On symmetries of $\mathcal{N} = (4, 4)$ sigma models on T^4

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ABSTRACT: Motivated by an analogous result for K3 models, we classify all groups of symmetries of non-linear sigma models on a torus T^4 that preserve the $\mathcal{N} = (4, 4)$ super-conformal algebra. The resulting symmetry groups are isomorphic to certain subgroups of the Weyl group of E_8, that plays a role similar to the Conway group for the case of K3 models. Our analysis heavily relies on the triality automorphism of the T-duality group $SO(4, 4, \mathbb{Z})$. As a byproduct of our results, we discover new explicit descriptions of K3 models as asymmetric orbifolds of torus CFTs.

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1 Introduction and summary

Non-linear sigma models on tori provide the basic examples of string theory compactifications. The study of the discrete groups of symmetries and the orbifolds of supersymmetric torus models, both in heterotic or type II superstring theory, have been the subject of intensive study since the eighties [1–3] (see also [4] and references therein). These models represent both a fruitful arena for the formal understanding of string compactifications and are of great interest from a phenomenological viewpoint.

In this paper, we focus on type II non-linear sigma models with target space four dimensional tori T^4 and classify the groups of symmetries that commute with the ('small') world-sheet $\mathcal{N} = (4,4)$ superconformal algebra. This analysis is mainly motivated by the relation between such torus models and non-linear sigma models on K3. There are few K3 models for which an explicit and complete description as conformal field theories is available; among them, torus orbifolds play a fundamental role. The interest in K3 superstring compactifications has received new impetus in the last few years with the discovery of the Mathieu moonshine phenomenon. This conjecture, originated from an observation of Eguchi, Ooguri and Tachikawa (EOT) in [5], proposes a connection between the elliptic genus of K3 and a finite sporadic simple group, the Mathieu group M_{24}. After the original EOT proposal, a considerable amount of evidence in favour of this conjecture has been compiled [6–10] and several different incarnations of the relationship between M_{24} and various string compactifications on K3 have been uncovered [11–19]. Despite the amount of work on the subject, however, no satisfactory explanation of this phenomenon has been provided so far.

One of the most natural approaches towards an interpretation of Mathieu Moonshine is the analysis of the groups of discrete symmetries of K3 models. Orbifolds of non-linear sigma models on T^4 provide some of the simplest examples for such an investigation [20–22]. In [23], all possible groups of discrete symmetries of K3 models that commute with the $\mathcal{N} = (4,4)$ superconformal algebra have been classified. This result can be thought of as a stringy analogue of Mukai's theorem in algebraic geometry [24, 25], where the groups of symplectic automorphisms of K3 surfaces were considered. The rather surprising outcome of this classification is that all such groups of symmetries are subgroups of the Conway group Co_0, the group of automorphisms of the Leech lattice. Although M_{24} itself is a subgroup of Co_0, it does not appear in the list of [23]. As a consequence, no K3 sigma model admits M_{24} as its group of symmetries. It was observed in [26] that, for almost all K3 models, the symmetry group is actually a subgroup of M_{24} and that most of the exceptions correspond to torus orbifolds. The analysis of [26] also showed the existence of a K3 model represented by an asymmetric orbifold of a torus model by a symmetry of order 5.

This amount of work on non-linear sigma models on K3 led to a rather paradoxical situation: in fact, although non-linear sigma models on T^4 are much better understood than K3 models, the classification of the groups of symmetries preserving the $\mathcal{N} = (4,4)$ superconformal algebra was known only for the latter. The goal of this paper is to fill in this gap. This completes the classification for all (known) models with $\mathcal{N} = (4,4)$ superconformal symmetry at central charge $c = \tilde{c} = 6$. Our main result is the following:

The group G of symmetries of a supersymmetric non-linear sigma model on T^4 that commutes with the (small) $\mathcal{N} = (4,4)$ superconformal algebra is

$$G = \left(U(1)^4 \times U(1)^4\right).G_0 \,,$$

where G_0 is one of the following finite subgroups of $SU(2) \times SU(2)$:

   – Geometric groups: $\mathcal{C}_2, \quad \mathcal{C}_4, \quad \mathcal{C}_6, \quad \mathcal{D}_8(2), \quad \mathcal{D}_{12}(3), \quad \mathcal{T}_{24}$

$$
- \text{ Non-geometric groups: } \begin{cases} \mathcal{T}_{24} \times_{C_3} \mathcal{T}_{24}, & \mathcal{I}_{120} \times_{\mathcal{I}_{120}} \bar{\mathcal{I}}_{120}, & \mathcal{O}_{48} \times_{\mathcal{O}_{48}} \bar{\mathcal{O}}_{48}, \\ \mathcal{D}_{12}(3) \times_{\mathcal{C}_4} \mathcal{D}_{12}(3), & \mathcal{D}_{24}(6) \times_{\mathcal{D}_{24}(6)} \mathcal{D}_{24}(6), \\ \mathcal{D}_8(4) \times_{D_4(2)} \mathcal{D}_8(4) \end{cases}
$$

(Here, $\mathcal{C}_n$ denotes the cyclic group of order n, $\mathcal{D}_{4n}(n)$ the binary dihedral group of order $4n$ and $\mathcal{T}_{24}$, $\mathcal{O}_{48}$, $\mathcal{I}_{120}$ the binary tetrahedral, octahedral and icosahedral groups, respectively. Finally, $\mathcal{L} \times_{\mathcal{F}} \mathcal{R}$ denotes a subgroup of $\mathcal{L} \times \mathcal{R} \subset SU(2) \times SU(2)$ of order $|\mathcal{L}||\mathcal{R}|/|\mathcal{F}|$. More details about the groups G_0 and the corresponding models are given in section 4 and appendix A).

This rather abstract description of the symmetry groups deserves some further clarification (see section 2 for details). The continuous normal subgroup $U(1)^4 \times U(1)^4$ is generated by the zero modes of four holomorphic and four antiholomorphic $u(1)$ currents ∂X^i and $\bar{\partial} X^i$, and contains in particular the group $U(1)^4$ of translations along the four directions on the torus. The group $G_0 = G/(U(1)^4 \times U(1)^4)$ includes the rotations (fixed point automorphism) of the target space. If G_0 is generated only by automorphisms of the target space, we say that the group is (purely) geometric. In this case, the requirement that the superconformal algebra is preserved implies that G_0 is a subgroup of $SU(2)$. The properties of the purely geometric groups are well-known and their classification follows from a theorem by Fujiki [27]. In general, the group G_0 might contain elements that act asymmetrically on the left- and the right-moving fields (non-geometric group). In this case, G_0 is a subgroup of the product of two $SU(2)$, acting separately on the left- and right-moving fields. The $SU(2)$ subgroup of geometric symmetries is embedded diagonally in this general $SU(2) \times SU(2)$. Both in the geometric and non-geometric case, G_0 must act by automorphisms on the Narain lattice $\Gamma^{4,4}$. This condition constrains G_0 to be a discrete subgroup of $SU(2) \times SU(2)$ that depends on the specific torus model.

In every torus model, the group G_0 contains a central involution that flips the sign of all target space coordinates. For a generic model, this is the only non-trivial element of G_0. The precise classification of all possible discrete subgroups G_0, or rather of the quotients $G_1 = G_0/\mathbb{Z}_2$ by this central involution, is better understood by considering the action on the lattices of D-brane charges. In particular, as discussed in section 3, the lattices $\Gamma^{4,4}_{\text{even}}$ and $\Gamma^{4,4}_{\text{odd}}$ of even and odd dimensional D-brane charges, are both isomorphic to the winding-momentum (Narain) lattice. This is a peculiarity of non-linear sigma models on four dimensional tori and is closely related to the triality automorphism of the group $\mathrm{Spin}(4,4)$. The role of triality for non-linear sigma models on T^4 was emphasized in [28, 29].

In section 4, using this triality, we obtain a useful and suggestive characterization of the groups G_1 as subgroups of $W^+(E_8)$, the group of even Weyl transformations of the E_8 root lattice. More precisely, the possible groups G_1 are precisely the subgroups $W^+(E_8)$ that fix pointwise a sublattice of E_8 of rank at least four. This result is the closest analogue of the classification theorem for K3 models: in this case, the possible groups of symmetries are exactly those subgroups of the Conway group Co_0 that fix pointwise a sublattice of the Leech lattice of rank at least four [23]. Whether the appearance of the Leech and of the

E_8 lattice and their group of automorphisms is just an accident or if it has some physical meaning is still an open question.

Finally, in section 5, we discuss the orbifolds of torus models by cyclic groups of symmetries. By construction, these orbifolds, if consistent, must enjoy a $\mathcal{N} = (4, 4)$ superconformal symmetry at central charge $c = \tilde{c} = 6$, so that they are necessarily non-linear sigma models on T^4 or K3. It is easy to see that the orbifolds by cyclic subgroups of $U(1)^4 \times U(1)^4$ always give rise to torus models. For each symmetry $g \in G$ with non-trivial image in G_0, we classify the nature (i.e., the topology of the target space) of the corresponding orbifold model. In this sense, it is useful to think of g as an element in a $\mathbb{Z}_2$ central extension $\mathbb{Z}_2.W^+(E_8)$ of the Weyl group of E_8. Indeed, the nature of the orbifold only depends on the conjugacy class of g within $\mathbb{Z}_2.W^+(E_8)$. As a consistency check of our analysis, we verify that the classification of torus orbifolds matches perfectly with the results of [26], where a similar investigation was considered for orbifolds of K3 models.

Some technical details about finite subgroups of $SU(2) \times SU(2)$ and some generalities on lattices are relegated to the appendices.

2 Generalities on torus models

In this section we review the main features of supersymmetric non-linear sigma models on T^4 and fix our notation and conventions.

Abstractly, a supersymmetric T^4-model is defined in terms of four left-moving $u(1)$ currents

$$j^a(z) = i\partial\phi^a(z)$$

and four free fermions $\psi^a(z)$, $a = 1, \ldots, 4$, their right-moving analogues $\tilde{j}^a(\bar{z}) = i\bar{\partial}\tilde{\phi}^a(\bar{z}), \tilde{\psi}^a(\bar{z})$, as well as 'exponential fields' $V_\lambda(z, \bar{z})$ labelled by the vectors λ in an even unimodular lattice $\Gamma^{4,4}$ of signature $(4, 4)$. The holomorphic fields have a mode expansion

$$j^a(z) = \sum_{n \in \mathbb{Z}} \alpha_n^a z^{-n-1} , \qquad \psi^a(z) = \sum_{r \in \mathbb{Z}+\nu} \psi_r^a z^{-r-\frac{1}{2}} ,$$

where $\nu = 1/2$ and $\nu = 0$ in the Neveu-Schwarz (NS) and Ramond (R) sectors, respectively, and the modes satisfy the (anti-)commutation relations

$$[\alpha_m^a, \alpha_n^b] = m\delta^{ab}\delta_{m,-n} \qquad \{\psi_r^a, \psi_s^b\} = \delta^{ab}\delta_{r,-s} ,$$

corresponding to the OPE

$$j^a(z)j^b(w) \sim \frac{\delta^{ab}}{(z-w)^2} , \qquad \psi^a(z)\psi^b(w) \sim \frac{\delta^{ab}}{(z-w)} . \tag{2.1}$$

The anti-holomorphic fields $\tilde{j}^a(\bar{z}), \tilde{\psi}^a(\bar{z})$, $a = 1, \ldots, 4$, and their modes $\tilde{\alpha}_n^a, \tilde{\psi}_r^a$ satisfy analogous relations. The chiral algebra contains, in addition to the $u(1)^4$ 'bosonic' algebra generated by j^a, $a = 1, \ldots, 4$, also a 'fermionic' $so(4)_1$ Kac-Moody algebra generated by

$:\psi^i(z)\psi^k(z):$, $1 \le i < k \le 4$, and 16 fields of weight $(3/2, 0)$ of the form $:\psi^i(z)j^k(z):$. In particular, any torus model contains a (small) $\mathcal{N} = (4, 4)$ superconformal algebra with central charge $c = \tilde{c} = 6$, which will be described in detail in the next section. The zero modes of the currents generate a continuous group of symmetries of the conformal field theory. In particular, we denote by

$$U(1)_L^4 , \qquad\qquad U(1)_R^4 , \tag{2.2}$$

the groups of symmetries generated by the zero modes j_0^a and $\tilde{j}_0^a$ of the left- and right-moving currents and by

$$SO(4)_{f,L} , \qquad\qquad SO(4)_{f,R} , \tag{2.3}$$

the groups generated by the zero modes $\psi_0^a \psi_0^b$ and $\tilde{\psi}_0^a \tilde{\psi}_0^b$, $1 \le a < b \le 4$.

The fields $V_\lambda(z, \bar{z})$ are the vertex operators associated with eigenstates $|\lambda\rangle$ of α_0^a and $\tilde{\alpha}_0^b$, $a, b = 1, \ldots, 4$, with eigenvalues λ_L^a and λ_R^b. In each model, the possible vectors of eigenvalues $\lambda \equiv (\vec{\lambda}_L; \vec{\lambda}_R) := (\lambda_L^1, \ldots, \lambda_L^4; \lambda_R^1, \ldots, \lambda_R^4)$ form an even self-dual lattice $\Gamma_{w-m}^{4,4} \equiv \Gamma^{4,4}$ (winding-momentum or Narain lattice) with signature $(4, 4)$ and quadratic form

$$\lambda \bullet \lambda := \vec{\lambda}_L^2 - \vec{\lambda}_R^2 = \sum_{a=1}^4 \left((\lambda_L^a)^2 - (\lambda_R^a)^2 \right) . \tag{2.4}$$

Explicitly, for each $\lambda \equiv (\vec{\lambda}_L; \vec{\lambda}_R) \in \Gamma_{w-m}^{4,4}$, the field V_λ, is defined as

$$V_{(\vec{\lambda}_L; \vec{\lambda}_R)}(z, \bar{z}) := :\exp\left[i \sum_{a=1}^4 \lambda_L^a \phi^a(z) + i \sum_{a=1}^4 \lambda_R^a \tilde{\phi}^a(\bar{z}) \right]: \sigma_{(\vec{\lambda}_L; \vec{\lambda}_R)} \tag{2.5}$$

where the operators σ_λ, $\lambda \in \Gamma_{w-m}^{4,4}$ obey

$$\sigma_\lambda \sigma_\mu = \epsilon(\lambda, \mu) \sigma_{\lambda+\mu} \quad \text{for all } \lambda, \mu \in \Gamma_{w-m}^{4,4}$$

for a suitable normalized 2-cocycle $\epsilon(\lambda, \mu) \in \{\pm 1\}$ satisfying

$$\epsilon(\lambda, \mu) = (-1)^{\lambda \bullet \mu} \epsilon(\mu, \lambda) ,$$
$$\epsilon(\lambda, \mu)\epsilon(\lambda + \mu, \nu) = \epsilon(\lambda, \mu + \nu)\epsilon(\mu, \nu) , \tag{2.6}$$
$$\epsilon(\lambda, 0) = \epsilon(0, \mu) = 1 .$$

These conditions determine the 2-cocycle ϵ up to a 2-coboundary $\frac{\xi(\lambda)\xi(\mu)}{\xi(\lambda+\mu)}$, for any $\xi : \Gamma_{w-m}^{4,4} \to \{\pm 1\}$ with $\xi(0) = 1$, which can be re-adsorbed in the definition of $V_\lambda(z, \bar{z})$. The OPE of the fields $V_\lambda(z, \bar{z})$ with the fermions is non-singular, while

$$j^a(z) V_\lambda(w, \bar{w}) \sim \frac{\lambda_L^a}{z - w} V_\lambda(w, \bar{w}) , \qquad a = 1, \ldots, 4, \tag{2.7}$$

$$\tilde{j}^a(\bar{z}) V_\lambda(w, \bar{w}) \sim \frac{\lambda_R^a}{\bar{z} - \bar{w}} V_\lambda(w, \bar{w}) , \qquad a = 1, \ldots, 4, \tag{2.8}$$

$$V_\lambda(z, \bar{z}) V_{\lambda'}(w, \bar{w}) \sim \epsilon(\lambda, \lambda') (z - w)^{\vec{\lambda}_L \cdot \vec{\lambda}_L'} (\bar{z} - \bar{w})^{\vec{\lambda}_R \cdot \vec{\lambda}_R'} \left(V_{\lambda+\lambda'}(w, \bar{w}) + \ldots \right) , \tag{2.9}$$

where $\lambda, \lambda' \in \Gamma^{4,4}_{\text{w--m}}$ and ... in the last equation denotes less singular terms in $(z - w)$ and $(\bar{z} - \bar{w})$.

The real vector space space $\Pi = \Gamma^{4,4}_{\text{w--m}} \otimes \mathbb{R} \cong \mathbb{R}^{4,4}$ splits into an orthogonal sum $\Pi = \Pi_L \oplus \Pi_R$ of a positive-definite subspace Π_L, spanned by vectors of the form $(\vec{v}_L, 0)$, and a negative-definite Π_R spanned by $(0, \vec{v}_R)$. Each torus model is uniquely determined by the relative position of Π_L and Π_R with respect to the lattice $\Gamma^{4,4}_{\text{w--m}}$ and we obtain the usual Narain moduli space

$$\mathcal{M}_{T^4} = O(4, \mathbb{R}) \times O(4, \mathbb{R}) \backslash O(4, 4, \mathbb{R}) / O(\Gamma^{4,4}) \ .$$

A concrete realisation of this CFT is given in terms of a supersymmetric non-linear sigma model on a torus $T^4 = \mathbb{R}^4 / L$ with a constant antisymmetric B-field B. We fix once and for all a set of orthonormal coordinates for the euclidean space $\mathbb{R}^4$, so that the geometry of the torus is encoded in the shape of the lattice $L \subset \mathbb{R}^4$. The four scalar fields $X^a(z, \bar{z}) = \phi^a(z) + \tilde{\phi}^a(\bar{z})$, $a = 1, \ldots, 4$, describing the target space coordinates, are related to the CFT currents by $j^a = i\partial X^a$, $\tilde{j}^a = i\bar{\partial}X^a$. The corresponding lattice of $(\alpha_0, \tilde{\alpha}_0)$-eigenvalues is given by

$$\left\{ (\vec{\lambda}_L, \vec{\lambda}_R) := \tfrac{1}{\sqrt{2}}(\vec{m} - B\vec{l} - \vec{l}, \vec{m} - B\vec{l} + \vec{l}) \ \mid \ \vec{l} \in L, \ \vec{m} \in L^* \right\} \cong \Gamma^{4,4}_{\text{w--m}} \ ,$$

where $L^* \subset \mathbb{R}^4$ denotes the dual lattice. The fermionic fields $\psi^a, \tilde{\psi}^a$ are the supersymmetric partners of the scalars X^a.

Each of the abstract conformal field theories, defined in terms of the relative position of Π_L and Π_R with respect to the lattice $\Gamma^{4,4}_{\text{w--m}}$, is reproduced by several different geometric descriptions in terms of non-linear sigma models. Technically, the choice of any such description amounts to choosing two sublattices Λ_{m} and Λ_{w} of $\Gamma^{4,4}_{\text{w--m}}$, such that $\Lambda_{\text{m}}^\perp = \Lambda_{\text{m}}$, $\Lambda_{\text{w}}^\perp = \Lambda_{\text{w}}$ (i.e., they are maximal isotropic sublattices) and $\Lambda_{\text{m}} \oplus \Lambda_{\text{w}} \cong \Gamma^{4,4}_{\text{w--m}}$. Different choices are related to one another by automorphisms of $\Gamma^{4,4}_{\text{w--m}}$ (interpreted as T-dualities).

In particular, the choice of Λ_{m} determines an isometry $\gamma : \Pi_L \to \Pi_R(-1)$, characterised by the property that $\gamma(\vec{\lambda}_L) = \vec{\lambda}_R$ for each $\lambda = (\vec{\lambda}_L, \vec{\lambda}_R) \in \Lambda_{\text{m}} \subset \Pi_L \oplus \Pi_R$ (here, $\Pi_R(-1)$ is obtained from Π_R by flipping the sign of the metric). Then one can define a positive definite lattice $L^* \subset \Pi_L \cong \Pi_R(-1) \cong \mathbb{R}^4$ of rank four by

$$L^* := \left\{ \vec{l}^* \in \mathbb{R}^4 \mid \frac{1}{\sqrt{2}}(\vec{l}^*, \vec{l}^*) \in \Lambda_{\text{m}} \right\} \ . \tag{2.10}$$

The dual lattice $L = (L^*)^* \subset \mathbb{R}^4$ is interpreted as the lattice of winding vectors, so that the CFT can be described as a non-linear sigma model on the torus $T^4 := \mathbb{R}^4 / L$.[1]

Finally, the choice of a constant background B-field is encoded in the choice of the lattice Λ_{w}. Indeed, for a given Λ_{m}, the most general isotropic sublattice $\Lambda_{\text{w}} \subset \Gamma^{4,4}_{\text{w--m}}$

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[1]From a different viewpoint, given the interpretation of a left-moving current j^a as a derivative $i\partial X^a$ of a scalar field, the isometry $\gamma : \Pi_L \to \Pi_R(-1)$ determines which right-moving current $\tilde{j}^a = \gamma(j^a)$ should correspond to $i\bar{\partial}X^a$.

satisfying $\Lambda_{\mathrm{m}} \oplus \Lambda_{\mathrm{w}} \cong \Gamma^{4,4}_{\mathrm{w-m}}$ is given by

$$\Lambda_{\mathrm{w}} = \{\frac{1}{\sqrt{2}}(-B\vec{l}+\vec{l}, -B\vec{l}-\vec{l}) \mid \vec{l} \in L\} \ .$$

Here, the linear map B is given by a real antisymmetric matrix $B_{ij} = -B_{ji}$ as

$$B\vec{l}_i := \sum_j B_{ij} \vec{l}^{*j} \ ,$$

in terms of a basis $\vec{l}_1, \ldots, \vec{l}_4$ of L and of the dual basis $\vec{l}^{*1}, \ldots, \vec{l}^{*4}$ of L^*. The matrix B_{ij}, and consequently the lattice Λ_{w}, is determined by the decomposition $\Gamma^{4,4}_{\mathrm{w-m}} \otimes \mathbb{R} = \Pi_L \oplus \Pi_R$ and by the choice of $\Lambda_{\mathrm{m}} \cong \Gamma^{4,4}_{\mathrm{w-m}}$, up to a shift by an integral antisymmetric matrix.

2.1 The 'small' $\mathcal{N} = 4$ superconformal algebra

In the previous section, we have mentioned that every supersymmetric non-linear sigma model on T^4 contains a 'small' $\mathcal{N} = (4,4)$ superconformal algebra. The holomorphic small $\mathcal{N} = 4$ algebra is generated by four supercurrents $G^{\pm}(z), G'^{\pm}(z)$ of weight $(3/2, 0)$, the stress energy tensor $T(z)$ and a $su(2)_1$ Kac-Moody algebra ('R-symmetry') whose currents $J^3(z), J^+(z), J^-(z)$ are contained in the 'fermionic' $so(4)_1$. Let us introduce the complex fields

$$\partial Z^{(1)}(z) := \frac{1}{\sqrt{2}}(j^1(z) + ij^3(z)) \ , \qquad \partial Z^{(1)*}(z) := \frac{1}{\sqrt{2}}(j^1(z) - ij^3(z)) \ , \qquad (2.11)$$

$$\partial Z^{(2)}(z) := \frac{1}{\sqrt{2}}(j^2(z) + ij^4(z)) \ , \qquad \partial Z^{(2)*}(z) := \frac{1}{\sqrt{2}}(j^2(z) - ij^4(z)) \ . \qquad (2.12)$$

and

$$\chi^1(z) := \frac{1}{\sqrt{2}}(\psi^1(z) + i\psi^3(z)) \ , \qquad \chi^{*1}(z) := \frac{1}{\sqrt{2}}(\psi^1(z) - i\psi^3(z)) \ , \qquad (2.13)$$

$$\chi^2(z) := \frac{1}{\sqrt{2}}(\psi^2(z) + i\psi^4(z)) \ , \qquad \chi^{*2}(z) := \frac{1}{\sqrt{2}}(\psi^2(z) - i\psi^4(z)) \ . \qquad (2.14)$$

Then, the holomorphic $\mathcal{N} = 4$ supercurrents are given by

$$G^+(z) = \sqrt{2}i \left(:\chi^{*1}(z)\partial Z^{(1)}(z): + :\chi^{*2}(z)\partial Z^{(2)}(z): \right) \ , \qquad (2.15)$$

$$G^-(z) = \sqrt{2}i \left(:\chi^1(z)\partial Z^{(1)*}(z): + :\chi^2(z)\partial Z^{(2)*}(z): \right) \ , \qquad (2.16)$$

$$G'^+(z) = \sqrt{2} \left(-:\chi^{*1}(z)\partial Z^{(2)*}(z): + :\chi^{*2}(z)\partial Z^{(1)*}(z): \right) \ , \qquad (2.17)$$

$$G'^-(z) = \sqrt{2} \left(:\chi^1(z)\partial Z^{(2)}(z): - :\chi^2(z)\partial Z^{(1)}(z): \right) \ , \qquad (2.18)$$

and the $su(2)_1$ 'R-symmetry' currents by

$$J^3(z) := \frac{1}{2} \left(:\chi^{*1}(z)\chi^1(z): + :\chi^{*2}(z)\chi^2(z): \right) \qquad (2.19)$$

$$J^+(z) := i :\chi^{*1}(z)\,\chi^{*2}(z): \ , \qquad J^-(z) := i :\chi^1(z)\,\chi^2(z): \ . \qquad (2.20)$$

We denote by $SU(2)_J$ and $SU(2)_{\tilde{J}}$ the R-symmetry groups, i.e. the groups of symmetries of the CFT generated by the zero modes $J_0^3, J_0^{\pm}$ and by their right-moving counterparts $\tilde{J}_0^3, \tilde{J}_0^{\pm}$. It is also useful to define the currents

$$A^3(z) := \frac{1}{2}\Big(:\chi^{*1}(z)\chi^1(z): - :\chi^{*2}(z)\chi^2(z):\Big) \tag{2.21}$$

$$A^+(z) := i:\chi^{*1}(z)\,\chi^2(z): , \qquad A^-(z) := i:\chi^1(z)\,\chi^{*2}(z): . \tag{2.22}$$

and their right-moving counterparts $\tilde{A}_0^3, \tilde{A}_0^{\pm}$. These currents form a second $su(2)_1$ affine algebra commuting with (2.19)–(2.20). The zero modes $A_0^3, A_0^{\pm}$ and $\tilde{A}_0^3, \tilde{A}_0^{\pm}$ generate the groups of symmetries $SU(2)_A$ and $SU(2)_{\tilde{A}}$, respectively, that commute with $SU(2)_J$ and $SU(2)_{\tilde{J}}$. Altogether, these groups generate the whole fermionic symmetries (2.3)

$$SO(4)_{f,L} = (SU(2)_J \times SU(2)_A)/(-1_J, -1_A) , \qquad SO(4)_{f,R} = (SU(2)_{\tilde{J}} \times SU(2)_{\tilde{A}})/(-1_{\tilde{J}}, -1_{\tilde{A}}) ,$$

where we notice that the central elements $(-1)^{J_0^3 + A_0^3} \in SU(2)_J \times SU(2)_A$ and $(-1)^{\tilde{J}_0^3 + \tilde{A}_0^3} \in SU(2)_{\tilde{J}} \times SU(2)_{\tilde{A}}$ act trivially in the NS-NS sector.

2.2 A first classification of symmetries

Our goal is to classify all possible groups G of symmetries of the OPE that fix the small $\mathcal{N} = (4, 4)$ superconformal algebra. We focus only on symmetries that act non-trivially on the NS-NS sector; the induced action on the R-R sector is discussed in section 3.1.

Let us consider first the symmetries that act trivially on the fermionic fields $\psi^a, \tilde{\psi}^a$ as well as on the bosonic currents $j^a, \tilde{j}^a$, $a = 1, \ldots, 4$ so that the $\mathcal{N} = (4, 4)$ algebra is automatically invariant. These symmetries act non-trivially only on the fields V_λ and the only linear transformations preserving the OPE (2.7)-(2.9) is

$$V_\lambda(z, \bar{z}) \mapsto e^{2\pi i(\vec{v}_L \cdot \vec{\lambda}_L - \vec{v}_R \cdot \vec{\lambda}_R)} V_\lambda(z, \bar{z}) = e^{2\pi i \sum_a (v_L^a j_0^a - v_R^a \tilde{j}_0^a)} V_\lambda(z, \bar{z}) , \tag{2.23}$$

for some $(\vec{v}_L, \vec{v}_R) \in (\Gamma^{4,4} \otimes \mathbb{R})/\Gamma^{4,4}$. Therefore, the normal subgroup of G fixing the fields $\psi^a, \tilde{\psi}^a, j^a, \tilde{j}^a$, $a = 1, \ldots, 4$, is the group $U(1)_L^4 \times U(1)_R^4$ generated by j_0^a and $\tilde{j}_0^a$ as in (2.2). Thus, G can always be written as a product

$$G = (U(1)_L^4 \times U(1)_R^4).G_0 , \tag{2.24}$$

where $G_0 := G/(U(1)_L^4 \times U(1)_R^4)$.

We have reduced our problem to the classification of the possible groups G_0 that act non-trivially on the fundamental $u(1)$ currents and fermions. In order to preserve the OPEs (2.1) and to fix the supercurrent

$$\frac{1}{\sqrt{2}}(G^+(z) + G^-(z)) = \sum_{a=1}^{4} :\psi^a(z)j^a(z): ,$$

and its right-moving analogue, a symmetry g must act by a simultaneous orthogonal transformation on the fermions and on the $u(1)$ currents, i.e.

$$j^a(z) \mapsto \sum_b (g_L)_{ab} j^b(z) , \qquad\qquad \tilde{j}^a(z) \mapsto \sum_b (g_R)_{ab} \tilde{j}^b(z) , \tag{2.25}$$

$$\psi^a(z) \mapsto \sum_b (g_L)_{ab} \psi^b(z) , \qquad\qquad \tilde{\psi}^a(z) \mapsto \sum_b (g_R)_{ab} \tilde{\psi}^b(z) , \tag{2.26}$$

where $g_L, g_R \in O(4, \mathbb{R})$ are real orthogonal matrices. We denote by $O(4)_L \times O(4)_R$ the group of transformations (g_L, g_R). In order for the $j^a V_\lambda$ OPE to be preserved, g must act on the fields V_λ by

$$V_\lambda(z, \bar{z}) \mapsto \xi_g(\lambda) V_{g^{-1}(\lambda)}(z, \bar{z}) , \qquad \lambda \in \Gamma^{4,4} , \tag{2.27}$$

where

$$g^{-1}(\vec{\lambda}_L, \vec{\lambda}_R) := ((g_L)^{-1} \vec{\lambda}_L, (g_R)^{-1} \vec{\lambda}_R) , \qquad (\vec{\lambda}_L, \vec{\lambda}_R) \in \Gamma^{4,4} \subset \Pi_L \oplus \Pi_R . \tag{2.28}$$

The function $\xi_g : \Gamma^{4,4} \to \{\pm 1\}$ satisfies

$$\epsilon(\lambda, \mu) = \epsilon(g^{-1}(\lambda), g^{-1}(\mu)) \frac{\xi_g(\lambda) \xi_g(\mu)}{\xi_g(\lambda + \mu)} , \tag{2.29}$$

for all $\lambda, \mu \in \Gamma^{4,4}$. Since both $\epsilon(g^{-1}(\lambda), g^{-1}(\mu))$ and $\epsilon(\lambda, \mu)$ satisfy (2.6), they must be related as in (2.29) for some function ξ_g. In turn, eq.(2.29) determines ξ_g up to composition with elements of order two in the group $U(1)_L^4 \times U(1)_R^4$ of (2.24).[2]

The transformations (2.25),(2.26) and (2.27) define a symmetry of the CFT if and only if the transformation (2.28) is an automorphism of the lattice $\Gamma^{4,4}$, i.e. an element of $O(\Gamma^{4,4})$. The condition that a certain pair (g_L, g_R) of orthogonal transformations of the spaces Π_L and Π_R of left- and right-moving momenta induces an automorphism of the lattice $\Gamma^{4,4}$ clearly depends on the moduli.

So far we have only required the transformation g to preserve the OPE and fix one holomorphic and one antiholomorphic supercurrent. In order for g to fix the whole $\mathcal{N} = (4, 4)$ superconformal algebra, a necessary condition is that the currents (2.19)-(2.20) be invariant under g. Notice that if $\det g_L = -1$, then the orthogonal transformation (2.26) exchanges the two $su(2)_1$ algebras (2.19)–(2.20) and (2.21)–(2.22). Therefore, it is sufficient to consider transformations (2.26) within $SO(4)_L \times SO(4)_R$. With this restriction, it is easy to see that the group of symmetries of the fermions fixing the currents (2.19)-(2.20) is the $SU(2)_A \times SU(2)_{\tilde{A}}$ group generated by the zero modes A_0^3, $A_0^\pm$ of the currents (2.21)–(2.22) and their right-moving analogues $\tilde{A}_0^3$, $\tilde{A}_0^\pm$.

It follows that the transformation $(g_L, g_R) \in O(4)_L \times O(4)_R$, acting on the fields as in (2.25), (2.26) and (2.27), fixes the $\mathcal{N} = (4, 4)$ superconformal algebra if and only if it belongs to a $SU(2)_L \times SU(2)_R$ subgroup of $O(4)_L \times O(4)_R$, whose action on the complex fields (2.11) and (2.13) is

$$\begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} \mapsto \rho(g_L) \begin{pmatrix} \chi^1 \\ \chi^2 \end{pmatrix} , \qquad \begin{pmatrix} \chi^{*1} \\ \chi^{*2} \end{pmatrix} \mapsto \rho(g_L)^* \begin{pmatrix} \chi^{*1} \\ \chi^{*2} \end{pmatrix} , \tag{2.30}$$

$$\begin{pmatrix} \partial Z^{(1)} \\ \partial Z^{(2)} \end{pmatrix} \mapsto \rho(g_L) \begin{pmatrix} \partial Z^{(1)} \\ \partial Z^{(2)} \end{pmatrix} , \qquad \begin{pmatrix} \partial Z^{(1)*} \\ \partial Z^{(2)*} \end{pmatrix} \mapsto \rho(g_L)^* \begin{pmatrix} \partial Z^{(1)*} \\ \partial Z^{(2)*} \end{pmatrix} , \tag{2.31}$$

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[2]Technically, eq.(2.6) determines the cohomology class of the cocycle ϵ in $H^2(\Gamma^{4,4}, \mathbb{Z}_2) \cong H^2(\mathbb{Z}_2^8, \mathbb{Z}_2) \cong \mathbb{Z}_2^{\binom{8}{2}}$, where $\Gamma^{4,4}$ is isomorphic to $\mathbb{Z}^8$ as an abelian group. Thus, $\epsilon(g(\lambda), g(\mu))$ must differ from $\epsilon(\lambda, \mu)$ by a 2-coboundary $d\xi_g$. The cochain ξ_g is determined up to a 1-cocycle, i.e. a homomorphism $\Gamma^{4,4} \to \mathbb{Z}_2^8$.

$$\begin{pmatrix} \tilde{\chi}^1 \\ \tilde{\chi}^2 \end{pmatrix} \mapsto \tilde{\rho}(g_R) \begin{pmatrix} \tilde{\chi}^1 \\ \tilde{\chi}^2 \end{pmatrix} , \qquad \begin{pmatrix} \tilde{\chi}^{*1} \\ \tilde{\chi}^{*2} \end{pmatrix} \mapsto \tilde{\rho}(g_R)^* \begin{pmatrix} \tilde{\chi}^{*1} \\ \tilde{\chi}^{*2} \end{pmatrix} , \qquad (2.32)$$

$$\begin{pmatrix} \bar{\partial} Z^{(1)} \\ \bar{\partial} Z^{(2)} \end{pmatrix} \mapsto \tilde{\rho}(g_R) \begin{pmatrix} \bar{\partial} Z^{(1)} \\ \bar{\partial} Z^{(2)} \end{pmatrix} , \qquad \begin{pmatrix} \bar{\partial} Z^{(1)*} \\ \bar{\partial} Z^{(2)*} \end{pmatrix} \mapsto \tilde{\rho}(g_R)^* \begin{pmatrix} \bar{\partial} Z^{(1)*} \\ \bar{\partial} Z^{(2)*} \end{pmatrix} . \qquad (2.33)$$

Here, $\rho(g_L)$ and $\tilde{\rho}(g_R)$ are $SU(2)$ matrices and $\rho(g_L)^*$ and $\tilde{\rho}(g_R)^*$ are their complex conjugate. It is straightforward to verify that these transformations leave all generators of the $\mathcal{N} = (4,4)$ superconformal algebra invariant.

It is also useful to give a description of the group $SU(2)_L \times SU(2)_R$ in terms of quaternions. Let $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ be imaginary units satisfying the usual quaternionic multiplication rule

$$\mathbf{ij} = -\mathbf{ji} = \mathbf{k} , \quad \mathbf{jk} = -\mathbf{kj} = \mathbf{i} , \quad \mathbf{ki} = -\mathbf{ik} = \mathbf{j} , \quad \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 .$$

We have the standard identification of the space of quaternions $\mathbb{H}$ with $\mathbb{R}^4$

$$\mathbb{H} = \{a_1 + a_2 \mathbf{i} + a_3 \mathbf{j} + a_4 \mathbf{k} \mid (a_1, \ldots, a_4) \in \mathbb{R}^4\} \cong \mathbb{R}^4 ,$$

so that we can think of $\Gamma^{4,4}$ as a lattice within $\mathbb{H} \oplus \mathbb{H}$ by

$$(\vec{\lambda}_L, \vec{\lambda}_R) \equiv \left(\lambda_L^1 + \lambda_L^2 \mathbf{i} + \lambda_L^3 \mathbf{j} + \lambda_L^4 \mathbf{k} , \; \lambda_R^1 + \lambda_R^2 \mathbf{i} + \lambda_R^3 \mathbf{j} + \lambda_R^4 \mathbf{k}\right) .$$

Furthermore, the group unit quaternions form a copy of $SU(2)$

$$SU(2) = \{\begin{pmatrix} a+id & b+ic \\ -b+ic & a-id \end{pmatrix} \mid a^2+b^2+c^2+d^2 = 1\} = \{a_1+a_2\mathbf{i}+a_3\mathbf{j}+a_4\mathbf{k} \in \mathbb{H} \mid a^2+b^2+c^2+d^2 = 1\} .$$

Under this identification, the pair $(g_L, g_R) \in SU(2)_L \times SU(2)_R$ can be regarded as a pair of unit quaternions and the action on the fields is simply given by (left) multiplication

$$\psi^1 + \psi^2 \mathbf{i} + \psi^3 \mathbf{j} + \psi^4 \mathbf{k} \quad \mapsto \quad g_L(\psi^1 + \psi^2 \mathbf{i} + \psi^3 \mathbf{j} + \psi^4 \mathbf{k}) , \qquad (2.34)$$

$$\tilde{\psi}^1 + \tilde{\psi}^2 \mathbf{i} + \tilde{\psi}^3 \mathbf{j} + \tilde{\psi}^4 \mathbf{k} \quad \mapsto \quad g_R(\tilde{\psi}^1 + \tilde{\psi}^2 \mathbf{i} + \tilde{\psi}^3 \mathbf{j} + \tilde{\psi}^4 \mathbf{k}) , \qquad (2.35)$$

$$j^1 + j^2 \mathbf{i} + j^3 \mathbf{j} + j^4 \mathbf{k} \quad \mapsto \quad g_L(j^1 + j^2 \mathbf{i} + j^3 \mathbf{j} + j^4 \mathbf{k}) , \qquad (2.36)$$

$$\tilde{j}^1 + \tilde{j}^2 \mathbf{i} + \tilde{j}^3 \mathbf{j} + \tilde{j}^4 \mathbf{k} \quad \mapsto \quad g_R(\tilde{j}^1 + \tilde{j}^2 \mathbf{i} + \tilde{j}^3 \mathbf{j} + \tilde{j}^4 \mathbf{k}) , \qquad (2.37)$$

$$V_\lambda \quad \mapsto \quad \xi_g(\lambda) V_{g^{-1}(\lambda)} , \qquad (2.38)$$

where

$$g(\vec{\lambda}_L, \vec{\lambda}_R) := (g_L \vec{\lambda}_L, g_R \vec{\lambda}_R) , \qquad (\vec{\lambda}_L, \vec{\lambda}_R) \in \Gamma^{4,4} \subset \mathbb{H} \oplus \mathbb{H} . \qquad (2.39)$$

Once again, eqs.(2.34)–(2.38) (or, equivalently, (2.30)–(2.33)) define a symmetry of the CFT if and only if the induced transformation (2.39) is an automorphism of the lattice $\Gamma^{4,4}$. This condition constrains the group G_0 to be a discrete subgroup of $SU(2)_L \times SU(2)_R$. We stress that the action (2.38) of $(g_L, g_R) \in G_0$ on the fields V_λ is only defined modulo $U(1)_L^4 \times U(1)_R^4$ transformations.

To summarize, we have found the following characterization of the group G:

Theorem 1 *The group G of symmetries of a supersymmetric non-linear sigma model on T^4 preserving a small $\mathcal{N} = (4,4)$ superconformal algebra generated by (2.15)–(2.20), is given by a product*

$$G = (U(1)_L^4 \times U(1)_R^4).G_0 \ .$$

The normal subgroup $U(1)_L^4 \times U(1)_R^4$ is generated by the zero modes of the bosonic currents $j^a, \tilde{j}^a$, $a = 1, \ldots, 4$, and the quotient $G_0 = G/(U(1)_L^4 \times U(1)_R^4)$ is the finite group of elements $(g_L, g_R) \in SU(2)_L \times SU(2)_R$ such that (2.39) is an automorphism of the winding-momentum lattice $\Gamma^{4,4}$. In particular, $(g_L, g_R) \in G_0$ acts as in (2.34)–(2.38) on the fields.

In other words, G_0 is the intersection of the compact group $SU(2)_L \times SU(2)_R$ and the discrete group $SO(\Gamma^{4,4})$, when both groups are embedded in the orthogonal group $SO(4,4,\mathbb{R})$ acting on $\Pi_L \oplus \Pi_R \cong \mathbb{R}^{4,4}$. Clearly, these embeddings and therefore the intersection G_0 depend on the moduli, i.e. on the relative position of the subspaces Π_L and Π_R with respect to the lattice $\Gamma^{4,4}$.

The group G_0 always contains a central $\mathbb{Z}_2$ subgroup generated by the involution $(-1,-1) \in SU(2)_L \times SU(2)_R$ that flips the signs of all bosonic currents and all fermions; this is indeed a symmetry of all torus models. On the other hand, a generic deformation of the model will break any symmetry $g \in G_0$ not contained in this central $\mathbb{Z}_2$ subgroup. We conclude that the generic symmetry group G is

$$G = (U(1)_L^4 \times U(1)_R^4) : \mathbb{Z}_2 \ .$$

In the following sections, we will give a complete classification of the possible discrete subgroups $G_0 \subset SU(2)_L \times SU(2)_R$ and of the corresponding torus models.

3 Ramond-Ramond sector, D-branes and triality

In this section, we consider the action of the symmetry group G on the Ramond-Ramond fields and on the lattice of D-brane charges. This will lead to a useful characterization of the possible symmetry groups G in section 4.1.

3.1 Representation of the symmetries on the Ramond-Ramond sector

The Ramond-Ramond sector of the sigma model forms a representation of the algebra of fermionic zero modes $\psi_0^a, \tilde{\psi}_0^a$. In particular, the ground states have conformal weight $(\frac{1}{4}, \frac{1}{4})$ and span a 16 dimensional space $\mathcal{H}$. A convenient basis is given by simultaneous eigenvectors

$$|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle \ , \qquad s_1, s_2, \tilde{s}_1, \tilde{s}_2 \in \{\pm \tfrac{1}{2}\} \ ,$$

of the generators $J_0^3, A_0^3, \tilde{J}_0^3, \tilde{A}_0^3$ of the 'fermionic' $su(2)_1$ algebras (2.19), (2.20), (2.21), (2.22), such that

$$J_0^3|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = (s_1 + s_2)|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle \ , \quad A_0^3|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = (s_1 - s_2)|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle \ ,$$

$$\tilde{J}_0^3|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = (\tilde{s}_1 + \tilde{s}_2)|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle \ , \quad \tilde{A}_0^3|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = (\tilde{s}_1 - \tilde{s}_2)|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle \ .$$

The left- and right-moving world-sheet fermion number operators are defined by

$$(-1)^{F_L} := (-1)^{J_0^3} = (-1)^{s_1+s_2} = 2\,\psi_0^1\psi_0^2\psi_0^3\psi_0^4\,, \tag{3.1}$$

$$(-1)^{F_R} := (-1)^{\tilde{J}_0^3} = (-1)^{\tilde{s}_1+\tilde{s}_2} = 2\,\tilde{\psi}_0^1\tilde{\psi}_0^2\tilde{\psi}_0^3\tilde{\psi}_0^4\,, \tag{3.2}$$

and are preserved by the 'fermionic' $SO(4)_{f,L} \times SO(4)_{f,R}$ transformations. We denote by Π_{even} and Π_{odd} the subspaces of R-R ground states with $(-1)^{F_L+F_R} = 1$ and $(-1)^{F_L+F_R} = -1$, respectively.

Let us consider the action of the group G on the R-R ground states. Clearly, the normal subgroup $U(1)_L^4 \times U(1)_R^4$ acts trivially on this space, so we can focus on the quotient G_0. An element $g = (g_L, g_R) \in G_0$ acts by an $SU(2)_A \times SU(2)_{\tilde{A}}$ transformation on the fermionic fields $\psi^a, \tilde{\psi}^a$ and the action is represented by some $SU(2)$ matrices $\rho(g_L)$, $\tilde{\rho}(g_R)$ as in (2.30)–(2.33). In order to preserve the OPE, g has to act by the same $SU(2)_A \times SU(2)_{\tilde{A}}$ transformation also on the R-R ground fields. Assume, for the sake of simplicity, that this transformation is contained in the Cartan torus generated by A_0^3 and $\tilde{A}_0^3$. If the eigenvalues of $\rho(g_L)$ and $\tilde{\rho}(g_R)$ are $\zeta_L, \zeta_L^{-1}, \zeta_R, \zeta_R^{-1}$, then the action of g in the R-R sector is given by

$$g|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = \zeta_L^{A_0^3}\zeta_R^{\tilde{A}_0^3}|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle = \zeta_L^{s_1-s_2}\zeta_R^{\tilde{s}_1-\tilde{s}_2}|s_1, s_2; \tilde{s}_1, \tilde{s}_2\rangle\,.$$

Thus, the eigenvalues of g on the eigenspaces of $(-1)^{F_L}$ and $(-1)^{F_R}$ are

$(-1)^{F_L}$	$(-1)^{F_R}$	Eigenvalues of g			
-1	-1	$1\,,$	$1\,,$	$1\,,$	$1\,,$
$+1$	$+1$	$\zeta_L\zeta_R\,,$	$\zeta_L^{-1}\zeta_R\,,$	$\zeta_L\zeta_R^{-1}\,,$	$\zeta_L^{-1}\zeta_R^{-1}$
$+1$	-1	$\zeta_L\,,$	$\zeta_L\,,$	$\zeta_L^{-1}\,,$	$\zeta_L^{-1}\,,$
-1	$+1$	$\zeta_R\,,$	$\zeta_R\,,$	$\zeta_R^{-1}\,,$	ζ_R^{-1}

Notice that the eigenspace with $(-1)^{F_L} = (-1)^{F_R} = -1$ is fixed by g. Furthermore, the central involution $(-1, -1) \in SU(2)_L \times SU(2)_R$ acts trivially on the whole space Π_{even} with positive total fermion number $(-1)^{F_L+F_R} = +1$. Thus, the group acting faithfully on Π_{even} is $G_1 = G_0/(-1, -1)$ which is a subgroup of $(SU(2)_L \times SU(2)_R)/(-1, -1)$. These conclusions are valid even when the action of g on the fermions is not generated by A_0^3 and $\tilde{A}_0^3$, since by a suitable conjugation within $SU(2)_A \times SU(2)_{\tilde{A}}$ one can always reduce g to this Cartan torus while preserving the eigenspaces of $(-1)^{F_L}$ and $(-1)^{F_R}$. As will be described in the following sections, it is easier to characterize the possible groups G of symmetries by considering their action on the eigenspace Π_{even} with $(-1)^{F_L+F_R} = +1$.

In the above derivation we have been slightly naive in the identification of the action of g on the R-R sector. In general, a $SO(4)_{f,L} \times SO(4)_{f,R}$ transformation on the fundamental fermions in the NS-NS sector determines the transformation on the R-R sector only up to a sign – the symmetries $(-1)^{J_0^3+A_0^3}$ and $(-1)^{\tilde{J}_0^3+\tilde{A}_0^3}$ act trivially on the NS-NS sector and by a minus sign on the R-R sector. Therefore, for a generic subgroup G of $SO(4)_{f,L} \times SO(4)_{f,R}$ acting on the NS-NS sector, the induced group acting on the R-R sector and preserving the OPE is a $\mathbb{Z}_2$-central extension of G.

However, the groups G_0 we are considering are actually contained in the subgroup $SU(2)_A \times SU(2)_{\tilde{A}}$ of $SO(4)_{f,L} \times SO(4)_{f,R}$ generated by $A_0^3, A_0^{\pm}, \tilde{A}_0^3, \tilde{A}_0^{\pm}$, so that their central extension in the R-R sector is always the trivial one $\mathbb{Z}_2 \times G_0$. Therefore, we do not lose any information if we simply focus on the component of $\mathbb{Z}_2 \times G_0$ that fixes the four R-R states with $(-1)^{F_L} = (-1)^{F_R} = -1$. An analogous condition was considered in the classification of the groups of symmetries of K3 models in [23].

3.2 D-branes and triality

Fundamental D-branes are defined, in a given geometric interpretation of the torus model, by imposing either Neumann or Dirichlet boundary conditions in each direction. Each of these boundary conditions preserve a particular bosonic $u(1)^4$ subalgebra of the original $u(1)_L^4 \oplus u(1)_R^4$. For example, consider a geometric realization of the model, as described in section 2, associated with a maximal isotropic sublattice $\Lambda_{\rm m} \equiv {\rm span}_{\mathbb{Z}}\{\lambda_1, \ldots, \lambda_4\} \subset \Gamma^{4,4}$, which determines the lattice L^* of space-time momenta on a torus $\mathbb{R}^4/L$ as in eq.(2.10). Then, the Dirichlet boundary conditions describing a D0-brane in the Ramond-Ramond sector are enforced by

$$(\vec{\lambda}_{iL} \cdot \alpha_n - \vec{\lambda}_{iR} \cdot \tilde{\alpha}_{-n})|\mu; \Lambda_{\rm m}, \eta\rangle\!\rangle = 0 \ , \qquad n \in \mathbb{Z}, \ i = 1, \ldots, 4 \ , \tag{3.3}$$

$$(\vec{\lambda}_{iL} \cdot \psi_r + i\eta\vec{\lambda}_{iR} \cdot \tilde{\psi}_{-r})|\mu; \Lambda_{\rm m}, \eta\rangle\!\rangle = 0 \ , \qquad r \in \mathbb{Z}, \ i = 1, \ldots, 4 \ . \tag{3.4}$$

Here, $\eta \in \{\pm 1\}$ and $\Lambda_{\rm m}$ determine which $\mathcal{N} = 4$ and which $u(1)^4$ subalgebras are preserved by the boundary conditions, $\mu \equiv \frac{1}{\sqrt{2}}(\vec{l}^*, \vec{l}^*) \in \Lambda_{\rm m}$, with $\vec{l}^* \in L^*$, labels the distinct representations of this preserved $u(1)^4$ subalgebra and $|\mu; \Lambda_{\rm m}, \eta\rangle\!\rangle$ denotes the Ishibashi state in the corresponding sector. The boundary state $\|a; \Lambda_{\rm m}, \eta\rangle\!\rangle$ is then obtained by a superposition of the Ishibashi states for the different values of $\mu \in \Lambda_{\rm m}$

$$\|a; \Lambda_{\rm m}, \eta\rangle\!\rangle = \mathcal{N} \sum_{\mu \in \Lambda_{\rm m}} e^{2\pi i(a \bullet \mu)}|\mu; \Lambda_{\rm m}, \eta\rangle\!\rangle \ .$$

The modulus $a \in (\Lambda_{\rm w} \otimes \mathbb{R})/\Lambda_{\rm w}$ represents the position of the D0-brane on the four-dimensional torus and $\mathcal{N}$ is a suitable normalization. The treatment of the NS-NS sector is analogous, the only difference being in the boundary conditions (3.4), where the fermion modes $\psi_r, \tilde{\psi}_{-r}$ are defined for $r \in \frac{1}{2} + \mathbb{Z}$. In a full ten dimensional superstring theory, space-time supersymmetric D-branes are obtained by tensoring these boundary states with the analogous D-brane states in the remaining 6 space-time directions and considering a suitable GSO projected combination of NS-NS and R-R contributions.

The T-duality group $O(\Gamma^{4,4})$ acts transitively on the set of maximal isotropic sublattices $\Lambda \subset \Gamma^{4,4}$, so that fundamental D-branes can be defined independently of the choice of a geometric interpretation, by enforcing the conditions (3.3) and (3.4) for a given such Λ. For example, if we extend the set of generators of Λ to a basis $\lambda_1, \ldots, \lambda_4, \hat{\lambda}_1, \ldots, \hat{\lambda}_4$ of $\Gamma^{4,4}$ such that

$$\lambda_i \bullet \lambda_j = \hat{\lambda}_i \bullet \hat{\lambda}_j = 0 \ , \qquad \lambda_i \bullet \hat{\lambda}_j = \delta_{ij} \ , \tag{3.5}$$

then the isotropic sublattice generated by $\hat{\lambda}_1, \ldots, \hat{\lambda}_4$ is associated with a D4-brane and the lattice generated by $\lambda_1, \lambda_2, \hat{\lambda}_3, \hat{\lambda}_4$ is associated with a D2-brane extended in the 3–4 directions.

The Ramond-Ramond charge of a D-brane corresponds to the ground state component of the boundary state $\|a; \Lambda, \eta\rangle\rangle$ and is completely determined by eq.(3.4) for $r = 0$ up to normalization, which in turn is fixed by the Cardy and factorisation conditions. Let us focus on the case $\eta = 1$ and use a simplified notation $\|\Lambda\rangle\rangle \equiv \|a; \Lambda, \eta = 1\rangle\rangle$, where we drop the dependence on the modulus a. We denote by Ψ_Λ the R-R ground state component of $\|\Lambda\rangle\rangle$. For $\eta = +1$, equations (3.4) read

$$c(\lambda)\|\Lambda\rangle\rangle = c(\lambda)\Psi_\Lambda = 0 , \qquad \text{for all } \lambda \in \Lambda \subset \Gamma^{4,4} ,$$

where the operators

$$c(v) := \vec{v}_L \cdot \psi_0 + i\vec{v}_R \cdot \tilde{\psi}_0 , \qquad v \equiv (\vec{v}_L, \vec{v}_R) \in \Gamma^{4,4} \otimes \mathbb{R} , \tag{3.6}$$

generate the real Clifford algebra associated with the space $\Gamma^{4,4} \otimes \mathbb{R}$, with anticommutation relations

$$\{c(v), c(w)\} = \vec{v}_L \cdot \vec{w}_L - \vec{v}_R \cdot \vec{w}_R = v \bullet w . \tag{3.7}$$

The overlap between two D-branes is given by

$$\langle\langle\Lambda'\|(-1)^{F_R+1}q^{\frac{L_0+\tilde{L}_0}{2}-\frac{c}{24}}\|\Lambda\rangle\rangle = \langle\Psi_{\Lambda'}|(-1)^{F_R+1}\Psi_\Lambda\rangle , \tag{3.8}$$

which defines a non-degenerate bilinear form of signature $(8, 8)$ on the lattice of D-brane charges. Since all $c(v)$ anticommute with the chirality operator

$$c(v)(-1)^{F_L+F_R} = -(-1)^{F_L+F_R}c(v) , \qquad v \in \Gamma^{4,4} ,$$

Ψ_Λ must have definite chirality. D-brane charges with opposite chiralities are orthogonal with respect to (3.8), so that the lattice of D-brane charges splits as an orthogonal sum $\Gamma^{4,4}_{\text{even}} \oplus_\perp \Gamma^{4,4}_{\text{odd}}$ of lattices isomorphic to $\Gamma^{4,4}$. In a geometric description of the model, chirality corresponds to the parity of D-brane dimensionalities and the lattices $\Gamma^{4,4}_{\text{even}}$ and $\Gamma^{4,4}_{\text{odd}}$ are identified with the even and odd integral cohomology groups $H^{\text{even}}(T^4, \mathbb{Z})$ and $H^{\text{odd}}(T^4, \mathbb{Z})$, and the bilinear form is the cup product.

Each of the 8-dimensional vector spaces

$$\Pi_{\text{even}} = \Gamma^{4,4}_{\text{even}} \otimes \mathbb{R} , \qquad\qquad \Pi_{\text{odd}} = \Gamma^{4,4}_{\text{odd}} \otimes \mathbb{R} ,$$

contains two orthogonal subspaces $\Pi^\pm_{\text{even}}$, $\Pi^\pm_{\text{odd}}$ with definite signature with respect to (3.8) corresponding to the $(-1)^{F_R+1}$-eigenspaces with signature ± 1. From now on, we denote by $\Gamma^{4,4}_{\text{w-m}}$ the lattice of winding-momenta (eigenvalues of $\alpha_0, \tilde{\alpha}_0$), to distinguish it from the lattices of D-brane charges, and set $\Pi_{\text{w-m}} := \Pi = \Gamma^{4,4}_{\text{w-m}} \otimes \mathbb{R}$.

From a formal point of view, this construction is completely symmetric among the three lattices $\Gamma^{4,4}_{\text{w-m}}$, $\Gamma^{4,4}_{\text{even}}$ and $\Gamma^{4,4}_{\text{odd}}$, in the sense that one can start from any of the three lattices to define the other two. More precisely, let us consider one of the three lattices, denoted by $\Gamma^{4,4}_1$ and define the Clifford algebra (3.7) based on the real vector space $\Pi_1 = \Gamma^{4,4}_1 \otimes \mathbb{R} \cong \mathbb{R}^{4,4}$. This Clifford algebra admits a representation as an algebra of real matrices over a 16-dimensional real space V. As usual, the action of the group $SO(\Pi_1) \cong SO(4, 4)$ on the

Clifford algebra generators induces a spinorial representation of $\mathrm{Spin}(4,4)$ on V. More precisely, V is the direct sum $V = \Pi_2 \oplus \Pi_3$ of two irreducible $\mathrm{Spin}(4,4)$ representations of opposite chiralities. Here, Π_2, Π_3 are the 8 dimensional eigenspaces of the involution (chirality operator)

$$(-1)^{F_L + F_R} = 4c(e_1) \cdots c(e_8) \ ,$$

with $e_1, \ldots, e_8$ any oriented basis of Π_1 with $e_i \bullet e_j = \pm \delta_{ij}$.

The action of $\mathrm{Spin}(4,4)$ preserves a symmetric bilinear form $(\cdot, \cdot)$ of signature $(4,4)$ on each of the spaces Π_2 and Π_3. In order to define this bilinear form, it is useful to choose a basis $\lambda_1, \ldots, \lambda_4, \hat{\lambda}_1, \ldots, \hat{\lambda}_4$ of $\Gamma_1^{4,4}$ with standard Gram matrix (3.5). Then, there exists a one dimensional space of states $\mu \in V$ satisfying

$$c(\lambda_i)\mu = 0 \ , \qquad i = 1, \ldots, 4 \ . \tag{3.9}$$

We choose a non-zero vector μ in this space and, as usual, we define a set of generators of V by acting on μ in all possible ways with the operators $c(\hat{\lambda}_i)$

$$\mu_{i_1 \ldots i_r} := c(\hat{\lambda}_{i_1}) \cdots c(\hat{\lambda}_{i_r})\mu \ , \qquad 1 \le i_1 < \ldots < i_r \le 4 \ , \tag{3.10}$$

and define the bilinear form $(\cdot, \cdot)$ by

$$(\mu_{i_1, \ldots, i_r}, \mu_{j_1, \ldots, j_s}) = \begin{cases} 0 & \text{if } r + s \ne 4, \\ \epsilon_{i_r, \ldots, i_1, j_1, \ldots, j_s} & \text{if } r + s = 4, \end{cases} \tag{3.11}$$

where ϵ_{ijkl} is the completely antisymmetric tensor with $\epsilon_{1234} = 1$. In particular,

$$(\mu, c(\hat{\lambda}_1)c(\hat{\lambda}_2)c(\hat{\lambda}_3)c(\hat{\lambda}_4)\mu) = 1 \ . \tag{3.12}$$

Notice that μ has definite chirality, which depends on the orientation of the basis $\lambda_1, \ldots, \lambda_4$, $\hat{\lambda}_1, \ldots, \hat{\lambda}_4$. Up to a possible exchange of λ_1 and $\hat{\lambda}_1$, we can assume that $(-1)^{F_L + F_R}\mu = +\mu$. Therefore, all vectors $\mu_{i_1, \ldots, i_r}$ of the basis have definite chirality $(-1)^r$, so that (3.11) defines a bilinear form of signature $(4,4)$ on both Π_2 and Π_3.

We define the lattice $\Gamma_2^{4,4} \oplus \Gamma_3^{4,4}$ as the $\mathbb{Z}$-span of the vectors (3.10), with $\Gamma_i^{4,4} \otimes \mathbb{R} = \Pi_i$, $i = 2, 3$. By (3.11), $\Gamma_2^{4,4}$ and $\Gamma_3^{4,4}$ are mutually orthogonal even unimodular lattices of signature $(4,4)$. Given any other basis $\lambda_1', \ldots, \lambda_4', \hat{\lambda}_1', \ldots, \hat{\lambda}_4'$, satisfying (3.5), we can define a vector μ' satisfying (3.9) and (upon possibly reordering the sets $\lambda_1', \ldots, \lambda_4'$ and $\hat{\lambda}_1', \ldots, \hat{\lambda}_4'$) (3.12). This vector μ' is determined by (3.9) and (3.12) up to a sign. Furthermore, μ' depends only on the maximal isotropic sublattice $\Lambda' \subset \Gamma_1^{4,4}$ spanned by $\lambda_1', \ldots, \lambda_4'$, since eq.(3.12) gives the same normalization for all choices of $\hat{\lambda}_1', \ldots, \hat{\lambda}_4'$ satisfying (3.5). One finds that μ' has integral product with all generators of the basis (3.10). Therefore, by self-duality, μ' must be also contained in $\Gamma_2^{4,4} \oplus \Gamma_3^{4,4}$. As a consequence, the definition of the lattices $\Gamma_2^{4,4}$ and $\Gamma_3^{4,4}$ does not depend on the choice of the basis $\lambda_1, \ldots, \lambda_4, \hat{\lambda}_1, \ldots, \hat{\lambda}_4$. Notice that also all vectors $\mu_{i_1, \ldots, i_r}$ in (3.10) are associated with some maximal isotropic sublattice of $\Gamma_1^{4,4}$, generated by a suitable subset of four mutually orthogonal vectors in the basis $\lambda_1, \ldots, \lambda_4, \hat{\lambda}_1, \ldots, \hat{\lambda}_4$.

A generic automorphism $g_1 \in SO^+(\Gamma_1^{4,4})$ of the lattice $\Gamma_1^{4,4}$ lifts to some $\hat{g} \in \mathrm{Spin}(4,4)$ transformation on Π_2 and Π_3. By (3.9) and (3.12), this transformation maps a vector $\mu \in \Gamma_2^{4,4} \oplus \Gamma_3^{4,4}$ associated with a maximal isotropic $\Lambda \subset \Gamma_1^{4,4}$, to a vector $\pm\mu'$ associated with $g(\Lambda)$. Thus, $\hat{g}$ maps the lattice $\Gamma_2^{4,4} \oplus \Gamma_3^{4,4}$ into itself and since it also preserves the chirality and the bilinear form on V, it must act by automorphisms g_2, g_3 on both $\Gamma_2^{4,4}$ and $\Gamma_3^{4,4}$.

The choice of an orthogonal decomposition $\Pi_1 = \Pi_1^+ \oplus \Pi_1^-$ into a positive and a negative definite oriented subspace yields the definition of the involutions

$$(-1)^{F_L} = 2c(e_1)\cdots c(e_4) \,, \qquad\qquad (-1)^{F_R} = 2c(\hat{e}_1)\cdots c(\hat{e}_4) \,,$$

where $e_1, \ldots, e_4$ and $\hat{e}_1, \ldots, \hat{e}_4$ are oriented orthonormal bases of Π_1^+ and Π_1^-. In turn, this leads to the splittings $\Pi_2 = \Pi_2^+ \oplus \Pi_2^-$ and $\Pi_3 = \Pi_3^+ \oplus \Pi_3^-$ into the eigenspaces of $(-1)^{F_R+1}$.

Finally, this construction leads to the definition of a triality

$$
\begin{aligned}
\Gamma_3^{4,4} \,\times\, \Gamma_1^{4,4} \,\times\, \Gamma_2^{4,4} \qquad &\to \qquad\qquad \mathbb{Z} \,, \\
(\mu_3, \quad \mu_1, \quad \mu_2) \qquad &\mapsto \qquad (\mu_3, c(\mu_1)\mu_2) \,,
\end{aligned}
\tag{3.13}
$$

i.e. a non-degenerate trilinear form,[3] which extends to a triality of real vector spaces $\Pi_3 \times \Pi_1 \times \Pi_2 \to \mathbb{R}$. In particular, each $\mu_1 \in \Gamma_1^{4,4}$ defines two linear maps $c(\mu_1)_{23} : \Gamma_2^{4,4} \to (\Gamma_3^{4,4})^* \cong \Gamma_3^{4,4}$ and $c(\mu_1)_{32} : \Gamma_3^{4,4} \to (\Gamma_2^{4,4})^* \cong \Gamma_2^{4,4}$, which are isomorphisms when $\mu_1 \bullet \mu_1 = \pm 2$. In the same fashion, for any permutation (i, j, k) of $(1, 2, 3)$, the triality (3.13) defines for each $\mu_i \in \Gamma_i^{4,4}$ a map $c(\mu_i)_{jk} : \Gamma_j^{4,4} \to (\Gamma_k^{4,4})^* \cong \Gamma_k^{4,4}$. A straightforward computation shows that the maps $c(\mu_i) := c(\mu_i)_{jk} \oplus c(\mu_i)_{kj}$, for all $\mu_i \in \Gamma_i^{4,4}$, generate an algebra of matrices on the space $\Pi_j \oplus \Pi_k$ that forms a representation of the Clifford algebra of $\Gamma_i^{4,4}$. The triality among vector spaces is preserved by the action of $\hat{g} \in \mathrm{Spin}(4,4)$

$$(g_3(\mu_3), g_1(\mu_1), g_2(\mu_2)) = (\mu_3, \mu_1, \mu_2) \,,$$

where g_1, g_2 and g_3 are the vector and the two spinor representations of $\hat{g}$, respectively. Therefore, the statement that an automorphism $g_1 \in O(\Gamma_1^{4,4})$ induces automorphisms $g_2 \in O(\Gamma_2^{4,4})$ and $g_3 \in O(\Gamma_3^{4,4})$ can also be 'symmetrized': for each permutation (i, j, k) of $(1, 2, 3)$ and for each $\hat{g} \in \mathrm{Spin}(4,4)$, g_j and g_k are automorphisms of $\Gamma_j^{4,4}$ and $\Gamma_k^{4,4}$ if and only if g_i is an automorphism of $\Gamma_i^{4,4}$. Furthermore, the description of the moduli space of torus models as parametrizing the relative position of an even unimodular lattice $\Gamma^{4,4}$ with respect to a positive and a negative definite subspaces Π^+, Π^- with $\Gamma^{4,4} \otimes \mathbb{R} = \Pi^+ \oplus_\perp \Pi^-$ holds for any interpretation of $\Gamma^{4,4}$ as the lattice of winding-momenta, even or odd-dimensional D-branes [28].

This 'democracy' among the three lattices is characteristic of four-dimensional torus models and is related to the triality automorphism of $\mathrm{Spin}(4,4)$, which permutes the vector and the two irreducible spinor representations.

---

[3] Non-degenerate here means that every non-zero vector in one of the three lattices determines a non-degenerate bilinear form on the other two.

4 Groups and models

In this section, we classify all possible groups of symmetries G preserving the $\mathcal{N} = (4, 4)$ superconformal algebra and describe the corresponding torus models. The main idea is to focus on the action of G on the lattice $\Gamma^{4,4}_{\text{even}}$ of even dimensional D-brane charges. By the triality described in section 3.2, this analysis is sufficient to reconstruct the action of G on the lattice $\Gamma^{4,4}_{\text{w–m}}$ of winding-momenta and on all the fields of the theory.

4.1 Symmetries of torus models

Let G be the group of symmetries of a torus model that preserves the small $\mathcal{N} = (4, 4)$ superconformal algebra. A discussed in section 3.1, the representation of G over the space Π_{even} of R-R ground states with positive chirality is given by a group $G_1 \cong G_0/(-1)$ of orthogonal transformations of Π_{even} that fix the subspace Π^+_{even} with $(-1)^{F_R+1} = +1$. Furthermore, G_1 must act by automorphisms on the lattice $\Gamma^{4,4}_{\text{even}}$ of even D-brane charges.

Conversely, let $G_1 \subset SO^+(\Gamma^{4,4}_{\text{even}}) \subset SO^+(4, 4)$ be a group of automorphisms of the lattice of even D-brane charges that fix the subspace $\Pi^+_{\text{even}} \subset \Pi_{\text{even}}$. Thus, G_1 acts faithfully by $SO(4)$ transformations on the space Π^-_{even} of R-R ground fields with $(-1)^{F_R+1} = -1$. The spin cover $\text{Spin}(4)$ of this $SO(4)$ group is the $SU(2)_A \times SU(2)_{\tilde{A}}$ group generated by the zero modes of the currents $A^3_0, A^\pm_0, \tilde{A}^3_0, \tilde{A}^\pm_0$. Therefore, the group G_1 is the representation on the R-R sector of some group $G_0 \subset SU(2)_L \times SU(2)_R$, acting on the NS-NS-fields as in (2.30)–(2.33). The group G_0 is a isomorphic to a double (spin) cover of G_1, i.e. to the preimage of $G_1 \subset SO(\Gamma^{4,4}_{\text{even}}) \subset SO(4, 4)$ under the spin covering $\text{Spin}(4, 4) \to SO(4, 4)$, and acts on the space $\Pi_{\text{w–m}}$ of winding-momenta in one of the irreducible spin representations. By the triality construction described in the previous section, since G_1 acts by automorphisms on the lattice $\Gamma^{4,4}_{\text{even}}$, then G_0 must act by automorphisms on the lattice $\Gamma^{4,4}_{\text{w–m}}$ of winding momenta. Therefore, $(U(1)^4_L \times U(1)^4_R).G_0$ is a group of symmetries of the torus model that preserves the $\mathcal{N} = (4, 4)$ algebra.

Thus, we have found an alternative description of the groups of symmetries G considered in section 2. In fact, as will be discussed below, there are two more equivalent characterizations of these groups:

Theorem 2 *Let G_1 be a subgroup of $SO(4, \mathbb{R})$ and let $\mathbb{Z}_2.G_1 \subseteq \text{Spin}(4)$ denote its preimage under the spin covering homomorphism $\text{Spin}(4) \to SO(4)$. The following properties are equivalent:*

1. *The group*
$$G \cong (U(1)^4_L \times U(1)^4_R).(\mathbb{Z}_2.G_1) \,,$$
   is the group of symmetries of a supersymmetric non-linear sigma model on T^4 that preserves a small $\mathcal{N} = (4, 4)$ superconformal algebra.

2. *G_1 is a subgroup of $SO^+(\Gamma^{4,4}_{even}) \subset SO^+(4, 4, \mathbb{R})$ that fixes pointwise a positive-definite subspace $\Pi^+_{even} \subset \Gamma^{4,4}_{even} \otimes \mathbb{R}$ of dimension four.*

3. $G_1 = SO_0(\Lambda)$, where Λ is an even positive-definite lattice of rank at most four and $SO_0(\Lambda) \subseteq SO(\Lambda)$ is the group of automorphisms of Λ that act trivially on the discriminant group Λ^*/Λ (Λ^* denotes the dual of Λ, see appendix B).

4. G_1 is the subgroup of $W^+(E_8)$, the group of even Weyl transformations of the E_8 lattice, fixing a sublattice of rank at least four.

The equivalence of (1) and (2) has been discussed above. Let us first show that $(2) \Leftrightarrow (3)$. Let G_1 be a subgroup of $SO^+(\Gamma^{4,4}_{\text{even}})$ that leaves the subspace Π^+_{even} pointwise fixed. Let us denote by $\Gamma^{G_1} \subseteq \Gamma^{4,4}_{\text{even}}$ the sublattice of vectors fixed by G_1

$$\Gamma^{G_1} := \{v \in \Gamma^{4,4}_{\text{even}} \mid g(v) = v \text{ for all } g \in G_1\} \,,$$

and by Γ_{G_1} its orthogonal complement in $\Gamma^{4,4}_{\text{even}}$

$$\Gamma_{G_1} := \{w \in \Gamma^{4,4}_{\text{even}} \mid w \bullet v = 0 \text{ for all } v \in \Gamma^{G_1}\} \,.$$

Since $\Pi^+_{\text{even}} \subseteq \Gamma^{G_1} \otimes \mathbb{R}$, it follows that Γ_{G_1} is an even negative-definite lattice of rank at most 4. Notice that Γ^{G_1} and Γ_{G_1} are primitive mutually orthogonal sub-lattices of $\Gamma^{4,4}_{\text{even}}$ and the direct sum $\Gamma^{G_1} \oplus \Gamma_{G_1}$ has maximal rank 8. Then, by the standard 'gluing' construction (see appendix B for details), there is an isomorphism $(\Gamma^{G_1})^*/\Gamma^{G_1} \xrightarrow{\cong} (\Gamma_{G_1})^*/\Gamma_{G_1}$ of discriminant groups that reverses the induced discriminant form and such that

$$\Gamma^{4,4}_{\text{even}} = \{(v,w) \in (\Gamma^{G_1})^* \oplus (\Gamma_{G_1})^* \mid [v] \cong [w]\} \,,$$

where for each x in $(\Gamma^{G_1})^*$ or $(\Gamma_{G_1})^*$, $[x]$ denotes the image of x in the corresponding discriminant group. Since G_1 acts trivially on the discriminant group $(\Gamma^{G_1})^*/\Gamma^{G_1}$, then it must act trivially also on $(\Gamma_{G_1})^*/\Gamma_{G_1}$ [30]. We conclude that G_1 is a group of automorphisms of the positive-definite even lattice $\Lambda \cong \Gamma_{G_1}(-1)$ of rank (at most) four that fixes the discriminant group Λ^*/Λ.

Vice-versa, given any even positive-definite lattice Λ of rank at most four, there is a primitive embedding of $\Lambda(-1)$ in $\Gamma^{4,4}_{\text{even}}$ and one can always find a positive definite four dimensional subspace $\Pi^+_{\text{even}} \subset \Gamma^{4,4}_{\text{even}} \otimes \mathbb{R}$ such that $\Lambda(-1) = \Gamma^{4,4}_{\text{even}} \cap (\Pi^+_{\text{even}})^\perp$. As explained in appendix B, the automorphisms of Λ that fix the discriminant group Λ^*/Λ extend to automorphisms of $\Gamma^{4,4}_{\text{even}}$ that fix pointwise the orthogonal space Π^+_{even}. This proves the equivalence of (2) and (3).

The proof that $(3) \Leftrightarrow (4)$ is completely analogous, with the lattice $\Gamma^{4,4}$ replaced by E_8. Let Λ be an even positive definite lattice of rank at most 4 and let $SO_0(\Lambda)$ denotes the group of (positive determinant) automorphisms of Λ that act trivially on the discriminant group Λ^*/Λ. By a theorem by Nikulin (Theorem 1.12.4 of [30]), any even positive definite lattice with rank at most four can be primitively embedded in E_8, the unique 8-dimensional positive-definite even unimodular lattice. Furthermore, every automorphism $g \in SO_0(\Lambda)$ can be extended to an automorphism of E_8 that fixes the orthogonal complement $\Lambda^\perp$ of Λ in E_8. Therefore, $SO_0(\Lambda)$ is a subgroup of the group $G_1 \subset W^+(E_8)$ that fixes the sublattice $\Lambda^\perp \subset E_8$.

Conversely, suppose that G_1 is the subgroup of $W^+(E_8)$ that stabilises pointwise a sublattice $\Lambda^\perp \equiv (E_8)^{G_1}$ of rank at least 4. Then, G_1 acts faithfully as a group of automorphisms of the orthogonal complement Λ of $\Lambda^\perp$. Furthermore, since G_1 fixes $(\Lambda^\perp)^*/\Lambda^\perp$, by the gluing construction it must also fix the discriminant group Λ^*/Λ of its orthogonal complement. We conclude that $G_1 \cong SO_0(\Lambda)$.

4.2 Characterization of the groups of symmetries

In this section, we classify all possible groups G_1 satisfying the equivalent properties (3) and (4) of Theorem 2.

Let Λ be an even positive definite lattice of rank $r \leq 4$ and let $SO_0(\Lambda)$ the group of positive determinant automorphisms that act trivially on the discriminant group Λ^*/Λ. By property (3), there is a torus model such that $G_1 \cong SO_0(\Lambda)$. We recall that Λ can always be primitively embedded in the E_8 lattice and that every $g \in SO_0(\Lambda)$ extends to an automorphism $\hat{g} \in W^+(E_8)$ that fixes the orthogonal complement of Λ. Thus, Λ can always be regarded as a primitive sublattice of E_8 and we can label each $g \in SO_0(\Lambda)$ by the $W^+(E_8)$ conjugacy class of $\hat{g}$. The group $W^+(E_8)$ contains 11 conjugacy classes whose elements fix a sublattice of rank at least 4 [31], see table 1.

Class	η_1	η_1^{-1}	η_2	η_2^{-1}	$\pm\zeta_L$	$\pm\zeta_L^{-1}$	$\pm\zeta_R$	$\pm\zeta_R^{-1}$	$o(\pm g_0)$	rk	det
1A	1	1	1	1	± 1	± 1	± 1	± 1	1,2	0	-
2B	1	1	-1	-1	$\pm i$	$\mp i$	$\pm i$	$\mp i$	4	2	4
3A	1	1	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$\pm e^{\frac{2\pi i}{3}}$	$\pm e^{-\frac{2\pi i}{3}}$	$\pm e^{\frac{2\pi i}{3}}$	$\pm e^{-\frac{2\pi i}{3}}$	3,6	2	3
2A	-1	-1	-1	-1	± 1	± 1	∓ 1	∓ 1	2	4	16
2E	-1	-1	-1	-1	± 1	± 1	∓ 1	∓ 1	2	4	16
3E	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	$\pm e^{\frac{2\pi i}{3}}$	$\pm e^{-\frac{2\pi i}{3}}$	± 1	± 1	3,6	4	9
4A	i	$-i$	i	$-i$	$\pm i$	$\mp i$	± 1	± 1	4,4	4	4
4C	i	$-i$	-1	-1	$\pm e^{\frac{\pi i}{4}}$	$\pm e^{-\frac{\pi i}{4}}$	$\mp e^{\frac{\pi i}{4}}$	$\mp e^{-\frac{\pi i}{4}}$	8	4	8
5A	$e^{\frac{2\pi i}{5}}$	$e^{-\frac{2\pi i}{5}}$	$e^{\frac{4\pi i}{5}}$	$e^{-\frac{4\pi i}{5}}$	$\pm e^{\frac{2\pi i}{5}}$	$\pm e^{-\frac{2\pi i}{5}}$	$\pm e^{\frac{4\pi i}{5}}$	$\pm e^{-\frac{4\pi i}{5}}$	5,10	4	5
6A	$e^{\frac{2\pi i}{6}}$	$e^{-\frac{2\pi i}{6}}$	-1	-1	$\pm e^{\frac{2\pi i}{3}}$	$\pm e^{-\frac{2\pi i}{3}}$	$\pm e^{\frac{2\pi i}{6}}$	$\pm e^{\frac{-2\pi i}{6}}$	6	4	4
6D	$e^{\frac{2\pi i}{3}}$	$e^{-\frac{2\pi i}{3}}$	-1	-1	$\pm e^{\frac{2\pi i}{12}}$	$\pm e^{-\frac{2\pi i}{12}}$	$\pm e^{\frac{10\pi i}{12}}$	$\pm e^{-\frac{10\pi i}{12}}$	12	4	12

Table 1: Conjugacy classes of elements $\hat{g} \in W^+(E_8)$ that fix a sublattice of rank at least 4. For each class, we list the the eigenvalues $\eta_1, \eta_2, \eta_1^{-1}, \eta_2^{-1}$ of the corresponding $g \in G_1$ on the space Π_{even}^-, the rank of $(\Gamma_{\text{even}}^{4,4})_{\langle g \rangle}$ and the determinant of $(1-g)$ on $(\Gamma_{\text{even}}^{4,4})_{\langle g \rangle}$. The corresponding symmetry $\pm g_0 \in G_0$ of the torus model is determined up to a sign. We list the eigenvalues $\pm\zeta_L, \pm\zeta_R, \pm\zeta_L^{-1}, \pm\zeta_R^{-1}$ (each with multiplicity 2) for the action of $\pm g_0$ over the fermions $\psi^a, \tilde{\psi}^a$ and the orders $o(\pm g_0)$ of g_0 and $-g_0$.

Let $\lambda_1, \ldots, \lambda_r$ be a basis of generators for Λ and $Q_{ij} := \lambda_i \cdot \lambda_j$ be the corresponding Gram matrix. Then $g \in SO_0(\Lambda)$ acts by

$$g(\lambda_i) = \sum_{j=1}^r g_{ij}\lambda_j , \qquad i = 1, \ldots, r ,$$

where g_{ij} is an integral matrix satisfying

$$g^t\, Q\, g = Q\ .$$

Let $u_1, \ldots, u_r$ be the dual basis of generators of Λ^*, given by

$$u_i := \sum_j Q_{ij}^{-1}\lambda_j \in \Lambda \otimes \mathbb{Q}\ , \tag{4.1}$$

so that $u_i \cdot \lambda_j = \delta_{ij}$. The action of g extends to $\Lambda \otimes \mathbb{Q}$ by linearity, so that

$$u_i - g(u_i) = \sum_j (Q_{ij}^{-1}\lambda_j - Q_{ij}^{-1}g(\lambda_j)) = \sum_{j=1}^r (Q^{-1}(1-g))_{ij}\lambda_j\ , \qquad i = 1, \ldots, r\ ,$$

and the condition that g acts trivially on Λ^*/Λ is equivalent to

$$(Q^{-1}(1-g))_{ij} \in \mathbb{Z}\ , \qquad \text{for all } i,j = 1, \ldots, r\ . \tag{4.2}$$

In particular,
$$\frac{\det(1-g)}{\det Q} \in \mathbb{Z}\ , \tag{4.3}$$

which puts a non-trivial constraint on the possible Gram matrices Q, provided that $\mathrm{rk}(1 - g) = r$. The determinant $\det(1 - g)$ and the rank $\mathrm{rk}(1 - g)$ for each possible conjugacy class in $W^+(E_8)$ are given in table 1. For each $W^+(E_8)$-conjugacy class of g, there are only few isomorphism classes of lattices Λ of rank $r = \mathrm{rk}(1 - g)$ and satisfying (4.3) (see [32]). In the following sections, we consider a case by case analysis of these lattices. First of all, in section 4.3 we consider the case where G_1 has no symmetry g with $\mathrm{rk}(1 - g) = 4$. In this case, we argue that the whole group G_0 is induced by geometric automorphisms of the target space in a suitable geometric interpretation of the CFT as a non-linear sigma model on T^4.

The lattices Λ admitting a symmetry g with $\mathrm{rk}(1 - g) = 4$ and the corresponding $W^+(E_8)$ conjugacy classes of the automorphisms in $SO_0(\Lambda)$ are given in table 2. The corresponding groups $G_1 = SO_0(\Lambda)$ and their double covering G_0 cannot be described as automorphisms of the target space in any geometric interpretation of the model. A detailed description of the entries of this table is provided in section 4.4.

When $1 - g$ has rank $\mathrm{rk}(1-g) = 4 = r$, then the lattice Λ is uniquely determined, up to isomorphism, by the $W^+(E_8)$ conjugacy class of $\hat g$. Indeed, in this case, Λ is the orthogonal complement in E_8 of the sublattice fixed by $\hat g$ and for any two elements $\hat g, \hat g' \in W^+(E_8)$ in the same conjugacy class, the corresponding lattices Λ and Λ' are related by a $W^+(E_8)$ transformation and are therefore isomorphic.

For a given lattice Λ of rank four and a given element $g \in SO_0(\Lambda)$, the conjugacy class of $\hat g$ in $W^+(E_8)$ can be determined, in most cases, by the eigenvalues of g on Λ. The only exceptions are the classes 2A and 2E, that have the same eigenvalues. To distinguish these

Lattice Λ	$\det Q$	$W^+(E_8)$ classes	$G_1 \cong SO_0(\Lambda)$	G_0	$|G_0|$
D_4	4	2A, 4A, 6A	$\pm[T_{12} \times_{C_3} T_{12}]$	$\mathcal{T}_{24} \times_{C_3} \mathcal{T}_{24}$	192
A_4	5	5A	$+[I_{60} \times_{I_{60}} \bar{I}_{60}]$	$\mathcal{I}_{120} \times_{\mathcal{I}_{120}} \bar{\mathcal{I}}_{120}$	120
$A_1 A_3$	8	4C	$+[O_{24} \times_{O_{24}} \bar{O}_{24}]$	$\mathcal{O}_{48} \times_{\mathcal{O}_{48}} \bar{\mathcal{O}}_{48}$	48
A_2^2	9	3E	$+[D_6(3) \times_{C_2} D_6(3)]$	$\mathcal{D}_{12}(3) \times_{\mathcal{C}_4} \mathcal{D}_{12}(3)$	36
$A_1^2 A_2$	12	6D	$+[D_{12}(6) \times_{D_{12}(6)} D_{12}(6)]$	$\mathcal{D}_{24}(6) \times_{\mathcal{D}_{24}(6)} \mathcal{D}_{24}(6)$	24
A_1^4	16	2E	$\pm[D_4(2) \times_{D_4(2)} D_4(2)]$	$\mathcal{D}_8(4) \times_{D_4(2)} \mathcal{D}_8(4)$	16

Table 2: Lattices Λ of rank 4 admitting automorphisms $g \in SO_0(\Lambda)$ with $\mathrm{rk}(1-g) = 4$. For each lattice, we give the determinant of the Gram matrix Q, the $W^+(E_8)$ classes of the elements g with $\mathrm{rk}(1-g) = 4$, the group $G_1 \cong SO_0(\Lambda)$, its $\mathbb{Z}_2$-central extension $G_0 = \mathbb{Z}_2.G_1$ and its order $|G_0|$. See section 4.4 and appendix A for a description of the lattices and groups.

cases, it is sufficient to notice that an automorphism $g \in SO_0(\Lambda)$ corresponds to the class 2A if and only if it is the square of some $h \in SO_0(\Lambda)$ of class 4A.

When $\det Q \in \{4, 5, 8, 9\}$, there is a unique isomorphism class of lattices Λ for each value of the determinant [32] and this determines the corresponding entries in table 2. When $\det Q = 12$ or $\det Q = 16$ there are two isomorphism classes of lattices. The analysis of $SO_0(\Lambda)$ in section 4.4 shows that the lattices in table 2 admit a symmetry in the corresponding $W^+(E_8)$ class, and this excludes the remaining isomorphism classes of lattices of determinant 12 and 16.

Finally, we argue that each of the six lattices in table 2 corresponds to a unique point in the moduli space $\mathcal{M}_{T^4}$. In other words, there is a unique torus model (up to dualities) realising each of the symmetry groups of table 2. In fact, the group G_1 of a torus model is isomorphic to $SO_0(\Lambda)$, with Λ one of the entries of table 2, if and only if the lattice $S := \Gamma_{\text{even}}^{4,4} \cap \Pi_{\text{even}}^-$ is isomorphic to $\Lambda(-1)$. Vice versa, each sublattice $S \subset \Gamma_{\text{even}}^{4,4}$ isomorphic to $\Lambda(-1)$ is associated with a unique torus model with symmetry group $G_1 \cong SO_0(\Lambda)$, defined by setting $\Pi_{\text{even}}^- = S \otimes \mathbb{R}$. Thus, we have to show that any two sublattices $S, S' \subset \Gamma_{\text{even}}^{4,4}$ with $S \cong \Lambda(-1) \cong S'$ are related by a $O(\Gamma_{\text{even}}^{4,4})$ transformation, so that the associated torus models are related by some T-duality. Let K and K' be the orthogonal complements in $\Gamma_{\text{even}}^{4,4}$ of S and S', respectively. Then, by the standard gluing construction, there are isomorphisms $\gamma : S^*/S \to K^*/K$ and $\gamma' : (S')^*/S' \to (K')^*/K'$ between the discriminant groups such that the induced discriminant forms are inverted (see appendix B)

$$q_S = -q_K \circ \gamma \,, \qquad q_{S'} = -q_{K'} \circ \gamma' \,, \tag{4.4}$$

and such that the lattice $\Gamma_{\text{even}}^{4,4}$ is given by

$$\Gamma_{\text{even}}^{4,4} = \{(v, w) \in S^* \oplus K^* \mid \gamma([v]) = [w]\} = \{(v, w) \in (S')^* \oplus (K')^* \mid \gamma'([v]) = [w]\} \,, \tag{4.5}$$

where $[v]$ and $[w]$ are the images of v and w in the corresponding discriminant groups. The existence of the isomorphisms γ and γ' implies, in particular, that K and K' must be positive definite lattices of rank 4 with the same determinant as Λ. In all six cases of table 2, one finds that γ, γ' exist if and only if $K \cong \Lambda \cong K'$ [32]. Furthermore, the isomorphisms

γ and γ' satisfying (4.4) are unique up to automorphisms of S, S', K, K'. It follows that we can always find isomorphisms $f_S : S^* \to (S')^*$ and $f_K : K^* \to (K')^*$ such that the induced isomorphisms of discriminant groups satisfy

$$f_K \circ \gamma = \gamma' \circ f_S .$$

By (4.5), this implies that $f_S \oplus f_K : S^* \oplus K^* \to (S')^* \oplus (K')^*$ restricts to an automorphism of $\Gamma_{\text{even}}^{4,4}$ and, by construction, this automorphism maps S to S'.

4.3   Models with purely geometric symmetry group

In this section, we will show that G_1 contains no elements with $\text{rk}(1-g) = 4$ if and only if the whole group $G_0 = \mathbb{Z}_2.G_1$ is induced by target space automorphisms in a suitable geometric interpretation of the torus model. We will also classify such purely geometric groups.

Suppose that $G_1 \subset SO^+(\Gamma_{\text{even}}^{4,4})$ contains only elements in class 1A, 2B or 3A of $W^+(E_8)$, so that $\text{rk}(1-g) < 4$ (see table 1). Let $\Gamma^{G_1} \subset \Gamma^{4,4}$ be the sublattice fixed by G_1, let Γ_{G_1} be its orthogonal complement and denote by $\Lambda = \Gamma_{G_1}(-1)$ the lattice with opposite quadratic form. Λ is an even positive definite lattice of rank at most 4 and such that $SO_0(\Lambda) \cong G_1$; furthermore, there is no vector of Λ fixed by all elements of G_1. We will show that under the above condition for G_1, Λ has rank at most three.

Let $\Lambda^{(2)}$ denote the sublattice of Λ generated by all its elements of squared length 2 (roots). Suppose first that $\Lambda^{(2)}$ has rank four. All root lattices of rank 4 are listed in Table 2, so that $\Lambda^{(2)}$ must be one of them. Now, Λ has the same rank as $\Lambda^{(2)}$ and its vectors have integral scalar product with all elements of $\Lambda^{(2)}$ so that $\Lambda^{(2)} \subseteq \Lambda \subseteq (\Lambda^{(2)})^*$. Furthermore, all the vectors of Λ of squared norm two are contained in $\Lambda^{(2)}$. For all root lattices $\Lambda^{(2)}$ in table 2, the only sublattices of $(\Lambda^{(2)})^*$ satisfying these properties are the root lattices themselves, so that necessarily $\Lambda = \Lambda^{(2)}$. But this contradicts the assumption that G_1 has only elements in the classes 1A, 2B and 3A. It follows that $\Lambda^{(2)}$ has rank at most three. Let us denote by $\hat{\Lambda} := \Lambda \cap (\Lambda^{(2)})^\perp$ its orthogonal complement.

For each $g \in SO_0(\Lambda)$, $g \neq 1$, let $\Lambda^g \subset \Lambda$ be the sublattice fixed by g and Λ_g its orthogonal complement in Λ. By eq.(4.3) and table 1, Λ_g has rank $\text{rk}(1-g) = 2$ and determinant 3 (for g in class 3A) or 4 (for g in class 2B), so that the only possibilities for the Gram matrix of Λ_g, up to isomorphism, are

$$Q_{A_2} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} , \qquad Q_{A_1^2} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} .$$

In both cases, the lattice Λ_g is generated by its elements of squared length 2, so that $\Lambda_g \subseteq \Lambda^{(2)}$ for all $g \in G_1$. It follows that, for all $g \in G_1$, $\hat{\Lambda}$ is orthogonal to Λ_g, so that it must be fixed by g. However, by construction, there is no sublattice fixed by all $g \in G_1$, so that $\hat{\Lambda} = 0$ and Λ has the same rank as $\Lambda^{(2)}$ and thus at most three.

As a consequence, whenever G_1 contains only elements g with $\text{rk}(1-g) < 4$, the lattice $\Gamma_{\text{even}}^{4,4}$ contains a primitive sublattice K of signature $(4,1)$ which is fixed by G_1. In turn, such a K always contains an even unimodular sublattice $\Gamma^{1,1}$ of signature $(1,1)$, generated by

some $\lambda_1, \lambda_2 \in K$ with $\lambda_1^2 = \lambda_2^2 = 0$ and $\lambda_1 \bullet \lambda_2 = 1$. Two such elements of $\Gamma_{\text{even}}^{4,4}$ correspond to two primitive maximal isotropic sublattices $\Lambda_1, \Lambda_2 \subset \Gamma_{\text{w-m}}^{4,4}$, with $\Lambda_1 \oplus \Lambda_2 \cong \Gamma_{\text{w-m}}^{4,4}$ (see section 3.2), i.e. to a geometric interpretation of the CFT as a non-linear sigma model on $T^4 \cong \mathbb{R}^4/L$, for some lattice $L \in \mathbb{R}^4$ and some B-field (see section 2). It follows that the group $G_0 = \mathbb{Z}_2.G_1$ acts by $SU(2)$ transformations (2.31),(2.33) with $g_L = g_R$ on the complex coordinates $Z^{(1)}, Z^{(2)}$ on T^4 defined by (2.11); furthermore, these transformations induce automorphisms of the lattice L and preserve the B-field.

The groups of automorphisms of complex tori of complex dimension 2 have been studied by Fujiki [27]. Upon a suitable choice of the B-field, each such automorphism induces a symmetry of the corresponding non-linear sigma model, preserving a $\mathcal{N} = (4, 4)$ superconformal algebra. In particular, G_0 corresponds to the group of automorphisms that fix a point of the torus. The analysis in [27] considers the action of these geometric automorphisms on the integral second cohomology group $H^2(T^4, \mathbb{Z})$; this is the exact geometric analogue of the action of $G_1 = G_0/(-1)$ on even D-brane charges. More precisely, for a given geometric interpretation of the CFT, the lattice $\Gamma_{\text{even}}^{4,4}$ can be identified with the even integral cohomology group $H^{\text{even}}(T^4, \mathbb{Z})$. Being the group of symmetries G_1 purely geometric, it fixes the degree zero and degree four components of $H^{\text{even}}(T^4, \mathbb{Z})$, so that $(\Gamma_{\text{even}}^{4,4})_{G_1}$ can be identified with $H^2(T^4, \mathbb{Z})_{G_1}$. Our analysis, therefore, naturally leads to the same result as in Lemma 3.3 and Theorem 6.4 of [27]. The list of possible lattices $\Lambda = (\Gamma_{\text{even}}^{4,4})_{G_1}(-1)$, their rank, the group $G_1 \cong SO_0(\Lambda)$ and its $\mathbb{Z}_2$-central extension G_0 are given in table 3. Notice that, since Λ has rank at most three, G_1 is actually a subgroup of

Λ	Rank	G_1	G_0	$\lvert G_0 \rvert$	Generators of G_0
0	0	1	$\mathcal{C}_2$	2	$(-1, -1)$
A_1^2	2	$\mathbb{Z}_2$	$\mathcal{C}_4$	4	$(\mathbf{i}, \mathbf{i})$
A_2	2	$\mathbb{Z}_3$	$\mathcal{C}_6$	6	$(\mathbf{e}_{\frac{1}{6}}, \mathbf{e}_{\frac{1}{6}})$
A_1^3	3	$\mathbb{Z}_2^2$	$\mathcal{D}_8(2)$	8	$(\mathbf{i}, \mathbf{i}), \quad (\mathbf{j}, \mathbf{j})$
$A_1 A_2$	3	$\mathbb{Z}_3 \times \mathbb{Z}_2$	$\mathcal{D}_{12}(3)$	12	$(\mathbf{j}, \mathbf{j}), \quad (\mathbf{e}_{\frac{1}{6}}, \mathbf{e}_{\frac{1}{6}})$
A_3	3	$Alt(4)$	$\mathcal{T}_{24}$	24	$(\mathbf{i}, \mathbf{i}), \quad (\boldsymbol{\omega}, \boldsymbol{\omega})$

Table 3: The possible lattices $\Lambda \cong (\Gamma_{\text{even}}^{4,4})_{G_1}$ with $\text{rk}\,\Lambda < 4$. For each such lattice, we report the rank, the group $G_1 = SO_0(\Lambda)$ of automorphisms fixing the discriminant group Λ^*/Λ, the double covering $G_0 = \mathbb{Z}_2.G_1$ and the order and generators of $G_0 \subset SU(2) \times SU(2)$. We use the notation $\mathbf{e}_r := \exp(2\pi \mathbf{i} r)$ and $\boldsymbol{\omega} = \frac{1}{2}(-1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$.

$SO(3)$ and its spin cover G_0 is a subgroup of $SU(2)$. This $SU(2)$ is diagonally embedded in the group $SU(2)_L \times SU(2)_R$ acting as in (2.30)–(2.33), that is $G_0 \subset SU(2)_L \times SU(2)_R$ is generated by elements of the form $(g_L, g_R) = (q, q)$ for some quaternions $q \in SU(2)$.

The torus models with a given symmetry group G_1, associated with a lattice $\Lambda = (\Gamma_{\text{even}}^{4,4})_{G_1}(-1)$ of rank r, form a sublocus of dimension $4(4-r)$ in the moduli space. Indeed, this is the dimension of the Grassmannian parametrizing the choice of a four dimensional positive-definite space Π_{even}^+ within the space $(\Gamma_{\text{even}}^{4,4})^{G_1} \otimes \mathbb{R}$ of signature $(4, 4 - r)$.

4.4 Models with non-geometric symmetries

As discussed in section 4.2, there are six torus models that admit symmetries preserving the $\mathcal{N} = (4, 4)$ algebra and with no geometric interpretation. They are characterized by the condition that $\Gamma_{\text{even}}^{4,4} \cap \Pi_{\text{even}}^{-} \cong \Lambda(-1)$, where Λ is one of the six lattices in table 2. In this section, we provide some more details about the lattices Λ and the corresponding models.

For each of these six cases, we first describe the lattice Λ: we provide the matrix $Q_{ij} := \lambda_i \cdot \lambda_j$ for a standard choice of generators $\lambda_1, \ldots, \lambda_4$ of Λ, the determinant $\det Q$, the structure of the discriminant group Λ^*/Λ and the values of the induced discriminant form $q : \Lambda^*/\Lambda \to \mathbb{Q}/2\mathbb{Z}$ on the generators of the discriminant group. We also provide a definition of $\Lambda \subset \mathbb{R}^4$ as a lattice in the space of quaternions $\mathbb{H} \cong \mathbb{R}^4$. Next, we describe the group $G_1 \cong SO_0(\Lambda)$ of automorphisms of Λ with positive determinant and that act trivially on the discriminant group. In all the cases we are interested in, Λ is the root lattice of some Lie algebra and $SO_0(\Lambda)$ is the group $W^+(\Lambda)$ of even Weyl transformations. Since G_1 is a finite subgroup of $SO(4) \cong (SU(2) \times SU(2))/(-1, -1)$, it can be described as $G_1 = \pm[L \times_F R]$ or $G_1 = +[L \times_F R]$, depending on whether $-1 \in G_1$ or not (see appendix A). We classify the elements of G_1 depending on their order and conjugacy class in $W^+(E_8)$ and for each such class we indicate the number of elements in G_1.

The group G_0 is the preimage of G_1 under the covering map $\text{Spin}(4) \to SO(4)$ and it can be easily determined as discussed in appendix A. We provide a set of generators (g_L, g_R) of $G_0 \subset SU(2) \times SU(2)$ in the form of a pair of quaternions. Then, the generators of G_1 are the $SO(4)$ transformations $[g_L, g_R] \equiv [-g_L, -g_R]$ acting by

$$[g_L, g_R] : \lambda \mapsto g_L \lambda \bar{g}_R , \qquad \lambda \in \Lambda \subset \mathbb{H} , \quad [g_L, g_R] \in G_1 ,$$

on the lattice $\Lambda = (\Gamma_{\text{even}}^{4,4})_{G_1}(-1)$.

Next, we provide a description of the winding-momentum lattice $\Gamma_{\text{w-m}}^{4,4}$ as a lattice in $\mathbb{R}^{4,4} \cong \Pi_L \oplus \Pi_R \cong \mathbb{H} \oplus \mathbb{H}$. In other words, we denote the elements of $\Gamma_{\text{w-m}}^{4,4}$ as pairs of quaternions $(q_L, q_R) \in \Gamma_{\text{w-m}}^{4,4}$, where $q_L \in \Pi_L \cong \mathbb{H}$, $q_R \in \Pi_R \cong \mathbb{H}$, so that the action of $(g_L, g_R) \in G_0 \subset SU(2) \times SU(2)$ is simply given by

$$(g_L, g_R) : (q_L, q_R) \mapsto (g_L q_L, g_R q_R) \qquad (q_L, q_R) \in \Gamma_{\text{w-m}}^{4,4}, \quad (g_L, g_R) \in G_0 ,$$

as in eq.(2.39). The action of (g_L, g_R) on the fields of the theory is given by (2.30)–(2.33). We also describe the sublattices $\Gamma_{\text{w-m}}^{4,4} \cap \Pi_L$ and $\Gamma_{\text{w-m}}^{4,4} \cap \Pi_R$ of purely holomorphic and purely anti-holomorphic winding-momenta. If these sublattices have positive rank, then the chiral algebra of the model is extended with respect to the one of a generic torus model.

Finally, we provide a possible geometric interpretation of the CFT as a non-linear sigma model with target space $\mathbb{R}^4/L$ and B-field B. The information about L and B is given in the form of a matrix whose columns $l_1, \ldots, l_4$ form a set generators for $L \subset \mathbb{R}^4$, the quadratic from $Q_{ij} = l_i \cdot l_j$ and the B-field B in the form of a real antisymmetric matrix, such that

$$\Gamma_{\text{w-m}}^{4,4} = \left\{ \frac{1}{\sqrt{2}} \left(n_i l^{i*} + w^i l_i + w^i B_{ij} l^{j*} , \, n_i l^{i*} - w^i l_i + w^i B_{ij} l^{j*} \right) \mid n_1, \ldots, n_4, w^1, \ldots w^4 \in \mathbb{Z} \right\} .$$

Here $l^{1*}, \ldots, l^{4*}$ is the basis of L^* dual to $l_1, \ldots, l_4$, given by $l^{i*} = (Q^{-1})^{ij} l_j$.

4.4.1 Lattice D_4

Λ_{D_4} is the root lattice of $so(8)$

$$Q_{D_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix}$$

$$\det Q = 4$$

$$\Lambda^*/\Lambda = \langle x, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$q(x) = q(y) = q(x+y) = 1 \mod 2\mathbb{Z}$$

Quaternionic:

$$\Lambda_{D_4} \cong \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \mid a, b, c, d \in \mathbb{Z}, a + b + c + d \in 2\mathbb{Z}\}$$

$$\Lambda_{D_4}^* \cong \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \mid (a, b, c, d) \in \mathbb{Z}^4 \text{ or } (\mathbb{Z} + \tfrac{1}{2})^4\} \cong \frac{1}{\sqrt{2}} \Lambda_{D_4} \ ,$$

Automorphism group $G_1 = SO_0(\Lambda_{D_4})$:

$$G_1 = \pm[T_{12} \times_{C_3} T_{12}]$$

$$|G_1| = 96$$

Order	1	2	2	3	4	6
$W^+(E_8)$ class	1A	2A	2B	3A	4A	6A
# elements	1	1	18	32	12	32

Symmetry group $G_0 = \mathcal{T}_{24} \times_{C_3} \mathcal{T}_{24}$, order $|G_0| = 192$, with generators

$$G_0 = \langle (1, \mathbf{i}), (1, \mathbf{j}), (\mathbf{i}, 1), (\mathbf{j}, 1), (\boldsymbol{\omega}, \boldsymbol{\omega}) \rangle \qquad \boldsymbol{\omega} = \frac{1}{2}(-1 + \mathbf{i} + \mathbf{j} + \mathbf{k})$$

Lattice of winding-momenta:

$$\Gamma_{\text{w-m}}^{4,4} = \left\{ \frac{1}{\sqrt{2}}(a_1 + a_2\,\mathbf{i} + a_3\,\mathbf{j} + a_4\,\mathbf{k}, \ b_1 + b_2\,\mathbf{i} + b_3\,\mathbf{j} + b_4\,\mathbf{k}) \mid a_i, b_i \in \mathbb{Z}, \right.$$

$$\left. \sum_i a_i \in 2\mathbb{Z}, \ a_1 - b_1 \equiv a_2 - b_2 \equiv a_3 - b_3 \equiv a_4 - b_4 \mod 2 \right\}$$

Purely left- and purely right-moving momenta:

$$\Gamma_{\text{w-m}}^{4,4} \cap \Pi_L \cong \Gamma_{\text{w-m}}^{4,4} \cap \Pi_R = \left\{ \sqrt{2}(a_1 + a_2\,\mathbf{i} + a_3\,\mathbf{j} + a_4\,\mathbf{k}) \mid (a_1, \ldots, a_4) \in \mathbb{Z}^4 \text{ or } (\mathbb{Z} + \tfrac{1}{2})^4 \right\} \cong \Lambda_{D_4}$$

Geometric description as torus model on $\mathbb{R}^4/L$ and B-field B

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad Q_L = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & -\tfrac{1}{2} & -\tfrac{1}{2} \\ 0 & 0 & 0 & 0 \\ \tfrac{1}{2} & 0 & 0 & -\tfrac{1}{2} \\ \tfrac{1}{2} & 0 & \tfrac{1}{2} & 0 \end{pmatrix}$$

4.4.2   Lattice A_4

Λ_{A_4} is the root lattice of $su(5)$

$$Q_{A_4} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\det Q = 5$$

$$\Lambda^*/\Lambda = \langle x \rangle \cong \mathbb{Z}_5$$

$$q(x) = q(4x) = \tfrac{4}{5}, \quad q(2x) = q(3x) = \tfrac{6}{5}$$

Quaternionic:

$$\Lambda_{A_4} = \{\sqrt{2}(a+b\,\mathbf{i}+c\,\boldsymbol{\omega}+d\,\boldsymbol{\iota}_I) \mid a,b,c,d \in \mathbb{Z}\} \qquad \boldsymbol{\omega} = \tfrac{1}{2}(-1+\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad \boldsymbol{\iota}_I = \tfrac{1}{2}(\mathbf{i}+\tfrac{-\sqrt{5}-1}{2}\mathbf{j}+\tfrac{\sqrt{5}-1}{2}\mathbf{k})$$

Automorphism group $G_1 = SO_0(\Lambda_{A_4})$:

$$G_1 = +[I_{60} \times_{I_{60}} \bar{I}_{60}] \cong Alt(5)$$

$$|G_1| = 60$$

Order	1	2	3	5
$W^+(E_8)$ class	$1A$	$2B$	$3A$	$5A$
# elements	1	15	20	24

Symmetry group $G_0 = \mathcal{I}_{120} \times_{\mathcal{I}_{120}} \bar{\mathcal{I}}_{120} \cong 2.Alt(5)$, order $|G_0| = 120$, with generators

$$G_0 = \big\langle\, (\boldsymbol{\omega}, \eta(\boldsymbol{\omega})), \quad (\boldsymbol{\iota}_I, \eta(\boldsymbol{\iota}_I)) \,\big\rangle, \qquad \boldsymbol{\omega} = \tfrac{1}{2}(-1+\mathbf{i}+\mathbf{j}+\mathbf{k}), \quad \boldsymbol{\iota}_I = \tfrac{1}{2}(\mathbf{i}+\tfrac{-\sqrt{5}-1}{2}\mathbf{j}+\tfrac{\sqrt{5}-1}{2}\mathbf{k})$$

The $\mathbb{Q}$-linear map η acts by $\sqrt{5} \mapsto -\sqrt{5}$, $\mathbf{i} \mapsto -\mathbf{i}$, $\mathbf{j} \mapsto -\mathbf{k}$, $\mathbf{k} \mapsto -\mathbf{j}$. It is an outer automorphism of $\mathcal{I}_{120}$. In particular, $\eta(\boldsymbol{\omega}) = \bar{\boldsymbol{\omega}}$ and $\eta(\boldsymbol{\iota}_I) = \bar{\boldsymbol{\iota}}_I$.

Lattice of winding-momenta:

$$\Gamma^{4,4}_{\text{w–m}} = \frac{\sqrt{2}}{5^{1/4}} \big\langle (1,1),\ (\mathbf{i}, \eta(\mathbf{i})),\ (\mathbf{k}, \eta(\mathbf{k})),\ (\boldsymbol{\omega}, \eta(\boldsymbol{\omega})),$$
$$(\boldsymbol{\iota}_I, \eta(\boldsymbol{\iota}_I)),\ (\mathbf{i}\boldsymbol{\iota}_I, \eta(\mathbf{i}\boldsymbol{\iota}_I)),\ (\mathbf{j}\boldsymbol{\iota}_I, \eta(\mathbf{j}\boldsymbol{\iota}_I)),\ (\boldsymbol{\omega}\boldsymbol{\iota}_I, \eta(\boldsymbol{\omega}\boldsymbol{\iota}_I)) \big\rangle_{\mathbb{Z}}$$

Purely left- and purely right-moving momenta: $\Gamma^{4,4}_{\text{w–m}} \cap \Pi_L = 0 = \Gamma^{4,4}_{\text{w–m}} \cap \Pi_R$

Geometric description as torus model on $\mathbb{R}^4/L$ and B-field B

$$L = \frac{\sqrt{2}}{5^{1/4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix} \qquad Q_L = \frac{2}{\sqrt{5}} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -\frac{1}{2} & -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{2} & 0 & -\frac{1}{2} & -\frac{1}{2} \\ \frac{3}{4} & \frac{1}{2} & 0 & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & -\frac{1}{4} & 0 \end{pmatrix}$$

4.4.3 Lattice $A_1 A_3$

$\Lambda_{A_1 A_3}$ is the root lattice of $su(2) \oplus su(4)$

$$Q_{A_1 A_3} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\det Q = 8$$

$$\Lambda^*/\Lambda = \langle x, y \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_4$$

$$q(y) = q(3y) = \tfrac{3}{4}, \quad q(2y) = 1, \quad q(ax + by) = \tfrac{a^2}{2} + q(by)$$

Quaternionic:

$$\Lambda_{A_1 A_3} = \{(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k})\boldsymbol{\omega} + \sqrt{2}e \mid a + b + c + d = 0, \quad a, b, c, d, e \in \mathbb{Z}\}$$

Automorphism group $G_1 = SO_0(\Lambda_{A_1 A_3})$:

$$G_1 \cong +[O_{24} \times_{O_{24}} \bar{O}_{24}] \cong Sym(4)$$

$$|G_1| = 24$$

Order	1	2	3	4
$W^+(E_8)$ class	1A	2B	3A	4C
# elements	1	9	8	6

Symmetry group $G_0 = \mathcal{O}_{48} \times_{\mathcal{O}_{48}} \bar{\mathcal{O}}_{48} \cong 2.Sym(4)$, order $|G_0| = 48$, with generators

$$G_0 = \langle (\boldsymbol{\omega}, \boldsymbol{\omega}), \quad (\boldsymbol{\iota}_O, -\boldsymbol{\iota}_O) \rangle, \qquad \boldsymbol{\omega} = \tfrac{1}{2}(-1 + \mathbf{i} + \mathbf{j} + \mathbf{k}), \quad \boldsymbol{\iota}_O = \frac{\mathbf{j} + \mathbf{k}}{\sqrt{2}}$$

Lattice of winding momenta:

$$\Gamma_{\text{w--m}}^{4,4} = 2^{-\frac{1}{4}} \left\langle (1, 1), (\mathbf{i}, \mathbf{i}), (\mathbf{j}, \mathbf{j}), (\boldsymbol{\omega}, \boldsymbol{\omega}), (\boldsymbol{\iota}_O, -\boldsymbol{\iota}_O), (\mathbf{i}\boldsymbol{\iota}_O, -\mathbf{i}\boldsymbol{\iota}_O), (\mathbf{j}\boldsymbol{\iota}_O, -\mathbf{j}\boldsymbol{\iota}_O), (\mathbf{k}\boldsymbol{\iota}_O, -\mathbf{k}\boldsymbol{\iota}_O) \right\rangle_{\mathbb{Z}}$$

Purely left- and purely right-moving momenta: $\Gamma_{\text{w--m}}^{4,4} \cap \Pi_L = 0 = \Gamma_{\text{w--m}}^{4,4} \cap \Pi_R$

Geometric description as torus model on $\mathbb{R}^4/L$ and B-field B

$$L = \frac{1}{2^{3/4}} \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & \sqrt{3} \end{pmatrix} \qquad Q_L = \frac{1}{\sqrt{2}} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ \frac{1}{3} & -\frac{1}{3} & 0 & 0 \end{pmatrix}$$

4.4.4 Lattice A_2^2

$\Lambda_{A_2^2}$ is the root lattice of $su(3) \oplus su(3)$

$$Q_{A_2^2} = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\det Q = 9$$

$$\Lambda^*/\Lambda = \langle x, y \rangle \cong \mathbb{Z}_3 \times \mathbb{Z}_3$$

$$q(x) = q(2x) = q(y) = q(2y) = \tfrac{2}{3},$$
$$q(ax + by) = q(ax) + q(by)$$

Quaternionic:

$$\Lambda_{A_2^2} = \{\sqrt{2}(a + b\,\mathbf{e}_{\frac{1}{3}} + c\mathbf{j} + d\,\mathbf{j}\,\mathbf{e}_{\frac{1}{3}}) | a, b, c, d \in \mathbb{Z}\} \qquad \mathbf{e}_{\frac{1}{3}} := \exp(2\pi\mathbf{i}/3)$$

Automorphism group $G_1 = SO_0(\Lambda_{A_2^2})$:

$$G_1 \cong +[D_6(3) \times_{C_2} D_6(3)] \cong (\mathbb{Z}_3 \times \mathbb{Z}_3).\mathbb{Z}_2$$

$$|G_1| = 18$$

Order	1	2	3	3
$W^+(E_8)$ class	1A	2B	3A	3E
# elements	1	9	4	4

Symmetry group $G_0 = \mathcal{D}_{12}(3) \times_{\mathcal{C}_4} \mathcal{D}_{12}(3)$, order $|G_0| = 36$, with generators

$$G_0 = \langle\, (1, \mathbf{e}_{\frac{1}{3}}),\ (\mathbf{e}_{\frac{1}{3}}, 1),\ (-\mathbf{j}, \mathbf{j})\, \rangle \qquad \mathbf{e}_{\frac{1}{3}} := \exp(2\pi\mathbf{i}/3) = -\frac{1}{2} + \frac{\sqrt{3}}{2}\mathbf{i}$$

Lattice of winding momenta:

$$\Gamma_{\text{w-m}}^{4,4} = \left\{ \sqrt{\frac{2}{3}}(a_1 + a_2\,\mathbf{e}_{\frac{1}{3}} + b_1\mathbf{j} + b_2\,\mathbf{j}\,\mathbf{e}_{\frac{1}{3}},\ \tilde{a}_1 + \tilde{a}_2\,\mathbf{e}_{\frac{1}{3}} + \tilde{b}_1\mathbf{j} + \tilde{b}_2\,\mathbf{j}\,\mathbf{e}_{\frac{1}{3}}) \mid a_i, b_i, \tilde{a}_i, \tilde{b}_i \in \mathbb{Z}, \right.$$
$$\left. a_1 + a_2 \equiv \tilde{a}_1 + \tilde{a}_2 \mod 3, \quad b_1 + b_2 \equiv -(\tilde{b}_2 + \tilde{b}_1) \mod 3 \right\}$$

Purely left- and purely right-moving momenta:

$$\Gamma_{\text{w-m}}^{4,4} \cap \Pi_{L,R} = \left\{ \sqrt{\tfrac{2}{3}}\big(a_1(1 - \mathbf{e}_{\frac{1}{3}}) + 3a_2 + b_1\mathbf{j}(1 - \mathbf{e}_{\frac{1}{3}}) + 3b_2\mathbf{j}\big) \mid a_1, a_2, b_1, b_2 \in \mathbb{Z} \right\} \cong \Lambda_{A_2^2}$$

Geometric description as torus model on $\mathbb{R}^4/L$ and B-field B

$$L = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} \end{pmatrix} \qquad Q_L = \frac{2}{3} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -\frac{2}{3} & 0 & 0 \\ \frac{2}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{2}{3} \\ 0 & 0 & \frac{2}{3} & 0 \end{pmatrix}$$

4.4.5 Lattice $A_1^2 A_2$

$\Lambda_{A_1^2 A_2}$ is the root lattice of $su(2) \oplus su(2) \oplus su(3)$

$$Q_{A_1^2 A_2} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

$$\det Q = 12$$

$$\Lambda^*/\Lambda = \langle x, y, z \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3$$

$$q(x) = q(y) = 1/2, \quad q(z) = q(2z) = \tfrac{2}{3},$$
$$q(ax + by + cz) = q(ax) + q(by) + q(cz)$$

Quaternionic:

$$\Lambda_{A_1^2 A_2} = \{\sqrt{2}(a + b\,\mathbf{i} + c\,\mathbf{j} + d\,\mathbf{j}\,\mathbf{e}_{\frac{1}{3}}) \mid a, b, c, d \in \mathbb{Z}\} \qquad \mathbf{e}_{\frac{1}{3}} = \exp(2\pi\mathbf{i}/3)$$

Automorphism group $G_1 = SO_0(\Lambda_{A_1^2 A_2})$:

$$G_1 = +[D_{12}(6) \times_{D_{12}(6)} D_{12}(6)]$$
$$\cong D_{12}(6) \cong \mathbb{Z}_2 \times Sym(3)$$
$$|G_1| = 12$$

Order	1	2	3	6
$W^+(E_8)$ class	1A	2B	3A	6D
# elements	1	7	2	2

Symmetry group $G_0 = \mathcal{D}_{24}(6) \times_{\mathcal{D}_{24}(6)} \mathcal{D}_{24}(6) \cong \mathcal{D}_{24}(6)$, order $|G_0| = 24$, with generators

$$G_0 = \langle\, (\mathbf{e}_{\frac{1}{3}}, \mathbf{e}_{\frac{1}{3}}), \quad (-\mathbf{j}, \mathbf{j}), \quad (-\mathbf{i}, \mathbf{i}) \rangle$$

Lattice of winding momenta:

$$\Gamma_{\text{w-m}}^{4,4} = 3^{-\frac{1}{4}} \Big(\big\langle (1,1),\ (\mathbf{i}, -\mathbf{i}),\ (\mathbf{e}_{\frac{1}{3}}, \mathbf{e}_{\frac{1}{3}}),\ (\mathbf{i}\,\mathbf{e}_{\frac{1}{3}}, -\mathbf{i}\,\mathbf{e}_{\frac{1}{3}}) \big\rangle_{\mathbb{Z}}$$
$$\oplus \big\langle (\mathbf{j}, -\mathbf{j}),\ (\mathbf{k}, \mathbf{k}),\ (\mathbf{j}\,\mathbf{e}_{\frac{1}{3}}, -\mathbf{j}\,\mathbf{e}_{\frac{1}{3}}),\ (\mathbf{k}\,\mathbf{e}_{\frac{1}{3}}, \mathbf{k}\,\mathbf{e}_{\frac{1}{3}}) \big\rangle_{\mathbb{Z}} \Big)$$

Purely left- and purely right-moving momenta: $\Gamma_{\text{w-m}}^{4,4} \cap \Pi_L = 0 = \Gamma_{\text{w-m}}^{4,4} \cap \Pi_R$

Geometric description as torus model on $\mathbb{R}^4/L$ and B-field B

$$L = \frac{\sqrt{2}}{3^{1/4}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad Q_L = \frac{2}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -\frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \\ 0 & 0 & -\frac{1}{2} & 0 \end{pmatrix}$$

4.4.6   Lattice A_1^4

$\Lambda_{A_1^4}$ is the root lattice of $su(2) \oplus su(2) \oplus su(2) \oplus su(2)$

$$Q_{A_1^4} = \begin{pmatrix} 2\ 0\ 0\ 0 \\ 0\ 2\ 0\ 0 \\ 0\ 0\ 2\ 0 \\ 0\ 0\ 0\ 2 \end{pmatrix} \qquad\qquad \begin{array}{c} \det Q = 16 \\[4pt] \Lambda^*/\Lambda = \langle x_1, \ldots, x_4 \rangle \cong \mathbb{Z}_2^4 \\[4pt] q(\sum_i a_i x_i) = \tfrac{1}{2} \sum_i a_i, \qquad a_i \in \{0,1\} \end{array}$$

Quaternionic:

$$\Lambda_{A_1^4} = \{\sqrt{2}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \mid a, b, c, d \in \mathbb{Z}\}$$

Automorphism group $G_1 = SO_0(\Lambda_{A_1^4})$:

$$G_1 \cong \pm[D_4(2) \times_{D_4(2)} D_4(2)] \cong \mathbb{Z}_2^3$$

$$|G_1| = 8$$

Order	1	2	2
$W^+(E_8)$ class	1A	2B	2E
# elements	1	6	1

Symmetry group $G_0 = \mathcal{D}_8(2) \times_{D_4(2)} \mathcal{D}_8(2)$, order $|G_0| = 16$, with generators

$$G_0 = \langle\ (1, -1), \quad (\mathbf{i}, \mathbf{i}), \quad (\mathbf{j}, \mathbf{j})\ \rangle$$

Lattice of winding momenta:

$$\Gamma^{4,4}_{\text{w-m}} = \left\{ \frac{1}{\sqrt{2}}(a_1 + a_2\,\mathbf{i} + a_3\,\mathbf{j} + +a_4\,\mathbf{k},\ b_1 + b_2\,\mathbf{i} + b_3\,\mathbf{j} + +b_4\,\mathbf{k}) \mid a_i, b_i \in \mathbb{Z},\ a_i + b_i \in 2\mathbb{Z} \right\}$$

Purely left- and purely right-moving momenta:

$$\Gamma^{4,4}_{\text{w-m}} \cap \Pi_L = \Gamma^{4,4}_{\text{w-m}} \cap \Pi_R = \{\sqrt{2}(a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}) \mid a, b, c, d \in \mathbb{Z}\} \cong \Lambda_{A_1^4}$$

Geometric description as torus model on $\mathbb{R}^4/\mathbb{Z}^4$ and B-field $B = 0$

$$L = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \end{pmatrix} \qquad Q_L = \begin{pmatrix} 1\ 0\ 0\ 0 \\ 0\ 1\ 0\ 0 \\ 0\ 0\ 1\ 0 \\ 0\ 0\ 0\ 1 \end{pmatrix} \qquad B = \begin{pmatrix} 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \\ 0\ 0\ 0\ 0 \end{pmatrix}$$

5 Twining genera and orbifolds

The elliptic genus of a theory with $\mathcal{N} = (4, 4)$ superconformal algebra is defined as

$$\phi(\tau, z) = \text{Tr}_{RR}(q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} y^{J_0^3} (-1)^{F_L + F_R}) , \qquad q := e^{2\pi i \tau}, \quad y := e^{2\pi i z} .$$

It is invariant under deformations and gets non-vanishing contributions only from right-moving BPS states, i.e. with right-moving conformal weight $\tilde{h} = \frac{c}{24}$. For non-linear sigma models on T^4, the elliptic genus vanishes, because the R-R BPS states form an even dimensional representation for the Clifford algebra of the right-moving fermionic zero modes $\tilde{\psi}_0^a$, and contributions with positive and negative $(-1)^{F_R}$ cancel each other exactly.

When the CFT has a group of symmetries G preserving the $\mathcal{N} = (4, 4)$ superconformal algebra, one can define the twining genus for each element $g \in G$ (which really depends on the conjugacy class of g) by

$$\phi_g(\tau, z) = \text{Tr}_{RR}(g q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} y^{J_0^3} (-1)^{F_L + F_R}) .$$

For a non-linear sigma model on T^4 or K3, ϕ_g is a weak Jacobi form of weight 0 and index 1 under a congruence subgroup of $SL(2, \mathbb{Z})$, which depends on g.

One can also construct the orbifold of the conformal field theory by the cyclic group $\langle g \rangle$, by introducing the g^i-twisted sectors, $i = 1, \ldots, N-1$, and including in the spectrum of the theory only the g-invariant subspace of all twisted and untwisted sectors. The orbifold is a consistent CFT, provided that the level-matching condition is satisfied, i.e. that the spin $h - \tilde{h}$ of the g^i-twisted fields takes values in $\frac{i}{N}\mathbb{Z}$, for all $i \in \mathbb{Z}$. By construction, the orbifold is a $\mathcal{N} = (4, 4)$ superconformal algebra at central charge $c = \tilde{c} = 6$, so that, if consistent, it must be a non-linear sigma model on T^4 or K3. These two cases can be distinguished by the elliptic genus of the orbifold, which is given by the formula

$$\phi^{orb}(\tau, z) = \frac{1}{N} \sum_{i,j=0}^{N-1} \phi_{g^i, g^j}(\tau, z) ,$$

where ϕ_{g^i, g^j} is defined as the g^j-twining trace over the g^i-twisted sector

$$\phi_{g^i, g^j}(\tau, z) = \text{Tr}_{RR, \ g^i\text{-twisted}}(g^j q^{L_0 - \frac{c}{24}} \bar{q}^{\tilde{L}_0 - \frac{\tilde{c}}{24}} y^{J_0^3} (-1)^{F_L + F_R}) .$$

In fact, in order to distinguish between a torus and a K3 model, it is sufficient to consider the value of the elliptic genus at $z = 0$ (the Witten index), which is a constant equal to the Euler number of the target space

$$\phi(\tau, z = 0) = \begin{cases} 0 & \text{for a torus model,} \\ 24 & \text{for a K3 model.} \end{cases}$$

In the rest of this section, we will compute the twining genera and determine the kind (either torus or K3 model) of the orbifold theory for all possible symmetries $g \in G$ of a non-linear sigma model on T^4.

First of all, if g acts trivially on the right-moving fermions $\tilde{\psi}^a$, $a = 1, \ldots, 4$, then the twining genus ϕ_g vanishes for the same reasons as for the elliptic genus. Furthermore, since the fermions are included in the spectrum of the orbifold theory, also the orbifold elliptic genus ϕ^{orb} vanishes. This implies that if $\langle g \rangle$ is a finite subgroup of $U(1)^4 \times U(1)^4$, then the orbifold by this group, if consistent, gives again a torus model. Therefore, it is sufficient to focus on the elements $g \in G$ with non-trivial image $g_0 \in G_0$ modulo $U(1)^4 \times U(1)^4$. It is convenient to label each $g_0 \in G_0$ by the $W^+(E_8)$ class of the corresponding $g' \in G_1 \cong G_0/\mathbb{Z}_2$ and by the sign of the eigenvalues of the lift g_0 when acting on the fermions $\psi^a, \tilde{\psi}^a$, $a = 1, \ldots, 4$. For example, $+3A$ denotes an element g_0 which is the lift to G_0 of some $g' \in G_1$ in the $W^+(E_8)$ class $3A$, with eigenvalues $+\zeta_L, +\zeta_L^{-1}, +\zeta_R, +\zeta_R^{-1}$ as listed in table 1. For the classes acting asymmetrically between left- and right-movers, we use a prime, as in $\pm 4A'$, to denote the lift g_0 where the left-moving eigenvalues $\pm \zeta_L, \pm \zeta_L^{-1}$ and the right-moving ones $\pm \zeta_R, \pm \zeta_R^{-1}$ are exchanged with respect to the ones listed in table 1.

For a given $g \in G$, whose corresponding image $g_0 \in G_0$ has eigenvalues $\zeta_L, \zeta_L^{-1}, \zeta_R, \zeta_R^{-1}$, the twining genus is given by

$$\phi_g(\tau, z) = \phi_g^{gd}(\tau, z)\phi_g^{osc}(\tau, z)\phi_g^{w-m}(\tau) \;,$$

where the contribution from the 16 R-R ground states is

$$\phi_g^{gd}(\tau, z) = y^{-1}(1 - \zeta_L y)(1 - \zeta_L^{-1} y)(1 - \zeta_R)(1 - \zeta_R^{-1}) = 2(1 - \Re(\zeta_R))(y^{-1} + y - 2\Re(\zeta_L)) \;,$$

the contribution from the left-moving fermionic and bosonic oscillators is

$$\phi_g^{osc}(\tau, z) = \prod_{n=1}^{\infty} \frac{(1 - \zeta_L y q^n)(1 - \zeta_L^{-1} y q^n)(1 - \zeta_L y^{-1} q^n)(1 - \zeta_L^{-1} y^{-1} q^n)}{(1 - \zeta_L q^n)^2 (1 - \zeta_L^{-1} q^n)^2} \;,$$

and the contribution from winding-momentum is

$$\phi_g^{w-m}(\tau) = \sum_{(\vec{\lambda}_L, 0) \in (\Gamma_{w-m}^{4,4} \cap \Pi_L)^{g_0}} \xi_g(\lambda) q^{\frac{\vec{\lambda}_L^2}{2}} \;.$$

Notice that $\phi_g^{w-m}(\tau)$ is the only factor of ϕ_g which depends on the specific element $g \in G$ rather than on its image g_0 in the quotient group G_0. Furthermore, $\phi_g^{w-m}(\tau) = 1$ unless there are non-zero purely left-moving vectors $(\vec{\lambda}_L, 0) \in \Gamma_{w-m}^{4,4} \cap \Pi_L$ that are fixed by g_0. This twining genus can be written in terms of eta and theta functions as

$$\phi_g(\tau, z) = (\zeta_R + \zeta_R^{-1} - 2)\frac{\vartheta_1(\tau, z)^2}{\eta(\tau)^6}\Theta_{\Lambda_L, v}(\tau) \;, \qquad \zeta_L = 1 \tag{5.1}$$

or

$$\phi_g(\tau, z) = (\zeta_R + \zeta_R^{-1} - 2)(\zeta_L + \zeta_L^{-1} - 2)\frac{\vartheta_1(\tau, z + r_L)\vartheta_1(\tau, z - r_L)}{\vartheta_1(\tau, r_L)\vartheta_1(\tau, -r_L)} \;, \qquad \zeta_L \neq 1 \tag{5.2}$$

where $r_L \in \mathbb{Q}/\mathbb{Z}$ is such that $\zeta_L = e^{2\pi i r_L}$ and

$$\vartheta_1(\tau, z) = -iq^{\frac{1}{8}}(y^{-\frac{1}{2}} - y^{\frac{1}{2}})\prod_{n=1}^{\infty}(1 - q^n)(1 - y q^n)(1 - y^{-1} q^n) \;,$$

g	$o(g)$	$\dim \mathcal{H}_U$	$\dim \mathcal{H}_{g^1} + \ldots + \dim \mathcal{H}_{g^{N-1}}$	$\dim \mathcal{H}$	$\mathrm{Tr}_{\mathbf{24}}(Q_g)$	Q_g
$-1A$	2	8	16	24	-8	$2B$
$+3A$	3	6	$9+9$	24	-3	$3C$
$\pm 2B$	4	6	$4+10+4$	24	-4	$4C$
$-3A$	6	6	$1+5+6+5+1$	24	-4	$6A$
$-4A$	4	4	$8+4+8$	24	0	$4D$
$+5A$	5	4	$5+5+5+5$	24	-1	$5C$
$-3E$	6	4	$4+2+8+2+4$	24	-2	$6L$
$\pm 6A$	6	4	$3+6+2+6+3$	24	-1	$6M$
$\pm 4C$	8	4	$2+3+2+6+2+3+2$	24	-2	$8G$
$-5A$	10	4	$1+3+1+3+4+3+1+3+1$	24	-2	$10F$
$\pm 6D$	12	4	$1+1+2+3+1+4+1+3+2+1+1$	24	-2	$12N$

Table 4: For each $W^+(E_8)$ conjugacy class of $g \in G_0$ with orbifold K3, we provide the order $o(g)$ in G_0, the dimension of the untwisted R-R ground sector $\mathcal{H}_U$, the dimensions of the g-invariant twisted sectors $\mathcal{H}_{g^k}$, $k = 1, \ldots, N = o(g)$, the total dimension of the space $\mathcal{H}$ of R-R ground states in the orbifold (which is always 24 for a K3 model), the trace of the quantum symmetry Q over $\mathcal{H}$ and the Co_0 class of Q_g, as reported in [26].

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1 - q^n) \ .$$

Furthermore, in the case $\zeta_L = 1$, i.e. when g fixes Π_L, we define the theta series

$$\Theta_{\Lambda_L, v}(\tau) := \sum_{(\vec{\lambda}_L, 0) \in \Lambda_L} q^{\frac{\lambda_L^2}{2}} e^{2\pi i \vec{v}_L \cdot \vec{\lambda}_L} \ ,$$

where $\Lambda_L := \Gamma_{\mathrm{w-m}}^{4,4} \cap \Pi_L$ and $v \equiv (\vec{v}_L, \vec{v}_R) \in (\Gamma_{\mathrm{w-m}}^{4,4} \otimes \mathbb{R})/\Gamma_{\mathrm{w-m}}^{4,4}$ determines an element of $U(1)_L^4 \times U(1)_R^4$ as in eq.(2.23).

From eqs.(5.1) and (5.2), we obtain

$$\phi_{g^k}(\tau, 0) = (1 - \zeta_L^k)(1 - \zeta_L^{-k})(1 - \zeta_R^k)(1 - \zeta_R^{-k}) \ .$$

In particular, when $\zeta_L = 1$ or $\zeta_R = 1$, the Witten index $\phi_g(\tau, 0)$ vanishes. Using

$$\phi_{g^i, g^j}(\tau, 0) = \phi_{g^{\gcd(i,j,N)}}(\tau, 0) \ , \qquad 1 \le i, j \le N \ ,$$

where N is the order of g, we can easily compute the Witten index of the orbifold theory. In particular, the orbifold is a torus model if g_0 is in one of the classes

$$+1A, \ \pm 2A, \ \pm 2E, \ +3E, \ +3E', \ +4A, \ +4A' \ ,$$

while the orbifold is a K3 model when g_0 is in any of the other classes. In the latter case, the orbifold K3 model always contains a 'quantum symmetry' Q_g that acts by $e^{\frac{2\pi i k}{N}}$ on the g^k-twisted sector. The original torus model can be re-obtained by taking the orbifold of

the K3 model by the quantum symmetry. In [26], all possible symmetries of K3 models whose orbifold is a torus were classified and labeled according to their conjugacy class in the Conway group Co_0. We can compare now this analysis with the results of the present paper. More precisely, since we are able to compute the dimension $\dim \mathcal{H}_{g^k} = \frac{1}{N} \sum_{i=1}^{N} \phi_{g^k, g^i}(\tau, 0)$ of the space of g^k-twisted R-R ground states, we can compare the eigenvalues of the quantum symmetries Q_g of the orbifold model with the eigenvalues expected for a Conway class in the 24-dimensional representation. The results, summarized in table 4, match perfectly with the expectations from [26].

Our argument implies that there are K3 models that are orbifolds of non-linear sigma models on T^4 by symmetries of order $2, 3, 4, 5, 6, 8, 10, 12$. The torus model with a symmetry of order 5, together with its orbifold K3 model, was studied in [26]; it turns out that this is the model of section 4.4.2 with symmetry group $G_0 = \mathcal{I}_{120} \times_{\mathcal{I}_{120}} \bar{\mathcal{I}}_{120}$. Furthermore, the torus model based on the lattice Λ_{D_4} of section 4.4.1 was considered in [33], where it was stressed that a symmetry in the class $-4A$ exists. Its orbifold is the K3 model with the 'largest symmetry group' $\mathbb{Z}_2^8 : M_{20}$ [33].

6 Conclusions

In this paper we consider supersymmetric non-linear sigma models with target space a four dimensional torus T^4 and classify all possible group of symmetries of such models that preserve the $\mathcal{N} = (4, 4)$ superconformal algebra. This results extends the analysis of [23] concerning non-linear sigma models on K3 to include all known models with $\mathcal{N} = (4, 4)$ superconformal symmetry at central charge 6.

One of the immediate by-products of our analysis is the description of new interesting torus models with a large amount of discrete symmetries. The new results concern mainly the models admitting left-right asymmetric symmetries with no geometric interpretation. It turns out that there are only few isolated points in the moduli space where such symmetries are realised.

Some of the corresponding torus models are the usual suspects: the simplest example is the product of four orthogonal circles at the self-dual radius, with vanishing B-field (section 4.4.6). Another well-known example is the model with target space $\mathbb{R}^4/D_4$, where D_4 is the root lattice of the $so(8)$ algebra (section 4.4.1); the discrete symmetries and some of the orbifolds of the latter model were recently investigated in [20, 33]. In these two models, the chiral algebra is much larger than the one at a generic point in the moduli space.

On the contrary, some of the new models considered in section 4 have the *smallest* possible chiral algebra for a non-linear sigma model on T^4. These cases include, in particular, the model with a symmetry of order 5, considered in [26]: we discover that its group of symmetries G_0 is actually isomorphic to the binary icosahedral group $\mathcal{I}_{120}$, and that the lattice of winding-momenta is most easily described in terms of the icosian ring of unit quaternions (section 4.4.2). Another interesting model has a symmetry group G_0 isomorphic to the binary dihedral group $\mathcal{D}_{24}(6)$ of order 24 and its lattice of winding-momenta is based on the Eisenstein integers (section 4.4.5).

Our results can find some useful applications to the study of toroidal and K3 super-string compactifications. In particular, it would be interesting to understand how this investigation is related to the groups of symmetries of heterotic strings compactified on T^4, since the supersymmetric side of these models is analogous to the chiral half of the type II models we consider.

In the context of the Mathieu moonshine phenomenon, the analysis of the present paper can lead to a generalization of the results of [34]. In this work, some Siegel modular forms were constructed as the multiplicative lifts of the twisted-twining genera of generalized Mathieu moonshine [11]. Many of these modular forms admit an interpretation as partition functions for 1/4 BPS states in four dimensional CHL models with $\mathcal{N} = 4$ space-time supersymmetry. In [34], it was observed that there exists a particular modular transformation, induced by electric-magnetic duality in the four dimensional CHL model, that exchanges the multiplicative lifts of twining genera in two distinct K3 models, related by an orbifold. An obvious generalization of the Siegel modular forms of [34] can be obtained by taking the multiplicative lifts of twining genera in torus models, such as the ones considered in section 5 of the present paper. In this case, one expects the 'electric-magnetic duality' to relate these forms to the multiplicative lifts of the twining genera in exceptional K3 models, i.e. the K3 models admitting a group of symmetries not in M_{24} [26]. It would be very interesting to investigate in detail this instance of electric-magnetic duality and to understand its physical meaning.

A somehow related open question concerns the existence of any kind of moonshine phenomenon for non-linear sigma models on T^4. By analogy with the K3 case, one would expect that the corresponding twining genera to reproduce some of the functions computed in section 5.[4]

The symmetries considered in the present paper can be described in terms of topological defects preserving a small $\mathcal{N} = (4, 4)$ superconformal algebra. The orbifold procedure relating the corresponding torus models to K3 has also a natural interpretation in terms of defects [35]. At the moment, only the topological defects of d-dimensional torus models that preserve the $u(1)^d \oplus u(1)^d$ current algebra have been classified [36]. In the same spirit, it is natural to ask whether the classification of this paper includes all possible defects preserving the $\mathcal{N} = (4, 4)$ algebra or if more possibilities exist. In the latter case, one might obtain new explicit descriptions of K3 models arising from the associated generalized orbifolds of torus models.

Acknowledgments

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A Discrete subgroups of $Spin(4)$ and $SO(4)$

For the classification of the discrete subgroups of $\text{Spin}(4) \cong SU(2) \times SU(2)$ and $SO(4) \cong \text{Spin}(4)/(-1,-1)$ we follow closely the treatment of [37]. We recall that $SU(2)$ can be described as the group of unit quaternions

$$SU(2) = \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \mid a^2 + b^2 + c^2 + d^2 = 1\} \,,$$

and its discrete subgroups are

$\mathcal{G} \subset SU(2)$	$|\mathcal{G}|$	Name	Image in $SO(3)$	Generators
$\mathcal{C}_n \begin{matrix}(\equiv 1C_n)\\(\equiv 2C_{n/2})\end{matrix}$	$\begin{matrix}\text{odd } n\\\text{even } n\end{matrix}$	cyclic	$\begin{matrix}C_n\\C_{n/2}\end{matrix}$	$\mathbf{e}_{1/n}$
$\mathcal{D}_{4n}(n) \equiv 2D_{2n}(n)$	$4n$	(binary) dihedral	$D_{2n}(n)$	$\mathbf{e}_{1/2n}, \mathbf{j}$
$\mathcal{T}_{24} \equiv 2T_{12}$	24	(binary) tetrahedral	T_{12}	$\mathbf{i}, \boldsymbol{\omega}$
$\mathcal{O}_{48} \equiv 2O_{24}$	48	(binary) octahedral	O_{24}	$\boldsymbol{\iota}_O, \boldsymbol{\omega}$
$\mathcal{I}_{120} \equiv 2I_{60}$	120	(binary) icosahedral	I_{60}	$\boldsymbol{\iota}_I, \boldsymbol{\omega}$

where

$$\mathbf{e}_r = \exp(2\pi\mathbf{i}r), \qquad \boldsymbol{\omega} = \frac{-1 + \mathbf{i} + \mathbf{j} + \mathbf{k}}{2}, \qquad \boldsymbol{\iota}_O = \frac{\mathbf{j} + \mathbf{k}}{\sqrt{2}}, \qquad \boldsymbol{\iota}_I = \frac{1}{2}(\mathbf{i} + \tfrac{-\sqrt{5}-1}{2}\mathbf{j} + \tfrac{\sqrt{5}-1}{2}\mathbf{k}).$$

The simplest subgroups of $SU(2) \times SU(2) \cong \text{Spin}(4)$ are just direct products $\mathcal{L} \times \mathcal{R}$, where $\mathcal{L}, \mathcal{R}$ are discrete subgroups of $SU(2)$. We denote by $[\mathcal{L} \times \mathcal{R}] \subset SO(4)$ the image of $\mathcal{L} \times \mathcal{R}$ under the covering map $\text{Spin}(4) \to (SU(2) \times SU(2))/(-1,-1) \cong SO(4)$. A generic element $[l, r] \equiv [-l, -r] \in [L \times R] \subset SO(4)$ acts on $\mathbb{R}^4 \cong \mathbb{H}$ by

$$[l, r] : q \mapsto lq\bar{r} \,, \qquad q \in \mathbb{H} \,,$$

where $\bar{r}$ denotes complex conjugation.

A generic discrete $\mathcal{G} \subset SU(2) \times SU(2)$ is a subgroup of a direct product group $\mathcal{G} \subseteq \mathcal{L} \times \mathcal{R}$. Any such subgroup $\mathcal{G}$ is determined by specifying two surjective homomorphisms

$$\alpha : \mathcal{L} \to \mathcal{F} \,, \qquad\qquad \beta : \mathcal{R} \to \mathcal{F} \,,$$

from $\mathcal{L}$ and $\mathcal{R}$ onto the same abstract group $\mathcal{F}$. Then, $\mathcal{G}$ is the group of pairs $(l, r) \in \mathcal{L} \times \mathcal{R}$ such that l and r have the same image in $\mathcal{F}$

$$\mathcal{G} = \{(l, r) \in \mathcal{L} \times \mathcal{R} \mid \alpha(l) = \beta(r)\} \,.$$

In particular, $\mathcal{L}_0 := \ker\alpha$ and $\mathcal{R}_0 := \ker\beta$ are (isomorphic to) the normal subgroups of $\mathcal{G}$ generated by elements of the form $(l, 1) \in \mathcal{G}$ and $(1, r) \in \mathcal{G}$, respectively, so that

$$\mathcal{L}_0 \times \mathcal{R}_0 \subseteq \mathcal{G} \subseteq \mathcal{L} \times \mathcal{R} \,.$$

For example, when $|\mathcal{F}| = 1$, we have $\mathcal{L}_0 = \mathcal{L}$, $\mathcal{R}_0 = \mathcal{R}$ and $\mathcal{G}$ is simply the direct product $\mathcal{L} \times \mathcal{R}$. At the opposite extreme, when $\mathcal{L} = \mathcal{F} = \mathcal{R}$ and α, β are the identity maps, then

$\mathcal{L}_0 = 1 = \mathcal{R}_0$ and $\mathcal{G}$ is also isomorphic to $\mathcal{F}$. The groups of purely geometric symmetries of torus models are of this kind.

In most cases, giving the groups $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{F}$ is sufficient to determine the homomorphisms α and β uniquely (up to conjugation of $\mathcal{L}$ and $\mathcal{R}$ in $SU(2)$ and up to automorphisms of $\mathcal{F}$). This justifies the notation $\mathcal{L} \times_{\mathcal{F}} \mathcal{R}$ for the corresponding group $\mathcal{G}$. When there are inequivalent choices for α, β, we denote the most obvious group by $\mathcal{L} \times_{\mathcal{F}} \mathcal{R}$ and the other ones by some decorations, such as $\mathcal{L} \times_{\mathcal{F}} \bar{\mathcal{R}}$. The precise structure of each group will be clear from the list of generators. For example, $\mathcal{I}_{120} \times_{\mathcal{I}_{120}} \mathcal{I}_{120}$ denotes the group where both α, β are the identity map, while $\mathcal{I}_{120} \times_{\mathcal{I}_{120}} \bar{\mathcal{I}}_{120}$ denotes the group where α is the identity map and β is a non-trivial outer automorphism of $\mathcal{I}_{120}$. The two groups are actually isomorphic as abstract groups (and isomorphic to $\mathcal{I}_{120}$), but they are not conjugated within $SU(2) \times SU(2)$.

For the subgroups of $SO(4) = (SU(2) \times SU(2))/(-1, -1)$ and $SO(4)/(-1) \cong SU(2)/(-1) \times SU(2)/(-1)$, the treatment is analogous. We denote the subgroups of $SO(4)/(-1)$ by $[L \times_F R]$, where $L, R, F \in SU(2)/(-1)$. Any such group has a lift to $SO(4)$, denoted by $\pm[L \times_F R]$, which contains the element $[1, -1] \equiv [-1, 1] \in SO(4)$ and is a double cover of $[L \times_F R]$. In particular, the elements of $\pm[L \times_F R]$ always come in pairs $[l, r]$, $[l, -r]$, with the same image in $[L \times_F R]$. In some cases, there exists an index two subgroup of $\pm[L \times_F R]$ which contains only one element (either $[l, r]$ or $[l, -r]$) for each pair; we denote this group by $+[L \times_F R]$. Notice that $+[L \times_F R] \subset SO(4)$ does not contain the element $[1, -1] \in SO(4)$ and is isomorphic, as an abstract group, to $[L \times_F R] \subset SO(4)/(-1)$.

For each finite $G \subset SO(4)$ it is easy to find its double covering $\mathcal{G} \subset \mathrm{Spin}(4)$:

$G \subset SO(4)$	$|G|$	Lift $\mathcal{G} \subset \mathrm{Spin}(4)$	$|\mathcal{G}|$
$\pm[L \times_F R]$	$2\frac{|L| \times |R|}{|F|}$	$\mathcal{L} \times_F \mathcal{R}$	$\frac{|\mathcal{L}| \times |\mathcal{R}|}{|F|}$
$+[L \times_F R]$	$\frac{|L| \times |R|}{|F|}$	$\mathcal{L} \times_{\mathcal{F}} \mathcal{R}$	$\frac{|\mathcal{L}| \times |\mathcal{R}|}{|\mathcal{F}|}$

Here, $\mathcal{L}, \mathcal{R}, \mathcal{F} \subset SU(2)$ are the lifts of $L, R, F \subset SO(3) \cong SU(2)/(-1)$.

B Lattices

In this appendix, we review some general properties of even lattices and their automorphisms, in particular the 'gluing' construction (see [38] and [30] for more details).

For any lattice $\Lambda \subset \mathbb{R}^n$, we denote by Λ^* its dual and by $\Lambda(-1)$ the lattice obtained by changing the sign of the quadratic form. We define the determinant of Λ to be the determinant $|\det Q|$ of the Gram matrix $Q_{ij} := \lambda_i \cdot \lambda_j$, for some basis $\lambda_1, \ldots, \lambda_r$ of generators of Λ. The determinant is independent of the choice of the basis. If Λ is integral, and in particular if it is even, then $\Lambda \subset \Lambda^*$ and we can define the *discriminant group* Λ^*/Λ. This is a finite abelian group of order $|\det Q|$.

Let us assume that Λ is even. The quadratic form $\Lambda \to 2\mathbb{Z}$ on Λ induces a quadratic form $\Lambda^* \to \mathbb{Q}$ on the dual lattice Λ^* and the discriminant form

$$q_\Lambda : \Lambda^*/\Lambda \to \mathbb{Q}/2\mathbb{Z} \, ,$$

on the discriminant group Λ^*/Λ.

A sublattice Λ' of Λ is called primitive if Λ/Λ' is a free group, i.e. if $\Lambda \cap (\Lambda' \otimes \mathbb{Q}) = \Lambda'$. Let Λ be a primitive sublattice of an even unimodular lattice Γ and let $\Lambda^\perp$ be its orthogonal complement $\Lambda^\perp = \Gamma \cap (\Lambda \otimes \mathbb{R})^\perp$. Then, there exists an isomorphism of discriminant groups

$$\gamma : \Lambda^*/\Lambda \xrightarrow{\cong} (\Lambda^\perp)^*/\Lambda^\perp ,$$

that inverts the discriminant quadratic form

$$q_{\Lambda^\perp} \circ \gamma = -q_\Lambda \mod 2\mathbb{Z} , \tag{B.1}$$

and such that

$$\Gamma = \{(v, w) \in \Lambda^* \oplus_\perp (\Lambda^\perp)^* \mid [w] = \gamma([v])\} \tag{B.2}$$

where for any $x \in \Lambda^*$ (respectively, $x \in (\Lambda^\perp)^*$) $[x]$ denotes the class of x in Λ^*/Λ (respectively, in $(\Lambda^\perp)^*/\Lambda^\perp$). Vice versa, for any pair of even lattices Λ and $\Lambda^\perp$ with an isomorphism $\gamma : \Lambda^*/\Lambda \xrightarrow{\cong} (\Lambda^\perp)^*/\Lambda^\perp$ satisfying (B.1), the lattice Γ defined by eq.(B.2) is even unimodular. This is called the 'gluing' construction. Clearly, a necessary condition for the isomorphism γ to exist is that the discriminant groups have the same order, i.e. the two lattices have the same determinant.

Any automorphism $g \in O(\Lambda)$ induces an automorphism of the dual Λ^* and of the discriminant group, preserving the discriminant form q_Λ. We denote by $O_0(\Lambda)$ the subgroup of automorphisms acting trivially on the discriminant group and by $SO(\Lambda)$ and $SO_0(\Lambda)$ the corresponding subgroups of orientation-preserving automorphisms. If Λ and $\Lambda^\perp$ are two primitive mutually orthogonal sublattices of the even unimodular lattice Γ, as above, then any element $g \in O_0(\Lambda)$ extends to an automorphism of Γ [30]. Indeed, g obviously extends to an automorphism of $\Lambda^* \oplus_\perp (\Lambda^\perp)^*$ by $(v, w) \mapsto (g(v), w)$ and since $[v] = [g(v)]$ for all $v \in \Lambda^*$ this automorphism preserves the sublattice Γ.

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