian position, are the observable quantities.[1] Thus, the interpretation of the density and velocity field is inherently linked with the proper identification of the Eulerian frame. In the Newtonian approximation the identification of the Eulerian frame is trivial, and the connection to the Lagrangian frame is given by the coordinate transformation (1). This identification is however non-trivial in the relativistic generalisation; there is generally no preferred coordinate system in GR, and as a consequence there is no single frame which could be labelled as Eulerian. As we shall see in the following, in GR there exists a class of coordinate systems which can be associated with a Eulerian frame. The essential idea here is to use the Newtonian correspondence from LPT to identify "a" Eulerian frame in GR, preferably a Eulerian frame accompanied with simple physical interpretations (which turns out to be the one associated with the Poisson gauge). Thus, fairly analogous to the Newtonian coordinate transformation (1) we define its relativistic counterpart to be

$$x^\mu(t, \boldsymbol{q}) = q^\mu + F^\mu(t, \boldsymbol{q}), \qquad (2)$$

where μ are the four space-time components (since it is the four-dimensional line element which is invariant in GR); $x^\mu \doteq (\tau, \boldsymbol{x})$ and $q^\mu \doteq (t, \boldsymbol{q})$ are the Eulerian and Lagrangian coordinates, respectively, and $F^\mu \doteq (L, \boldsymbol{F})$ is the relativistic displacement field. Thus, the displacement field now consists not only of a spatial but also of a temporal part. This is nothing but the statement that space and time are on an equal footing in GR. Physically it means that space and time will mix due to the non-linear clustering. The coordinate transformation (2) is the central building block to formulate a relativistic LPT. Our procedure to obtain a relativistic LPT can be summarised as follows:

(1.) find a relativistic solution in a synchronous/comoving coordinate system;

(2.) identify the corresponding frame to be Lagrangian;

(3.) use Eq. (2) to find F^μ and the metric perturbations in the "new" coordinate system with coordinates x^μ; and to

(4.) identify the very coordinate system to be a Eulerian frame, if the metric potentials and the displacement field agree with Newtonian results (at least) in the weak-field limit.

We shall use this procedure to find all Eulerian frames.

One important application of this paper is certainly related to N-body simulations. Some investigations have been made on Newtonian N-body simulations and its compatibility with GR [26, 27]. An explicit recipe to interpret N-body results with respect to GR at linear order in CPT was first given in Ref. [28]. It is also known that GR yields an initial constraint for the density field beyond leading order [22, 29–31], although its interpretation and practical implementation is still in its beginning [22, 25]. Here we seek to gain further understanding of this issue. Moreover, we report the occurence of an additional non-linear constraint coming from GR, which only affects the velocity field and not the density field at second order. Specifically, we obtain a non-zero transverse component in the Lagrangian displacement field which is the result of the non-linear motion of the fluid particle. In the Eulerian frame this phenomenon appears as a non-linear frame-dragging.

In general, the occurence of a non-zero transverse component in the relativistic Lagrangian displacement field is expected to happen at some order in perturbation theory, even within the restrictive class of an irrotational motion in an Eulerian coordinate system—a restriction we also consider here. Indeed, similar considerations within the Newtonian limit of LPT with equivalent initial conditions were studied in detail (e.g. [11, 12, 32]), and a non-zero transverse displacement field was found at third-order in Newtonian LPT. This transverse displacement field can be interpreted as a fictitious force, very similar to the Coriolis force, induced through a non-inertial particle motion [32]. The transverse displacement field therefore corrects the motion of the CDM particle, and it is thus essential to include it in the analysis—neglecting it would formally lead to wrong results[2] as has been shown in the Newtonian analysis of Ref. [33]. The same is obviously true for the general relativistic treatment.

This *paper* is organised as follows. In section II we review the metric up to second order in a synchronous coordinate system, which was obtained in Refs. [20–22]. This metric serves as the starting point for the current investigation. Explicitly, the synchronous/comoving metric contains vector and tensor perturbations which are excited from scalar perturbations at the linear level. The coordinate transformation to a Poissonian coordinate system, described in section III, will then be used to obtain a physical interpretation of the perturbations in the observer's frame. In section IV we explain how to solve the coordinate transformation with an iterative treatment, and we also define useful operators which are needed in the latter. Then, we report the first-order and second-order results of the transformation in sections V and VI, respectively. In Section VII we report a proce-

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[1] Here we neglect geometrical and dynamical distortions coming from the propagation of the photons in a clumpy and expanding Universe. We also neglect biasing effects in this *paper*.

[2] The neglection of the transverse displacement field thus yields a violation of the irrotationality condition, i.e., in the Newtonian limit $\boldsymbol{\nabla}_{\boldsymbol{x}} \times \boldsymbol{u} \neq \boldsymbol{0}$, where $\boldsymbol{u}$ is the particle velocity.

dure to identify all possible Eulerian gauges. Explicitly, we find three (non-trivial) gauge choices which can be associated with a Eulerian frame, and we clarify their physical interpretations. We summarise and conclude afterwards in Section VIII. We also wish to highlight the appendix where we relate our findings to the Newtonian approximation.

II. THE METRIC PERTURBATIONS IN A SYNCHRONOUS COORDINATE SYSTEM

In this section we review the relativistic solution in a synchronous/comoving coordinate system, which is defined to be the Lagrangian frame.[3] Here we only introduce our conventions and report the final result for an irrotational CDM component up to second order; explicit calculations can be found in e.g. Ref. [17], in particular Ref. [20] for the tetrad formalism, and Refs. [21, 22] for the gradient expansion technique.

The corresponding comoving/synchronous line element for the Lagrangian frame is

$$ds^2 = -dt^2 + a^2(t)\,\gamma_{ij}(t, \boldsymbol{q})\,dq^i dq^j\,, \tag{3}$$

where t is the proper time of the CDM particles and $\boldsymbol{q}$ are comoving/Lagrangian coordinates, constant for each pressureless and irrotational CDM fluid element; $a(t)$ is the cosmological scale factor which is proportional to $t^{2/3}$ for an Einstein-de Sitter (EdS) universe. Summation over repeated indices is implied—for latin indices from 1 to 3, and for greek indices from 0 to 3. Inflation predicts at linear order the initial seed metric

$$k_{ij} = \delta_{ij}\left[1 + \frac{10}{3}\Phi(\boldsymbol{q})\right]\,, \tag{4}$$

where $\Phi(\boldsymbol{q})$ is the primordial Newtonian potential, given at initial time t_0. In our case $\Phi(\boldsymbol{q})$ is just a Gaussian field, and it is directly related to Bardeen's gauge-invariant potential [1]. Here and in the following a ", i" denotes a differentiation with respect to Lagrangian coordinate q_i.

Solving the Einstein equations and the Bianchi identities with the use of the initial seed metric k_{ij} and some iterative technique, we obtain for an EdS universe up to second order

$$\gamma_{ij}(t, \boldsymbol{q}) = \delta_{ij}\left(1 + \frac{10}{3}\Phi\right) + 3a(t)t_0^2\left[\Phi_{,ij}\left(1 - \frac{10}{3}\Phi\right) - 5\Phi_{,i}\Phi_{,j} + \frac{5}{6}\delta_{ij}\Phi_{,l}\Phi_{,l}\right]$$
$$- \left(\frac{3}{2}\right)^2 \frac{3}{7}a^2(t)t_0^4\left[4\Phi_{,ll}\Phi_{,ij} - \delta_{ij}\left(\Phi_{,ll}\Phi_{,mm} - \Phi_{,lm}\Phi_{,lm}\right)\right] + \left(\frac{3}{2}\right)^2 \frac{19}{7}a^2(t)t_0^4\,\Phi_{,li}\Phi_{,lj} + \chi_{ij}(t, \boldsymbol{q})\,, \tag{5}$$

where we have retained only the fastest growing mode solutions.[4] The divergence-free and trace-free tensor $\chi_{ij}(t, \boldsymbol{q})$ is of order Φ^2, and it results from the magnetic part of the Weyl tensor—its explicit form is not needed in the following but see e.g. [17, 20]. Note that χ_{ij} is not determined by the gradient expansion of Refs. [21–23]. The inclusion of χ_{ij} in the following does not change our conclusions, and we just include it for the sake of generality.

III. THE COORDINATE TRANSFORMATION

To obtain the relativistic displacement field and the perturbations in the Eulerian frame we perform a coordinate transformation. We transform the result (5) written in the synchronous/comoving gauge with coordinates $(t, \boldsymbol{q})$,

$$ds^2 = g_{\mu\nu}(t, \boldsymbol{q})\,dq^\mu dq^\nu = -dt^2 + a^2(t)\gamma_{ij}(t, \boldsymbol{q})\,dq^i dq^j\,, \tag{6}$$

to the Poisson gauge with coordinates $(\tau, \boldsymbol{x})$ and corresponding metric (τ is *not* the conformal time)

$$ds^2 = g_{\tilde{\mu}\tilde{\nu}}(\tau, \boldsymbol{x})\,dx^{\tilde{\mu}}dx^{\tilde{\nu}} = -\left[1 + 2A(\tau, \boldsymbol{x})\right]d\tau^2 + 2a(\tau)w_i(\tau, \boldsymbol{x})\,d\tau dx^i + a^2(\tau)\left\{\left[1 - 2B(\tau, \boldsymbol{x})\right]\delta_{ij} + S_{ij}(\tau, \boldsymbol{x})\right\}dx^i dx^j\,. \tag{7}$$

A, B, w_i, and S_{ij} are supposed to be small perturbations. The tensor S_{ij} is traceless, i.e. $S^i_{\ i} = 0$. The Poisson gauge is defined via [3, 34, 37]

$$\partial^{x_i} w_i = 0\,,$$
$$\partial^{x_i} S_{ij} = \partial^{x_i} S_{ij}^{\mathrm{T}} = 0\,, \qquad \text{(gauge conditions).} \quad (8)$$

These conditions hold also in the perturbative sense. The two frames are related by the coordinate transformation

$$x_\mu(t, \boldsymbol{q}) = q_\mu + F_\mu(t, \boldsymbol{q})\,, \qquad (9)$$

with

$$x_\mu = \begin{pmatrix} \tau \\ \boldsymbol{x} \end{pmatrix}, \qquad q_\mu = \begin{pmatrix} t \\ \boldsymbol{q} \end{pmatrix}, \qquad F_\mu = \begin{pmatrix} L \\ \boldsymbol{F} \end{pmatrix}, \quad (10)$$

where $L(t, \boldsymbol{q})$ and $\boldsymbol{F}(t, \boldsymbol{q})$ are supposed to be small perturbations. $\boldsymbol{F}$ is the spatial part of the relativistic Lagrangian displacement field, and L is the time perturbation—in the case of the Poisson gauge, L is the velocity potential of the CDM particle (i.e., this is generally not true for other Eulerian gauges, see section VII). Note explicitly, that L contains only the potential part of the particle's velocity, thus the full 3-velocity field is given by the time-derivative of the 3-displacement field, i.e., $\boldsymbol{u} = a\,\partial \boldsymbol{F}/\partial t$. We decompose $\boldsymbol{F}$ into a curl-free and divergence-free vector field

$$\boldsymbol{F}(t, \boldsymbol{q}) = \boldsymbol{F}^{\|}(t, \boldsymbol{q}) + \boldsymbol{F}^{\perp}(t, \boldsymbol{q})\,, \qquad (11)$$

and without loss of generality we choose to decompose it with respect to the Lagrangian coordinate system with corresponding coordinates q^μ.

General covariance requires the invariance of the line element $\mathrm{d}s^2$, and thus:

$$g_{\mu\nu}(t, \boldsymbol{q}) = \frac{\partial x^{\tilde\mu}}{\partial q^\mu} \frac{\partial x^{\tilde\nu}}{\partial q^\nu} g_{\tilde\mu\tilde\nu}(\tau, \boldsymbol{x})\,. \qquad (12)$$

We shall solve the above general coordinate transformation perturbatively, whilst expanding all fields and dependences.

IV. ITERATIVE SOLUTION SCHEME AND USEFUL PROJECTION OPERATORS

The general coordinate transformation (12) gives separate equations for the space-space, space-time and time-time parts, which can be used to constrain the parameters $(A, B, w_i, S_{ij}, L, F_i)$. We solve these equations order by order. Formally, each small quantity is expanded in a series, i.e.

$$A = \epsilon A^{(1)} + \epsilon^2 A^{(2)} + \dots\,,$$
$$B = \epsilon B^{(1)} + \epsilon^2 B^{(2)} + \dots\,,$$

etc., where ϵ is supposed to be a small dimensionless parameter. The primordial potential Φ is of order ϵ. For convenience we truncate the coordinate transformation of the metrics, Eq. (12), up to second order and suppress the perturbation parameter ϵ in the following. After some manipulations we find for Eq. (12)

$$\gamma_{ij}(t, \boldsymbol{q}) \simeq -\frac{L_{,i} L_{,j}}{a^2} + 2\frac{L_{,(i} w_{j)}}{a} + \delta_{ij}\left[1 - 2B(\tau, \boldsymbol{x}) + \frac{4L(t, \boldsymbol{q})}{3t} + \frac{2L^2}{9t^2} - \frac{8BL}{3t}\right]$$
$$+ 2F_{(i,j)}(t, \boldsymbol{q})\left(1 - 2B + \frac{4L}{3t}\right) + F_{l,i} F_{l,j} + S_{ij}(\tau, \boldsymbol{x}) + 2F_{l,(i} S_{j)l}\,, \qquad (13)$$

$$0 \simeq -(1 + 2A)\,L_{,i} - L_{,i}\frac{\partial L}{\partial t} + a^2(t)\left[1 - 2B + \frac{4L}{3t}\right]\frac{\partial F_i(t, \boldsymbol{q})}{\partial t} + a^2 S_{im}\frac{\partial F_m}{\partial t} + a^2 F_{l,i}\frac{\partial F_l}{\partial t}$$
$$+ a(t)\,w_i(\tau, \boldsymbol{x})\left[1 + \frac{2L}{3t} + \frac{\partial L}{\partial t}\right]\,, \qquad (14)$$

$$-1 \simeq -1 - 2A(\tau, \boldsymbol{x}) - 2\frac{\partial L(t, \boldsymbol{q})}{\partial t} - 4A\frac{\partial L}{\partial t} - \left(\frac{\partial L}{\partial t}\right)^2 + 2a\,w_l\frac{\partial F_l}{\partial t} + a^2\frac{\partial F_l}{\partial t}\frac{\partial F_l}{\partial t}\,. \qquad (15)$$

We have suppressed some dependences where there is no confusion, i.e., dependences in second-order terms can be interchanged, and the resulting error is only of third order.

We solve Eqs. (13)–(15) with an iterative technique. For that purpose the following decomposition valid for any tensor T_{ij} is useful [24]

$$T_{ij} = \frac{\delta_{ij}}{3}\hat{Q} + \left(\partial_i\partial_j - \frac{\delta_{ij}}{3}\boldsymbol{\nabla}^2\right)\hat{T}^{\|} + 2\hat{T}^{\perp}_{(i,j)} + \hat{T}^{\mathrm{T}}_{ij}, \quad (16)$$

where $\hat{Q}$ is the trace of T_{ij}, $\hat{T}^{\perp}_i$ is a divergence-free vector, and for the transverse traceless tensor we have $\partial^j\hat{T}^{\mathrm{T}}_{ij} =$

0. It is then straightforward to define the corresponding projection operators

$$\hat{T}^{\parallel} = \frac{3}{2} \frac{\partial^i \partial^j}{\nabla^2 \nabla^2} T_{ij} - \frac{1}{2} \frac{1}{\nabla^2} \hat{Q} \,,$$
$$\varepsilon^{kli} \hat{T}^{\perp}_{i,l} = \frac{1}{\nabla^2} \varepsilon^{kli} \partial^j \partial^l T_{ij} \,, \tag{17}$$

where ε^{kli} is the Levi-Civita symbol, and $1/\nabla^2$ is the inverse Laplacian.

With the above we can extract the relevant information from Eqs. (13)–(15) to obtain

1. the Lagrangian displacement field $\boldsymbol{F} = \boldsymbol{F}^{\parallel} + \boldsymbol{F}^{\perp}$. The operator $\hat{T}^{\parallel}$ applied to Eq. (13) constrains the longitudinal part of the displacement field $\boldsymbol{F}^{\parallel}$, whereas $\varepsilon^{kli} \hat{T}^{\perp}_{i,l}$ constrains its transverse part, $\boldsymbol{F}^{\perp}$.

2. The divergence of Eq. (14) constrains the time perturbation L.

3. The scalar perturbation B is obtained by the trace-part of Eq. (13).

4. The curl of Eq. (14) constrains the vector perturbation $\boldsymbol{w}$.

5. From Eq. (15) we obtain the scalar perturbation A.

In the following we solve the coordinate transformation with that procedure, order by order.

Before proceeding it is worthwile to compare this procedure with other methods in the literature. One crucial extension in this procedure compared to the one in [21, 22] is the consistent inclusion of the transverse displacement field $\boldsymbol{F}^{\perp}$ in the coordinate transformation. $\boldsymbol{F}^{\perp}$ has to be included since $\partial^i S_{ij} \equiv 0$, but the same is *generally not true* for the divergence of Eq. (13). Thus, $\boldsymbol{F}^{\perp}$ absorbes the transverse part of Eq. (13), and relates it to $\boldsymbol{w}$ via Eq. (14). Indeed, the following identity is valid at least up to second order: $\boldsymbol{w} \equiv -a \, \partial \boldsymbol{F}^{\perp}/\partial t$. In Refs. [21, 22] the coordinate transformation is performed from the synchronous gauge to the Newtonian gauge— instead to the Poisson gauge which is the generalisation of the Newtonian gauge. In the Newtonian gauge vector and tensor perturbations are set to zero by hand, thus $\boldsymbol{w} := \boldsymbol{0}$ and so is $\boldsymbol{F}^{\perp}$. This is however rather accidental, and there is generally no reason to discard the transverse component of the displacement field.

V. SOLUTIONS UP TO FIRST ORDER IN THE POISSON GAUGE

With the use of the above recipe and with the gauge conditions (8), we obtain up to first order for Eq. (9)

$$F^{(1)}_{\mu}(t, \boldsymbol{q}) = \begin{pmatrix} L^{(1)} \\ \boldsymbol{F}^{(1)\parallel} + \boldsymbol{F}^{(1)\perp} \end{pmatrix} \,, \tag{18}$$

with

$$L^{(1)}(t, \boldsymbol{q}) = \Phi(\boldsymbol{q}) \, t \,, \tag{19}$$

$$F^{(1)\parallel}_i(t, \boldsymbol{q}) = \frac{3}{2} a(t) t_0^2 \, \partial_{q_i} \Phi(\boldsymbol{q}) \,, \tag{20}$$

$$F^{(1)\perp}_i(t, \boldsymbol{q}) = 0 \,, \tag{21}$$

and for the scalar, vector and tensor perturbations respectively

$$A^{(1)}(\tau, \boldsymbol{x}) = B^{(1)}(\tau, \boldsymbol{x}) = -\Phi(\boldsymbol{x}) \,,$$
$$w^{(1)}_i(\tau, \boldsymbol{x}) = 0 \,, \tag{22}$$
$$S^{(1)}_{ij}(\tau, \boldsymbol{x}) = 0 \,.$$

$F^{(1)}_i$ is the displacement field in the Zel'dovich approximation [7]; since it is purely longitudinal the particle's trajectory is just along the overall potential flow. $L^{(1)}$ is the peculiar velocity potential of the CDM particle, and the perturbations $A^{(1)}$ and $B^{(1)}$ in the Poisson gauge, Eq. (22), are in agreement with the weak-field limit of general relativity. Thus, we recover the Newtonian approximation at linear order. Note that we have interchanged in (22) the dependence of Φ such that $\boldsymbol{q} \to \boldsymbol{x}$, which is only valid up to first order. At second order we simply have to Taylor expand the dependence $\Phi(\boldsymbol{q}) \simeq \Phi(\boldsymbol{x} - \boldsymbol{F}^{(1)})$, in accordance with the coordinate transformation (9).

VI. SOLUTIONS UP TO SECOND ORDER IN THE POISSON GAUGE

Similar considerations can be made up to second order. We obtain the second-order quantities

$$F^{(2)}_{\mu}(t, \boldsymbol{q}) = \begin{pmatrix} L^{(2)} \\ \boldsymbol{F}^{(2)\parallel} + \boldsymbol{F}^{(2)\perp} \end{pmatrix} \,, \tag{23}$$

with

$$L^{(2)}(t, \boldsymbol{q}) = \frac{3}{4} t^{5/3} t_0^{4/3} \Phi_{,l} \Phi_{,l} - \frac{9}{7} t^{5/3} t_0^{4/3} \frac{1}{\nabla^2_{\boldsymbol{q}}} \mu_2$$
$$- \frac{7}{6} t \Phi^2 + 4tC \,, \tag{24}$$
$$F^{(2)\parallel}_i(t, \boldsymbol{q}) = - \left(\frac{3}{2} \right)^2 \frac{3}{7} a^2(t) t_0^4 \frac{1}{\nabla^2_{\boldsymbol{q}}} \partial_{q_i} \mu_2$$
$$- 5a(t) t_0^2 \partial_{q_i} \Phi^2 + 6a(t) t_0^2 \partial_{q_i} C \,, \tag{25}$$
$$F^{(2)\perp}_i(t, \boldsymbol{q}) = 6a(t) t_0^2 R_i \,, \tag{26}$$

and the scalar, vector and tensor perturbations respectively up to second order (for convenience we include the first-order perturbations)

$$A(\tau, \boldsymbol{x}) \simeq \phi_{\mathrm{N}} - 4\overline{C} \,, \qquad B(\tau, \boldsymbol{x}) \simeq \phi_{\mathrm{N}} + \frac{8}{3} \overline{C} \,,$$
$$w_i(\tau, \boldsymbol{x}) = -4\tau^{1/3} t_0^{2/3} \overline{R}_i \,, \qquad S^{(2)}_{ij}(\tau, \boldsymbol{x}) = \overline{\chi}_{ij} \,, \tag{27}$$

with

$$\mu_2(t, \boldsymbol{q}) \equiv \frac{1}{2} \left(\Phi_{,ll} \Phi_{,mm} - \Phi_{,lm} \Phi_{,lm} \right), \tag{28}$$

$$C(t, \boldsymbol{q}) \equiv \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{q}}^2 \boldsymbol{\nabla}_{\boldsymbol{q}}^2} \left[\frac{3}{4} \Phi_{,ll} \Phi_{,mm} + \Phi_{,m} \Phi_{,llm} + \frac{1}{4} \Phi_{,lm} \Phi_{,lm} \right], \tag{29}$$

$$R_i(t, \boldsymbol{q}) \equiv \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{q}}^2 \boldsymbol{\nabla}_{\boldsymbol{q}}^4} \left[\Phi_{,il} \Phi_{,mml} - \Phi_{,lli} \Phi_{,mm} + \Phi_{,i} \Phi_{,llmm} - \Phi_{,m} \Phi_{,mlli} \right], \tag{30}$$

$$\phi_{\mathrm{N}}(\tau, \boldsymbol{x}) \equiv -\Phi(\boldsymbol{x}) + \frac{3}{2} a t_0^2 \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{x}}^2} \left[\frac{5}{7} \Phi_{|ll} \Phi_{|mm} + \Phi_{|l} \Phi_{|lmm} + \frac{2}{7} \Phi_{|lm} \Phi_{|lm} \right]. \tag{31}$$

The definitions of $\overline{C}$, $\overline{R}_i$, etc. are identical to C, R_i, etc., but with dependences and derivatives interchanged to $(\tau, \boldsymbol{x})$ rather than $(t, \boldsymbol{q})$. We denote spatial derivatives with respect to the Eulerian coordinate x_i with a slash. In A and B we have neglected terms proportional to Φ^2 which are not enhanced by spatial gradients.

The first term in Eq. (25) contains the second-order improvement from Newtonian LPT, whereas the remnant terms are the same relativistic corrections as in [21, 22]. Equation (24) is the velocity potential of the displacement field. The transverse part of the displacement field, Eq. (26), together with the corresponding $\boldsymbol{w}$ is one of our main results:

$$\boldsymbol{w} = -a \frac{\partial \boldsymbol{F}^{\perp}}{\partial t}. \tag{32}$$

This equation (which holds only up to second order) clearly indicates the gravitomagnetic origin of the frame dragging: The frame dragging vector potential $\boldsymbol{w}$ is directly related to the transverse part of the fluid's velocity.[5]

Since initial conditions are set within the linear regime, where the transverse displacement field is (at least perturbatively) suppressed, the above shows that the frame dragging will grow as soon as non-linearities will form. Therefore, the frame dragging gets enhanced by the non-linearities in the onwarding gravitational evolution. This argument is also valid for the linear frame-dragging which we not consider here since we study only the evolution of irrotational fluids.

Equation (27) contains the result of the above coordinate transformation, as seen in the observer's frame. The expression ϕ_{N}, Eq. (31), matches exactly Newtonian perturbation theory (see the appendix), whereas the remnant terms in A and B denote relativistic corrections which are proportional to the non-local kernel C, Eq. (29). These results agree with the treatment of Ref. [22] in the Newtonian gauge, and therefore generalises their results to the inclusion of vector and tensor perturbations.

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[5] Note that the peculiar velocity of the CDM particle is $\boldsymbol{u} = a \, \partial \boldsymbol{x}/\partial t \equiv a \, \partial \boldsymbol{F}/\partial t$, where the last step follows from $\boldsymbol{x}(t, \boldsymbol{q}) = \boldsymbol{q} + \boldsymbol{F}(t, \boldsymbol{q})$.

A. Origin of the transverse displacement field

At second order the spatial coordinate transformation between the Eulerian and Lagrangian frame is not entirely longitudinal anymore, i.e., the Lagrangian displacement field aquires a non-zero transverse part. Physically, the transverse displacement field is needed to correct for the actual direction of the particle's motion. Technically, the occurence of a transverse displacement field is expected to happen at some order since the coordinate transformation (and thus the fluid's motion as well) is non-linear and non-inertial.

The transverse displacement field is by definition a vector perturbation. It is important to note that the vector perturbation is not generated through the coordinate transformation itself, but it is already non-zero in the synchronous coordinate system. To see this we apply the second operator in (17) on the 3-metric γ_{ij} (see Eq. (5)), i.e., $\varepsilon^{kli} \partial^j \partial^l \gamma_{ij} \equiv \boldsymbol{\nabla}^2 \varepsilon^{kli} \hat{\gamma}_{i,l}^{\perp}$. Then we find the divergenceless vector

$$\hat{\gamma}_i^{\perp} = -5 a t_0^2 R_i + \left(\frac{3}{2} \right)^2 a^2 t_0^4 Q_i, \tag{33}$$

with R_i same as in Eq. (30), and

$$Q_i = \frac{1}{\boldsymbol{\nabla}^2 \boldsymbol{\nabla}^2} \Big[\Phi_{,lmi} \Phi_{,lmjj} - \Phi_{,lmm} \Phi_{,ljji} \\ + \Phi_{,li} \Phi_{,lmmjj} - \Phi_{,lm} \Phi_{,lmjji} \Big]. \tag{34}$$

Since $\hat{\gamma}_i^{\perp}$ is given in a synchronous/comoving coordinate system, where the velocity of the CDM particle is by definition zero, only the transformation to an observer's frame leads to a physical interpretation of the divergenceless vector. The physical interpretation is that the transverse displacement field appears as a non-linear frame dragging in the Eulerian frame.

VII. ARE THERE OTHER EULERIAN FRAMES?

Within the Newtonian approximation there exists only one Eulerian frame. In GR, however, the situation is generally more complicated, just because there are so many

possibilities to choose the coordinate system (i.e., the gauge). One would naturally ask whether other coordinate systems can be identified to be Eulerian (we will also define what we consider as a Eulerian frame). Indeed, we will show in the following that there exist three Eulerian coordinate systems, but we argue that the Poissonian coordinate system is accompanied with the easiest physical interpretation. Thus, the Poissonian coordinate system is a preferred Eulerian frame.

For convenience we restrict to first-order perturbations in the following, and we leave a full second-order treatment for future investigations. We can then neglect vector and tensor perturbations because they are of second-order. As before we define the synchronous/comoving coordinate system with coordinates $(t, \boldsymbol{q})$ to be associated with the Lagrangian frame. The metric perturbations in the Lagrangian frame read

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\Big[\delta_{ij}\Big(1 + \frac{10}{3}\Phi\Big) \\ + 3a(t)t_0^2\Phi_{,ij}\Big]\mathrm{d}q^i\mathrm{d}q^j. \quad (35)$$

Consider the first-order coordinate transformation from the unique Lagrangian frame to *some* Eulerian frame:

$$x^\mu(t, \boldsymbol{q}) = q^\mu + F^\mu(t, \boldsymbol{q}), \quad (36)$$

with

$$x_\mu = \begin{pmatrix} \tau \\ \boldsymbol{x} \end{pmatrix}, \qquad q_\mu = \begin{pmatrix} t \\ \boldsymbol{q} \end{pmatrix}, \qquad F_\mu = \begin{pmatrix} L \\ \boldsymbol{F} \end{pmatrix}, \quad (37)$$

where the corresponding generic scalar-metric of the Eulerian frame—firstly without any gauge-fixing, is

$$\mathrm{d}s^2 = -\big[1 + 2A\big]\mathrm{d}\tau^2 + 2aw_{,i}\,\mathrm{d}\tau\mathrm{d}x^i \\ + a^2\left\{[1 - 2B]\,\delta_{ij} + 2h_{,ij}\right\}\mathrm{d}x^i\mathrm{d}x^j. \quad (38)$$

We follow Ref. [26] and calculate the proper time between two events along a worldline, which reads

$$\int \sqrt{-\mathrm{d}s^2} = \int \mathrm{d}\tau \sqrt{1 + 2A - 2aw_{,i}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} - a^2[(1 + 2B)\delta_{ij} + 2h_{,ij}]\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}} \\ \simeq \int \mathrm{d}\tau \left(1 + A - aw_{,i}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} - \frac{a^2}{2}\delta_{ij}\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\frac{\mathrm{d}x^j}{\mathrm{d}\tau}\right) \equiv \int \mathrm{d}\tau L. \quad (39)$$

In the last steps we only kept terms of $\mathcal{O}(g\frac{\mathrm{d}x^i}{\mathrm{d}\tau})$, where $g \in \{A, B, w, h, \mathrm{d}x^i/\mathrm{d}\tau\}$. The proper time is only extremal if the integrand L satisfies the Euler-Lagrange equation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial L}{\partial\left(\frac{\mathrm{d}x^i}{\mathrm{d}\tau}\right)} = \frac{\partial L}{\partial x^i}. \quad (40)$$

We then obtain up to first order [26]

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(aw_{,i} + a^2\frac{\mathrm{d}x_i}{\mathrm{d}\tau}\right) = -A_{,i} \quad \text{(no gauge fixing)}. \quad (41)$$

This looks almost like Newton's law of motion which is at first-order

$$\frac{\mathrm{d}}{\mathrm{d}\tau}a^2\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = \Phi_{,i} \qquad \text{(Euler equation)}. \quad (42)$$

Again, we did not specify a gauge yet in equation (41). Our aims at this stage are

- to find all possible gauges which lead exactly to the Euler equation (42) at linear order,

- to establish the weak-field limit between the Lagrangian and Eulerian frame.

The former implies that we are only interested in particle trajectories which appear Newtonian-like in the weak-field limit. The latter implies that we have to encode the spatial information of the particle trajectory in the Zel'dovich displacement field. Thus, we set $F_i^{(1)} = \frac{3}{2}at_0^2\Phi_{,i}$ for the coordinate transformation (36), but leave the temporal perturbation L firstly unfixed. Studying the coordinate transformation for the Lagrangian and Eulerian metrics

$$g_{\mu\nu}(t, \boldsymbol{q}) = \frac{\partial x^{\tilde\mu}}{\partial q^\mu}\frac{\partial x^{\tilde\nu}}{\partial q^\nu}g_{\tilde\mu\tilde\nu}(\tau, \boldsymbol{x}), \quad (43)$$

we find only three non-trivial gauge choices which satisfy the above conditions. We thus identify three Eulerian gauges:

(1.) The Newtonian/longitudinal (NL) gauge [1] with: $A \neq 0$, $B \neq 0$, $w = 0$ and $h = 0$.

(2.) The spatially Euclidean (SE) gauge [26] with: $A \neq 0$, $B = 0$, $w \neq 0$ and $h = 0$.

(3.) The synchronous-shear (SS) gauge with: $A = 0$, $B \neq 0$, $w \neq 0$ and $h = 0$.

Here we summarise our findings for the perturbations at first order and discuss them briefly. As mentioned

above the spatial displacement field is for all of these gauges the Zel'dovich displacement field, $F_i^{(1)} = \frac{3}{2}at_0^2\Phi_{,i}$, which immediately fixes $\mathrm{d}x_i/\mathrm{d}\tau \equiv \mathrm{d}F_i/\mathrm{d}\tau$ in the Euler-Lagrange equation (41) as well.

(1.) *Newtonian/longitudinal gauge.* The perturbations in the NL gauge read $A_{\mathrm{NL}} = B_{\mathrm{NL}} = -\Phi$, and the temporal part of the 4-displacement field is $L_{\mathrm{NL}} = \Phi t$. As above, L_{NL} is the velocity potential of the fluid particle, and thus yield a simple physical interpretation of the time-part of the 4-displacement field. Since $w_{\mathrm{N}} = 0$ the Euler-Lagrange equation (41) yields the Euler equation (42), where the cosmological potential on the RHS is solely given by the time perturbation $A_{\mathrm{NL}} \equiv -\Phi$. Note that the Poisson gauge reduces to the Newtonian gauge in the scalar sector.

(2.) *Spatially Euclidean gauge.* The SE gauge was recently discussed in Refs. [26, 35] and is in particular interesting since it does not contain any perturbations in the space-space part of the metric. Thus, the 3-geometry appears Euclidean. The non-zero perturbations are $A_{\mathrm{SE}} = -5/2\Phi$, $w_{\mathrm{SE}} = 3/(2a)\Phi\tau$, and the temporal perturbation is $L_{\mathrm{SE}} = 5/2\Phi t$. Plugging these values into the Euler-Lagrange equation (41), we realise that not only A_{SE} is the cosmological potential but the combination $A_{\mathrm{SE}} + \frac{\mathrm{d}}{\mathrm{d}\tau}aw_{\mathrm{SE}} \equiv -\Phi$. Thus, the particles still move according to Newton's law of motion, but the cosmological potential receives a non-zero contribution from w_{SE}. This feature generally complicates the latter physical interpretation because w_{SE} sources (already at linear order) a perturbation in the expansion rate; additionally, w_{SE} sources the shear as well.[6]

(3.) *Synchronous-shear gauge.* In contrast to the SE gauge, where the perturbations in the space-space part of the metric is zero, the perturbations in the SS gauge are only zero in the temporal part of the metric. The non-zero perturbations read $B_{\mathrm{SS}} = -5/3\Phi + 2L(\boldsymbol{x})/(3\tau)$, $w_{\mathrm{SS}} = [L(\boldsymbol{x}) - \Phi\tau]/a$, and $L_{\mathrm{SS}} \equiv L(\boldsymbol{x})$ is constant in time. The SS gauge has therefore a residual gauge freedom. In Ref. [26] the constant L_{SS} was fixed such that the density and velocity matches exactly results from Newton theory at linear order; they called this specific choice the Newtonian matter gauge. In Ref. [31] the constant L_{SS} was set to zero, and they called it the Eulerian gauge. Independent of the specific choice of L_{SS}, the Euler-Lagrange equation (41) yield the Euler equation within the SS gauge, where the cosmological potential is entirely given in terms of $\frac{\mathrm{d}}{\mathrm{d}\tau}aw_{\mathrm{SS}} \equiv -\Phi$. Similarly to the SE gauge, the SS gauge is flawed with difficulties in the physical interpretation, since the non-zero w_{SS} distorts the Hubble diagrams and also sources cosmic shear. For recent discussions about such issues see Refs. [26, 27].

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[6] The expansion rate and the shear can be defined as the trace and the trace-less part of the extrinsic curvature, respectively [1, 26].

VIII. SUMMARY AND CONCLUSIONS

We found a Eulerian-Lagrangian correspondence within general relativity, which can be used to study the evolution of scalar, vector and tensor perturbations. We have identified the relativistic displacement field $F^\mu(t, \boldsymbol{q})$ in terms of the coordinate/gauge transformation

$$x^\mu(t, \boldsymbol{q}) = q^\mu + F^\mu(t, \boldsymbol{q}), \qquad (44)$$

where $x^\mu = (\tau, \boldsymbol{x})$ are the Eulerian coordinates and $q^\mu = (t, \boldsymbol{q})$ the Lagrangian coordinates. Our starting point is the second-order synchronous metric $\gamma_{ij}(t, \boldsymbol{q})$, given in Eq. (5), which describes the gravitational evolution of an irrotational dust component in an EdS Universe. The reported synchronous metric can be obtained e.g. from the gradient expansion technique [21, 22] or from the tetrad formalism [19, 20]. We have then performed a coordinate transformation to the Poissonian coordinate system

$$\begin{aligned}
\mathrm{d}s^2 = &-\left[1 + 2A\right]\mathrm{d}\tau^2 + 2aw_i\,\mathrm{d}\tau\mathrm{d}x^i \\
&+ a^2\left\{\left[1 - 2B\right]\delta_{ij} + S_{ij}\right\}\mathrm{d}x^i\mathrm{d}x^j,
\end{aligned} \qquad (45)$$

where the perturbations A, B, $\boldsymbol{w}$ and S_{ij} can be found in Eqs. (27)–(31). The 4-displacement field is

$$F^\mu(t, \boldsymbol{q}) = \begin{pmatrix} L^{(1)} + L^{(2)} \\ \boldsymbol{F}^{(1)} + \boldsymbol{F}^{(2)} \end{pmatrix}, \qquad (46)$$

where the respective quantities on the RHS can be found in Eqs. (18) and (23). In the Poissonian coordinate system we identify the weak-field limit for the cosmological potential, and the occurence of known results from Newtonian Eulerian/standard perturbation theory up to second order (cf. appendix). The spatial part of F^μ is the displacement field from Newtonian LPT plus additional relativistic corrections. We also find a transverse part in the spatial displacement field (26) which does not have any Newtonian counterpart (at that order), and it was not reported earlier in the literature. The temporal part of F^μ is the velocity potential from Newtonian LPT plus additional relativistic corrections.

Since we identify known results from Eulerian/standard perturbation theory in the Poissonian coordinate system and since we can relate these results to the synchronous/comoving coordinate system via the Lagrangian displacement field, we conclude that the Poissonian coordinate system can be associated with a Eulerian frame of reference. This has two important consequences. Firstly, the density and velocity in the Poissonian coordinate system have a physical significance in the sense, that the gauge-dependent nature of the density and velocity can be associated with their frame-dependent origin. Stated in another way, since we are able to identify the Poissonian coordinate system with a Eulerian frame of reference, we deduce that the relativistic corrections of the density and velocity

are not gauge artefacts but real and thus measurable for an observer who is in the Eulerian frame at rest. Secondly, our results indicate that the generator of the above coordinate transformation has a direct physical significance, i.e., the generator of the coordinate transformation is the trajectory of the CDM particle. The Lagrangian and Eulerian frames are separated in terms of the displacement field, and these frames move apart from each other according to the fluid's velocity. The reported transverse part in the spatial displacement field yields a non-linear frame-dragging as seen in the Eulerian frame, since the transverse displacement field sources the frame-dragging vector potential $\boldsymbol{w}$ in the Poissonian coordinate system (45).

Our results can be directly incorporated in Newtonian N-body simulations. The reported relativistic corrections appear as non-linear constraints which influence the particle's trajectory at any time during the simulation/gravitational evolution. Since these relativistic corrections are small with respect to the Newtonian bulk part, we think that the Newtonian approximation should be sufficient at weakly non-linear scales. However, the relativistic corrections influence the initial statistics of the density and velocity field especially at scales close to the horizon, as was recently shown in [22, 28]. Thus, the relativistic corrections should be included for generating initial conditions of Newtonian N-body simulations, preferably in terms of the relativistic displacement field as suggested here. Explicitly, the CDM particles are displaced from their initial grid positions according to the spatial displacement field $\boldsymbol{F}(\tau, \boldsymbol{q})$ (note that we use the Eulerian time τ to account for the initial time on the numerical grid [22]). Similarly, the peculiar velocity of the CDM particle at initial time is given by $\boldsymbol{u}(\tau, \boldsymbol{q}) = a(\tau)\partial \boldsymbol{F}(\tau, \boldsymbol{q})/\partial\tau$, and $\boldsymbol{F}$ contains the aforementioned longitudinal and transverse component. The transverse displacement field does not affect the (initial) density field but the (initial) velocity field. Physically, the transverse displacement field corrects for the direction of motion of the CDM particle, and neglecting it

would formally yield wrong initial statistics for the velocity field. Technically, its practical implementation for N-body simulations is straightforward, and exisiting schemes just have to be complemented; explicit recipes to obtain initial displacements and velocities for N-body simulations can be found in [22, 36].

Finally, in Section VII, we have formulated a procedure to find all possible Eulerian gauges. For simplicity we restricted in this part of our analysis to the scalar sector at linear order, and we shall generalise our findings in a forthcoming project. We found that only three gauges yield Newtonian-like trajectories together with the Zel'dovich displacement field (i.e., the weak field limit for the Eulerian and Lagrangian frames): (1) the Newtonian/longitudinal gauge [1] which corresponds to the scalar sector of the Poisson gauge, (2) the spatially Euclidean gauge [26], and (3) the synchronous-shear gauge. We argued that option (1) is preferred since it comes with the easiest interpretation. Options (2) and (3), on the other hand, induce non-trivial perturbations in the trace-part and the trace-less part of the extrinsic curvature, and thus yield distortions to the Hubble diagrams and to the shear, respectively. Phenomenologically, such dominant distortions to the Hubble diagrams can be associated with the gravitational lensing [27], hence options (2) and (3) might be preferred gauge choices in investigations which involve ray-tracing techniques.

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Appendix A: Comparison with the Newtonian treatment

In this appendix (which is based on [38]) we wish to relate our results to the Newtonian approximation. Let $\boldsymbol{x}$ denote the comoving coordinate defined by the rescaling of the physical coordinate $\boldsymbol{r}$ by the cosmic scale factor $a(t)$ ($\equiv (t/t_0)^{2/3}$ for an EdS universe), where t is the cosmic time. The Eulerian equations of motions for self–gravitating dust are governed by momentum conservation, mass conservation and the Poisson equation which are respectively

$$\frac{\partial}{\partial t}\left[a(t)\,\boldsymbol{u}(t,\boldsymbol{x})\right] + \left[\boldsymbol{u}(t,\boldsymbol{x})\cdot\boldsymbol{\nabla}_{\boldsymbol{x}}\right]\boldsymbol{u}(t,\boldsymbol{x}) = -\boldsymbol{\nabla}_{\boldsymbol{x}}\phi(t,\boldsymbol{x})\,,\tag{A1}$$

$$a(t)\frac{\partial\delta(t,\boldsymbol{x})}{\partial t} + \boldsymbol{\nabla}_{\boldsymbol{x}}\cdot\left\{[1+\delta(t,\boldsymbol{x})]\,\boldsymbol{u}(t,\boldsymbol{x})\right\} = 0\,,\tag{A2}$$

$$\boldsymbol{\nabla}_{\boldsymbol{x}}^2\phi(t,\boldsymbol{x}) = \frac{3}{2}H^2(t)\,a^2(t)\,\delta(t,\boldsymbol{x})\,,\tag{A3}$$

where $\boldsymbol{u} = a\,\partial\boldsymbol{x}/\partial t$ is the peculiar velocity of the fluid particle, $H = 2/(3t)$ for an EdS universe, ϕ is the cosmological potential and the density contrast $\delta(t,\boldsymbol{x})$ sepa-

rates the local variation of the mass density $\rho(t,\boldsymbol{x})$ from a global background $\overline{\rho}(t)$: $\rho(t,\boldsymbol{x}) = \overline{\rho}(t)[1+\delta(t,\boldsymbol{x})]$. Furthermore, we demand an irrotational fluid motion: $\boldsymbol{\nabla}_{\boldsymbol{x}}\times\boldsymbol{u} = \boldsymbol{0}$.

A convenient way to solve the above set of equations is to use the Newtonian LPT (e.g. [9, 12, 13] and references in [6]). In Newtonian LPT, the observer follows the trajectories of the individual fluid elements, where each trajectory is encoded in the time–integrated displacement field $\boldsymbol{\Psi}$. (To avoid confusion with the relativistic displacement field, we label the Newtonian one with $\boldsymbol{\Psi}$ instead of $\boldsymbol{F}$). The coordinate mapping from the fluid particles' initial position $\boldsymbol{q}$ plus its gravitationally induced displacement is then given by

$$\boldsymbol{x}(t) = \boldsymbol{q} + \boldsymbol{\Psi}(t,\boldsymbol{q})\,.\tag{A4}$$

The displacement field contains all the dynamical information of the system, and the fluid displacement automatically obeys mass conservation by the relation

$$\delta(t,\boldsymbol{x}) = \frac{1}{\det[\delta_{ij}+\Psi_{i,j}]} - 1\,,\tag{A5}$$

with the Jacobian of the transformation $J = \det[\delta_{ij}+\Psi_{i,j}]$, where "$,j$" denotes a spatial differentiation w.r.t. Lagrangian coordinate q_j, and $i,j,\ldots = 1\ldots3$. In LPT the above relation replaces the mass conservation (A2), where the neglection of an integration constant δ_0 can always be justified in the Newtonian limit, i.e., by a proper set of initial conditions, or by using a different set of Lagrangian coordinates, or by the assumption of an initial quasi–homogeneity, see Ref. [13].

In Newtonian LPT the system (A1)–(A3), together with the irrotationality constraint is solved with a perturbative ansatz for the displacement field $\boldsymbol{\Psi}$, which is supposed to be a small quantity:

$$\boldsymbol{\Psi}(t,\boldsymbol{q}) = \sum_{i=1}^{\infty}\boldsymbol{\Psi}^{(i)}(t,\boldsymbol{q})\,.\tag{A6}$$

Usually, one utilises Newtonian LPT within a restricted class of initial conditions where only one initial data has to be given [9] (this class is of the Zel'dovich type [7]). Then, the initial data at time t_0 is given by the initial gravitational potential $\Phi(t_0,\boldsymbol{q})$ (up to some arbitrary constants) only, which is supposed to be smooth and of order 10^{-5}. Solving the above in NLPT up to second order one finds for the fastest growing solutions [13]:

$$\begin{aligned}\Psi_i(t,\boldsymbol{q}) = &\left(\frac{3}{2}\right)a(t)\,t_0^2\,\Phi_{,i}(t_0,\boldsymbol{q})\\ &-\left(\frac{3}{2}\right)^2\frac{3}{7}a^2(t)\,t_0^4\frac{\partial}{\partial q_i}\frac{1}{\boldsymbol{\nabla}_{\boldsymbol{q}}^2}\mu_2(t_0,\boldsymbol{q}) + \mathcal{O}(\Phi^3)\,,\end{aligned}\tag{A7}$$

where $1/\boldsymbol{\nabla}_{\boldsymbol{q}}^2$ is the inverse Laplacian, and $\mu_2(t_0,\boldsymbol{q}) = 1/2(\Phi_{,ll}\Phi_{,mm}-\Phi_{,lm}\Phi_{,lm})$. Now, what is the effect on

the Poisson equation, specifically, what is the relation between the cosmological potential $\phi(t, \boldsymbol{x})$ and the initial gravitational potential Φ? To see this we plug Eq. (A5) into the Poisson equation (A3), i.e.,

$$\boldsymbol{\nabla}_{\boldsymbol{x}}^2 \phi(t, \boldsymbol{x}) = \frac{2}{3} \frac{a^2(t)}{t^2} \left(\frac{1}{\det[\delta_{ij} + \Psi_{i,j}]} - 1 \right), \qquad \text{(A8)}$$

and with the use of the second-order displacement field (A7) we Taylor expand the RHS. Then we obtain

$$\boldsymbol{\nabla}_{\boldsymbol{x}}^2 \phi(t, \boldsymbol{x}) = -\Phi_{,ll}(t_0, \boldsymbol{q}) - \frac{6}{7} a(t) \, t_0^2 \, \mu_2(t_0, \boldsymbol{q})$$
$$+ \frac{3}{2} a(t) \, t_0^2 \, \Phi_{,ll}(t_0, \boldsymbol{q}) \, \Phi_{,mm}(t_0, \boldsymbol{q}) + \mathcal{O}(\Phi^3). \qquad \text{(A9)}$$

Note that the LHS is an Eulerian quantity, whereas the expressions on the RHS depend on Lagrangian coordinates and Lagrangian derivatives. We expand the dependences and interchange the derivatives (we denote "$|_i$" for the differentiation w.r.t. Eulerian coordinate x_i) on the RHS, and finally multiply the whole equation with a $1/\boldsymbol{\nabla}_{\boldsymbol{x}}^2$. Then we have

$$\phi(t, \boldsymbol{x}) = -\Phi(t_0, \boldsymbol{x}) + \frac{3}{4} a(t) \, t_0^2 \, \Phi_{|l}(t_0, \boldsymbol{x}) \, \Phi_{|l}(t_0, \boldsymbol{x})$$
$$+ \frac{15}{7} a(t) \, t_0^2 \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{x}}^2} \overline{\mu}_2(t_0, \boldsymbol{x}), \qquad \text{(A10)}$$

with $\overline{\mu}_2(t_0, \boldsymbol{x})$ analogue to $\mu_2(t_0, \boldsymbol{q})$ but the dependences and derivatives are w.r.t. $\boldsymbol{x}$. The above has been obtained in reference [23] (though their approach differs from ours). To see its connection to the 'Newtonian literature' we expand the second term on the RHS with $\boldsymbol{\nabla}_{\boldsymbol{x}}^2 / \boldsymbol{\nabla}_{\boldsymbol{x}}^2$ which leads to

$$\phi(t, \boldsymbol{x}) = -\Phi(t_0, \boldsymbol{x}) + \frac{3}{2} a(t) \, t_0^2 \, \mathcal{F}^{-2} \{F_2(t_0, \boldsymbol{x})\}, \quad \text{(A11)}$$

where we have defined

$$\mathcal{F}^{-2} \{F_2(t_0, \boldsymbol{x})\} = \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{x}}^2} \left[\frac{5}{7} \Phi_{|ll} \, \Phi_{|mm} + \Phi_{|l} \, \Phi_{|lmm} \right.$$
$$\left. + \frac{2}{7} \Phi_{|lm} \, \Phi_{|lm} \right]. \quad \text{(A12)}$$

This is nothing but the result expected from standard perturbation theory (SPT) up to second order (see for example Eq. (45) in [6]). Equation (A10) or Eq. (A11) can be interpreted as follows: At leading order the cosmological potential is just proportional to the initial gravitational potential, whereas at second order the temporal

extrapolation of the initial tidal field leads to an "evolving" cosmological potential. Note that expression (A11) is identical with (31), where the latter was obtained in the relativistic coordinate transformation (12).

Similar considerations can be made for the peculiar fluid velocity. We connect the fluid velocity to the initial gravitational potential. Up to second order in conventional Newtonian LPT the fluid motion is purely potential in the Lagrangian frame [11, 12], so we are allowed to introduce a (peculiar) velocity potential $\mathcal{S}$ such that

$$\boldsymbol{u}(t, \boldsymbol{x}) = \frac{\boldsymbol{\nabla}_{\boldsymbol{x}} \mathcal{S}(t, \boldsymbol{x})}{a(t)} \equiv \boldsymbol{\nabla}_{\boldsymbol{r}} \mathcal{S}, \qquad \text{(A13)}$$

and plug it into the Euler equation (A1). The very equation can then be integrated w.r.t. $\boldsymbol{x}$ and it yields the Bernoulli equation [39–41] (it is equivalent to the non–relativistic Hamilton–Jacobi equation, see e.g. [23])

$$\frac{\partial}{\partial t} \mathcal{S}(t, \boldsymbol{x}) + \frac{1}{2a^2(t)} \left[\boldsymbol{\nabla}_{\boldsymbol{x}} \mathcal{S}(t, \boldsymbol{x}) \right]^2 = -\phi(t, \boldsymbol{x}), \quad \text{(A14)}$$

where ϕ is explicitly given in Eq. (A10) up to second order. Here we have set an integration constant $c(t)$ to zero since it can always be absorbed into the velocity potential by replacing $\mathcal{S} \to \mathcal{S} + \int c(t) \, \mathrm{d}t$; so it does not affect the flow [41].

We solve the above differential equation with a recursive technique, assuming the usual series hierarchy within SPT. Then we obtain for the peculiar–velocity potential

$$\mathcal{S}(t, \boldsymbol{x}) = \Phi(t_0, \boldsymbol{x}) \, t - \frac{3}{4} t_0^{4/3} t^{5/3} \, \Phi_{|l} \, \Phi_{|l} - \frac{9}{7} t_0^{4/3} t^{5/3} \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{x}}^2} \overline{\mu}_2$$
$$\equiv \Phi(t_0, \boldsymbol{x}) \, t - \frac{3}{2} t_0^{4/3} t^{5/3} \mathcal{F}^{-2} \{G_2(t_0, \boldsymbol{x})\}, \quad \text{(A15)}$$

with

$$\mathcal{F}^{-2} \{G_2(t_0, \boldsymbol{x})\} = \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{x}}^2} \left[\frac{3}{7} \Phi_{|ll} \, \Phi_{|mm} + \Phi_{|l} \, \Phi_{|lmm} \right.$$
$$\left. + \frac{4}{7} \Phi_{|lm} \, \Phi_{|lm} \right], \qquad \text{(A16)}$$

or interchanging the dependences and derivatives to be Lagrangian

$$\mathcal{S}(t, \boldsymbol{q}) = \Phi(t_0, \boldsymbol{q}) \, t + \frac{3}{4} t_0^{4/3} t^{5/3} \, \Phi_{,l} \, \Phi_{,l} - \frac{9}{7} t_0^{4/3} t^{5/3} \frac{1}{\boldsymbol{\nabla}_{\boldsymbol{q}}^2} \mu_2.$$
$$\text{(A17)}$$

Again, this is the second–order result for the velocity potential from SPT [6]. The expression (A17) is identical with the non-relativistic part in the time perturbation L, see Eq. (24).

In summary, we have calculated the non-relativistic perturbations ϕ and $\mathcal{S}$, which agree exactly with their counterparts in the Poissonian metric (see section VIII).