

Dissertation
submitted to the
Combined Faculties of the Natural Sciences and Mathematics
of the Ruperto-Carola-University of Heidelberg. Germany
for the degree of
Doctor of Natural Sciences

Put forward by
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born in: Minsk, Belarus
Oral examination: July 9th, 2014

Dynamics of an electron spin
in
strong classical and quantized electromagnetic fields

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Zusammenfassung

Die Bewegung eines Elektrons, das mit einem starken elektromagnetischen Feld wechselwirkt, wird in dieser Arbeit mit besonderer Betonung auf den Freiheitsgrad des Spins untersucht. Es wird gezeigt, dass der Hamilton-Operator dieses Systems mit Hilfe der kanonischen Transformationen der Feldvariablen in zwei Teile getrennt werden kann, nämlich die Wechselwirkung des Elektrons mit der Einzelmode des elektromagnetischen Felds und mit dessen Fluktuationen. Eine störungstheoretische Behandlung dieser Fluktuationen erlaubt es, die Bedingungen für die Anwendbarkeit der Einzelmodenapproximation zu bestimmen. Darüber hinaus wird die zeitliche Entwicklung des Spins in dieser Näherung untersucht. Zusätzlich zu den schnellen Spinoszillationen mit der Laserfrequenz kann eine zweite Zeitskala identifiziert werden, die durch die Ausstrahlung und Absorption der Feldquanten verursacht wird, die so genannte “collapse” und “revival” Dynamik. Dieser Effekt ist bei gegenwärtig verfügbaren Intensitäten von 10^{18} W/cm² beobachtbar. Danach wird der Fall eines starken Felds betrachtet, bei dem die Fluktuationen vernachlässigbar sind. Dabei untersuchen wir die Asymmetrien in der Streuung der Elektronen, hervorgerufen durch die Elektronpolarisation sowie die Dauer des Laserimpulses, und geben die optimalen Bedingungen zur Beobachtung dieser Asymmetrie an.

Abstract

The electron motion in the presence of a strong classical and quantized pulse of an electromagnetic field is studied with a special emphasis on the spin degree of freedom. It is shown that the Hamiltonian of this system can be separated into two parts with the help of canonical transformations of the field variables, namely the interaction between an electron and a single-collective-mode of the field and fluctuations relatively to this collective-mode. The application of perturbation theory to the fluctuations allows the conditions of applicability of the single-mode approximation for the quantized external field to be formulated. Furthermore, within this approximation the electron spin evolution is investigated. In addition to fast spin oscillations at the laser frequency, a second time scale is identified due to the intensity-dependent emissions and absorptions of field quanta, that is collapse and revival dynamics. The effect is observable at the experimentally feasible intensity of 10^{18} W/cm². After this, we switch to the regime of higher intensities, when the fluctuations of the external field can be neglected. We investigate the asymmetries in the electron scattering arising due to the electron polarization and pulse duration, and constrain the optimal conditions for the asymmetry observation.

Within the framework of this thesis, the following articles were published in refereed journals:

- O. D. Skoromnik, I. D. Feranchuk, and C. H. Keitel
Collapse-and-revival dynamics of strongly laser-driven electrons
Phys. Rev. A. **87**, 052107 (2013).
- O. D. Skoromnik and I. D. Feranchuk
Justification of the single-mode approximation for a finite-duration laser pulse interacting with an electron
accepted J.Phys.B (2014).

The following articles are in preparation for publication in refereed journals:

- O. D. Skoromnik, K. Z. Hatsagortsyan, and C. H. Keitel
Spin dependent non-linear Compton scattering
in preparation.

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Notation

Three-dimensional

Three-dimensional tensor indices are denoted by Latin letters i, k, l, \dots taking the values x, y, z or $1, 2, 3$.

Three-dimensional vector is denoted by bold letter \mathbf{a} .

The three-dimensional volume element is d^3x or $d\mathbf{x}$.

Four-dimensional

Four-dimensional tensor indices are denoted by Greek letters λ, μ, ν, \dots , taking the values $0, 1, 2, 3$.

A metric has signature -2 . The metric tensor $g_{\mu\nu} = \text{diag}\{1, -1, -1, -1\}$.

Components of a four vector are stated in the form $a^\mu = (a^0, \mathbf{a})$.

To simplify the formulas, the index is often omitted in writing the components of a four vector.

The scalar products of four vectors are written as $(a \cdot b)$ or $a \cdot b$; $(a \cdot b) \equiv \sum_{\mu=0}^3 a_\mu b^\mu = a^0 b^0 - \mathbf{a} \cdot \mathbf{b}$.

The summation over repeated indices is assumed.

The radius vector is $x^\mu = (t, \mathbf{r})$. The 4-volume element is d^4x or $d\mathbf{x}$.

The operator of differentiation with respect to the coordinate x^μ is $\partial_\mu = \partial/\partial x^\mu$.

The fully antisymmetric 4-tensor Levi-Civita is $\epsilon^{\mu\nu\gamma\delta}$, with $\epsilon^{0123} = -\epsilon_{0123} = -1$.

The four-dimensional delta function $\delta^4(a) = \delta(a^0)\delta(\mathbf{a})$.

Operators

Operators are denoted by letters in straight font \mathbf{A} or for vector operators with bold straight font \mathbf{p} .

The commutator of two operators \mathbf{A} and \mathbf{B} is $[\mathbf{A}, \mathbf{B}] = \mathbf{AB} - \mathbf{BA}$.

The Hermitian conjugate operator is \mathbf{A}^\dagger .

Dirac's equation

The Dirac matrices are α and β , with $\alpha^2 = \beta^2 = 1$; $\alpha\beta + \beta\alpha = 0$ and $\alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}$. The covariant notation is $\gamma^0 = \beta$, $\gamma = \beta\alpha$; $(\gamma^0)^2 = 1$ and $(\gamma^i)^2 = -1$. $(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$.

The product of a four vector a with the Dirac matrices is $\hat{a} = (\gamma \cdot a) \equiv \sum_\mu \gamma^\mu a_\mu$.

Units

The natural units are used, with $\hbar = c = 1$. In this units $e^2 = 1/137$.

Constants

Velocity of light $c = 2.997925 \cdot 10^{10} \text{ cm/s}$.

Unit of charge $e = 4.80310^{-9}$ CGS electrostatic units.

Electron mass $m = 9.11 \cdot 10^{-28} \text{ g}$.

Planck's constant $\hbar = 1.055 \cdot 10^{-34} \text{ J} \cdot \text{s}$.

Fine structure constant $\alpha = e^2/\hbar c$; $1/\alpha = 137.04$.

Classical electron radius $r_e = e^2/mc^2 = 2.818 \cdot 10^{-13} \text{ cm}$.

Compton wavelength of the electron $\hbar/mc = 3.862 \cdot 10^{-11} \text{ cm}$.

Electron rest energy $mc^2 = 0.511 \cdot 10^6 \text{ eV} = 2.606 \cdot 10^{10} \text{ cm}^{-1}$.

Electron volt $1 \text{ eV} = 1.602 \cdot 10^{-19} \text{ J} = 0.510 \cdot 10^5 \text{ cm}^{-1}$.

Other

Coulomb law $V(r) = \alpha/r$.

Fourier expansion

$$f(\mathbf{r}) = \int f(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{r}} \frac{d\mathbf{k}}{(2\pi)^3}, \quad f(\mathbf{k}) = \int f(\mathbf{r}) e^{-i\mathbf{k} \cdot \mathbf{r}} d\mathbf{x}.$$

Introduction

Quantum electrodynamics (QED) is considered as one of the most successful gauge field theories available today [1]. Firstly, among the four fundamental forces the electromagnetic interaction can be experimentally accessed in the easiest way. Secondly, compared to other quantum field theories, the interaction Hamiltonian between electromagnetic and electron-positron fields is precisely known [2]. Finally, quantum electrodynamics includes only two constants: the electron charge e and mass m , which can be measured nowadays with an extremely high accuracy. For example, in a recent experiment [3] the electron mass was determined with the amazingly small relative error of $3 \cdot 10^{-11}$. The accurate knowledge of the electron mass enables to compare the measured value of the electron g -factor in heavy hydrogen-like ions with bound state QED calculations, thus providing a unique framework for tests of fundamental physics.

Another possibility for tests of theoretical predictions is connected with the invention of very powerful lasers [4–7]. The HERCULES [8] laser facility provides pulses with peak intensity of 10^{22} W/cm² and pulse duration of 30 fs. Moreover, even higher intensities up to 10^{25} W/cm² are expected at the forthcoming ELI and HiPER experiments [9, 10]. It is clear that such strong external fields¹ can not be treated by means of perturbation theory in the light-matter interactions. Luckily, the exact solution of the Dirac equation in the field of a plane electromagnetic wave is known [11], allowing us to switch to the Furry representation of QED [12]. In this picture the external field is treated in a non-perturbative way and only the interaction with the secondary-quantized vacuum field is regarded as a perturbation.

This approach was employed for the first time in the pioneering works of Reiss, Ritus and Nikishov [13–17]. They studied the action of a monochromatic plane electromagnetic wave with frequency ω on Compton effect and electron-positron pair creation. It was shown that these processes dependence on the external field can be characterized via two dimensionless parameters ξ and χ . The first parameter ξ determines the strength of an electromagnetic field and does not contain Planck's constant. It is equal to the work, referred to m , performed by the field of strength E over the wavelength², i.e., $\xi = eE/(m\omega)$. It can also be represented as the ratio of the field work over the Compton wavelength to the energy ω of the field quantum. In the limit of small $\xi \ll 1$ the most probable processes are those where only few photons participate. For example, the standard (linear) Compton effect includes only two electrons and two photons. The first-order correction to the linear case is proportional to ξ^2 . When ξ increases, the probability to absorb more than one photon from the external field becomes larger and the process turns out to be multi-quantum. Consequently, for the correct description we have to

¹The field intensity $3.5 \cdot 10^{16}$ W/cm² corresponds to electric field amplitudes of the same order as the Coulomb field in atoms at the first Bohr radius.

² $\xi \sim 1$ corresponds to 10^{18} W/cm² for ω in the visible spectral region.

switch to the Furry representation in which matrix elements have non-linear dependence on the field. Therefore we can state that the parameter ξ defines non-linearity. The second parameter χ contains Planck's constant \hbar and is equal to the work, in the units of m , performed by the field over the Compton wavelength in the particle rest frame. This parameter is responsible for the magnitude of quantum effects. In the non-linear case, i.e., when $\xi \gg 1$ the quantum effects are the largest when $\chi \sim 1$, while in the linear one they are optimal when $\chi \sim \xi$.

This ground breaking investigation was performed as early as in 1964, only four years later after the realization of the first laser by Maiman [18]. At those days the field intensity did not even reach 10^{14} W/cm², therefore, all experimentally accessible processes were in the regime of $\xi \ll 1$, where perturbation theory is applicable. The situation, changed only with the appearance of the chirped pulse amplification technique [19] in 1985, when the threshold of 10^{15} W/cm² was achieved. Since then, the laser intensity has constantly grown, to face recently the above mentioned high values. However, such extreme field intensities are accessed not in the continuous laser operation regime, but in a pulsed one. This motivated a lot of research aimed at reexamining results obtained by Ritus and investigating the effects coming from the pulse shape and duration.

For example, the influence of the pulse duration on the process of Compton scattering in the absence of polarization effects is investigated in refs. [20–29] by means of the analyses of the emitted photon spectra. Two cases of an external field are distinguished: the case of a short and single laser pulse and a sequence of such short laser pulses. Comparison of these two approaches shows that for pulses which include only few oscillations of the laser field the emitted photon spectra look very similar and do not coincide with the case when an external field is a monochromatic plane wave, in particular asymmetries in the angular distribution are present. Nevertheless, when the pulse duration increases all three scenarios of the external field provide the same radiation spectra.

Further attempts to describe Compton scattering in the presence of a short intense pulse were made, namely the calculation of the second-order processes in the absence of polarization effects in a monochromatic plane wave [30] and in a short laser pulse [31]. It was shown that the total decay rate of the non-linear process in the limit of a very long laser pulse and in the regime $\xi \sim 1$ can be up to 30 % larger than the perturbative one. Moreover, the angular distributions are altered significantly in the non-linear case, thus highlighting that the non-perturbative description is necessary. The comparison of the monochromatic plane wave case with the short laser pulse one shows, that the positions of the spectral peaks are significantly shifted into the region of larger frequencies. We should also mention here that the analysis of the emitted photon spectra allows one to determine the carrier envelope phase of the external laser pulse [32].

In analogy with Compton scattering the reexamination of the processes of laser assisted bremsstrahlung [33], pair production by the photon (Breit-Wheeler process) [34–40] and by the field of a nucleus (Bethe-Heitler process) [41–44] in the presence of an external laser pulse was performed. The main focus is on the investigation of the influence of the external field polarization and carrier-envelope phase of the laser pulse on the energy spectra of the created electrons and positrons. For instance, it was shown that the probability of the Breit-Wheeler process is the largest at the linear polarization of the external field. Furthermore, the change of the carrier-envelope phase of the laser pulse can vary the angular distributions of electrons or positrons from being the symmetric to the asymmetric.

Apart from the above mentioned works, another growing field is the investigation of the

role of spin in strong field processes. Here the publications are limited to the semi-classical calculation of spin radiation [45] and the scattering of an electron by a standing wave (Kapitza-Dirac effect) [46, 47] and to the QED treatment of the polarization of the electrons through Compton scattering in a field of a monochromatic plane wave [48–55]. In the latter case, it was shown that the self-polarization of the electrons exists only for non-head-on geometries and can achieve a high value up to 65%, however, the time needed for the polarization is very large, of the order of one second, hence is not accessible with presently available laser fields. The influence of the laser pulse duration is investigated only in the two works [56, 57]. In the first one, non-linear Compton scattering by a linearly polarized laser pulse of a finite duration is analyzed, with a focus on the spin effects of the target electrons only. It was shown that only in small parts of the emission spectra the influence of spin can be significant. While in the second work, only one regime of the scattering process was presented and was stated that the spin probability is larger at smaller ξ .

In all the above mentioned references the external electromagnetic field was considered as a classical plane wave and its quantum fluctuations were generally assumed to be negligible for interactions with very strong fields. Therefore, it is of interest to formulate a question about the influence of quantum fluctuations on the system dynamics. Among all known exact solutions of the Dirac equation related to laser pulses [11, 58–75], one describes the electron motion in the single-mode quantized field [61]. This allows one to study QED processes also in the presence of a quantized field. The issue was addressed in a few articles, in particular the effect of the quantized field on non-linear Compton scattering [76–78], on the electron motion in a magnetic field [79], on the dynamics of Rydberg atoms [80, 81] and the investigation of entanglement between an electron and a photon [82]. In the case of Compton scattering [77] the frequency of the emitted photon has corrections due to the depletion of the external field mode, i.e., the number of photons in the mode in the initial state is not equal to the number of photons in the final state. Consequently, in the emitted photon spectra additional harmonics will appear, that is a purely quantum effect.

At the same time, it is well known from quantum optics that the dynamics of an atom in a comparably weaker and resonant laser field depends on the quantum fluctuations of the field. One of the most interesting phenomena of this kind is the collapse-revival effect in the evolution of the Jaynes-Cummings model [83–85] for a two-level atom. This effect was predicted theoretically [86–89] and later observed experimentally [90, 91]. Its qualitative explanation and analytical description were also given in [92–96]. It was shown that the evolution of the population of the atomic states is characterized by two time scales. The first time scale is the period of Rabi oscillations, while the second, slower one is defined by the collapse and revival times of the populations being associated with the absorption and emission of field quanta. An electron can, on one hand, be considered as a two-level system with regard to the spin space. On the other hand, the electron has no other internal quantum degrees of freedom such as the atom. For that reason, the question arises, whether the observation of collapse-revival dynamics for laser-driven electrons is feasible.

In analogy with an atom, qualitatively different time scales can be selected in the evolution of a quasi-energy³ electron state $\psi_p^{(e)}(\mathbf{r}, t)$ with definite quasi-momentum p . One of these

³When an electron interacts with an external field, only the total four momentum of the system is a physical quantity. The term quasi-momentum (energy) is defined as the difference between the full momentum (energy) of the system and the momentum of the external field.

scales is defined by the frequency ω of the electromagnetic field. Another scale is defined by the coherence time T_c , i.e., the length on which the radiation is emitted by the electron. The characteristic time T_c is inversely proportional to the probability w_c of a photon emission per unit of time. An approximation for T_c was found in [15]: $T_c \sim w_c^{-1} \approx 2T/\xi^2$. The existence of two time scales is expected to be observable when the interaction time between an electron and a field T_{int} satisfies $T_{\text{int}} > T_c \gg T$, which can be fulfilled for realizable parameters of the laser pulse. Consequently, the collapses and revivals in the electron dynamics are expected to be observable in this regime.

Another issue that was not investigated in the literature is connected with the fact, that the arbitrary wave packet of the electromagnetic field can not be expressed as a function of one coordinate and time (see 4). In this connection, it is quite obvious that electromagnetic pulses exist, which can not be treated as a plane electromagnetic waves. For example, Gaussian beams [97] reflect the situation of strongly focused laser pulses. As a consequence, the wave packet is not described by the plane-wave exact solutions [11, 61].

The usual laser pulse is characterized by its intensity I , duration τ and focusing area S . The parameters τ and S determine the frequency spread and angular divergence respectively, and are connected with the total pulse energy $W = IS\tau$. Accordingly, the estimations of these quantities must be obtained in order to answer the question about the applicability of the single-mode approximation for the description of a realistic laser-matter interaction.

An analogous problem in quantum optics is the analysis of the evolution of an atom interacting with a resonant field in a cavity. Although different aspects of the atom-field interaction were discussed through this model (see for example, [85] and references therein), in most works the external field was assumed to be a single-mode cavity eigenstate. Few extensions to the model were proposed. One of them is based on the inclusion of the losses of the resonant mode through a lossy cavity [98–108]. Another one is the generalization for the case of few discrete modes interacting with an atom [109–113]. However, models taking into account a superposition of modes in the vicinity of a resonant one have been discussed only recently [114–119].

The just described analysis of the literature connected with strong field QED motivates the research direction of the present thesis. Namely, we will address three issues:

- Formulation of the applicability conditions of the single-mode approximation for the description of realistic laser pulses.
- The influence of the quantized nature of a driving electromagnetic field on the evolution of the electron spin.
- The dependence of non-linear Compton scattering on the electron polarization and duration of the laser pulse.

Accordingly the work is organized in the following way:

In chapter I we investigate exact solutions of the Dirac equation for an electron in the field of a plane electromagnetic wave. The classical electromagnetic field corresponds to the Volkov's solution [11], while the single-mode and quantized field corresponds to the Berson's one [61]. Berson's solution was for the first time derived for the coordinate representation of the creation and annihilation operators of the quantized field, which is not convenient for most calculations. Therefore, we obtained the exact solution of the Dirac equation with a potential given by the

single-mode quantized field in abstract form of the creation and annihilation operators. After this we answer the question about the applicability of the plane wave approximation for the description of realistic laser pulses. In order to achieve that we include in the Dirac equation all modes of the quantized field. Next, performing Bogolubov's [120] canonical transformation of the field variables the Hamiltonian of the system is separated into two parts. The first one describes the interaction between an electron and a single-collective-mode of the field and coincides with the Dirac equation with a single mode (thus exactly solvable). The properties of this mode are defined by the superposition of the modes corresponding to the pulse wave packet. The second part of the Hamiltonian takes into account the field fluctuations relatively to the collective-mode. The single-mode approximation is applicable if the influence of fluctuations on the collective-mode is small. Therefore, we build a perturbation theory over fluctuations and find the conditions on pulse duration, intensity and focusing for which the single-mode approximation is valid.

In chapter II we use the results of chapter I and investigate the influence of the quantum nature of the electromagnetic field on the electron spin. We assume that before the interaction starts the external electromagnetic field is in a coherent state and the electron is a free particle. However, the exact solution of the Dirac equation in a single-mode quantized field does not address such a situation, since it describes the interaction of the electron with the field being in a Fock state. Therefore, the wave function of the system is decomposed in a linear combination of solutions of the Dirac equation and coefficients of this combination are found from the above mentioned initial conditions. Next, in the limit of a strong external field, i.e., for a large occupation number of the external field mode, the expectation value of the electron spin is calculated. This is performed with the help of the saddle point and modified cumulant methods. Two time scales are identified in the problem. The first one is defined by the frequency of the external field, while the second one is associated with the dispersion of the coherent state occupation number. The second time scale represents the so called collapse and revival dynamics of the electron spin. After this, the possibility of an experimental observation of collapses and revivals is discussed.

In contrast to chapter II, chapter III describes the interaction of an electron with the external field of such a strength, that its fluctuations can be neglected. We focus our attention on the influence of the spin on non-linear Compton scattering in short laser pulses. In this section the Furry [12] representation is used for the description of the scattering processes, i.e., the external field is treated in a non-perturbative way by means of the exact solution of the Dirac equation described in chapter I, while the interaction with a vacuum electromagnetic field is treated via perturbation theory. The combined analytical and numerical scheme of calculations is built, which permits to investigate all possible scattering scenarios, for example, the dependence on the field strength and pulse duration and different scattering geometries.

Finally we provide conclusion and outlook.

Chapter I

Exact solutions of the Dirac equation in classical and quantized electromagnetic fields

1 Exact solution of the Dirac equation in a classical electromagnetic field

For the description of many quantum electrodynamical (QED) processes in the presence of an external field the Furry representation [12] for the electron states is used. The application of this representation to the calculation of the QED effects in the presence of a strong electromagnetic field is possible as the exact solution for an electron in the field of an electromagnetic plane wave exist [11].

The four potential of the plane wave A with a wave four vector k ($k^2 = 0$) depends only on the field phase $\phi = (k \cdot x)$, so $A = A(\phi)$, and satisfies the Lorentz gauge condition $\partial_\mu A^\mu = 0$. In such a field the exact solution of the Dirac equation

$$(i\gamma^\mu \partial_\mu - \gamma^\mu e A_\mu(\phi) - m)\psi = 0 \quad (1.1)$$

is

$$\psi_p = \left[1 + \frac{e}{2(k \cdot p)} \hat{k} \hat{A} \right] \frac{u(p)}{\sqrt{2p^0}} e^{iS}, \quad (1.2)$$

where

$$S = -(p \cdot x) - \int^{(k \cdot x)} \left[\frac{e}{(k \cdot p)} (p \cdot A) - \frac{e^2}{2(k \cdot p)} A^2 \right] d\phi. \quad (1.3)$$

The constant bispinor $u(p)$ is the same bispinor as the bispinor amplitude of the free electron's plane wave and is normalized via the condition $\bar{u}u = 2m$. We note here that the conditions $\hat{k}\hat{k} = 0$ and $\hat{k}\hat{A} = -\hat{A}\hat{k}$ are important for the solution procedure.

The wave functions (1.2) satisfy the normalization condition (see [17])

$$\frac{1}{(2\pi)^3} \int \psi_p^\dagger \psi_p d\mathbf{r} = \frac{1}{(2\pi)^3} \int \bar{\psi}_{p'} \gamma^0 \psi_p d\mathbf{r} = \delta(\mathbf{p}' - \mathbf{p}). \quad (1.4)$$

The knowledge of the wave function gives the possibility to calculate all observable quantities of the system. For example, the current density four vector is

$$j^\mu = \frac{1}{p^0} \left[p^\mu - eA^\mu + k^\mu \left(\frac{e(p \cdot A)}{(k \cdot p)} - \frac{e^2 A^2}{2(k \cdot p)} \right) \right]. \quad (1.5)$$

It is crucial to emphasize, that the Volkov solution (1.2) is a quasi-classical solution: the electron is described quantum mechanically and the external field is described classically. However, the electromagnetic field is essentially the quantum object and consequently the solution (1.2) need to be modified for the fully quantum mechanical description.

2 Dirac equation with a multi-mode quantized electromagnetic field

The system of equations of quantum electrodynamics contains a set of the Dirac and Maxwell equations for the secondary-quantized electron-positron wave function and the four potential operator of the field. If a single-particle approximation [2] is used such system can be reduced to one equation for the state vector and has the form

$$i \frac{\partial \Psi}{\partial t} = \left(\sum_k \omega_k \mathbf{a}_k^\dagger \mathbf{a}_k + \alpha \cdot \left(\mathbf{p} - \sum_k \frac{e \mathbf{e}(\mathbf{k})}{\sqrt{2\omega_k V}} (\mathbf{a}_k e^{i\mathbf{k} \cdot \mathbf{r}} + \mathbf{a}_k^\dagger e^{-i\mathbf{k} \cdot \mathbf{r}}) \right) + \beta m \right) \Psi. \quad (2.1)$$

Equation (2.1) includes the energy of the quantized electromagnetic field, the energy of the electron and the energy of the interaction. Here we used notations for a normalization volume V , a photon wave vector \mathbf{k} , a frequency ω_k and a polarization vector $\mathbf{e}(\mathbf{k})$, photon annihilation and creation operators \mathbf{a}_k and \mathbf{a}_k^\dagger of the mode \mathbf{k} , an electron charge e and mass m .

The system's state vector Ψ is a bispinor and is a realization of the irreducible representation of the Lorentz group [121]. The photon creation and annihilation operators act in the Hilbert space of the occupation numbers and satisfy the commutator relation $[\mathbf{a}, \mathbf{a}^\dagger] = 1$. We consider that the electromagnetic field is in the Lorenz gauge, which employs $\mathbf{k} \cdot \mathbf{e}(\mathbf{k}) = 0$.

Equation (2.1) can also be written in covariant form if the transformation $\Psi = e^{-i \sum_k \omega_k t \mathbf{a}_k^\dagger \mathbf{a}_k} \psi$ is used, thus obtaining the covariant Dirac equation:

$$(i\gamma^\mu \partial_\mu - \gamma^\mu e A_\mu - m) \psi = 0, \quad (2.2)$$

with the four potential of the field

$$A_\mu = \sum_k \frac{e_\mu(k)}{\sqrt{2\omega_k V}} (\mathbf{a}_k e^{-i(k \cdot x)} + \mathbf{a}_k^\dagger e^{i(k \cdot x)}), \quad e^0 = 0.$$

In the following paragraphs we will find the exact solution of equation (2.2) in the single-mode approximation and formulate the conditions of its applicability. In order to do so we need to perform several additional transformations. Firstly, the electron coordinates can be excluded from the field operators with the transformation $\psi = e^{i \sum_k (k \cdot x) \mathbf{a}_k^\dagger \mathbf{a}_k} \chi$. As a result, the operators are changed as follows:

$$i\gamma^\mu \partial_\mu \rightarrow i\gamma^\mu \partial_\mu - \sum_k \gamma^\mu k_\mu \mathbf{a}_k^\dagger \mathbf{a}_k, \quad \mathbf{a}_k \rightarrow \mathbf{a}_k e^{i(k \cdot x)}, \quad \mathbf{a}_k^\dagger \rightarrow \mathbf{a}_k^\dagger e^{-i(k \cdot x)},$$

and equation (2.2) leads to

$$\left(i\hat{\partial} - \sum_k \hat{k} \mathbf{a}_k^\dagger \mathbf{a}_k - \sum_k \hat{b}(\mathbf{k})(\mathbf{a}_k + \mathbf{a}_k^\dagger) - m \right) \chi = 0, \quad (2.3)$$

where $b_\mu(\mathbf{k}) = ee_\mu(\mathbf{k}) / \sqrt{2V\omega_k}$ and $e_0(\mathbf{k}) = 0$. With the transformation $\chi = e^{-i(q \cdot x)} \varphi$, the coordinate dependence is separated from the field and spin degrees of freedom

$$(\hat{q} - m - H) = \left(\hat{q} - m - \sum_k \hat{k} \mathbf{a}_k^\dagger \mathbf{a}_k - \sum_k \hat{b}(\mathbf{k})(\mathbf{a}_k + \mathbf{a}_k^\dagger) \right) \varphi = 0, \quad (2.4)$$

where the four vector q can be considered as the total momentum of the system [61]. The final form of equation (2.4) will be used below.

3 Exact solution of the Dirac equation in a single-mode quantized electromagnetic field

The accurate analytic solution of the Dirac equation in a single-mode quantized electromagnetic field was found by Berson [61] in 1969. However, that solution was obtained in a coordinate (Bargmann) representation for the creation and annihilation operators of the field. In what follows we show that it is possible to find an analogous solution directly in operator form.

In 2 we found that the total four momentum of the electron and the field satisfies the equation (2.4). In order to obtain the equation with the only one mode of the field we drop all sums over k in equation (2.4), thus yielding

$$H\phi \equiv (\hat{q} - \hat{k} \mathbf{a}^\dagger \mathbf{a} - \hat{b}(\mathbf{a} + \mathbf{a}^\dagger) - m) \phi = 0. \quad (3.1)$$

To solve equation (3.1), the photon and spin variable should be separated, which can be performed by means of the transformation

$$\phi = U\varphi, \quad U = e^{l\hat{k}\hat{b}(\mathbf{a} + \mathbf{a}^\dagger)},$$

with a constant l that is to be defined later on.

In the Lorentz gauge, the value $(k \cdot b) = 0$ leads to

$$\hat{b}\hat{k} + \hat{k}\hat{b} = 2(k \cdot b) = 0, \quad \hat{k}\hat{k} = k^2 = 0, \quad U = e^{l\hat{k}\hat{b}(\mathbf{a} + \mathbf{a}^\dagger)} = 1 + l\hat{k}\hat{b}(\mathbf{a} + \mathbf{a}^\dagger).$$

Calculating the operator $H' = U^{-1}HU$, one can find

$$\begin{aligned} & (\hat{q} - \hat{k} \mathbf{a}^\dagger \mathbf{a} + l(\mathbf{a} + \mathbf{a}^\dagger)(2\hat{b}(q \cdot k) - 2\hat{k}(q \cdot b)) - \\ & - \hat{b}(\mathbf{a} + \mathbf{a}^\dagger) - 2l^2(\mathbf{a} + \mathbf{a}^\dagger)^2(q \cdot k)b^2\hat{k} - m + l(\mathbf{a} + \mathbf{a}^\dagger)^2 2(q \cdot k)b^2\hat{k}) \varphi \\ & = 0. \end{aligned} \quad (3.2)$$

If we choose

$$l = 1/(2(q \cdot k)),$$

the terms linear in \hat{b} are canceled and equation (3.2) changes to

$$H'\varphi = \left(\hat{q} - m - \hat{k} \left[\mathbf{a}^\dagger \mathbf{a} + \frac{(q \cdot b)}{(q \cdot k)} (\mathbf{a} + \mathbf{a}^\dagger) - \frac{b^2}{2(q \cdot k)} (\mathbf{a} + \mathbf{a}^\dagger)^2 \right] \right) \varphi = 0. \quad (3.3)$$

The operator H' is diagonalized via the following transformations

$$\varphi = DB\Theta, \quad D = e^{\alpha \mathbf{a}^\dagger - \alpha^* \mathbf{a}}, \quad B = e^{-\frac{\eta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})}, \quad (3.4)$$

with the parameters α and η . Here, as well known, the operator D shifts a and a^\dagger by the complex numbers α and α^* , respectively:

$$D^{-1} \mathbf{a} D = \mathbf{a} + \alpha, \quad D^{-1} \mathbf{a}^\dagger D = \mathbf{a}^\dagger + \alpha^*.$$

The operator B transforms the operators a and a^\dagger as follows

$$\begin{aligned} B^{-1} \mathbf{a} B &= \mathbf{a} \operatorname{ch} \eta + \mathbf{a}^\dagger \operatorname{sh} \eta, \\ B^{-1} \mathbf{a}^\dagger B &= \mathbf{a} \operatorname{sh} \eta + \mathbf{a}^\dagger \operatorname{ch} \eta. \end{aligned}$$

These parameters are defined by the condition that the operator $H_1 = B^{-1} D^{-1} H' D B$ transforms to a diagonal form. This leads to

$$\begin{aligned} H_1 \Theta &= \left(\hat{q} - \hat{k} \left(\sqrt{1 - \frac{2b^2}{(q \cdot k)}} (\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2}) - \frac{1}{2} - \frac{(q \cdot b)^2}{(q \cdot k)} \frac{1}{(q \cdot k) - 2b^2} \right) - m \right) \Theta = 0; \\ \alpha &= -\frac{(q \cdot b)}{(q \cdot k)} \frac{1}{1 - 2b^2/(q \cdot k)}, \quad \operatorname{ch} \eta = \frac{1}{2} \left(\sqrt{\kappa} + \frac{1}{\sqrt{\kappa}} \right), \quad \kappa = \frac{1}{\sqrt{1 - \frac{2b^2}{(q \cdot k)}}}. \end{aligned} \quad (3.5)$$

The eigenvector of equation (3.5) can be represented in the form

$$(\hat{p}_n - m) \Theta = 0, \quad \Theta = u(p_n) |n\rangle, \quad (3.6)$$

where $|n\rangle$ is the state vector of the harmonic oscillator, $u(p_n)$ is the constant bispinor which satisfies the same equation as in the case of the free electron, and the vector p_n depends on the quantum number n as follows

$$p_n = q - k \left(\sqrt{1 - \frac{2b^2}{(q \cdot k)}} (n + \frac{1}{2}) - \frac{1}{2} - \frac{(q \cdot b)^2}{(q \cdot k)^2} \frac{1}{1 - 2b^2/(q \cdot k)} \right). \quad (3.7)$$

As a result of all these transformations, the wave function of the electron in the single-mode quantized field has the following form

$$\psi_{qn} = N e^{-i(q \cdot x) + i \mathbf{a}^\dagger \mathbf{a} (k \cdot x)} \left(1 + \frac{\hat{k} \hat{b}}{2(q \cdot k)} (\mathbf{a} + \mathbf{a}^\dagger) \right) e^{\alpha(\mathbf{a}^\dagger - \mathbf{a})} e^{-\frac{\eta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})} u(p_n) |n\rangle, \quad (3.8)$$

where N is a normalization constant.

The vector p_n satisfies

$$p_n^2 - m^2 = 0, \quad (3.9)$$

which is a consequence of equation (3.6), and the four-vector q is the total moment of the system.

The wave function (3.8) coincides with Berson's solution [61] if the Bargmann representation

$$a = \frac{1}{\sqrt{2}}(x + \frac{\partial}{\partial x}); \quad a^\dagger = \frac{1}{\sqrt{2}}(x - \frac{\partial}{\partial x}), \quad (3.10)$$

is used for the operators, with x being the field variable.

It was also shown in [61] that if the field operators are changed to the classical values $a \approx a^\dagger \approx \beta$, the wave function (3.8) coincides with Volkov's solution [1] for a classical field,

$$A_\mu = \frac{e_\mu}{\sqrt{2\omega V}} 2\beta \cos(k \cdot x). \quad (3.11)$$

The wave functions ψ_{qn} satisfy the orthogonality condition

$$\frac{1}{(2\pi)^3} \int \psi_{q'n'}^\dagger \psi_{qn} d\mathbf{r} = N^2 2\varepsilon_n \delta_{n'n} \delta(\mathbf{q}' - \mathbf{q}), \quad (3.12)$$

where $\varepsilon_n = \sqrt{p_n^2 + m^2}$. Thus the normalization constant N can be chosen as for the free electron $N = 1/\sqrt{2\varepsilon_n}$.

4 Wave packet of an electromagnetic field

In the next paragraphs we will investigate the question of the applicability conditions of the plane wave approximation. This question arises due to the fact that the arbitrary wave packet of an electromagnetic field can not be written as the function of $(k \cdot x) = \omega(t - z)$. Let us explicitly demonstrate it. For this purpose, we proceed with the usual procedure of the wave packet's construction [122]:

$$u(\mathbf{r}, t) = \int d\mathbf{k} C(\mathbf{k} - \mathbf{k}_0) e^{i\mathbf{k} \cdot \mathbf{r} - i\omega t}, \quad (4.1)$$

with \mathbf{k}_0 ($k_0^2 = \omega_0^2$) the central wave vector of the wave packet. The substitution $\mathbf{k} = \mathbf{q} + \mathbf{k}_0$, the selection of the coordinate system with z axis directed along \mathbf{k}_0 and the decomposition of the frequency in a Taylor series up to the second order leads to

$$\begin{aligned} u(\mathbf{r}, t) &= e^{i\mathbf{k}_0 \cdot \mathbf{r} - i\omega_0 t} \int d\mathbf{q} C(\mathbf{q}) \exp \left\{ i\mathbf{q} \cdot \mathbf{r} - i \frac{\mathbf{k}_0 \cdot \mathbf{q}}{\omega_0} t - \frac{it}{2\omega_0} \left(q^2 - \frac{(\mathbf{k}_0 \cdot \mathbf{q})^2}{\omega_0^2} \right) \right\} \\ &= e^{i\omega_0(t-z)} \int d\mathbf{q} C(\mathbf{q}) \exp \left\{ iq_x x + iq_y y - iq_z(t-z) - \frac{it}{2\omega_0} (q_x^2 + q_y^2) \right\}. \end{aligned} \quad (4.2)$$

From here, we can conclude, that the arbitrary wave packet can not be written as the function of $\omega(t - z)$. Moreover, the term proportional to the $q_x^2 + q_y^2$ is responsible for the spreading, thus is always present in the field.

The Volkov solution of the Dirac equation [11] in which the four potential depends only on the field phase $(k \cdot x) = \omega(t - z)$ is a solution in the field of a plane electromagnetic wave. In Ref. [61] was shown, that if the external field is the single-mode quantized and is in a coherent state, then the expansion of the quantized solution of the Dirac equation near the quasi-classical value \bar{n} of the photon number, leads to the Volkov solution. This means that the single-mode approximation is indeed the plane wave approximation.

5 An approximating single-mode Hamiltonian and the canonical transformation for its diagonalization

In paragraph 3 we have found the exact solution of the Dirac equation in the operator form for the case when an electron interacts only with one mode of the field. However, the experimental realization of the external field is based on the application of laser pulses which have a finite duration and a transversal spread. This corresponds to the electron's interaction with a multi-mode wave packet with a non-zero spectral width and angular divergence. Therefore, it is of great importance to find the accurate conditions of the applicability of the single-mode approximation, in order to compare theoretical predictions and experimental results. For this purpose we employ the method of approximating Hamiltonian described in detail in reference [120]. The idea of this method consists in the separation of equation (2.4) into two parts, namely the single-collective-mode of the field and fluctuations relative to this collective mode. Consequently, the single-mode approximation can be applicable if the influence of this fluctuations on the system dynamics is small. Parts of this chapter were published as [123].

Experimentally available laser pulses, can be described by a quasi-monochromatic wave packet with the central frequency ω_0 and the wave vector $\mathbf{k}_0 = \omega_0 \mathbf{n}$ (\mathbf{n} is a unit vector) with corresponding spreads in the solid angle $\Delta\Omega$:

$$\delta\omega \sim \frac{1}{\tau}, \quad \delta\mathbf{k}_0 \approx \omega_0^2 \Delta\Omega \sim \frac{1}{S}, \quad (5.1)$$

characterizing by the duration τ and spacial width S of the laser pulse. The non-monochromaticity will be characterized via two dimensionless parameters

$$\sigma_2 = \frac{\delta\omega}{\omega_0} \approx \frac{1}{\omega_0 \tau}, \quad \sigma_1 = \frac{\delta\mathbf{k}_0}{k_0^2} \approx \frac{1}{\omega_0^2 S}. \quad (5.2)$$

For highly intense pulses, as those considering in the following all modes within the volume $\Delta = \delta\omega \delta\mathbf{k}_0$ in the \mathbf{k} -space are highly populated and correspond to the large quantum numbers $n_{\mathbf{k}}$ of the field state vector.

Since the non-vanishing modes of the quantized external field are inside the small volume Δ in \mathbf{k} -space the total field Hamiltonian H in equation (2.4) can be written as:

$$\begin{aligned} H &\equiv H_A + H_1 + H_2, \\ H_A &= \sum_{\mathbf{k} < \Delta} [\hat{k}_0 \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} + \hat{b}_0 (\mathbf{a}_{\mathbf{k}} + \mathbf{a}_{\mathbf{k}}^\dagger)] + \sum_{\mathbf{k} > \Delta} \hat{k} \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}}, \end{aligned} \quad (5.3)$$

where constant four vectors k_0 , b_0 and the small volume Δ in \mathbf{k} -space near k_0 are the variational parameters of the approximating Hamiltonian and will be defined later. The sums $\sum_{\mathbf{k} < \Delta}$ and $\sum_{\mathbf{k} > \Delta}$ mean summation inside and outside the volume Δ respectively and the operators $H_{1,2}$ are determined identically from equation (2.4)

$$\begin{aligned} H_1 &= \sum_{\mathbf{k} < \Delta} [(\hat{k} - \hat{k}_0) \mathbf{a}_{\mathbf{k}}^\dagger \mathbf{a}_{\mathbf{k}} + (\hat{b}(\mathbf{k}) - \hat{b}_0) (\mathbf{a}_{\mathbf{k}} + \mathbf{a}_{\mathbf{k}}^\dagger)], \\ H_2 &= \sum_{\mathbf{k} > \Delta} \hat{b}(\mathbf{k}) (\mathbf{a}_{\mathbf{k}} + \mathbf{a}_{\mathbf{k}}^\dagger). \end{aligned} \quad (5.4)$$

By the definition in ref. [120], the approximating Hamiltonian H_A should quantitatively describe the system, be accurately diagonalizable and the perturbations due to the operators $H_{1,2}$ need to be small. For the diagonalization of H_A let us utilize the method of canonical transformations, which was introduced by Bogolubov and Tyablikov for the polaron problem in the strong field limit [124]. For this purpose we go back to the coordinate representation in (5.3):

$$\begin{aligned} H_A &= \frac{1}{2}\hat{k}_0 \sum_{k<\Delta} (\mathbf{p}_k^2 + \mathbf{q}_k^2) + \hat{b}_0 \sqrt{2} \sum_{k<\Delta} \mathbf{q}_k + \sum_{k>\Delta} \hat{k} \mathbf{a}_k^\dagger \mathbf{a}_k, \\ \mathbf{q}_k &= \frac{\mathbf{a}_k + \mathbf{a}_k^\dagger}{\sqrt{2}}, \quad \mathbf{p}_k = -i \frac{\partial}{\partial \mathbf{q}_k} = -i \frac{\mathbf{a}_k - \mathbf{a}_k^\dagger}{\sqrt{2}}. \end{aligned} \quad (5.5)$$

Following Bogolubov [124], we introduce the collective variable Q in which all field modes are added coherently and the “relative” field variables y_k which define quantum fluctuations relative to the collective mode

$$Q = \sum_{k<\Delta} \mathbf{q}_k, \quad y_k = \mathbf{q}_k - \frac{1}{N}Q, \quad \mathbf{q}_k = y_k + \frac{1}{N}Q, \quad \sum_{k<\Delta} y_k = 0, \quad N = \sum_{k<\Delta} 1, \quad (5.6)$$

where $N \gg 1$ is equal to the number of modes in the volume Δ . The transformation of the momentum operators is calculated according to its definition [124]:

$$\mathbf{p}_k = -i \frac{\partial}{\partial \mathbf{q}_k} = -i \left\{ \frac{\partial Q}{\partial \mathbf{q}_k} \frac{\partial}{\partial Q} + \sum_{\Delta f} \frac{\partial y_f}{\partial \mathbf{q}_k} \frac{\partial}{\partial y_f} \right\}. \quad (5.7)$$

Calculation of the derivatives with the help of (5.6) gives the generalized momenta:

$$\mathbf{p}_k = \mathbf{P} + \mathbf{p}_{yk}, \quad \sum_{\Delta k} \mathbf{p}_{yk} = 0, \quad \mathbf{P} = -i \frac{\partial}{\partial Q}, \quad \mathbf{p}_{yk} = -i \frac{\partial}{\partial y_k} + \frac{i}{N} \sum_{\Delta f} \frac{\partial}{\partial y_f}. \quad (5.8)$$

Insertion of (5.8) and (5.6) into the Hamiltonian (5.5) leads to the separation of the collective coordinates, the fluctuation operators y_k and the “external” variables \mathbf{a}_k and \mathbf{a}_k^\dagger , corresponding to k outside the Δ volume:

$$H_A = \frac{1}{2}\hat{k}_0 \left[\frac{1}{N}Q^2 + NP^2 \right] + \hat{b}_0 \sqrt{2}Q + \frac{1}{2}\hat{k}_0 \sum_{k<\Delta} (\mathbf{p}_{yk}^2 + y_k^2) + \sum_{k>\Delta} \hat{k} \mathbf{a}_k^\dagger \mathbf{a}_k. \quad (5.9)$$

We can now quantize the collective and “relative” variables by introducing the new set of creation and annihilation operators

$$\begin{aligned} Q &= \frac{\sqrt{N}}{\sqrt{2}}(A + A^\dagger), \quad P = -i \frac{1}{\sqrt{2N}}(A - A^\dagger), \quad [A, A^\dagger] = 1, \\ y_k &= \frac{1}{\sqrt{2}}(\tilde{c}_k + \tilde{c}_k^\dagger), \quad \mathbf{p}_{yk} = -i \frac{1}{\sqrt{2}}(\tilde{c}_k - \tilde{c}_k^\dagger), \\ \tilde{c}_k &= \mathbf{a}_k - \frac{1}{N} \sum_{f<\Delta} \mathbf{a}_f, \quad [\mathbf{a}_k, \mathbf{a}_{k_1}^\dagger] = \delta_{kk_1}, \quad [\tilde{c}_k, \tilde{c}_{k_1}^\dagger] = \delta_{kk_1} + \frac{1}{N}. \end{aligned} \quad (5.10)$$

With the accuracy of $\sim 1/N$ the Hamiltonian (5.9) transforms into

$$\begin{aligned} H_A &= \hat{k}_0 A^\dagger A + \hat{b}_0 \sqrt{N}(A + A^\dagger) + \hat{k}_0 \sum_{k < \Delta} \tilde{c}_k^\dagger \tilde{c}_k + \sum_{k > \Delta} \hat{k} a_k^\dagger a_k \equiv H_{sm} + H_f + H_e, \\ H_{sm} &= \hat{k}_0 A^\dagger A + \hat{b}_0 \sqrt{N}(A + A^\dagger), \quad H_f = \hat{k}_0 \sum_{k < \Delta} \tilde{c}_k^\dagger \tilde{c}_k, \quad H_e = \sum_{k > \Delta} \hat{k} a_k^\dagger a_k. \end{aligned} \quad (5.11)$$

where the operators are written in the normal form and the energy of “vacuum oscillations” is not taken into account. In this representation the operator H_{sm} which corresponds to the single-mode approximation is completely separated from the contributions defined by the fluctuation operator H_f and by the external modes operator H_e . Therefore the state vector of the system in the zeroth approximation is represented as the product:

$$|\Psi^{(0)}\rangle = |\Psi_A\rangle |\{n_f\}\rangle |\{n_e\}\rangle, \quad \tilde{c}_k^\dagger \tilde{c}_k |n_k^f\rangle = n_k^f |n_k^f\rangle, \quad a_k^\dagger a_k |n_k^e\rangle = n_k^e |n_k^e\rangle, \quad (5.12)$$

where $|n_k^f\rangle$ defines the state of the “fluctuations”, $|n_k^e\rangle$ is the state of the “external” modes of the electromagnetic field not interacting with an electron, and $|\Psi_A\rangle$ describes the state of the electron interacting with a collective mode of the field. $|\Psi_A\rangle$ satisfies the equation:

$$\begin{aligned} \{\hat{q}^{(0)} - m - H_A\} |\Psi_A\rangle &= \{(\hat{q}^{(0)} - \hat{a}) - m - [\hat{k}_0(A^\dagger A + f) + \hat{b}_0 \sqrt{N}(A + A^\dagger)]\} |\Psi_A\rangle = 0, \\ a_\mu &= \sum_{k > \Delta} k_\mu n_k^e, \quad f = \sum_{k < \Delta} n_k^f. \end{aligned} \quad (5.13)$$

The Hamiltonian H_A in equation (5.13) up to the constant four vectors a_μ and $f_\mu = k_{0\mu} f$ coincides with the Dirac equation with only one mode of the field and can be diagonalized ³:

$$\begin{aligned} |\Psi_{An}(a, f)\rangle &= C_1 \left(1 + \frac{\hat{k}_0 \hat{b}_0 \sqrt{N}}{2(z \cdot k_0)} (A + A^\dagger) \right) Su(p_n) |n\rangle, \\ S &= e^{\alpha(A^\dagger - A)} e^{-\frac{\eta}{2}(A^2 - A^{\dagger 2})}, \quad A^\dagger A |n\rangle = n |n\rangle, \\ p_n &= z - k_0 \left(\sqrt{1 - \frac{2Nb_0^2}{(z \cdot k_0)}} (n + 1/2) - 1/2 + f - \frac{(z \cdot b_0)^2 N}{(z \cdot k_0)^2} \frac{1}{1 - 2Nb_0^2/(z \cdot k_0)} \right), \\ \alpha &= -\frac{(z \cdot b_0) \sqrt{N}}{(z \cdot k_0)} \frac{1}{1 - 2Nb_0^2/(z \cdot k_0)}, \quad \cosh \eta = \frac{1}{2} \left(\sqrt{\varkappa} + \frac{1}{\sqrt{\varkappa}} \right), \quad \varkappa = \frac{1}{\sqrt{1 - \frac{2Nb_0^2}{(z \cdot k_0)}}}, \end{aligned} \quad (5.14)$$

where $z_\mu = q_\mu^{(0)} - a_\mu$ and $u(p_n)$ is a bispinor which coincides with a free Dirac bispinor.

6 Corrections to the approximating Hamiltonian

In 5 we built the eigenvectors (5.12) of the approximating Hamiltonian H_A and assumed that they define the main contribution to the solution of the initial equation (2.4). However, we have not determined the optimal parameters $(\Delta, \hat{k}_0, \hat{b}_0)$ of the approximating Hamiltonian and not found the conditions for the wave packet when the single-mode approximation can be applied. In order to do so, in accordance with the method defined in ref. [124] one should now consider

the corrections given by the operators $H_{1,2}$ from (5.4). The first-order correction to the system's energy determines the optimal parameters $(\Delta, \hat{k}_0, \hat{b}_0)$ of the approximating Hamiltonian (5.3). The second-order correction defines the width and lifetime of the collective mode and gives the conditions on the wave packet intensity and duration, thus determining the single-mode approximation.

We build the corresponding perturbation theory for the initial equation (2.4) by inserting a formal parameter λ in (2.4)

$$\{\hat{q} - m - H_A\} |\Psi\rangle = \lambda(H_1 + H_2)|\Psi\rangle, \quad (6.1)$$

and representing a solution in a form of a series [124]:

$$|\Psi\rangle = |\Psi^{(0)}\rangle + \lambda|\Psi^{(1)}\rangle + \dots, \quad q = q^{(0)} + \lambda q^{(1)} + \dots \quad (6.2)$$

From equation (6.2) the first two orders of the perturbation can be found:

$$\begin{aligned} (\hat{q}^{(0)} - m - H_A)|\Psi^{(0)}\rangle &= 0, \\ \hat{q}^{(1)}|\Psi^{(0)}\rangle + (\hat{q}^{(0)} - m - H_A)|\Psi^{(1)}\rangle &= (H_1 + H_2)|\Psi^{(0)}\rangle. \end{aligned} \quad (6.3)$$

The first equation coincides with equation (5.13) and has the set of eigenvectors (5.14) which form a full and orthogonal basis in a Hilbert space.

The closest field state to the experiment is a coherent state. Therefore, we suppose that the electromagnetic field is described by the wave packet:

$$|\Xi\rangle = C \exp \left\{ \sum_k [u_k \mathbf{a}_k^\dagger - u_k^* \mathbf{a}_k] \right\} |0\rangle, \quad \mathbf{a}_k |0\rangle = 0, \quad (6.4)$$

which is a set of coherent states with the amplitudes u_k . This wave packet is localized in \mathbf{k} -space near the momentum \mathbf{k}_0 and can be modeled with the Gaussian distribution:

$$u_k = e^{-\frac{\mathbf{k}_\perp^2}{2\sigma_1^2\omega_0^2}} e^{-\frac{(\omega-\omega_0)^2}{2\sigma_2^2\omega_0^2}}, \quad \mathbf{k} = \mathbf{k}_\perp + \omega \frac{\mathbf{k}_0}{\omega_0}, \quad \mathbf{k}_\perp \cdot \mathbf{k}_0 = 0, \quad (6.5)$$

where σ_2 and σ_1 determine a frequency and an angular spread in the laser pulse respectively. The constant C in (6.1) for a pulse of an intensity I , a transversal width S and a duration τ can be obtained from the normalization on the full pulse energy W :

$$\begin{aligned} W &= IS\tau = \langle \Xi | \sum_k \omega_k \mathbf{a}_k^\dagger \mathbf{a}_k | \Xi \rangle = C^2 \frac{V}{8\pi^3} \int d\omega d\mathbf{k}_\perp \omega |u_k|^2 = C^2 \frac{V}{8\pi^3} \omega_0^4 \pi^{3/2} \sigma_1^2 \sigma_2, \\ C &= \sqrt{\frac{8\pi^{3/2} IS\tau}{V\sigma_1^2 \sigma_2 \omega_0^4}}. \end{aligned} \quad (6.6)$$

The wave packet (6.4) can be expanded in a series over the full set of states (5.14):

$$|\Xi\rangle = \sum_{n,a,f} C_n(a, f) |\Psi_A(n, a, f)\rangle, \quad (6.7)$$

with coefficients $C_n(a, f)$, which depend not only on the collective mode quantum number n , but also on the “fluctuating” and the “external” modes quantum numbers f and a respectively. This linear combination can be used for the description of QED processes (non-linear Compton scattering, electron-positron pair creation) in Furry picture, taking into account the realistic duration and angular spread of the laser pulse. In the following we show that the dependencies on f and a can be neglected if the \mathbf{k} -space volume Δ is chosen in a consistent way.

We can estimate the contribution of the various terms in the Hamiltonian H_A using the state (6.4):

$$\begin{aligned} \langle \Xi | \omega_0 \mathbf{A}^\dagger \mathbf{A} | \Xi \rangle &\approx \langle \Xi | \sum_{\mathbf{k} < \Delta} \omega_0 \mathbf{a}_k^\dagger \mathbf{a}_k | \Xi \rangle = \frac{\omega_0 C^2 V}{(2\pi)^3} \int_{\mathbf{k} < \Delta} d\mathbf{k} |u_k|^2 = \\ &= \frac{8\pi^{3/2} I S \tau^2}{(2\pi)^3} \left(\int_0^{\frac{\Delta_1}{\sigma_1 \omega_0}} dt e^{-t^2} \right)^2 \int_0^{\frac{\Delta_2}{\sigma_2 \omega_0}} du e^{-u^2} = I S \tau \Phi^3(\delta), \end{aligned} \quad (6.8)$$

where we assumed that the volume Δ in \mathbf{k} -space can be written as $\Delta = \Delta_1^2 \Delta_2 = \delta^3 \sigma_1^2 \sigma_2 \omega_0^3$. Here δ is dimensionless parameter that will be defined below and $\Phi(z) = 2/\sqrt{\pi} \int_0^z e^{-t^2} dt$ is the error function. Other terms in H_A are calculated in a similar way:

$$\begin{aligned} N &= \frac{V}{8\pi^3} \delta^3 \sigma_1^2 \sigma_2 \omega_0^3, \\ \langle \Xi | f | \Xi \rangle &= \langle \Xi | \sum_{\mathbf{k} < \Delta} \left[\mathbf{a}_k^\dagger - \frac{1}{N} \sum_{\mathbf{l} < \Delta} \mathbf{a}_l^\dagger \right] \left[\mathbf{a}_k - \frac{1}{N} \sum_{\mathbf{l} < \Delta} \mathbf{a}_l \right] | \Xi \rangle \\ &= C^2 \sum_{\mathbf{k} < \Delta} \left[u_k^* - \frac{1}{N} \sum_{\mathbf{l} < \Delta} u_l^* \right] \left[u_k - \frac{1}{N} \sum_{\mathbf{l} < \Delta} u_l \right] \\ &= C^2 \frac{V}{(2\pi)^3} \left(\int d\mathbf{k} |u_k|^2 - \frac{V}{(2\pi)^3 N} \left| \int d\mathbf{k} u_k \right|^2 \right) = I S \tau \Phi^3(\delta) \left(1 - \frac{2^3 \pi^{\frac{3}{2}}}{\delta^3} \frac{\Phi^6(\frac{\delta}{\sqrt{2}})}{\Phi^3(\delta)} \right), \\ \langle \Xi | a | \Xi \rangle &\leq \langle \Xi | a_0 | \Xi \rangle = \langle \Xi | \sum_{\mathbf{k} > \Delta} \omega_0 \mathbf{a}_k^\dagger \mathbf{a}_k | \Xi \rangle = I S \tau (1 - \Phi^3(\delta)), \end{aligned} \quad (6.9)$$

The contribution due to the “fluctuating” modes in (6.9) is defined by the value $\langle \Xi | f | \Xi \rangle$ and is equal to zero if the parameter δ is chosen as the solution of the equation

$$1 - \frac{2^3 \pi^{\frac{3}{2}}}{\delta^3} \frac{\Phi^6(\frac{\delta}{\sqrt{2}})}{\Phi^3(\delta)} = 0, \quad \delta \approx 3.54. \quad (6.10)$$

It is evident that the actual value of this parameter depends on the laser pulse form but in any case it can be calculated in a similar way.

The contributions of the “external” pulse modes to H_A can be neglected because they are defined by the value

$$\frac{\langle \Xi | a_0 | \Xi \rangle}{\langle \Xi | \omega_0 \mathbf{A}^\dagger \mathbf{A} | \Xi \rangle} = \frac{(1 - \Phi^3(\delta))}{\Phi^3(\delta)} \approx 1.64 \cdot 10^{-6}, \quad (6.11)$$

when δ is found from (6.10).

A similar estimation $(1 - \Phi^3(\delta)) \approx 10^{-6}$, defines the difference between the energy accumulated in the collective mode (6.8) and the total energy of the laser pulse (6.6). It means that if the parameter Δ is chosen as

$$\Delta \approx (3.54)^3 \sigma_1^2 \sigma_2 \omega_0^3, \quad (6.12)$$

the values a and f can be omitted in the operator H_A that corresponds to the vacuum of the “fluctuating” and “external” modes. In this case H_A can be considered as the single-mode Hamiltonian in the zeroth order (6.3).

Now we determine the parameters $\hat{k}_0, \hat{b}_0, \omega_0$ of this Hamiltonian. Let us consider the first order equation in (6.3). Using its projection on the state vector $\langle \Psi^{(0)} |$ (we pay attention to the fact, that for the correct perturbation theory for the Dirac equation the eigenvalue $\langle \Psi^{(0)} |$ of a zero-order is not a hermitian conjugate to $|\Psi^{(0)}\rangle$ but is the Dirac conjugate i.e. $\langle \Psi^{(0)} | = (|\Psi^{(0)}\rangle)^\dagger \gamma^0$):

$$\langle \Psi^{(0)} | \hat{q}^1 | \Psi^{(0)} \rangle = \langle \Psi^{(0)} | (H_1 + H_2) | \Psi^{(0)} \rangle. \quad (6.13)$$

According to Bogolubov [124] the stable solution of the initial equation (2.4) exists if the first-order correction to the eigenvalue is equal to zero and this condition allows one to find the unknown parameters. In our case it leads to two equations

$$\langle \Psi^{(0)} | H_1 | \Psi^{(0)} \rangle = 0; \quad \langle \Psi^{(0)} | H_2 | \Psi^{(0)} \rangle = 0, \quad (6.14)$$

because the operators H_1, H_2 refer to different variables. The details of the calculations of the expectation values of the Hamiltonian H_1 can be found in Appendix A. The result reads

$$\begin{aligned} \langle H_1 \rangle = & n_0 \left(\kappa + \frac{1}{\kappa} \right) \left(\sum_{k < \Delta} \frac{k}{N} - k_0 \right) \cdot p_n \\ & + 2 \frac{2\alpha\kappa n_0 + \alpha \left(\kappa + \frac{1}{\kappa} \right) n_0}{\sqrt{N}(z \cdot k_0)} \left((b_0 \cdot p_n) \sum_{k < \Delta} (k \cdot k_0) - (k_0 \cdot p_n) \sum_{k < \Delta} (k \cdot b_0) \right) \\ & - \frac{n_0^2 (\kappa + 1/\kappa) \kappa + n_0^2 (\kappa - 1/\kappa) \frac{\kappa}{2}}{(z \cdot k_0)^2} \sum_{k < \Delta} b_0^2 (k \cdot k_0) (k_0 \cdot p_n) \\ & + \frac{4\alpha}{\sqrt{N}} \left(\sum_{k < \Delta} b(\mathbf{k}) - N b_0 \right) \cdot p_n + \frac{4\kappa n_0}{(z \cdot k_0)} \left\{ \left(N b_0^2 - \sum_{k < \Delta} (b_0 \cdot b(\mathbf{k})) \right) (k_0 \cdot p_n) \right. \\ & \left. + (b_0 \cdot p_n) \sum_{k < \Delta} (b(\mathbf{k}) \cdot k_0) \right\} - \frac{12\alpha\kappa n_0 \sqrt{N}}{(z \cdot k_0)^2} b_0^2 \sum_{k < \Delta} (b(\mathbf{k}) \cdot k_0) (k_0 \cdot p_n), \end{aligned} \quad (6.15)$$

The calculation of the average value of the Hamiltonian H_2 is performed in exactly the same way

$$\begin{aligned} \langle H_2 \rangle &= \langle 0^f | \langle n_k^e | \langle \Psi_A | H_2 | \Psi_A \rangle | n_k^e \rangle | 0^f \rangle = \sum_{k > \Delta} \langle \Psi_A | \hat{b}(\mathbf{k}) | \Psi_A \rangle \langle \Xi | (a_k + a_k^\dagger) | \Xi \rangle \\ &= \sum_{k > \Delta} \left(2b(\mathbf{k}) \cdot p_n + \frac{4\alpha \sqrt{N}}{(z \cdot k_0)} (b(\mathbf{k}) \cdot k_0 b_0 \cdot p_n \right. \\ &\quad \left. - b_0 \cdot b(\mathbf{k}) k_0 \cdot p_n) - \frac{2\kappa n_0 N}{(z \cdot k_0)^2} b_0^2 b(\mathbf{k}) \cdot k_0 k_0 \cdot p_n \right) \langle \Xi | (a_k + a_k^\dagger) | \Xi \rangle. \end{aligned} \quad (6.16)$$

As was stated above, according to ref. [124] the first corrections to the approximating Hamiltonian H_A are equal to zero. This gives a condition for determination of the variational parameters Δ , b_0 , k_0 of the Hamiltonian H_A . Therefore, if we choose

$$\omega_0 = \frac{1}{N} \sum_{k < \Delta} \omega_k, \quad k_0 = \frac{1}{N} \sum_{k < \Delta} k, \quad b_0 = \frac{1}{N} \sum_{k < \Delta} b(\mathbf{k}), \quad (6.17)$$

the average of the Hamiltonian $\langle H_1 \rangle$ turns into zero. The average of the Hamiltonian H_2 vanishes according to symmetry consideration as it is not bilinear over polarization vectors $\sum_{k > \Delta} b(\mathbf{k})$. The physical meaning of this choice is that the collective single-mode corresponds to an average over the modes of a quasi-monochromatic wave packet.

We have determined the variational parameters of the approximating Hamiltonian and consequently can proceed with the estimation of the field intensity for which the single-mode approximation is valid. For this purpose, the second-order correction

$$E_0^{(2)} = - \sum_{E_0 \neq E_{0\alpha}} \frac{|\langle E_0 | H_1 | E_{0\alpha} \rangle|^2}{E_{0\alpha} - E_0},$$

to the system's energy needs to be calculated. Here H_1 is the perturbation operator

$$\begin{aligned} H_1 = & \sum_{k < \Delta} \frac{1}{2} (\hat{k} - \hat{k}_0) \left(\mathbf{P}^2 + \frac{\mathbf{Q}^2}{N^2} \right) + \sum_{k < \Delta} \frac{1}{2} (\hat{k} - \hat{k}_0) (\mathbf{p}_{yk}^2 + \mathbf{y}_k^2) \\ & + \sum_{k < \Delta} (\hat{k} - \hat{k}_0) \left(\mathbf{P} \mathbf{p}_{yk} + \mathbf{Q} \frac{\mathbf{y}_k}{N} \right) + \sum_{k < \Delta} \sqrt{2} (\hat{b}(\mathbf{k}) - \hat{b}_0) \left(\mathbf{y}_k + \frac{\mathbf{Q}}{N} \right). \end{aligned} \quad (6.18)$$

As it can be seen from equation (6.18) only the last two terms contribute to $E_0^{(2)}$. The state $|E_{0\alpha}\rangle$ is the wave function

$$|0_f n_0\rangle = \frac{u(p)}{\sqrt{2\epsilon}} S |n_0\rangle |0_f\rangle, \quad (6.19)$$

where ϵ is the electron's energy, n_0 is the number of quanta in the “collective” mode, $|0_f\rangle$ is the state of the field fluctuations and for simplicity we neglected the term proportional to $\hat{k}\hat{b}$. Let us rewrite the Hamiltonian H_1 in the variables A , A^\dagger of the “collective” mode and \tilde{c}_k , \tilde{c}_k^\dagger of the fluctuations respectively

$$\begin{aligned} H_1 = & (\hat{k} - \hat{k}_0) \left(-\frac{1}{2\sqrt{N}} (A - A^\dagger) (\tilde{c}_k - \tilde{c}_k^\dagger) + (A + A^\dagger) \frac{(\tilde{c}_k + \tilde{c}_k^\dagger)}{2\sqrt{N}} \right) \\ & + \sqrt{2} (\hat{b}(\mathbf{k}) - \hat{b}_0) \left(\frac{\tilde{c}_k + \tilde{c}_k^\dagger}{\sqrt{2}} + \frac{A + A^\dagger}{\sqrt{2N}} \right). \end{aligned} \quad (6.20)$$

Now we can calculate the transition matrix element $\langle 0_f n_0 | H_1 | n_f n \rangle$:

$$\begin{aligned} \langle 0_f n_0 | H_1 | n_f n \rangle = & \frac{\bar{u}(p)(\hat{k} - \hat{k}_0)u(p)}{2\epsilon\sqrt{N}} \langle n_0 | S^\dagger A^\dagger S | n \rangle \delta_{1_f, n_f} \\ & + \frac{\bar{u}(p)(\hat{b}(\mathbf{k}) - \hat{b}_0)u(p)}{2\epsilon} \left[\delta_{1_f, n_f} \delta_{n_0, n} + \frac{1}{\sqrt{N}} \langle n_0 | S^\dagger (A + A^\dagger) S | n \rangle \delta_{0_f, n_f} \right]. \end{aligned} \quad (6.21)$$

The use of the transformation (A.4) of the creation and annihilation operators of the “collective” mode by the operator S and the calculation of the averages in a spin space yields

$$\begin{aligned} \langle 0_f n_0 | H_1 | n_f n \rangle &= \frac{((p \cdot k) - (p \cdot k_0)) \delta_{1_f, n_f}}{\epsilon \sqrt{N}} \left[\frac{1}{2} \left(\sqrt{\kappa} + \frac{1}{\sqrt{\kappa}} \right) \sqrt{n_0} \delta_{n_0-1, n} \right. \\ &+ \left. \frac{1}{2} \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) \sqrt{n_0 + 1} \delta_{n_0+1, n} \right] + \frac{(b(\mathbf{k}) \cdot p) - (b_0 \cdot p)}{\epsilon} \left[\delta_{1_f, n_f} \delta_{n_0, n} \right. \\ &+ \left. \frac{\sqrt{\kappa} (\sqrt{n_0 + 1} \delta_{n_0+1, n} + \sqrt{n_0} \delta_{n_0-1, n})}{\sqrt{N}} \delta_{0_f, n_f} \right]. \end{aligned} \quad (6.22)$$

Consequently we can write down the second-order correction to the system’s energy

$$\begin{aligned} E_0^{(2)} &\approx \sum_{k < \Delta} \frac{|(v \cdot k) - (v \cdot k_0)|^2}{4N} \left(\frac{\kappa_+^2 n_0}{\omega_0 - \omega_k} - \frac{\kappa_-^2 (n_0 + 1)}{\omega_k + \omega_0} \right) \\ &+ \sum_{k < \Delta} |(v \cdot b(\mathbf{k})) - (v \cdot b_0)|^2 \left(\frac{1}{\omega_k} - \frac{\kappa}{\omega_0 N} \right), \end{aligned} \quad (6.23)$$

where $\kappa_+ = \sqrt{\kappa} + 1/\sqrt{\kappa}$, $\kappa_- = \sqrt{\kappa} - 1/\sqrt{\kappa}$ and $v_\mu = p_\mu/\epsilon = (1, \mathbf{v})$ and \mathbf{v} is the velocity of an electron.

Equation (6.23) has four terms but only two are important. The term inversely proportional to the frequency difference $\omega_0 - \omega_k$ describes a resonance and defines the frequency shift and the lifetime of the collective mode. The term inversely proportional to ω_k defines the fluctuations arising due to the interaction between an electron and an external field. The remaining two terms can be neglected as the second one is not a resonance and the fourth one is inversely proportional to the normalization volume V ($N \sim V$ and $b \sim 1/\sqrt{V}$).

In order to perform a summation in \mathbf{k} -space we firstly fix a coordinate system. Let the z -axis be directed along \mathbf{k}_0 , the x -axis along \mathbf{v}_\perp - the velocity component perpendicular to the \mathbf{k}_0 . The details of the summations over \mathbf{k} can be found in appendix B. The second-order correction to the energy of the system reads

$$\begin{aligned} E_0^{(2)} &= \frac{n_0 \omega_0 \pi v_\perp^2}{4} \left(\lambda \tan^{-1} 4\lambda - \frac{1}{4} + \frac{i\pi}{2} \cdot \begin{cases} \lambda, & \lambda > 1 \\ \frac{1}{\lambda}, & \lambda < 1 \end{cases} \right) \\ &+ e^2 \frac{\delta^5 \sigma_1^2 \sigma_2 \omega_0 (\sigma_2^2 (v^2 - v_z^2)^2 + 16 \sigma_1^2 v_\perp^2 v_z^2)}{96(2\pi)^3 v^2}, \end{aligned} \quad (6.24)$$

where $\lambda = \sigma_2/(\delta \sigma_1^2)$.

7 Conditions of the applicability of the single-mode approximation

The first term in (6.24) is proportional to the same quantum number n_0 as the energy of the “collective” mode. Its real part defines the shift $\Delta\omega_0$ and the imaginary part defines the width

$\Gamma/2$ of the collective mode. Therefore, one can rely on the single-mode approximation if these values are small in comparison with ω_0 :

$$\begin{aligned} \frac{\Delta\omega_0}{\omega_0} &= \frac{\pi v_\perp^2}{4} \left[\lambda \tan^{-1} 4\lambda - \frac{1}{4} \right] \ll 1, \\ \frac{\Gamma}{\omega_0} &= \frac{\pi^2 v_\perp^2}{4} \cdot \begin{cases} \lambda, & \lambda > 1 \\ \frac{1}{\lambda}, & \lambda < 1 \end{cases} \ll 1. \end{aligned} \quad (7.1)$$

As was stated above, the single-mode approximation is valid when the change in $E_0^{(2)}$ due to the fluctuations of the quantum field are small with comparison to the ground state energy of the “collective” mode (6.8). These fluctuations are defined by the last term in equation (6.24). This leads to the additional parameter

$$\mu = e^2 \frac{\delta^5 \sigma_1^2 \sigma_2 \omega_0 v_\perp^2 (\sigma_2^2 v_\perp^2 + 16 \sigma_1^2 v_z^2)}{96(2\pi)^3 v^2 I S \tau \Phi^3(\delta)} \ll 1, \quad (7.2)$$

which defines the lowest pulse intensity for which the single-mode approximation can be used.

Modern lasers can reach nowadays high intensities [5–7, 9, 10] up to 10^{22} W/cm² with a pulse duration of about 30 fs. Let us estimate the parameters (7.1) and (7.2) for an intensity $I = 10^{22}$ W/cm², photon frequency $\omega = 7.8 \cdot 10^4$ cm⁻¹ (a corresponding wavelength of 800 nm), pulse duration $\tau = 8.7 \cdot 10^{-4}$ cm⁻¹ (corresponding to 30 fs) and focusing $S = 10^{-8}$ cm².

The physical parameters σ_1 and σ_2 are connected with the characteristics of the laser pulse by equation (5.2), and their numerical value for the above I, S, τ is equal to

$$\sigma_1 = 0.127, \quad \sigma_2 = 0.014. \quad (7.3)$$

An electron beam always has angular divergence $\Delta\theta$ and $v_\perp \sim \Delta\theta \sim 1/\gamma$, where γ is the electron’s gamma factor. Therefore, for the moderately relativistic electrons we can consider that $v_\perp \leq \sigma_1$.

By plugging the numerical values in equations (7.1) and (7.2) one obtains

$$\mu \sim 10^{-28}, \quad \frac{\Delta\omega}{\omega_0} \sim 6 \cdot 10^{-4}, \quad (7.4)$$

$$\frac{\Gamma}{\omega_0} \sim 0.01, \quad (7.5)$$

and we can conclude that the single-mode approximation is applicable.

As can be seen from equation (7.5) the parameter μ and frequency shift are very small values for the intensities in the strong field QED range, i.e., $10^{16} - 10^{22}$ W/cm² and pulse duration of 30 fs. This means that the electron mainly interacts with the collective single-mode. The influence of the fluctuations is suppressed.

However, the most important parameter, which can limit the applicability of the single-mode approximation is indeed independent on the intensity, but depends on the pulse duration. It determines the width of the collective mode Γ/ω_0 , equation (7.5). The decrease of the pulse duration from 30 fs to 3 fs, will increase its value by one order. The physical meaning of this

result, corresponds to the situation that for the really short laser pulses, the collective field mode does not have sufficient time for its formation.

Concluding we can state, that the applicability of the single-mode approximation is mainly limited not by the pulse intensity, but rather by its duration and focusing size. Therefore, for a particular spectral distribution of the external laser pulse one should estimate σ_1 and σ_2 , then insert their values together with the transversal electron velocity v_\perp into equations (7.1-7.2) and make the conclusion about the applicability of the single-mode approximation.

Chapter II

Collapse-and-revival dynamics of an electron in a quantized electromagnetic field

8 Evolution of an electron state vector in a quantized electromagnetic field

When the experimental realization of the external electromagnetic field is based on the application of laser pulses of high intensity and certain duration, such that the single-mode approximation can be applied, it is commonly accepted that this external field can be considered as classical. This leads to the use of Volkov solution for external electron lines in the Furry picture of QED. However, the electromagnetic field is a system having an infinite number degrees of freedom and there is no conservation law for the number of particles. Therefore, its possible states include states with an arbitrary number of particles. For this reason, one might consider that the electron behavior in a quantized field will be different from the behavior in a classical one. In order to investigate this feature we will apply the exact solutions obtained in Chapter I for the calculation of electron's observables. Parts of the present chapter have been published in Refs. [125].

Let us first find the evolution of the state vector of the electron and the quantized field. We will start with the fix of the time reference. It is natural to connect it with the moment $t = 0$ when the electron passes the boundary of the laser pulse. This means that the system state vector at $t = 0$ is described by a free electron wave function and the field by a coherent state [95]

$$\psi_0 = e^{ip_0 \cdot (r-r_0)} \frac{u(p_0)}{\sqrt{2\varepsilon_0}} |\beta\rangle, \quad |\beta\rangle = \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle e^{-\beta^2/2}, \quad (8.1)$$

where $|\beta\rangle$ is a coherent state of the field, p_0 the electron momentum, $u(p_0)$ a constant bispinor normalized with the condition

$$\bar{u}(p_0) \gamma^0 u(p_0) = 2\varepsilon_0,$$

and the vector r_0 defines the initial phase of the electron state.

Let us now use a linear combination of the solutions (3.8)

$$\Psi(x) = \int d\mathbf{q} \sum_n C_{q,n} \psi_{qn}(x); \quad x = (t, \mathbf{x}) \quad (8.2)$$

in order to satisfy the initial condition

$$e^{ip_0(r-r_0)} \frac{u(p_0)}{\sqrt{2\varepsilon_0}} |\beta\rangle = \Psi|_{t=0}. \quad (8.3)$$

The wave function $\Psi(x)$ indeed defines the evolution of the system's state vector with correct initial conditions. Therefore, we need to calculate the unknown coefficients $C_{q,n}$. As the wave functions ψ_{qn} are orthogonal, we find that

$$C_{q,n} = \frac{1}{(2\pi)^3} \int d\mathbf{r} \psi_{qn}^\dagger e^{ip_0 \cdot \mathbf{r}} \frac{u(p_0)}{\sqrt{2\varepsilon_0}} |\beta\rangle. \quad (8.4)$$

Inserting the wave function ψ_{qn}^\dagger into equation (8.4), one obtains

$$\begin{aligned} C_{qn} = & \frac{1}{(2\pi)^3} \int d\mathbf{r} e^{-i(\mathbf{q}-\mathbf{p}_0) \cdot \mathbf{r}} \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \left(\gamma^0 \left(1 + \frac{2\alpha}{2(\mathbf{q} \cdot \mathbf{k})} \right) \langle n | \mathbf{S}_q^\dagger e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{a}^\dagger \mathbf{a}} e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} | 0 \rangle \right. \\ & \left. + \frac{\hat{b} \hat{k} \gamma^0 \sqrt{\kappa}}{2(\mathbf{q} \cdot \mathbf{k})} \left(\sqrt{n+1} \langle n+1 | \mathbf{S}_q^\dagger e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{a}^\dagger \mathbf{a}} e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} | 0 \rangle + \sqrt{n} \langle n-1 | \mathbf{S}_q^\dagger e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{a}^\dagger \mathbf{a}} e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} | 0 \rangle \right) \right) \frac{u(p_0)}{\sqrt{2\varepsilon_0}}, \end{aligned} \quad (8.5)$$

where

$$S_q^\dagger = e^{\frac{\eta_q}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})} e^{-\alpha_q(\mathbf{a}^\dagger - \mathbf{a})}, \quad \kappa = \frac{1}{\sqrt{1 - 2b^2/(\mathbf{q} \cdot \mathbf{k})}}, \quad \alpha_q = -\frac{(\mathbf{q} \cdot \mathbf{b})}{(\mathbf{q} \cdot \mathbf{k})} \frac{1}{1 - 2b^2/(\mathbf{q} \cdot \mathbf{k})}, \quad (8.6)$$

and coherent state $|\beta\rangle = e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} |0\rangle$. The index q of the quantities α_q and η_q indicates their dependence on q .

The problem of calculating the coefficients $C_{q,n}$ reduces to that of calculating a matrix element of the type

$$\langle n | \mathbf{S}_q^\dagger e^{i\mathbf{k} \cdot \mathbf{r} \mathbf{a}^\dagger \mathbf{a}} e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} | 0 \rangle.$$

For this, we need the representation of the exponential of a sum of operators in normal form. The normal form of the operator of the coherent state is [95]

$$e^{\beta \mathbf{a}^\dagger - \beta^* \mathbf{a}} = e^{-|\beta|^2/2} e^{\beta \mathbf{a}^\dagger} e^{-\beta^* \mathbf{a}} = e^{|\beta|^2/2} e^{-\beta^* \mathbf{a}} e^{\beta \mathbf{a}^\dagger}. \quad (8.7)$$

The decomposition of the exponent with the second power of the creation and annihilation operators is [11]

$$e^{\frac{\eta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})} = e^{-\frac{1}{2} \text{th} \eta \mathbf{a}^{\dagger 2}} e^{-\ln \text{ch} \eta (\mathbf{a}^\dagger \mathbf{a} + \frac{1}{2})} e^{\frac{1}{2} \text{th} \eta \mathbf{a}^2}. \quad (8.8)$$

This decomposition is possible since the three operators \mathbf{a}^2 , $\mathbf{a}^{\dagger 2}$ and $\mathbf{a}^\dagger \mathbf{a}$ form a closed algebra. Taking into account the transformation of the creation and annihilation operators with an operator \mathbf{S}^\dagger ,

$$\mathbf{S}^\dagger (\mathbf{a} + \mathbf{a}^\dagger) \mathbf{S} = \sqrt{\kappa} (\mathbf{a} + \mathbf{a}^\dagger) + 2\alpha, \quad (8.9)$$

the action of the creation operator \mathbf{a}^\dagger on the left bra vector $\langle 0|$ yields zero and the harmonic oscillator state vector is connected with the vacuum by $\langle n| = \langle 0|\mathbf{a}^n / \sqrt{n!}$, and we obtain

$$\langle n|e^{\frac{\eta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})}e^{-\alpha(\mathbf{a}^\dagger - \mathbf{a})}e^{i\mathbf{k} \cdot \mathbf{r}\mathbf{a}^\dagger}e^{\beta\mathbf{a}^\dagger - \beta^*\mathbf{a}}|0\rangle = \frac{e^{-|\theta|^2/2 + \alpha\theta - \alpha^2/2 + 1/2(\theta - \alpha)^2\text{th}\eta}}{\sqrt{\text{ch}\eta}}\langle 0|\frac{((\mathbf{a} - \text{th}\eta\mathbf{a}^\dagger) + \frac{\theta - \alpha}{\text{ch}\eta})^n}{\sqrt{n!}}|0\rangle \quad (8.10)$$

In order to calculate the vacuum average in (8.10), we replace the power n by the n th derivative of the exponent

$$\langle 0|\frac{((\mathbf{a} - \text{th}\eta\mathbf{a}^\dagger) + \frac{\theta - \alpha}{\text{ch}\eta})^n}{\sqrt{n!}}|0\rangle = \frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}\langle 0|e^{x(\mathbf{a} - \text{th}\eta\mathbf{a}^\dagger) + x\frac{\theta - \alpha}{\text{ch}\eta}}|0\rangle\Big|_{x=0} = \quad (8.11)$$

$$\frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}e^{x\frac{\theta - \alpha}{\text{ch}\eta}}\langle 0|e^{x(\mathbf{a} - \text{th}\eta\mathbf{a}^\dagger)}|0\rangle\Big|_{x=0} = \frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}e^{x\frac{\theta - \alpha}{\text{ch}\eta} - x^2\text{th}\eta/2}\Big|_{x=0}. \quad (8.12)$$

Selecting the full square of the variable x and changing variables, we obtain

$$\frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}e^{x\frac{\theta - \alpha}{\text{ch}\eta} - x^2\text{th}\eta/2}\Big|_{x=0} = \frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}e^{-[(\sqrt{\frac{\text{th}\eta}{2}}x - \frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}})^2 - \frac{(\theta - \alpha)^2}{\text{sh}2\eta}]} \Big|_{x=0},$$

$$y = \sqrt{\frac{\text{th}\eta}{2}}x - \frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}}, \quad x = 0, \quad y = -\frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}}, \quad \frac{d^n}{dx^n} = \left(\frac{\text{th}\eta}{2}\right)^{\frac{n}{2}} \frac{d^n}{dy^n}, \quad (8.13)$$

$$\frac{1}{\sqrt{n!}}\frac{d^n}{dx^n}e^{-[(\sqrt{\frac{\text{th}\eta}{2}}x - \frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}})^2 - \frac{(\theta - \alpha)^2}{\text{sh}2\eta}]} \Big|_{x=0} = \frac{e^{\frac{(\theta - \alpha)^2}{\text{sh}2\eta}}}{\sqrt{n!}}\left(\frac{\text{th}\eta}{2}\right)^{\frac{n}{2}} \frac{d^n}{dy^n}e^{-y^2}\Big|_{y=-\frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}}}. \quad (8.14)$$

Using the definition of Hermitian polynomials,

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2},$$

we finally get

$$\langle n|\mathbf{S}^\dagger e^{-i\mathbf{k} \cdot \mathbf{r}\mathbf{a}^\dagger}e^{\beta\mathbf{a}^\dagger - \beta^*\mathbf{a}}|0\rangle = \frac{e^{-|\theta|^2/2 + \alpha\theta - \alpha^2/2 + 1/2(\theta - \alpha)^2\text{th}\eta}}{\sqrt{\text{ch}\eta}} \frac{1}{\sqrt{n!}}\left(\frac{\text{th}\eta}{2}\right)^{\frac{n}{2}} H_n\left(\frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}}\right). \quad (8.15)$$

$$C_{q,n} = \frac{\bar{u}(p_n)\gamma^0 u(p_0)}{\sqrt{2\varepsilon_n}\sqrt{2\varepsilon_0}}\left(1 + \frac{2\alpha}{(q \cdot k)}\right)M_{q,n} + \frac{\bar{u}(p_n)\hat{b}\hat{k}\gamma^0 u(p_0)}{\sqrt{2\varepsilon_n}\sqrt{2\varepsilon_0}} \frac{\sqrt{\kappa}(\sqrt{n+1}M_{q,n+1} + \sqrt{n}M_{q,n-1})}{2(q \cdot k)} \quad (8.16)$$

where the matrix element $M_{q,n}$ is calculated via

$$M_{q,n} = \frac{1}{(2\pi)^3} \int d\mathbf{r} e^{-i(q - p_0) \cdot (\mathbf{r} - \mathbf{r}_0)} \frac{e^{-|\theta|^2/2 + \alpha\theta - \alpha^2/2 + 1/2(\theta - \alpha)^2\text{th}\eta}}{\sqrt{\text{ch}\eta}} \frac{1}{\sqrt{n!}}\left(\frac{\text{th}\eta}{2}\right)^{\frac{n}{2}} H_n\left(\frac{\theta - \alpha}{\sqrt{2\text{th}\eta\text{ch}\eta}}\right), \quad (8.17)$$

$\theta = \beta e^{i\mathbf{k} \cdot \mathbf{r}}$, and β is a coherent state parameter.

For that reason, the wave function (8.2) with the coefficients (8.16) describes exactly the evolution of the system consisting of a relativistic electron in a linearly polarized single-mode quantized field.

9 Strong field limit

In the previous paragraph 8 we have obtained an electron's wave function evolution in a field of an arbitrary strength being a coherent state. The results of a previous paragraph simplifies essentially if the external field is strong, i.e. the number of photons $n = \beta^2$ is a large value.

Let us estimate the characteristic parameters of the problem. Consider a strong laser field, with density of photons [14]

$$\rho = \frac{n}{V},$$

with n being the number of photons and V a normalization volume. For real system parameters, the limits

$$V \rightarrow \infty; n \rightarrow \infty; n/V \rightarrow \text{const.} \quad (9.1)$$

should be considered and all other terms inversely proportional to a power of V can be neglected.

In spite of the fact that the photon energy is small compared to the electron energy $\omega/\varepsilon \ll 1$, the total momentum of the field $\mathbf{k}n$ can be compared with the momentum of the electron $\mathbf{p}_0 \sim \mathbf{k}n$, because the photon number n is large.

Taking into account the condition (9.1) the parameters α , $\text{ch}\eta$, $\text{sh}\eta$, $\text{th}\eta$ in the state vector (8.2) simplify to:

$$\begin{aligned} \alpha &= -\frac{(q \cdot b)}{(q \cdot k)} \frac{1}{1 - \frac{2b^2}{(q \cdot k)}} \approx -\frac{(q \cdot b)}{(q \cdot k)}, \quad \varkappa = \frac{1}{\sqrt{1 - \frac{2b^2}{(q \cdot k)}}} \approx 1, \\ \text{ch}\eta &= \frac{1}{2} \sqrt[4]{1 - \frac{2b^2}{(q \cdot k)}} \left(\frac{\sqrt{1 - \frac{2b^2}{(q \cdot k)}} + 1}{\sqrt{1 - \frac{2b^2}{(q \cdot k)}}} \right) \approx 1, \quad \text{th}\eta = \frac{1 - \sqrt{1 - \frac{2b^2}{(q \cdot k)}}}{1 + \sqrt{1 - \frac{2b^2}{(q \cdot k)}}} \approx \text{sh}\eta \approx \eta \approx \frac{b^2}{2(q \cdot k)}. \end{aligned} \quad (9.2)$$

It is also important to find the dispersion relation for the zero component q^0 of the four-vector q that is given by the equation

$$p_n^2 - m^2 = 0. \quad (9.3)$$

Substituting into equation (9.3) the connection between the vector p_n and q , the dispersion relation can be found:

$$\begin{aligned} (q^0)^2 - 2q^0 \tilde{k}^0 - (q^2 - 2q \cdot \tilde{\mathbf{k}} + m^2) &= 0, \\ \tilde{k} &= k^0 \left(\sqrt{1 - \frac{2b^2}{(q \cdot k)}} \left(n + \frac{1}{2} \right) - \frac{1}{2} - \frac{(q \cdot b)^2}{(q \cdot k)^2} \frac{1}{1 - 2b^2/(q \cdot k)^2} \right). \end{aligned} \quad (9.4)$$

Then the solution of the quadratic equation gives the required zero component of the four-vector q in the limits (9.1)

$$q^0 = \omega n + \sqrt{m^2 + (q - \mathbf{k}n)^2}. \quad (9.5)$$

10 Electron spin as an observable quantity

It is well known from quantum optics that the dynamics of an atom in a comparably weaker and resonant laser field depends on the quantum fluctuations of the field. One of the most interesting phenomena of this kind is the collapse–revival effect in the evolution of the Jaynes-Cummings model [83–85] for a two-level atom. This effect was predicted theoretically [86–89, 92–96] and later observed experimentally [90, 91]. It was shown that the evolution of the population of the atomic states is characterized by two time scales. The first time scale is the period of the Rabi oscillations while the second slower one is defined by the collapse and revival times of the populations being associated with the absorption and emission of the field quanta.

An electron can be considered as a two-level system with regard to spin space. For that reason, the questions of the feasibility of collapse–revival dynamics for laser-driven electrons arises.

As was already stated in the introduction, qualitatively different time scales can be selected in the evolution of a quasi-energy electron state $\psi_p^{(e)}(\mathbf{r}, t)$ [126] with definite quasi-momentum p . One of these scales is defined by the frequency ω of the electromagnetic field:

$$T = \frac{2\pi}{\omega} \approx 10^{-4} \text{ cm},$$

for the photon energy $\omega \approx 1 \text{ eV}$.

Another scale is defined by the coherence time T_c which was introduced in [14, 15]. The characteristic time T_c is inversely proportional to the probability w_c of a photon emission per unit of time by an electron which is in a Volkov state. It equals the distance in which the uncertainty of the phase for the Volkov wave function changes by 2π . An approximation for this time was found in [15]:

$$T_c \sim w_c^{-1} \approx \frac{2}{\xi^2} T; \quad \xi = \frac{ea}{m} \leq 1,$$

where ξ is an invariant parameter which characterizes the “strength” of the electromagnetic field [121], a is the amplitude of the electromagnetic field potential, and e and m are the electronic charge and mass, respectively. It is well known that the value $\xi \approx 0.35$ corresponds to the intensity of electromagnetic field $I_0 \approx 10^{18} \text{ W/cm}^2$ when $\omega \approx 1 \text{ eV}$.

The existence of two time scales is expected to be observable when the travel time of an electron in the field $L = T_{\text{int}}$ (that is, the time of interaction between an electron and the field) satisfies

$$L > T_c \gg T,$$

which can be fulfilled for realizable parameters of the laser pulse.

In order to analyze the influence of quantum effects on the system dynamics, we consider the electron spin four-vector, which is defined via [121]

$$s^\mu(\mathbf{x}, t) = \frac{\langle \psi | \gamma^0 \gamma^5 \gamma^\mu \delta(\mathbf{x} - \mathbf{r}') | \psi \rangle}{\langle \psi | \psi \rangle}. \quad (10.1)$$

To calculate the average value of the spin (10.1), one should perform the averaging in spin space. For this purpose, the density matrix of the free electron is used,

$$\rho = u(p) \otimes \bar{u}(p) = \frac{1}{2}(\hat{p} + m)(1 - \gamma^5 \hat{a}), \quad (10.2)$$

where a is the four-vector that differs from the four-vector s by the normalization $a = \frac{\varepsilon}{m}s$ [121]. In the case of a free electron, the four-vector s^μ has components

$$s = \frac{m}{\varepsilon} \zeta + \frac{\mathbf{p}(\mathbf{p} \cdot \zeta)}{\varepsilon(\varepsilon + m)}, \quad s^0 = \frac{\mathbf{p} \cdot \zeta}{\varepsilon}, \quad (10.3)$$

where ζ is the electron spin in the rest frame

$$\zeta = \langle \sigma \rangle,$$

σ are the Pauli matrices, \mathbf{p} is the electron momentum, and ε the electron energy, which satisfies

$$\varepsilon^2 = \mathbf{p}^2 + m^2.$$

Now we recall the expression for the spin in the quasi-classical limit, which follows from the Volkov solution of the Dirac equation 1. The calculation of the spin using definition (10.2) with the use of the density matrix (10.3) yields

$$\langle s \rangle = a \frac{m}{\varepsilon} + \frac{me}{\varepsilon(k \cdot p)} (\mathbf{k}(A \cdot a) - A(\mathbf{k} \cdot a)) - \frac{me^2}{2\varepsilon(k \cdot p)^2} \mathbf{k}(k \cdot a)A^2. \quad (10.4)$$

The time dependence in (10.4) is contained only in the four-potential A , which is a periodic function (3.11). In this case, the components of the electron spin include parts which oscillate at the frequency of the classical field. If one averages the expression (10.4), the linear terms in A vanish and the mean value becomes

$$\langle \bar{s} \rangle = a \frac{m}{\varepsilon} - \frac{me^2}{2\varepsilon(k \cdot p)^2} \mathbf{k}(k \cdot a)\bar{A}^2, \quad (10.5)$$

where the bar on the top of the variable denotes averaging over the initial phases of the electron in the beam when entering into the area of space with the field.

As follows from equation (10.5), the observation of the change in spin dynamics caused by the influence of the field is possible only if the amplitude A_0 of the four-potential is comparable with the electron energy ε . This amplitude is connected with the average number of photons through

$$A_0 = \frac{\sqrt{2\bar{n}}}{\sqrt{V\omega}}.$$

This means that the field quantum number \bar{n} should be large, and this corresponds to the limits of strong field 9.

11 Cumulant method for the matrix element calculation

When the calculations of different matrix elements of the type $\langle n' | e^{\lambda A} | n \rangle$, with λ being a small parameter, need to be performed, the cumulant method can be applied [127]. It is based on perturbative expansion of a matrix, with the accuracy defined, by the number of cumulants and transitions close to the diagonal. For example, the expansion for N cumulants and transitions from the main diagonal (n, n) to the diagonals $(n, n \pm 1)$ and $(n, n \pm 2)$ looks like

$$\langle n' | e^{\lambda A} | n \rangle = \delta_{nn'} e^{\sum_{k=1}^N \lambda^k K_{0k}} + \lambda B_{\pm} \delta_{(n \pm 1)n'} e^{\sum_{k=1}^N \lambda^k K_{\pm 1k}} + \lambda C_{\pm} \delta_{(n \pm 2)n'} e^{\sum_{k=1}^N \lambda^k K_{\pm 2k}}, \quad (11.1)$$

where K_{0k} , $K_{\pm 1k}$, $K_{\pm 2k}$, B_{\pm} and C_{\pm} are cumulant coefficients. If we decompose the left and right hand sides of equation (11.1) and equate terms with the same powers of λ , the cumulant parameters can be found as:

$$\begin{aligned} K_{01} &= 0; \quad K_{02} = \frac{1}{2} \langle n | A^2 | n \rangle; \quad B_{\pm} = \langle n \pm 1 | A | n \rangle; \quad K_{11} = \frac{1}{2} \langle n \pm 1 | A^2 | n \rangle; \\ C_{\pm} &= \langle n \pm 2 | A | n \rangle; \quad K_{21} = \frac{1}{2} \langle n \pm 2 | A^2 | n \rangle. \end{aligned} \quad (11.2)$$

Hence, the matrix element with the second-order accuracy in parameter λ is

$$\begin{aligned} \langle n' | e^{\lambda A} | n \rangle &\approx \delta_{nn'} e^{\frac{1}{2} \langle n | A^2 | n \rangle} + \langle n + 1 | A | n \rangle \delta_{(n+1)n'} e^{\frac{1}{2} \langle n+1 | A^2 | n \rangle} + \langle n - 1 | A | n \rangle \delta_{(n-1)n'} e^{\frac{1}{2} \langle n-1 | A^2 | n \rangle} \\ &+ \langle n + 2 | A | n \rangle \delta_{(n+2)n'} e^{\frac{1}{2} \langle n+2 | A^2 | n \rangle} + \langle n - 2 | A | n \rangle \delta_{(n-2)n'} e^{\frac{1}{2} \langle n-2 | A^2 | n \rangle}. \end{aligned} \quad (11.3)$$

In order to show how the method works, we will apply it to the calculation of the matrix element of a type

$$\langle n' | e^{\frac{\Delta}{2}(a^2 - a^{\dagger 2})} e^{-\frac{\delta}{\sqrt{x}}(a^{\dagger} - a)} | n \rangle, \quad (11.4)$$

where Δ and δ are small quantities and are functions of a small parameter η . On the first step, we represent the multiplication of two exponentials by one exponential.

Imagine an operator function of a parameter τ

$$E(\tau) = \exp[\tau(a + b)]. \quad (11.5)$$

If operators a and b commutes, then $E(\tau) = \exp(\tau a) \exp(\tau b)$. The additional dependence on the commutators can be shown by introducing an operator K

$$E(\tau) = \exp(\tau b) K \exp(\tau a). \quad (11.6)$$

The ordering of multiples in (11.6) can be arbitrary, however for every order the equation for K will be different. By differentiating equation (11.6) by τ the equation for K can be obtained:

$$\frac{\partial K}{\partial \tau} = \exp(-\tau b) a \exp(\tau b) K - K a. \quad (11.7)$$

By setting $\tau = 0$ we specify an initial condition

$$K(0) = 1. \quad (11.8)$$

If operators \mathbf{a} and \mathbf{b} are such that the majority of commutators are equal to zero, equation (11.7) can be solved. For example, if $[\mathbf{a}, \mathbf{b}] = c$, then it is the only non-vanishing commutator and

$$\exp(-\tau\mathbf{b})\mathbf{a}\exp(\tau\mathbf{b}) = \mathbf{a} - \tau[\mathbf{b}, \mathbf{a}],$$

yielding well known Glauber's formula

$$\exp[\tau(\mathbf{a} + \mathbf{b})] = \exp(\tau\mathbf{a})\exp(\tau\mathbf{b})\exp\left(\frac{\tau^2}{2}[\mathbf{b}, \mathbf{a}]\right). \quad (11.9)$$

If commutators of a type $[\mathbf{b}, [\mathbf{b} \dots [\mathbf{b}, \mathbf{a}], \dots]]$ are not equal to zero, i.e., they are all commute with each other and with \mathbf{a} , the equation can be easily solved, giving

$$\mathbf{E}(\tau) = \exp(\tau\mathbf{b})\exp\left[\int_0^\tau dt \exp(-t\mathbf{b})\mathbf{a}\exp(t\mathbf{b})\right]. \quad (11.10)$$

Even if all commutators are not equal to zero, a case exist, when a simple solution can be obtained. If we consider operators \mathbf{a} , \mathbf{b} and all their commutators as a set, and this set has finite number of elements, then commuting all elements sequentially with each other, after some time, we obtain the same elements of a set. Consequently, the problem of decomposing exponent is reduced, to the decomposition over the elements of this set and finding the decomposition coefficients. For example, if operators satisfy the identities

$$[\mathbf{a}, \mathbf{b}] = c, \quad [\mathbf{a}, c] = -\lambda\mathbf{a}, \quad [\mathbf{b}, c] = \lambda\mathbf{b}, \quad (11.11)$$

our set consists of three elements \mathbf{a} , \mathbf{b} and c . Therefore we seek a solution in a form

$$\mathbf{E}(\tau) = \exp(\alpha(\tau)\mathbf{b})\exp(\beta(\tau)c)\exp(\gamma(\tau)\mathbf{a}), \quad (11.12)$$

where α , β and γ are unknown functions of τ . Differentiating this equation over τ , we obtain

$$\mathbf{a} + \mathbf{b} = \alpha'(\tau)\mathbf{b} + \beta'(\tau)\exp(\alpha\mathbf{b})c\exp(-\alpha\mathbf{b}) + \gamma'(\tau)\exp(\alpha\mathbf{b})\exp(\beta c)\mathbf{a}\exp(-\beta c)\exp(-\alpha\mathbf{b}).$$

By using commutation relations and comparing coefficients near corresponding operators we find the system of equations

$$\gamma' = \exp(-\lambda\beta), \quad \beta' = \alpha, \quad \alpha' = 1 - \lambda\frac{\alpha^2}{2}, \quad (11.13)$$

solution of which is

$$\alpha = \gamma = \sqrt{\frac{2}{\lambda}}\text{th}\left(\sqrt{\frac{\lambda}{2}}\tau\right), \quad \beta = \frac{2}{\lambda}\ln\text{ch}\left(\sqrt{\frac{\lambda}{2}}\tau\right). \quad (11.14)$$

Therefore we found a decomposition of the exponent over the elements of a set.

Let us come back to the operator (11.4) and calculate the commutator of the two operators

$$\left[\frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}), -\frac{\delta}{\sqrt{\kappa}}(\mathbf{a}^\dagger - \mathbf{a})\right] = \frac{\Delta\delta}{\sqrt{\kappa}}(\mathbf{a}^\dagger - \mathbf{a}), \quad (11.15)$$

and therefore the commutators of the type

$$\left[\left[\left[\frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}), -\frac{\delta}{\sqrt{\kappa}}(\mathbf{a}^\dagger - \mathbf{a}) \right], \frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}) \right], \dots, \frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}) \right],$$

are not zero. This gives the possibility for the application of the formula (11.10), yielding

$$\exp \left\{ -\frac{\delta}{\sqrt{\kappa}}(\mathbf{a}^\dagger - \mathbf{a}) + \frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}) \right\} = \exp \frac{\Delta}{2}(\mathbf{a}^2 - \mathbf{a}^{\dagger 2}) \exp \left\{ -\left(\frac{\delta}{\sqrt{\kappa}} + \frac{\delta\Delta}{\sqrt{2\kappa}} \right)(\mathbf{a}^\dagger - \mathbf{a}) \right\} \quad (11.16)$$

The parameters δ and Δ are small quantities and are functions of some other parameter η , consequently their product is the quantity of a higher order in the parameter η . For this reason, we can consider that the operators $-\delta/\sqrt{\kappa}(\mathbf{a}^\dagger - \mathbf{a})$ and $\Delta/2(\mathbf{a}^2 - \mathbf{a}^{\dagger 2})$ commute with each other.

Now we can calculate cumulant coefficients

$$\begin{aligned} \langle n | \mathbf{A}^2 | n \rangle &= -\frac{2\delta^2 n}{\kappa} - \frac{\Delta^2 n^2}{2}, & \langle n \pm 1 | \mathbf{A}^2 | n \rangle &= -\frac{\delta\Delta}{\sqrt{\kappa}} n^{\frac{3}{2}}, \\ \langle n \pm 2 | \mathbf{A}^2 | n \rangle &= \frac{\delta^2 n}{\kappa}, & \langle n \mp 1 | \mathbf{A} | n \rangle &= \pm \frac{\delta\sqrt{n}}{\sqrt{\kappa}} \left(1 - \frac{\Delta}{2} \right), \\ \langle n \mp 2 | \mathbf{A} | n \rangle &= \pm \frac{\Delta n}{2}. \end{aligned} \quad (11.17)$$

Here we pay attention to the fact, that despite the second order of parameters δ and Δ is small, these quantities are multiplied by the quantum number n , which is the large value. Consequently, the unknown matrix element becomes

$$\begin{aligned} \langle n' | S_{q'}^\dagger S_q | n \rangle &= \delta_{n,n'} e^{-\frac{\delta^2 n}{\kappa} - \frac{\Delta^2 n^2}{4}} + \delta_{n-1,n'} \frac{\delta\sqrt{n}}{\sqrt{\kappa}} e^{-\frac{\delta\Delta}{2\sqrt{\kappa}} n^{\frac{3}{2}}} - \delta_{n+1,n'} \frac{\delta\sqrt{n}}{\sqrt{\kappa}} e^{-\frac{\delta\Delta}{2\sqrt{\kappa}} n^{\frac{3}{2}}} \\ &+ \delta_{n-2,n'} \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2\kappa}} - \delta_{n+2,n'} \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2\kappa}}. \end{aligned} \quad (11.18)$$

12 Electron spin evolution in a quantized field

In the previous paragraphs we have found how the electron state vector evolve in a quantized field, being in the closest to the experimental realization state, i.e., in the coherent state. We have formulated the cumulant method for the matrix elements evaluation and introduced a motivation for the calculation of the observable value of the electron's spin. In this paragraph we will calculate the observable value of the spin being in a strong coherent field with the number of photons $\bar{n} = \beta^2$.

Let us use the spin definition (10.1) with the wave function (8.2):

$$\begin{aligned} \langle S^\mu \rangle &= \frac{1}{\sum_{n''} \int d\mathbf{q}'' |C_{\mathbf{q}'', n''}|^2} \int d\mathbf{q} d\mathbf{q}' \sum_{n'} \sum_n e^{i(\mathbf{q}' - \mathbf{q})x} \langle n' | \mathbf{S}_{q'}^\dagger \frac{\bar{u}(p_{n'})}{\sqrt{2\varepsilon_{n'}}} \left[1 + \frac{(\mathbf{a} + \mathbf{a}^\dagger)}{2(\mathbf{q}' \cdot \mathbf{k})} \hat{\mathbf{b}} \hat{\mathbf{k}} \right] \\ &\cdot \gamma^5 \gamma^\mu \left[1 + \frac{(\mathbf{a} + \mathbf{a}^\dagger)}{2(\mathbf{q} \cdot \mathbf{k})} \hat{\mathbf{k}} \hat{\mathbf{b}} \right] \frac{u(p_n)}{\sqrt{2\varepsilon_n}} S_q | n \rangle C_{\mathbf{q}', n'}^* C_{\mathbf{q}, n}. \end{aligned} \quad (12.1)$$

In order to carry out the calculations in equation (12.1) the averages over the field variables, the averages in a spin space, the summation over n -s and integrations over q -s need to be performed. We start the evaluation of (12.1) by calculating the matrix element between the field states

$$T_{n'n} = \langle n' | S_{q'}^\dagger \frac{\bar{u}(p_{n'})}{\sqrt{2\varepsilon_{n'}}} \left[1 + \frac{(a + a^\dagger)}{2(q' \cdot k)} \hat{b} \hat{k} \right] \gamma^5 \gamma^\mu \left[1 + \frac{(a + a^\dagger)}{2(q \cdot k)} \hat{k} \hat{b} \right] \frac{u(p_n)}{\sqrt{2\varepsilon_n}} S_q | n \rangle, \quad (12.2)$$

or, expanding the brackets,

$$\begin{aligned} T_{n'n} = & \gamma^5 \gamma^\mu \langle n' | S_{q'}^\dagger S_q | n \rangle + \frac{\gamma^5 \gamma^\mu \hat{k} \hat{b}}{2(q \cdot k)} \left[\sqrt{\kappa} \langle n' | S_{q'}^\dagger S_q (a + a^\dagger) | n \rangle + 2\alpha \langle n' | S_{q'}^\dagger S_q | n \rangle \right] \\ & + \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu}{2(q' \cdot k)} \left[\sqrt{\kappa'} \langle n' | (a + a^\dagger) S_{q'}^\dagger S_q | n \rangle + 2\alpha' \langle n' | S_{q'}^\dagger S_q | n \rangle \right] \\ & + \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b}}{4(q' \cdot k)(q \cdot k)} \left(\sqrt{\kappa'} \sqrt{\kappa} \langle n' | (a + a^\dagger) S_{q'}^\dagger S_q (a + a^\dagger) | n \rangle \right. \\ & + 4\alpha\alpha' \langle n' | S_{q'}^\dagger S_q | n \rangle + 2\alpha \sqrt{\kappa'} \langle n' | (a + a^\dagger) S_{q'}^\dagger S_q | n \rangle \\ & \left. + 2\alpha' \sqrt{\kappa} \langle n' | S_{q'}^\dagger S_q (a + a^\dagger) | n \rangle \right). \end{aligned} \quad (12.3)$$

The evaluation of the matrix element (12.2) is reduced to the calculation of the matrix element $\langle n' | S_{q'}^\dagger S_q | n \rangle$ with $n \neq n'$ in the general case. The method of the calculation of the matrix element of such a type was developed in 11. Therefore, the result of the evaluation is

$$\begin{aligned} \langle n' | S_{q'}^\dagger S_q | n \rangle = & \delta_{n,n'} e^{-\frac{\delta^2 n}{\kappa} - \frac{\Delta^2 n^2}{4}} + \frac{\delta \sqrt{n}}{\sqrt{\kappa}} e^{-\frac{\delta \Delta}{2\sqrt{\kappa}} n^{\frac{3}{2}}} (\delta_{n-1,n'} - \delta_{n+1,n'}) + \\ & + \frac{n\Delta}{2} e^{\frac{\delta^2 n}{2\kappa}} (\delta_{n-2,n'} - \delta_{n+2,n'}), \end{aligned} \quad (12.4)$$

where $\delta_{n,n'}$ is the Kronecker symbol, $\delta = \alpha_{q'} - \alpha_q$, and $\Delta = \eta_{q'} - \eta_q$. The matrix elements with additional creation and annihilation operators in (12.3), for example, $\langle n' | S_{q'}^\dagger S_q (a + a^\dagger) | n \rangle$, can be obtained from equation (12.4) by shifting indices, multiplying by the corresponding power of n , and leaving the terms in which n changes by no more than two.

Insertion of equation (12.3) into equation (12.4) with the use of the results of the 9 of a strong field limit, yields

$$\begin{aligned} \langle s^\mu \rangle = & \frac{1}{\sum_n \int d\mathbf{q} |C_{q,n}|^2} \int d\mathbf{q} d\mathbf{q}' \sum_n e^{i(q'-q)x} \frac{\bar{u}(p_{n'})}{\sqrt{2\varepsilon_{n'}}} \\ & \times \left\{ C_{q',n}^* C_{q,n} \left(\gamma^5 \gamma^\mu + \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b}}{2(q' \cdot k)(q \cdot k)} n \right) e^{-\delta^2 n - \frac{\Delta^2 n^2}{4}} \right. \\ & + C_{q',n-1}^* C_{q,n} \left[\delta \sqrt{n} e^{-\frac{\delta \Delta}{2} n^{\frac{3}{2}}} \left(\gamma^5 \gamma^\mu + \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b}}{2(q' \cdot k)(q \cdot k)} n \right) \right. \\ & \quad \left. + \left(e^{-\delta^2 n - \frac{\Delta^2 n^2}{4}} + \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \right) \left(\frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu}{2(q' \cdot k)} + \frac{\gamma^5 \gamma^\mu \hat{k} \hat{b}}{2(q \cdot k)} \right) \sqrt{n} \right] \\ & \left. + C_{q',n+1}^* C_{q,n} \left[-\delta \sqrt{n} e^{-\frac{\delta \Delta}{2} n^{\frac{3}{2}}} \left(\gamma^5 \gamma^\mu + \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b}}{2(q' \cdot k)(q \cdot k)} n \right) \right. \right. \end{aligned}$$

$$\begin{aligned}
& + \left(e^{-\delta^2 n - \frac{\Delta^2 n^2}{4}} - \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \right) \left(\frac{\hat{b}\hat{k}\gamma^5\gamma^\mu}{2(q' \cdot k)} + \frac{\gamma^5\gamma^\mu\hat{k}\hat{b}}{2(q \cdot k)} \right) \sqrt{n} \Big] \\
& + C_{q',n-2}^* C_{q,n} \left[\left(e^{-\delta^2 n - \frac{\Delta^2 n^2}{4}} + \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \right) \frac{\hat{b}\hat{k}\gamma^5\gamma^\mu\hat{k}\hat{b}}{4(q' \cdot k)(q \cdot k)} n \right. \\
& \quad \left. + \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \gamma^5\gamma^\mu + \delta \sqrt{n} e^{-\frac{\delta\Delta}{2} n^{\frac{3}{2}}} \left(\frac{\hat{b}\hat{k}\gamma^5\gamma^\mu}{2(q' \cdot k)} + \frac{\gamma^5\gamma^\mu\hat{k}\hat{b}}{2(q \cdot k)} \right) \sqrt{n} \right] \\
& + C_{q',n+2}^* C_{q,n} \left[\left(e^{-\delta^2 n - \frac{\Delta^2 n^2}{4}} - \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \right) \frac{\hat{b}\hat{k}\gamma^5\gamma^\mu\hat{k}\hat{b}}{4(q' \cdot k)(q \cdot k)} n - \frac{\Delta n}{2} e^{\frac{\delta^2 n}{2}} \gamma^5\gamma^\mu \right. \\
& \quad \left. - \delta \sqrt{n} e^{-\frac{\delta\Delta}{2} n^{\frac{3}{2}}} \left(\frac{\hat{b}\hat{k}\gamma^5\gamma^\mu}{2(q' \cdot k)} + \frac{\gamma^5\gamma^\mu\hat{k}\hat{b}}{2(q \cdot k)} \right) \sqrt{n} \right] \Bigg\} \frac{u(p_n)}{\sqrt{2\varepsilon_n}}. \tag{12.5}
\end{aligned}$$

Further simplifications of (12.5) are possible in the limit of large n , when the coefficients $C_{q,n}$ (8.11) can be estimated using the asymptotic of the Hermitian polynomials [128] and Stirling's formula for the factorial (see appendix C), i.e.

$$\lim_{n \rightarrow \infty, x \rightarrow \infty} H_n(x) \rightarrow 2^n x^n, \quad n! \sim \sqrt{2\pi n} n^{n+1/2} e^{-n}. \tag{12.6}$$

Insertion of the approximation (12.6) into the coefficients $C_{q,n}$ yields

$$\begin{aligned}
C_{q,n} & \approx \frac{(2\pi)^{-1/4}}{(2\pi)^3} \left(\frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^0 \frac{u(p_0)}{\sqrt{2\varepsilon_0}} + \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \hat{b}\hat{k}\gamma^0 \frac{u(p_0)}{\sqrt{2\varepsilon_0}} \frac{\sqrt{n}}{2(q \cdot k)} \right) M_{q,n}, \\
M_{q,n} & = \int d\mathbf{r} \exp \left\{ -i(\mathbf{q} - \mathbf{p}_0)(\mathbf{r} - \mathbf{r}_0) - |\theta|^2/2 + \alpha\theta - \alpha^2/2 + \frac{1}{2}(\theta - \alpha)^2\eta \right. \\
& \quad \left. - \frac{1}{2}\left(n + \frac{1}{2}\right) \ln n + \frac{n}{2} + n \ln(\theta - \alpha) \right\}. \tag{12.7}
\end{aligned}$$

Equation (12.7) contains various products of the coefficients $C_{q,n}$ and the complex conjugate $C_{q',n'}^*$, involving various combinations of n and n' . These products can be written in the general form as

$$e^{i(q'-q) \cdot x} C_{q',n+l}^* C_{q,n} = \frac{(2\pi)^{-\frac{1}{2}}}{(2\pi)^6} A_{q'}^* A_q \int d\mathbf{r}' d\mathbf{r} e^{\Phi_l(t, \mathbf{x}, \mathbf{q}, \mathbf{q}', \mathbf{r}, \mathbf{r}', n)}, \tag{12.8}$$

where the phase function is equal to

$$\begin{aligned}
\Phi_l(t, \mathbf{x}, \mathbf{q}, \mathbf{q}', \mathbf{r}, \mathbf{r}', n) & = it(q_{n+l}^0 - q_n^0) - i(\mathbf{q}' - \mathbf{q}) \cdot \mathbf{x} + i(\mathbf{q}' - \mathbf{p}_0)(\mathbf{r}' - \mathbf{r}_0) \\
& - i(\mathbf{q} - \mathbf{p}_0)(\mathbf{r} - \mathbf{r}_0) - \beta^2 + \beta \left(\alpha' e^{-i\mathbf{k} \cdot \mathbf{r}'} + \alpha e^{i\mathbf{k} \cdot \mathbf{r}} \right) - \left(n + \frac{1}{2} \right) \ln n \\
& + n + n \left(\ln \beta^2 - i\mathbf{k}(\mathbf{r}' - \mathbf{r}) - \frac{1}{\beta} \left(\alpha' e^{i\mathbf{k} \cdot \mathbf{r}'} + \alpha e^{-i\mathbf{k} \cdot \mathbf{r}} \right) \right) \\
& + \frac{\beta^2}{2} \left(\eta' e^{-2i\mathbf{k} \cdot \mathbf{r}'} + \eta e^{2i\mathbf{k} \cdot \mathbf{r}} \right) - \frac{l}{2} \ln n + l(\ln \beta - i\mathbf{k} \cdot \mathbf{r}') \tag{12.9}
\end{aligned}$$

with A_q being the non-oscillating amplitude factor and index l runs through $\{0, 1, 2\}$.

The evaluation of (12.5) for the average value of the spin will be carried out in two steps. At the first step, we will calculate the integrals over the variables \mathbf{r} , \mathbf{r}' , \mathbf{q} , \mathbf{q}' , and sum over the field quantum number n . At the second step, we average the matrix element over a spin space.

The integrations over the variables \mathbf{r} and \mathbf{r}' in equation (12.8) will be performed in the coordinate system where z and z' axes directed along the \mathbf{k} vector. Therefore, any vectors can be written in the form $\mathbf{y} = \mathbf{y}_\perp + \mathbf{y}_z$, with \mathbf{y}_z being directed along z , z' and \mathbf{y}_\perp being directed perpendicular to the latter. Then the integrations over \mathbf{r}_\perp and \mathbf{r}'_\perp give rise to the product of δ -functions $\delta(\mathbf{q}'_\perp - \mathbf{p}_{0\perp})\delta(\mathbf{q}_\perp - \mathbf{p}_{0\perp})$, which removes the integration over \mathbf{q}'_\perp and \mathbf{q}_\perp and leads to the conservation law

$$\mathbf{q}_\perp = \mathbf{q}'_\perp = \mathbf{p}_{0\perp}. \quad (12.10)$$

Then the phase function (12.9) transforms to

$$\begin{aligned} \Phi(q_z, q'_z, z, z', n) = & it(q_{n\pm l}^{0'} - q_n^0) - i(q'_z - q_z)z_i + i(q'_z - p_{0z})z' - i(q_z - p_{0z})z \\ & - \beta^2 + \beta(\alpha' e^{-i\omega z'} + \alpha e^{i\omega z}) - \left(n + \frac{1}{2}\right) \ln n + n \\ & + n \left(\ln \beta^2 - i\omega(z' - z) - \frac{1}{\beta} (\alpha' e^{i\omega z'} + \alpha e^{-i\omega z}) \right) \\ & + \frac{\beta^2}{2} (\eta' e^{-2ikz'} + \eta e^{2ikz}) - \frac{l}{2} \ln n + l(\ln \beta - i\omega z'), \\ q_n^0 = & \omega n + \sqrt{p_{0\perp}^2 + (q_z - \omega n)^2 + m^2}; \quad z_i = \frac{\mathbf{x}\mathbf{k}}{\omega} - z_0. \end{aligned} \quad (12.11)$$

The change of variables

$$q_z - \omega n \rightarrow q_z; \quad q_z \rightarrow q_z + \omega n.$$

modifies the phase:

$$\begin{aligned} \Phi(q_z, q'_z, z, z', n) = & i\omega l t + it \left(\sqrt{p_{0\perp}^2 + (q'_z - \omega l)^2 + m^2} - \sqrt{p_{0\perp}^2 + q_z^2 + m^2} \right) \\ & - i(q'_z - q_z)z_i + i(q'_z z' - q_z z) + i(\omega n - p_{0z})(z' - z) \\ & - \beta^2 + \beta(\alpha' e^{-i\omega z'} + \alpha e^{i\omega z}) - \left(n + \frac{1}{2}\right) \ln n + n \\ & + n \left(\ln \beta^2 - i\omega(z' - z) - \frac{1}{\beta} (\alpha' e^{i\omega z'} + \alpha e^{-i\omega z}) \right) \\ & + \frac{\beta^2}{2} (\eta' e^{-2ikz'} + \eta e^{2ikz}) - \frac{l}{2} \ln n + l(\ln \beta - i\omega z'). \end{aligned} \quad (12.12)$$

Now one should average over the coordinate z_i , which corresponds to the averaging over the initial electron coordinate \mathbf{r}_0 in the uniform electron beam in real experiments. The averaging results in an additional δ -function, $\delta(q_z - q'_z)$, which removes the integration over q'_z and leads to the conservation law

$$q_z = q'_z.$$

Let us estimate the values of δ and Δ in equation (12.5) after the integrations have been already performed. Using the definition

$$\delta = \alpha_{q'} - \alpha_q = \frac{\partial \alpha}{\partial \mathbf{q}}(\mathbf{q}' - \mathbf{q}) + \frac{\partial \alpha}{\partial n}l, \quad \Delta = \eta_{q'} - \eta_q = \frac{\partial \eta}{\partial \mathbf{q}}(\mathbf{q}' - \mathbf{q}) + \frac{\partial \eta}{\partial n}l, \quad (12.13)$$

these values are equal to zero within the considered accuracy because $\mathbf{q}' = \mathbf{q}$ and the derivatives over n also vanish, i.e.

$$\begin{aligned} \alpha &= -\frac{(q \cdot b)}{(q \cdot k)} = \frac{\mathbf{q}_\perp \cdot \mathbf{b}}{k(\sqrt{p_{0\perp}^2 + m^2 + q_z^2} - q_z)}, \quad \frac{\partial \alpha}{\partial n} = 0, \\ \eta &= \frac{b^2}{2(q \cdot k)} = \frac{b^2}{2k(\sqrt{p_{0\perp}^2 + m^2 + q_z^2} - q_z)}, \quad \frac{\partial \eta}{\partial n} = 0. \end{aligned} \quad (12.14)$$

The spin (12.1) then transforms to

$$\begin{aligned} \langle s^\mu(t) \rangle &= \frac{1}{\sum_n \int d\mathbf{q} |C_{q,n}|^2} \int dq_z dz' dz \sum_n |A_q|^2 \left\{ e^{\Phi_0} \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 \gamma^\mu \frac{u(p_n)}{\sqrt{2\varepsilon_n}} \right. \\ &\quad + \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \frac{\hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b}}{4(q \cdot k)^2} n \frac{u(p_n)}{\sqrt{2\varepsilon_n}} (e^{\Phi_0} + e^{\Phi_{-2}} + e^{\Phi_2}) \\ &\quad \left. + \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} (\hat{b} \hat{k} \gamma^5 \gamma^\mu + \gamma^5 \gamma^\mu \hat{k} \hat{b}) \frac{\sqrt{n}}{2(q \cdot k)} \frac{u(p_n)}{\sqrt{2\varepsilon_n}} (e^{\Phi_{-1}} + e^{\Phi_1}) \right\}. \end{aligned} \quad (12.15)$$

The main contributions to the sum over the field quantum number n arise from values of n near the quasi-classical value $\bar{n} \gg 1$. This gives the possibility of changing the summation over n to an integration over the complex variable. Then this integral in the complex plane can be evaluated using the saddle point method (see Appendix C). According to the method, the saddle point is defined by the first derivative of the phase

$$\begin{aligned} \Phi(q_z, q'_z, z, z', n) &= i\omega t l + it(\sqrt{p_{0\perp}^2 + (q_z - \omega l)^2 + m^2} - \sqrt{p_{0\perp}^2 + q_z^2 + m^2}) + iq_z(z' - z) \\ &\quad - ip_{0z}(z' - z) - \beta^2 + \beta(\alpha' e^{-i\omega z'} + \alpha e^{i\omega z}) + \frac{\beta^2}{2}(\eta' e^{-2ikz'} + \eta e^{2ikz}) \\ &\quad - \left(n + \frac{1}{2}\right) \ln n + n + n \left(\ln \beta^2 - \frac{1}{\beta} (\alpha' e^{i\omega z'} + \alpha e^{-i\omega z}) \right) \mp \frac{l}{2} \ln n \\ &\quad \pm l(\ln \beta - i\omega z'). \end{aligned} \quad (12.16)$$

This gives the equation for the determination of the saddle point

$$\frac{\partial \Phi}{\partial n} = -\frac{\alpha}{\beta} (e^{ikz'} + e^{-ikz}) - \ln n + \ln \beta^2 = 0. \quad (12.17)$$

The value $\beta = \sqrt{\bar{n}}$ is a large number, and therefore we can build a solution of equation (12.17) as a series over parameter $1/\beta$. With the accuracy $o(1/\beta^2)$ one obtains the saddle point n_0 as

$$n_0 = \beta^2 - \alpha\beta(\cos kz' + \cos kz) - i\alpha\beta(\sin kz' - \sin kz). \quad (12.18)$$

The zeroth order solution for n_0 , which is equal to β^2 , yields the quasi-classical limit. The first order corrections, proportional to the parameter $\beta\alpha$, give rise to the desired quantum effects. The substitution of equation (12.18) into the phase (12.16) gives

$$\begin{aligned} \Phi_l(n_0) = ikl t \left(1 - \frac{q_z}{\sqrt{p_{0\perp}^2 + m^2 + q_z^2}} \right) + i(q_z - p_{0z})(z' - z) - 2i\alpha\beta(\sin kz' - \sin kz) \\ + \frac{\eta\beta^2}{2}(\cos 2kz' + \cos 2kz) - i\frac{\eta\beta^2}{2}(\sin 2kz' - \sin 2kz) - \ln\beta - ilkz', \end{aligned} \quad (12.19)$$

and the integrals in the expression for the spin have the form

$$\int dndq_z dz' dz \sum_n \left\{ A_q^* A_q e^{\Phi_l(n_0) + \frac{\Phi''_{nn}}{2!}(n-n_0)^2} \right\}, \quad (12.20)$$

where Φ''_{nn} denotes the second derivative calculated at the saddle point n_0 .

The phase $\Phi_l(n_0)$ has linear and second order terms in β . We now show that the terms in the phase $\Phi_l(n_0)$ which are quadratic in the coherent state parameter β can be neglected for intensities up to including 10^{18} W/cm².¹ For this purpose, let us estimate the absolute values of $\beta\alpha$ and $\eta\beta^2$. If we choose the density of photons $\rho = 10^{20}$ cm⁻³, the photon frequency $\omega = 10^5$ cm⁻¹, and $\gamma = \varepsilon_q/m = 10$, then the values of the products are

$$\beta\alpha \approx e_0 \frac{\sqrt{\rho}\theta}{\sqrt{\omega}\omega(\gamma^{-2} + \theta^2)} \sim 10^3 \quad (12.21)$$

and

$$\beta^2\eta \approx e_0^2 \frac{\rho}{\omega^2 m \gamma(\gamma^{-2} + \theta^2)} \sim 10^{-1}, \quad (12.22)$$

where e_0 is the charge of the electron and θ is the angle between \mathbf{k} and \mathbf{p} (in this case, the Doppler effect has its maximum value) i.e.

$$\theta \sim \gamma^{-1} \ll 1.$$

This allows us to neglect the second order terms in comparison with the first order ones in (12.19).

The integrals over z , z' , and q_z in the expression (12.15) have the following form

$$\int dz dz' dq_z e^{\Phi_l(n_0)}, \quad l = \{0, \pm 1, \pm 2\}, \quad (12.23)$$

and can be calculated analytically. For example, examining a general form of the integrals and leaving only linear terms in β one obtains

$$A = \int_{-\infty}^{\infty} e^{iQz} e^{-ia \sin kz} dz. \quad (12.24)$$

¹Note, for intensities higher than 10^{18} W/cm² the parameters $\alpha\beta$ and $\eta\beta^2$ are comparable. That is why the inclusion of $\eta\beta^2$ leads to the replacement of Bessel functions by generalized Bessel functions in the sum (12.28). In this case an analytical evaluation of the sum (12.28) is impossible and numerical methods should be used instead.

We now separate the interval of integration into parts. Each part has the length 2π . Then (12.24) transforms to

$$A = \sum_{m=-\infty}^{\infty} \int_{-\pi/k}^{\pi/k} e^{iQ(z+2\pi m/k)} e^{-ia \sin kz} dz = \sum_{m=-\infty}^{\infty} e^{iQ2\pi m/k} \int_{-\pi/k}^{\pi/k} e^{iQz} e^{-ia \sin kz} dz. \quad (12.25)$$

The first exponential in (12.25) has the representation [129]

$$\sum_{m=-\infty}^{\infty} e^{iQ2\pi m/k} = \sum_{p=-\infty}^{\infty} \delta\left(\frac{Q}{k} - p\right) = k \sum_{p=-\infty}^{\infty} \delta(Q - pk). \quad (12.26)$$

Therefore, for A , we obtain

$$A = \sum_{p=-\infty}^{\infty} \delta(Q - pk) J_p^*(a). \quad (12.27)$$

The application of (12.27) to the integrals will give the desired expression:

$$\int dz dz' dq_z e^{\Phi_l} = (2\pi)^2 \sum_{u=-\infty}^{\infty} e^{iklt \left(1 - \frac{p_{0z} + kl + ku}{\sqrt{p_{0\perp}^2 + m^2 + (p_{0z} + kl + ku)^2}}\right)} J_{-u}(-2\alpha\beta) J_{u+l}(2\alpha\beta). \quad (12.28)$$

The summation over u in (12.28) can be evaluated via the Euler-Maclaurin formula [128], where the sum is replaced by an integral. The use of the integration formula of Bessel functions by their index

$$\begin{aligned} \int_{-\infty}^{\infty} a^{-\mu-x} b^{-\nu+x} e^{icx} J_{\mu+x}(a) J_{\nu-x}(b) dx &= \left[\frac{2 \cos\left(\frac{c}{2}\right)}{a^2 e^{-\frac{1}{2}ic} + b^2 e^{\frac{1}{2}ic}} \right]^{\frac{\mu}{2} + \frac{\nu}{2}} e^{i\frac{c}{2}(\nu-\mu)} \\ &\cdot J_{\mu+\nu} \left(\left[2 \cos\left(\frac{c}{2}\right) (a^2 e^{-\frac{1}{2}ic} + b^2 e^{\frac{1}{2}ic}) \right]^{\frac{1}{2}} \right), \end{aligned} \quad (12.29)$$

yields

$$\int dz dz' dq_z e^{\Phi_l} = (2\pi)^2 (-i)^l e^{iklt \left(1 - \frac{2p_{0z} + kl}{2\varepsilon_0}\right)} J_l(4\alpha\beta \sin \frac{k^2 lt}{2\varepsilon_0}). \quad (12.30)$$

The norm of the coefficients $C_{q,n}$ can be calculated in the same way as the calculation of the product of $C_{q,n}$ -coefficients. This coincides with the integral of $\int dz dz' dq_z e^{\Phi_0}$ and leads to

$$\sum_n \int d\mathbf{q} |C_{q,n}|^2 = (2\pi)^2 |A_q|^2 \int d\mathbf{n} e^{-\ln \beta + \frac{\Phi''_{00}}{2!} (n-n_0)^2}. \quad (12.31)$$

Therefore, when we insert the norm into the spin expression (12.15), it cancels the integral over n and e^{Φ_0} .

The last step is to calculate the average in spin space. For this purpose, the zeroth order value for n_0 can be inserted into all pre-exponential terms. This gives the expression for the averages

$$\frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 \gamma^\mu \frac{u(p_n)}{\sqrt{2\varepsilon_n}}; \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b} \frac{u(p_n)}{\sqrt{2\varepsilon_n}}; \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} (\hat{b} \hat{k} \gamma^5 \gamma^\mu + \gamma^5 \gamma^\mu \hat{k} \hat{b}) \frac{u(p_n)}{\sqrt{2\varepsilon_n}}. \quad (12.32)$$

Inserting the density matrix of the electron

$$\rho = u(p) \otimes \bar{u}(p) = \frac{1}{2}(\hat{p} + m)(1 - \gamma^5 \hat{a}), \quad (12.33)$$

into equations (12.32), we obtain

$$\begin{aligned} \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 \gamma^\mu \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= \frac{1}{\varepsilon_n} \text{Sp} \rho \gamma^5 \gamma^\mu; \\ \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b} \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= -\frac{k^\mu b^2}{\varepsilon_n} \text{Sp} \rho \gamma^5 \hat{k}; \\ \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 (\hat{b} \hat{k} \gamma^\mu + \gamma^\mu \hat{k} \hat{b}) \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= \frac{1}{\varepsilon_n} \text{Sp} (k^\mu \rho \gamma^5 \hat{b} - b^\mu \rho \gamma^5 \hat{k}). \end{aligned} \quad (12.34)$$

By using the results of paragraph D, i.e., taking into account the fact that the trace of the product of the gamma matrices is nonzero only for even numbers of matrices, $(\gamma^5)^2 = 1$ and γ^5 anti-commutes with γ^μ one obtains

$$\begin{aligned} \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 \gamma^\mu \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= \frac{m}{\varepsilon_n} a^\mu; \\ \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \hat{b} \hat{k} \gamma^5 \gamma^\mu \hat{k} \hat{b} \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= -\frac{mk^\mu b^2}{\varepsilon_n} (a \cdot k); \\ \frac{\bar{u}(p_n)}{\sqrt{2\varepsilon_n}} \gamma^5 (\hat{b} \hat{k} \gamma^\mu + \gamma^\mu \hat{k} \hat{b}) \frac{u(p_n)}{\sqrt{2\varepsilon_n}} &= \frac{m}{\varepsilon_n} (k^\mu (a \cdot b) - b^\mu (a \cdot k)). \end{aligned} \quad (12.35)$$

Combining all together, we get the mean value of the electron spin four vector as

$$\begin{aligned} \langle s^\mu(t) \rangle &= \frac{m}{\varepsilon_0} a_0^\mu - \frac{m}{\varepsilon_0} k^\mu (a_0 \cdot k) \frac{\beta^2 b^2}{(p_0 \cdot k)^2} (1 + \text{Re} \Pi_2) + \left[\frac{m}{\varepsilon_0} \frac{\beta}{(p_0 \cdot k)} (k^\mu (a_0 \cdot b) - b^\mu (a_0 \cdot k)) \right] 2\text{Re} \Pi_1, \\ \Pi_l &= (-i)^l e^{i\omega l t \left(1 - \frac{p_{0z}}{\varepsilon_0}\right)} J_l(4\alpha\beta \sin \frac{\omega^2 l t}{2\varepsilon_0}), \quad l = 1, 2, \end{aligned} \quad (12.36)$$

where $\varepsilon_0 = \sqrt{p_0^2 + m^2}$, a_0 is the initial four-vector of the electron spin, $p_0 = (\varepsilon_0, \mathbf{p}_0)$, and J_l the Bessel function of the order l .

For the analysis of the electron polarization, we also need to calculate the electron current density $j^\mu(\mathbf{x}, t) = \langle \psi | \gamma^\mu \delta(\mathbf{x} - \mathbf{r}') | \psi \rangle / \langle \psi | \psi \rangle$ averaged over the initial coordinates of the electron, \mathbf{r}_0 . This value can be evaluated in the same way as for the electron spin density. Here we present only the final result.

$$j^\mu(t) = \frac{p^\mu}{\varepsilon_0} - \frac{\beta b^\mu}{\varepsilon_0} 2\text{Re} \Pi_1 + \frac{k^\mu}{\varepsilon_0} \left(\frac{\beta(b \cdot p)}{(k \cdot p)} 2\text{Re} \Pi_1 - \frac{\beta^2 b^2}{(k \cdot p)} (1 + \text{Re} \Pi_2) \right). \quad (12.37)$$

13 Quasi-classical limit

In the previous paragraph, we have calculated the observable value of an electron's spin in the case when an external field is quantized, i.e., including quantum fluctuations of the field. We

have stated that the solution for the saddle point (12.18) can be represented as a series in the inverse powers of the coherent state parameter β , with the zero order solution $n_0 = \beta^2$ being a quasi-classical limit. Therefore, in this case the quantum spin vector s must coincide with the corresponding quasi-classical Volkov one, when the former is not averaged in time. This corresponds to the quantum case when the averaging over the initial coordinate z_i in the phase of the expression (12.17) is not performed.

In order to show this correspondence we use the saddle point method to evaluate the sum over the field quantum number n without averaging over z_i . As before, this leads to the same equation for n (12.17), but in this case we preserve only the leading term in β , which is equal to

$$n_0 = \beta^2,$$

and coincides with the quasi-classical limit. This means that the values $\alpha\beta$ and $\eta\beta^2$ are equal to zero.

We now change the variable q'_z to $q'_z - \omega l$ in (12.16). This leads to the integrals

$$\begin{aligned} I \sim & e^{i\omega l(t-z_i)-\ln\beta} \cdot \int dz' dq'_z e^{it\sqrt{p_{0\perp}^2+q_z'^2+m^2}-iq'_z z_i+iq'_z z'-ip_{0z} z'} \\ & \cdot \int dz dq_z e^{-it\sqrt{p_{0\perp}^2+q_z^2+m^2}+iq_z z_i-iq_z z+ip_{0z} z} e^{r(\delta,\Delta)}, \end{aligned} \quad (13.1)$$

where $e^{r(\delta,\Delta)}$ represents additional terms appearing in the cumulant expansion. They are independent of z and z' , and thus are not relevant here.

The integrals over z and z' yield the product of delta functions $\delta(q_z - p_{0z})\delta(q'_z - p_{0z})$, which cancels the integration over q_z and q'_z and removes $r(\delta, \Delta)$. The average in spin space is carried out analogously to the previous calculation for the derivation of equation (12.36).

Therefore, the average value of the spin four-vector can be expressed through the integrals

$$I \sim e^{i\omega l(t-z_i)},$$

which exactly leads to the quasi-classical Volkov value without time averaging.

14 Description of the final electron polarization

The interaction process of the electron with the field can be understood in the following way. For $t < 0$, the electron is free, such that it is described by the free solution of the Dirac equation and the field is in a coherent state β , with the average number of photons $\bar{n} = \beta^2$.

At $t = 0$, the electron crosses the border of the field beam and the interaction starts. We suppose that the boundaries of the field beam are rather sharp. However, the real boundary transition width is not of zero width, and carrying out the above averaging over the initial coordinate r_0 corresponds to an averaging over this width. This procedure is widely used in scattering theory and is described in great detail in reference [122]. During the interaction, the system of the electron and the field is described by the wave function (8.2). The electron interacts with the field during the time

$$t_0 \sim \frac{d}{v_0 \sin \theta}, \quad (14.1)$$

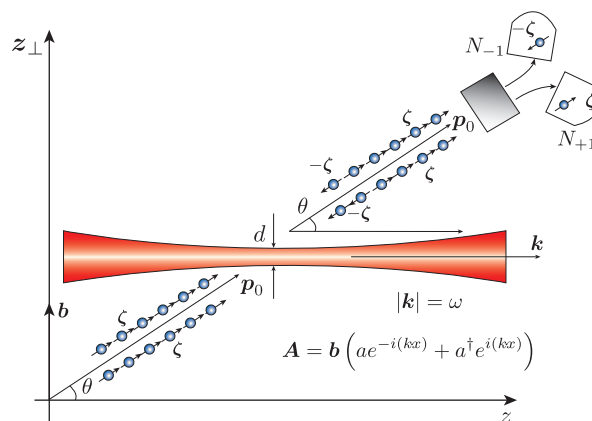


Figure 14.1 – Electron motion in a single-mode quantized field. \mathbf{p}_0 is the initial momentum of the electron, \mathbf{b} the polarization vector of the field, the z -axis is directed along the wave vector \mathbf{k} , \mathbf{a} and \mathbf{a}^\dagger are, respectively, the annihilation and creation operators, ω is the frequency of field, and N_{+1} (N_{-1}) the number of particles with helicity equal to 1 (-1).

where d is the “thickness” of the laser pulse, v_0 the velocity of the electron, and θ the angle between \mathbf{k} and \mathbf{v}_0 (see Figure 14.1).

At $t = t_0$, the interaction is turned off, the electron becomes free and the detector² can measure the spin $s(t_0)$. A change in the interaction time t_0 will lead to different spin values. This time can be changed in two ways. The first is to change the energy of the electron. The larger the electron energy, the less time it spends in the field. The second way is to change the angle θ . The closer the angle θ is to π , the more time the electron interacts with the field.

The polarization of the electron is characterized by its helicity—the projection of the spin s on the direction of its momentum $\nu = \mathbf{p}_0/p_0$. The eigenvalues of the helicity operator for a free electron are ± 1 . According to the general rules of quantum mechanics, the expectation value of the helicity operator can be written as

$$\langle \Sigma \cdot \nu \rangle = p_+ \cdot 1 + p_- \cdot (-1), \quad (14.2)$$

with p_1 and p_{-1} being the probabilities of observing the electron with helicity $+1$ and -1 , respectively. Usually, one considers at the beginning the situation of p_1 being unity and p_{-1} zero, and the interest is how p_1 and p_{-1} are modified due to the interaction.

However, in a real experiment, there is no single electron, but rather, a beam of N electrons. If an initial electron beam was fully polarized, some quantity of electrons with opposite polarization should appear after the interaction is finished. Namely, the number N_{-1} of such electrons is equal to

$$N_{-1} = p_{-1}N \quad (14.3)$$

if at the initial moment of time $t = 0$ the helicity of the electron was equal to $+1$.

To calculate the polarization, we choose a coordinate system with z -axis directed along \mathbf{k} . It is further assumed that the initial electron momentum \mathbf{p}_0 has the angle θ with the z -axis and that

²Here we assume that the detector tracks solely the electrons that possess initial momentum p_0 while in [55], the electrons scattered by the field are tracked (the number of such an electrons is small).

the field is linearly polarized and directed perpendicularly to the z -axis (Fig. 14.1). We assume that at the initial moment of time, the vector ζ (the average spin in the electron's rest frame) is directed along the momentum in the laboratory frame, such that $\mathbf{p} \cdot \zeta = p$, which corresponds to the helicity of the electron's being +1.

In this coordinate system,

$$\begin{aligned} p_0 &= (\varepsilon_0, 0, p_0 \sin \theta, p_0 \cos \theta), & k &= (\omega, 0, 0, \omega), \\ b &= (0, 0, b, 0), & \zeta &= (0, \sin \theta, \cos \theta), \\ a_0 &= \frac{\varepsilon_0}{m} s_0 = \frac{\varepsilon_0}{m} \left(\frac{p_0}{\varepsilon_0}, 0, \sin \theta, \cos \theta \right). \end{aligned} \quad (14.4)$$

If one uses the definitions (14.4), the four-products in (12.36) can be found as follows:

$$\begin{aligned} \frac{m}{\varepsilon_0} (a_0 \cdot k) &= \omega(v_0 - \cos \theta), & \frac{m}{\varepsilon_0} (a_0 \cdot b) &= -b \sin \theta, \\ (p_0 \cdot k) &= \omega \varepsilon_0 (1 - v_0 \cos \theta), & \beta \alpha &= -\beta \frac{(q \cdot b)}{(p_0 \cdot k)} = \frac{\beta b \varepsilon_0}{\varepsilon_0 \omega} \frac{v_0 \sin \theta}{1 - v_0 \cos \theta}, \end{aligned} \quad (14.5)$$

where the velocity of the electron is $v_0 = p_0/\varepsilon_0$ and the total momentum of the system $q = p_0 + k\beta^2$.

In order to find the helicity of the electron after the interaction with the field, we project the spin vector s onto the direction of the electron momentum $\nu = \mathbf{p}_0/p_0 = (0, \sin \theta, \cos \theta)$:

$$\begin{aligned} s \cdot \nu &= s_0 \cdot \nu + \mathbf{k} \cdot \mathbf{p}_0 (s_0 \cdot k) \frac{\beta^2 b^2}{(p_0 \cdot k)^2} (1 + \text{Re } \Pi_2) \\ &\quad + \frac{\beta}{(p_0 \cdot k)} (\mathbf{k} \cdot \mathbf{p}_0 (s_0 \cdot b) - \mathbf{b} \cdot \nu (s_0 \cdot k)) 2 \text{Re } \Pi_1. \end{aligned} \quad (14.6)$$

Taking into account that the scalar products are equal to

$$s_0 \cdot \nu = 1, \quad \mathbf{k} \cdot \nu = \omega \cos \theta, \quad \mathbf{b} \cdot \nu = b \sin \theta \quad (14.7)$$

and inserting them into (14.6), we find

$$s \cdot \nu = 1 + \frac{\xi^2}{\gamma^2} \frac{(v_0 - \cos \theta) \cos \theta}{(1 - v_0 \cos \theta)^2} (1 + \text{Re } \Pi_2) - 2 \frac{\xi}{\gamma} \frac{v_0 \sin \theta}{1 - v_0 \cos \theta} \text{Re } \Pi_1, \quad (14.8)$$

where

$$\begin{aligned} \text{Re } \Pi_1 &= \sin(\omega t (1 - v_0 \cos \theta)) J_1 \left(4 \frac{\xi}{\gamma} \frac{\varepsilon_0}{\omega} \frac{v_0 \sin \theta}{1 - v_0 \cos \theta} \sin \left(\frac{\omega}{2\varepsilon_0} \omega t \right) \right), \\ \text{Re } \Pi_2 &= -\cos(2\omega t (1 - v_0 \cos \theta)) J_2 \left(4 \frac{\xi}{\gamma} \frac{\varepsilon_0}{\omega} \frac{v_0 \sin \theta}{1 - v_0 \cos \theta} \sin \left(\frac{\omega}{\varepsilon_0} \omega t \right) \right) \end{aligned}$$

and the dimensionless parameter

$$\xi = \frac{\beta b}{m} \quad (14.9)$$

was introduced. However, to find the observable quantity which is the polarization, equation (14.8) should be normalized by the probability of finding an electron at the observation point,

i.e., divide by $j^0(t)$, the zeroth component of the current density vector, defined in (12.37). Then the observable value of the polarization amounts to

$$\frac{\mathbf{s} \cdot \boldsymbol{\nu}}{j^0} = \frac{1 + \xi^2 f - 2\xi g}{1 + \xi^2 f_1 - 2\xi g}, \quad (14.10)$$

where

$$f = \frac{(v_0 - \cos \theta) \cos \theta}{(1 - v_0 \cos \theta)^2 \gamma^2} (1 + \operatorname{Re} \Pi_2), \quad g = \frac{v_0 \sin \theta \operatorname{Re} \Pi_1}{(1 - v_0 \cos \theta) \gamma}, \quad f_1 = \frac{(1 + \operatorname{Re} \Pi_2)}{(1 - v_0 \cos \theta) \gamma^2}.$$

Equation (14.10) describes the dependence of the electron polarization on the interaction time t . Using equations (14.2), (14.10), and the condition $p_1 + p_{-1} = 1$, the probabilities of finding the electron in the transmitted (non scattered) beam in various polarization states can be calculated as follows

$$p_1 = \frac{1}{2} + \frac{1}{2} \frac{1 + \xi^2 f - 2\xi g}{1 + \xi^2 f_1 - 2\xi g}, \quad p_{-1} = \frac{1}{2} - \frac{1}{2} \frac{1 + \xi^2 f - 2\xi g}{1 + \xi^2 f_1 - 2\xi g}. \quad (14.11)$$

These probabilities should be compared with the Volkov probabilities written in the same variables,

$$p_{v,1} = \frac{1}{2} + \frac{1}{2} \frac{1 + \xi^2 f_v}{1 + \xi^2 f_{1v}}, \quad p_{v,-1} = \frac{1}{2} - \frac{1}{2} \frac{1 + \xi^2 f_v}{1 + \xi^2 f_{1v}}, \quad (14.12)$$

where $f_v = (v_0 - \cos \theta) \cos \theta / ((1 - v_0 \cos \theta)^2 \gamma^2)$ and $f_{1v} = 1 / ((1 - v_0 \cos \theta) \gamma^2)$.

One can see that the probabilities depend on time in the quantum case, unlike those of Volkov's solution. This means that the quantum fluctuations can change the system dynamics similarly to what takes place for the two level atom.

Let us find the parameter which governs the slow oscillations in the time evolution in (14.8). For this purpose, we investigate the system evolution for small times t . In this case, the sine inside the Bessel functions can be expanded in its Taylor series and one finds that the amplitude of the quantum fluctuations is defined by the parameter ξ that corresponds to the parameter x , which was introduced by Ritus in his work [14].

15 Collapse-and-revival dynamics of an electron spin

Modern lasers can reach nowadays high intensities [5–7, 9, 10] up to 10^{22} W/cm² with a pulse duration of about 30 fs. For our concrete analysis, we choose the intensity $I = 10^{18}$ W/cm², and the photon frequency $\omega = 7.8 \cdot 10^4$ cm⁻¹, i.e. a corresponding wavelength of 800 nm. As was already mentioned, the interaction time can be adapted either via the electron's energy or by changing the electron's path in the light pulse such as e.g. by varying the entrance angle θ . Figure 15.1 displays the probability of finding the particles with the flipped polarization in the electron beam as a function of the equal entrance and exit angles θ between \mathbf{p}_0 and \mathbf{k} .

As can be seen from the graphs, when the interaction time increases, corresponding to larger angles, characteristic structures of the probabilities appear. In addition to fast oscillations at the frequency of field ω , there are slow oscillations governed by the parameter ξ . These oscillations occur around the mean value, which corresponds to the quasi-classical Volkov case. It should be

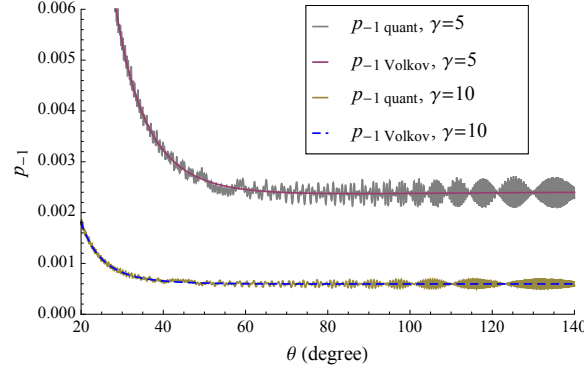


Figure 15.1 – The probability of finding the electron with flipped polarization as a function of the angle θ for an intensity $I = 10^{18} \text{ W/cm}^2$, a frequency of the photon $\omega = 7.8 \cdot 10^4 \text{ cm}^{-1}$ (corresponding to wavelength of 800 nm), an initial probability $p_{-1} = 0$, and γ values of the electron equal to 5 and 10.

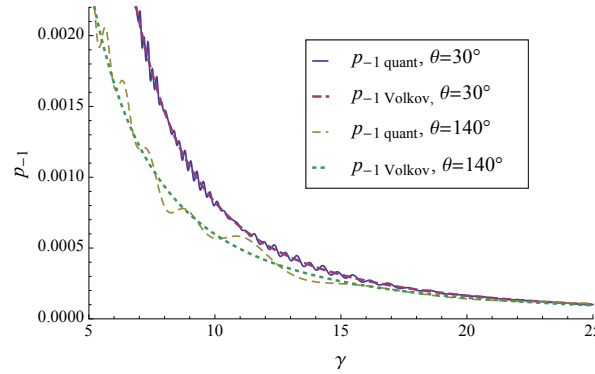


Figure 15.2 – The probability of finding an electron with flipped polarization as a function of γ for an intensity $I = 10^{18} \text{ W/cm}^2$, a frequency of the photon $\omega = 7.8 \cdot 10^4 \text{ cm}^{-1}$ (corresponding to wavelength of 800 nm), an initial probability $p_{-1} = 0$, and values of the entrance angle θ equal to 30 and 140 degrees.

noted that this special structure appears only when the field is considered as a quantum object: it can not appear in the quasi-classical case.

Figure 15.2 shows the dependence of the probability of finding an electron with flipped polarization as a function of the electron energy for the two values of fly-in angle θ equal to 30 and 140 degrees. As in the previous case, there are oscillations which appear due to the quantum nature of the electromagnetic field. The average of these oscillations represents the quasi-classical Volkov probability. Figure 15.3 shows the dependence of the probability of finding an electron with a flipped polarization as a function of the dimensionless parameter ξ , with the entrance angle θ equal to 140 degrees and two values of the gamma factor of the electron equal to 5 and 10. As in Fig. 15.1 and 15.2, the probability oscillates near the quasi-classical Volkov value. Since probability p_{-1} is inversely proportional to γ , the spin flip will have larger values for less relativistic electrons. However, p_{-1} is also proportional to ξ and the interaction time T_{int} decreases for large ξ and γ , such that there is trade-off of the various parameters.

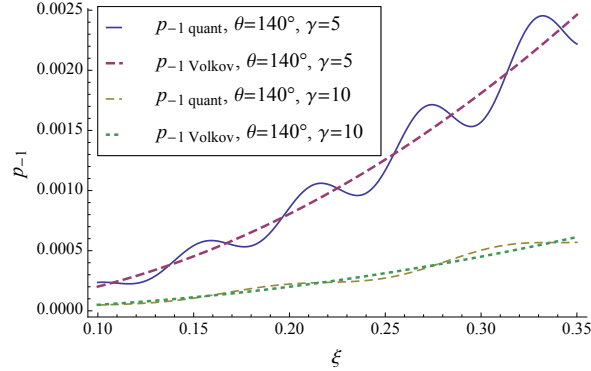


Figure 15.3 – The probability of finding an electron with flipped polarization as a function of ξ for the entrance angle θ equal to 140 degrees, a frequency of the photon $\omega = 7.8 \cdot 10^4 \text{ cm}^{-1}$ (corresponding to wavelength of 800 nm), an initial probability $p_{-1} = 0$, and values of the γ factor of the electron equal to 5 and 10. The parameter ξ equal to 0.1 corresponds to the intensity $8.4 \cdot 10^{16} \text{ W/cm}^2$ and ξ equal to 0.35 to the intensity $1.0 \cdot 10^{18} \text{ W/cm}^2$.

In real experiments an electron beam involves a spread in both initial energy and direction. We ensured that an uncertainty of one percent in energy and in the angular distribution does not change the displayed probabilities in above figures visibly. We emphasize from the order of magnitude in the probabilities in Figs. 15.1 & 15.2 that the number of electrons should be well above 1000 and that mutual interactions shall be avoided with appropriate densities.

Note that the atomic two-level system possesses an analogous behavior. In this case, the level population is considered instead of the electron polarization. The fast oscillations in the population inversion essentially depend on the Rabi frequency while for an electron in a quantized field, it can be linked to the oscillations at the frequency of the electromagnetic field ω . The structure of the slow oscillations is reminiscent of the “collapse–revival” effect for inverted populations, according to which the population inversion of the two-level system vanishes but after some time revives again. This effect is purely quantum mechanical and can not be found in a quasi-classical analysis.

At the end we want to address the influence of the Compton effect on the collapse-revival effect, because there always is low energy photon emission in which the emerging electron is measured as having unchanged momentum. For that reason the “momentum unchanged” channel will be contaminated by Compton electrons. The number of such electrons, estimated via Klein-Nishina formula [2] for the employed parameters, does not exceed 10^{-6} of the total number of electrons in the beam.

Chapter III

Spin-dependent Compton scattering in an intense classical electromagnetic field

16 Furry picture in quantum electrodynamics

In the second chapter we assumed that the single-mode approximation is valid and applied the exact solution of the Dirac equation to study the collapse-revival dynamics of an electron in the field of a quantized plane electromagnetic wave in a non-perturbative way. We have found important peculiarities of the evolution of the electron spin, which have a special structure: fast oscillations at the frequency of the field ω , and slow oscillations that correspond to the collapse–revival effect. The slow oscillations are governed by the invariant parameter ξ . This special structure appears due to the quantum nature of the electromagnetic field. We proposed the possibility of the observation of the collapses and revivals for the field intensity $I = 10^{18} \text{ W/cm}^2$ which corresponds to the parameter $\xi \sim 0.35$. In this section, however we will neglect the quantum fluctuations of the external field and investigate the limit $\xi \gg 1$, with the description of QED processes in Furry representation [12]. In what follows we will revise this approach.

When electrons, positrons and photons interact with each other, the stationary states of this system change, moreover the number of particles changes as well. Consequently, this brings us to the secondary-quantization of the fields. This means, that the wave function of the system does not only depend on the coordinates and time, but is an operator acting in a space of the occupation numbers. Then for the description of the scattering processes the “interaction” picture is used. If the Hamiltonian of the system can be introduced as a sum $H = H_0 + H_i$ of the interaction Hamiltonian H_i and the free Hamiltonian H_0 , then in the “interaction” picture the evolution of the state vectors is governed by the interaction Hamiltonian and operators (F is an arbitrary operator) are time dependent and governed by the free Hamiltonian:

$$\begin{aligned} i \frac{\partial}{\partial t} \Phi(t) &= H_i^{(i)} \Phi(t), \\ \frac{\partial}{\partial t} F &= i[H_0, F]. \end{aligned} \tag{16.1}$$

The unitary operator $\mathbf{S}(t, t_0)$, which connects operators in the Heisenberg representation and

operators in the interaction representation satisfies the same equation as $\Phi(t)$

$$\begin{aligned} i \frac{\partial}{\partial t} \mathbf{S}(t, t_0) &= H_i^{(i)} \mathbf{S}(t, t_0), \quad \mathbf{S}(t_0, t_0) = 1, \\ \Phi(t) &= \mathbf{S}(t, t_1) \Phi(t_1). \end{aligned} \quad (16.2)$$

We call the operator $\mathbf{S}(t, t_0)$ the evolution operator or S-matrix. The interaction operator in QED is $H_i = - \int N(j^\mu A_\mu) dx$, where $j^\mu = ie/2[\bar{\psi}(x), \gamma^\mu \psi(x)]$ is the electron-positron current density, N is the normal ordering operator and A^μ is the operator of an electromagnetic field.

After the evolution operator is specified, we suppose that the adiabatic hypothesis is valid, i.e., it is considered that the interaction between the fields is slowly turned off at $t = \pm\infty$, which corresponds to that the state vector of the system in the interaction representation coincides with the state vector in the Heisenberg representation

$$\Phi(-\infty) = \Phi_i.$$

The state vector $\Phi(+\infty) = \mathbf{S}(+\infty, -\infty)\Phi(-\infty)$ can be decomposed as a superposition of free field states

$$\Phi(+\infty) = \sum_f a_f \Phi_f, \quad (16.3)$$

This gives a possibility to find the probability of a transition of the system of electrons, photons and positrons from the state i to state f :

$$a_f = (\Phi_f, \mathbf{S}\Phi_i) \equiv \langle f | \mathbf{S} | i \rangle. \quad (16.4)$$

By obtaining equation (16.4) we considered that the interaction can be switched off adiabatically. This means, that the interaction Hamiltonian

$$H_i = 0 \quad \text{when } t = \pm\infty. \quad (16.5)$$

However, independently on the distance between the interacting particles we can not consider the condition (16.5) to be fulfilled, as the interaction always exist between the electron-positron and the vacuum electromagnetic fields. The only formal possibility to vanish H_i is to set the charge of an electron to zero. At the same time, the change of e will lead to the change of the electron mass, which is the energy of the ground state of the interacting fields. We have no reason to consider that this mass is equal to the mass of an electron which does not interact with an electromagnetic field (“bare” electron). Therefore, the condition (16.5) coincide to the transition from real physical particles to the ideal “bare” particles, with a different mass spectrum. In order to overcome this difficulty the renormalization procedure of an electron charge and a mass need to be introduced [2].

Concluding, the application of the S-matrix formalism (16.4) to the calculation of the quantum electrodynamical processes is based on the secondary-quantization of the fields, on the “interaction” representation of state vectors and operators, on the applicability of the adiabatic hypothesis and the renormalization procedure.

Equation (16.2) is an operator equation for the determination of the transition operator \mathbf{S} and can not be solved in the general case. However, the current density operator j^μ is proportional

to the electron charge, which plays a role of the interaction constant between the fields. This constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.0388}$$

is a small value, and consequently we can try to find a solution in a form of a series in α [2]:

$$\mathbf{S}(t, t_0) = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n T(H_i(t_1)H_i(t_2) \dots H_i(t_n)), \quad (16.6)$$

or

$$\mathbf{S}(t, t_0) = T \left(\exp \left(-i \int_{t_0}^t H_i(t') dt' \right) \right), \quad (16.7)$$

where T is a time ordering operator. The time ordering operator places operators by increasing time, for example

$$T(H_i(t_1)H_i(t_2)) = \begin{cases} H_i(t_1)H_i(t_2), & t_2 < t_1 \\ H_i(t_2)H_i(t_1), & t_1 < t_2 \end{cases}. \quad (16.8)$$

Let us introduce the N-product of operators. The action of N on the product of creation and annihilation operators results in the creation operators are always on the left side from the annihilation operators. For example:

$$N(a_1 a_2^\dagger a_3^\dagger a_4) = \delta a_2^\dagger a_3^\dagger a_1 a_4,$$

where δ is equal to $(-1)^n$, with n being a permutation power for the Fermi operators and 1 for the Bose operators.

The time ordering can be decomposed using Wick theorem [130]:

Wick theorem 1. A T-ordering product of the field operators is equal to the sum of N-products where operators are contracted in all possible ways:

$$\begin{aligned} T(a_1 a_2 a_3 \dots a_n) &= N(a_1 a_2 \dots a_n) + N(a_1^\bullet a_2^\bullet a_3 \dots a_n) + N(a_1^\bullet a_2 a_3^\bullet \dots a_n) + \dots \\ &+ N(a_1^\bullet a_2 a_3 \dots a_n^\bullet). \end{aligned} \quad (16.9)$$

Wick theorem 2. A mixed T-ordering is equal to the sum of N-products, in which operators are contracted in all possible ways, except of contractions inside the same N-product:

$$\begin{aligned} T(N(a_1 a_2) N(a_3 a_4) \dots N(a_{n-1} a_n)) &= N(a_1 a_2 a_3 \dots a_n) + N(a_1^\bullet a_2^\bullet a_3 a_4 \dots a_{n-1} a_n) \\ &+ N(a_1^\bullet a_2 a_3 a_4^\bullet \dots a_{n-1} a_n) + \dots + N(a_1^\bullet a_2 a_3 a_4 \dots a_{n-1} a_n^\bullet). \end{aligned} \quad (16.10)$$

The contractions $a_1^\bullet a_2^\bullet$, $a_3^\bullet a_4^\bullet$, ..., $a_{n-1}^\bullet a_n^\bullet$ are not in the sum (16.10).

The contractions between field operators are defined in the following way

$$\begin{aligned}
A_\mu^\bullet(x)A_\nu^\bullet(x') &= g_{\mu\nu}D(x-x'), \\
\Psi_\alpha^\bullet(x)\Psi_\beta^\bullet(x') &= 0, \\
\bar{\Psi}_\alpha^\bullet(x)\bar{\Psi}_\beta^\bullet(x') &= 0, \\
\Psi_\alpha^\bullet(x)\bar{\Psi}_\beta^\bullet(x') &= S_{\alpha\beta}(x-x'), \\
\bar{\Psi}_\beta^\bullet(x)\Psi_\alpha^\bullet(x') &= -S_{\alpha\beta}(x-x'),
\end{aligned} \tag{16.11}$$

where

$$D(x) = \frac{-i}{(2\pi)^4} \int \frac{e^{i(k \cdot x)}}{k^2 - i0} d^4k \text{ and } S_{\alpha\beta} = \frac{-i}{(2\pi)^4} \int \frac{\hat{p} + m}{p^2 + m^2 - i0} e^{i(p \cdot x)} d^4p \tag{16.12}$$

and α, β are the spinor indices.

The series for the S-matrix is a product of different field operators in the “interaction” picture taken in a spacial points x_1, x_2, \dots, x_n . Therefore the matrix elements of these operators need to be calculated. In order to do so, we mention, that the field operators in the “interaction” picture satisfy the same equation of motion as in the case of free fields and can be decomposed in series

$$\begin{aligned}
A^\mu(x) &= \sum_{k,\lambda} \frac{1}{\sqrt{2V\omega}} \left(e_\lambda^\mu \mathbf{c}_{k\lambda} e^{-i(k \cdot x)} + e_\lambda^{*\mu} \mathbf{c}_{k\lambda}^\dagger e^{i(k \cdot x)} \right), \\
\Psi(x) &= \sum_{p,\mu} \frac{1}{\sqrt{2V\epsilon}} \left(u^\mu(p) \mathbf{a}_{p\mu} e^{-i(p \cdot x)} + \bar{u}^\mu(-p) \mathbf{b}_{p\mu}^\dagger e^{i(p \cdot x)} \right), \\
\bar{\Psi}(x) &= \sum_{p,\mu} \frac{1}{\sqrt{2V\epsilon}} \left(\bar{u}^\mu(p) \mathbf{a}_{p\mu}^\dagger e^{i(p \cdot x)} + u^\mu(-p) \mathbf{b}_{p\mu} e^{-i(p \cdot x)} \right),
\end{aligned} \tag{16.13}$$

where $\mathbf{c}_{k\lambda}^\dagger$ is the creation and $\mathbf{c}_{k\lambda}$ is the annihilation operators of a photon with the wave vector k and polarization λ ; analogously operators $\mathbf{a}_{p\mu}, \mathbf{a}_{p\mu}^\dagger$ are the annihilation and creation operators of an electron with the momentum p and helicity μ , and, at last, $\mathbf{b}_{p\mu}, \mathbf{b}_{p\mu}^\dagger$ are the annihilation and creation operators of the positron with the momentum p and helicity μ .

From here, it is not difficult to find the non-vanishing matrix elements

$$\begin{aligned}
\langle 0_{k\lambda} | A^\mu(x) | 1_{k\lambda} \rangle &= \frac{1}{\sqrt{2V\omega}} e_\lambda^\mu e^{-i(k \cdot x)}, & \langle 1_{k\lambda} | A^\mu(x) | 0_{k\lambda} \rangle &= \frac{1}{\sqrt{2V\omega}} e_\lambda^{*\mu} e^{i(k \cdot x)}, \\
\langle 0_{p\mu}^+ | \Psi(x) | 1_{p\mu}^+ \rangle &= \frac{1}{\sqrt{2V\epsilon}} u^\mu(p) e^{-i(p \cdot x)}, & \langle 1_{p\mu}^+ | \bar{\Psi}(x) | 0_{p\mu}^+ \rangle &= \frac{1}{\sqrt{2V\epsilon}} \bar{u}^\mu(p) e^{i(p \cdot x)}, \\
\langle 0_{p\mu}^- | \bar{\Psi}(x) | 1_{p\mu}^- \rangle &= \frac{1}{\sqrt{2V\epsilon}} \bar{u}^\mu(-p) e^{-i(p \cdot x)}, & \langle 1_{p\mu}^- | \Psi(x) | 0_{p\mu}^- \rangle &= \frac{1}{\sqrt{2V\epsilon}} u^\mu(-p) e^{i(p \cdot x)}.
\end{aligned} \tag{16.14}$$

Equation (16.6), (16.11) and (16.14) are the basis of the perturbation theory in QED, which is relativistically invariant.

Above we considered that when t tends to $\pm\infty$ particles become free. However, the S-matrix formalism gives possibility to solve problems when particles are moving in an external electromagnetic field (Furry representation) and interact with a secondary-quantized electromagnetic field.

Let the particles move in the external field $A_\mu^{(e)}$. Then the secondary-quantized wave functions $\psi(x)$, $\bar{\psi}(x)$ of an electron and a positron can be decomposed in a series over exact solutions of the Dirac equation in the external field $A_\mu^{(e)}$:

$$\begin{aligned}\psi(x) &= \sum_s \mathbf{a}_s \psi_s^{(+)}(x) + \sum_r \mathbf{b}_r^\dagger \psi_r^{(-)}(x), \\ \bar{\psi}(x) &= \sum_s \mathbf{a}_s^\dagger \bar{\psi}_s^{(+)}(x) + \sum_r \mathbf{b}_r \bar{\psi}_r^{(-)}(x),\end{aligned}\tag{16.15}$$

where indices $+$ and $-$ denote states with positive and negative energies respectively.

The matrix elements in this case change and look like

$$\begin{aligned}\langle 0_s^+ | \psi(x) | 1_s^+ \rangle &= \psi_s^{(+)}(x), & \langle 1_s^+ | \bar{\psi}(x) | 0_s^+ \rangle &= \bar{\psi}_s^{(+)}(x), \\ \langle 0_r^- | \bar{\psi}(x) | 1_r^- \rangle &= \bar{\psi}_r^{(-)}(x), & \langle 1_r^- | \bar{\psi}(x) | 0_r^- \rangle &= \psi_r^{(-)}(x).\end{aligned}\tag{16.16}$$

An external field is weak in many cases and can be considered via perturbation theory, through the inclusion in the potential $A'_\mu(x) = A_\mu(x) + A^{(e)}(x)$. For this reason the electron's current $j^\mu(x)$ will interact also with the external field and we can still use the S-matrix formalism. In this case the decomposition of the secondary-quantized wave functions $\psi(x)$ and $\bar{\psi}(x)$ is build over the eigenfunctions of a free particle.

There is also an intermediate case, when a part of an external field is considered as a strong and is included through the exact solution of the Dirac equation and another weak part can be considered as a small perturbation. Then as in a previous case the decomposition (16.15) for the electron-positron wave functions is used and in addition a vector potential becomes the sum $A'_\mu(x) = A_\mu(x) + A^{(e)}(x)$.

17 Orthogonality of Volkov solutions

The orthogonality of Volkov's solutions is important for the construction of the perturbation theory. Following [15], we will represent here an explicit prove as some papers [131, 132] contain the very strange statement that the Volkov solutions are not orthogonal and consequently the conclusions that the Hamiltonian is non-Hermitian.

In order to do so, we chose the coordinate system with the z -axis be directed along \mathbf{k} . In this coordinate system four vectors of the problem are

$$\begin{aligned}k &= (\omega, 0, 0, \omega), & A &= (0, A_x, A_y, 0), \\ p &= (\epsilon, \mathbf{p}_\perp, p_z), & \mathbf{p}_\perp &= (p_x, p_y, 0),\end{aligned}\tag{17.1}$$

and consequently scalar products in the Volkov wave function looks like:

$$\begin{aligned}(k \cdot p) &= \omega(\epsilon - p_z) = \omega p^-, & (p \cdot A) &= -\mathbf{p}_\perp \cdot \mathbf{A}, \\ A^2 &= -\mathbf{A}^2, & (k \cdot x) &= \omega(t - z) = \omega \eta.\end{aligned}\tag{17.2}$$

By plugging (17.2) into equation (1.2) one obtains

$$\psi_{p\mu} = \left[1 + \frac{e\hat{k}\hat{A}}{2(k \cdot p)} \right] \frac{u^\mu(p)}{\sqrt{2p^0}} \exp \left\{ i \left(\mathbf{p}_\perp \cdot \mathbf{x} - p^- \frac{t+z}{2} - \int_0^\eta d\eta' \frac{(\mathbf{p}_\perp - e\mathbf{A})^2 + m^2}{2p^-} \right) \right\}. \tag{17.3}$$

As can be seen from equation (17.3), the field term in the exponent does not depend on the x and y coordinates, but depends on the t and z . For this reason, during investigation of the orthogonality condition we will be able to integrate in x, y plane, which yields two dimensional delta function $\delta(p'_x - p_x)\delta(p'_y - p_y)$:

$$\begin{aligned} \int d\mathbf{r} \psi_{p'\mu'}^\dagger(\mathbf{r}) \psi_{p\mu}(\mathbf{r}) &= (2\pi)^2 \delta(p'_x - p_x) \delta(p'_y - p_y) \\ &\times \int_{-\infty}^{\infty} dz \frac{\bar{u}^{\mu'}(p')}{\sqrt{2p'^0}} \left[1 + \frac{e\hat{A}\hat{k}}{2\omega p'^-} \right] \gamma^0 \left[1 + \frac{e\hat{k}\hat{A}}{2\omega p^-} \right] \frac{u^\mu(p)}{\sqrt{2p^0}} \\ &\times \exp \left\{ -i(p'^- - p^-) \frac{t+z}{2} - i \frac{p'^- - p^-}{2p'^- p^-} \int_0^\eta d\eta' \left((\mathbf{p}_\perp - e\mathbf{A})^2 + m^2 \right) \right\}. \end{aligned} \quad (17.4)$$

Next we make a variable change

$$\begin{aligned} u &= z + \frac{1}{p'^- p^-} \int_0^\eta d\eta' \left((\mathbf{p}_\perp - e\mathbf{A})^2 + m^2 \right), \\ dz &= \frac{du}{1 - ((\mathbf{p}_\perp - e\mathbf{A})^2 + m^2) / (p'^- p^-)}, \end{aligned} \quad (17.5)$$

and equation (17.4) transforms to

$$\begin{aligned} (2\pi)^2 \delta(p'_x - p_x) \delta(p'_y - p_y) &\int_{-\infty}^{\infty} du \frac{\bar{u}^{\mu'}(p')}{\sqrt{2p'^0}} \left[1 + \frac{e\hat{A}\hat{k}}{2\omega p'^-} \right] \gamma^0 \left[1 + \frac{e\hat{k}\hat{A}}{2\omega p^-} \right] \frac{u^\mu(p)}{\sqrt{2p^0}} \\ &\times \frac{\exp \{ -i(p'^- - p^-)(t+u)/2 \}}{1 - ((\mathbf{p}_\perp - e\mathbf{A})^2 + m^2) / (p'^- p^-)}. \end{aligned} \quad (17.6)$$

Then with a very lengthy and cumbersome calculation it is possible to show that the average value from spinors exactly cancels the denominator of the equation (17.6). Consequently, we can integrate over u , which yields $\delta(p'^- - p^-)$. For this reason

$$\int d\mathbf{r} \psi_{p'\mu'}^\dagger(\mathbf{r}) \psi_{p\mu}(\mathbf{r}) = (2\pi)^3 \delta(p'_x - p_x) \delta(p'_y - p_y) \delta(p'^- - p^-) \frac{p^-}{p^0} u^{\mu'\dagger}(p) u^\mu(p) \delta_{\mu'\mu}. \quad (17.7)$$

18 Characteristic parameters of an electron and a field system

In this section we will revise one more time the characteristic parameters of strong field QED. The field strength is characterized through the invariant parameter [15]

$$\xi = \frac{eA}{m}, \quad (18.1)$$

where A is the amplitude of the potential of an electromagnetic field. This parameter does not contain Planck's constant and is equal to the work referred to m , performed by the field over the wavelength. It can also be represented as the ratio of the field work over the Compton wavelength to the energy ω of the field quantum.

In the limit of small ξ the most probable processes, are those where only few photons participate. For example, standard (linear) Compton effect includes only two electrons and two photons. From a mathematical perspective, the decomposition of a secondary-quantized wave functions ψ and $\bar{\psi}$ over the plane wave states takes place, and all the matrix elements are calculated using equations (16.14). The first correction to the linear case is proportional to ξ^2 . In the limit $\xi \ll 1$ a strong external electromagnetic field is reduced to a single photon.

When ξ increases, the probability to absorb more than one photon from the external field becomes larger and the process turns out to be multi-quantum. Equation (16.14) must be replaced by equation (16.16), transition amplitudes which are connected with the matrix elements (16.16) have non-linear dependence on the field. Therefore we can state, that the parameter ξ defines non-linearity.

The second parameter

$$\chi = \frac{e \sqrt{-F_{\mu\nu} F^{\mu\delta} p^\nu p_\delta}}{m^3}, \quad (18.2)$$

contains the Planck's constant \hbar and is equal to the work, in the units of m , performed by the field over the Compton wavelength in the particle rest system or equivalently $\chi = E/E_{\text{cr}}$ in a particle rest frame, where the critical field $E_{\text{cr}} = m^2/e$. This field performs a work m over a Compton wavelength $\lambda_c = 1/m$. The parameter χ is responsible for the magnitude of the quantum effects. In the non-linear case, i.e., when $\xi \gg 1$ the quantum effects are the largest when $\chi \sim 1$, while in the linear case they are optimal when $\chi \sim \xi$. We note here, that in the head-on collision of a bunch of relativistic electrons with an intense laser plane wave, the parameter χ can be represented as

$$\chi = 2\xi \frac{\omega}{m} \gamma, \quad (18.3)$$

where γ is the electron gamma factor, $\gamma = \epsilon/m$ and ω is the wave frequency.

Whenever the scattering processes in the presence of a strong laser field are analyzed, the validity of the single-mode approximation must be verified [7], i.e. we have conditions on the pulse duration, focusing and the field intensity.

19 Non-linear Compton effect transition amplitude in a short laser pulse

In the first chapter we investigated the exact solution of the Dirac equation in the field of a plane wave. In [16] we have shown how the secondary-quantized wave function can be expanded over these exact solutions. Consequently, we can obtain the first non-vanishing S-matrix element:

$$\begin{aligned} \mathbf{S} &= \int d^4x \mathbf{T} \left(\mathbf{N} \left(j^\mu(x) A_\mu(x) \right) \right) = ie \int d^4x \mathbf{N} \left(\bar{\psi}(x) \gamma^\mu \psi(x) A_\mu(x) \right) \\ &= ie \mathbf{N} \left[\left(\sum_s \mathbf{a}_s^\dagger \bar{\psi}_s^{(+)}(x) + \sum_r \mathbf{b}_r \bar{\psi}_r^{(-)}(x) \right) \gamma^\mu \left(\sum_s \mathbf{a}_s \psi_s^{(+)}(x) + \sum_r \mathbf{b}_r^\dagger \psi_r^{(-)}(x) \right) \right. \\ &\quad \left. \times \sum_{k\lambda} \frac{\sqrt{4\pi}}{\sqrt{2V\omega}} \left(e_{\lambda\mu} \mathbf{c}_{k\lambda} e^{-i(k \cdot x)} + e_{\lambda\mu}^* \mathbf{c}_{k\lambda}^\dagger e^{i(k \cdot x)} \right) \right]. \end{aligned} \quad (19.1)$$

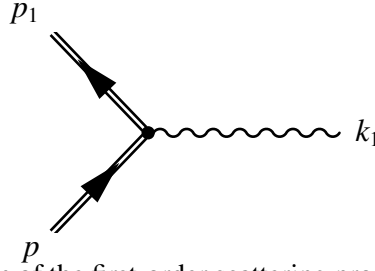


Figure 19.1 – Feynman diagram of the first-order scattering process in the field of a plane electromagnetic wave.

From equation (19.1) we conclude, that there are few possible decay processes of the first order, for example:

- An electron-positron pair creation by a photon.
- An electron-positron pair annihilation into a photon.
- An electron decay into photon and electron.
- A positron decay into photon and positron.

All these processes are described by a single diagram, which is depicted on Fig. 19.1. We pay attention, that contrary to the free case, a photon emission is possible due to the presence of the plane wave.

We are interested in the case of Compton scattering, i.e., when in the initial state was a single electron $|p\mu\rangle$ with the momentum p and helicity μ , which decayed into the final state of an electron $|p_1\mu_1\rangle$ and a photon $|k_1\lambda_1\rangle$, with the four vector k_1 and polarization λ_1 . By using formulas (16.16) for the matrix elements, we find the expression for the S-matrix

$$S_{fi} = -ie \sqrt{4\pi} \int \bar{\psi}_{\mu_1 p_1}(x) \frac{\hat{e}^* e^{i(k_1 \cdot x)}}{\sqrt{2V\omega_1}} \psi_{\mu p}(x) d^4x \quad (19.2)$$

where $\psi_{\mu p}(x)$ are the Volkov wave functions, which were defined in 1 and \hat{e}^* is the polarization vector of the emitted photon.

For the evaluation of (19.2) the model for the vector potential of the external field need to be specified. We choose the circular polarization with an envelop function:

$$A^\mu(x) = \frac{m\xi}{e} f(\phi) (\mathfrak{A}_1^\mu \cos \phi + \mathfrak{A}_2^\mu \sin \phi), \quad (19.3)$$

where ϕ is the phase of the wave $\phi = (k \cdot x)$, k is the four wave vector $k = (\omega, \mathbf{k})$, $f(\phi)$ is the envelope function, \mathfrak{A}_1 and \mathfrak{A}_2 are the constant four vectors, which satisfy the conditions

$$(\mathfrak{A}_1 \cdot \mathfrak{A}_2) = 0, \quad (\mathfrak{A}_1 \cdot \mathfrak{A}_1) = (\mathfrak{A}_2 \cdot \mathfrak{A}_2) = -1, \quad (k \cdot \mathfrak{A}_1) = (k \cdot \mathfrak{A}_2) = 0. \quad (19.4)$$

We will normalize the Volkov wave functions not on the momentum, but rather on the differences $p^- \equiv p^0 - p^3$, see 17:

$$\begin{aligned} \int d\mathbf{x} \psi_{\mu_1 p_1}^\dagger \psi_{\mu p} &= (2\pi)^3 \delta(p_1^1 - p^1) \delta(p_1^2 - p^2) \delta(p_1^- - p^-) \frac{p^-}{p^0} u^{\mu_1 \dagger}(p) u^\mu(p) \\ &= (2\pi)^3 \delta(p_1^1 - p^1) \delta(p_1^2 - p^2) \delta(p_1^3 - p^3) \theta(p_1^0 - p^0) u^{\mu_1 \dagger}(p) u^\mu(p) \end{aligned} \quad (19.5)$$

where θ is the Heaviside function. There are two sets of variables in equation (19.5), which are $x^- = x^0 - x^3$, x^1 , x^2 , x^3 and x^0 , x^1 , x^2 , x^3 . The Jacobian of this transformation is equal to 1.

The usual procedure with the replacement of δ -functions with a corresponding volume leads to

$$\begin{aligned}\delta(p_1^1 - p^1)\delta(p_1^2 - p^2)\delta(p_1^- - p^-) &\longrightarrow \frac{V}{(2\pi)^3}, \\ \delta(p_1^1 - p^1)\delta(p_1^2 - p^2)\delta(p_1^3 - p^3) &\longrightarrow \frac{V}{(2\pi)^3}.\end{aligned}\quad (19.6)$$

Consequently the normalization of the Volkov wave function in the final volume V can be represented as

$$\int_{V^-} |\psi_{\mu p}|^2 d\mathbf{x} = 2Vp^-. \quad (19.7)$$

Let us come back to the calculation of the transition matrix element (19.2). By plugging the definition (1.2) of the Volkov wave functions into equation (19.2) one obtains

$$\begin{aligned}S_{fi} = -\frac{ie\sqrt{4\pi}}{\sqrt{8V^3p_1^-p^-\omega_1}} \int \bar{u}^{\mu_1}(p_1) &\left(\hat{e}^* + \frac{e}{2(k \cdot p)} \hat{e}^* \hat{k} \hat{A} + \frac{e}{2(k \cdot p_1)} \hat{A} \hat{k} \hat{e}^* \right. \\ &\left. + \frac{e^2}{4(k \cdot p_1)(k \cdot p)} \hat{A} \hat{k} \hat{e}^* \hat{k} \hat{A} \right) u^\mu(p) e^{iS - iS' + i(k_1 \cdot x)} d\mathbf{x}.\end{aligned}\quad (19.8)$$

The evaluation of equation (19.8) is convenient to perform in the coordinate system with the z axis be directed along \mathbf{k} . In this system the scalar product $(k \cdot x) = \omega(t - z) \equiv \eta$. Consequently, we can write down the phase in (19.8) as

$$\begin{aligned}i((k_1 \cdot x) - S_1 + S) = i(\omega_1 + \epsilon_1 - \epsilon)t - i(k_{1z} + p_{1z} - p_z)z - i((\mathbf{k}_1 + \mathbf{p}_1 - \mathbf{p}) \cdot \mathbf{r})_\perp \\ + im\xi \left(\frac{(p_1 \cdot \mathfrak{A}_1)}{(k \cdot p_1)} - \frac{(p \cdot \mathfrak{A}_1)}{(k \cdot p)} \right) \int^\eta f(\varphi) \cos \varphi d\varphi \\ + im\xi \left(\frac{(p_1 \cdot \mathfrak{A}_2)}{(k \cdot p_1)} - \frac{(p \cdot \mathfrak{A}_2)}{(k \cdot p)} \right) \int^\eta f(\varphi) \sin \varphi d\varphi \\ + im^2\xi^2 \left(\frac{1}{(k \cdot p_1)} - \frac{1}{(k \cdot p)} \right) \int^\eta f^2(\varphi) d\varphi,\end{aligned}\quad (19.9)$$

where the \perp abbreviation means the scalar product in the transversal to the z axis plane, i.e., $(\mathbf{a} \cdot \mathbf{b})_\perp = a_x b_x + a_y b_y$.

The four potential of the external field depends only on the variable η , thus the integration in (19.8) over \perp direction can be performed, yielding the two dimensional delta function:

$$(2\pi)^2 \delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp). \quad (19.10)$$

The change of variables

$$\eta = \omega(t - z), \quad dz = -\frac{1}{\omega} d\eta, \quad z = t - \frac{1}{\omega} \eta \quad (19.11)$$

allows us to rewrite the transition matrix element in the following way

$$\begin{aligned}
S_{fi} = & \frac{ie\sqrt{4\pi}(2\pi)^2\delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp)}{\sqrt{8V^3p_1^-p^-}\omega_1\omega} \int dt d\eta \bar{u}^{\mu_1}(p_1) \left\{ \hat{e}^* + \frac{m\xi f(\eta)}{2(k \cdot p)} \hat{e}^* \hat{k} (\hat{\mathfrak{U}}_1 \cos \eta + \hat{\mathfrak{U}}_2 \sin \eta) \right. \\
& + \frac{m\xi f(\eta)}{2(k \cdot p_1)} (\hat{\mathfrak{U}}_1 \cos \eta + \hat{\mathfrak{U}}_2 \sin \eta) \hat{k} \hat{e}^* + \frac{m^2 \xi^2 f^2(\eta)(k \cdot e^*)}{2(k \cdot p_1)(k \cdot p)} \hat{k} \left. \right\} u^\mu(p) \\
& \times \exp \left[i(k_1^- + p_1^- - p^-)t + i \frac{k_{1z} + p_{1z} - p_z}{\omega} \eta \right. \\
& \left. + i m \xi \left(\alpha_1 \int^\eta f(\varphi) \cos \varphi d\varphi + \alpha_2 \int^\eta f(\varphi) \sin \varphi d\varphi \right) + i m^2 \xi^2 \beta \int^\eta f^2(\varphi) d\varphi \right],
\end{aligned} \tag{19.12}$$

where

$$\begin{aligned}
\alpha_1 &= \left(\frac{(p_1 \cdot \mathfrak{U}_1)}{(k \cdot p_1)} - \frac{(p \cdot \mathfrak{U}_1)}{(k \cdot p)} \right), \quad \alpha_2 = \left(\frac{(p_1 \cdot \mathfrak{U}_2)}{(k \cdot p_1)} - \frac{(p \cdot \mathfrak{U}_2)}{(k \cdot p)} \right), \\
\beta &= \left(\frac{1}{(k \cdot p_1)} - \frac{1}{(k \cdot p)} \right).
\end{aligned}$$

As follows from equation (19.12) the integration over t can be performed, which gives the delta function

$$(2\pi)\delta(k_1^- + p_1^- - p^-).$$

We now define a set of integrals

$$\begin{aligned}
f_0 &= \int d\eta \exp[i\Phi(\eta)], & f_{11} &= \int d\eta f(\eta) \cos \eta \exp[i\Phi(\eta)], \\
f_{21} &= \int d\eta f(\eta) \sin \eta \exp[i\Phi(\eta)], & f_{12} &= \int d\eta f^2(\eta) \cos^2 \eta \exp[i\Phi(\eta)], \\
f_{22} &= \int d\eta f^2(\eta) \sin^2 \eta \exp[i\Phi(\eta)],
\end{aligned} \tag{19.13}$$

where

$$f_{12} + f_{22} = \int d\eta f^2(\eta) \exp[i\Phi(\eta)]$$

and

$$\begin{aligned}
\Phi(\eta) &= \frac{k_{1z} + p_{1z} - p_z}{\omega} \eta + m\xi \left(\alpha_1 \int^\eta f(\varphi) \cos \varphi d\varphi + \alpha_2 \int^\eta f(\varphi) \sin \varphi d\varphi \right) \\
&+ m^2 \xi^2 \beta \int^\eta f^2(\varphi) (\sin^2 \varphi + \cos^2 \varphi) d\varphi.
\end{aligned} \tag{19.14}$$

The use of these notations gives the compact form of the transition matrix element

$$\begin{aligned}
S_{fi} = & \frac{ie\sqrt{4\pi}(2\pi)^3\delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp)\delta(k_1^- + p_1^- - p^-)}{\sqrt{8V^3p_1^-p^-}\omega_1\omega} \bar{u}^{\mu_1}(p_1) \left\{ \hat{e}^* f_0 \right. \\
& + \frac{m\xi}{2(k \cdot p)} \hat{e}^* \hat{k} (\hat{\mathfrak{U}}_1 f_{11} + \hat{\mathfrak{U}}_2 f_{21}) + \frac{m\xi}{2(k \cdot p_1)} (\hat{\mathfrak{U}}_1 f_{11} + \hat{\mathfrak{U}}_2 f_{21}) \hat{k} \hat{e}^* \\
& \left. + \frac{m^2 \xi^2 (f_{12} + f_{22})(k \cdot e^*)}{2(k \cdot p_1)(k \cdot p)} \hat{k} \right\} u^\mu(p)
\end{aligned} \tag{19.15}$$

20 The role of the adiabatic switch off of an electromagnetic field

Let us come to the analysis of the integrals which are defined by equation (19.13). These integrals can be divided into two groups, namely f_0 and the rest. The shape function $f(\eta)$ vanishes when $\eta \rightarrow \pm\infty$, consequently the integrals of the second group are well convergent when $\eta \rightarrow \pm\infty$. We show that if the field is adiabatically switched off and on, the integral f_0 can be expressed as a liner combination of the $f_{11}, f_{12}, f_{21}, f_{22}$.

The application of the adiabatic hypothesis to the integral corresponds to the following mathematical procedure

$$\begin{aligned} \int_{-\infty}^{\infty} d\Phi(\eta) e^{i\Phi(\eta)} &= \int_{-\infty}^0 d\Phi(\eta) e^{i\Phi(\eta)} + \int_0^{\infty} d\Phi(\eta) e^{i\Phi(\eta)} \\ &= \int_{-\infty}^0 d\Phi(\eta) e^{(i+\lambda)\Phi(\eta)} + \int_0^{\infty} d\Phi(\eta) e^{(i-\lambda)\Phi(\eta)} \\ &= \frac{e^{(i+\lambda)\Phi(\eta)}}{i+\lambda} \Big|_{\Phi(\eta)=-\infty}^{\Phi(\eta)=0} + \frac{e^{(i-\lambda)\Phi(\eta)}}{i-\lambda} \Big|_{\Phi(\eta)=0}^{\Phi(\eta)=\infty}, \end{aligned} \quad (20.1)$$

where λ is a small parameter. Consequently, the regularized right hand side of equation (20.1) is equal to zero.

On the other hand, this integral can be written as

$$\int_{-\infty}^{\infty} d\Phi(\eta) e^{i\Phi(\eta)} = \int_{-\infty}^{\infty} d\eta \frac{d\Phi}{d\eta} e^{i\Phi(\eta)}. \quad (20.2)$$

The calculation of the phase derivative yields

$$\frac{d\Phi(\eta)}{d\eta} = \frac{k_{1z} + p_{1z} - p_z}{\omega} + m\xi(\alpha_1 f(\eta) \cos \eta + \alpha_2 f(\eta) \sin \eta) + m^2 \xi^2 \beta f^2(\eta). \quad (20.3)$$

By plugging equation (20.3) into equation (20.2) one obtains

$$\frac{k_{1z} + p_{1z} - p_z}{\omega} f_0 + m\xi(\alpha_1 f_{11} + \alpha_2 f_{21}) + m^2 \xi^2 \beta(f_{12} + f_{22}) = 0. \quad (20.4)$$

Consequently, f_0 is expressed as a linear combination of the other functions

$$f_0 = -\omega \frac{m\xi(\alpha_1 f_{11} + \alpha_2 f_{21}) + m^2 \xi^2 \beta(f_{12} + f_{22})}{k_{1z} + p_{1z} - p_z}. \quad (20.5)$$

The adiabatic hypothesis plays a very important role, as it gives the possibility to introduce the regularization procedure. Moreover, when the field is switched off, the Volkov solution must coincide with the solution of the free Dirac equation. From here, we can immediately conclude, that the normalization of the wave functions does not change, and Volkov solutions are orthogonal, leading to the correct construction of the perturbation theory (The strict proof is given in 17).

The physical meaning of this result corresponds to the situation that the interaction always takes place in the finite volume of space.

21 The decay rate of the first-order process in the field of a plane electromagnetic wave

In the previous paragraph we have obtained the transition matrix element for the non-linear Compton effect and have shown that it is proportional to the three delta functions which define the process kinematics. For the arbitrary first-order decay process it can be written in the general form as

$$S_{fi} = ie \sqrt{4\pi} \frac{(2\pi)^3 \delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp) \delta(k_1^- + p_1^- - p^-)}{\sqrt{8V^3 p_1^- p^- \omega_1 \omega}} M_{fi}, \quad (21.1)$$

where M_{fi} is called the amplitude. The probability of the process $W_{i \rightarrow f}$ is proportional to the absolute value square of S_{fi} and consequently proportional to the square of delta functions. We can replace one of these delta functions by

$$\delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp) \delta(k_1^- + p_1^- - p^-) \rightarrow \frac{V}{(2\pi)^3}, \quad (21.2)$$

thus obtaining

$$|S_{fi}|^2 = \frac{e^2 (2\pi)^3 \delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp) \delta(k_1^- + p_1^- - p^-)}{2V^2 p_1^- p^- \omega_1 \omega^2} |M_{fi}|^2 \quad (21.3)$$

In the case of the scattering both final and initial states belong to the continuous spectrum and the interest is to find the probabilities of the final particles being in the range $d^3 p_1$ and $d^3 k_1$. Consequently, we need to multiply $|S_{fi}|^2$ by the number of states $d^3 k_1 V / (2\pi)^3$, $d^3 p_1 V / (2\pi)^3$ in the intervals $d^3 p_1$ and $d^3 k_1$ respectively. This yields the following expression for the probability $dW_{i \rightarrow f}$:

$$dW_{i \rightarrow f} = \frac{e^2 d^3 p_1 d^3 k_1}{16\pi^2 p_1^- p^- \omega_1 \omega^2} \delta(\mathbf{k}_1^\perp + \mathbf{p}_1^\perp - \mathbf{p}^\perp) \delta(k_1^- + p_1^- - p^-) |M_{fi}|^2. \quad (21.4)$$

When the scattering is considered in the absence of the external field the transition probability is proportional to the four dimensional delta function $\delta^{(4)}(\sum p_i - \sum p_f)$, and consequently its square is proportional to the finite interaction time T . In our case, the fourth delta function can not be selected from the amplitude, therefore the finite time dependence is included in $|M_{fi}|^2$.

We pay attention to the fact that the differential $d^3 p_1$ of the final fermion is written in the new variables x^- , x^1 , x^2 , x^3 . This gives the possibility to consider p^0 and p^3 as the independent variables when integrating the probability over the final states.

Integrating over one of the momenta in (21.4) removes delta function. We will distinguish two cases. In the first case the final photons are tracked, therefore the integration over $d^3 p_1$ need to be performed. The second case corresponds to the integration over $d^3 k_1$ and to the tracking of final fermion respectively. This yields the connection between variables of the final particles

$$\begin{cases} p_{1x} = p_x - k_{1x} \\ p_{1y} = p_y - k_{1y} \\ p_{1z} = \omega_1 + \epsilon_1 - \epsilon - k_{1z} + p_z \end{cases}, \quad \begin{cases} k_{1x} = p_x - p_{1x} \\ k_{1y} = p_y - p_{1y} \\ k_{1z} = \omega_1 + \epsilon_1 - \epsilon - p_{1z} + p_z \end{cases}. \quad (21.5)$$

We have shown how the probability of the first-order process can be expressed through an amplitude. However, absolute value square of the amplitude has not been evaluated. In what follows we show how this calculation can be performed.

The amplitude of the process is expressed in the general form as

$$M_{fi} = \bar{u}_f Q u_i, \quad (21.6)$$

where indices f and i denote a final and an initial states respectively and consequently the square of the amplitude reads

$$|M_{fi}|^2 = (\bar{u}_f Q u_i)(\bar{u}_i \bar{Q} u_f) = u_{i\alpha} \bar{u}_{i\alpha'} \bar{Q}_{\alpha'\beta} u_{f\beta} \bar{u}_{f\beta'}. \quad (21.7)$$

Here $\bar{Q} = \gamma^0 Q^\dagger \gamma^0$, α and β are spinor indices. The electron's polarization is characterized via spin four vector defined in equation (10.3) with the corresponding density matrix (10.2):

$$\begin{aligned} \rho &= u(p) \otimes \bar{u}(p) = \frac{1}{2}(\hat{p} + m)(1 - \gamma^5 \hat{a}), \\ a &= \zeta + \frac{\mathbf{p}(\mathbf{p} \cdot \zeta)}{m(\varepsilon + m)}, \quad a^0 = \frac{\mathbf{p} \cdot \zeta}{m}. \end{aligned} \quad (21.8)$$

Therefore, we can replace $u_{i\alpha} \bar{u}_{i\alpha'}$ with the density matrix $\rho_{i\alpha\alpha'}$.

However, for the determination of the final electron polarization we can formulate two problems: the first is to find the probability of an electron with a certain polarization and the second is to determining the polarization of a final electron or, equivalently, determining its density matrix.

The first question corresponds to the certain measurement type, which is performed with a detector, selecting a state with a definite polarization. This polarization is characterized through a polarization vector a_d (detector polarization). Consequently the replacement of $u_{i\alpha} \bar{u}_{i\alpha'}$ to the corresponding density matrix can be performed, giving

$$|M_{fi}|^2 \longrightarrow \text{Sp}\{\rho_i \bar{Q} \rho_d Q\} \quad (21.9)$$

Let us come back to the second question of the determination of the density matrix of a final electron. If the electron is in a final state with the bispinor u_f , then the probability to measure a certain polarization a_d by a detector is proportional to $|\bar{u}_d u_f|^2$. We can replace the bispinors with the corresponding density matrices

$$u_d \otimes \bar{u}_d \rightarrow \rho_d, \quad u_f \otimes \bar{u}_f \rightarrow \rho_f.$$

Therefore this probability is proportional to $\text{Sp}\{\rho_f \rho_d\}$. On the other hand it must be equal to the probability (21.9):

$$\text{Sp}\{\rho_f \rho_d\} \sim \text{Sp}\{\rho_i \bar{Q} \rho_d Q\}. \quad (21.10)$$

Equation (21.10) allows to find the density matrix of the final electron, because ρ_f is defined only by the final spin four vector a_f , which reads

$$a_f^\mu = \frac{1}{2m} \frac{\text{Sp}\{\rho_i \bar{Q}(\hat{p}_f + m)\gamma^5 \gamma^\mu(\hat{p}_f + m)Q\}}{\text{Sp}\{\rho_i \bar{Q}(\hat{p}_f + m)Q\}}. \quad (21.11)$$

From equation (21.11) the density matrix of the final electron can be easily found:

$$\rho_f = \frac{(\hat{p}_f + m)Q\rho_i\bar{Q}(\hat{p}_f + m)Q}{\text{Sp}\{\rho_i\bar{Q}(\hat{p}_f + m)Q\}}. \quad (21.12)$$

The formulas (21.9), (21.11) and (21.12) are sufficient for the probability calculation involving electrons. However, M_{fi} contains the polarization vectors of the final photons and $|M_{fi}|^2$ is bilinear over them. In the general case, $|M_{fi}|^2$ can be written as

$$|M_{fi}|^2 = \text{Sp}\{\rho_i\bar{Q}'^\mu\rho_dQ'^\nu\}e_\mu e_\nu^*, \quad (21.13)$$

where he have calculated the average in a spin space according to (21.9) and selected polarization vectors e_μ, e_μ^* from the matrices Q and \bar{Q} . The matrices without polarization vectors are denoted with primes.

As in the electron's case we can formulate two problems: the first is to find the probability that the emitted photon will have a certain polarization and the finding of the polarization of the emitted photon itself (finding its density matrix).

The solution of the first problem corresponds to the replacement

$$e_\mu e_\nu^* \longrightarrow \rho_{\mu\nu}^p,$$

where the photon density matrix is defined in the following way [2]:

$$\begin{aligned} \rho_{\mu\nu}^p = & -\frac{1}{2}g_{\mu\nu} + \frac{\xi_1}{2}(u_\mu^{(1)}u_\nu^{(2)} + u_\mu^{(2)}u_\nu^{(1)}) - i\frac{\xi_2}{2}(u_\mu^{(1)}u_\nu^{(2)} - u_\mu^{(2)}u_\nu^{(1)}) \\ & + \frac{\xi_3}{2}(u_\mu^{(1)}u_\nu^{(1)} + u_\mu^{(2)}u_\nu^{(2)}), \end{aligned} \quad (21.14)$$

and four vectors $u_\mu^{(1)}, u_\mu^{(2)}$ satisfy the conditions

$$(u^{(1)})^2 = (u^{(2)})^2 = -1, \quad (u^{(1)} \cdot u^{(2)}) = 0, \quad (k \cdot u^{(1)}) = (k \cdot u^{(2)}) = 0.$$

From these considerations we can write down the absolute value square of the amplitude as

$$|M_{fi}|^2 = \text{Sp}\{\rho_i\bar{Q}'^\mu\rho_dQ'^\nu\}\rho_{\mu\nu}^p. \quad (21.15)$$

The solution of the second problem can be performed in the same way as for the electron. This yields the density matrix of the emitted photon, which reads

$$\rho_f^{p\mu\nu} = \frac{\text{Sp}\{\rho_i\bar{Q}'^\mu\rho_dQ'^\nu\}}{\sum_\mu \text{Sp}\{\rho_i\bar{Q}'^\mu\rho_dQ'^\mu\}}. \quad (21.16)$$

22 Evaluation of the decay rate for the non-linear Compton scattering

In this section we will apply the rules which were formulated in paragraph 21 for the evaluation of the decay rate and provide comparison of the numerical results with the other works.

Let us specify all quantities which have appeared in the decay probability (21.4). First of all, we fix a coordinate system by directing the z -axis along \mathbf{k} , Figure 22.1. Consequently the angles of the spherical coordinate system will be counted from \mathbf{k} .

We will distinguish two cases of the final states. In the first case the detector will track photons and in the second one it will track electrons. This corresponds to the integration over d^3p_1 and over d^3k_1 in the first and second cases respectively. Consequently we have two set of conservation laws defined by equation (21.5). In the coordinate system depicted on Figure 22.1 these conservation laws can be rewritten as

$$\begin{cases} k_{1x} = -p_1 \sin \theta \cos \varphi \\ k_{1y} = p \sin \theta_i - p_1 \sin \theta \sin \varphi \\ k_{1z} = \omega_1 + \epsilon_1 - \epsilon - p_1 \cos \theta + p \cos \theta_i \end{cases}, \quad \begin{cases} p_{1x} = -\omega_1 \sin \theta \cos \varphi \\ p_{1y} = p \sin \theta_i - \omega_1 \sin \theta \sin \varphi \\ p_{1z} = \omega_1 + \epsilon_1 - \epsilon - \omega_1 \cos \theta + p \cos \theta_i \end{cases}. \quad (22.1)$$

In order to find the frequency ω_1 of the emitted photon, which is expressed through the electron variable or the energy of electron expressed through the photon ones, the condition that the particles lie on their mass shells must be used:

$$k_{1x}^2 + k_{1y}^2 + k_{1z}^2 = \omega_1^2, \quad p_{1x}^2 + p_{1y}^2 + p_{1z}^2 + m^2 = \epsilon_1^2. \quad (22.2)$$

With the help of the conservation law (22.1) these equations can be easily solved, yielding

$$\begin{aligned} \omega_1 = \frac{1}{2} & \left(p^2 - 2pp_1 \sin \theta \sin \theta_i \sin \phi + 2p_1 \cos \theta (-p \cos \theta_i + \epsilon - \epsilon_1) \right. \\ & \left. - 2p \cos \theta_i (\epsilon - \epsilon_1) + p_1^2 + \epsilon^2 - 2\epsilon\epsilon_1 + \epsilon_1^2 \right) / \left(-p \cos \theta_i + p_1 \cos \theta + \epsilon - \epsilon_1 \right), \quad (22.3) \\ \epsilon_1 = & \left(-p\omega_1 \sin \theta \sin \theta_i \sin \phi - \omega_1 \cos \theta (p \cos \theta_i + \omega_1 - \epsilon) \right. \\ & \left. - p \cos \theta_i (\epsilon - \omega_1) + \omega_1^2 + \epsilon^2 - \omega_1 \epsilon \right) / \left(\omega_1 \cos \theta - p \cos \theta_i - \omega_1 + \epsilon \right). \end{aligned}$$

The presence of δ function in the decay rate shows that it is non-vanishing only in that points where the conservation law takes place. By integrating over d^3k_1 or d^3p_1 we eliminate δ function and obtain relations (22.3), which must satisfy additional conditions

$$\omega_1 \geq 0, \quad \epsilon_1 \geq m. \quad (22.4)$$

If (22.4) are not fulfilled than the decay rate is equal to zero. The relations (22.4) truncate unphysical processes and coincide to the situation that the frequency of the emitted photon can not be negative and the energy of the final electron must be greater than its rest energy. Consequently, we obtained the boundaries for the final electron (photon) momentum and its angles, which coincide with the physical result.

For the description of the final electron polarization the spin quantization axis can be chosen by two different ways. According to the first, it is chosen along the final electron momentum in the laboratory frame. Consequently, the initial and final electron spins read

$$\zeta = \frac{\mathbf{p}}{|\mathbf{p}|} \mu, \quad \zeta_1 = \frac{\mathbf{p}_1}{|\mathbf{p}_1|} \mu_1. \quad (22.5)$$

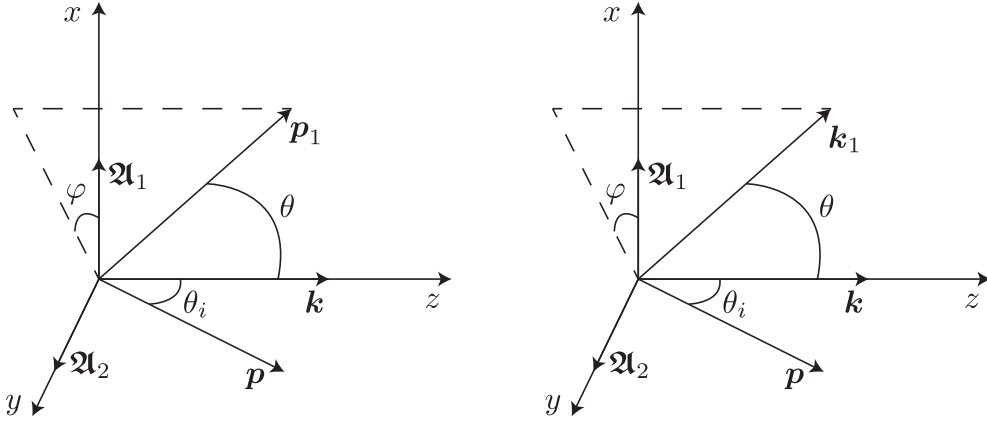


Figure 22.1 – Coordinate systems in which the decay rate is evaluated. Left pane: detector tracks final electrons. Right pane: detector tracks final photons.

The spin flip occur when $\mu_1 = -\mu$.

The second way of the description corresponds to the situation when the final electron spin quantization axis is chosen along initial electron momentum in the laboratory frame. In this case the spins look like

$$\begin{aligned} \zeta &= \frac{\mathbf{p}}{|\mathbf{p}|} \zeta, & \zeta_1 &= \frac{\mathbf{p}}{|\mathbf{p}|} \zeta_1, \\ \zeta &= \mu, & \zeta_1 &\neq \mu_1. \end{aligned} \quad (22.6)$$

The spin flip occur when $\zeta_1 = -\zeta$.

The second description is more convenient from the experimental point of view as we look how the same quantum number changes during the scattering process. For this reason we will follow this way in subsequent calculations.

When the evaluation of the trace according to equation (21.13) is performed, the electron density matrix $\rho = 1/2(\hat{p} + m)(1 - \gamma^5 \hat{a})$, with the spin four vector a is used. The vector a is connected with the three vector ζ (see equation (10.3)) as

$$a = \left(\frac{\mathbf{p} \cdot \zeta}{m}, \zeta + \frac{\mathbf{p}(\mathbf{p} \cdot \zeta)}{m(\epsilon + m)} \right). \quad (22.7)$$

The decay rate includes different scalar products, which must be expressed in our coordinate system. For this purpose, we will write down all four vectors, included in equation (21.4). For the electron tracking case

$$\begin{aligned} \mathbf{p} &= (\epsilon, 0, p \sin \theta, p \cos \theta), & \mathbf{k} &= (\omega, 0, 0, \omega), \\ \mathbf{p}_1 &= (\epsilon_1, p_1 \sin \theta \cos \varphi, p_1 \sin \theta \sin \varphi, p_1 \cos \theta), \\ \mathbf{k}_1 &= (\omega_1, k_{1x}, k_{1y}, k_{1z})|_{k_1} \text{ expressed through the conservation law,} \\ \mathbf{a} &= \left(\frac{p\zeta}{m}, 0, \frac{\epsilon\zeta}{m} \sin \theta_i, \frac{\epsilon\zeta}{m} \cos \theta_i \right), \\ \mathbf{a}_1 &= \left(\frac{p_1\zeta}{m}, \frac{p_1\zeta}{m(\epsilon_1 + m)} p_1 \sin \theta \cos \varphi, \zeta_1 \sin \theta_i \right) \end{aligned} \quad (22.8)$$

$$\begin{aligned}
& + \frac{p_{1\zeta}}{m(\epsilon_1 + m)} p_1 \sin \theta \sin \varphi, \zeta_1 \cos \theta_i + \frac{p_{1\zeta}}{m(\epsilon_1 + m)} p_1 \cos \theta \Big), \\
p_{1\zeta} &= \zeta_1 \sin \theta_i p_1 \sin \theta \sin \varphi + \zeta_1 \cos \theta_i p_1 \cos \theta, \\
\mathfrak{A}_1 &= (0, 1, 0, 0), \quad \mathfrak{A}_2 = (0, 0, 1, 0).
\end{aligned}$$

For the photon tracking case vectors $k, p, a, \mathfrak{A}_1, \mathfrak{A}_2$ remain the same. For the other quantities, one obtains

$$\begin{aligned}
k_1 &= (\omega_1, \omega_1 \sin \theta \cos \varphi, \omega_1 \sin \theta \sin \varphi, \omega_1 \cos \theta), \\
p_1 &= (\epsilon_1, p_{1x}, p_{1y}, p_{1z})|_{p_1 \text{ expressed through the conservation law}}, \\
a_1 &= \left(\frac{p_{1\zeta}}{m}, -\frac{p_{1\zeta}}{m(\epsilon_1 + m)} \omega_1 \sin \theta \cos \varphi, \zeta_1 \sin \theta_i + \frac{p_{1\zeta}}{m(\epsilon_1 + m)} (p \sin \theta_i - \omega_1 \sin \theta \sin \varphi), \right. \\
& \quad \left. \zeta_1 \cos \theta_i + \frac{p_{1\zeta}}{m(\epsilon_1 + m)} (\omega_1 + \epsilon_1 - \epsilon - \omega_1 \cos \theta + p \cos \theta_i) \right), \\
p_{1\zeta} &= \zeta_1 \sin \theta_i (p \sin \theta_i - \omega_1 \sin \theta \sin \varphi) + \zeta_1 \cos \theta_i (\omega_1 - \epsilon_1 - \epsilon - \omega_1 \cos \theta + p \cos \theta_i).
\end{aligned} \tag{22.9}$$

Equations (22.8) and (22.9) give possibility to find all scalar products which will appear during traces calculation in the decay rate.

Let us specify the model for the external electromagnetic field. We will distinguish two cases of different polarization, namely the linear and the circular ones. The linear polarization case is the limit of circular polarization when the quantities $\alpha_2, f_{21}, f_{22}, \mathfrak{A}_2$ are set to zero in equations (19.13), (19.14), (19.14) and only $\cos^2 \varphi$ is left in the phase Φ in the term proportional to β^2 .

As was already mentioned in the introduction, we are investigating the case of a single laser pulse with a shape function, which usually is chosen in a form [28, 133]:

$$f(\varphi) = e^{-\alpha \varphi^2}, \quad f(\varphi) = \sin^2 \left(\frac{\varphi}{N} \right). \tag{22.10}$$

Some other works [27] model an external short pulse of an electromagnetic field not by an introducing a shape function on a top of a cosine or sine, but rather by changing the vector potential to

$$A = \frac{m\xi}{e} \mathfrak{A}_1 \operatorname{sech} \varphi. \tag{22.11}$$

Such a choice of the external field is valid only for the very short pulses, with the length of one period of laser oscillation.

In our description we will use the exponential shape function in order to model the finite pulse duration. The constant alpha in the argument of the exponent will give the number of periods of the laser pulse. The first three graphs on Figure 22.2 represent the different pulse lengths, starting from ten periods (corresponds to $\alpha = 0.01$) and finishing with a very short pulse, which contains only one period (corresponds to $\alpha = 4$). The last graph represents the case of $\operatorname{sech} \varphi$ shape function.

Let us come to the description of the numeric procedure for the calculation of the integrals $f_{11}, f_{12}, f_{21}, f_{22}$. As for the last integral f_0 , it is represented as a linear combination of

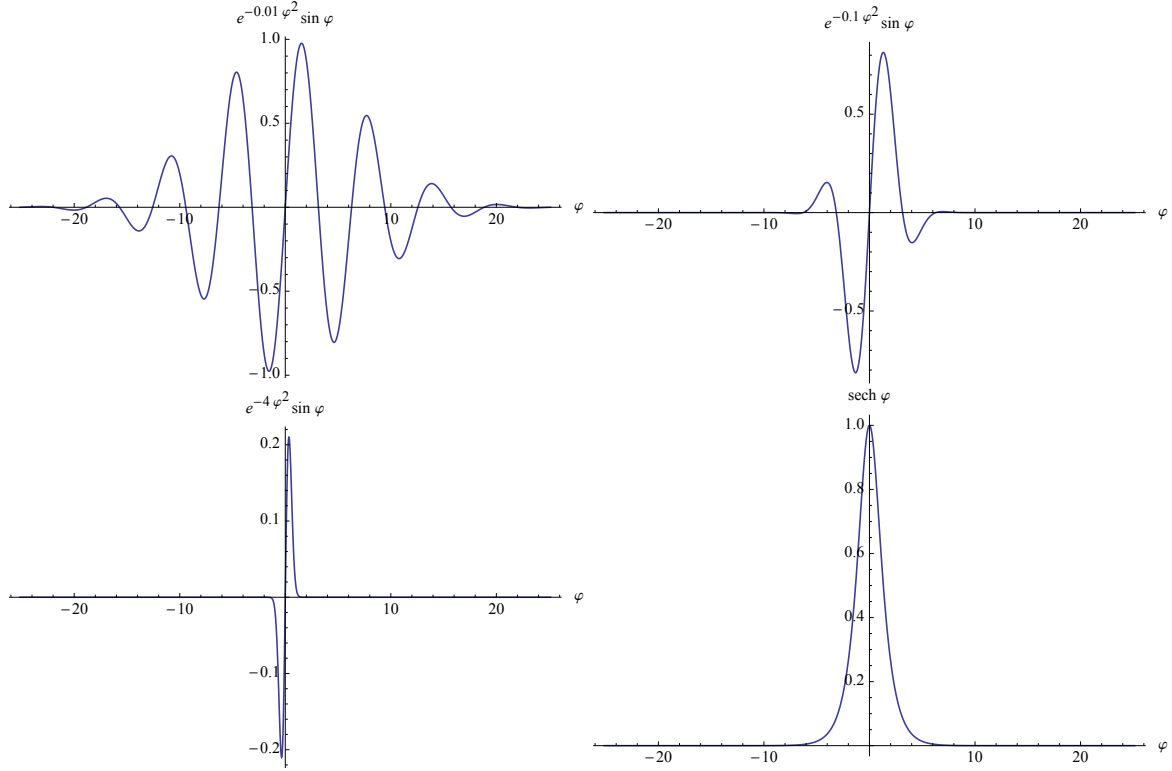


Figure 22.2 – The dependence of the y component of the vector potential as a function of the field phase φ for the different pulse length.

$f_{11}, f_{12}, f_{21}, f_{22}$, which was shown in 20. We will use the numerical method proposed by D. Levin [134]. First of all, as the pre-exponential terms include the rapidly decreasing function $\exp -\alpha\varphi^2$, the limits of the integration can be replaced from infinity to some final value, which will be chosen from the stability considerations of the numerical scheme. For this reason, we can rewrite this integrals in a form

$$I = \int_a^b f(x) e^{iq(x)} dx. \quad (22.12)$$

We also note here, that the derivative of the phase $\Phi(\eta)$, defined in equation (20.3), which is in the present abbreviations is equal to $q(x)$, satisfies the condition $|q'(x)| \gg 1$.

The Levin method is based on the fact, that if $f(x)$ were of the form $f(x) = iq'(x)p(x) + p'(x)$, then the integral could be evaluated directly as

$$I = \int_a^b (iq'(x)p(x) + p'(x)) e^{iq(x)} dx = \int_a^b \frac{d}{dx} (p(x) e^{iq(x)}) dx = p(b) e^{iq(b)} - p(a) e^{iq(a)}. \quad (22.13)$$

Therefore, equation

$$f(x) = iq'(x)p(x) + p'(x), \quad (22.14)$$

can be considered as a differential equation for $p(x)$ and any solution can be used for evaluat-

ing I . The general solution of equation (22.14) is equal to

$$p(x) = e^{-iq(x)} \left[\int_a^x f(t) e^{iq(t)} dt + c \right]. \quad (22.15)$$

In the case of an arbitrary $f(x)$ and $q'(x)$, $p(x)$ is as oscillatory as the integrand of I . However, if $f(x)$ and $q'(x)$ are slowly oscillatory, it is possible to prove a theorem [134] that there exist a slowly oscillatory solution $p_0(x)$ of (22.14). Consequently, all the difficulties of the evaluation of the highly oscillatory integrand I have been overcome.

Next, according to the method, the collocation approximation for $p_0(x)$ is introduced as

$$p_n(x) = \sum_{k=1}^n \alpha_k u_k(x), \quad (22.16)$$

where $\{u_k\}_{k=1}^n$ are some linearly independent basis functions, with the coefficients $\{\alpha_k\}_{k=1}^n$ determined from the solution of the system of linear equations

$$\sum_{k=1}^n \alpha_k u'_k(x_j) + iq'(x_j) \sum_{k=1}^n \alpha_k u_k(x_j) = f(x_j), \quad j = 1, 2, \dots, n. \quad (22.17)$$

Here $\{x_j\}_{j=1}^n$ are regularly distributed points of the mesh in $[a, b]$.

After the unknown collocation coefficients were determined, the approximation to the integral can be computed as

$$I_n = \sum_{k=1}^n \alpha_k u_k(b) e^{iq(b)} - \sum_{k=1}^n \alpha_k u_k(a) e^{iq(a)}. \quad (22.18)$$

Now we are ready to formulate the algorithm of the numerical calculation of the decay rate:

1. We start from the trace evaluation in symbolic form

$$\text{Sp}\{\rho_i \bar{Q}^\mu \rho_d Q^\nu\} \rho_{\mu\nu}^p, \quad (22.19)$$

where

$$\begin{aligned} Q^\mu &= \gamma^\mu f_0 + \frac{m\xi}{2(p \cdot k)} \gamma^\mu \hat{k} (f_{11} \hat{\mathfrak{A}}_1 + f_{21} \hat{\mathfrak{A}}_2) + \frac{m\xi}{2(p_1 \cdot k)} (f_{11} \hat{\mathfrak{A}}_1 + f_{21} \hat{\mathfrak{A}}_2) \hat{k} \gamma^\mu \\ &\quad + \frac{m^2 \xi^2 k^\mu (f_{12} + f_{21})}{2(p \cdot k)(p_1 \cdot k)} \hat{k} \\ \bar{Q}^\nu &= \gamma^0 (Q^\nu)^\dagger \gamma^0, \end{aligned}$$

and the density matrices of the initial and final electrons and emitted photons are defined by equations (21.16) and (21.14) respectively.

2. We chose the spin quantization axis along the initial electron momentum and plug-in in the equation (22.19) all four vectors, defined by equations (22.8) and (22.9).

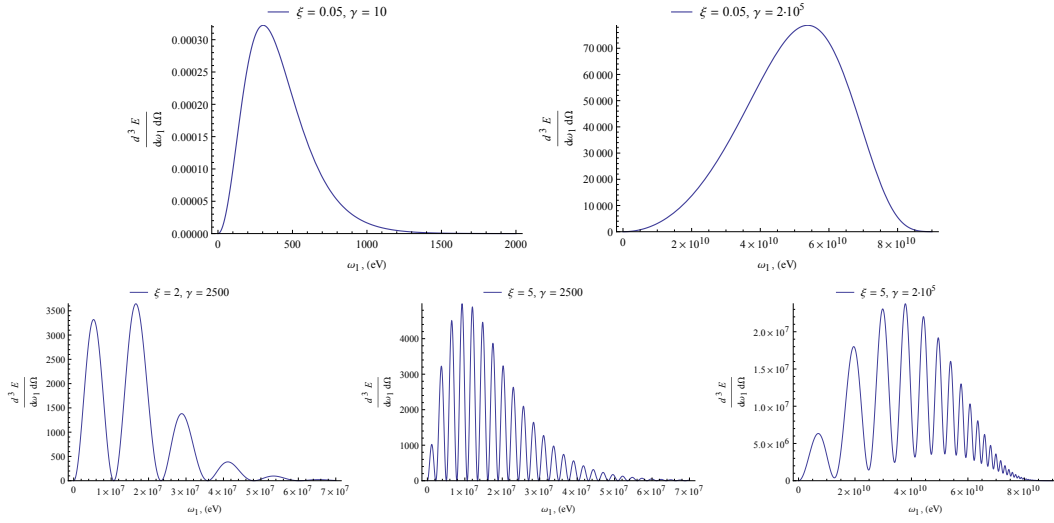


Figure 22.3 – The emitted photon energy spectra as a function of the emitted photon frequency ω_1 , for the different values of the field strength ξ and electron gamma factor γ . The case of a linear polarization of the external field, sech φ shaped vector potential and averaging over the initial and final electron spins.

3. All four products are calculated, consequently obtaining some cumbersome analytic expression $F(p_1, \theta, \varphi, \zeta_1, \omega, \xi, \mathfrak{U}_1, \mathfrak{U}_2, \alpha, p, \theta_i, \zeta)$, which depends on the momentum p_1 , angles θ, φ and polarization ζ_1 of the initial electron (photon), frequency ω , polarizations $\mathfrak{U}_1, \mathfrak{U}_2$, the strength ξ and duration α of the external field, the momentum p , entrance angle θ_i and polarization ζ of the initial electron.
4. The numerical values of all parameters are inserted into F .
5. A numerical integration with the Levin method is performed.

As the above described algorithm represents a big amount of non-trivial numerical calculations, it need to be validated on already known configuration of parameters. For this reason, we calculated the emitted photon spectra, which were investigated in the work [27] for the case of:

- A final particle which is tracked is the photon.
- The external field polarization is chosen to be linear, which is described by the four vector $\mathfrak{U}_2 = (0, 0, 1, 0)$.
- The external field dependence on its phase is modeled by sech φ .
- The summation over initial and final electron polarizations is performed, which corresponds to the

$$\rho_{i(f)} = \frac{1}{2}(\hat{p}_{i(f)} + m)$$

density matrix of the initial (final) electron.

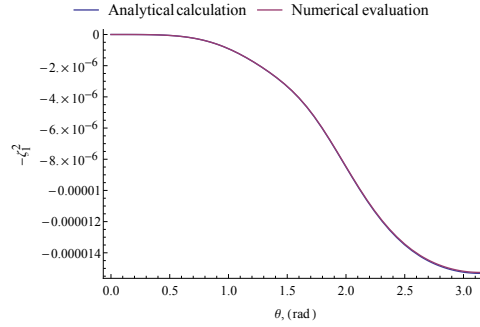


Figure 22.4 – Validation of the numerical algorithm by comparison of the dependence of the square of the absolute value of the electron final polarization on the angle θ between \mathbf{k} and \mathbf{k}_1 .

- The averaging over the final photon polarization is carried out. In this case the photon density matrix is equal to $-g_{\mu\nu}/2$.
- The initial electron counter propagates with the external field (head-on collision).
- θ angle of the emitted photon is equal to zero.

The numerical spectra are depicted on Figure 22.3. The comparison of our results with the literature results [27] shows perfect agreement.

In the verification of the results in the previous case we have averaged over initial and final electron spins. Consequently, our algorithm was not checked for the case when this procedure was not performed. For this reason, we have calculated the ordinary Compton effect and compared our numerical result with the analytical expression in this case. In this connection we mention here, that the final electron will have a polarization only in the case of a circularly polarized initial photon. The differential cross section of an unpolarized electron on a circularly polarized photon reads [2]:

$$d\sigma = \frac{r_0^2}{2} \frac{\omega_1^2}{\omega^2} F d\Omega + \frac{r_0^2}{4} \frac{\omega_1^2}{\omega^2} \mathbf{g}_1 \cdot \boldsymbol{\zeta}_d \xi_2 d\Omega, \quad (22.20)$$

where

$$F = \frac{\omega}{\omega_1} + \frac{\omega_1}{\omega} - (1 - \xi_3) \sin^2 \theta,$$

$r_0 = \alpha/m$ is the classical electron radius, ξ_2, ξ_3 are the second and the third Stokes parameter of the initial photon, θ is the angle between \mathbf{k} , \mathbf{k}_1 and

$$\mathbf{g}_1 = -\frac{1}{m}(1 - \cos \theta) \left[\mathbf{k} \cos \theta + \mathbf{k}_1 - (1 + \cos \theta) \frac{\omega + \omega_1}{\omega - \omega_1 + 2m} (\mathbf{k} - \mathbf{k}_1) \right].$$

Equation (22.20) gives possibility to find polarization of the final electron:

$$\zeta_1 = \frac{\mathbf{g}_1}{F} \xi_2. \quad (22.21)$$

In the case of the pure state $\zeta_1^2 = 1$. However, during the scattering process there is no guarantee that the final electron state is a pure one and consequently $\zeta_1^2 \leq 1$.

In order to compare this analytical result with the one, produced by a numerical procedure we calculated the electron spin four vector a_f according to equation (21.11). The scalar product of a_f with itself is equal to $-\zeta_1^2$. The result of a comparison is plotted as a function of the angle θ between \mathbf{k} and \mathbf{k}_1 on the Figure 22.4. As can be seen from the graph the results coincide exactly.

23 Polarization effects in non-linear Compton scattering in a short laser pulse

In the previous section we have described the numerical algorithm for the calculation of the decay rate and verified the results of the numerical evaluation for the limiting well known cases. In what follows we calculate numerically the decay rate and asymmetry of scattering for polarization dependent Compton effect in a short laser pulse.

Let us specify the parameters regime in which the values of the asymmetry are expected to be the largest. On one hand, from standard vacuum QED calculations it is well known (22.21), that the final electron will be polarized only if the initial photon is circularly polarized. As the limit $\xi \ll 1$ coincides with the perturbative one, we can expect that in the non-linear case the polarization of the final electron will have the largest value for the circular polarization of the external field. On the other hand, the parameter χ should be of the order of one. This follows from the analysis of non-linear Compton scattering in the case of monochromatic plane wave, because for $\xi \gg 1$ the quantum effects are the largest when $\chi \sim 1$ [15]. Consequently, this motivates the following experimental situation, for the scattering calculation:

- We consider a head-on collision of the electron and laser beam, i.e., $\mathbf{k} \cdot \mathbf{p} = -kp$.
- The space-time dependence of the vector potential is given as

$$A_x \sim \frac{e^{-\alpha\varphi^2}}{1+2n}(\cos \varphi + n \cos 2\varphi),$$

$$A_y \sim \frac{e^{-\alpha\varphi^2}}{1+2n}(\sin \varphi + n \sin 2\varphi).$$

The parameter $\alpha = \{0.01, 0.1\}$ specify pulse length and coincides to ten and one laser oscillations respectively. The parameter n represents the relative intensity of the second harmonic, which inclusion will be explained later.

- The initial beam of electrons have polarization either directed or oppositely-directed along the initial electron momentum. We denote +1 and -1 the first and the second situations respectively.
- The final electron polarization is directed in the opposite direction to the initial polarization. For example, if the initial polarization is +1 then we are interested in the final polarization -1 and vice versa.

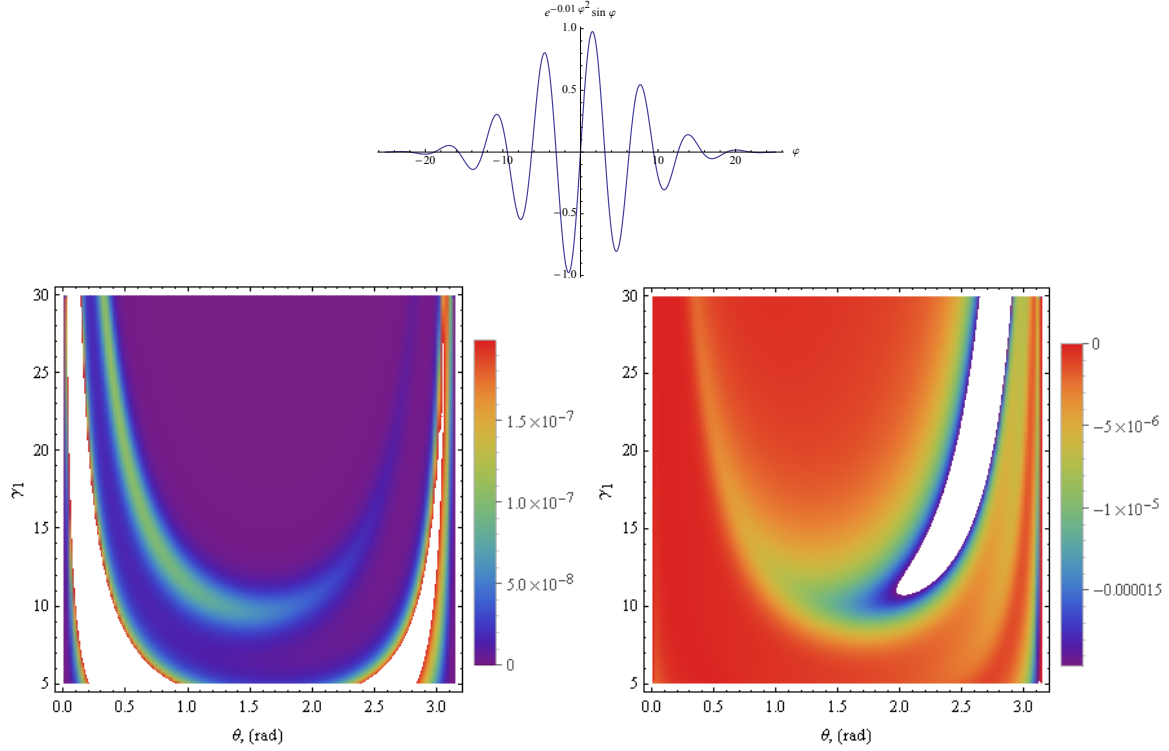


Figure 23.1 – (Left pane) The decay rate for the non-linear Compton scattering as a function of the final electron γ_1 and θ angle of the spherical coordinate system, for the parameters $\xi = 10$ and the field is circularly polarized, $\gamma = 10^4$ (corresponds to $\chi = 0.4$), φ angle of the final electron is equal to zero, the pulse duration is equal to approximately ten periods ($\alpha = 0.01$). (Right pane) The asymmetry of scattering Δ as a function of the final electron γ_1 and θ angle of the spherical coordinate system for the same parameters as on the left pane. White color represents the probability value larger than 10^{-7} and asymmetry value larger than 10^{-4} .

- All energies will be measured as a fraction of an electron mass m . Consequently, the initial and final electrons have gamma factors γ and γ_1 respectively.
- The dimensionless density Γ of the decay rate is plotted:

$$\frac{dW}{d^3p} = \frac{\alpha}{m^3} \Gamma(\zeta, \zeta_1), \quad \alpha \text{ is a fine structure constant.}$$

- The influence of the spin on the scattering process will be characterized by the asymmetry of the scattering [2]:

$$\Delta = \frac{\Gamma(1, -1) - \Gamma(-1, 1)}{\Gamma(1, -1) + \Gamma(-1, 1)}.$$

- The laser frequency is chosen to be $\omega = 1 \text{ eV} = 1.96 \cdot 10^{-6} m$, which corresponds to the wavelength 1250 nm.

- We average over the final photon polarization, i.e. $\rho_{\mu\nu}^p = -g_{\mu\nu}/2$.
- As was already discussed in 18 the quantum effects take maximum value when parameter χ is of the order of one and the field is strong $\xi \gg 1$. From this consideration we chose $\chi = 0.4$. For example, this corresponds to $\gamma = 10^4$ of the initial electron and $\xi = 10$ of the external field. We note here that the range $\xi \leq 100$ is experimentally available [4]. The cases when $\chi \gg 1$ and $\chi \ll 1$ may not be considered as in the first case the more probable process is the electron-positron pair production, while the second one corresponds to the linear Compton effect [15].

The results of the numerical evaluation of the decay rate and the scattering asymmetry are depicted on Figure 23.1, 23.2 and 23.3 as a function of the final electron gamma factor and the angle θ between \mathbf{k} and \mathbf{p}_1 for the different pulse lengths. As can be seen from Figure 23.1 for the decay rate we can identify the two regions, namely the central part of the graph and two spikes of much higher value than the central part. These spikes are located near the angles close to 0 and π . This means that the majority of the particles will be scattered on the angles close to 0 or π . Moreover, the number of particles for the γ_1 in the depicted range is even a little bit larger, than the number of electrons which are scattered at the angles near π .

The physical meaning of this result can be understood in the following way: when the parameter χ is close to unity (in our case $\chi = 0.4$) the influence of the quantum effects is the largest. Therefore, it is very probable that the electron will be stopped in the field, which means that it emits a very energetic photon. If this happens, then the electron will be moving in the field starting from zero velocity. In order to understand why the electron moves along the \mathbf{k} vector, we can solve the classical equations of motion in a field of a plane electromagnetic wave. According to the solution [135], the electron velocity will be increasing in the direction of \mathbf{k} vector. For this reason we can observe the first spike near $\theta = 0$.

The second spike near $\theta = \pi$ corresponds to the emission of photons of small ω . In order to understand why the number of the electrons which scatter at the angles close to zero is larger than the number of electrons which scatters at π we can look at the emission spectra, depicted at Figure 22.3. As can be seen from the graph, for large γ the emission of photons with higher frequency is more probable, which leads to the larger number of electrons, which scatters at angles close to 0.

The electron decay is essentially the non-linear process, therefore, the number of electrons, which are scattered at different angles exist. However, the number of these electrons is small.

Let us now come to the analysis of the asymmetry of the scattering. As can be seen from the Figure 23.1 the asymmetry is always negative and form a “sword” like structure at the angular range $\theta = \{2, 2.8\}$. Unfortunately, the maximal value of the asymmetry is located in the region of the non-maximal decay rate, consequently leading to the problem in the observation. This result coincide with the one, obtained in the reference [55] as there it is shown, that despite the fact, that the asymmetry is a large value, its observation is difficult as in these regions of space the decay rate is very small.

Figures 23.2 and 23.3 represent the same dependencies as Figure 23.1 but for different pulse durations. According to the figures 23.2 and 23.3, the absolute value of the asymmetry growth when the pulse duration is decreasing. The intermediate pulse length ($\alpha = 0.1$) shows the increase in the asymmetry value of one order in comparison with the long pulse. Further decrease of the pulse duration ($\alpha = 4$) leads to the even larger values of the asymmetry. However, in

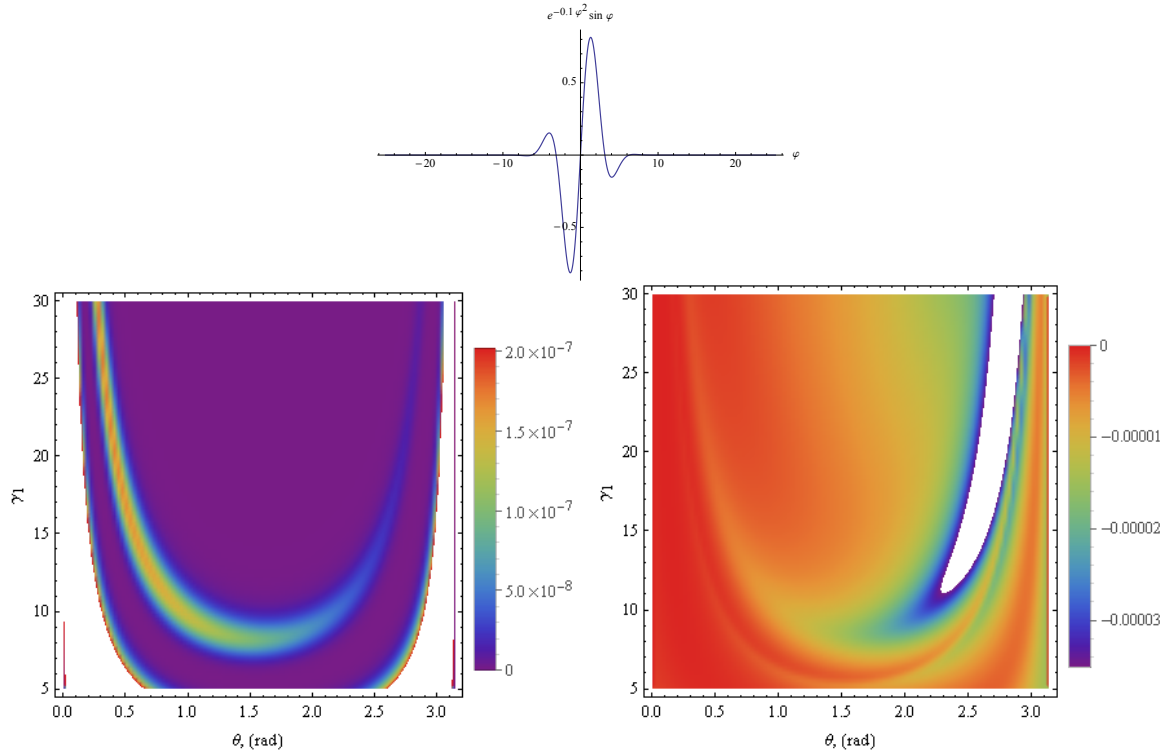


Figure 23.2 – (Left pane) The decay rate for the non-linear Compton scattering as a function of the final electron γ_1 and θ angle of the spherical coordinate system, for the parameters $\xi = 10$ and the field is circularly polarized, $\gamma = 10^4$ (corresponds to $\chi = 0.4$), φ angle of the final electron is equal to zero, the pulse duration is equal to approximately one period ($\alpha = 0.1$). (Right pane) The asymmetry of scattering Δ as a function of the final electron γ_1 and θ angle of the spherical coordinate system for the same parameters as on the left pane. White color represents the probability value larger than 10^{-7} and asymmetry value larger than 10^{-4} .

the region of space where the decay rate is non-vanishing the asymmetry is not equal to zero only in a very narrow angular range, where it changes sign. Consequently, as all detectors have finite angular resolution the averaging over this range will lead to the almost zero result. The remaining regions of space where the asymmetry is a large value coincide with the ones of the zero decay rate. Therefore, we can conclude, that for an extremely short pulses it is practically impossible to observe the asymmetry.

From the above analysis we could understand the general properties of the asymmetry, namely

- The asymmetry value increases with the decrease of the pulse duration. The optimal duration coincides with the approximately one period of the pulse length.
- The large values of the asymmetry lie in the region of space in which the decay rate has its minimum.

Let us try to explain this result. For this purpose we plot (see Figure 23.4) the trajectory of the electric (magnetic) field vector in the plane which is perpendicular to the vector \mathbf{k} , for

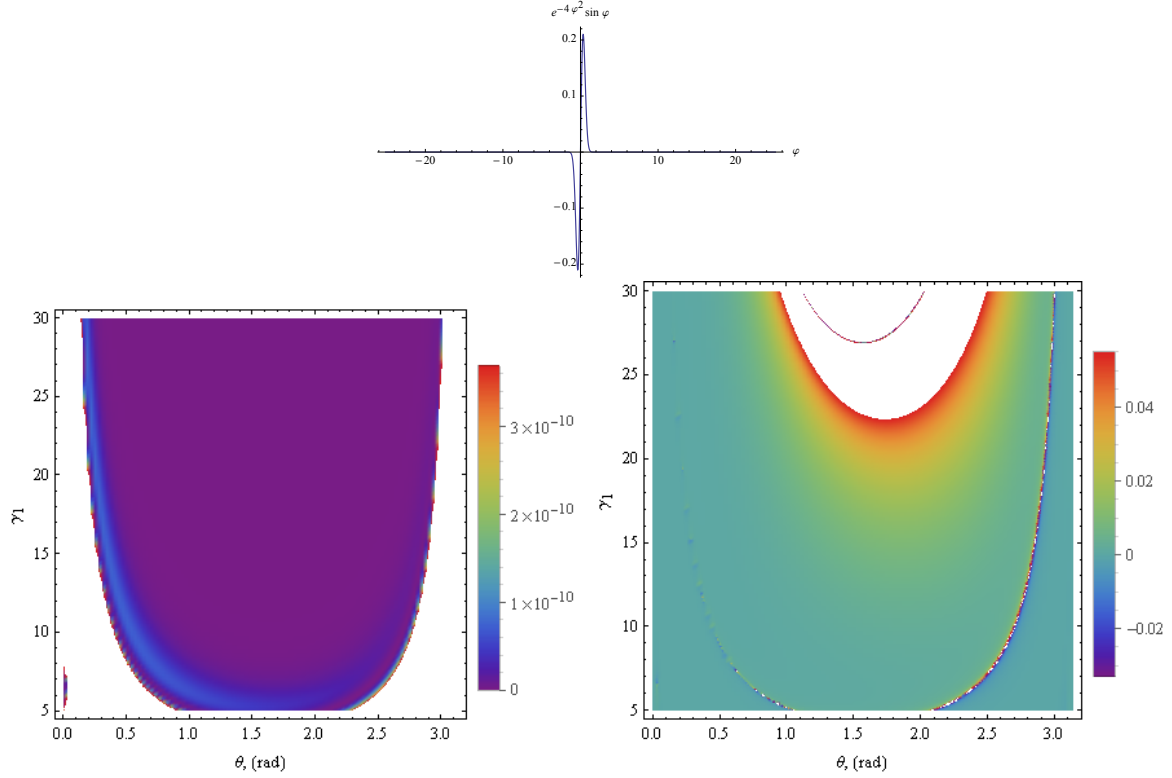


Figure 23.3 – (Left pane) The decay rate for the non-linear Compton scattering as a function of the final electron γ_1 and θ angle of the spherical coordinate system, for the parameters $\xi = 10$ and the field is circularly polarized, $\gamma = 10^4$ (corresponds to $\chi = 0.4$), φ angle of the final electron is equal to zero and the extremely short laser pulse ($\alpha = 4$). (Right pane) The asymmetry of scattering Δ as a function of the final electron γ_1 and θ angle of the spherical coordinate system for the same parameters as on the left pane. White color represents the probability value larger than 10^{-10} and asymmetry value larger than 10^{-2} .

different pulse durations. As can be seen from Figure 23.4 for a long pulse (Pane (a)) within one turn of the field vector, its magnitude in the upper direction is almost equal to the one in the down direction. This difference increases as the vector rotates. For this reason, the part of the electron energy when the spin is directed along the field is almost equal to the energy when the spin is oppositely-directed with the field. Therefore, the asymmetry between these states is small (when the difference is large the field is almost equal to zero). This picture perfectly coincides with the one, depicted on Figure 23.1.

Contrary to the case of a long pulse, for the short pulse (Pane (b)) within a single turn, the difference in the magnitudes of the field vectors is already a large value. Consequently, the asymmetry growth for a short pulse as in the agreement with Figure 23.2.

From the above analysis, we can conclude that for the optimal observation of the asymmetry the pulse duration should be of the order of one period. However, such short pulses with a very high field strength ($\xi \sim 10$) are not experimentally available. For this reason, we suggest to use a longer pulse with a broken symmetry. One way to implement this is to add the second harmonic into the field. According to Figure 23.4, (Pane (c)), the inclusion of the second harmonic breaks

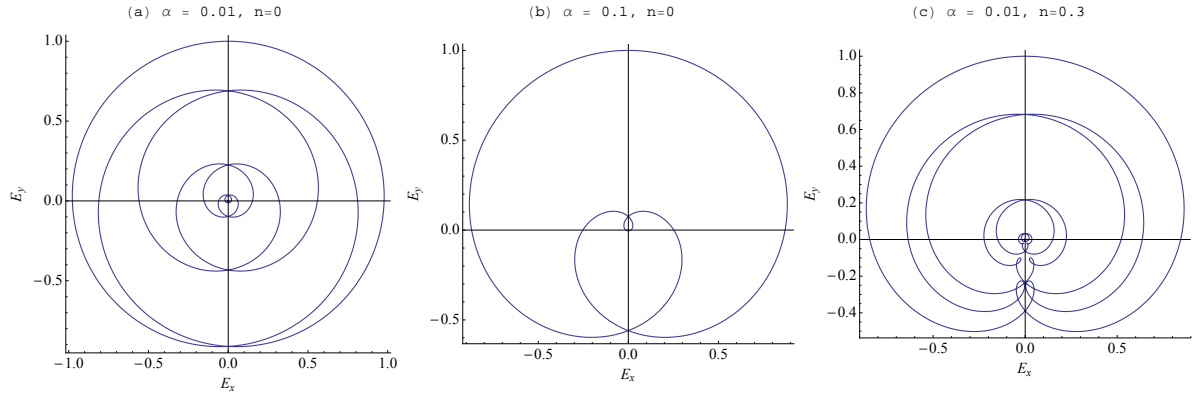


Figure 23.4 – The rotation of the electric (magnetic) field vector in the plane, which is perpendicular to the \mathbf{k} . (Pane (a)) Long pulse ($\alpha = 0.01$), without the second harmonic. (Pane (b)) Long pulse ($\alpha = 0.1$), without the second harmonic. (Pane (c)) Long pulse ($\alpha = 0.01$), with the second harmonic $n = 0.3$.

the field symmetry. The calculation of the asymmetry (Figure 23.5) proves this result. Therefore we achieved the larger value of the asymmetry, for a longer pulse length.

We have also investigated the influence of the field strength on the asymmetry. As follows from Figure 23.6 the increase of the field strength ξ from 10 to 100 decreases the magnitude of the asymmetry by one order. This can be understood in a way that when the field is very strong, the electron spin will be mainly directed along the field [55], consequently, the longitudinal asymmetry will be small. Therefore, the absolute value of the asymmetry decreases with the increase of the field strength.

Finally, we can formulate the optimal conditions for the asymmetry observation:

1. The pulse duration need to be of the order of one period. If such a short pulse is unavailable, the second harmonic can be included in the field, breaking the symmetry in the field vector rotation.
2. The absolute value of the asymmetry decreases with an increase of the field strength ξ .
3. The parameter χ should be of the order of one. Together with the remark 2, one can chose $\gamma \sim 10^4$ and $\xi \sim 10$.

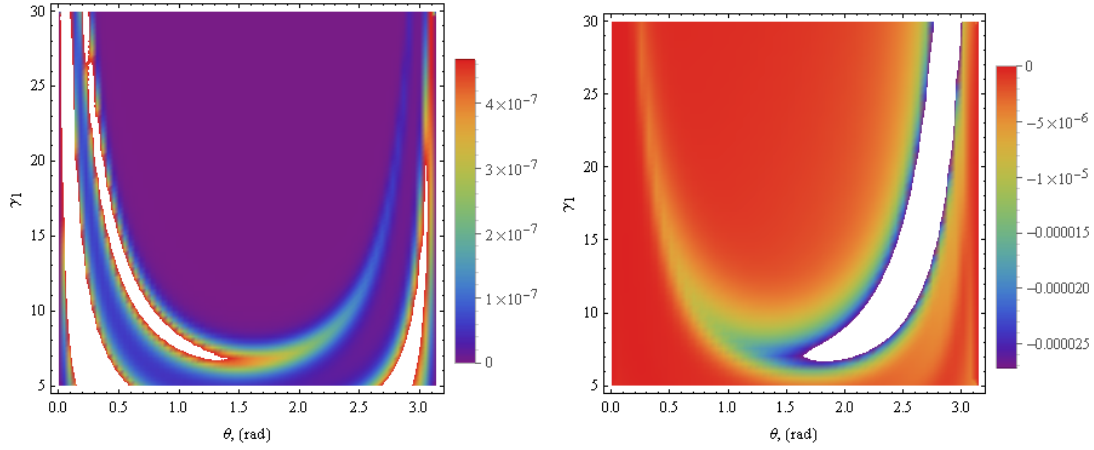


Figure 23.5 – (Left pane) The decay rate for the non-linear Compton scattering as a function of the final electron γ_1 and θ angle of the spherical coordinate system, for the parameters $\xi = 10$ and the field is circularly polarized, $\gamma = 10^4$ (corresponds to $\chi = 0.4$), φ angle of the final electron is equal to zero and the long laser pulse ($\alpha = 0.01$) with a broken symmetry. The symmetry is broken with the help of the second harmonic ($n = 1$). (Right pane) The asymmetry of scattering Δ as a function of the final electron γ_1 and θ angle of the spherical coordinate system for the same parameters as on the left pane. White color represents the probability value larger than 10^{-7} and asymmetry value larger than 10^{-4} .

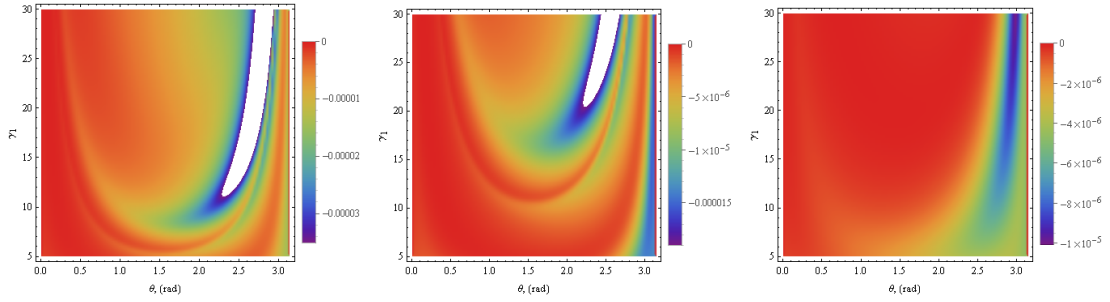


Figure 23.6 – The asymmetry of scattering Δ as a function of the final electron γ_1 and θ angle of the spherical coordinate system for the parameters $\gamma = 10^4$, φ angle of the final electron is equal to zero, the short laser pulse ($\alpha = 0.01$) and the field is circularly polarized. (Left pane) $\xi = 10$, (Middle pane) $\xi = 20$, (Right pane) $\xi = 100$.

Conclusion

The main purpose of this thesis consisted in the search of purely quantum effects, which do not exist in classical physics, in the interactions between an electron and a strong external electromagnetic field. The literature analysis presented in the introduction suggested to look in the direction connected with the quantized nature of the external field and with the spin of the electron. Consequently, we started in chapter I from the investigation of the exact solutions of the Dirac equation in the presence of an external single-mode quantized electromagnetic field. We formulated this solution in the most convenient form for calculations, namely with the use of the abstract representation of the creation and annihilation operators of the electromagnetic field. At this stage it was already clear that the plane wave approximation can not describe all possible laser pulses. For this reason, the conditions for the pulse duration, focussing size and field intensity must exist, which should justify the single-mode approximation. Starting from one particle sector of QED for the electron and including all modes of the external field in the Dirac equation we were able to separate the Hamiltonian of the system into two parts, that is to say, the single-collective mode of the field and fluctuations relative to the former. This was achieved with the help of the modified Bogolubov's canonical transformation of the field variables. It was shown that the properties of the single-collective mode of the field are related with the physical parameters of the laser pulse. Next, by building the perturbation theory over the fluctuations we were able to specify the three dimensionless conditions which specify the validity of the single-mode approximation. The first two conditions determine the relation of the frequency shift and collective-mode width to the driven field frequency, while the last one is defined as the relation between the energy of the fluctuation and the energy of the collective-mode. The single-mode (plane wave) approximation is proven to be valid when these ratios are small.

In chapter II we employed the results of chapter I, i.e., the operator form of the exact solution of the Dirac equation, for the calculation of the electron spin dynamics in the single-mode and quantized field. We determined the evolution of the electron state for the situation when in the initial moment of time the field was in the coherent state and the electron was free. This was done by forming a linear combination of the exact solutions of the Dirac equation and determining the coefficients of this linear combination from the initial condition. After the coefficients were determined, the expectation value of the electron spin in the limit of a strong field has been calculated. During evaluation the saddle point and modified cumulant methods were used. It was found that, when the interaction time between the electron and the field increases the special structure in the probability appears, namely collapse-revival dynamics. Finally, the possibility of the experimental observation of the effect was discussed.

During the discussion in chapter II we considered large, but not extreme, values of laser intensity. In this regime the quantum fluctuations can change the dynamics of the system. In

contrast to this, in chapter III we switched to the regime of utmost fields and investigated the influence of the spin on Compton scattering. We developed a joint analytical-numerical scheme of calculations of decay rates and compared our numerical results with those, known from the literature. This allowed us to verify the calculation strategy. On the next step we evaluated the decay rate as a function of pulse duration and formulated the optimal conditions of the scattering asymmetry observation: the quantum non-linearity parameter χ should be of the order of one and the pulse duration need to be small, e.g., contains few laser oscillations. The parameter χ dependence on the field intensity and gamma factor of the electron gives optimal values of $\xi \sim 10$ and $\gamma \sim 10^4$. We also noted, that the pulses of such a strength and duration of the order of one period are still not experimentally available. Therefore, it is suggested to add the second harmonic to the driven circularly polarized field, thus breaking the symmetry in the electric (magnetic) field rotation, which is equivalent to a short laser pulse.

Concluding we want briefly address the new results, obtained for the first time in the present thesis:

- The conditions of the applicability of the single-mode approximation are formulated.
- The collapse-revival dynamics of the electron spin is found for large intensities of the order 10^{18} W/cm².
- The influence of the laser pulse duration on spin-dependent Compton scattering is investigated and parameters for the optimal observation of the scattering asymmetry are suggested.

Outlook

The further extension of the present work can be presented in the following ways:

1. The method of the selection of the collective mode can be applied for the description of the interaction between two electrons and a laser pulse. This will allow to calculate the entanglement between electron spins. The change of parameters of the laser pulse will provide a possible control of the entangled electron states.
2. We have shown that during the interaction of the electron with a focussed laser pulse of a final duration a single-collective mode and fluctuations can be selected. The interaction with the single-collective-mode leads to the Volkov or Berson solutions of the Dirac equation, which are then used in Furry representation for the calculation of various QED processes. However, the interaction of the electron with fluctuating modes will give corrections to the scattering cross sections which can be taken into account with the method developed in this thesis.
3. The quantum nature of the synchrotron radiation leads to the self-polarization effect of an electron in undulator [136]. If the electron bunch passes through the laser radiation on one section of the cyclotron ring the self polarization can be amplified. Therefore, it is of great interest to apply the developed formalism for the description of the electron motion in strong combined laser and magnetic fields.

Appendix A

Calculation of the first order correction to the approximating Hamiltonian

We start with the calculation of the average of the Hamiltonian H_1 . For this purpose we need to rewrite it in the variables of a “collective” mode A, A^\dagger and “fluctuations” $\tilde{C}_k, \tilde{C}_k^\dagger$.

$$\begin{aligned}
H_1 &= \sum_{k<\Delta} (\hat{k} - \hat{k}_0) \mathbf{a}_k^\dagger \mathbf{a}_k + (\hat{b}(\mathbf{k}) - \hat{b}_0)(\mathbf{a}_k + \mathbf{a}_k^\dagger) = \sum_{k<\Delta} (\hat{k} - \hat{k}_0) \frac{1}{2} (\mathbf{p}_k^2 + \mathbf{q}_k^2) + \sqrt{2}(\hat{b}(\mathbf{k}) - \hat{b}_0) \mathbf{q}_k \\
&= \sum_{k<\Delta} (\hat{k} - \hat{k}_0) \frac{1}{2} \left(\mathbf{P}^2 + 2\mathbf{P} \mathbf{p}_{yk} + \mathbf{p}_{yk}^2 + \mathbf{y}_k^2 + \frac{2\mathbf{Q} \mathbf{y}_k}{N} + \frac{\mathbf{Q}^2}{N^2} \right) + \sum_{k<\Delta} \sqrt{2}(\hat{b}(\mathbf{k}) - \hat{b}_0) \left(\mathbf{y}_k + \frac{1}{N} \mathbf{Q} \right) \\
&= \sum_{k<\Delta} \frac{1}{2} (\hat{k} - \hat{k}_0) \left(\mathbf{P}^2 + \frac{\mathbf{Q}^2}{N^2} \right) + \sum_{k<\Delta} \frac{1}{2} (\hat{k} - \hat{k}_0) (\mathbf{p}_{yk}^2 + \mathbf{y}_k^2) \\
&\quad + \sum_{k<\Delta} (\hat{k} - \hat{k}_0) \left(\mathbf{P} \mathbf{p}_{yk} + \mathbf{Q} \frac{\mathbf{y}_k}{N} \right) + \sum_{k<\Delta} \sqrt{2}(\hat{b}(\mathbf{k}) - \hat{b}_0) \left(\mathbf{y}_k + \frac{\mathbf{Q}}{N} \right). \tag{A.1}
\end{aligned}$$

The expectation value of H_1 , computed with respect to the ground state of fluctuations (5.11) is equal to

$$\begin{aligned}
\langle H_1 \rangle &= \langle 0^f | \langle n_k^e | \langle \Psi_A | H_1 | \Psi_A \rangle | n_k^e \rangle | 0^f \rangle \\
&= \langle \Psi_A | A^\dagger A \left(\sum_{k<\Delta} \frac{\hat{k}}{N} - \hat{k}_0 \right) + \frac{(A^\dagger + A)}{\sqrt{N}} \left(\sum_{k<\Delta} \hat{b}(\mathbf{k}) - N\hat{b}_0 \right) | \Psi_A \rangle. \tag{A.2}
\end{aligned}$$

By exploiting the definition of $|\Psi_A\rangle$ in equation (A.2), one obtains

$$\begin{aligned}
\langle H_1 \rangle &= \langle n | \bar{u}(p_n) S^\dagger \left(A^\dagger A \left(\sum_k \frac{\hat{k}}{N} - \hat{k}_0 \right) + \frac{\hat{b}_0 \hat{k}_0 \sum_k \hat{k}}{2\sqrt{N}(z \cdot k_0)} (A^\dagger + A) A^\dagger A + A^\dagger A (A + A^\dagger) \frac{\sum_k \hat{k} \hat{k}_0 \hat{b}_0}{2\sqrt{N}(z \cdot k_0)} \right. \\
&\quad \left. - \frac{\sqrt{N}(A + A^\dagger) A^\dagger A \hat{b}_0 \hat{k}_0 \hat{k}_0 + \frac{\hat{b}_0 \hat{k}_0 \sum_k \hat{k} \hat{k}_0 \hat{b}_0}{4(z \cdot k_0)^2} (A + A^\dagger) A^\dagger A (A + A^\dagger) \right) S u(p_n) | n \rangle \\
&\quad + \langle n | \bar{u}(p_n) S^\dagger \left(\frac{(A + A^\dagger)}{\sqrt{N}} \left(\sum_k \hat{b}(\mathbf{k}) - N\hat{b}_0 \right) + \frac{\hat{b}_0 \hat{k}_0 (\sum_k \hat{b}(\mathbf{k}) - N\hat{b}_0)}{2(z \cdot k_0)} (A + A^\dagger)^2 \right. \\
&\quad \left. + \frac{(\sum_k \hat{b}(\mathbf{k}) - N\hat{b}_0) \hat{k}_0 \hat{b}_0}{2(z \cdot k_0)} (A + A^\dagger)^2 + \frac{\sqrt{N} \hat{b}_0 \hat{k}_0 (\sum_k \hat{b}(\mathbf{k}) - N\hat{b}_0) \hat{k}_0 \hat{b}_0}{4(z \cdot k_0)^2} (A + A^\dagger)^3 \right) S u(p_n) | n \rangle, \tag{A.3}
\end{aligned}$$

where $S = e^{\alpha(A^\dagger - A)} e^{-\frac{\eta}{2}(A^2 - A^{\dagger 2})}$. Next we calculate the average of the field variables, taking into account the transformation law of A and A^\dagger by operator S :

$$\begin{aligned} S^\dagger A S &= \frac{1}{2} \left(\sqrt{\kappa} + \frac{1}{\sqrt{\kappa}} \right) A + \frac{1}{2} \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) A^\dagger + \alpha, \\ S^\dagger A^\dagger S &= \frac{1}{2} \left(\sqrt{\kappa} + \frac{1}{\sqrt{\kappa}} \right) A^\dagger + \frac{1}{2} \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}} \right) A + \alpha. \end{aligned} \quad (\text{A.4})$$

Parameter κ was defined in equation (5.14). Therefore, we can find how the combination of A and A^\dagger in equation (A.3) transforms, for example:

$$\begin{aligned} A^\dagger A &\rightarrow \frac{1}{2} \left(\kappa + \frac{1}{\kappa} \right) + \frac{1}{4} \left(\kappa - \frac{1}{\kappa} \right) (A^2 + A^{\dagger 2}) + \alpha \sqrt{\kappa} (A + A^\dagger) + \beta, \\ (A + A^\dagger) A^\dagger A &\rightarrow \frac{\sqrt{\kappa}}{2} \left(\kappa + \frac{1}{\kappa} \right) (A + A^\dagger) A^\dagger A \\ &\quad + \frac{\sqrt{\kappa}}{4} \left(\kappa - \frac{1}{\kappa} \right) (A + A^\dagger) (A^2 + A^{\dagger 2}) + \alpha \kappa (A + A^\dagger)^2 \\ &\quad + \beta \sqrt{\kappa} (A + A^\dagger) + \alpha \left(\kappa + \frac{1}{\kappa} \right) A^\dagger A + \frac{\alpha}{2} \left(\kappa - \frac{1}{\kappa} \right) (A^2 + A^{\dagger 2}) \\ &\quad + 2\alpha^2 \sqrt{\kappa} (A + A^\dagger) + 2\alpha\beta, \\ A^\dagger A (A + A^\dagger) &\rightarrow \frac{\sqrt{\kappa}}{2} \left(\kappa + \frac{1}{\kappa} \right) A^\dagger A (A + A^\dagger) \\ &\quad + \frac{\sqrt{\kappa}}{4} \left(\kappa - \frac{1}{\kappa} \right) (A^2 + A^{\dagger 2}) (A + A^\dagger) + \alpha \kappa (A + A^\dagger)^2 \\ &\quad + \beta \sqrt{\kappa} (A + A^\dagger) + \alpha \left(\kappa + \frac{1}{\kappa} \right) A^\dagger A + \frac{\alpha}{2} \left(\kappa - \frac{1}{\kappa} \right) (A^2 + A^{\dagger 2}) \\ &\quad + 2\alpha^2 \sqrt{\kappa} (A + A^\dagger) + 2\alpha\beta, \\ (A + A^\dagger) A^\dagger A (A + A^\dagger) &\rightarrow \frac{\kappa}{2} \left(\kappa + \frac{1}{\kappa} \right) (A + A^\dagger) A^\dagger A (A + A^\dagger) \\ &\quad + \frac{\kappa}{4} \left(\kappa - \frac{1}{\kappa} \right) (A + A^\dagger) (A^2 + A^{\dagger 2}) (A + A^\dagger) \\ &\quad + \alpha \kappa^{3/2} (A + A^\dagger)^3 + (\beta \kappa + 4\alpha^2 \kappa) (A + A^\dagger)^2 \\ &\quad + \alpha \sqrt{\kappa} \left(\kappa + \frac{1}{\kappa} \right) (A^\dagger A (A + A^\dagger) + (A + A^\dagger) A^\dagger A) \\ &\quad + \frac{\alpha \sqrt{\kappa}}{2} \left(\kappa - \frac{1}{\kappa} \right) ((A + A^\dagger) (A^2 + A^{\dagger 2}) + (A^2 + A^{\dagger 2}) (A + A^\dagger)) \\ &\quad + (4\alpha\beta \sqrt{\kappa} + 4\alpha^3 \sqrt{\kappa}) (A + A^\dagger) \\ &\quad + 2\alpha^2 \left(\kappa + \frac{1}{\kappa} \right) A^\dagger A + \alpha^2 \left(\kappa - \frac{1}{\kappa} \right) (A^2 + A^{\dagger 2}) + 4\alpha^2 \beta, \\ A + A^\dagger &\rightarrow \sqrt{\kappa} (A + A^\dagger) + 2\alpha \\ (A + A^\dagger)^2 &\rightarrow \kappa (A + A^\dagger)^2 + 4\alpha \sqrt{\kappa} (A + A^\dagger) + 4\alpha^2 \end{aligned}$$

$$(A + A^\dagger)^3 \rightarrow \kappa^{3/2}(A + A^\dagger)^3 + 6\alpha\kappa(A + A^\dagger)^2 + 12\alpha^2 \sqrt{\kappa}(A + A^\dagger) + 8\alpha^3 \quad (\text{A.5})$$

where $\beta = \left(\alpha^2 + \frac{1}{4} \left(\sqrt{\kappa} - \frac{1}{\sqrt{\kappa}}\right)^2\right)$.

According to the definition, the collective mode has a high intensity, i.e. it is highly populated, with the quantum number $n = n_0$ being a large value. Therefore, the averages with respect to the field variables are

$$\begin{aligned} \langle n_0 | S^\dagger (A + A^\dagger) S | n_0 \rangle &= 2\alpha, \\ \langle n_0 | S^\dagger (A + A^\dagger)^2 S | n_0 \rangle &= \kappa(2n_0 + 1) + 4\alpha^2, \\ \langle n_0 | S^\dagger (A + A^\dagger)^3 S | n_0 \rangle &= 6\alpha\kappa(2n_0 + 1) + 8\alpha^3, \\ \langle n_0 | S^\dagger A^\dagger A S | n_0 \rangle &= \frac{n_0}{2} \left(\kappa + \frac{1}{\kappa} \right) + \beta, \\ \langle n_0 | S^\dagger (A + A^\dagger) A^\dagger A S | n_0 \rangle &= \langle n_0 | S^\dagger A^\dagger A (A + A^\dagger) S | n_0 \rangle \\ &= \alpha\kappa(2n_0 + 1) + \alpha \left(\kappa + \frac{1}{\kappa} \right) n_0 + 2\alpha\beta, \\ \langle n_0 | S^\dagger (A + A^\dagger) A^\dagger A (A + A^\dagger) S | n_0 \rangle &= \frac{\kappa}{2} \left(\kappa + \frac{1}{\kappa} \right) (2n_0^2 + n_0 + 1) + \frac{\kappa}{4} \left(\kappa - \frac{1}{\kappa} \right) (2n_0^2 + 2n_0) \\ &\quad + (\beta\kappa + 4\alpha\kappa)(2n_0 + 1) + 2\alpha^2 \left(\kappa + \frac{1}{\kappa} \right) n_0 + 4\alpha^2\beta. \end{aligned} \quad (\text{A.6})$$

The last step is to calculate the averages in the Dirac spin space. For this purpose we will employ the electron's density matrix $u(p) \otimes \bar{u}(p) = \rho = 1/2(\hat{p} + m)(1 - \gamma^5 \hat{a})$, with p and a being its four momentum and polarization, respectively. For example,

$$u_\beta(p_n) \bar{u}_\alpha(p_n) \left(\frac{1}{N} \sum_{k < \Delta} \hat{k} - \hat{k}_0 \right)_{\alpha\beta} = \rho_{\beta\alpha} \left(\frac{1}{N} \sum_{k < \Delta} \hat{k} - \hat{k}_0 \right)_{\alpha\beta} = \text{Sp} \left(\rho \left(\frac{1}{N} \sum_{k < \Delta} \hat{k} - \hat{k}_0 \right) \right). \quad (\text{A.7})$$

Therefore the calculated values of the traces are

$$\begin{aligned} \text{Sp} \left(\rho \left(\frac{1}{N} \sum_{k < \Delta} \hat{k} - \hat{k}_0 \right) \right) &= \sum_{k < \Delta} \frac{2k \cdot p_n}{N} - 2k_0 \cdot p_n, \\ \text{Sp} \left(\rho \hat{b}_0 \hat{k}_0 \sum_{k < \Delta} \hat{k} \right) &= \sum_{k < \Delta} (2im\sigma^{ab_0kk_0} + 2k \cdot k_0 b_0 \cdot p_n - 2k \cdot b_0 k_0 \cdot p_n), \\ \text{Sp} \left(\rho \sum_{k < \Delta} \hat{k} \hat{k}_0 \hat{b}_0 \right) &= \sum_{k < \Delta} (-2im\sigma^{ab_0kk_0} + 2k \cdot k_0 b_0 \cdot p_n - 2k \cdot b_0 k_0 \cdot p_n), \\ \text{Sp} \left(\rho \hat{b}_0 \hat{k}_0 \sum_{k < \Delta} \hat{k} \hat{k}_0 \hat{b}_0 \right) &= -4b_0^2 \sum_{k < \Delta} k \cdot k_0 k_0 \cdot p_n, \\ \text{Sp} \left(\rho \left(\sum_{k < \Delta} \hat{b}(k) - N\hat{b}_0 \right) \right) &= 2 \sum_{k < \Delta} b(k) \cdot p_n - 2Nb_0 \cdot p_n, \\ \text{Sp} \left(\rho \hat{b}_0 \hat{k}_0 \left(\sum_{k < \Delta} \hat{b}(k) - N\hat{b}_0 \right) \right) &= 2 \left(-im \sum_{k < \Delta} \sigma^{ab_0k_0b(k)} + b_0^2 N k_0 \cdot p_n \right) \end{aligned} \quad (\text{A.8})$$

$$\begin{aligned}
& + \sum_{k < \Delta} b(\mathbf{k}) \cdot k_0 b_0 \cdot p_n - \sum_{k < \Delta} b_0 \cdot b(\mathbf{k}) k_0 \cdot p_n \Big), \\
\text{Sp} \left(\rho \left(\sum_{k < \Delta} \hat{b}(\mathbf{k}) - N \hat{b}_0 \right) \hat{k}_0 \hat{b}_0 \right) &= 2 \left(i m \sum_{k < \Delta} \sigma^{ab_0 k_0 b(\mathbf{k})} + b_0^2 N k_0 \cdot p_n \right. \\
& \left. + \sum_{k < \Delta} b(\mathbf{k}) \cdot k_0 b_0 \cdot p_n - \sum_{k < \Delta} b_0 \cdot b(\mathbf{k}) k_0 \cdot p_n \right), \\
\text{Sp} \left(\rho \hat{b}_0 \hat{k}_0 \left(\sum_{k < \Delta} \hat{b}(\mathbf{k}) - N \hat{b}_0 \right) \hat{k}_0 \hat{b}_0 \right) &= -4 b_0^2 \sum_{k < \Delta} b(\mathbf{k}) \cdot k_0 k_0 \cdot p_n,
\end{aligned}$$

where σ^{abcd} means the contraction of absolutely antisymmetric four-tensor Levi-Civita with any four vectors a, b, c, d . For example, $\sigma^{abcd} = \sum_{\alpha\beta\gamma\delta} \epsilon^{\alpha\beta\gamma\delta} a_\alpha b_\beta c_\gamma d_\delta$ or $\sigma^{0abc} = \sum_{\alpha\beta\gamma} \epsilon^{0\alpha\beta\gamma} a_\alpha b_\beta c_\gamma$.

Inserting (A.6) and (A.8) into (A.3) we find the average value of the Hamiltonian

$$\begin{aligned}
\langle H_1 \rangle &= n_0 \left(\kappa + \frac{1}{\kappa} \right) \left(\sum_{k < \Delta} \frac{k}{N} - k_0 \right) \cdot p_n \\
&+ 2 \frac{2\alpha\kappa n_0 + \alpha \left(\kappa + \frac{1}{\kappa} \right) n_0}{\sqrt{N}(z \cdot k_0)} \left((b_0 \cdot p_n) \sum_{k < \Delta} (k \cdot k_0) - (k_0 \cdot p_n) \sum_{k < \Delta} (k \cdot b_0) \right) \\
&- \frac{n_0^2 (\kappa + 1/\kappa) \kappa + n_0^2 (\kappa - 1/\kappa) \frac{\kappa}{2}}{(z \cdot k_0)^2} \sum_{k < \Delta} b_0^2 (k \cdot k_0) (k_0 \cdot p_n) \\
&+ \frac{4\alpha}{\sqrt{N}} \left(\sum_{k < \Delta} b(\mathbf{k}) - N b_0 \right) \cdot p_n + \frac{4\kappa n_0}{(z \cdot k_0)} \left\{ \left(N b_0^2 - \sum_{k < \Delta} (b_0 \cdot b(\mathbf{k})) \right) (k_0 \cdot p_n) \right. \\
&\left. + (b_0 \cdot p_n) \sum_{k < \Delta} (b(\mathbf{k}) \cdot k_0) \right\} - \frac{12\alpha\kappa n_0 \sqrt{N}}{(z \cdot k_0)^2} b_0^2 \sum_{k < \Delta} (b(\mathbf{k}) \cdot k_0) (k_0 \cdot p_n),
\end{aligned} \tag{A.9}$$

where only the leading terms in n_0 are left.

Appendix B

Calculation of the integrals in the second order correction to the approximating Hamiltonian

The first term in equation (6.23) is

$$\sum_{k < \Delta} \frac{|(\omega_k - \omega_0) - (\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v}|^2}{\omega_0 - \omega_k} = \frac{V}{(2\pi)^3} \int_{\Delta} d\mathbf{k} \left((\omega_0 - \omega_k) + 2(\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v} + \frac{|(\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v}|^2}{\omega_0 - \omega_k} \right) = \frac{V}{(2\pi)^3} \int_{\Delta} d\mathbf{k} \frac{|(\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v}|^2}{\omega_0 - \omega_k} \quad (\text{B.1})$$

The first two integrals in equation (B.1) are equal to zero as integrals of an odd function over a symmetric interval. Now, we change a variable \mathbf{k} to $\mathbf{k}_0 + \mathbf{q}$. After expansion of the scalar product in (B.1), the terms linear in \mathbf{q} will also not contribute as they are the odd functions. For this reason, one obtains

$$\begin{aligned} \frac{V}{(2\pi)^3} \int_{\Delta} d\mathbf{k} \frac{|(\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v}|^2}{\omega_0 - \omega_k} &= \frac{V}{(2\pi)^3} \int_{\Delta} d\mathbf{q} \frac{(\mathbf{q} \cdot \mathbf{v})^2 (\omega_0 + \omega_k)}{\omega_0^2 - \omega_k^2} \\ &= \frac{2\omega_0 V}{(2\pi)^3} \int_{\Delta} d\mathbf{q} \frac{(\mathbf{q} \cdot \mathbf{v})^2}{\omega_0^2 - \omega_k^2} = -\frac{2\omega_0 V}{(2\pi)^3} \int_{\Delta} d\mathbf{q} \frac{v_{\perp}^2 q_x^2 + v_z^2 q_z^2}{2\omega_0 q_z + q_z^2 + q_x^2 + q_y^2}. \end{aligned} \quad (\text{B.2})$$

The denominator of equation (B.2) is equal to zero for $q_{z,1} = -2\omega_0$, $q_{z,2} = -(q_x^2 + q_y^2)/2\omega_0$ and needs to be regularized via [2]

$$\frac{1}{u} = P \frac{1}{u} - i\pi \delta(u), \quad (\text{B.3})$$

with the symbol P being the principal value. Insertion of (B.3) into (B.2) gives

$$\begin{aligned} &- \frac{2\omega_0 V}{(2\pi)^3} \int_{\Delta} d\mathbf{q} \frac{v_{\perp}^2 q_x^2 + v_z^2 q_z^2}{(q_z - q_{z,1})(q_z - q_{z,2})} \\ &+ \frac{2i\pi V \omega_0}{(2\pi)^3} \int_{\Delta} d\mathbf{q} (v_{\perp}^2 q_x^2 + v_z^2 q_z^2) \delta((q_z - q_{z,1})(q_z - q_{z,2})) \\ &= -\frac{V v_{\perp}^2}{(2\pi)^3} \int_{\Delta} d\mathbf{q} \frac{q_x^2}{q_z + \frac{q_x^2 + q_y^2}{2\omega_0}} + \frac{i\pi V v_{\perp}^2}{(2\pi)^3} \int_{\Delta} d\mathbf{q} q_x^2 \delta\left(q_z + \frac{q_x^2 + q_y^2}{2\omega_0}\right). \end{aligned} \quad (\text{B.4})$$

While obtaining (B.4) we took into account that $(q_z - q_{z,1})(q_z - q_{z,2}) = 2\omega_0(q_z - q_{z,2})$ and the terms proportional to v_z^2 can be neglected, as the integration takes place near zero, leading to $q_z^2 \sim q_\perp^4$. The integration in the first term is performed in a polar coordinate system, yielding:

$$\begin{aligned} & - \frac{Vv_\perp^2}{(2\pi)^3} \int_\Delta d\mathbf{q} \frac{q_x^2}{q_z + \frac{q_x^2 + q_y^2}{2\omega_0}} = - \frac{Vv_\perp^2}{(2\pi)^3} \int dq_z dq_\perp d\phi \frac{q_\perp^3 \cos^2 \phi}{q_z + \frac{q_\perp^2}{2\omega_0}} \\ & = - \frac{\pi Vv_\perp^2}{(2\pi)^3} \left(\frac{\Delta_1^2 \Delta_2 \omega_0}{8} - \frac{\Delta_2^2 \omega_0^2}{2} \tan^{-1} \frac{4\Delta_2 \omega_0}{\Delta_1^2} \right) \end{aligned} \quad (\text{B.5})$$

The integration of the second term with respect to q_z yields a Heaviside function $\theta(\omega_0 \Delta_2 - q_x^2 - q_y^2)$ as $q_z = -(q_x^2 + q_y^2)/2\omega_0 > -\Delta_2/2$. Therefore,

$$\frac{i\pi Vv_\perp^2}{(2\pi)^3} \int d\mathbf{q} q_x^2 \delta\left(q_z + \frac{q_x^2 + q_y^2}{2\omega_0}\right) = \frac{i\pi^2 Vv_\perp^2}{(2\pi)^3} \cdot \begin{cases} \frac{(\omega_0 \Delta_2)^2}{4}, & \sqrt{\omega_0 \Delta_2} < \Delta_1 \\ \frac{\Delta_1^4}{4}, & \sqrt{\omega_0 \Delta_2} > \Delta_1 \end{cases}, \quad (\text{B.6})$$

Combining together (B.6) and (B.5)

$$\begin{aligned} \frac{V}{(2\pi)^3} \int_\Delta d\mathbf{k} \frac{|(\mathbf{k} - \mathbf{k}_0) \cdot \mathbf{v}|^2}{\omega_0 - \omega_k} &= \frac{\pi Vv_\perp^2 \omega_0^4 \delta^2}{(2\pi)^3} \left(\frac{\delta \sigma_1^2 \sigma_2}{8} - \frac{\sigma_2^2}{2} \tan^{-1} \frac{4\sigma_2}{\delta \sigma_1^2} \right) \\ &+ \frac{i\pi^2 Vv_\perp^2 \omega_0^4}{(2\pi)^3} \cdot \begin{cases} \delta^2 \sigma_2^2, & \sqrt{\frac{\sigma_2}{\delta}} < \sigma_1 \\ \delta^4 \sigma_1^4, & \sqrt{\frac{\sigma_2}{\delta}} > \sigma_1 \end{cases}. \end{aligned} \quad (\text{B.7})$$

At last, we come to the calculation of the remaining correction in equation (B.1) from the fluctuations:

$$\sum_{k < \Delta} \frac{|\mathbf{v}(\mathbf{b}(\mathbf{k}) - \mathbf{b}_0)|^2}{\omega_k} = \frac{e^2}{2V} \sum_{k < \Delta, \alpha} \frac{|\mathbf{v} \cdot (\mathbf{e}_{k,\alpha} / \sqrt{\omega_k} - \mathbf{e}_{0,\alpha} / \sqrt{\omega_0})|^2}{\omega_k}. \quad (\text{B.8})$$

In order to calculate the integral in equation (B.8), we introduce the polarization vectors

$$\begin{aligned} \mathbf{e}_{0,2} &= \frac{\mathbf{k}_0 \times \mathbf{v}}{\omega_0 v}, & \mathbf{e}_{0,1} &= \frac{\mathbf{e}_{0,2} \times \mathbf{k}_0}{\omega_0} = \frac{(\mathbf{k}_0 \times \mathbf{v}) \times \mathbf{k}_0}{\omega_0^2 v} = \frac{\mathbf{v} \omega_0^2 - \mathbf{k}_0 (\mathbf{k}_0 \cdot \mathbf{v})}{\omega_0^2 v}, \\ \mathbf{e}_{k,2} &= \frac{\mathbf{k} \times \mathbf{v}}{\omega_k v}, & \mathbf{e}_{k,1} &= \frac{\mathbf{e}_{k,2} \times \mathbf{k}}{\omega_k} = \frac{(\mathbf{k} \times \mathbf{v}) \times \mathbf{k}}{\omega_k^2 v} = \frac{\mathbf{v} \omega_k^2 - \mathbf{k} (\mathbf{k} \cdot \mathbf{v})}{\omega_k^2 v}. \end{aligned} \quad (\text{B.9})$$

Here we pay attention to the fact that $\mathbf{v} \cdot \mathbf{e}_{k,2} = \mathbf{v} \cdot \mathbf{e}_{0,2} = 0$. Insertion of equation (B.9) into equation (B.8) gives

$$\frac{e^2}{2(2\pi)^3 \omega_0} \int d\mathbf{k} \left| v \left(\frac{1}{\sqrt{\omega_k}} - \frac{1}{\sqrt{\omega_0}} \right) - \frac{1}{v} \left(\frac{(\mathbf{k} \cdot \mathbf{v})^2}{\omega_k^{5/2}} - \frac{(\mathbf{k}_0 \cdot \mathbf{v})^2}{\omega_0^{5/2}} \right) \right|^2. \quad (\text{B.10})$$

The change of variable \mathbf{k} to $\mathbf{k}_0 + \mathbf{q}$ and the decomposition of the differences in brackets up to the first order in Taylor series in \mathbf{q} bring us to the final result

$$\begin{aligned} \frac{e^2}{2(2\pi)^3\omega_0} \int d\mathbf{q} \left| \left(-\frac{v}{2\omega_0^{5/2}} + \frac{5(\mathbf{k}_0 \cdot \mathbf{v})^2}{2v\omega_0^{9/2}} \right) \mathbf{k}_0 \cdot \mathbf{q} - \frac{2\mathbf{k}_0 \cdot \mathbf{v}}{v\omega_0^{5/2}} \mathbf{v} \cdot \mathbf{q} \right|^2 \\ = e^2 \frac{\delta^5 \sigma_1^2 \sigma_2 \omega_0 \left(\sigma_2^2 (v^2 - v_z^2)^2 + 16\sigma_1^2 v_\perp^2 v_z^2 \right)}{96(2\pi)^3 v^2}. \end{aligned} \quad (\text{B.11})$$

Combining all together, we finally obtain the second-order correction to the system's energy

$$\begin{aligned} E_0^{(2)} &= \frac{n_0 \omega_0 \pi v_\perp^2}{4} \left(\lambda \tan^{-1} 4\lambda - \frac{1}{4} + \frac{i\pi}{2} \cdot \begin{cases} \lambda, & \lambda > 1 \\ \frac{1}{\lambda}, & \lambda < 1 \end{cases} \right) \\ &+ e^2 \frac{\delta^5 \sigma_1^2 \sigma_2 \omega_0 \left(\sigma_2^2 (v^2 - v_z^2)^2 + 16\sigma_1^2 v_\perp^2 v_z^2 \right)}{96(2\pi)^3 v^2}, \end{aligned} \quad (\text{B.12})$$

where $\lambda = \sigma_2/(\delta\sigma_1^2)$.

Appendix C

Saddle point method

In the present appendix we will explain saddle point method [137] for an asymptotic integral evaluation.

Imagine a function of a complex variable z , which is represented as an integral

$$J(z) = \int_C e^{zf(t)} dt, \quad (\text{C.1})$$

where the integrand goes to zero at the ends of the contour C . Such form of the integral is commonly used in the theory of special functions. For instance, the integral representation $\int_0^\infty e^{-\tau} \tau^z d\tau$ of the gamma function $\Gamma(z+1)$, with the variable change $\tau = tz$, transforms into

$$\Gamma(z+1) = z^{z+1} \int_0^\infty e^{-tz} t^z dt = z^{z+1} \int_0^\infty e^{z(\ln t - t)} dt, \quad (\text{C.2})$$

and consequently, the function $\Gamma(z+1)/z^{z+1}$ coincides with (C.1).

When $|z| \rightarrow \infty$, it leads to some very rapid oscillations in the value of the integrand. If z is complex, or if $f(t)$ is complex along some parts of the contour C , the imaginary part of $zf(t)$ will generally increase together with the increase of $|z|$. Therefore, $i\text{Im}[zf(t)]$ will rapidly oscillate, with increasing frequency. Consequently, large positive values are almost completely cancelled by large negative values on the integration contour. As the contour of the integration can be chosen in an arbitrary way, we can try to deform a contour, in order to minimize the oscillations.

In general case, the contour goes through the regions where $\text{Re}(zf(t))$ is positive and where it is negative. In the regions where $\text{Re}(zf(t))$ is positive the integrand is larger, and it is most desirable to reduce oscillations. Therefore, we search a contour on which the real part is the largest and imaginary part is constant. For this reason, we can write the integrand as

$$J(z) = \int_C e^{zf(t)} dt = e^{\text{Im}[zf(t)]} \int_C e^{\text{Re}[zf(t)]} dt. \quad (\text{C.3})$$

The extremum of $\text{Re}[zf(t)]$ is defined by the first derivative of $f(t)$, i.e.,

$$f'(t_0) = 0. \quad (\text{C.4})$$

The surface in the neighborhood of t_0 must be flat, as it cannot be a maximum or a minimum, because of the theorem of the maximum of the absolute value of a function of a complex

variable. The point t_0 must therefore be a saddle point. By this consideration the contour of integration can be chosen in two ways. The first one is when the value of $zf(t)$ increases at the end of the contour, however this choice can not correspond to the convergent integral; hence the only possible way is to chose a contour such that it decreases as rapidly as possible at its ends, so most of the value of the integral comes form the neighborhood of t_0 . For example, a good analogy is a mountain path. The method works better with the increase of $|z|$, as the maximum of the “mountain path” becomes sharper and sharper.

Let us illustrate the above discussion. In the neighborhood of t_0 we can expand $f(t)$ in a Taylor series:

$$f(t) = f(t_0) + f'' \frac{(t - t_0)^2}{2} + \dots \quad (\text{C.5})$$

The path should be chosen such that the integrand decreases exponentially. The change of the variable

$$\tau = \sqrt{e^{i(\pi+\phi)} f''(t_0)} (t - t_0), \quad z = |z| e^{i\phi}, \quad (\text{C.6})$$

leads to

$$J(z) \sim \frac{e^{zf(t_0)}}{\sqrt{e^{i(\pi+\phi)} f''(t_0)}} \int_C e^{-\frac{1}{2}|z|^2 \tau^2} d\tau. \quad (\text{C.7})$$

From here can be seen, as $|z|$ increases, the exponential becomes sharper and less of the contour becomes important. For sufficiently large $|z|$ is decreasing and at some point becomes effectively zero. Therefore, contour can be chosen along the real axis from $-\infty$ to ∞ . Integration in (C.7) for $|z| \rightarrow \infty$ yields

$$J(z) \sim e^{zf(t_0)} \sqrt{\frac{2\pi}{ze^{i\pi} f''(t_0)}}. \quad (\text{C.8})$$

For a gamma function $f(t) = \ln t - t$ and $f'(t_0) = 1/2 - 1$. Consequently, $t_0 = 1$, $f(t_0) = -1$ and $f''(t_0) = -1$. The variable $\tau = (t - 1)e^{i\phi/2}$. Hence, we obtain Stirling's formula

$$\Gamma(z + 1) \xrightarrow{z \rightarrow \infty} \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z}. \quad (\text{C.9})$$

Equation (C.8) is the leading term in an asymptotic series. In order to obtain the next terms in a series we include a phase of z in $f(t)$, thus considering z as a real.

Let us make a substitution in equation (C.5)

$$f(t) = f(t_0) - w^2, \quad (\text{C.10})$$

The variable w is a real as $\text{Im}[f(t)] = \text{Im}[f(t_0)]$. By plugging (C.10) into $J(z)$, one obtains

$$J(z) = e^{zf(t_0)} \int_C e^{-zw^2} dt = e^{zf(t_0)} \int_C e^{-zw^2} \frac{dt}{dw} dw. \quad (\text{C.11})$$

We also assume that the integration direction for w is from $-\infty$ to $+\infty$:

$$J(z) = e^{zf(t_0)} \int_{-\infty}^{+\infty} e^{-zw^2} \frac{dt}{dw} dw. \quad (\text{C.12})$$

In order to perform evaluation in equation (C.12) the derivative dt/dw need to be computed. For this purpose, we expand it in a power series

$$\frac{dt}{dw} = \sum_{n=0}^{\infty} a_n w^n, \quad (\text{C.13})$$

where according to equation (C.10) only even powers of w will enter. The substitution of (C.13) into (C.12) gives the asymptotic series

$$J(z) \xrightarrow{z \rightarrow \infty} e^{zf(t_0)} \sqrt{\frac{\pi a_0^2}{z}} \sum_{n=0}^{\infty} \left(\frac{a_{2n}}{a_0} \right) \frac{\Gamma\left(n + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{z}\right)^n. \quad (\text{C.14})$$

The coefficients a_n are determined by [137]:

$$a_n = \frac{1}{n!} \left\{ \frac{d^n}{dx^n} \left[\sum_p A_p x^p \right] \right\}_{x=0}^{-\frac{n}{2}-\frac{1}{2}}, \quad (\text{C.15})$$

$$\left[\frac{f(t_0) - f(t)}{(t - t_0)^2} \right] = \sum_p A_p (t - t_0)^p.$$

The application of relations (C.14) and (C.15) to the $\Gamma(z + 1)$ yields

$$\left[\frac{t - 1 - \ln t}{(t - 1)^2} \right] = \frac{1}{2} - \frac{t - 1}{3} + \frac{(t - 1)^2}{4} - \dots, \quad (\text{C.16})$$

and consequently $A_n = (-1)^n / (n+2)$ and $a_0 = 1/\sqrt{2}$, $a_2/a_0 = 1/6$ and $a_4/a_0 = 1/216$. Therefore the asymptotic series for the gamma function looks like

$$\Gamma(z + 1) = \sqrt{2\pi} z^{z+\frac{1}{2}} e^{-z} \left[1 + \frac{1}{12z} + \frac{1}{288z^2} + \dots \right]. \quad (\text{C.17})$$

The saddle point method is widely used in the theory of the special functions, for example for finding the asymptotic behavior of the Bessel and related functions.

Appendix D

Traces of gamma matrices

In order to calculate the decay rates or cross section, with the rules specified in the previous paragraph we need to calculate the traces of gamma matrices. Here we briefly revise this procedure [121].

We start from the anti-commutation relation of gamma matrices

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2g^{\alpha\beta}, \quad (D.1)$$

$$\gamma^\alpha \gamma^5 + \gamma^5 \gamma^\alpha = 0, \quad (\gamma^5)^2 = 1. \quad (D.2)$$

Let us prove few statements:

- Trace of the odd number of gamma matrices is equal to zero. As follows from equation (D.2)

$$\begin{aligned} \text{Sp} \underbrace{\gamma^\alpha \gamma^\beta \dots \gamma^\sigma}_{\text{even number}} &= \text{Sp} \gamma^\alpha \gamma^\beta \dots \gamma^\sigma \gamma^5 \gamma^5 = -\text{Sp} \gamma^5 \gamma^\alpha \gamma^\beta \dots \gamma^\sigma \gamma^5 = \\ &= -\text{Sp} \gamma^\alpha \gamma^\beta \dots \gamma^\sigma \gamma^5 \gamma^5 = -\text{Sp} \gamma^\alpha \gamma^\beta \dots \gamma^\sigma = 0. \end{aligned}$$

- $\text{Sp} \gamma^\alpha \gamma^\beta = 4g^{\alpha\beta}$

$$\text{Sp} \gamma^\alpha \gamma^\beta = \text{Sp}(2g^{\alpha\beta} - \gamma^\beta \gamma^\alpha) = 8g^{\alpha\beta} - \text{Sp} \gamma^\alpha \gamma^\beta.$$

Therefore

$$\text{Sp} \hat{a} \hat{b} = 4(a \cdot b).$$

- $\text{Sp} \gamma^\alpha \gamma^\beta \gamma^\rho \gamma^\sigma = 4(g^{\alpha\beta} g^{\rho\sigma} - g^{\alpha\rho} g^{\beta\sigma} + g^{\alpha\sigma} g^{\beta\rho})$

The proof is exactly the same as in previous case.

- $\text{Sp} \gamma^5 = 0$

$$\text{Sp} \gamma^5 = \text{Sp} \gamma^5 \gamma^0 \gamma^0 = -\text{Sp} \gamma^0 \gamma^5 \gamma^0 = \text{Sp} \gamma^5 \gamma^0 \gamma^0 = -\text{Sp} \gamma^5.$$

- $\text{Sp}\gamma^5\gamma^\alpha\gamma^\beta = 0$

$$\text{Sp}\gamma^5\gamma^\alpha\gamma^\beta = \text{Sp}\gamma^5\frac{1}{2}(\gamma^\alpha\gamma^\beta + \gamma^\beta\gamma^\alpha) + \text{Sp}\gamma^5\frac{1}{2}(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha)$$

Next we will use the relation

$$\gamma^5(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha) = i\epsilon^{\alpha\beta\rho\sigma}\gamma_\sigma\gamma_\rho.$$

Therefore

$$\text{Sp}\gamma^5\gamma^\alpha\gamma^\beta = g^{\alpha\beta}\text{Sp}\gamma^5 + \frac{i}{2}\epsilon^{\alpha\beta\rho\sigma}\text{Sp}\gamma_\rho\gamma_\sigma = 2i\epsilon^{\alpha\beta\rho\sigma}g_{\rho\sigma} = 0.$$

- $\text{Sp}\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\gamma^5 = -4i\epsilon^{\alpha\beta\rho\sigma}.$

The trace in the left hand side is symmetrical over indices permutation

$$\text{Sp}\gamma^\alpha\gamma^\beta\gamma^\rho\gamma^\sigma\gamma^5 = \text{Sp}(2g^{\alpha\beta} - \gamma^\beta\gamma^\alpha)\gamma^\rho\gamma^\sigma\gamma^5 = -\text{Sp}\gamma^\beta\gamma^\alpha\gamma^\rho\gamma^\sigma\gamma^5 = \dots$$

By using the definition of $\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3$ we obtain

$$\text{Sp}\gamma^0\gamma^1\gamma^2\gamma^3\gamma^5 = 4i = -4i\epsilon^{0123}$$

- $\text{Sp}\gamma^{\alpha_1}\gamma^{\alpha_2}\dots\gamma^{\alpha_n} = g^{\alpha_1\alpha_2}\text{Sp}\gamma^{\alpha_3}\dots\gamma^{\alpha_n} - g^{\alpha_1\alpha_3}\text{Sp}\gamma^{\alpha_2}\dots\gamma^{\alpha_n} + \dots g^{\alpha_1\alpha_n}\text{Sp}\gamma^{\alpha_2}\gamma^{\alpha_3}\dots$

This relation gives possibility to decrease the number of gamma matrices inside the trace from n to $n-2$. The proof of this relation is performed by using anti-commutation relation (D.1) as was done in the case of two and four matrices.

Bibliography

- [1] S. Weinberg. *Scientific American*, 231:50, 1974.
- [2] A. I. Akhiezer and V. B. Berestetskij. *Quantum Electrodynamics*. Nauka, 3rd edition, 1969.
- [3] S. Sturm, F. Kohler, J. Zatorski, A. Wagner, Z. Harman, G. Werth, W. Quint, C. H. Keitel, and K. Blaum. High-precision measurement of the atomic mass of the electron. *Nature*, 506(7489):467–470, 02 2014.
- [4] Gerard A. Mourou, Toshiki Tajima, and Sergei V. Bulanov. Optics in the relativistic regime. *Rev. Mod. Phys.*, 78:309–371, Apr 2006.
- [5] V. Bagnoud, B. Aurand, A. Blazevic, S. Borneis, C. Bruske, B. Ecker, U. Eisenbarth, J. Fils, A. Frank, E. Gaul, S. Goette, C. Haefner, T. Hahn, K. Harres, H.-M. Heuck, D. Hochhaus, D. Hoffmann, D. Javorková, H.-J. Kluge, T. Kuehl, S. Kunzer, M. Kreutz, T. Merz-Mantwill, P. Neumayer, E. Onkels, D. Reemts, O. Rosmej, M. Roth, T. Stoehlker, A. Tauschwitz, B. Zielbauer, D. Zimmer, and K. Witte. Commissioning and early experiments of the phelix facility. *Applied Physics B: Lasers and Optics*, 100:137–150, 2010.
- [6] Ed Gerstner. Laser physics: Extreme light. *Nature*, 446(7131):16–18, 03 2007.
- [7] V. Yanovsky, V. Chvykov, G. Kalinchenko, P. Rousseau, T. Planchon, T. Matsuoka, A. Maksimchuk, J. Nees, G. Cheriaux, G. Mourou, and K. Krushelnick. Ultra-high intensity- 300-tw laser at 0.1 hz repetition rate. *Opt. Express*, 16(3):2109–2114, Feb 2008.
- [8] Hercules laser facility. <http://cuos.engin.umich.edu/researchgroups/hfs/facilities/hercules-petawatt-laser/>, 2014.
- [9] ELI, Extreme Light Infrastructure. <http://www.extreme-light-infrastructure.eu/>, 2014.
- [10] HiPER, High Power Laser Energy Research. <http://www.hiper-laser.org/>, 2014.
- [11] D. M. Volkov. *JETP*, 7:1286, 1937.
- [12] W. H. Furry. On bound states and scattering in positron theory. *Phys. Rev.*, 81:115–124, Jan 1951.
- [13] A. I. Nikishov and V. I. Ritus. *Sov. Phys. JETP*, 19:529, 1964.

- [14] N. B. Narozhny, A. I. Nikishov, and V. I. Ritus. *Sov. Phys. JETP*, 20:622, 1965.
- [15] V. I. Ritus. Quantum effects of the interaction of elementary particles with an intense electromagnetic field. *Journal of Russian Laser Research*, 6:497–617, 1985. 10.1007/BF01120220.
- [16] A.I. Nikishov. Problems of intense external-field intensity in quantum electrodynamics. *Journal of Soviet Laser Research*, 6(6):619–717, 1985.
- [17] Howard R. Reiss. A convergent perturbation expansion in first-quantized electrodynamics. *Journal of Mathematical Physics*, 3(3):387–395, 1962.
- [18] T. H. Maiman. Stimulated optical radiation in ruby. *Nature*, 187(4736):493–494, 08 1960.
- [19] Donna Strickland and Gerard Mourou. Compression of amplified chirped optical pulses. *Optics Communications*, 55(6):447 – 449, 1985.
- [20] Lowell S. Brown and T. W. B. Kibble. Interaction of intense laser beams with electrons. *Phys. Rev.*, 133:A705–A719, Feb 1964.
- [21] Milton D. Slaughter. Electron-laser pulse scattering. *Phys. Rev. D*, 11:1639–1648, Mar 1975.
- [22] C. Bamber, S. J. Boege, T. Koffas, T. Kotseroglou, A. C. Melissinos, D. D. Meyerhofer, D. A. Reis, W. Ragg, C. Bula, K. T. McDonald, E. J. Prebys, D. L. Burke, R. C. Field, G. Horton-Smith, J. E. Spencer, D. Walz, S. C. Berridge, W. M. Bugg, K. Shmakov, and A. W. Weidemann. Studies of nonlinear QED in collisions of 46.6 GeV electrons with intense laser pulses. *Phys. Rev. D*, 60:092004, Oct 1999.
- [23] P. Panek, J.Z. Kamiński, and F. Ehlotzky. X-ray generation by Compton scattering of elliptically polarized laser light at relativistic radiation powers. *Optics Communications*, 213(1–3):121 – 128, 2002.
- [24] A. Di Piazza, K. Z. Hatsagortsyan, and C. H. Keitel. Quantum radiation reaction effects in multiphoton Compton scattering. *Phys. Rev. Lett.*, 105:220403, Nov 2010.
- [25] T. Heinzl, D. Seipt, and B. Kämpfer. Beam-shape effects in nonlinear Compton and Thomson scattering. *Phys. Rev. A*, 81:022125, Feb 2010.
- [26] D. Seipt and B. Kämpfer. Nonlinear Compton scattering of ultrashort intense laser pulses. *Phys. Rev. A*, 83:022101, Feb 2011.
- [27] F. Mackenroth and A. Di Piazza. Nonlinear Compton scattering in ultrashort laser pulses. *Phys. Rev. A*, 83:032106, Mar 2011.
- [28] K. Krajewska and J. Z. Kamiński. Compton process in intense short laser pulses. *Phys. Rev. A*, 85:062102, Jun 2012.

- [29] V. N. Nedoroshta, A. I. Voroshilo, and S. P. Roshchupkin. Resonant scattering of a photon by an electron in the moderately-strong-pulsed laser field. *Phys. Rev. A*, 88:052109, Nov 2013.
- [30] Erik Lötstedt and Ulrich D. Jentschura. Nonperturbative treatment of double Compton backscattering in intense laser fields. *Phys. Rev. Lett.*, 103:110404, Sep 2009.
- [31] F. Mackenroth and A. Di Piazza. Nonlinear double Compton scattering in the ultrarelativistic quantum regime. *Phys. Rev. Lett.*, 110:070402, Feb 2013.
- [32] F. Mackenroth, A. Di Piazza, and C. H. Keitel. Determining the carrier-envelope phase of intense few-cycle laser pulses. *Phys. Rev. Lett.*, 105:063903, Aug 2010.
- [33] Erik Lötstedt. Laser-assisted bremsstrahlung and electron-positron pair creation in relativistic laser fields. *AIP Conference Proceedings*, 1153(1):75–82, 2009.
- [34] Tim-Oliver Müller and Carsten Müller. Spin correlations in nonperturbative electron–positron pair creation by petawatt laser pulses colliding with a tev proton beam. *Physics Letters B*, 696(3):201 – 206, 2011.
- [35] K. Krajewska and J. Z. Kamiński. Correlations in laser-induced electron-positron pair creation. *Phys. Rev. A*, 84:033416, Sep 2011.
- [36] K. Krajewska and J. Z. Kamiński. Phase effects in laser-induced electron-positron pair creation. *Phys. Rev. A*, 85:043404, Apr 2012.
- [37] K. Krajewska and J. Z. Kamiński. Breit-Wheeler process in intense short laser pulses. *Phys. Rev. A*, 86:052104, Nov 2012.
- [38] S. Meuren, C. H. Keitel, and A. Di Piazza. Polarization operator for plane-wave background fields. *Phys. Rev. D*, 88:013007, Jul 2013.
- [39] Huayu Hu, Carsten Müller, and Christoph H. Keitel. Complete QED theory of multi-photon trident pair production in strong laser fields. *Phys. Rev. Lett.*, 105:080401, Aug 2010.
- [40] S. Meuren and A. Di Piazza. Quantum electron self-interaction in a strong laser field. *Phys. Rev. Lett.*, 107:260401, Dec 2011.
- [41] V. P. Yakovlev. Electron-positron pair production by a strong electromagnetic wave in the field of a nucleus. *Sov. Phys. JETP*, 22(223), 1966.
- [42] A. Di Piazza, A. I. Milstein, and C. Müller. Polarization of the electron and positron produced in combined coulomb and strong laser fields. *Phys. Rev. A*, 82:062110, Dec 2010.
- [43] A. Di Piazza and A. I. Milstein. Angular distribution of high-energy e^+e^- photoproduction close to the end of the spectrum at large momentum transfer. *Phys. Rev. A*, 85:042107, Apr 2012.

- [44] K. Krajewska, C. Müller, and J. Z. Kamiński. Bethe-Heitler pair production in ultrastrong short laser pulses. *Phys. Rev. A*, 87:062107, Jun 2013.
- [45] M. W. Walser, D. J. Urbach, K. Z. Hatsagortsyan, S. X. Hu, and C. H. Keitel. Spin and radiation in intense laser fields. *Phys. Rev. A*, 65:043410, Apr 2002.
- [46] Sven Ahrens, Heiko Bauke, Christoph H. Keitel, and Carsten Müller. Kapitza-Dirac effect in the relativistic regime. *Phys. Rev. A*, 88:012115, Jul 2013.
- [47] Sven Ahrens, Heiko Bauke, Christoph H. Keitel, and Carsten Müller. Spin dynamics in the Kapitza-Dirac effect. *Phys. Rev. Lett.*, 109:043601, Jul 2012.
- [48] V.G. Bagrov, Yu.I. Klimenko, and V.R. Khalilov. Influence of induced processes on the spin dynamics of an electron moving in a plane wave. *Soviet Physics Journal*, 25(10):932–936, 1982.
- [49] I.M. Ternov, A.I. Studenikin, and A.M. Khapaev. Determination of the radiation polarization of charges. *Soviet Physics Journal*, 33(1):45–48, 1990.
- [50] D.Yu. Ivanov, G.L. Kotkin, and V.G. Serbo. Complete description of polarization effects in emission of a photon by an electron in the field of a strong laser wave. *The European Physical Journal C - Particles and Fields*, 36(1):127–145, 2004.
- [51] D.Yu. Ivanov, G.L. Kotkin, and V.G. Serbo. Complete description of polarization effects in $e^+ e^-$ - pair production by a photon in the field of a strong laser wave. *The European Physical Journal C - Particles and Fields*, 40(1):27–40, 2005.
- [52] G.L. Kotkin, S.I. Polityko, and V.G. Serbo. Polarization of final electrons in the Compton effect. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 405(1):30 – 38, 1998.
- [53] G.L. Kotkin, H. Perlt, and V.G. Serbo. Polarization of high-energy electrons traversing a laser beam. *Nuclear Instruments and Methods in Physics Research Section A: Accelerators, Spectrometers, Detectors and Associated Equipment*, 404(2–3):430 – 436, 1998.
- [54] G. L. Kotkin, V. G. Serbo, and V. I. Telnov. Electron (positron) beam polarization by Compton scattering on circularly polarized laser photons. *Phys. Rev. ST Accel. Beams*, 6:011001, Jan 2003.
- [55] Dmitry V. Karlovets. Radiative polarization of electrons in a strong laser wave. *Phys. Rev. A*, 84:062116, Dec 2011.
- [56] Madalina Boca, Victor Dinu, and Viorica Florescu. Spin effects in nonlinear Compton scattering in a plane-wave laser pulse. *Nuclear Instruments and Methods in Physics Research Section B: Beam Interactions with Materials and Atoms*, 279(0):12 – 15, 2012. Proceedings of the Fifth International Conference on Elementary Processes in Atomic Systems Belgrade, Serbia, 21-25 June 2011.
- [57] K. Krajewska and J. Z. Kamiński. Spin effects in nonlinear Compton scattering in ultra-short linearly-polarized laser pulses. *Laser and Particle Beams*, 31:503–513, 9 2013.

- [58] Sándor Varró. New exact solutions of the Dirac equation of a charged particle interacting with an electromagnetic plane wave in a medium. *Laser Physics Letters*, 10(9):095301, 2013.
- [59] Sándor Varró. A new class of exact solutions of the Klein–Gordon equation of a charged particle interacting with an electromagnetic plane wave in a medium. *Laser Physics Letters*, 11(1):016001, 2014.
- [60] A.A. Borghardt and D. Ya. Karpenko. Fundamental solution of the Volkov problem (characteristic representation). *Journal of Nonlinear Mathematical Physics*, 5(4):357–363, 1998.
- [61] I. Berson. *Soviet Physics JETP*, 29(5):871, 1969.
- [62] I. Berson and J. Valdmanis. Electron in the field of two monochromatic electromagnetic waves. *Journal of Mathematical Physics*, 14(10):1481–1484, 1973.
- [63] P. Filipowicz. Relativistic electron in a quantised plane wave. *Journal of Physics A: Mathematical and General*, 18(10):1675, 1985.
- [64] V. Bagrov and D. Gitman. Non-Volkov solutions for a charge in a plane wave. *Annalen der Physik*, 14(8):467–478, 2005.
- [65] V.G. Bagrov, D.M. Gitman, M.C. Baldiotti, and A.D. Levin. Spin equation and its solutions. *Annalen der Physik*, 14(11-12):764–789, 2005.
- [66] Madalina Boca. On the properties of the Volkov solutions of the Klein–Gordon equation. *Journal of Physics A: Mathematical and Theoretical*, 44(44):445303, 2011.
- [67] I. M. Ternov and V. G. Bagrov. On coherent states of relativistic particles. *Annalen der Physik*, 495(1):1–9, 1983.
- [68] V.G. Bagrov, I.L. Bukhbinder, D.M. Gitman, and P.M. Lavrov. Coherent states of an electron in a quantized electromagnetic wave. *Theoretical and Mathematical Physics*, 33(3):1111–1116, 1977.
- [69] V.G. Bagrov, N.N. Byzov, and S.V. Sorokin. Motion of a classical charged particle with spin in the external field of a plane electromagnetic wave. *Soviet Physics Journal*, 24(11):1045–1048, 1981.
- [70] Peter J. Redmond. Solution of the Klein-Gordon and Dirac equations for a particle with a plane electromagnetic wave and a parallel magnetic field. *Journal of Mathematical Physics*, 6(7):1163–1169, 1965.
- [71] J. Bergou and F. Ehlotzky. Relativistic quantum states of a particle in an electromagnetic plane wave and a homogeneous magnetic field. *Phys. Rev. A*, 27:2291–2296, May 1983.
- [72] Madalina Boca, Viorica Florescu, and Mihai Gavrila. Generalized space-translated Dirac and Pauli equations for superintense laser-atom interactions. *Phys. Rev. A*, 85:023425, Feb 2012.

- [73] V.G. Bagrov, P.V. Bozrikov, and D.M. Gitman. Electron in the field of a plane quantized electromagnetic wave. *Theoretical and Mathematical Physics*, 14(2):150–156, 1973.
- [74] D.I. Abakarov and V.P. Oleinik. Electron in the field of a quantized electromagnetic wave and in a homogeneous magnetic field. *Theoretical and Mathematical Physics*, 12(1):673–679, 1972.
- [75] M.V. Fedorov and A.E. Kazakov. An electron in a quantized plane wave and in a constant magnetic field. *Zeitschrift für Physik*, 261(3):191–202, 1973.
- [76] A. D. Gazazyan and B. G. Sherman. Effects of quantum coherence of electromagnetic radiation in interactions with electrons. *Sov. Phys. JETP*, 69(1):38–45, July 1989.
- [77] J Bergou and S Varro. Nonlinear scattering processes in the presence of a quantised radiation field. ii. relativistic treatment. *Journal of Physics A: Mathematical and General*, 14(9):2281, 1981.
- [78] O.B. Prepelitsa. Quantum-electrodynamic processes in a strong nonclassical electromagnetic field. *Journal of Experimental and Theoretical Physics*, 85(5):838–845, 1997.
- [79] Piotr Filipowicz and Jan Mostowski. Quantum revivals in the motion of an electron in a magnetic field. *Physics Letters A*, 86(6–7):356 – 358, 1981.
- [80] V.A. Kovarsky. Superhigh frequency modulation of light by Rydberg atoms interacting with the electromagnetic field squeezed vacuum. *Physics Letters A*, 212(4):195 – 200, 1996.
- [81] P R Sharapova and O V Tikhonova. Dynamics of ionisation and entanglement in the 'atom + quantum electromagnetic field' system. *Quantum Electronics*, 42(3):199, 2012.
- [82] S Varró. Entangled photon–electron states and the number-phase minimum uncertainty states of the photon field. *New Journal of Physics*, 10(5):053028, 2008.
- [83] E. T. Jaynes and F. W. Cummings. *Proc. Inst. Elect. Eng.*, 51, 1963.
- [84] F. W. Cummings. Stimulated emission of radiation in a single mode. *Phys. Rev.*, 140:A1051–A1056, Nov 1965.
- [85] Bruce W. Shore and Peter L. Knight. The Jaynes-Cummings model. *Journal of Modern Optics*, 40(7):1195–1238, 1993.
- [86] J. H. Eberly, N. B. Narozhny, and J. J. Sanchez-Mondragon. Periodic spontaneous collapse and revival in a simple quantum model. *Phys. Rev. Lett.*, 44:1323–1326, May 1980.
- [87] N. B. Narozhny, J. J. Sanchez-Mondragon, and J. H. Eberly. Coherence versus incoherence: Collapse and revival in a simple quantum model. *Phys. Rev. A*, 23:236–247, Jan 1981.
- [88] Gerhard Rempe and Herbert Walther. Sub-poissonian atomic statistics in a micromaser. *Phys. Rev. A*, 42:1650–1655, Aug 1990.

- [89] M. Fleischhauer and W. P. Schleich. Revivals made simple: Poisson summation formula as a key to the revivals in the Jaynes-Cummings model. *Phys. Rev. A*, 47:4258–4269, May 1993.
- [90] Gerhard Rempe, Herbert Walther, and Norbert Klein. Observation of quantum collapse and revival in a one-atom maser. *Phys. Rev. Lett.*, 58:353–356, Jan 1987.
- [91] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J. M. Raimond, and S. Haroche. Quantum rabi oscillation: A direct test of field quantization in a cavity. *Phys. Rev. Lett.*, 76:1800–1803, Mar 1996.
- [92] I. D. Feranchuk and A.V.Leonov. *Physics Letters A*, 373:517, 2009.
- [93] I. D. Feranchuk and A.V. Leonov. Strong field effects in the evolution of a two-level system. *Physics Letters A*, 375(3):385 – 389, 2011.
- [94] I. D. Feranchuk and A. V. Leonov. Resonant modification of the Rabi oscillations of a two-level system. *Physics Letters A*, 373(45):4113 – 4116, 2009.
- [95] M. O. Scully and M. S. Zubairy. *Quantum Optics*. Cambridge University Press, Cambridge, 1997.
- [96] P Filipowicz. Quantum revivals in the Jaynes-Cummings model. *Journal of Physics A: Mathematical and General*, 19(18):3785, 1986.
- [97] J. Durnin, J. J. Miceli, and J. H. Eberly. Diffraction-free beams. *Phys. Rev. Lett.*, 58:1499–1501, Apr 1987.
- [98] Subir Sachdev. Atom in a damped cavity. *Phys. Rev. A*, 29:2627–2633, May 1984.
- [99] S. M. Barnett and P. L. Knight. Dissipation in a fundamental model of quantum optical resonance. *Phys. Rev. A*, 33:2444–2448, Apr 1986.
- [100] R. R. Puri and G. S. Agarwal. Collapse and revival phenomena in the Jaynes-Cummings model with cavity damping. *Phys. Rev. A*, 33:3610–3613, May 1986.
- [101] P. Filipowicz, J. Javanainen, and P. Meystre. Theory of a microscopic maser. *Phys. Rev. A*, 34:3077–3087, Oct 1986.
- [102] R. R. Puri and G. S. Agarwal. Finite- Q cavity electrodynamics: Dynamical and statistical aspects. *Phys. Rev. A*, 35:3433–3449, Apr 1987.
- [103] J. R. Kukliński and J. L. Madajczyk. Strong squeezing in the Jaynes-Cummings model. *Phys. Rev. A*, 37:3175–3178, Apr 1988.
- [104] A. H. Castro Neto and A. O. Caldeira. Quantum dynamics of an electromagnetic mode in a cavity. *Phys. Rev. A*, 42:6884–6893, Dec 1990.
- [105] J. Eiselt and H. Risken. Quasiprobability distributions for the Jaynes-Cummings model with cavity damping. *Phys. Rev. A*, 43:346–360, Jan 1991.

- [106] J. I. Cirac, H. Ritsch, and P. Zoller. Two-level system interacting with a finite-bandwidth thermal cavity mode. *Phys. Rev. A*, 44:4541–4551, Oct 1991.
- [107] Tran Quang, P. L. Knight, and V. Buek. Quantum collapses and revivals in an optical cavity. *Phys. Rev. A*, 44:6092–6096, Nov 1991.
- [108] J. Gea-Banacloche. Jaynes-Cummings model with quasiclassical fields: the effect of dissipation. *Phys. Rev. A*, 47:2221–2234, Mar 1993.
- [109] Shih-Chuan Gou. Quantum behavior of a two-level atom interacting with two modes of light in a cavity. *Phys. Rev. A*, 40:5116–5128, Nov 1989.
- [110] Amitabh Joshi and R. R. Puri. Characteristics of rabi oscillations in the two-mode squeezed state of the field. *Phys. Rev. A*, 42:4336–4342, Oct 1990.
- [111] A. S. Parkins. Resonance fluorescence of a two-level atom in a two-mode squeezed vacuum. *Phys. Rev. A*, 42:6873–6883, Dec 1990.
- [112] W. K. Lai, V. Buek, and P. L. Knight. Dynamics of a three-level atom in a two-mode squeezed vacuum. *Phys. Rev. A*, 44:6043–6056, Nov 1991.
- [113] A. M. Abdel-Hafez. Degenerate and nondegenerate two-mode normal squeezing in a two-level atom and two-mode system. *Phys. Rev. A*, 45:6610–6614, May 1992.
- [114] M. Khanbekyan, L. Knöll, A. A. Semenov, W. Vogel, and D.-G. Welsch. Quantum-state extraction from high- q cavities. *Phys. Rev. A*, 69:043807, Apr 2004.
- [115] C. Di Fidio, W. Vogel, M. Khanbekyan, and D.-G. Welsch. Photon emission by an atom in a lossy cavity. *Phys. Rev. A*, 77:043822, Apr 2008.
- [116] Bing He, Qing Lin, and Christoph Simon. Cross-Kerr nonlinearity between continuous-mode coherent states and single photons. *Phys. Rev. A*, 83:053826, May 2011.
- [117] A. Wickenbrock, M. Hemmerling, G. R. M. Robb, C. Emary, and F. Renzoni. Collective strong coupling in multimode cavity QED. *Phys. Rev. A*, 87:043817, Apr 2013.
- [118] D. M. Kennes, O. Kashuba, M. Pletyukhov, H. Schoeller, and V. Meden. Oscillatory dynamics and non-Markovian memory in dissipative quantum systems. *Phys. Rev. Lett.*, 110:100405, Mar 2013.
- [119] Saleh Rahimi-Keshari, Thomas Kiesel, Werner Vogel, Samuele Grandi, Alessandro Zavatta, and Marco Bellini. Quantum process nonclassicality. *Phys. Rev. Lett.*, 110:160401, Apr 2013.
- [120] N. N. Bogolubov, B.I.Sadovnikov, and A.S.Shumovsky. *Mathematical Methods in Statistical Mechanics of Model Systems*. Nauka, 1989.
- [121] L. D. Landau and E. M. Lifschitz. *Quantum Electrodynamics*. Pergamon, 1st edition, 1971.

- [122] M. L. Goldberger and K. M. Watson. *Collision Theory*. John Wiley & Sons, 1st edition, 1964.
- [123] O. D. Skoromnik and I. D. Feranchuk. Justification of the single-mode approximation for a finite-duration laser pulse interacting with an electron. *Journal of Physics B: Atomic, Molecular and Optical Physics*, 2014.
- [124] N. N. Bogolubov. *Aspects of Polaron Theory*. World Scientific, 2008.
- [125] O. D. Skoromnik, I. D. Feranchuk, and C. H. Keitel. Collapse-and-revival dynamics of strongly laser-driven electrons. *Phys. Rev. A*, 87:052107, May 2013.
- [126] Y. B. Zelídovich. *Sov. Phys. Usp*, 18:79, 1975.
- [127] R. Kubo. *JPSJ*, 17:1100, 1962.
- [128] E. T. Whittaker and G. N. Watson. *A Course of Modern Analysis*. Cambridge University Press, fourth edition, 1927. Reprinted 1990.
- [129] L. D. Landau and E. M. Lifschitz. *Statistical physics*. Nauka, 3rd edition, 1976.
- [130] G. C. Wick. The evaluation of the collision matrix. *Phys. Rev.*, 80:268–272, Oct 1950.
- [131] Oldwig Von Roos. Frequency shift in high-intensity Compton scattering. *Phys. Rev.*, 150:1112–1118, Oct 1966.
- [132] Z. Fried, A. Baker, and D. Korff. Comments on intensity-dependent frequency shift in Compton scattering and its possible detection. *Phys. Rev.*, 151:1040–1048, Nov 1966.
- [133] Madalina Boca and Viorica Florescu. Nonlinear Compton scattering with a laser pulse. *Phys. Rev. A*, 80:053403, Nov 2009.
- [134] D. Levin. Procedures for computing one- and two-dimensional integrals of functions with rapid irregular oscillations. *Math. Comp.*, 38:531–538, 1982.
- [135] L. D. Landau and E. M. Lifschitz. *The Classical Theory of Fields*. Pergamon, 2nd edition, 1962.
- [136] A. A. Sokolov and I. M. Ternov. *Doklady Acad. Nauk USSR*, 153:1052, 1963.
- [137] P. M. Morse and H. Feshbach. *Methods of Theoretical Physics*. McGraw-Hill, New York, 1953.

Acknowledgments

I would like to thank people without whom this work would not be possible.

I would like to thank Honorarprof. Dr. Christoph H. Keitel who accepted me as a PhD student in his division and provided excellent working conditions. Moreover, I deeply indebted for very fruitful discussion that we had during the work. I especially thank his patience while correcting my english writing.

I express the greatest thanks to Prof. Dr. Ilya Feranchuk for all the discussions and support that I had during my PhD. I am very lucky to have him as teacher. Thank you.

I acknowledge Dr. habil. Karen Z. Hatsagortsyan for all his interest and time that he invested in my work.

I thank Prof. Manfred Salmhofer for agreeing to write the second review for this thesis.

I thank Prof. Kurt Roth and Dr. Daniel Fischer for agreeing to be in my PhD exam committee.

I express my gratitude to Stefano M. Cavaletto, Enderalp Yakaboylu, Norman Neitz, Nikolai Belov, Sven Augustin, Matteo Tamburini, Sebastian Meuren, Alberto Benedetti for proof reading thesis or its parts.

I thank many members of the theory division of the Max-Planck Institute for Nuclear Physics: Antonino Di Piazza, Adriana Palffy, Natalia Oreshkina, Heiko Bauke, Naveen Kumar, Jacek Zatorski, Michael Klaiber, Norman Neitz, Anton Wllert, Qurrat-ul-Ain Gulfam.

I thank secretaries of the theory division and international office Sibel Babacan and Ludmila Holmach.

I want to thank my family and especially my grandfather for constant motivation and encouragement. Without him I would never finish this thesis.

I want to thank my wife Tanya, for her love, patience and brace.