CERN-TH/96-322
IFUP-TH/96-65
hep-th/9701146

EXPANDING AND CONTRACTING UNIVERSES IN THIRD QUANTIZED STRING COSMOLOGY

A. Buonanno[(a)(b)], M. Gasperini[(c)(d)], M. Maggiore[(a)(b)] and C. Ungarelli[(a)(b)]

[(a)] *Dipartimento di Fisica, Università di Pisa, Piazza Torricelli 2, I-56100 Pisa, Italy*
[(b)] *Istituto Nazionale di Fisica Nucleare, Sezione di Pisa, Pisa, Italy*
[(c)] *Theory Division, CERN, CH-1211 Geneva 23, Switzerland*
[(d)] *Dipartimento di Fisica Teorica, Università di Torino, Via P. Giuria 1, 10125 Turin, Italy*

Abstract

We discuss the possibility of quantum transitions from the string perturbative vacuum to cosmological configurations characterized by isotropic contraction and decreasing dilaton. When the dilaton potential preserves the sign of the Hubble factor throughout the evolution, such transitions can be represented as an anti-tunnelling of the Wheeler–De Witt wave function in minisuperspace or, in a third-quantization language, as the production of pairs of universes out of the vacuum.

To appear in **Class. Quantum Grav.**

CERN-TH/96-322
November 1996

At very early times, according to the standard cosmological scenario, the Universe is expected to approach a Planckian, quantum gravity regime where a classical description of the spacetime manifold is no longer appropriate. A possible quantum description of the Universe, in that regime, is based on the Wheeler–De Witt (WDW) wave function [1, 2], generally defined on the superspace spanned by all three-dimensional geometric configurations. In that context it becomes possible to compute, with an appropriate model of (mini)superspace, the probability distribution of a given cosmological configuration versus an appropriate "state" parameter (for instance the cosmological constant Λ). The results, however, are in general affected by operator-ordering ambiguities, and are also strongly dependent on the boundary conditions [3]–[5] imposed on the solutions of the WDW equation.

String theory has recently motivated the study of a cosmological scenario in which the Universe starts from the string perturbative vacuum and evolves through an initial, "pre-big bang" phase [6], characterized by an accelerated growth of the curvature and of the gauge coupling $g = e^{\phi/2}$ (ϕ is the dilaton field). In such a context, the WDW equation is obtained from the low-energy string effective action [7]–[9], and has no operator ordering ambiguities [7] since the ordering is uniquely fixed by the duality symmetries of the action. Also, the boundary conditions are determined by the choice of the perturbative vacuum as the initial state for the cosmological evolution.

According to the lowest-order effective action, the classical evolution from the perturbative vacuum necessarily leads the background to a singularity, and the transition to the present decelerated "post-big bang" configuration is impossible, for any realistic type of (local) dilaton potential [10]. With an appropriate potential, however, the transition may become allowed at the quantum level even if, for the same potential, it remains classically forbidden. This effect was discussed in previous papers [7], in which the WDW equation was applied to compute the transition probability between two duality-related pre- and post-big bang cosmological phases.

The string perturbative vacuum is, in general, a higher-dimensional state, and the initial growth of the dilatonic coupling g requires, according to the lowest-order action, a large enough number of expanding dimensions. For instance, in a Bianchi-type I background with d expanding and n contracting isotropic spatial dimensions, the growth of g requires [6] $d + \sqrt{d+n} > n$, which cannot be satisfied by $d = 3$, in particular, in the ten-dimensional superstring vacuum. With a monotonic evolution of the scale factor, this represents another obstruction to a smooth transition to our present, dimensionally reduced Universe.

The aim of this paper is to show that the initial perturbative vacuum is not inconsistent, at the quantum level, with a final contracting cosmological configuration, when we add to the lowest-order action an appropriate dilaton potential (such as the simple one induced by an effective cosmological constant). In particular, for a WDW potential which is translationally invariant in minisuperspace, along the direction parametrized by the scale factor, and for which the sign of the Hubble factor is classically conserved during the whole evolution, the cosmological contraction corresponds to a pure quantum effect. It can be described as an "anti-tunnelling" of the WDW wave function from the string perturbative vacuum or, in a third quantization [11] language, as a production of "pairs of universes" (one expanding, the other contracting) out of the third quantized vacuum. Such a process requires the identification of the time-like coordinate in minisuperspace with the direction parametrized by the shifted dilaton $\overline{\phi}$ (see below), and is complementary to the process of spatial reflection of the wave function, which describes transitions from pre- to post-big bang configurations [7].

We shall adopt, in this paper, the minisuperspace model already discussed in [7], based on the tree-level, lowest-order in α', string effective action [12]. Working in the simplifying assumption that only the metric and the dilaton contribute non-trivially to the background, in d isotropic spatial dimensions, the corresponding action can be written as

$$S = -\frac{1}{2\,\lambda_s^{d-1}} \int d^{d+1}x \sqrt{|g|}\, e^{-\phi} \left(R + \partial_\mu\phi\partial^\mu\phi + V\right). \tag{1}$$

Here $\lambda_s = (\alpha')^{1/2}$ is the fundamental string length parameter governing the higher-derivative expansion of the action, and V is a (possibly non-perturbative) dilaton potential. By using the parametrization appropriate to an isotropic, spatially flat cosmological background:

$$g_{\mu\nu} = \mathrm{diag}\left(N^2(t), -a^2(t)\delta_{ij}\right), \qquad a = \exp\left[\beta(t)/\sqrt{d}\right], \qquad \phi = \phi(t), \tag{2}$$

and assuming spatial sections of finite volume, the action can be expressed in the convenient form

$$S = \frac{\lambda_s}{2} \int dt\, \frac{e^{-\overline{\phi}}}{N} \left(\dot{\beta}^2 - \dot{\overline{\phi}}^2 - N\,V\right), \tag{3}$$

where $\overline{\phi}$ is the shifted dilaton:

$$\overline{\phi} = \phi - \log \int d^dx/\lambda_s^d - \sqrt{d}\,\beta\,. \tag{4}$$

The variation with respect to N then leads to the Hamiltonian constraint

$$\Pi_\beta^2 - \Pi_{\overline{\phi}}^2 + \lambda_s^2 \, V(\beta, \overline{\phi}) \, e^{-2\overline{\phi}} = 0 \ , \tag{5}$$

where $\Pi_\beta, \Pi_{\overline{\phi}}$ are the (dimensionless) canonical momenta (in the gauge $N = 1$):

$$\Pi_\beta = \frac{\delta S}{\delta \dot{\beta}} = \lambda_s \, \dot{\beta} \, e^{-\overline{\phi}}, \qquad \qquad \Pi_{\overline{\phi}} = \frac{\delta S}{\delta \dot{\overline{\phi}}} = -\lambda_s \, \dot{\overline{\phi}} \, e^{-\overline{\phi}}. \tag{6}$$

When $V = 0$, the classical solutions of the action (3) describing the phase of accelerated pre-big bang evolution are characterized by two duality-related branches [6], defined in the negative time range:

$$t < 0, \quad a = a_0(-t)^{\mp 1/\sqrt{d}}, \quad \overline{\phi} - \phi_0 = -\ln(-t) = \pm\beta, \quad \Pi_\beta = \pm k = \text{const}, \quad \Pi_{\overline{\phi}} = \mp \Pi_\beta < 0 \tag{7}$$

(k, a_0 and ϕ_0 are integration constants). For the upper-sign branch the metric is expanding ($\Pi_\beta > 0$), and the curvature scale $\dot{\beta}^2$ and the string coupling $g(t)$ are growing, starting asymptotically from the perturbative vacuum, the state with flat metric ($\dot{\beta} = 0 = \dot{\overline{\phi}}$) and vanishing coupling constant ($\phi = -\infty$, $g = 0$). The lower-sign branch corresponds instead to a contracting configuration ($\Pi_\beta < 0$), in which the coupling $g(t)$ is decreasing. In the presence of a constant dilaton potential, $V = \Lambda = \text{const}$, the accelerated pre-big bang solutions are again characterized by two branches [13]:

$$t < 0, \qquad a = a_0 \left[\tanh(-t\sqrt{\Lambda}/2)\right]^{\mp 1/\sqrt{d}}, \qquad \overline{\phi} - \phi_0 = -\ln\sinh\left(-t\sqrt{\Lambda}\right), \qquad \Pi_{\overline{\phi}} < 0, \tag{8}$$

which are respectively expanding with growing dilaton (upper sign, $\Pi_\beta > 0$) and contracting with decreasing dilaton (lower sign, $\Pi_\beta < 0$). In this case both branches are dominated, in the low-curvature regime, by the contribution of a positive cosmological constant Λ. The initial perturbative vacuum is replaced by a configuration with flat metric and linearly evolving dilaton ($\dot{\beta} = 0$, $\dot{\phi} = \text{const}$), another well-known string theory background [14] (exact solution to all orders in the α' expansion). Near the singularity ($t \to 0_-$), however, the contribution of Λ becomes negligible, and the solution (8) asymptotically approaches that of eq. (7).

In this paper we shall assume that an effective cosmological constant Λ is generated non-perturbatively in the strong coupling, Planckian regime, and we shall use the WDW equation to discuss the possibility of transitions, induced by Λ, from the perturbative vacuum to a final configuration with contracting metric and decreasing dilaton. We shall consider, in particular, the case in which the effective dilaton potential can be approximated by the Heaviside step function θ as $V(\beta, \overline{\phi}) = \Lambda\ \theta(\overline{\phi})$. The corresponding WDW equation, in the minisuperspace spanned by β and $\overline{\phi}$, is obtained from the Hamiltonian constraint (5) through the differential representation $\Pi = -i\nabla$:

$$\left[\partial^2_{\overline{\phi}} - \partial^2_\beta + \lambda_s^2\, \Lambda\, \theta(\overline{\phi})\, e^{-2\overline{\phi}}\right]\ \Psi = 0\,. \tag{9}$$

The momentum along the β axis is conserved,

$$[\Pi_\beta, H] = 0, \qquad\qquad \Pi_\beta = \lambda_s \dot{\beta} e^{-\overline{\phi}} = k = \text{const}, \tag{10}$$

and the general solution of the WDW equation can be factorized as $\Psi_k(\overline{\phi}, \beta) = \psi_k(\overline{\phi}) e^{ik\beta}$.

Note that we have assumed a potential V depending explicitly only on $\overline{\phi}$ because the classical evolution of the scale factor, in that case, is monotonic, and no contracting configuration can be eventually obtained, classically, if we start from the isotropic perturbative vacuum. From a quantum-mechanic point of view, however, the situation is different. Indeed, if we assign to $\overline{\phi}$ the role of time-like coordinate, eq. (9) is formally equivalent to a Klein–Gordon equation with time-dependent mass term. The solution ψ_k is a linear combination of plane waves for $\overline{\phi} < 0$, and of Bessel functions [15] $J_{\pm\nu}(z)$, of imaginary index $\nu = ik$ and argument $z = \lambda_s \sqrt{\Lambda} e^{-\overline{\phi}}$, for $\overline{\phi} > 0$. In particular, the functions

$$\Psi_k^{(\pm)} = \frac{e^{ik\beta}}{\sqrt{4\pi k}}\, e^{\mp ik\overline{\phi}}\,, \qquad\qquad \overline{\phi}\ <\ 0, \tag{11}$$

$$\Psi_k^{(\pm)} = \frac{e^{ik\beta}}{\sqrt{4\pi k}} \left(\frac{z_0}{2}\right)^{\mp\nu} \Gamma(1 \pm \nu)\, J_{\pm\nu}(z)\,, \qquad\qquad \overline{\phi}\ >\ 0, \tag{12}$$

where $z_0 = \lambda_s \sqrt{\Lambda}$ and Γ is the Euler function, provide orthonormal sets of solutions with respect to the Klein-Gordon scalar product

$$(\Psi^1, \Psi^2) = -i \int d\beta\, \Psi^1(\beta, \overline{\phi})\ \overleftrightarrow{\partial}_{\overline{\phi}}\ \Psi^{2*}(\beta, \overline{\phi})\ . \tag{13}$$

We shall fix the boundary conditions by imposing that, for $\overline{\phi} < 0$, the Universe is represented by the wave function

$$\Psi_{Ik}(\beta, \overline{\phi} < 0) = \frac{1}{\sqrt{4\pi k}}\, e^{ik(\beta - \overline{\phi})}, \tag{14}$$

corresponding to a state of growing dilaton and accelerated pre-big bang expansion from the perturbative vacuum, with $\Pi_\beta = -\Pi_{\overline{\phi}} = k > 0$ according to eq. (7). The eigenvalue k of Π_β parametrizes the initial state in the space of all classical configurations (7). For $\overline{\phi} > 0$ the wave function is uniquely determined by the matching conditions for Ψ and $\partial_{\overline{\phi}}\Psi$ at $\overline{\phi} = 0$, in terms of the functions (12), as

$$\Psi_{IIk}(\beta, \overline{\phi} > 0) = A_k^+ \, \Psi_k^{(+)} + A_k^- \, \Psi_k^{(-)}, \tag{15}$$

where

$$A_k^\pm = \frac{i\, z_0}{2\, k} \left(\frac{z_0}{2}\right)^{\pm ik} \Gamma(1 \mp ik) \left[\pm J'_{\mp ik}(z_0) \mp \frac{i\, k}{z_0}\, J_{\mp ik}(z_0)\right] \tag{16}$$

(a prime denotes differentiation of the Bessel functions with respect to their argument). Given a pure initial state $\Psi_I^{(+)}$ of "positive frequency" k, the final state is thus a mixture of "positive" and "negative" frequency modes, $\Psi_{II}^{(+)}$ and $\Psi_{II}^{(-)}$, satisfying asymptotically the conditions

$$\lim_{\overline{\phi}\to\infty} \Psi_{II}^{(\pm)}(\beta, \overline{\phi}) = \Psi_\infty^{(\pm)}(\beta, \overline{\phi}) \sim e^{ik(\beta \mp \overline{\phi})},$$
$$\Pi_\beta \Psi_\infty^{(\pm)} = -i\partial_\beta \Psi_\infty^{(\pm)} = k\Psi_\infty^{(\pm)}, \qquad \Pi_{\overline{\phi}}\Psi_\infty^{(\pm)} = -i\partial_{\overline{\phi}}\Psi_\infty^{(\pm)} = \mp\Pi_\beta\Psi_\infty^{(\pm)}. \tag{17}$$

The mixing is determined by the coefficients $A_k^\pm$, satisfying the standard Bogoliubov normalization condition $|A_k^+|^2 - |A_k^-|^2 = 1$. In a second quantization context, it is well known that such a mixing describes a process of pair production [16], the negative energy mode being associated to an antiparticle state of positive energy and opposite spatial momentum. It thus seems correct to interpret the above splitting of the WDW wave function, in a third quantization context [11], as the production of a pair of universes, with quantum numbers $\{\Pi_\beta, \Pi_{\overline{\phi}}\}$, corresponding to positive energy ($\Pi_{\overline{\phi}} < 0$) and opposite momentum along the spacelike direction β. One of the two universes is isotropically expanding ($\Pi_\beta > 0$), with growing dilaton; the "anti-universe" is isotropically contracting ($\Pi_\beta < 0$), with decreasing dilaton. Both configurations evolve towards the curvature singularity of the classical pre-big bang solution (8). However, while the growing dilaton state corresponds to a continuous classical evolution from the perturbative vacuum, no smooth connection to such vacuum is possible, classically, for the state with decreasing dilaton.

It is important to stress that, as long as $V = V(\overline{\phi})$ and, consequently, Π_β is conserved, a third-quantized production of universes is only possible provided we assign the role of time-like coordinate to $\overline{\phi}$, and the potential satisfies $V(\overline{\phi})e^{-2\overline{\phi}} \to 0$ for $\overline{\phi} \to +\infty$ (in order to identify, asymptotically, positive and negative frequency modes). The pairs of universes are produced in the limit of large positive $\overline{\phi}$, so that we cannot describe in this context a transition to post-big bang cosmological configurations, which are instead characterized by $\overline{\phi} < 0$. A quantum description of the transition from pre- to post-big bang requires in fact the interpretation of β as the time-like axis, as discussed in [7]. In that case, a third quantized production of pairs becomes possible only if Π_β is not conserved, namely if V depends also on β.

For the process considered in this paper, the probability is controlled by $|A_k^-|^2$, which determines the expectation number pairs of universes produced in the final state. The production probability is negligible when $|A_k^-| \ll 1$; it has the typical probability of a vacuum fluctuation effect when $|A_k^-| \sim |A_k^+| \sim 1$; finally, when $|A_k^-| \sim |A_k^+| \gg 1$, the initial wave function is parametrically amplified [17] and the probability is large. In our case, the interesting parameter characterizing the process, besides Λ, is the portion of proper spatial volume $\Omega = a^d \int d^dx$ undergoing the transition. Considering, in particular, $d = 3$ spatial dimensions, and using the definitions of k and $\overline{\phi}$, the initial momentum k can be conveniently expressed as $k = \sqrt{3}\Omega_s g_s^{-2}\lambda_s^{-3}$, where $g_s = \exp(\phi_s/2)$ and Ω_s are, respectively, the value of the coupling and of the proper spatial volume evaluated at the string scale $t = t_s$, when $H \equiv \dot{\beta}/\sqrt{3} = \lambda_s^{-1}$. By exploiting the properties of the Bessel functions, we can then express the asymptotic limits of the Bogoliubov coefficients (16) in terms of the physical parameters Ω_s and Λ. We obtain, at fixed $\Omega_s/(g_s^2\lambda_s^3) = 1$,

$$|A^+|^2 - 1 \simeq |A^-|^2 \simeq \frac{1}{48}\,\Lambda^2\,\lambda_s^4\,, \qquad \Lambda \ll \lambda_s^{-2}, \tag{18}$$

$$|A^+|^2 \simeq |A^-|^2 \simeq \sqrt{\Lambda\,\lambda_s^2}\;\frac{\cosh(\sqrt{3}\,\pi) - \sin(2\sqrt{\Lambda\,\lambda_s^2})}{4\sqrt{3}\,\sinh(\sqrt{3}\,\pi)}\,, \qquad \Lambda \gg \lambda_s^{-2}, \tag{19}$$

and, at fixed $\Lambda\lambda_s^2 = 1$,

$$|A^+|^2 \simeq |A^-|^2 \simeq \frac{g_s^4\,\lambda_s^6}{12\,\Omega_s^2}\,|J_0'(1)|^2\,, \qquad |J_0'(1)| \simeq 0.44, \qquad \Omega_s \ll g_s^2\lambda_s^3 \tag{20}$$

(the limit $\Omega_s \gg g_s^2\lambda_s^3$ cannot be performed because the quantum process is confined to the region of large $\overline{\phi}$).

The quantum production of universes in a state with non-vanishing cosmological constant Λ is thus strongly suppressed for small values of Λ, while it is favoured in the opposite limit of large Λ and of proper volumes that are small in string units, in qualitative agreement with previous results [7], and also with the general approach to quantum cosmology based on tunnelling boundary conditions [4, 5]. Instead of a "tunnelling from nothing", however, this quantum production of expanding and contracting universes can be seen as an "anti-tunnelling from the string perturbative vacuum" of the WDW wave function. Indeed, the asymptotic expansion of the solution (14), (15),

$$\begin{aligned} \overline{\phi} \to +\infty \quad , \qquad & \psi \sim A_{in} e^{-ik\overline{\phi}} + A_{ref} e^{ik\overline{\phi}}, \\ \overline{\phi} \to -\infty \quad , \qquad & \psi \sim A_{tr} e^{-ik\overline{\phi}}, \end{aligned} \tag{21}$$

describes formally a scattering process along $\overline{\phi}$, in which the expanding universe corresponds to the incident part of the wave function, the contracting anti-universe to the reflected part, and the initial vacuum to the transmitted part. In the parametric amplification regime of eqs. (19) and (20), where $|A^+| \sim |A^-| \gg 1$, the reflection coefficient $R = |A_{ref}|^2/|A_{in}|^2$ is approximately 1, and the Bogoliubov coefficient $|A^-|$, which controls the probability of pair production, becomes the inverse of the tunnelling coefficient $T = |A_{tr}|^2/|A_{in}|^2$:

$$|A^-|^2 = \frac{|A_{ref}|^2}{|A_{tr}|^2} = \frac{R}{T} \simeq \frac{1}{T}. \tag{22}$$

In view of future applications, we have also computed numerically the Bogoliubov coefficients $A^\pm$ by discretizing the WDW equation with the explicit method [18], and using the routine Fast Fourier Transform [19]. A computer simulation, in which the pair production process is graphically represented by the scattering and reflection of an initial wave packet, has given results in complete agreement with the analytic computation (16).

In conclusion, we have shown in this paper that it is not impossible, in a quantum cosmology context, to nucleate universes in a state characterized by isotropic contraction and decreasing dilaton. The process can be described as the production from the vacuum of universe–anti-universe pairs in the strong coupling regime, triggered by the presence of an effective cosmological constant. When $V = V(\overline{\phi})$ and Π_β is conserved, the pair-production process requires the identification of $\overline{\phi}$ as time-like coordinate in minisuperspace, while the transition from pre- to post-big bang configurations requires the complementary choice of β as the time-like axis.

The validity of our analysis is limited by the very crude approximation (the step potential) adopted to modellize the time-evolution of the non-perturbative dilaton potential. Also, an appropriate potential should depend on ϕ (not on $\overline{\phi}$ as assumed in this paper); in that case, however, the transition from expansion to contraction may be allowed also classically (in an appropriate limit), and is represented in minisuperspace as a reflection [20] (instead of an anti-tunnelling) of the wave function. In spite of these limitations, the analysis of this paper confirms that the WDW approach provides an adequate framework for a consistent formulation of quantum string cosmology, with the boundary conditions uniquely prescribed by the choice of the initial perturbative vacuum.

Acknowledgements

We are grateful to Vittorio de Alfaro and Roberto Ricci for discussions and clarifying comments. Special thanks are due to Gabriele Veneziano for a careful reading of the manuscript and for helpful suggestions.

References

[1] De Witt B S 1967 Phys. Rev. 160 1113; Wheeler J A 1968 *Battelle Rencontres* ed De Witt C and Wheeler J A (Benjamin, New York).

[2] For a recent review on quantum cosmology see Vilenkin A 1996 *String gravity and physics at the Planck energy scale* ed Sanchez N and Zichichi A (Kluwer Acad. Pub., Dordrecht) p 345.

[3] Hartle J B and Hawking S W 1983 Phys. Rev. D28 2960; Hawking S W 1984 Nucl. Phys. B239 257; Hawking S W and Page D N 1986 Nucl. Phys. B264 185.

[4] Linde A D 1984 Sov. Phys. JETPT 60 211; Lett. Nuovo Cimento 39 401; Zel'dovich Y B and Starobinski A A 1984 Sov. Astron. Lett. 10 135; Rubakov V A 1984 Phys. Lett. B148 280.

[5] Vilenkin A 1984 Phys. Rev. D30 509, 1986 D33 3560, 1988 D37 888.

[6] Gasperini M and Veneziano G 1993 Astropart. Phys. 1 317; Mod. Phys. Lett. A8 3701; 1994 Phys. Rev. D50 2519. An updated collection of papers on the pre-big bang scenario is available at `http://www.to.infn.it/teorici/gasperini/`.

[7] Gasperini M Maharana J and Veneziano G 1996 Nucl. Phys. B472 349; Gasperini M and Veneziano G 1996 Gen. Rel. Grav. 28 1301.

[8] Kehagias A A and Lukas A 1996 Nucl. Phys. B477 549.

[9] Lidsey J E 1996 *Inflationary and deflationary branches in extended pre-big bang cosmology* (gr-qc/9605017) Phys. Rev. D (in press).

[10] Brustein R and G. Veneziano G 1994 Phys. Lett. B329 429; Kaloper N Madden R and Olive K A 1995 Nucl. Phys. B452 677; 1996 Phys. Lett. B371 34; Easther R Maeda K and Wands D 1996 Phys. Rev. D53 4247.

[11] Rubakov V A 1988 Phys. Lett. B214 503; Kozimirov N and Tkachev I I 1988 Mod. Phys. Lett. A4 2377; McGuigan M 1988 Phys. Rev. D38 3031, 1989 D39 2229; 1990 Phys. Rev. D41 418.

[12] See for instance Metsaev R R and Tseytlin A A 1987 Nucl. Phys. B293 385.

[13] Veneziano G 1991 Phys. Lett. B265 387.

[14] Myers R C 1987 Phys. Lett. B199 371.

[15] Abramowicz M and Stegun I A 1972 *Handbook of Mathematical Functions* (Dover, New York).

[16] See for instance Birrel N and Davies P 1982 *Quantum fields in curved space* (Cambridge University Press, Cambridge).

[17] Grishchuk L P 1975 Sov. Phys. JEPT 40 409; Starobinski A A 1979 JEPT Letters 30 682.

[18] Mitchell A R and Griffiths D F 1980 *The finite difference method in partial differential equations* (John Wiley and Sons Ltd., Chichester).

[19] Press W H et al 1992 *Numerical recipes in Fortran: the art of scientific computing* (Cambridge University Press, Cambridge).

[20] Ricci R et al 1996 in preparation.