

Reduction of the two-body dynamics to a one-body description in classical electrodynamics

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Abstract

We discuss the mapping of the conservative part of two-body electrodynamics onto that of a test charged particle moving in some external electromagnetic field, taking into account recoil effects and relativistic corrections up to second post-Coulombian order. Unlike the results recently obtained in general relativity, we find that in classical electrodynamics it is not possible to implement the matching without introducing external parameters in the effective electromagnetic field. Relaxing the assumption that the effective test particle moves in a flat spacetime provides a feasible way out.

I. INTRODUCTION

Recently, a novel approach to the two-body problem in general relativity has been introduced [1]. The main motivation of that investigation rests on better understanding the late dynamical evolution of a coalescing binary system made of compact bodies of comparable masses, such as black holes and/or neutron stars. In fact, these astrophysical systems are among the most promising candidate sources for the detection of gravitational-waves with the future terrestrial interferometers such as the Laser Interferometric Gravitational Wave Observatory (LIGO) and Virgo. The basic idea pursued in [1], in part inspired by some results obtained in quantum electrodynamics [2,3], was to map the conservative two-body dynamics (henceforth denoted as the “real” dynamics) onto an effective one-body one, where a test particle moves in an effective external metric. As long as radiation reaction effects are not taken into account, the effective metric is just a deformation of the Schwarzschild metric with deformation parameter $\nu = \mu/M$, where μ is the reduced mass of the binary system and M its total mass. The “effective” description should be viewed as a way of re-summing in a non-perturbative manner the badly convergent post-Newtonian-expanded dynamics of the “real” description. The results in [1] were restricted to the second post-Newtonian level (2PN) and the analysis was mainly focused on the conservative part of the dynamics. More recently, a feasible way of incorporating radiation reaction effects has been proposed [4] and the extension of the aforesaid approach to 3PN order has been investigated [5].

The purpose of the present paper is to test the robustness of the basic idea underlying the mapping of the two-body problem onto an effective one-body one, by applying it to classical electrodynamics. We limit to the conservative part of the dynamics of the bound states of two charged particles, up to second post-Coulombian order (2PC), and we take into account recoil effects. We investigate the possibility of describing the exchange of energies between the two bodies in the “real” problem through an “effective” auxiliary description, where a test particle moves in some external effective electromagnetic field. Generically, we expect that this electromagnetic field will be a deformation of the Coulomb potential with deformation parameter $\nu = \mu/M$, where μ is the usual reduced mass of the two charged particles and M the total mass of the system. We shall see that the matching is also possible introducing in the effective description either a ν -dependent vector potential or a deformed flat metric with deformation parameter ν .

As already mentioned, the idea of reducing the relativistic two-body dynamics onto a relativistic one-body one was originally introduced in quantum electrodynamics. In particular, in [2] the authors, taking into account recoil effects, resummed in the eikonal approximation the “crossed-ladder” Feynman diagrams for the scattering of two relativistic particles and mapped the one-body relativistic Balmer formula onto the two-body relativistic one. This method gives the correct quantum energy levels at least up to 1PC order, but some of the centrifugal barrier effects have to be added by hand. Todorov et al. [3] developed a more systematic approach, based on the Lyppmann-Schwinger quasi-potential equation, which also gives correct results for the quantum energy levels, including the main parts of the radiative effects of the Lamb shift. Nevertheless, this last approach [3] rests on some choices for the quasi-potential equation which are not very well justified and introduces in the effective description various energy-dependent quantities. In the following, whenever it is possible, we will compare our results in classical electrodynamics with the previous analysis for the corresponding quantum problem. Finally, note that, the aim of this paper is not to obtain new results with respect to the quantum energy-levels of the bound states of a two-body charged system, which is well known to be a hard problem [6]. On the other hand, the present work wants to investigate, in the context of classical electrodynamics, the basic idea of reducing the two-body dynamics onto a one-body one, recently introduced in general relativity [1].

The outline of the paper is as follows. In Section II we review the relativistic two-body problem up to 2PC order and summarize its dynamics in a coordinate-invariant manner evaluating, within the Hamilton-Jacobi framework, the “energy-levels” of the bound states. In Section III we introduce the “effective” one-body description and define the “rules” needed to map the “real” onto the “effective” problem. Then, in Sections III A, III B and III C we analyze three feasible manners of implementing the matching. Finally, Section IV summarizes our main conclusions.

II. TWO-BODY DYNAMICS UP TO SECOND POST-COULOMBIAN ORDER

It was realized long ago that, in relativistic dynamics, if the position variables that are used to describe a system of charged interacting particles are the coordinates associated to

a Lorentz frame ¹, then all higher time derivatives must appear in the Lagrangian [8]. To get an “ordinary” Lagrangian it is necessary to introduce canonical position variables different from the Lorentz coordinates [8]. At 2PC order the acceleration dependent Lagrangian was originally derived by Golubenkov and Smorodinskii [10]. If one eliminates in that Lagrangian the higher time derivatives by using the equation of motion of lower orders then, as pointed out in [8,11], one does not obtain the correct equations of motion in a Lorentz frame. To eliminate correctly the accelerations one can use the method of “redefinition of position variables”, introduced by Damour and Schäfer in [9], which consists in appealing to a contact transformation induced by a change of coordinates from the Wheeler-Feynman coordinate system (Lorentz frame) [12] to a well defined asymptotically inertial frame [13]. More explicitly, the acceleration dependent Lagrangian at 2PC order is given by [9]:

$$\tilde{\mathcal{L}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{v}_1, \mathbf{v}_2, \mathbf{a}_1, \mathbf{a}_2) = \tilde{\mathcal{L}}_0 + \frac{1}{c^2} \tilde{\mathcal{L}}_2 + \frac{1}{c^4} \tilde{\mathcal{L}}_4, \quad (2.1)$$

with

$$\tilde{\mathcal{L}}_0 = \frac{1}{2} m_1 \mathbf{v}_1^2 + \frac{1}{2} m_2 \mathbf{v}_2^2 - \frac{e_1 e_2}{R}, \quad (2.2)$$

$$\tilde{\mathcal{L}}_1 = \frac{1}{8} m_1 \mathbf{v}_1^4 + \frac{1}{8} m_2 \mathbf{v}_2^4 + \frac{e_1 e_2}{2R} [\mathbf{v}_1 \cdot \mathbf{v}_2 + (\tilde{\mathbf{n}} \cdot \mathbf{v}_1)(\tilde{\mathbf{n}} \cdot \mathbf{v}_2)], \quad (2.3)$$

$$\begin{aligned} \tilde{\mathcal{L}}_4 = & \frac{1}{16} m_1 \mathbf{v}_1^6 + \frac{1}{16} m_2 \mathbf{v}_2^6 - \frac{e_1 e_2}{8} \left\{ R [3(\mathbf{a}_1 \cdot \mathbf{a}_2) - (\tilde{\mathbf{n}} \cdot \mathbf{a}_1)(\tilde{\mathbf{n}} \cdot \mathbf{a}_2)] + 2[(\mathbf{v}_1 \cdot \mathbf{a}_2)(\tilde{\mathbf{n}} \cdot \mathbf{v}_1) \right. \\ & \left. - (\mathbf{v}_2 \cdot \mathbf{a}_1)(\tilde{\mathbf{n}} \cdot \mathbf{v}_2)] + (\tilde{\mathbf{n}} \cdot \mathbf{a}_1) [\mathbf{v}_2^2 - (\tilde{\mathbf{n}} \cdot \mathbf{v}_2)^2] - (\tilde{\mathbf{n}} \cdot \mathbf{a}_2) [\mathbf{v}_1^2 - (\tilde{\mathbf{n}} \cdot \mathbf{v}_1)^2] \right. \\ & \left. + \frac{1}{R} [\mathbf{v}_1^2 \mathbf{v}_2^2 - 2(\mathbf{v}_1 \cdot \mathbf{v}_2)^2 - \mathbf{v}_1^2 (\tilde{\mathbf{n}} \cdot \mathbf{v}_2)^2 - \mathbf{v}_2^2 (\tilde{\mathbf{n}} \cdot \mathbf{v}_1)^2 + 3(\tilde{\mathbf{n}} \cdot \mathbf{v}_1)^2 (\tilde{\mathbf{n}} \cdot \mathbf{v}_2)^2] \right\}, \quad (2.4) \end{aligned}$$

where $\mathbf{R} = \mathbf{z}_1 - \mathbf{z}_2$, $\tilde{\mathbf{n}} = \mathbf{R}/R$, $\mathbf{v}_i = \dot{\mathbf{z}}_i$ and $\mathbf{a}_i = \dot{\mathbf{v}}_i$. In [9], Damour and Schäfer after having critically discussed and clarified the various results previously derived in the literature [14], worked out the contact transformations,

$$\mathbf{q}_1 = \mathbf{z}_1 - \frac{1}{c^4} \frac{e_1 e_2}{4 m_1} \left\{ (\tilde{\mathbf{n}} \cdot \mathbf{v}_2) \mathbf{v}_2 + \tilde{\mathbf{n}} \left[\frac{1}{2} ((\tilde{\mathbf{n}} \cdot \mathbf{v}_2)^2 - \mathbf{v}_2^2) + \frac{e_1 e_2}{m_2 R} \right] \right\}, \quad (2.5)$$

$$\mathbf{q}_2 = \mathbf{z}_2 + \frac{1}{c^4} \frac{e_1 e_2}{4 m_2} \left\{ (\tilde{\mathbf{n}} \cdot \mathbf{v}_1) \mathbf{v}_1 + \tilde{\mathbf{n}} \left[\frac{1}{2} ((\tilde{\mathbf{n}} \cdot \mathbf{v}_1)^2 - \mathbf{v}_1^2) + \frac{e_1 e_2}{m_1 R} \right] \right\}, \quad (2.6)$$

¹For coordinates belonging to a Lorentz frame we mean coordinates which transform as linear representation of the Poincarè group [9].

which allow to eliminate the accelerations appearing in Eqs. (2.2)–(2.4). Hence, the final acceleration independent Lagrangian at 2PC order is given by [9]:

$$\mathcal{L}(\mathbf{q}_1, \mathbf{q}_2, \dot{\mathbf{q}}_1, \dot{\mathbf{q}}_2) = \mathcal{L}_0 + \frac{1}{c^2} \mathcal{L}_2 + \frac{1}{c^4} \mathcal{L}_4, \quad (2.7)$$

with

$$\mathcal{L}_0 = \frac{1}{2} m_1 \dot{\mathbf{q}}_1^2 + \frac{1}{2} m_2 \dot{\mathbf{q}}_2^2 - \frac{e_1 e_2}{q}, \quad (2.8)$$

$$\mathcal{L}_2 = \frac{1}{8} m_1 \dot{\mathbf{q}}_1^4 + \frac{1}{8} m_2 \dot{\mathbf{q}}_2^4 + \frac{e_1 e_2}{2q} [\dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_2 + (\mathbf{n} \cdot \dot{\mathbf{q}}_1)(\mathbf{n} \cdot \dot{\mathbf{q}}_2)], \quad (2.9)$$

$$\begin{aligned} \mathcal{L}_4 = & \frac{1}{16} m_1 \dot{\mathbf{q}}_1^6 + \frac{1}{16} m_2 \dot{\mathbf{q}}_2^6 - \frac{e_1 e_2}{8q} \left\{ \dot{\mathbf{q}}_1^2 \dot{\mathbf{q}}_2^2 - 2(\dot{\mathbf{q}}_1 \cdot \dot{\mathbf{q}}_2)^2 + 3(\mathbf{n} \cdot \dot{\mathbf{q}}_1)^2 (\mathbf{n} \cdot \dot{\mathbf{q}}_2)^2 \right. \\ & - (\mathbf{n} \cdot \dot{\mathbf{q}}_1)^2 \dot{\mathbf{q}}_2^2 - (\mathbf{n} \cdot \dot{\mathbf{q}}_2)^2 \dot{\mathbf{q}}_1^2 + \frac{e_1 e_2}{m_2 q} [\dot{\mathbf{q}}_1^2 - 3(\mathbf{n} \cdot \dot{\mathbf{q}}_1)^2] \\ & \left. + \frac{e_1 e_2}{m_1 q} [\dot{\mathbf{q}}_2^2 - 3(\mathbf{n} \cdot \dot{\mathbf{q}}_2)^2] - \frac{2(e_1 e_2)^2}{m_1 m_2 q^2} \right\}, \end{aligned} \quad (2.10)$$

where $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ and $\mathbf{n} = \mathbf{q}/q$. Applying the Legendre transformation to \mathcal{L} , we derive (in full agreement with [9])

$$\mathcal{H}(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) = \mathcal{H}_0 + \frac{1}{c^2} \mathcal{H}_2 + \frac{1}{c^4} \mathcal{H}_4, \quad (2.11)$$

where

$$\mathcal{H}_0 = \frac{1}{2} \left(\frac{\mathbf{p}_1^2}{m_1} + \frac{\mathbf{p}_2^2}{m_2} \right) + \frac{e_1 e_2}{q}, \quad (2.12)$$

$$\mathcal{H}_2 = -\frac{1}{8} \left(\frac{\mathbf{p}_1^4}{m_1^3} + \frac{\mathbf{p}_2^4}{m_2^3} \right) - \frac{e_1 e_2}{2m_1 m_2 q} [\mathbf{p}_1 \cdot \mathbf{p}_2 + (\mathbf{n} \cdot \mathbf{p}_1)(\mathbf{n} \cdot \mathbf{p}_2)], \quad (2.13)$$

$$\begin{aligned} \mathcal{H}_4 = & \frac{1}{16} \left(\frac{\mathbf{p}_1^6}{m_1^5} + \frac{\mathbf{p}_2^6}{m_2^5} \right) + \frac{e_1 e_2}{m_1 m_2 q} \left\{ \frac{3(\mathbf{n} \cdot \mathbf{p}_1)^2 (\mathbf{n} \cdot \mathbf{p}_2)^2}{8m_1 m_2} - \frac{\mathbf{p}_1^2 (\mathbf{n} \cdot \mathbf{p}_2)^2}{8m_1 m_2} - \frac{\mathbf{p}_2^2 (\mathbf{n} \cdot \mathbf{p}_1)^2}{8m_1 m_2} \right. \\ & + \frac{1}{4} [(\mathbf{n} \cdot \mathbf{p}_1)(\mathbf{n} \cdot \mathbf{p}_2) + (\mathbf{p}_1 \cdot \mathbf{p}_2)] \left(\frac{\mathbf{p}_1^2}{m_1^2} + \frac{\mathbf{p}_2^2}{m_2^2} \right) - \frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)^2}{4m_1 m_2} + \frac{\mathbf{p}_1^2 \mathbf{p}_2^2}{8m_1 m_2} \\ & \left. + \frac{e_1 e_2}{q} \left(\frac{\mathbf{p}_1^2}{m_1} + \frac{\mathbf{p}_2^2}{m_2} \right) - \frac{(e_1 e_2)^2}{4q^2} \right\}. \end{aligned} \quad (2.14)$$

Let us denote

$$M = m_1 + m_2, \quad \mu = \frac{m_1 m_2}{M}, \quad \nu = \frac{\mu}{M}, \quad (2.15)$$

where the parameter ν takes values between 0 and 1/4, corresponding to the test mass limit and the equal mass case, respectively. Henceforth, we shall limit to the dynamics of the

bound states generated by the two charged bodies, therefore $e_1 e_2 < 0$ and we pose the coupling constant $\alpha = -e_1 e_2 > 0$. In the center of mass frame we have $\mathbf{P} = \mathbf{p}_1 = -\mathbf{p}_2$ and introducing the following reduced variables

$$\widehat{\mathcal{H}} = \frac{\mathcal{H}}{\mu}, \quad \mathbf{p} = \frac{\mathbf{P}}{\mu}, \quad \widehat{t} = \frac{\mu t}{\alpha}, \quad r = \frac{\mu q}{\alpha}, \quad (2.16)$$

we can re-write the Hamiltonian, Eq. (2.11), in the more convenient form

$$\begin{aligned} \widehat{\mathcal{H}}(\mathbf{r}, \mathbf{p}) &= \frac{1}{2} \mathbf{p}^2 - \frac{1}{r} - \frac{1}{8c^2} (1 - 3\nu) \mathbf{p}^4 - \frac{1}{2c^2} \frac{\nu}{r} [\mathbf{p}^2 + (\mathbf{n} \cdot \mathbf{p})^2] \\ &\quad - \frac{1}{8c^4} \frac{1}{r} [3\nu^2 (\mathbf{n} \cdot \mathbf{p})^4 + \nu (3\nu - 2) \mathbf{p}^4 + 2\nu (\nu - 1) \mathbf{p}^2 (\mathbf{n} \cdot \mathbf{p})^2] \\ &\quad + \frac{1}{16c^4} (1 - 5\nu + 5\nu^2) \mathbf{p}^6 + \frac{1}{4c^4} \frac{\nu}{r^2} \mathbf{p}^2 + \frac{1}{4c^4} \frac{\nu}{r^3}. \end{aligned} \quad (2.17)$$

The above Hamiltonian is invariant under time translations and space rotations. We denote the two conserved quantities, that is the centre-of-mass non-relativistic energy and angular momentum, by

$$\widehat{\mathcal{H}}(\mathbf{r}, \mathbf{p}) = \widehat{\mathcal{E}}^{\text{NR}} = \frac{\mathcal{E}_{\text{c.m.}}^{\text{NR}}}{\mu}, \quad \mathbf{r} \wedge \mathbf{p} = \mathbf{j} = \frac{\mathcal{J}_{\text{c.m.}}}{\alpha}. \quad (2.18)$$

In the following we pose $\mathcal{E}^{\text{NR}} \equiv \mathcal{E}_{\text{c.m.}}^{\text{NR}}$ and $\mathcal{J} \equiv \mathcal{J}_{\text{c.m.}}$. Using the Hamilton-Jacobi formalism, we can summarize in a coordinate-invariant manner the two-charge dynamics by evaluating the ‘‘energy-levels’’ of the system. Introducing the reduced Hamilton principal-function \widehat{S} , defined by $(\partial \widehat{S} / \partial \mathbf{r}) = \mathbf{p}$, separating the time and angular coordinates and restricting to the planar motion, we can write

$$\widehat{S} = -\widehat{\mathcal{E}}^{\text{NR}} \widehat{t} + j \varphi + \widehat{S}_r(r, \widehat{\mathcal{E}}^{\text{NR}}, j). \quad (2.19)$$

Solving the Hamilton-Jacobi equation $\widehat{\mathcal{H}}(\mathbf{r}, \mathbf{p}) = \widehat{\mathcal{E}}^{\text{NR}}$ with respect to $(d\widehat{S}_r/dr) = p_r = \mathbf{n} \cdot \mathbf{p}$, using $\mathbf{p}^2 = (\mathbf{n} \cdot \mathbf{p})^2 + \mathbf{j}^2/r^2$, we get

$$\widehat{S}_r(r, \widehat{\mathcal{E}}^{\text{NR}}, j) = \int dr \sqrt{\mathcal{R}(r, \widehat{\mathcal{E}}^{\text{NR}}, j)}, \quad (2.20)$$

where \mathcal{R} is a polynomial of the fifth order in $1/r$, explicitly given by:

$$\mathcal{R}(r, \widehat{\mathcal{E}}^{\text{NR}}, j) = A + \frac{2B}{r} + \frac{C}{r^2} + \frac{D_1}{r^3} + \frac{D_2}{r^4} + \frac{D_3}{r^5}, \quad (2.21)$$

with

$$A = 2\widehat{\mathcal{E}}^{\text{NR}} + \frac{1}{c^2} (1 - 3\nu) (\widehat{\mathcal{E}}^{\text{NR}})^2 + \frac{1}{c^4} \nu (4\nu - 1) (\widehat{\mathcal{E}}^{\text{NR}})^3, \quad (2.22)$$

$$B = 1 + \frac{1}{c^2} (1 - \nu) \widehat{\mathcal{E}}^{\text{NR}} + \frac{1}{c^4} \frac{\nu}{2} (2\nu - 1) (\widehat{\mathcal{E}}^{\text{NR}})^2, \quad (2.23)$$

$$C = -j^2 + \frac{1}{c^2} (1 + \nu), \quad (2.24)$$

$$D_1 = -\frac{1}{c^2} \nu j^2 - \frac{1}{c^4} \nu^2 j^2 \widehat{\mathcal{E}}^{\text{NR}} + \frac{1}{c^4} \frac{\nu}{2} (4\nu - 1), \quad (2.25)$$

$$D_2 = -\frac{3}{c^4} \nu^2 j^2, \quad (2.26)$$

$$D_3 = +\frac{3}{4c^4} \nu^2 j^4. \quad (2.27)$$

For our purposes we need to compute the reduced radial action variable

$$i_r^{\text{real}}(\widehat{\mathcal{E}}^{\text{NR}}, j) = \frac{2}{2\pi} \int_{r_{\min}}^{r_{\max}} dr \sqrt{\mathcal{R}(r, \widehat{\mathcal{E}}^{\text{NR}}, j)}. \quad (2.28)$$

To evaluate the above integral we use the formula (3.9) of Ref. [7], derived by performing a complex contour integration. The result for the radial action variable $\mathcal{I}_R^{\text{real}} = \alpha i_r^{\text{real}}$ reads:

$$\begin{aligned} \mathcal{I}_R^{\text{real}}(\mathcal{E}^{\text{NR}}, \mathcal{J}) &= \frac{\alpha \mu^{1/2}}{\sqrt{-2\mathcal{E}^{\text{NR}}}} \left[1 - \frac{1}{4}(\nu - 3) \frac{\mathcal{E}^{\text{NR}}}{\mu c^2} - \frac{1}{32} (5 - 6\nu - 3\nu^2) \left(\frac{\mathcal{E}^{\text{NR}}}{\mu c^2} \right)^2 \right] \\ &\quad - \mathcal{J} + \frac{\alpha^2}{c^2 \mathcal{J}} \left(\frac{1}{2} - \frac{\nu \mathcal{E}^{\text{NR}}}{2 \mu c^2} \right) + \frac{1}{8} (1 - 6\nu) \frac{\alpha^4}{c^4 \mathcal{J}^3}. \end{aligned} \quad (2.29)$$

Finally, to get the ‘‘energy-levels’’ we solve the above equation in terms of the relativistic energy $\mathcal{E}^{\text{R}} = \mathcal{E}^{\text{NR}} + M c^2$. Introducing the Delaunay action variable $\mathcal{N} = \mathcal{I}_R^{\text{real}} + \mathcal{J}$, we get:

$$\begin{aligned} \mathcal{E}^{\text{R}}(\mathcal{N}, \mathcal{J}) &= M c^2 - \frac{1}{2} \frac{\alpha^2 \mu}{\mathcal{N}^2} + \frac{1}{c^2} \alpha^4 \mu \left[-\frac{1}{2} \frac{1}{\mathcal{J} \mathcal{N}^3} + \frac{1}{8} (3 - \nu) \frac{1}{\mathcal{N}^4} \right] + \frac{1}{c^4} \alpha^6 \mu \left[-\frac{3}{8} \frac{1}{\mathcal{J}^2 \mathcal{N}^4} \right. \\ &\quad \left. + \frac{1}{16} (-5 + 3\nu - \nu^2) \frac{1}{\mathcal{N}^6} + \frac{1}{4} (3 - 2\nu) \frac{1}{\mathcal{J} \mathcal{N}^5} + \frac{1}{8} (6\nu - 1) \frac{1}{\mathcal{J}^3 \mathcal{N}^3} \right]. \end{aligned} \quad (2.30)$$

At 0PC order we recover the well known result of the degeneracy of the energy-levels in the Coulomb problem. Let us observe that at 1PC order, identifying \mathcal{N}/\hbar with the principal quantum-number and \mathcal{J}/\hbar with the total angular-momentum quantum-number, we obtain that Eq. (2.30) gives, e.g., the correct bound-state energies of the singlet states of the positronium [2,3] ($e_1 = -e_2$ and $m_1 = m_2$) in the (classical) limit $\mathcal{J}/\hbar \gg 1$. Moreover, within the approximation $\mathcal{J}/\hbar \gg 1$, our method captures all the centrifugal barrier shifts that have to be added by hand in [2]. However, we cannot recover from Eq. (2.30) the correct quantum energy-levels at 2PC level, because at this order radiation reaction effects should have been

taken into account. Indeed, in electrodynamics they enter at 1.5PC order, with a dipole-type interaction. Only if we limit to systems with $e_1/m_1 = e_2/m_2$, we can postpone radiation reaction effects at the quadrupole order, which means at 2.5 PC level. In the present work we are interested in the conservative part of the bound states dynamics, hence we do not make the restriction $e_1/m_1 = e_2/m_2$. The radiative corrections which contribute to the main part of the Lamb shift have been evaluated in [3], using the quasi-potential approach, and are of the order $\alpha^5 \log \alpha$. Corrections of the order α^5 , α^6 , $\alpha^6 \log \alpha$ have also been partially obtained in the literature for some quantum bound states of positronium and muonium [6].

III. “EFFECTIVE” ONE-BODY DESCRIPTION

The basic idea of the present work is to map the “real” two-body dynamics, described in the previous section, to an “effective” dynamics of a test particle of mass m_0 and charge e_0 , moving in an external electromagnetic field. The action for the test particle is given by:

$$S_{\text{eff}} = \int \left(-m_0 c ds_0 + \frac{1}{c} e_0 A_{\mu}^{\text{eff}}(z) dz^{\mu} \right), \quad (3.1)$$

where $A_{\text{eff}}^{\mu} = (\Phi_{\text{eff}}, \mathbf{A}_{\text{eff}})$. It is straightforward to derive that the effective Hamiltonian satisfies the well known equation

$$\frac{(\mathcal{H}_{\text{eff}} - e_0 \Phi_{\text{eff}})^2}{c^2} = m_0^2 c^2 + \left(\mathbf{p} - \frac{e_0}{c} \mathbf{A}_{\text{eff}} \right)^2. \quad (3.2)$$

The effective electromagnetic field A_{eff}^{μ} will be constructed in the form of an expansion in the dimensionless parameter $\alpha_0/(m_0 c^2 R)$, where $\alpha_0 = e_0^2$ is the coupling constant and $\alpha_0/(m_0 c^2)$ is the classical charge radius of m_0 . Hence, we pose:

$$\Phi_{\text{eff}}(R) = \frac{e_0 \phi_0}{R} \left[1 + \phi_1 \frac{\alpha_0}{m_0 c^2 R} + \phi_2 \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2 + \dots \right], \quad (3.3)$$

$$\mathbf{A}_{\text{eff}}(R) = \frac{e_0 \mathbf{a}}{c R} \left[a_0 + a_1 \frac{\alpha_0}{m_0 c^2 R} + \dots \right], \quad (3.4)$$

where ϕ_0, ϕ_1, ϕ_2 and a_0, a_1 are dimensionless parameters and \mathbf{a} is a vector with the dimension of a velocity. All these unknown coefficients will be fixed by the matching between the “real” and the “effective” description. Note that, in the above equations the variable R stands for the effective radial coordinate and differs from the real separation R used in Sec. II. Moreover, in Eqs. (3.3), (3.4) we have indicated only the terms we shall need up to 2PC order.

The dynamics of the one-body problem can be described, in a coordinate-invariant manner, in the Hamilton-Jacobi framework, by considering the “energy-levels” of the bound states of the particle m_0 in the external electromagnetic field. The Hamilton-Jacobi equation can be obtained from Eq. (3.2) posing $\mathcal{H}_{\text{eff}} = \mathcal{E}_0$ and introducing the Hamilton principal-function $\partial S_{\text{eff}}/\partial \mathbf{R} = \mathbf{p}$. Limiting to the motion in the equatorial plane ($\theta = \pi/2$) we can separate the variables, writing

$$S_{\text{eff}} = -\mathcal{E}_0 t + \mathcal{J}_0 \varphi + S_R^0(R, \mathcal{E}_0, \mathcal{J}_0), \quad (3.5)$$

where \mathcal{E}_0 and $\mathcal{J}_0 \equiv |\mathcal{J}_0|$ are the conserved energy and angular momentum defined by Eq. (3.1). The effective radial action variable reads

$$\mathcal{I}_R^{\text{eff}} = \frac{2}{2\pi} \int_{R_{\text{min}}}^{R_{\text{max}}} dR \frac{dS_R^0}{dR}. \quad (3.6)$$

Like in the two-body description we can derive the “energy-levels” of the “effective” one-body problem. They can be written as ²:

$$\begin{aligned} \mathcal{E}_0(\mathcal{N}_0, \mathcal{J}_0) = m_0 c^2 - \frac{1}{2} \frac{m_0 \alpha_0^2}{\mathcal{N}_0^2} + \frac{1}{c^2} \alpha_0^4 m_0 \left(\frac{\mathcal{E}_{3,1}}{\mathcal{J}_0 \mathcal{N}_0^3} + \frac{\mathcal{E}_{4,0}}{\mathcal{N}_0^4} \right) \\ + \frac{1}{c^4} \alpha_0^6 m_0 \left[\frac{\mathcal{E}_{3,3}}{\mathcal{J}_0^3 \mathcal{N}_0^3} + \frac{\mathcal{E}_{4,2}}{\mathcal{J}_0^2 \mathcal{N}_0^4} + \frac{\mathcal{E}_{5,1}}{\mathcal{J}_0 \mathcal{N}_0^5} + \frac{\mathcal{E}_{6,0}}{\mathcal{N}_0^6} \right], \end{aligned} \quad (3.7)$$

where $\mathcal{N}_0 = \mathcal{I}_R^{\text{eff}} + \mathcal{J}_0$ and $\mathcal{E}_{i,j}$ are combinations of the coefficients ϕ_0, ϕ_1, ϕ_2 and a_0, a_1 given in Eqs. (3.3), (3.4).

Let us now define the rules to match the “real” to the “effective” problem. Like in [1], we find very natural sticking with the following relations between the adiabatic invariants:

$$\mathcal{N} = \mathcal{N}_0, \quad \mathcal{J} = \mathcal{J}_0. \quad (3.8)$$

However, the way the “energy-levels”, Eq. (2.30) and Eq. (3.7), are related is more subtle. If we simply identify $\mathcal{E}_0(\mathcal{J}_0, \mathcal{N}_0) = \mathcal{E}^{\text{R}}(\mathcal{J}, \mathcal{N}) + (m_0 - M) c^2$, and impose that the mass of the

²Note that, if a vector potential is present, the energy-levels could also depend on the magnetic number \mathcal{J}_z^0 . In the present paper when dealing with a vector potential (see Sec. IIIB) we shall assume that the source of the magnetic field is the angular momentum, hence the magnetic field will be perpendicular to the plane of motion. This choice implicitly assumes $\mathcal{J}_0 \equiv \mathcal{J}_z^0$.

effective test particle coincides with the reduced mass, i.e. $m_0 = \mu$, we obtain that already at 1PC order it is impossible to reduce the two-body dynamics to a one-body description. Hence, following [1] we assume that there is a one-to-one mapping between the “real” and the “effective” energy-levels of the general form:

$$\frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} = \frac{\mathcal{E}^{\text{NR}}}{\mu c^2} \left[1 + \alpha_1 \frac{\mathcal{E}^{\text{NR}}}{\mu c^2} + \alpha_2 \left(\frac{\mathcal{E}^{\text{NR}}}{\mu c^2} \right)^2 \right], \quad (3.9)$$

where α_1 and α_2 are unknown coefficients that will be fixed by the matching. Given the aforesaid “rules”, we shall investigate in the subsequent sections three feasible ways the mapping can be implemented. The diverse descriptions differ by the choice of the effective electromagnetic field and the spacetime metric.

A. Effective scalar potential depending on the energy

In this section we study the possibility of reducing the two-body dynamics to a one-body one introducing, in the “effective” description, the scalar potential Φ_{eff} displayed in Eq. (3.3), and assuming that the vector potential \mathbf{A}_{eff} is zero. In this case the derivative of the radial Hamilton principal-function is given by:

$$\frac{dS_R^0}{dR} = 2m_0 \mathcal{E}_0^{\text{NR}} - 2m_0 e_0 \Phi_{\text{eff}} - \frac{\mathcal{J}_0^2}{R^2} + \frac{(\mathcal{E}_0^{\text{NR}})^2}{c^2} + \frac{e_0^2 \Phi_{\text{eff}}^2}{c^2} - \frac{2e_0 \mathcal{E}_0^{\text{NR}} \Phi_{\text{eff}}}{c^2}, \quad (3.10)$$

where we have introduced the non-relativistic energy $\mathcal{E}_0^{\text{NR}} = \mathcal{E}_0^{\text{R}} - m_0 c^2$. Plugging the above expression in Eq. (3.6) we get:

$$\begin{aligned} \mathcal{I}_R^{\text{eff}}(\mathcal{E}_0^{\text{NR}}, \mathcal{J}_0) &= \frac{\alpha_0 m_0^{1/2}}{\sqrt{-2\mathcal{E}_0^{\text{NR}}}} \left[-\phi_0 - \frac{3\phi_0}{4} \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} + \frac{5\phi_0}{32} \left(\frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \right)^2 \right] - \mathcal{J}_0 \\ &+ \frac{\alpha_0^2}{\mathcal{J}_0 c^2} \left[\frac{\phi_0^2}{2} - \phi_0 \phi_1 - \phi_0 \phi_1 \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \right] + \frac{1}{8} \frac{\alpha_0^4}{\mathcal{J}_0^3 c^4} [\phi_0^4 - 12\phi_0^3 \phi_1 + 8\phi_0^2 \phi_2 + 4\phi_0^2 \phi_1^2]. \end{aligned} \quad (3.11)$$

Identifying Eq. (3.11) with Eq. (2.29), assuming $m_0 = \mu$ and using Eqs. (3.8), (3.9) we obtain the equations for the unknowns $\phi_0, \phi_1, \phi_2, a_0, a_1$ and α_1 and α_2 . In particular, at 0PC order we have

$$-\phi_0 \alpha_0 = \alpha, \quad (3.12)$$

and we find quite natural to pose $\phi_0 = -1$, that is $e_0^2 = \alpha_0 = \alpha = -e_1 e_2$. The equations at 1PC level are:

$$-\phi_0 \alpha_0 (2\alpha_1 - 3) = \alpha (\nu - 3), \quad \alpha_0^2 (\phi_0^2 - 2\phi_0 \phi_1) = \alpha^2, \quad (3.13)$$

while at 2PC order they read:

$$-\phi_0 \alpha_0 (5 - 12\alpha_1 - 12\alpha_1^2 + 16\alpha_2) = \alpha (5 - 6\nu - 3\nu^2), \quad (3.14)$$

$$\alpha_0^4 (\phi_0^4 + 4\phi_0^2 \phi_1^2 - 12\phi_0^3 \phi_1 + 8\phi_0^2 \phi_2) = \alpha^4 (1 - 6\nu), \quad (3.15)$$

$$\phi_0 \phi_1 \alpha_0^2 = \frac{\nu}{2} \alpha^2. \quad (3.16)$$

Let us notice that at 1PC order, Eq. (3.13) gives $\alpha_1 = \nu/2$ and $\phi_1 = 0$. Then at 2PC order one can solve Eqs. (3.14) and (3.15) in terms of α_2 and ϕ_2 , obtaining $\alpha_2 = 0$ and $\phi_2 = -3\nu/4$, but Eq. (3.16) is inconsistent. To solve this incompatibility we are obliged to introduce another parameter in the “effective” description. A simple possibility is to suppose that the diverse coefficients that appear in the effective scalar potential depend on an external parameter E_{ext} , having the dimension of an energy, that is:

$$\phi_0(E_{\text{ext}}) = \phi_0^{(0)} + \phi_0^{(2)} \frac{E_{\text{ext}}}{m_0 c^2} + \phi_0^{(4)} \left(\frac{E_{\text{ext}}}{m_0 c^2} \right)^2, \quad (3.17)$$

$$\phi_1(E_{\text{ext}}) = \phi_1^{(0)} + \phi_1^{(2)} \frac{E_{\text{ext}}}{m_0 c^2}, \quad (3.18)$$

$$\phi_2(E_{\text{ext}}) = \phi_2^{(0)}. \quad (3.19)$$

We find that in order to implement the matching with the “real” description the parameter E_{ext} should be fixed equal to the “effective” non-relativistic energy, i.e. $E_{\text{ext}} \equiv \mathcal{E}_0^{\text{NR}}$. In more details, the introduction of an energy dependence in the coefficients ϕ_0, ϕ_1, ϕ_2 reshuffles the c^{-2} expansion of Eq. (3.11), modifying the Eqs. (3.13)–(3.16) and allowing to solve in many ways the constraint equations. The simplest solution is envisaged by requiring that the energy-dependence enters only at 2PC order in the coefficient ϕ_1 . In this case, the solution reads:

$$\phi_0^{(0)} = -1, \quad \phi_0^{(2)} = 0, \quad \phi_0^{(4)} = 0, \quad (3.20)$$

$$\phi_1^{(0)} = 0, \quad \phi_1^{(2)} = -\frac{\nu}{2}, \quad \phi_2^{(0)} = -\frac{3}{4}\nu, \quad (3.21)$$

$$\alpha_1 = \frac{\nu}{2}, \quad \alpha_2 = 0. \quad (3.22)$$

To summarize, we have succeeded in mapping the two-body dynamics onto the one of a test particle of mass $m_0 = \mu$ moving in the external scalar potential:

$$\Phi_{\text{eff}}(R, E_{\text{ext}}) = -\frac{e_0}{R} \left[1 - \frac{\nu}{2} \left(\frac{E_{\text{ext}}}{m_0 c^2} \right) \left(\frac{\alpha_0}{m_0 c^2 R} \right) - \frac{3\nu}{4} \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2 \right], \quad (3.23)$$

where $E_{\text{ext}} \equiv \mathcal{E}_0^{\text{NR}}$. We have found that the matching is implemented relating the “real” and “effective” energy-levels by the formula:

$$\frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} = \frac{\mathcal{E}^{\text{NR}}}{\mu c^2} \left[1 + \frac{\nu}{2} \frac{\mathcal{E}^{\text{NR}}}{\mu c^2} \right], \quad (3.24)$$

which, as noticed in [1], gives the following relation between the real total relativistic energy \mathcal{E} and the effective relativistic energy \mathcal{E}_0 :

$$\frac{\mathcal{E}_0}{m_0 c^2} \equiv \frac{\mathcal{E}^2 - m_1^2 c^4 - m_2^2 c^4}{2m_1 m_2 c^4}. \quad (3.25)$$

The above equation has a rather interesting property. In the limit $m_1 \ll m_2$ the effective energy of the effective particle equals the energy of the particle 1 in the rest frame of particle 2 (and reciprocally if $m_2 \ll m_1$). Moreover, the result (3.25) coincides with the one derived in Ref. [2] in the context of quantum electrodynamics. We find quite remarkable that our way of relating the “real” and “effective” energy-levels agrees with the one introduced in [2]. Nevertheless, we consider the dependence on the energy of the effective scalar potential, Eq. (3.23), quite unsatisfactory, though envisaged by Todorov et al. [3] in the quasi-potential approach. Indeed, in our context the presence of an external parameter in the scalar potential obscures the nature of the mapping and complicates the possibility of incorporating radiation reaction effects. Certainly, this cannot be achieved straightforwardly in the way suggested in [4] for the gravitational case.

As a final remark, let us note that if we were using the effective description introduced in the quasi-potential approach by Todorov et al. [3], we should have considered a test particle with effective mass, m_{eff} , and effective energy, \mathcal{E}_{eff} , given by:

$$m_{\text{eff}}(\mathcal{E}_{\text{real}}) = \frac{m_1 m_2 c^2}{\mathcal{E}_{\text{real}}}, \quad \mathcal{E}_{\text{eff}} \equiv \frac{\mathcal{E}_{\text{real}}^2 - m_1^2 c^4 - m_2^2 c^4}{2\mathcal{E}_{\text{real}}}. \quad (3.26)$$

We have investigated the possibility of introducing an energy dependence in the effective mass of the test particle, but we found that, in this case, it is not possible to overcome the inconsistency in the matching equations that raised at 2PC order. A way out could be to introduce also an energy dependence in the effective coupling α_{eff} , but we find this possibility not very appealing.

B. Effective vector potential depending on the angular momentum

We have seen in the previous section that, at 2PC level, in order to cope with an inconsistency of the constraint equations, we were obliged to introduce an external parameter in the coefficients of the scalar potential. In this section we shall investigate the possibility of overcoming the above inconsistency by introducing, in the “effective” description, a scalar potential Φ_{eff} , independent of any external parameter, and a vector potential \mathbf{A}_{eff} which will depend on an external vector \mathbf{J}_{ext} . In order to implement the matching, we have found that it is sufficient to limit to the following form of the vector potential (see Eq. (3.4)):

$$\mathbf{A}_{\text{eff}} = \frac{e_0 (\mathbf{J}_{\text{ext}} \wedge \mathbf{R})}{m_0 c R^3} \left[a_0 + a_1 \frac{\alpha_0}{m_0 c^2 R} + \dots \right], \quad (3.27)$$

where \mathbf{J}_{ext} is supposed to be perpendicular to the plane of motion³. In the Hamilton-Jacobi framework, restricting to $\theta = \pi/2$, we have

$$\mathbf{p} = \frac{\partial S_{\text{eff}}}{\partial \mathbf{R}} = \hat{\mathbf{e}}_R \frac{\partial S_{\text{eff}}}{\partial R} + \hat{\mathbf{e}}_\varphi \frac{1}{R} \frac{\partial S_{\text{eff}}}{\partial \varphi}, \quad (3.28)$$

where $\hat{\mathbf{e}}_R$ and $\hat{\mathbf{e}}_\varphi$ are vectors of the orthonormal basis. Due to the particular choice of the vector \mathbf{J}_{ext} we made, the following equation holds:

$$\mathbf{A}_{\text{eff}} = \frac{e_0 J_{\text{ext}} \hat{\mathbf{e}}_\varphi}{m_0 c R^2} \left[a_0 + a_1 \frac{\alpha_0}{m_0 c^2 R} + \dots \right], \quad (3.29)$$

where $J_{\text{ext}} = |\mathbf{J}_{\text{ext}}|$. Finally, using $\partial S_{\text{eff}}/\partial \varphi = \mathcal{J}_0$ (see Eq. (3.5)), we get:

$$\mathbf{p} \cdot \mathbf{A}_{\text{eff}} = \frac{e_0 J_{\text{ext}} \mathcal{J}_0}{m_0 c R^3} \left[a_0 + a_1 \frac{\alpha_0}{m_0 c^2 R} + \dots \right], \quad \mathbf{A}_{\text{eff}}^2 = \frac{e_0^2 J_{\text{ext}}^2 a_0^2}{m_0^2 c^2 R^4} + \dots \quad (3.30)$$

Note the crucial fact that, with the very special choice of the vector potential we made, $\mathbf{p} \cdot \mathbf{A}_{\text{eff}}$ does not depend on \mathbf{p}_R . Plugging the above expressions in the Hamilton-Jacobi equation, Eq. (3.2), with $\mathcal{H}_{\text{eff}} = \mathcal{E}_0^{\text{NR}} + m_0^2 c^2$ we obtain:

$$\begin{aligned} \frac{dS_R^0}{dR} = & 2m_0 \mathcal{E}_0^{\text{NR}} - 2m_0 e_0 \Phi_{\text{eff}} - \frac{\mathcal{J}_0^2}{R^2} + \frac{(\mathcal{E}_0^{\text{NR}})^2}{c^2} + \frac{e_0^2 \Phi_{\text{eff}}^2}{c^2} - \frac{2e_0 \mathcal{E}_0^{\text{NR}} \Phi_{\text{eff}}}{c^2} \\ & + \frac{2\mathcal{J}_0 J_{\text{ext}}}{R^2} \left[a_0 \frac{\alpha_0}{m_0 c^2 R} + a_1 \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2 \right] - \frac{J_{\text{ext}}^2 a_0^2}{R^2} \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2, \end{aligned} \quad (3.31)$$

³Note that, with this choice of the vector potential the magnetic field will be perpendicular to the plane of motion.

where Φ_{eff} is given by Eq. (3.3). Evaluating the radial action variable (see Eq. (3.6)) we finally get:

$$\begin{aligned} \mathcal{I}_R^{\text{eff}}(\mathcal{E}_0^{\text{NR}}, \mathcal{J}_0, J_{\text{ext}}) &= \frac{\alpha_0 m_0^{1/2}}{\sqrt{-2\mathcal{E}_0^{\text{NR}}}} \left[-\phi_0 - \frac{3\phi_0}{4} \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} + \frac{5\phi_0}{32} \left(\frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \right)^2 \right] - \mathcal{J}_0 + \frac{\alpha_0^2}{\mathcal{J}_0 c^2} \left[\frac{\phi_0^2}{2} \right. \\ &- \phi_0 \phi_1 - \phi_0 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} + \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \left(-\phi_0 \phi_1 - \phi_0 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} + a_1 \frac{J_{\text{ext}}}{\mathcal{J}_0} + a_0^2 \frac{J_{\text{ext}}^2}{\mathcal{J}_0^2} \right) \left. \right] + \frac{1}{8} \frac{\alpha_0^4}{\mathcal{J}_0^3 c^4} \left[\phi_0^4 \right. \\ &- 12\phi_0^3 \phi_1 + 8\phi_0^2 \phi_2 + 4\phi_0^2 \phi_1^2 + 24\phi_0^2 \phi_1 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} - 12\phi_0^3 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} + 12\phi_0^2 a_1 \frac{J_{\text{ext}}}{\mathcal{J}_0} + 24\phi_0^2 a_0^2 \frac{J_{\text{ext}}^2}{\mathcal{J}_0^2} \left. \right]. \end{aligned} \quad (3.32)$$

Let us impose that the above equation coincides with the analogous expression for the “real” description, given by Eq. (2.29). Assuming $m_0 = \mu$ and using Eqs. (3.8), (3.9) we derive the new constraint equations to be satisfied. At 0PC order we still have $-\phi_0 \alpha_0 = \alpha$, and we pose $\phi_0 = -1$, while at 1PC level we get:

$$-\phi_0 \alpha_0 (2\alpha_1 - 3) = \alpha (\nu - 3), \quad \alpha_0^2 \left(\phi_0^2 - 2\phi_0 \phi_1 - 2\phi_0 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} \right) = \alpha^2. \quad (3.33)$$

The first equation in (3.33) gives $\alpha_1 = \nu/2$, while the second one is automatically satisfied if we make the rather natural requirement that either the Coulomb potential does not have any correction at 1PC order ($\phi_1 = 0$) or the vector potential enters only at the next Coulombian order ($a_0 = 0$). Finally, the 2PC order constraints read:

$$-\phi_0 \alpha_0 (5 - 12\alpha_1 - 12\alpha_1^2 + 16\alpha_2) = \alpha (5 - 6\nu - 3\nu^2), \quad (3.34)$$

$$\begin{aligned} \alpha_0^4 \left(\phi_0^4 + 4\phi_0^2 \phi_1^2 - 12\phi_0^3 \phi_1 + 8\phi_0^2 \phi_2 + 24\phi_0^2 \phi_1 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} \right. \\ \left. - 12\phi_0^3 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} + 24\phi_0^2 a_0^2 \frac{J_{\text{ext}}^2}{\mathcal{J}_0^2} + 12\phi_0^2 a_1 \frac{J_{\text{ext}}}{\mathcal{J}_0} \right) = \alpha^4 (1 - 6\nu), \end{aligned} \quad (3.35)$$

$$\alpha_0^2 \left(\phi_0 \phi_1 + \phi_0 a_0 \frac{J_{\text{ext}}}{\mathcal{J}_0} - a_1 \frac{J_{\text{ext}}}{\mathcal{J}_0} - a_0^2 \frac{J_{\text{ext}}^2}{\mathcal{J}_0^2} \right) = \nu \frac{\alpha^2}{2}. \quad (3.36)$$

Plugging the results obtained at 0PC and 1PC order in Eqs. (3.34)–(3.36) and assuming that the external vector J_{ext} coincides with the constant of motion \mathcal{J}_0 , we end up with the unique, rather simple solution:

$$\phi_2 = 0, \quad a_1 = -\frac{\nu}{2}, \quad \alpha_2 = 0. \quad (3.37)$$

In conclusion, in this Section we have obtained that at 2PC order it is possible to reduce the two-charge dynamics to the one of a test particle moving in an effective electromagnetic

field described by a Coulomb potential $\Phi_{\text{eff}}(R) = -e_0/R$ and a vector potential dependent on the external vector \mathbf{J}_{ext} ($\equiv \mathcal{J}_0$):

$$\mathbf{A}_{\text{eff}}(R, \mathbf{J}_{\text{ext}}) = -\frac{\nu}{2} \frac{e_0 \alpha_0}{m_0^2 c^3} \frac{(\mathbf{J}_{\text{ext}} \wedge \mathbf{R})}{R^4}. \quad (3.38)$$

Moreover, quite remarkably, we have found, under rather natural assumptions, that the one-to-one mapping between the “real” and the “effective” energy-levels is still given by the formula (3.25). However, as already discussed at the end of the previous section, the fact that the electromagnetic field still has to depend on external parameters is not very desirable. In the next section we shall investigate a feasible way out.

C. Effective metric

So far we have seen that in order to succeed in reducing the two-body dynamics onto a one-body description we were obliged to introduce external parameters, which have been identified either with the energy or the angular momentum of the test particle m_0 . This result is not very appealing, especially when we want to incorporate radiation reaction effects. A possible way out would be to relax the hypothesis that in the one-body description the test particle move in a flat spacetime. The effective spacetime metric should be viewed as an effective way of describing the global exchange of energy between the two charged particles in the “real” description.

The most general spherical symmetric metric written in Schwarzschild gauge has the form:

$$ds_{\text{eff}}^2 = -A(R) c^2 dt^2 + B(R) dR^2 + R^2 (d\theta^2 + \sin^2 \theta d\varphi^2), \quad (3.39)$$

where the coefficients $A(R)$ and $B(R)$ are given as an expansion in the dimensionless parameter $\alpha_0/(m_0 c^2 R)$, that is:

$$A(R) = 1 + A_1 \frac{\alpha_0}{m_0 c^2 R} + A_2 \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2 + A_3 \left(\frac{\alpha_0}{m_0 c^2 R} \right)^3 + \dots, \quad (3.40)$$

$$B(R) = 1 + B_1 \frac{\alpha_0}{m_0 c^2 R} + B_2 \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2 + \dots. \quad (3.41)$$

The reduction, from the two-body problem to the one-body one, can simply be implemented assuming that in the “effective” description only the scalar potential Φ_{eff} is different from zero. In this case the derivative of the Hamilton principal-function reads:

$$\frac{dS_R^0}{dR} = \frac{B(R)}{c^2 A(R)} (\mathcal{E}_0 + m_0 c^2 - e_0 \Phi_{\text{eff}})^2 - \frac{B(R)}{R^2} \mathcal{J}_0^2 - B(R) m_0^2 c^2, \quad (3.42)$$

and for the radial action variable we derive:

$$\begin{aligned} \mathcal{I}_R^{\text{eff}}(\mathcal{E}_0^{\text{NR}}, \mathcal{J}_0) &= \frac{\alpha_0 m_0^{1/2}}{\sqrt{-2\mathcal{E}_0^{\text{NR}}}} \left[\mathcal{A} + \mathcal{B} \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} + \mathcal{C} \left(\frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \right)^2 \right] - \mathcal{J}_0 \\ &+ \frac{\alpha_0^2}{\mathcal{J}_0 c^2} \left[\mathcal{D} + \mathcal{E} \frac{\mathcal{E}_0^{\text{NR}}}{m_0 c^2} \right] + \frac{\alpha_0^4}{\mathcal{J}_0^3 c^4} \mathcal{F}, \end{aligned} \quad (3.43)$$

where the various coefficients can be written explicitly as:

$$\mathcal{A} = -\phi_0 - \frac{1}{2} A_1, \quad (3.44)$$

$$\mathcal{B} = -\frac{3}{4} \phi_0 + \left(B_1 - \frac{7}{8} A_1 \right), \quad (3.45)$$

$$\mathcal{C} = \frac{5}{32} \phi_0 + \left(\frac{B_1}{4} - \frac{19}{64} A_1 \right), \quad (3.46)$$

$$\mathcal{D} = \phi_0 \left(-\phi_1 - \frac{B_1}{2} + A_1 \right) + \frac{1}{2} \phi_0^2 - \frac{1}{4} A_1 B_1 + \frac{A_1^2}{2} - \frac{A_2}{2}, \quad (3.47)$$

$$\mathcal{E} = \phi_0 \left(-\phi_1 + A_1 - \frac{B_1}{2} \right) + A_1^2 - A_2 - \frac{1}{2} A_1 B_1 - \frac{B_1^2}{8} + \frac{B_2}{2}, \quad (3.48)$$

$$\begin{aligned} \mathcal{F} &= \frac{1}{64} (24A_1^4 - 48A_1^2 A_2 + 8A_2^2 + 16A_1 A_3 - 8A_1^3 B_1 + 8A_1 A_2 B_1 - A_1^2 B_1^2 + 4A_1^2 B_2) \\ &+ \frac{\phi_0}{16} (-16\phi_1 A_1^2 + 24A_1^3 + 8\phi_2 A_1 + 8\phi_1 A_2 - 32A_1 A_2 + 8A_3 + 4\phi_1 A_1 B_1 \\ &- 8A_1^2 B_1 + 4A_2 B_1 - A_1 B_1^2 + 4A_1 B_2) + \frac{\phi_0^3}{4} (-6\phi_1 + 4A_1 - B_1) + \frac{\phi_0^4}{8} \\ &+ \frac{\phi_0^2}{16} (8\phi_1^2 - 40\phi_1 A_1 + 32A_1^2 + 16\phi_2 - 20A_2 + 8\phi_1 B_1 - 10A_1 B_1 - B_1^2 + 4B_2). \end{aligned} \quad (3.49)$$

The above expressions coincide with the ones obtained in pure general relativity [1], once the limit $\phi_0 \rightarrow 0$ is considered and α_0 is identified with the analogous quantity in the gravitational case, i.e. with $Gm_1 m_2$ (G is the Newton constant). Let us now equate the “real”, Eq. (2.29) and the “effective”, Eq. (3.43), radial action variables, assuming that the following relations hold: $\mathcal{J}_0 = \mathcal{J}$, $m_0 = \mu$ and Eq. (3.9). At 0PC order we get the constraint $\alpha_0(-\phi_0 - A_1/2) = \alpha$ which can be naturally fulfilled imposing that $A_1 = 0$ and posing $\phi_0 = -1$, as above. At 1PC level we derive:

$$-2\phi_0 \alpha_0 (2\alpha_1 - 3) + \alpha_0 (7A_1 - 8B_1 - 2A_1 \alpha_1) = 2\alpha (\nu - 3), \quad (3.50)$$

$$2\alpha_0^2 (\phi_0^2 + \phi_0 (2A_1 - B_1 - 2\phi_1)) + \alpha_0^2 (2A_1^2 - 2A_2 - A_1 B_1) = 2\alpha^2. \quad (3.51)$$

If we demand that at this order the scalar potential and the effective metric do not differ from the Coulomb potential and the flat spacetime metric, respectively, i.e. we pose $\phi_1 = 0, A_2 = 0, B_1 = 0$, we find that Eq. (3.50) gives $\alpha_1 = \nu/2$ while Eq. (3.51) is automatically satisfied. Inserting these values in the constraint equations at 2PC order and imposing that there are no corrections to the Coulomb potential at this order ($\phi_2 = 0$) we obtain the unique simple solution:

$$\alpha_2 = 0, \quad A_3 = \nu, \quad B_2 = -\nu. \quad (3.52)$$

Hence, we have found that with the introduction of an effective metric we are not obliged to introduce in the electromagnetic field any dependence on external parameters, neither the energy nor the angular momentum. Moreover, up to 2PC order we find that there is no need of modifying the Coulomb scalar potential, i.e. $\Phi_{\text{eff}}(R) = -e_0/R$ and the “energy-levels” of the real and “effective” description are still related by Eq. (3.25). Finally, the external spacetime metric is simply given by:

$$A(R) = 1 + \nu \left(\frac{\alpha_0}{m_0 c^2 R} \right)^3, \quad B(R) = 1 - \nu \left(\frac{\alpha_0}{m_0 c^2 R} \right)^2. \quad (3.53)$$

IV. CONCLUSIONS

In this paper we have analysed the application of a new approach to studying the relativistic dynamics of the bound states of two classical charged particles, with comparable masses, interacting electromagnetically. The key idea, originally introduced investigating the two-body problem in general relativity [1], has been to map the “real” two-body problem onto the one of a test particle moving in an external electromagnetic field.

We have found that the matching can be implemented imposing the following rather natural “rules”: i) the adiabatic invariants \mathcal{N} and \mathcal{J} in the two descriptions have to be identified; ii) the reduced mass of the “real” system, μ , has to coincide with the mass of the effective particle, m_0 , and iii) the energy axis between the two problems has to be transformed. Let us note immediately that, a bottom-line of our results has been that, in all the three cases considered (see Sec. III A, III B and III C), we have found quite naturally that the energy axis, between the two descriptions, has to change in such a way that the

effective energy of the effective particle coincides with the energy of the particle 1 in the rest frame of particle 2 in the limit $m_1 \ll m_2$ (and vice versa) (see Eq. (3.25)).

Nevertheless, contrary to the results obtained in general-relativity [1], the requirements i), ii) and iii) envisaged above, do not fix uniquely the external electromagnetic field, with which the effective test particle m_0 interacts. In fact, we have found that, in order to overcome an inconsistency in the constraint equations which define the matching, we had to introduce an external parameter either in the scalar potential, Eq. (3.23), or in the vector potential, Eq. (3.38). These parameters have to be identified with the non-relativistic energy and the angular momentum of the effective test particle m_0 , respectively. As pointed out above and in Ref. [1], the dependence of the effective electromagnetic field on some external parameter makes the mapping between the two descriptions quite awkward and complicates the inclusion of radiation reaction effects. A possible solution of this issue is to relax the hypothesis that the test particle moves in a flat spacetime. Indeed, in this case we have found that the conditions i), ii) and iii) fix rather naturally the external scalar potential and the effective metric. They provide, up to 2PC order, an effective Coulomb potential and a rather simple ν -deformed flat metric (see Eq. (3.53)).

Once the matching has been successfully defined, to have a complete knowledge of the “real” dynamics through the auxiliary “effective” one, we can construct, like in [1], the canonical transformation which relates the variables of the relative motion in the “real” description, to the coordinates and momenta of the test particle in the “effective” problem. However, this calculation goes beyond the scope of the present paper.

Finally, a last remark. In Sec. III B we have introduced a vector potential in the effective description in such a way that the source of the magnetic field is the angular momentum of the system. This study suggests the investigation, in the general relativity context [1], of relaxing the hypothesis of mapping the “real” two-body dynamics onto the one of a test particle moving in a deformed Schwarzschild spacetime. Indeed, it could well be possible to match the two problems appealing to an effective deformed Kerr spacetime.

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REFERENCES

- [1] A. Buonanno and T. Damour, Phys. Rev. **D 59**, 084006 (1999).
- [2] E. Brézin, C. Itzykson and J. Zinn-Justin, Phys. Rev. **D 1**, 2349 (1970); C. Itzykson and J.B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1980).
- [3] I.T. Todorov, Phys. Rev. **D3**, 2351 (1971); V.A. Rizov, I.T. Todorov and B.L. Aneva, Nucl. Phys. **B98**, 447 (1975); I.T. Todorov, *Quasi-potential approach to the two-body problem in quantum field theory*, in *Properties of Fundamental Interactions 9C*, edited by A. Zichichi (Editrice Compositori, Bologna, 1973), pp. 951-979; A. Maheswari, E.R. Nissimov and I.T. Todorov, Lett. Math. Phys. **5**, 359 (1981).
- [4] A. Buonanno and T. Damour, Phys. Rev. **D 62**, 064015 (2000).
- [5] T. Damour, P. Jaranowski and G. Schäfer, Phys. Rev. **D** (to be published) [gr-qc/0005034].
- [6] R. Karplus and A. Klein, Phys. Rev. **D 87**, 848 (1952); R. Barbieri and E. Remiddi, Phys. Lett. **B 65**, 258 (1976); Vu K. Cung, A. Devoto and T. Fulton, Phys. Rev. **A 19**, 1886 (1978); J. Schwinger, *Particles, Sources and Fields*, (Addison-Wesley, Redwood City, California, 1989).
- [7] T. Damour and G. Schäfer, Nuovo Cimento **101**, 127 (1988).
- [8] E.H. Kerner, J. Math. Phys. **3**, 35 (1962); *ibid.* **6**, 1218 (1965).
- [9] T. Damour and G. Schäfer, Phys. Rev. **D 37**, 1099 (1988); J. Math. Phys. **32**, 127 (1991).
- [10] V.N. Golubenkov and Ya. A. Smorodinskii, Zh. Eksp. Teor. Fiz. **31**, 330 (1956) [Sov. Phys. JETP **4**, 442 (1957)].
- [11] T. Damour, in *Gravitational Radiation*, edited by N. Deruelle and T. Piran (North-Holland, Amsterdam, 1983).
- [12] J.A. Wheeler and R.P. Feynman, Rev. Mod. Phys. **21**, 425 (1949).
- [13] G. Schäfer, Phys. Lett. **A 100**, 128 (1984).

- [14] L.D. Landau and E.M. Lifshitz, *The Classical Theory of Fields*, 4th revised English Ed. (Pergamon, New York, 1975); B.M. Barker and R.F. O'Connell, *Ann. Phys.* **129**, 358 (1980); *Can. J. Phys.* **58**, 1659 (1980); X. Jaén, J. Llosa and A. Molina, *Phys. Rev. D* **34**, 2302 (1986).