ROUGH SOLUTIONS OF EINSTEIN VACUUM EQUATIONS IN CMCSH GAUGE

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ABSTRACT. In this paper, we consider very rough solutions to Cauchy problem for the Einstein vacuum equations in CMC spacial harmonic gauge, and obtain the local well-posedness result in $H^s, s > 2$. The novelty of our approach lies in that, without resorting to the standard paradifferential regularization over the rough, Einstein metric $\mathbf{g}$, we manage to implement the commuting vector field approach to prove Strichartz estimate for geometric wave equation $\Box_{\mathbf{g}}\phi = 0$ directly.

1. Introduction

In mathematical relativity, a fundamental question is to find a four dimensional Lorentz metric $\mathbf{g}$ that satisfies the vacuum Einstein equation

$$(1.1) \qquad \mathbf{Ric}(\mathbf{g}) = 0.$$

Since the equation is diffeomorphic invariant, certain gauge should be fixed before solving it. There exist extensive works on (1.1) under the wave condition gauge or the constant mean curvature gauge.

In [3] Andersson and Moncrief consider the vacuum Einstein equation (1.1) under the so-called constant mean curvature and spatial harmonic coordinate (CMCSH) gauge condition. To set up the framework, let Σ be a 3-dimensional compact, connected and orientable smooth manifold, and let $\mathcal{M} := \mathbb{R} \times \Sigma$. Let $t : \mathcal{M} \to \mathbb{R}$ be the projection on the first component and let $\Sigma_t := \{t\} \times \Sigma$ be the level sets of t. One may construct solutions of (1.1) by considering Lorentz metrics $\mathbf{g}$ of form

$$\mathbf{g} = -n^2 dt \otimes dt + g_{ij}(dx^i + Y^i dt) \otimes (dx^j + Y^j dt)$$

with suitable determination of the scalar function n, the vector field $Y := Y^j \partial_j$ and the Riemannian metric $g := g_{ij} dx^i \otimes dx^j$ on Σ. In order for ∂_t to be time-like, it is necessary to have $n^2 - g_{ij}Y^iY^j > 0$. Let $\mathbf{T}$ be the time-like normal to Σ_t, then

$$\partial_t = n\mathbf{T} + Y.$$

We call n the lapse function and Y the shift vector field.

Let $\hat{g}$ be a fixed smooth Riemannian metric on Σ with Levi-Civita connection $\widehat{\nabla}$ and Christoffel symbol $\hat{\Gamma}_{ij}^k$. Let Γ_{ij}^k denote the Christoffel symbol with respect to g, we may introduce the vector field $U = U^l \partial_l$ with

$$U^l := g^{ij}(\Gamma_{ij}^l - \hat{\Gamma}_{ij}^l).$$

Let k be the second fundamental form of Σ_t in $\mathcal{M}$, i.e. $k_{ij} = -\frac{1}{2}\mathcal{L}_{\mathbf{T}}\mathbf{g}_{ij}$. The solution of (1.1) constructed in [3] is to find the pair (g, k) such that they satisfy the CMCSH condition

$$(1.2) \qquad \mathrm{Tr}k := g^{ij}k_{ij} = t \quad \text{and} \quad U^j = 0$$

and the vacuum Einstein evolution equations

$$(1.3) \qquad \partial_t g_{ij} = -2nk_{ij} + \mathcal{L}_Y g_{ij}$$

$$(1.4) \qquad \partial_t k_{ij} = -\nabla_i \nabla_j n + n(R_{ij} + \mathrm{Tr}k k_{ij} - 2k_{im}k_j^m) + \mathcal{L}_Y k_{ij}$$

with the constraint equations

$$(1.5) \qquad R - |k|^2 + (\mathrm{Tr}k)^2 = 0 \quad \text{and} \quad \nabla_i \mathrm{Tr}k - \nabla^j k_{ij} = 0.$$

It has been shown in [3] that for initial data $(g^0, k^0) \in H^s \times H^{s-1}$ with $s > 5/2$ satisfying the constraint equation (1.5) with $t_0 := \mathrm{Tr}k^0 < 0$, the Cauchy problem for the system (1.2)–(1.5) is locally well-posed. In particular, there is a time $T_* > 0$ depending on $\|g^0\|_{H^s}$ and $\|k^0\|_{H^{s-1}}$ such that the Cauchy problem has a unique solution defined on $[t_0 - T_*, t_0 + T_*] \times \Sigma$. We should mention that, for the solution constructed in this way, the lapse function n and the shift vector field Y satisfy the elliptic equations

$$(1.6) \qquad -\Delta n + |k|^2 n = 1$$

and

$$(1.7) \qquad \Delta Y^i + R_j^i Y^j = \left(-2nk^{jl} + 2\nabla^j Y^l \right) U_{jl}^i + 2\nabla^j n k_j^i - \nabla^i n k_j^j,$$

where U_{jl}^i is the tensor defined by

$$(1.8) \qquad U_{jl}^i := \Gamma_{jl}^i - \hat{\Gamma}_{jl}^i.$$

It is natural to ask under what minimal regularity on the initial data the CM-CSH Cauchy problem (1.2)–(1.5) is locally well-posed. In this paper we prove the following result which shows the well-posedness of the problem when the initial data is in $H^s \times H^{s-1}$ with $s > 2$.

Theorem 1.1 (Main Theorem). *For any $s > 2$, $t_0 < 0$ and $M_0 > 0$, there exist positive constants T_*, M_1 and M_2 such that the following properties hold true:*

(i) *For any initial data set (g^0, k^0) satisfying (1.5) with $t_0 := \mathrm{Tr}k^0 < 0$ and $\|g^0\|_{H^s(\Sigma_{t_0})} + \|k^0\|_{H^{s-1}(\Sigma_{t_0})} \le M_0$, there exists a unique solution $(g, k) \in C(I_*, H^s \times H^{s-1}) \times C^1(I_*, H^{s-1} \times H^{s-2})$ to the problem (1.2)–(1.5);*

(ii) *There holds*
$$\|\widehat{\nabla}g, k\|_{L^2_{I_*} L^\infty_x} + \|\widehat{\nabla}g, k\|_{L^\infty_{I_*} H^{s-1}} \le M_1;$$

(iii) *For $2 < r \le s$, and for each $\tau \in I_*$ the linear equation*
$$\begin{cases} \Box_{\mathbf{g}} \psi = 0, & (t, x) \in I_* \times \Sigma \\ \psi(\tau, \cdot) = \psi_0 \in H^r(\Sigma), & \partial_t \psi(\tau, \cdot) = \psi_1 \in H^{r-1}(\Sigma) \end{cases}$$

admits a unique solution $\psi \in C(I_*, H^r) \times C^1(I_*, H^{r-1})$ satisfying the estimates
$$\|\psi\|_{L^\infty_t H^r} + \|\partial_t \psi\|_{L^\infty_t H^{r-1}} \le M_2 \|\psi_0, \psi_1\|_{H^r \times H^{r-1}}$$

and
$$\|\boldsymbol{D}\psi\|_{L^2_t L^\infty_x} \le M_2 \|(\psi_0, \psi_1)\|_{H^r \times H^{r-1}};$$

where $I_* := [t_0 - T_*, t_0 + T_*]$.

We actually obtain a stronger result than Theorem 1.1, which is contained in Theorem 4.1.

1.1. **Review and Motivation.** Since the pioneer work of Choquet-Bruhat [6], there has been extensive work on the well-posedness of quasilinear wave equation

$$(1.9) \qquad \begin{cases} \Box_{g(\phi)}\phi := \partial_t^2 \phi - g^{ij}(\phi)\partial_i\partial_j\phi = N(\phi, \partial\phi), \\ \phi|_{t=0} = \phi_0, \quad \partial_t\phi|_{t=0} = \phi_1 \end{cases}$$

in $\mathbb{R}^{n+1}$, where the symmetric matrix $g^{ij}(\phi)$ is positive definite and smooth as a function of ϕ, and the function $N(\phi, \partial\phi)$ is smooth in its arguments and is quadratic in $\partial\phi$. In view of the energy estimate

$$(1.10) \qquad \|\partial\phi(t)\|_{H^{s-1}} \lesssim \|\partial\phi(0)\|_{H^{s-1}} \cdot \exp\left(\int_0^t \|\partial\phi(\tau)\|_{L_x^\infty} d\tau\right),$$

the Sobolev embedding and a standard iteration argument, the classical result of Hughes-Kato-Marsden [8] of well-posedness in the Sobolev space H^s follows for any $s > \frac{n}{2} + 1$, where the estimate on $\|\partial\phi\|_{L_t^\infty L_x^\infty}$ is heavily relied. To improve the classical result, it is crucial to get a good estimate on $\|\partial\phi\|_{L_t^1 L_x^\infty}$. This is naturally reduced to deriving the Strichartz estimate for the wave operator $\Box_{g(\phi)}$ which has rough coefficients since $g^{ij}(\phi)$ depend on the solution ϕ and thus at most have as much regularity as ϕ. The first important breakthrough was achieved by Bahouri-Chemin [5] and by Tataru [21] using parametrix constructions. They obtained the well-posedness of (1.9) in H^s with $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{4}$ by establishing a Strichartz estimate of the following form, if $\partial g \in L_t^2 L_x^\infty$,

$$\|\partial\phi\|_{L_I^2 L_x^\infty} \le c(\|\phi_0\|_{H^{\frac{n}{2}+\frac{1}{2}+\sigma}} + \|\phi_1\|_{H^{\frac{n}{2}-\frac{1}{2}+\sigma}})$$

with a loss of $\sigma > \frac{1}{4}$, for solutions to linearized equations. This well-posedness result was later improved to $s > \frac{n}{2} + \frac{1}{2} + \frac{1}{6}$ in [23].

The next important progress was made by Klainerman in [9] where a vector field approach was developed to establish the Strichartz estimate. This approach was further developed by Klainerman-Rodnianski in [11] where they successfully improved the local well-posedness of (1.9) in $\mathbb{R}^{3+1}$ to the Sobolev space H^s with $s > 2 + \frac{2-\sqrt{3}}{2}$. Due to the limited regularity of the coefficients, the paradifferential localization procedure in [5, 22, 23] was adopted in [11] to consider the Strichartz estimate for solutions of linearized wave equation $\Box_{g_{\le\lambda^a}}\psi = 0$ for some $0 < a \le 1$, where $g_{\le\lambda^a} := S_{\lambda^a}(g(S_{\lambda^a}(\phi)))$ is the truncation of $g(\phi)$ at the frequency level λ^a. Here $S_\lambda := \sum_{\mu\le\lambda} P_\mu$ and P_λ is the Littlewood-Paley projector with frequency $\lambda = 2^k$ defined for any function f by

$$(1.11) \qquad P_\lambda f(x) = f_\lambda(x) = \int e^{-ix\cdot\xi}\zeta(\lambda^{-1}\xi)\hat{f}(\xi)d\xi$$

with ζ being a smooth function supported in the shell $\{\xi : \frac{1}{2} \le |\xi| \le 2\}$ satisfying $\sum_{k\in\mathbb{Z}} \zeta(2^k\xi) = 1$ for $\xi \ne 0$. We refer to [20, 16] for detailed properties of Littlewood-Paley decompositions. With the help of a $\mathcal{T}\mathcal{T}^*$ argument, such Strichartz estimate was reduced to the dispersive estimate for solutions of $\Box_{g_{\le\lambda^a}}\psi = 0$ with frequency localized initial data. It was then further reduced to deriving the boundedness of Morewatz type energy for $\partial\psi$ and its higher derivatives. To derive these energy estimates requires us to control deformation tensor of Morawetz vector field, which involves the Ricci coefficients relative to the Lorentzian metric $-dt^2 + (g_{\le\lambda^a})_{ij}dx^i \otimes dx^j$. Since **Ric** appears crucially in the structure equations for Ricci coefficients, deep observations and techniques were developed to control **Ric** relative to the smoothed metric, such as taking advantage of the facts that

the coefficients g themselves verify equations of the form (1.9), and the observation that $\mathbf{R}_{44}$, the tangential component of $\mathbf{Ric}$ along null hypersurfaces, has better structure. For Einstein vacuum equation under the wave coordinates gauge, the local well-posedness were obtained in H^s for any $s > 2$ in [12, 13, 14]. The core progress which enables the improvement from $s > 2 + \frac{2-\sqrt{3}}{2}$ to $s > 2$ was made in [14] by showing that the Ricci tensor relative to the frequency-truncated metric $\mathbf{h} := \mathbf{g}_{\leq\lambda}$ does not deviate from 0 to a harmful level; the decay rate of $\mathbf{Ric}(\mathbf{h})$ and its derivatives were proven to be sufficiently strong in terms of λ. However, similar estimates for $\mathbf{Ric}(\mathbf{h})$ can hardly be obtained for (1.9). The sharp local well-posedness for type (1.9) in H^s with $s > 2$ was achieved by Smith and Tataru in [19] based on the wave packet parametrix construction, with an innovative application to represent the nontruncated metric $\mathbf{g}$. The particular structure of $\mathbf{R}_{44}$ observed in [11] also played important role to control the geometry of null surface. The local well-posedness with $s = 2$ for Einstein vacuum equation was conjectured by Klainerman in [10]. Recently we learn that significant progress has been achieved for this so-called L^2 curvature conjecture [17].

A reduction to consider $\Box_{\mathbf{g}_{\leq\lambda^a}}\psi = 0$, with $0 < a \leq 1$ appeared in all the above mentioned work. This regularization on metric is used to phase-localize the solution, and in most of the works, to balance the differentiability on coefficients required either by parametrix construction or by energy method. Such a regularization on metric, nevertheless, poses major technical baggage, in particular, to carry out the vector field approach in Einstein vacuum spacetime, since $\mathbf{Ric}(\mathbf{g}_{\leq\lambda})$ no longer vanishes. The analysis in [14] on the defected Ricci tensor and its derivatives is a very delicate procedure, which relies crucially on full force of $\partial\mathbf{h}$, hence, on their non-smoothed counter part $\partial\mathbf{g}$ as well. One particular issue tied to CMCSH gauge itself arises due to the lack of control on $\mathbf{D_T}Y$, the time derivative of the shift vector field. Although $\mathbf{D_T}Y$ satisfies an elliptic equation, that equation is not good enough to provide a valid control on $\mathbf{D_T}Y$ even in terms of L^2-norm. The loss of control over some components of $\partial\mathbf{g}$ becomes a serious hurdle in recovering the decay for $\mathbf{Ric}(\mathbf{h})$ and its derivatives. The potential issue on Ricci defect forces us to abandon the frequency truncation on metric.

The important aspect of our analysis is to implement the vector field approach directly in the non-smoothed Einstein spacetime $(\mathcal{M}, \mathbf{g})$ to establish the Strichartz estimate with an arbitrarily small loss for the linearized problem $\Box_{\mathbf{g}}\psi = 0$. This confirms that, due to the better behavior of $\mathbf{Ric}$, Einstein metric is in nature "smooth" enough to implement the vector field approach without the truncation on $\mathbf{g}$ in Fourier space, and leads to the H^s well-posedness result with $s > 2$ for Einstein equation.

Note that in [11], [12]-[14], deriving the bounded Morawetz type energy of derivatives of ψ, with $\Box_{\mathbf{g}_{\leq\lambda^a}}\psi = 0$, is the main building block to obtain the dispersive estimate for $P\partial_t\psi$ required by Strichartz estimate. This procedure relies on $H^\sigma, \sigma > \frac{1}{2}$ norm of curvature, which is impossible to be obtained relative to the rough and non-smoothed metric. Our strategy is to derive the dispersive estimate merely using Morawetz type energy for ψ itself. The analysis to control such energy is mainly focused on the Ricci coefficients relative to Einstein metric. Although part of such analysis benefits from $\mathbf{Ric} = \mathbf{0}$, the crucial estimates such as strichartz type norm $\|\hat{\chi}\|_{L_t^2 L_x^\infty}$ require the bound of $H^\sigma, \sigma > 1/2$ for $\widehat{\nabla}k$ and $\widehat{\nabla}^2 g$ if only the classic L_x^∞ Calderson-Zygmund inequality is employed. We solve this problem by modifing

Calderson Zygmund inequality followed by taking advantage of the extra differentialbity for $\widehat{\nabla}g, k$ that can be obtained by Strichartz estimates. The difficulty coming from $\mathbf{D_T}Y$ still penetrates in key steps in the vector fields approach, where all components of $\partial\mathbf{g}$ were typically involved. We exclude such term by modifying the standard treatments including modifying energy momentum tensor, refining $\mathcal{T}\mathcal{T}^*$ argument and curvature decomposition into more invariant fashion.

Our approach can be directly applied for reproducing $H^{2+\epsilon}$ result for Einstien equations in wave coordinates gauge. It actually works better under wave coordinates since $\mathbf{D_T}Y$ can be well controlled in this situation. Steps which are involved with getting around this term in CMCSH gauge, such as energy estimate, estimate for flux, $\mathcal{T}\mathcal{T}^*$ argument and decompositions for curvature, take simpler and more straightforward form in wave coordinate gauge. Thus our approach gives a vast simplification over the the methodology in [12]-[14].

1.2. **Outline of the proof.** According to [13, 19], in order to complete the proof of Theorem 1.1 it suffices to show that for any $s > 2$ there exist two positive constants C and T depending on $\|g\|_{H^s(\Sigma_0)}$ and $\|k\|_{H^{s-1}(\Sigma_0)}$ such that

$$(1.12) \qquad \|g\|_{L_t^\infty H^s(I\times\Sigma)} + \|k\|_{L_t^\infty H^{s-1}(I\times\Sigma)} \le C,$$

where $I := [t_0 - T, t_0 + T]$. We achieve this by a bootstrap argument. That is, we first make the bootstrap assumption

$$(\text{BA1}) \qquad \int_{t_0-T}^{t_0+T} \|\widehat{\nabla}g,\, k,\, \widehat{\nabla}Y,\, \widehat{\nabla}n\|_{L^\infty(\Sigma_t)}dt \le B_1,$$

where, for any Σ-tangent tensor F, we will use $\|F\|_{L^\infty(\Sigma_t)}$ to denote its L^∞-norm with respect to the Riemannian metric g on Σ_t. We then show that (BA1) and some auxiliary bootstrap assumptions imply (1.12). We prove these bootstrap assumptions can be improved for small but universal $T > 0$.

We will only work on the time interval $[t_0, t_0 + T]$ since the same procedure applies to the time interval $[t_0 - T, t_0]$ by simply reversing the time. In view of (BA1) and elliptic estimates, we derive in Section 2 better estimates for $\widehat{\nabla}Y$ and $\widehat{\nabla}n$. That is, we show that, for any $1 < b < 2$, there holds

$$\|\widehat{\nabla}Y, \widehat{\nabla}n\|_{L_{[t_0,t_0+T]}^b L_x^\infty} \le C$$

which improves the estimates for $\widehat{\nabla}n$ and $\widehat{\nabla}Y$ in (BA1) with T sufficiently small. In order to improve the estimates for $\widehat{\nabla}g$ and k, we establish the core estimates in Theorem 1.1(ii) by showing that

$$\|\widehat{\nabla}g, k\|_{L_{[t_0,t_0+T]}^2 L_x^\infty} \le CT^\delta,$$

for some $\delta > 0$. Here we briefly describe the ideas behind the proof.

1.2.1. *Step 1. Energy estimates and flux.* In Sections 2 and 3, we derive (1.12) under bootstrap assumptions. We also derive for the scalar solution of homogeneous geometric wave equation $\Box_{\mathbf{g}}\phi = 0$, the energy estimate

$$\|\partial\phi\|_{H^{s-1}} \lesssim \|\phi(0)\|_{H^s} + \|\partial\phi(0)\|_{H^{s-1}}.$$

The typical argument relies on the estimate of $\|\partial\mathbf{g}\|_{L_t^1 L_x^\infty}$, including the one for $\mathbf{D_T}Y$ in this norm. In view of (1.7), $\mathbf{D_T}Y$ satisfies the elliptic equation

$$\Delta\nabla_{n\mathbf{T}}Y^i + R_j^i\nabla_{n\mathbf{T}}Y^j - 2U_{mp}{}^i\nabla^m\nabla_{n\mathbf{T}}Y^p = -n(\operatorname{curl}H)_j^i Y^j + \mathbf{g}\cdot\nabla\tilde{\pi}\cdot\tilde{\pi},$$

where $\tilde{\pi}$ denotes components of $\partial \mathbf{g}$ excluding $\partial_t Y$. Due to the appearance of the term $R^i_j \nabla_{n\mathbf{T}} Y^j$, this elliptic equation is not good enough to provide valid control for $\mathbf{D_T} Y$ unless the space metric g has negative sectional curvature. In order to avoid the difficulty coming from $\mathbf{D_T} Y$, in Section 3 we derive the energy estimate by considering the first order hyperbolic system

$$(1.13) \qquad \begin{cases} \partial_t u - \widehat{\nabla}_Y u = nv + F_u \\ \partial_t v - \widehat{\nabla}_Y v = n\hat{\Delta} u + F_v \end{cases}$$

for the pairs $(u, v) = (g, -2k), (k, E)$ and $(\phi, e_0\phi)$ with corresponding remainder terms (F_u, F_v), where, for any Σ tangent tensor F,

$$(1.14) \qquad\qquad\qquad \hat{\Delta} F := g^{ij} \widehat{\nabla}_i \widehat{\nabla}_j F.$$

Consistent with these energy estimates, we also obtain the $L^\infty_t H^{s-\frac{1}{2}}$ and $L^1_t H^s$ estimates of $\widehat{\nabla} n, \widehat{\nabla} Y, \mathbf{D_T} n$ with the help of elliptic equations (1.6) and (1.7).

However, to derive the flux estimate for $\widehat{\nabla} g, k, P_\mu k$, and $P_\mu \widehat{\nabla} g$, we still have to rely on the second order hyperbolic system of k and $\widehat{\nabla} g$, both of which contain the time derivative of the shift vector field. This issue is solved in Section 3.2 by introducing a modified energy momentum tensor.

As the major technicality to carry out energy estimates in fractional Sobolev space and dyadic flux, a series of more delicate commutator estimates are established in Appendix III (Section 10) on the Littlewood Paley projection and the rough metric, particularly to handle the decreased differentiability of coefficients.

1.2.2. *Step 2. Reduction to dyadic Strichartz estimates on frequency dependent time intervals.* By using the Littlewood-Paley decomposition, it is easy to reduce the proof of Theorem 1.1 to establishing for sufficiently large λ the estimates

$$(1.15) \qquad \|P_\lambda \widehat{\nabla} g, P_\lambda k\|_{L^2_I L^\infty_x} \lesssim \lambda^{-\delta} |I|^{\frac{1}{2} - \frac{1}{q}} \|\widehat{\nabla} g, k\|_{H^{s-1}(\Sigma_0)}$$

and

$$(1.16) \qquad \|P_\lambda \partial \phi\|_{L^2_I L^\infty_x} \lesssim \lambda^{-\delta} |I|^{\frac{1}{2} - \frac{1}{q}} \|\widehat{\nabla} \phi, e_0 \phi\|_{H^{s-1}(\Sigma_0)}$$

for any solution ϕ of the equation $\Box_\mathbf{g} \phi = 0$, where $I = [t_0, t_0 + T]$, $q > 2$ is sufficiently close to 2, and $\delta > 0$ is sufficiently close to 0.

We reduce the proof of (1.15) and (1.16) to Strichartz estimates on small time intervals. We pick a sufficiently small $\epsilon_0 > 0$ and partition $[t_0, t_0 + T]$ into disjoint union of subintervals $I_k := [t_{k-1}, t_k]$ of total number $\lesssim \lambda^{8\epsilon_0}$ with the properties that

$$(1.17) \qquad |I_k| \lesssim \lambda^{-8\epsilon_0} T \quad \text{and} \quad \|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^2_{I_k} L^\infty_x} \leq \lambda^{-4\epsilon_0}.$$

To explain our approach, we take the derivation of (1.16) as an example. We consider on each I_k the Strichartz norm for $P_\lambda \partial \phi$. By commuting P_λ with $\Box_\mathbf{g}$ we have

$$\Box_\mathbf{g} P_\lambda \phi = F_\lambda,$$

where $F_\lambda = [\Box_\mathbf{g}, P_\lambda] \phi$ which can be treated as phase-localized at level of λ in certain sense although it is not frequency-localized. We use $W(t, s)$ to denote the operator that sends (f_0, f_1) to the solution of $\Box_\mathbf{g} \psi = 0$ satisfying the initial conditions

$\psi(s) = f_0$ and $\partial_t \psi(s) = f_1$ at the time s. Using Duhamel principle followed by differentiation, we can represent $P_\lambda \partial \phi$ as

$$(1.18) \qquad P_\lambda \partial \phi(t) = \partial W(t, t_{k-1}) P_\lambda \phi[t_{k-1}] + \int_s^t \partial W(t, s)(0, F_\lambda(s)) ds,$$

where we used the convention $\phi[t] := (\phi(t), \partial_t \phi(t))$. Running a $\mathcal{T}\mathcal{T}^*$ argument leads to Strichartz estimate for one dyadic piece of $\partial \psi$,

$$(1.19) \qquad \|P_\lambda \partial \psi\|_{L^q_{I_k} L^\infty_x} \lesssim \lambda^{\frac{3}{2} - \frac{1}{q}} \|\psi[0]\|_{H^1},$$

where $q > 2$ is sufficiently close to 2.

Similar procedure was used in [12] for $\Box_{\mathbf{g}_{\leq \lambda}} \phi = 0$. Observe that the solution of this homogeneous wave equation is frequency-localized at the level of λ if the data is localized in Fourier space at the dyadic shell $\{\xi : \frac{\lambda}{2} \leq |\xi| \leq 2\lambda\}$. Therefore, the dyadic Strichartz estimates (1.19) can be applied directly to the representation of $P_\lambda \partial \phi$. Since we will work for the metric $\mathbf{g}$ without frequency truncation, the corresponding operator $W(t, s)$ does not preserve the frequency-localized feature of data. The Strichartz estimate for $\partial W(t, t_{k-1}) P_\lambda \phi[t_{k-1}]$ is no longer expected to be obtained directly from (1.19). We solve this problem in Section 4 by modifying (1.18) with the help of the reproducing property of the Littlewood-Paley projections, i.e. $P_\lambda = \tilde{P}_\lambda \tilde{P}_\lambda$, as follows,

$$(1.20) \qquad P_\lambda \partial \phi(t) = \tilde{P}_\lambda \partial W(t, t_{k-1}) \tilde{P}_\lambda \phi[t_{k-1}] + \int_s^t \tilde{P}_\lambda \partial W(t, s)(0, F_\lambda(s)) ds.$$

This makes it possible to apply (1.19). The effort then goes into piecing together the result of dyadic Strichartz estimates over intervals I_k with the help of (1.17). This trick would have successfully reduced the main estimates to dyadic strichartz estimate for the solution of $\Box_{\mathbf{g}} \phi = 0$ on one sub-interval I_k, had the term of $\mathbf{D_T} Y$ not appeared in F_λ. We then refine (1.20) further by modifying the application of Duhamel principle.

1.2.3. *Step 3. Reduction to dispersive estimates and boundedness theorem.* By rescaling coordinates as $(t, x) \to ((t - t_{k-1})/\lambda, x/\lambda)$, we need only to consider (1.19) on $[0, t_*] \times \Sigma$ with $t_* \leq \lambda^{1-8\epsilon_0} T$. In view of a $\mathcal{T}\mathcal{T}^*$ argument, this essentially relies on the dispersive estimate

$$(1.21) \qquad \|P\mathbf{D_T} W(t, s) I[s]\|_{L^\infty_x} \lesssim \left((1 + |t - s|)^{-\frac{2}{q}} + d(t) \right) \sum_{k=0}^m \|\widehat{\nabla}^k I[s]\|_{L^1_x}$$

with initial data $I[s] = (\psi(s), \mathbf{D_T} \psi(s))$ for all $0 < s \leq t_*$, where m is a positive integer, and $d(t)$ is a function satisfying $\|d\|_{L^{\frac{q}{2}}} \lesssim 1$ for $q > 2$ sufficiently close to 2.

Let $\{\chi_J\}$ be a suitable partition of unity on Σ supported on balls of radius 1 in rescaled coordinates. We localize the solution of $\Box_{\mathbf{g}} \psi = 0$ by writing $\psi(t, x) = \sum_J \psi_J(t, x)$, where $\psi_J(t, x)$ is the solution of $\Box_{\mathbf{g}} \psi_J = 0$ with the initial data $\psi_J[\tau_0] = \chi_J \cdot \psi[\tau_0]$. We then reduce the derivation of (1.21) to proving that

$$(1.22) \qquad \|P\mathbf{D_T} \phi(t)\|_{L^\infty_x} \leq \left(\frac{1}{(1 + |t - \tau_0|)^{\frac{2}{q}}} + d(t) \right) \sum_{k=0}^{m-2} \|\widehat{\nabla}^k \phi[\tau_0]\|_{L^2},$$

with ϕ the solution of $\Box_{\mathbf{g}} \phi = 0$ and with data supported within a unit ball at Σ_{τ_0}. It then suffices to consider (1.22) on $\mathcal{J}_0^+$, the causal future of the support of χ_J from $t = \tau_0 \approx 1$, where one can introduce optical function u whose level sets

are null cones C_u. Thus $\mathcal{J}_0^+$ can be foliated by $S_{t,u} := C_u \cap \Sigma_t$ and a null frame $\{L, \underline{L}, e_1, e_2\}$ can be naturally defined, where $e_A, A = 1, 2$, are tangent to $S_{t,u}$. Using these vector fields and $\underline{u} = 2t - u$, one can introduce the Morawetz vector field $K = \frac{1}{2}n(u^2\underline{L} + \underline{u}^2 L)$. Consequently, for any function f, one can introduce the generalized energy

$$\tilde{Q}[f](t) := \int_\Sigma \bar{Q}(K, \mathbf{T})[f],$$

where $\bar{Q}(K, \mathbf{T})[f]$ is defined by applying $X = K, Y = \mathbf{T}, \Omega = 4t$ to

$$(1.23) \qquad \bar{Q}(X, Y)[f] = Q(X, Y)[f] + \frac{1}{2}\Omega f Y(f) - \frac{1}{4}f^2 Y(\Omega)$$

with $Q_{\mu\nu}$ being the standard energy momentum tensor

$$Q_{\mu\nu} := Q[f]_{\mu\nu} = \partial_\mu f \partial_\nu f - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{\alpha\beta}\partial_\alpha f \partial_\beta f).$$

The typical energy method gives

$$(1.24) \qquad \tilde{Q}[f](t) - \tilde{Q}[f](\tau_0) = -\frac{1}{2}\int_{\mathcal{J}_0^+} {}^{(K)}\bar{\pi}_{\alpha\beta}Q[f]_{\alpha\beta} + \int_{\Sigma \times I}\Box_\mathbf{g} f \cdot Kf + l.o.t,$$

where, for any vector field X, the deformation tensor ${}^{(X)}\pi_{\alpha\beta} := \mathcal{L}_X\mathbf{g}_{\alpha\beta}$ and ${}^{(K)}\bar{\pi}_{\alpha\beta} := {}^{(K)}\pi_{\alpha\beta} - 4t\mathbf{g}_{\alpha\beta}$. By applying (1.24) to $f = \mathbf{D_T}\phi$, we consider to bound generalized energy $\tilde{Q}[\mathbf{D_T}\phi]$ in terms of their initial values at $t = \tau_0 \approx 1$. Due to one bad term contained in $\Box_\mathbf{g}\mathbf{D_T}\phi = [\Box_\mathbf{g}, \mathbf{D_T}]\phi$, the estimate of $\tilde{Q}[\mathbf{D_T}\phi]$ has to be coupled with $\tilde{Q}[\mathbf{D}_Z\phi]$ with Z either $\underline{L}$ or e_A, for which we need to control $\|\mathbf{D}^{(Z)}\pi\|_{L_t^1 L_x^\infty}$ and $\int_0^{t_*}\sup_u \|\mathbf{D}^{(Z)}\pi\|_{L^2(S_{t,u})}dt$. Since $\mathbf{D}^{(Z)}\pi$ contains curvature terms, such estimates relative to non-smoothed metric can only be obtained under the assumption of $H^{\frac{5}{2}+\epsilon}$ on data. Similar regularity issue occurs for the estimates required for $\mathbf{D}^{(\mathbf{T})}\pi$ due to the integration by part argument employed to handle that bad term. Therefore, we no longer expect to obtain the boundedness of the conformal energy for any derivative of ϕ, including the one for $\mathbf{D_T}\phi$.

Our strategy is to control $\|P\mathbf{D_T}\phi(t)\|_{L_x^\infty}$ merely in terms of $\tilde{Q}[\phi](t)$, with certain loss of decay rate and with error incorporated into $d(t)$ in (1.22). With ϖ a cut-off function whose support is in a so-called exterior region, our treatment concerning the harder part, $P(\varpi\mathbf{D_T}\phi)$, starts with writing it as $P(\varpi\mathbf{D_T}\phi) = P(\varpi L\phi) - P(\varpi N\phi)$ with N the unit outward normal vector fields on $S_{t,u} \subset \Sigma$. The first term is controlled by Bernstein inequality and $\tilde{Q}[\phi](t)$. The second term is treated in view of

$$(1.25) \qquad P(\varpi N\phi) = \varpi N^l \partial_l P\phi + [P, \varpi N^l]\partial_l\phi.$$

The first term of (1.25) is then related to $\tilde{Q}[\phi]$ with the help of Sobolev embedding and commutator estimates. By using the machinery developed in Section 10, the treatment on the commutators involved in both terms in (1.25) is reduced to estimating $\|\partial(\varpi N)\|_{L_x^\infty}$. Note that ∂N can be expressed as $\mathbf{g} \cdot (\chi, \zeta, \widehat{\nabla}g, k)$, thus we need to establish estimates on $L_t^{\frac{q}{2}} L_x^\infty$, $q > 2$ of Ricci coefficients $\hat{\chi}, \zeta$ and L^∞ estimate on $\mathrm{tr}\chi$. The components of ${}^{(K)}\bar{\pi}_{\alpha\beta}$ in (1.24) involve ${}^{(\mathbf{T})}\pi$, χ, ζ and other Ricci coefficients as well. By assuming suitable control on Ricci coefficients, the proof of boundedness theorem is given in Section 5. We accomplish this step by showing that (1.22) holds true with $m = 3$.

1.2.4. Step 4. L^{2+} type flux and Ricci coefficients. The control of Ricci coefficients consistent with H^2 Einstein metrics has been studied in [12, 24, 26], where a set of estimates concerning $\operatorname{tr}\chi, \hat{\chi}, \zeta, \underline{\zeta}$ was achieved in terms of curvature flux, combined with flux of k if null hypersurface is foliated by S_t, level sets of t. Bearing the flavor of these works, in the situation when $H^{2+\epsilon}$ estimates for $\mathbf{g}$ can be established, we first manage to gain from the extra differentiability of metric a slightly stronger flux type control for $\widehat{\nabla}g, k$. We then obtain a stronger set of estimates on Ricci coefficients in terms of the L^{2+} type flux, which contains the $L_u^\infty L_t^2 L_\omega^\infty$-norms for $\hat{\chi}, \zeta$ and corresponding estimates for $\mu, \operatorname{tr}\chi - \frac{2}{n(t-u)}$. This enables us to carry out delicate analysis such as the standard $L^p, 1 < p \leq \infty$ type Calderon Zygmund inequality on null hypersurfaces under rough metric. In this procedure, thanks to working directly in vacuum spacetime, we no longer encounter the technical baggage in [12]–[14] posed by defected $\mathbf{Ric}(\mathbf{h})$. Nevertheless, this set of estimates is far from sufficient to control $\|\cdot\|_{L_t^1 L_u^\infty L_\omega^\infty}$ norm of deformation tensor ${}^{(K)}\pi$, which relies on estimates for $\hat{\chi}, \zeta$ in this norm. The idea of Klainerman-Rodnianski [13] is to rely on the Hodge system such as the equation of (6.44)

$$\operatorname{div}\hat{\chi} = \frac{1}{2}\nabla\kern-0.6em\raise0.3ex\hbox{/}\,\operatorname{tr}\chi - \beta + \cdots.$$

The key points of deriving a strichartz type norm for $\hat{\chi}$ lie in:
 (1) To obtain a decomposition of the form $\beta = \nabla\kern-0.6em\raise0.3ex\hbox{/}\,\tilde{\pi} + \cdots$ so that

$$\operatorname{div}\hat{\chi} = \nabla\kern-0.6em\raise0.3ex\hbox{/}\,(\operatorname{tr}\chi - \frac{2}{n(t-u)}) + \nabla\kern-0.6em\raise0.3ex\hbox{/}\,\tilde{\pi} + \cdots$$

with $\tilde{\pi}$ being certain components of $\partial\mathbf{g}$. The new difficulty arising in our situation is to exclude the time derivative of shift in the decomposition for the null curvature component of β. This issue is settled in Section 8.
 (2) To employ Calderon-Zygmund theorem for Hodge system $\operatorname{div}F = \nabla\kern-0.6em\raise0.3ex\hbox{/}\,G + e$

(1.26) $\qquad \|F\|_{L^\infty(S)} \lesssim \|G\|_{L_x^\infty(S)} \ln(2 + r^{\frac{3}{2}-\frac{2}{p}}\|\nabla\kern-0.6em\raise0.3ex\hbox{/}\,G\|_{L^p(S)}) + r^{1-\frac{2}{p}}\|e\|_{L^p(S)},$

where $S = S_{t,u}$. Relative to a regularized metric $\mathbf{g}_{\leq\lambda}$, using (1.26) to estimate $\|F(t,\cdot)\|_{L_x^\infty}$ only leads to a loss of $\ln\lambda$, which can be easily balanced after integration in a frequency dependent time subinterval. However, relative to non-smoothed metric, the application of (1.26) relies on the norm of $\sup_u \|\nabla\kern-0.6em\raise0.3ex\hbox{/}\,\tilde{\pi}\|_{L^{2+}(S_{t,u})}$, which requires the boundedness of the $H^{\frac{3}{2}+}$ norm of $\widehat{\nabla}g$ and k that can only be achieved under the assumption of $H^s, s > \frac{5}{2}$ on data. We fix this problem by squeezing a bit more differentiability out of $\widehat{\nabla}g, k$ in $L_t^2 L_x^\infty$, which can be achieved by Strichartz estimate via a bootstrap argument. To implement this idea, we establish a modified Calderon-Zygmund inequality relative to the rough metric, which can be seen in Section 9.

2. H^2 estimates

We first derive some simple consequences of (BA1) that will be used throughout this paper.

Let X be an arbitrary vector field on Σ. We use $|X|_g$ and $|X|_{\hat{g}}$ to denote the lengths of X measured by g and $\hat{g}$ respectively. It then follows from (1.3) that

$$\partial_t(|X|_g^2) = Y^m \widehat{\nabla}_m g_{ij} X^i X^j - 2n k_{ij} X^i X^j + (g_{im}\widehat{\nabla}_j Y^m + g_{mj}\widehat{\nabla}_i Y^m) X^i X^j.$$

Therefore

$$\left|\partial_t|X|_g^2\right| \leq \left(2|\widehat{\nabla}Y| + |Y|_g|\widehat{\nabla}g| + 2n|k|\right)|X|_g^2.$$

In view of (1.6) and the maximum principle, we can derive that $0 < n \leq C$, where C is a constant depending only on t_0; see [25, Section 2]. Recall that $|Y|_g \leq n$. We thus have

$$\left|\partial_t|X|_g^2\right| \leq C\left(|\widehat{\nabla}Y| + |\widehat{\nabla}g| + |k|\right)|X|_g^2.$$

This together with the bootstrap assumption (BA1) implies $C^{-1}|X|_{g(t_0)} \leq |X|_{g(t)} \leq C|X|_{g(t_0)}$. Since $g(t_0)$ and $\hat{g}$ are always equivalent on compact Σ, we therefore have

$$(2.1) \qquad\qquad C^{-1}\hat{g} \leq g \leq C\hat{g}, \quad \text{on } [t_0, t_0 + T] \times \Sigma$$

for some universal constant $C > 0$.[1] This equivalence between g and $\hat{g}$ on each Σ_t gives us the freedom to use g or $\hat{g}$ to measure the length of any Σ-tangent tensor.

Using (2.1) and (BA1), we can follow the arguments in [25, Sections 2 and 3] to derive that

$$(2.2) \qquad C^{-1} < n < C, \quad Q(t) \lesssim C, \quad \|H, E, \mathrm{Ric}\|_{L_x^2} \leq C, \quad \|\pi, \mathbf{D}_{n\mathbf{T}}n\|_{H^1} \leq C$$

$$(2.3) \qquad \|\nabla^3 n\|_{L_x^2} + \|\nabla^2 \mathbf{D}_{n\mathbf{T}}n\|_{L_x^2} \lesssim \|k\|_{L_x^\infty},$$

where π is the deformation tensor of $\mathbf{T}$ with components k and $\nabla \log n$, E and H are the electric and magnetic parts of spacetime curvature defined by $E_{ij} = \mathbf{R}_{0i0j}$ and $H_{ij} = {}^\star\mathbf{R}_{0i0j}$ respectively, and $Q(t)$ is the Bel-Robinson energy defined by

$$Q(t) = \int_\Sigma \left(|E|_g^2 + |H|_g^2\right) d\mu_g.$$

As a consequence of (2.3), we have

$$(2.4) \qquad\qquad \|\nabla n, \mathbf{D}_{nT}n\|_{L_x^\infty} \lesssim 1 + \|k\|_{L_x^\infty}^{\frac{3}{2}-\frac{3}{p}}, \quad 3 < p \leq 6.$$

Let us fix the convention that $F * G$ denotes contraction by g and $\cdot$ denotes either usual multiplication or contraction by $\hat{g}$.

Lemma 2.1. *Under the spatial harmonic gauge, the shift vector field Y satisfies the equation*

$$(2.5) \qquad\qquad \hat{\Delta}Y = \pi * U + \pi * \pi + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot Y + g^3 \cdot \hat{R} \cdot Y$$

where $\hat{\Delta}$ is defined in (1.14), U is defined in (1.8), and $\hat{R}$ is the Riemannian curvature with respect to $\hat{g}$.

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[1]We will always use C to denote a universal constant that depends only on the constant in the bootstrap assumptions, information on $g(t_0)$ and $\|(g,k)\|_{H^s \times H^{s-1}(\Sigma_{t_0})}$. For two quantities Φ and Ψ we will use $\Phi \lesssim \Psi$ to mean that $\Phi \leq C\Psi$ for some universal constant C.

Proof. Straightforward calculation shows for any vector field Y and tensor F that

$$\nabla_j Y^i = \widehat{\nabla}_j Y^i + U^i_{jq} Y^q, \quad \nabla_j F^i_m = \widehat{\nabla}_j F^i_m + U^i_{jp} F^p_m - U^p_{jm} F^i_p.$$

In view of the spatial harmonic gauge condition $U^i := g^{jl} U^i_{jl} = 0$, we obtain

$$g^{mj} \nabla_m \nabla_j Y^i = g^{mj} \widehat{\nabla}_m \widehat{\nabla}_j Y^i + g^{mj} \widehat{\nabla}_m U^i_{jq} Y^q + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q.$$

Recall the identity

$$(2.6) \qquad\qquad R_{jl} = \hat{R}_{jl} + \widehat{\nabla}_i U^i_{jl} - \widehat{\nabla}_j U^i_{il} + U^p_{jl} U^i_{pi} - U^p_{il} U^i_{pj}$$

which can be verified directly, we can obtain

$$\Delta Y^i + R^i_p Y^p = \hat{\Delta} Y^i + \Omega^i_p Y^p + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q$$
$$+ U \cdot U \cdot g \cdot Y + \hat{R} \cdot g \cdot Y,$$

where

$$\Omega^i_p = g^{mj} \widehat{\nabla}_m U^i_{jp} + \left(\widehat{\nabla}_m U^m_{pk} - \widehat{\nabla}_p U^m_{mk} \right) g^{ki}.$$

By using the expression of U, the commutation formula and the gauge condition $U^p = g^{ij} U^p_{ij} = 0$ we have

$$\Omega^i_p = g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + \frac{1}{2} g^{ki} g^{ml} \left(\widehat{\nabla}_m \widehat{\nabla}_p g_{kl} - \widehat{\nabla}_p \widehat{\nabla}_m g_{kl} \right)$$
$$+ \frac{1}{2} g^{ki} g^{ml} \left(\widehat{\nabla}_m \widehat{\nabla}_p g_{kl} + \widehat{\nabla}_p \widehat{\nabla}_l g_{mk} - \widehat{\nabla}_p \widehat{\nabla}_k g_{ml} \right)$$
$$= g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + g \cdot g \cdot g \cdot \hat{R}.$$

Thus

$$\Delta Y^i + R^i_p Y^p = \hat{\Delta} Y^i + (g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + g^3 \cdot \hat{R}) \cdot Y + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q$$
$$+ g^{mj} U^i_{mp} U^p_{jq} Y^q.$$

Combining this with (1.7) gives

$$\hat{\Delta} Y^i + 2 g^{mj} U^i_{jq} \widehat{\nabla}_m Y^q + g^{mj} U^i_{mp} U^p_{jq} Y^q$$
$$= -2 n k^{mj} U^i_{mj} + 2 \nabla^m Y^j U^i_{mj} + 2 \nabla^m n k^i_m - \nabla^i n k^m_m$$
$$+ g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot Y + g^3 \cdot \hat{R} \cdot Y.$$

Since $\nabla^m Y^l U^i_{ml} = g^{mj} \nabla_j Y^l U^i_{ml} = g^{mj} \widehat{\nabla}_j Y^l U^i_{ml} + g^{mj} U^p_{jq} U^i_{mp} Y^q$, we obtain

$$\hat{\Delta} Y^i = g^{mj} U^i_{mp} U^p_{jq} Y^q - 2 n k^{mp} U^i_{mp} + 2 \nabla^m n k^i_m - \nabla^i n k^m_m$$
$$+ g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot Y + g^3 \cdot \hat{R} \cdot Y$$

which is the desired equation. $\qquad\qquad\square$

Lemma 2.2. *For any Σ-tangent tensor field F, on each Σ_t there holds*

$$\|\widehat{\nabla}^2 F\|_{L^2_x} \lesssim \|\hat{\Delta} F\|_{L^2_x} + \|\widehat{\nabla} g \cdot \widehat{\nabla} F\|_{L^2_x} + \|\widehat{\nabla} F\|_{L^2_x} + \|F\|_{L^2_x}.$$

Proof. Let du_g denote the volume form induced by g on Σ_t. Then, under the spacial harmonic gauge, there holds $\widehat{\nabla}_i(g^{ij}d\mu_g) = 0$ (see [3, Page 3]). Thus, by integration by part, we have

$$\int_\Sigma |\widehat{\nabla}^2 F|_g^2 d\mu_g = \int_\Sigma g^{ij}g^{pq}\widehat{\nabla}_i\widehat{\nabla}_p F^l \widehat{\nabla}_j\widehat{\nabla}_q F_l d\mu_g$$

$$= -\int_\Sigma \left(\widehat{\nabla}_p F^l \hat{\Delta}\widehat{\nabla}_q F_l g^{pq} + g^{ij}\widehat{\nabla}_i g^{pq}\widehat{\nabla}_p F^l \widehat{\nabla}_j\widehat{\nabla}_q F_l\right) d\mu_g$$

Here and throughout the paper we will $\hat{g}$ to raise and lower the indices in tensors. It is easy to check the following commutator formula

(2.7) $\hat{\Delta}\widehat{\nabla}_q F_l - \widehat{\nabla}_q\hat{\Delta}F_l = g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 F + g \cdot \hat{R} \cdot \widehat{\nabla}F + g \cdot \widehat{\nabla}\hat{R} \cdot F.$

Therefore we can derive that

$$\int_\Sigma |\widehat{\nabla}^2 F|_g^2 d\mu_g = \int_\Sigma \left(-g^{pq}\widehat{\nabla}_p F^l \widehat{\nabla}_q\hat{\Delta}F_l + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}F \cdot \widehat{\nabla}^2 F\right.$$

$$\left. + (g \cdot \hat{R} \cdot \widehat{\nabla}F + g \cdot \widehat{\nabla}\hat{R} \cdot F)\widehat{\nabla}F\right) d\mu_g$$

where the first term is $\int_\Sigma \hat{\Delta}F^l \hat{\Delta}F_l d\mu_g$ by integration by part. $\square$

Lemma 2.3. *On each Σ_t there hold*

(2.8) $\|\widehat{\nabla}Y\|_{L_x^2} \lesssim \|\widehat{\nabla}g\|_{L_x^2} + 1,$

(2.9) $\|\widehat{\nabla}^2 Y\|_{L_x^2} \lesssim \|(\pi, \widehat{\nabla}Y, \widehat{\nabla}g) \cdot \widehat{\nabla}g\|_{L_x^2} + \|\widehat{\nabla}g\|_{L_x^2} + 1,$

$$\|\widehat{\nabla}^3 Y\|_{L_x^2} \lesssim \|\widehat{\nabla}Y, \widehat{\nabla}g\|_{H^1}(\|\widehat{\nabla}g\|_{L_x^\infty}^{\frac{4}{3}}\|\widehat{\nabla}g\|_{L_x^2}^{\frac{2}{3}} + \|\widehat{\nabla}g\|_{L_x^\infty})$$

(2.10) $+ (\|\widehat{\nabla}^2 g\|_{L_x^2} + 1) \cdot \|\widehat{\nabla}g, \pi\|_{L_x^\infty} + \|\widehat{\nabla}g\|_{L_x^2} + 1.$

Proof. Consider (2.8) first. By using (2.5) and $\widehat{\nabla}_j(g^{ij}d\mu_g) = 0$, we have

$$\|\widehat{\nabla}Y\|_{L_x^2}^2 \approx \int g^{ij}\widehat{\nabla}_i Y^l \widehat{\nabla}_j Y_l d\mu_g = \int -\hat{\Delta}Y^l \cdot Y_l d\mu_g$$

$$\lesssim \|\pi \cdot \pi\|_{L_x^1} + \|\widehat{\nabla}g \cdot (\widehat{\nabla}g, \pi)\|_{L_x^1} + 1.$$

In view of (2.2), we thus obtain (2.8).

Next, by using (2.5) we have

$$\|\hat{\Delta}Y\|_{L_x^2} \lesssim \|(\pi, \widehat{\nabla}g) \cdot \widehat{\nabla}g\|_{L_x^2} + \|\pi\|_{L_x^4}^2 + 1.$$

It then follows from Lemma 2.2 that

$$\|\widehat{\nabla}^2 Y\|_{L_x^2} \lesssim \|(\pi, \widehat{\nabla}Y, \widehat{\nabla}g) \cdot \widehat{\nabla}g\|_{L_x^2} + \|\widehat{\nabla}Y\|_{L_x^2} + 1.$$

We thus obtain (2.9) in view of (2.8).

Finally we derive (2.10). From (2.5) and (2.7) we first have

$$\hat{\Delta}\widehat{\nabla}Y = \widehat{\nabla}\hat{\Delta}Y + [\hat{\Delta}, \widehat{\nabla}]Y$$

$$= \widehat{\nabla}\pi \cdot \widehat{\nabla}g + \pi \cdot \widehat{\nabla}^2 g + \pi \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g + \pi \cdot \widehat{\nabla}\pi + \pi \cdot \pi \cdot \widehat{\nabla}g$$

$$+ \widehat{\nabla}g \cdot \widehat{\nabla}^2 g \cdot Y + \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot Y + \widehat{\nabla}g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}Y + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 Y$$

(2.11) $+ g \cdot \hat{R} \cdot \widehat{\nabla}Y + g \cdot \widehat{\nabla}\hat{R} \cdot Y + \widehat{\nabla}(g^3 \cdot \hat{R} \cdot Y).$

It is easy to see that the last three terms involving $\hat{R}$ can be bounded by $\lesssim 1 + \|\widehat{\nabla}g\|_{L_x^2}$. Note that

$$\|\pi \cdot \pi \cdot \widehat{\nabla}g\|_{L_x^2} \lesssim \|\widehat{\nabla}g\|_{L_x^\infty}\|\pi\|_{L_x^4}^2 \lesssim \|\widehat{\nabla}g\|_{L_x^\infty},$$

$$\|(\widehat{\nabla}Y, \widehat{\nabla}g, \pi) \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g\|_{L_x^2} \lesssim (\|\widehat{\nabla}g, \widehat{\nabla}Y\|_{H^1} + \|\pi\|_{L_x^6})\|\widehat{\nabla}g\|_{L_x^2}^{\frac{2}{3}}\|\widehat{\nabla}g\|_{L_x^\infty}^{\frac{4}{3}}.$$

Thus, using Lemma 2.2, we can obtain (2.10). $\qquad\square$

2.1. **Energy estimate for $\widehat{\nabla}g$.** In order to proceed further, besides (BA1) we also need the following bootstrap assumption

(BA2) $$\|\widehat{\nabla}g\|_{L_{[0,T]}^2 L_x^\infty} + \|k\|_{L_{[0,T]}^2 L_x^\infty} \leq B_0.$$

which is a stronger version for the corresponding part in (BA1). The verification of (BA1) and (BA2) will be carried out in Section 4.

We first introduce some conventions. For any 2-tensors u and v we define

$$\langle u, v \rangle := \hat{g}^{ik}\hat{g}^{jl}u_{ij}v_{kl} \quad \text{and} \quad \langle \widehat{\nabla}u, \widehat{\nabla}v \rangle_g := g^{ij}\langle \widehat{\nabla}_i u, \widehat{\nabla}_j v \rangle.$$

We will use $|u|^2 := \langle u, u \rangle$ and $|\widehat{\nabla}u|_g^2 := \langle \widehat{\nabla}u, \widehat{\nabla}v \rangle_g$.

In the following we will derive some estimates on $\widehat{\nabla}g$ and the derivatives on Y. By using the formula

$$\mathcal{L}_Y u_{ij} = \widehat{\nabla}_Y u_{ij} + u_{im}\widehat{\nabla}_j Y^m + u_{mj}\widehat{\nabla}_i Y^m$$

for any 2-tensor u and the formula under the spatial harmonic gauge,

(2.12) $$R_{ij} = -\frac{1}{2}\hat{\Delta}g_{ij} + \widehat{\mathrm{R}}_{ij} + g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}g$$

we can derive from (1.3) and (1.4) that the 2-tensors $u := g$ and $v := -2k$ satisfy the hyperbolic system (1.13) with F_u and F_v given symbolically by

(2.13) $$F_u = u \cdot \widehat{\nabla}Y \quad \text{and} \quad F_v = 2\nabla^2 n + n \cdot k * k + k * \widehat{\nabla}Y.$$

From (1.13), the commutation formula (2.7), we can derive

(2.14) $$\partial_t \widehat{\nabla}u - \widehat{\nabla}_Y \widehat{\nabla}u = n\widehat{\nabla}v + F_{\widehat{\nabla}u}$$

(2.15) $$\partial_t \widehat{\nabla}v - \widehat{\nabla}_Y \widehat{\nabla}v = n\hat{\Delta}\widehat{\nabla}u + F_{\widehat{\nabla}v}$$

where

$$F_{\widehat{\nabla}u} = \widehat{\nabla}Y \cdot \widehat{\nabla}u + Y \cdot \hat{R} \cdot u + \widehat{\nabla}n \cdot v + \widehat{\nabla}F_u,$$

$$F_{\widehat{\nabla}v} = \widehat{\nabla}Y \cdot \widehat{\nabla}v + Y \cdot \hat{R} \cdot v + \widehat{\nabla}n\hat{\Delta}u + \widehat{\nabla}F_v$$
$$+ n(g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 u + g \cdot \hat{R} \cdot \widehat{\nabla}u + g \cdot \widehat{\nabla}\hat{R} \cdot u).$$

It is straightforward to derive that

$$\begin{cases} \widehat{\nabla}^2 F_u = \widehat{\nabla}^2 Y \cdot \widehat{\nabla}g + g \cdot \widehat{\nabla}^3 Y + \widehat{\nabla}Y \cdot \widehat{\nabla}^2 g, \\ \widehat{\nabla}F_v = g \cdot \widehat{\nabla}^2 Y \cdot k + 2\nabla^3 n + g \cdot \widehat{\nabla}k \cdot \widehat{\nabla}Y + \widehat{\nabla}(ng \cdot k \cdot k) \end{cases}$$

and

$$|\widehat{\nabla}F_{\widehat{\nabla}u}, F_{\widehat{\nabla}v}| \leq |\widehat{\nabla}^2 Y \cdot (k, \widehat{\nabla}g)| + |\nabla^3 n, \widehat{\nabla}^3 Y| + |(\widehat{\nabla}Y, \nabla n, k, \widehat{\nabla}g) \cdot \widehat{\nabla}(\widehat{\nabla}g, k)|$$
$$+ |\widehat{\nabla}n \cdot k \cdot (\widehat{\nabla}g, k)| + |\widehat{\nabla}g \cdot k \cdot \widehat{\nabla}Y| + |\widehat{\nabla}g \cdot k \cdot k| + |\nabla^2 n \cdot k|$$
(2.16) $$+ |k| + |\widehat{\nabla}Y| + |\widehat{\nabla}g| + 1.$$

In order to derive the estimates, we use the energy introduced in [3, Section 2]

$$(2.17) \qquad \mathcal{E}^{(0)}(t) = \mathcal{E}^{(0)}(u,v)(t) := \frac{1}{2} \int_{\Sigma} \left(|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2 \right) d\mu_g$$

with $u = g$ and $v = -2k$.

Proposition 2.4. *Under the bootstrap assumption (BA1), there holds*

$$\sup_{[t_0, t_0+T]} \|\widehat{\nabla} g\|_{L^2(\Sigma_t)} \le C.$$

Proof. Recall that for any vector fields Z tangent to Σ_t and any scalar function f there holds $\int_{\Sigma_t} \mathcal{L}_Z(f d\mu_g) = \int_{\Sigma_t} \text{div}(f Z) d\mu_g = 0$. Therefore

$$\partial_t \mathcal{E}^{(0)}(t) = \frac{1}{2} \int_{\Sigma_t} (\partial_t - \mathcal{L}_Y) \left\{ (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2) d\mu_g \right\}$$

$$= \int_{\Sigma_t} \left[\langle u, \partial_t u - \widehat{\nabla}_Y u \rangle + \langle v, \partial_t v - \widehat{\nabla}_Y v \rangle + g^{ij} \langle \widehat{\nabla}_i u, (\partial_t - \widehat{\nabla}_Y) \widehat{\nabla}_j u \rangle \right] d\mu_g$$

$$+ \frac{1}{2} \int_{\Sigma_t} (\partial_t g^{ij} - \mathcal{L}_Y g^{ij}) \langle \widehat{\nabla}_i u, \widehat{\nabla}_j u \rangle d\mu_g$$

$$+ \frac{1}{2} \int_{\Sigma_t} (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2)(\partial_t - \mathcal{L}_Y)(d\mu_g)$$

By using (1.3) we have $\partial_t g^{ij} - \mathcal{L}_Y g^{ij} = 2n k^{ij}$ and $(\partial_t - \mathcal{L}_Y)(d\mu_g) = -n \text{Tr} k d\mu_g$. These two identities together with (1.13), (2.14) and (2.15) give

$$\partial_t \mathcal{E}^{(0)}(t) = \int_{\Sigma_t} \left(n\langle u, v \rangle + \langle u, F_u \rangle + \langle v, F_v \rangle + n k^{ij} \widehat{\nabla}_i u \widehat{\nabla}_j u \right) d\mu_g$$

$$+ \int_{\Sigma_t} g \cdot \left(\widehat{\nabla} F_u \cdot \widehat{\nabla} u + Y \cdot \hat{R} \cdot u \cdot \widehat{\nabla} u + \widehat{\nabla} Y \cdot \widehat{\nabla} u \cdot \widehat{\nabla} u \right) d\mu_g$$

$$- \frac{1}{2} \int_{\Sigma_t} n \text{Tr} k (|u|^2 + |\widehat{\nabla} u|_g^2 + |v|^2) \} d\mu_g.$$

In view of the bounds on n, $|Y|$ and g, we can derive that

$$(2.18) \quad \partial_t \mathcal{E}^{(0)}(t) \lesssim \left(\|k, \widehat{\nabla} Y\|_{L^\infty} + 1 \right) \mathcal{E}^{(0)}(t) + \|\widehat{\nabla} F_u\|_{L^2} \|\widehat{\nabla} u\|_{L^2} + \|v\|_{L^2} \|F_v\|_{L^2}.$$

By using Lemma 2.3 we have

$$\|\widehat{\nabla} F_u\|_{L^2} \le \|\widehat{\nabla} Y\|_{L^\infty} \|\widehat{\nabla} g\|_{L^2} + \|\widehat{\nabla}^2 Y\|_{L^2} \lesssim (\|\widehat{\nabla} Y, \widehat{\nabla} g, \pi\|_{L^\infty} + 1) \|\widehat{\nabla} g\|_{L^2} + 1.$$

and

$$\|F_v\|_{L^2} \lesssim \|\nabla^2 n\|_{L^2} + \|k\|_{L^4}^2 + \|k\|_{L^6} \|\widehat{\nabla} Y\|_{L^3} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L^2} + \|\widehat{\nabla} Y\|_{L^2}$$

$$\lesssim (\|\pi, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty} + 1) \|\widehat{\nabla} g\|_{L^2} + 1.$$

Therefore

$$\partial_t \mathcal{E}^{(0)}(t) \lesssim \left(\|\pi, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty(\Sigma_t)} + 1 \right) \mathcal{E}^{(0)}(t) + 1.$$

This together with the bootstrap assumption (BA1) gives $\mathcal{E}^{(0)}(t) \le \mathcal{E}^{(0)}(t_0) + 1$ for all $t \in [t_0, t_0 + T]$. The proof is thus complete. $\qquad \square$

We now consider the energy $\mathcal{E}^{(1)}(t) = \mathcal{E}^{(0)}(\widehat{\nabla}u, \widehat{\nabla}v)$. From (2.18) it follows easily that

$$\partial_t \mathcal{E}^{(1)}(t) \lesssim \left(\|k, \widehat{\nabla}Y\|_{L_x^\infty} + 1 \right) \mathcal{E}^{(1)}(t) + \left(\|\widehat{\nabla}F_{\widehat{\nabla}u}\|_{L_x^2} + \|F_{\widehat{\nabla}v}\|_{L_x^2} \right) \sqrt{\mathcal{E}^{(1)}(t)}.$$

In view of (2.16), we derive

$$\begin{aligned}
\|\widehat{\nabla}F_{\widehat{\nabla}u}, F_{\widehat{\nabla}v}\|_{L_x^2} &\lesssim \|\widehat{\nabla}^2 Y\|_{L_x^2} \|\widehat{\nabla}g, k\|_{L_x^\infty} + \|\widehat{\nabla}^3 Y\|_{L_x^2} + \|\nabla^3 n\|_{L_x^2} \\
&\quad + \|\widehat{\nabla}Y, \widehat{\nabla}n, k, \widehat{\nabla}g\|_{L_x^\infty} \|\widehat{\nabla}(\widehat{\nabla}g, k)\|_{L_x^2} + \|\widehat{\nabla}g\|_{L_x^6} \|\widehat{\nabla}Y\|_{L_x^\infty} \|k\|_{L_x^3} \\
&\quad + \|\widehat{\nabla}g\|_{L_x^\infty} \|k\|_{L_x^4}^2 + \|\pi\|_{L_x^6}^3 + \|\nabla^2 n\|_{L_x^2} \|k\|_{L_x^\infty} \\
&\quad + \|k\|_{L_x^2} + \|\widehat{\nabla}Y\|_{L_x^2} + \|\widehat{\nabla}g\|_{L_x^2} + 1.
\end{aligned}$$

In view of (2.2), (2.3), Proposition 2.4, and (2.8) and (2.10) in Lemma 2.3, we then have

$$\begin{aligned}
\|\widehat{\nabla}F_{\widehat{\nabla}u}, F_{\widehat{\nabla}v}\|_{L_x^2} &\lesssim \left(\|\widehat{\nabla}g, k, \widehat{\nabla}Y, \widehat{\nabla}n\|_{L_x^\infty} + \|\widehat{\nabla}g\|_{L_x^\infty}^{\frac{4}{3}} \right) \left(\|\widehat{\nabla}^2 g\|_{L_x^2} + 1 \right) \\
&\quad + \|\widehat{\nabla}Y\|_{H^1} \left(\|\widehat{\nabla}g, k\|_{L_x^\infty} + \|\widehat{\nabla}g\|_{L_x^\infty}^{\frac{4}{3}} + 1 \right).
\end{aligned}$$

Using this estimate, the inequality $\|\widehat{\nabla}^2 g\|_{L_x^2} \leq \sqrt{\mathcal{E}^{(1)}(t)}$, and the Young's inequality, we can obtain

$$\begin{aligned}
\partial_t \mathcal{E}^{(1)}(t) &\lesssim \left(1 + \|k, \widehat{\nabla}n, \widehat{\nabla}g, \widehat{\nabla}Y\|_{L_x^\infty} + \|k, \widehat{\nabla}g\|_{L_x^\infty}^2 \right) \mathcal{E}^{(1)}(t) \\
&\quad + \|k, \widehat{\nabla}n, \widehat{\nabla}g, \widehat{\nabla}Y\|_{L_x^\infty} + \|\widehat{\nabla}g\|_{L_x^2}^2 + \|\widehat{\nabla}^2 Y\|_{L_x^2}^3 + 1
\end{aligned}$$

In view of (BA1) and (BA2), it follows easily that

$$\mathcal{E}^{(1)}(t) \lesssim \mathcal{E}^{(1)}(t_0) + 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^3.$$

This in particular implies that

(2.19) $$\|\widehat{\nabla}^2 g\|_{L^2(\Sigma_t)} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^{3/2}.$$

On the other hand, it follows from (2.9), (2.2), and Proposition 2.4 that

$$\begin{aligned}
\|\widehat{\nabla}^2 Y\|_{L_x^2} &\lesssim \|\widehat{\nabla}Y \cdot \widehat{\nabla}g\|_{L_x^2} + \|\widehat{\nabla}g\|_{L_x^4}^2 + \|\pi\|_{L_x^4}^2 + 1 \\
&\lesssim \|\widehat{\nabla}Y\|_{L_x^6}^{\frac{1}{2}} \|\widehat{\nabla}Y\|_{L_x^2}^{\frac{1}{2}} \|\widehat{\nabla}g\|_{L_x^6} + \|\widehat{\nabla}g\|_{L_x^4}^2 + 1.
\end{aligned}$$

Using $\|\widehat{\nabla}Y\|_{L_x^6} \lesssim \|\widehat{\nabla}^2 Y\|_{L_x^2} + 1$ and (2.8) we can obtain

$$\|\widehat{\nabla}^2 Y\|_{L_x^2} \lesssim \|\widehat{\nabla}g\|_{L_x^6}^2 + 1 \lesssim \|\widehat{\nabla}^2 g\|_{L_x^2}^2 + 1.$$

This together with (2.19) gives

(2.20) $$\|\widehat{\nabla}^2 Y\|_{L_x^2} \lesssim 1 + \|\widehat{\nabla}^2 Y\|_{L_t^3 L_x^2}^3.$$

Integrating with respect t over $[t_0, t_0 + T]$ yields

$$\|\widehat{\nabla}^2 Y\|_{L_{[t_0, t_0+T]}^3 L_x^2} \lesssim T^{\frac{1}{3}} \left(1 + \|\widehat{\nabla}^2 Y\|_{L_{[t_0, t_0+T]}^3 L_x^2}^3 \right).$$

Therefore we can choose a small but universal $T > 0$ such that $\|\widehat{\nabla}^2 Y\|_{L_{[t_0, t_0+T]}^3 L_x^2} \leq C$ for some universal constant C. Consequently, by using (2.19) and (2.20) we can obtain (2.21) and (2.22) in the following result.

Proposition 2.5. *Under the bootstrap assumption (BA1) and (BA2), there hold*

$$(2.21) \qquad\qquad \|g\|_{H^2(\Sigma_t)} + \|\widehat{\nabla}k\|_{L^2(\Sigma_t)} \le C,$$

$$(2.22) \qquad\qquad \|\widehat{\nabla}^2 Y\|_{L^2(\Sigma_t)} + \|\widehat{\nabla}Y\|_{L^2(\Sigma_t)} \le C,$$

$$(2.23) \qquad\qquad \|e_0(\widehat{\nabla}g)\|_{L^2(\Sigma_t)} + \|\partial_t \widehat{\nabla}g\|_{L^2(\Sigma_t)} \le C$$

for all $t \in [t_0, t_0+T]$ with $T > 0$ being a universal number, where, for any Σ_t-tangent tensor field F, we use the notation $e_0(F) := n^{-1}(\partial_t F - \widehat{\nabla}_Y F)$.

Proof. It remains only to prove (2.23). We use (2.14), (2.21) and (2.22) to deduce that

$$\|e_0(\widehat{\nabla}g)\|_{L^2} \lesssim \|\widehat{\nabla}k\|_{L^2} + \|\widehat{\nabla}Y\|_{L^6}\|\widehat{\nabla}g\|_{L^3} + \|\widehat{\nabla}^2 Y\|_{L^2}$$
$$+ \|\widehat{\nabla}n\|_{L^4}\|k\|_{L^4} + \|Y\|_{L^\infty}\|g\|_{L^2} \lesssim 1.$$

Finally, in view of (2.21) we obtain $\|\partial_t \widehat{\nabla}g\|_{L^2} \lesssim \|e_0(\widehat{\nabla}g)\|_{L^2} + \|\widehat{\nabla}^2 g\|_{L^2} \lesssim 1.$ □

Lemma 2.6. *Under the bootstrap assumptions (BA1) and (BA2), for $3 < p \le 6$ there hold*

$$(2.24) \qquad\qquad \|\widehat{\nabla}^3 Y\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{L^\infty} + 1,$$

$$(2.25) \qquad\qquad \|\widehat{\nabla}Y\|_{L^\infty} \lesssim \|k, \widehat{\nabla}g\|_{L^\infty}^{3/2-3/p} + 1.$$

Proof. In view of Proposition 2.5, (2.2), and Lemma 2.2, we obtain from (2.11) that

$$\|\widehat{\nabla}^3 Y\|_{L^2} \lesssim \|\widehat{\nabla}\pi, \widehat{\nabla}g\|_{L^2_x}\|\widehat{\nabla}g, \pi\|_{L^\infty_x} + \|\widehat{\nabla}^2 Y \cdot \widehat{\nabla}g\|_{L^2} + \|\widehat{\nabla}g, k\|_{L^\infty_x} + 1.$$

We may write

$$\|\widehat{\nabla}^2 Y \cdot \widehat{\nabla}g\|_{L^2} \lesssim \|\widehat{\nabla}^2 Y\|_{L^3}\|\widehat{\nabla}g\|_{L^6} \lesssim \|\widehat{\nabla}^2 Y\|_{L^3}\|\widehat{\nabla}^2 g\|_{L^2}.$$

Applying the Sobolev type inequality ([25, Lemma 2.5]) to $\|\widehat{\nabla}^2 Y\|_{L^3}$, and using (2.2), (2.3), (2.21) and (2.22), we can obtain (2.24). Finally we can use the Sobolev embedding given in [25, Lemma 2.6] to conclude (2.25). □

3. $H^{2+\epsilon}$ estimates

In this section, under the bootstrap assumptions (BA1) and (BA2), we will establish $H^{1+\epsilon}$ type energy estimates for $k, \widehat{\nabla}g$ and $\mathbf{D}\phi$ with ϕ being solutions of homogeneous wave equation $\Box_{\mathbf{g}}\phi = 0$. We will also obtain the $H^{\frac{3}{2}+\epsilon}$ and $H^{2+\epsilon}$ estimates for $\widehat{\nabla}n, \widehat{\nabla}Y, ne_0(n)$ simultaneously. As the main building block of this section, established in Appendix III are a series of product estimates in fractional Sobolev spaces and estimates for commutators between the Littlewood-Paley projections P_μ and the rough coefficients.

For simplicity of exposition, we fix some conventions. We will use $\tilde{\pi}$ to denote any term from the set $\widehat{\nabla}Y, \widehat{\nabla}n, k, \widehat{\nabla}g$ and $e_0(n)$, where $e_0(n) = n^{-1}(\partial_t n - \widehat{\nabla}_Y n)$ as defined before. It follows from Proposition 2.5 and (2.2) that $\|\tilde{\pi}\|_{H^1} \le C$. We also introduce the error terms

$$(3.1) \qquad\qquad \text{err}_1 = \mathbf{g} \cdot \tilde{\pi} \cdot \widehat{\nabla}\tilde{\pi}, \qquad \text{err}_2 = \mathbf{g} \cdot \tilde{\pi} \cdot \tilde{\pi} \cdot \tilde{\pi},$$

where $\mathbf{g}$ denotes any product of the components of n, g and Y. We denote by $\text{err}(\hat{R})$ any term involving $\hat{R}$ and its derivatives satisfying

$$(3.2) \qquad\qquad \|\text{err}(\hat{R})\|_{H^1(\Sigma_t)} \le C$$

for all $t \in [t_0, t_0 + T]$.

Proposition 3.1. *For $0 < \epsilon < 1/2$ there hold*

(3.3) $\|\Lambda^{1/2+\epsilon}(\widehat{\nabla}^2 n, \widehat{\nabla}(ne_0(n)), \widehat{\nabla}^2 Y)\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1,$

(3.4) $\|\Lambda^{\epsilon}(\widehat{\nabla}^3 n, \widehat{\nabla}^2(ne_0(n)), \widehat{\nabla}^3 Y)\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{L^{\infty}}\|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1$

and, for the error type terms defined in (3.1), there hold

(3.5) $\|\Lambda^{\epsilon} err_1\|_{L^2} \lesssim \|k, \widehat{\nabla}g\|_{L^{\infty}}(\|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1) + 1,$

(3.6) $\|\Lambda^{\epsilon} err_2\|_{L^2} \lesssim \|\widehat{\nabla}g, k\|_{H^{1+\epsilon}} + 1.$

Proof. For any scalar function f it is easy to derive the commutation formula

(3.7) $[\Delta, \nabla_{n\mathbf{T}}]f = -2nk_i^l \nabla_l \nabla^i f - \nabla^i n k_i^l \nabla_l f.$

To obtain the estimates of $\widehat{\nabla}n$ and $ne_0(n)$, we first use (1.6) and (3.7) to derive the identities

(3.8) $\hat{\Delta}\widehat{\nabla}n = \widehat{\nabla}(n|k|_g^2) + g\hat{R} \cdot \widehat{\nabla}n,$

(3.9) $\hat{\Delta}(ne_0(n)) = ne_0(n|k|_g^2) - 2nk_a^l \nabla_l \nabla^a n - \nabla^a n \nabla_l n k_a^l.$

In view of (3.8), we have

(3.10) $\hat{\Delta}\widehat{\nabla}n = n\widehat{\nabla}k \cdot k \cdot g + (\widehat{\nabla}n, k, \widehat{\nabla}g)^3 \cdot (n, g) + \mathrm{err}(\hat{R}).$

It then follows from (10.3) that

$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}\widehat{\nabla}n\|_{L^2} \lesssim \|\widehat{\nabla}k\|_{H^{\epsilon}}\|ng \cdot k\|_{H^1} + \|(\widehat{\nabla}n, k, \widehat{\nabla}g)^3 \cdot (n, g)\|_{L^2} + \|\mathrm{err}(\hat{R})\|_{L^2}.$

By using (10.30) and $\|\tilde{\pi}\|_{H^1} \leq C$, we can conclude that

(3.11) $\|\Lambda^{1/2+\epsilon}\widehat{\nabla}^2 n\|_{L^2} \lesssim \|\widehat{\nabla}k\|_{H^{\epsilon}} + 1.$

In view of (3.9) and (1.13), we have

(3.12) $\hat{\Delta}(ne_0(n)) = (n\hat{\Delta}g + \nabla^2 n)ng \cdot k + \mathbf{g} \cdot \tilde{\pi} \cdot \tilde{\pi} \cdot \tilde{\pi}.$

Thus, with the help of (10.3) and (3.11), it follows

$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}(ne_0(n))\|_{L^2} \lesssim (\|\widehat{\nabla}^2 g\|_{H^{\epsilon}} + \|\widehat{\nabla}^2 n\|_{H^{\epsilon}})\|\mathbf{g} \cdot k\|_{H^1} + \|\mathbf{g}|\tilde{\pi}|^3\|_{L^2}$

$\lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla}k\|_{H^{\epsilon}} + 1$

which implies, in view of (10.30) and $\|\tilde{\pi}\|_{H^1} \leq C$, that

(3.13) $\|\Lambda^{1/2+\epsilon}\widehat{\nabla}(ne_0(n))\|_{L^2} \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla}k\|_{H^{\epsilon}} + 1.$

Next we use (2.11) and (10.3) to obtain

$\|\Lambda^{-1/2+\epsilon}\hat{\Delta}\widehat{\nabla}Y\|_{L^2} \lesssim \|\widehat{\nabla}(\pi, \widehat{\nabla}g, \widehat{\nabla}Y)\|_{H^{\epsilon}} + \|\mathrm{err}(\hat{R})\|_{L^2}.$

This together with (3.11), (10.30), $\|\tilde{\pi}\|_{H^1} \leq C$ and the interpolation inequality gives

(3.14) $\|\Lambda^{1/2+\epsilon}\widehat{\nabla}^2 Y\|_{L^2} \lesssim \|\widehat{\nabla}^2 Y\|_{H^{\epsilon}} + \|\widehat{\nabla}^2 g, \widehat{\nabla}k\|_{H^{\epsilon}} + 1 \lesssim \|\widehat{\nabla}^2 g, \widehat{\nabla}k\|_{H^{\epsilon}} + 1.$

Combining the estimates (3.11), (3.13) and (3.14), we therefore complete the proof of (3.3).

As a byproduct of (3.3), we have

(3.15) $\|\Lambda^{\epsilon}\widehat{\nabla}\tilde{\pi}\|_{L^2} \lesssim \|k, \widehat{\nabla}g\|_{H^{1+\epsilon}} + 1.$

It then follows from (10.7) and (3.15) that

$$(3.16) \qquad \|\Lambda^\epsilon(\tilde\pi \cdot \widehat\nabla \tilde\pi)\|_{L^2} \lesssim \|\tilde\pi\|_{L^\infty} \|\tilde\pi\|_{H^{1+\epsilon}} \lesssim (\|k, \widehat\nabla g\|_{H^{1+\epsilon}} + 1)\|\tilde\pi\|_{L^\infty}.$$

By Lemma 10.2 and (3.15), we have

$$(3.17) \qquad \|\Lambda^\epsilon(\tilde\pi \cdot \tilde\pi \cdot \tilde\pi)\|_{L^2_x} \lesssim \|\tilde\pi\|^2_{H^1} \|\tilde\pi\|_{H^{1+\epsilon}} \lesssim \|k, \widehat\nabla g\|_{H^{1+\epsilon}} + 1.$$

To treat the factor $\mathbf{g}$ in the definition (3.1), in view of $\|\mathbf{g}\|_{H^2} \leq C$ in Proposition 2.5, using Lemma 10.11, and (3.16) and (3.17), we thus obtain (3.5) and (3.6).

Finally, we consider (3.4) with the help of (10.29). Let $F = ne_0(n), \widehat\nabla n, \widehat\nabla Y$. Then it follows from (2.2) and (2.22) that $\|F\|_{H^1} \lesssim 1$. Moreover, the elliptic equations (3.10), (3.12) and (2.11) can be written symbolically as $\hat\Delta F = \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat R)$. In view of (3.5), (3.6), and the definition of $\mathrm{err}(\hat R)$, we thus obtain (3.4). $\qquad\qquad\qquad\square$

3.1. First order hyperbolic systems.

3.1.1. *Energy estimates.* We consider a pair of tensors (u, v) satisfying the first order hyperbolic system

$$(3.18) \qquad \begin{cases} \partial_t u - \widehat\nabla_Y u = nv + F_u \\ \partial_t v - \widehat\nabla_Y v = n\hat\Delta u + F_v. \end{cases}$$

Note that for (u, v) satisfying (3.18), the pair $(U_1, V_1) = (\widehat\nabla u, \widehat\nabla v)$ satisfies an equation system of the form of (3.18) with

$$(3.19) \qquad \begin{cases} F_{U_1} = \widehat\nabla Y^m \widehat\nabla_m u + \widehat\nabla n \cdot v + \widehat\nabla F_u + Y \cdot \hat R \cdot u \\ F_{V_1} = \widehat\nabla Y^m \widehat\nabla_m v + \widehat\nabla n \cdot \hat\Delta u + n\hat R \cdot \widehat\nabla u + Y \cdot \hat R \cdot v + \widehat\nabla F_v + n\widehat\nabla(\hat R \cdot u) \end{cases}$$

where the last term in F_{U_1} and F_{V_1} can be dropped in the case that (u, v) is a pair of scalar functions.

We also can check that the pair of functions $(U^\mu, V^\mu) := (P_\mu u, P_\mu v)$ satisfies (3.18) with F_{U^μ} and F_{V^μ} given by [2]

$$(3.20) \qquad \begin{cases} F_{U^\mu} = [P_\mu, Y^m]\partial_m u + [P_\mu, n]v + P_\mu F_u, \\ F_{V^\mu} = [P_\mu, ng]\widehat\nabla^2_{ij} u + P_\mu F_v + [P_\mu, Y^m]\partial_m v. \end{cases}$$

Thus, it is easy to check that $(U^\mu_1, V^\mu_1) := (P_\mu \widehat\nabla u, P_\mu \widehat\nabla v)$ satisfies the system (3.18) with $F_{U^\mu_1}$ and $F_{V^\mu_1}$ given by

$$(3.21) \qquad \begin{cases} F_{U^\mu_1} = [P_\mu, Y^m]\partial_m \widehat\nabla u + [P_\mu, n]\widehat\nabla v + P_\mu F_{U_1}, \\ F_{V^\mu_1} = [P_\mu, ng]\widehat\nabla^2_{ij}\widehat\nabla u + [P_\mu, Y^m]\partial_m \widehat\nabla v + P_\mu F_{V_1}. \end{cases}$$

Lemma 3.2. *Let $0 < \epsilon < 1/2$. Then for F_{U^μ} and F_{V^μ} defined by (3.20) there hold the estimates*

(3.22)
$$\|\mu^{\frac{1}{2}+\epsilon} F_{U^\mu}\|_{l^2_\mu L^2_x} + \|\mu^{-\frac{1}{2}+\epsilon}\widehat\nabla F_{U^\mu}\|_{l^2_\mu L^2_x} \lesssim \|\widehat\nabla n, \widehat\nabla Y\|_{H^1}\|\widehat\nabla u, v\|_{H^\epsilon} + \|\mu^{\frac{1}{2}+\epsilon} P_\mu F_u\|_{l^2_\mu L^2_x},$$

---

[2]We remark that the precise form of the first term in F_{V^μ} should be $[P_\mu, ng\widehat\nabla^2]u$ which consists of $[P_\mu, ng]\widehat\nabla^2 u$ and $ng[P_\mu, \hat\Gamma]\partial u$. The latter is of much lower order, which not only can be treated similar to the first term, but also can be done in a much easier way since $\hat\Gamma$ is smooth. Thus, we will omit this term for ease of exposition.

$$\|\mu^\epsilon \widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L_x^2} + \|\mu^{1+\epsilon} F_{U^\mu}\|_{l_\mu^2 L_x^2}$$

(3.23)
$$\lesssim \|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla} u, v\|_{L_x^2} + \|\widehat{\nabla} n, \widehat{\nabla} Y\|_{L_x^\infty} \|\widehat{\nabla} u, v\|_{H^\epsilon} + \|\mu^\epsilon \widehat{\nabla} P_\mu F_u\|_{l_\mu^2 L_x^2},$$

and

(3.24)
$$\|\mu^\epsilon F_{V^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}(ng), \widehat{\nabla} Y\|_{L_x^\infty} \|\widehat{\nabla} u, v\|_{H^\epsilon} + \|\mu^\epsilon P_\mu F_v\|_{l_\mu^2 L_x^2}.$$

Proof. (3.23) follows from (10.21), (3.24) follows from (10.13), and (3.22) follows from (10.14) and (10.15). $\qquad\square$

Lemma 3.3. *For $0 < \epsilon < 1/2$ there hold*

(3.25)
$$\|\mu^{1+\epsilon} \widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}^2 F_u\|_{H^\epsilon} + \mathcal{I}(\epsilon, u, v),$$

(3.26)
$$\|\mu^\epsilon F_{V_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim (\|\widehat{\nabla} g\|_{L_x^\infty} + 1)\|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} F_v\|_{H^\epsilon} + \mathcal{I}(\epsilon, u, v) + \|\widehat{\nabla} u, v\|_{H^\epsilon},$$

where

$$\mathcal{I}(\epsilon, u, v) = \|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|_{L_x^2} + \|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L_x^\infty} \|\widehat{\nabla} u, v\|_{H^{1+\epsilon}}.$$

Proof. The first part of (3.25) follows from (10.23) and (3.20). In order to prove the second part of (3.25), we may use the same argument for deriving (3.23) to obtain
$$\|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu F_{U_1}\|_{l_\mu^2 L_x^2}.$$
In view of (3.19), we apply (10.26) to obtain

$$\|\mu^\epsilon \widehat{\nabla} P_\mu F_{U_1}\|_{l_\mu^2 L_x^2} \lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu \widehat{\nabla} F_u\|_{l_\mu^2 L_x^2} + \|\Lambda^\epsilon \widehat{\nabla}(Y \cdot \hat{R} \cdot u)\|_{L^2}$$
$$\lesssim \mathcal{I}(\epsilon, u, v) + \|\mu^\epsilon \widehat{\nabla} P_\mu \widehat{\nabla} F_u\|_{l_\mu^2 L_x^2}.$$

Combining the above two estimates, we therefore obtain the second part of (3.25).

Next we prove (3.26). We first apply Lemma 10.6 to derive that

(3.27)
$$\|\mu^\epsilon F_{V_1^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}(ng)\|_{L^\infty} \|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} Y\|_{L^\infty} \|\widehat{\nabla} v\|_{H^\epsilon} + \|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2}.$$

By using Lemma 10.11 we have

$$\|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla} Y\|_{L_x^\infty} \|\widehat{\nabla} v\|_{H^\epsilon} + \|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla} v\|_{L_x^2} + \|\widehat{\nabla} n\|_{L_x^\infty} \|\hat{\Delta} u\|_{H^\epsilon}$$
$$+ \|\widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\hat{\Delta} u\|_{L_x^2} + \|n\hat{R} \cdot \widehat{\nabla} u\|_{H^\epsilon} + \|Y\hat{R} \cdot v\|_{H^\epsilon} + \|\widehat{\nabla} F_v\|_{H^\epsilon}.$$

With the help of Lemma 10.11 and $\|\tilde{\pi}\|_{H^1} \le C$, we obtain

$$\|\hat{\Delta} u\|_{H^\epsilon} \lesssim \|\mu^\epsilon g^{ij} P_\mu \widehat{\nabla}_{ij}^2 u\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon [P_\mu, g^{ij}] \widehat{\nabla}_{ij}^2 u\|_{l_\mu^2 L_x^2}$$
$$\lesssim \|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} g\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla}^2 u\|_{L_x^2} \lesssim \|\widehat{\nabla}^2 u\|_{H^\epsilon},$$
$$\|n\hat{R} \cdot \widehat{\nabla} u\|_{H^\epsilon} \lesssim \|\mu^\epsilon [P_\mu, n\hat{R}] \widehat{\nabla} u\|_{l_\mu^2 L_x^2} + \|\widehat{\nabla} u\|_{H^\epsilon}$$
$$\lesssim \|\widehat{\nabla} u\|_{H^\epsilon} + \|\widehat{\nabla}(n\hat{R})\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla} u\|_{L_x^2} \lesssim \|\widehat{\nabla} u\|_{H^\epsilon}$$

Similarly, we have with the help of $\|\widehat{\nabla} Y\|_{H^1} \le C$ that $\|Y \cdot \hat{R} \cdot v\|_{H^\epsilon} \lesssim \|v\|_{H^\epsilon}$. Therefore

$$\|\mu^\epsilon P_\mu F_{V_1}\|_{l_\mu^2 L_x^2} \lesssim \mathcal{I}(\epsilon, u, v) + \|\widehat{\nabla} F_v\|_{H^\epsilon} + \|\widehat{\nabla} u\|_{H^\epsilon} + \|v\|_{H^\epsilon}.$$

Combining this estimate with (3.27) we thus obtain (3.26). $\qquad\square$

In the following we will derive the estimates on $\|\widehat{\nabla}^2 g\|_{H^\epsilon}$ and $\|\widehat{\nabla}k\|_{H^\epsilon}$. Recall the energy $\mathcal{E}^{(0)}(u,v)$ defined in (2.17). Let P_μ be the Littlewood-Paley projection with frequency size μ, we can introduce

$$\mathcal{E}^{(1)}_\mu(t) = \mathcal{E}^{(1)}_\mu(u,v) := \mathcal{E}^{(0)}(P_\mu\widehat{\nabla}u, P_\mu\widehat{\nabla}v),$$

and the energy

$$(3.28)\qquad \mathcal{E}^{(1+\epsilon)}(u,v)(t) := \sum_{\mu>1}\mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(u,v)(t) + \sum_{i=0}^{1}\mathcal{E}^{(i)}(u,v)(t).$$

In view of (2.18) we can derive that

(3.29)
$$\partial_t\mathcal{E}^{(1)}_\mu(t) \leq \left(\|k,\widehat{\nabla}Y\|_{L^\infty}+1\right)\mathcal{E}^{(1)}_\mu(t) + \|\widehat{\nabla}F_{U^\mu_1}\|_{L^2}\|\widehat{\nabla}U^\mu_1\|_{L^2} + \|V^\mu_1\|_{L^2}\|F_{V^\mu_1}\|_{L^2}.$$

Hence, by the Cauchy-Schwartz inequality, we obtain

$$\partial_t\left(\sum_\mu\mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)\right) \leq \left(\|k,\widehat{\nabla}Y\|_{L^\infty}+1\right)\sum_\mu\mu^{2\epsilon}\mathcal{E}^{(1)}_\mu(t)$$

$$(3.30)\qquad\qquad + \|\mu^\epsilon\widehat{\nabla}F_{U^\mu_1}\|_{l^2_\mu L^2}\|\mu^\epsilon\widehat{\nabla}U^\mu_1\|_{l^2_\mu L^2} + \|\mu^\epsilon V^\mu_1\|_{l^2_\mu L^2}\|\mu^\epsilon F_{V^\mu_1}\|_{l^2_\mu L^2}.$$

We will apply (3.30) to the pair $(u,v) = (g,-2k)$ and (2.13) to derive energy estimates. Lemma 3.3 will be used to estimate the terms $\|\mu^\epsilon\widehat{\nabla}F_{U^\mu_1}\|_{l^2_\mu L^2}$ and $\|\mu^\epsilon F_{V^\mu_1}\|_{l^2_\mu L^2}$ which involve terms related to F_u and F_v. The following result gives such estimates.

Lemma 3.4. *For $0 < \epsilon < 1/2$ there hold*

$$(3.31)\qquad \|\Lambda^\epsilon\widehat{\nabla}^2 F_u\|_{L^2_x} + \|\Lambda^\epsilon\widehat{\nabla}F_v\|_{L^2_x} \lesssim (\|\widehat{\nabla}g,\widehat{\nabla}Y,k,\widehat{\nabla}n\|_{L^\infty_x}+1)\|\widehat{\nabla}g,k\|_{H^{1+\epsilon}},$$

$$(3.32)\qquad \|\Lambda^{\frac{1}{2}+\epsilon}\widehat{\nabla}F_u\|_{L^2_x} \lesssim \|\widehat{\nabla}g,k\|_{H^{1+\epsilon}}+1.$$

Proof. Recall F_u and F_v from (2.13). By straightforward calculation, symbolically we have

$$\widehat{\nabla}^2 F_u = g\cdot\widehat{\nabla}^3 Y + \mathrm{err}_1 + \mathrm{err}_2,\qquad \widehat{\nabla}F_v = \widehat{\nabla}\nabla^2 n + \mathrm{err}_1 + \mathrm{err}_2.$$

where err_1 and err_2 denote the terms introduced in (3.1). Then (3.31) follows from Proposition 3.1. Applying Lemma 10.13 to $F = g$ and $G = \widehat{\nabla}Y$, and using (3.3) and (2.22) we obtain

$$\|\Lambda^{\epsilon+\frac{1}{2}}\widehat{\nabla}(g\cdot\widehat{\nabla}Y)\|_{L^2_x} \lesssim \|\widehat{\nabla}g\|_{H^{1+\epsilon}}\|\widehat{\nabla}Y\|_{H^1} + \|g\|_{L^\infty}\|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+\epsilon}} \lesssim \|\widehat{\nabla}g,k\|_{H^{1+\epsilon}}+1$$

which gives (3.32). $\qquad\square$

Proposition 3.5. *For $0 < \epsilon \leq s-2$ there holds*

$$(3.33)\qquad \|\widehat{\nabla}^2 g(t)\|_{H^\epsilon} + \|\widehat{\nabla}k(t)\|_{H^\epsilon} \leq C$$

and for any pair (u,v) satisfying (3.18) there holds

(3.34)
$$\mathcal{E}^{(1+\epsilon)}(u,v)(t) \lesssim \mathcal{E}^{(1+\epsilon)}(t_0) + \int_{t_0}^{t}\left(\|\widehat{\nabla}u\|_{H^{1+\epsilon}}\|\widehat{\nabla}F_u\|_{H^{1+\epsilon}} + \|v\|_{H^{1+\epsilon}}\|F_v\|_{H^{1+\epsilon}}\right).$$

Proof. Now we consider the energy defined by (3.28) for the pair $(u, v) = (g, -2k)$ by using (3.30). In view of (2.3), Propositions 2.4 and Proposition 2.5, we have

$$(3.35) \qquad\qquad \mathcal{E}^{(1)} + \mathcal{E}^{(0)} \leq C.$$

Combining this fact with (3.3), we have

$$\mathcal{I}(\epsilon, u, v) \lesssim \left(\|\widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty} + 1 \right) \sqrt{\mathcal{E}^{(1+\epsilon)}(u, v)}.$$

This together with Lemma 3.3, Lemma 3.4 and (3.30) implies

$$\partial_t \left(\sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t) \right) \leq \left(\|k, \widehat{\nabla} Y\|_{L^\infty} + 1 \right) \sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t)$$
$$+ \left(\|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty} + 1 \right) \mathcal{E}^{(1+\epsilon)}(u, v).$$

which combined with (3.35) gives

$$(3.36) \quad \partial_t \left(\sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t) \right) \lesssim \left(\|k, \widehat{\nabla} g, \widehat{\nabla} Y, \widehat{\nabla} n\|_{L^\infty} + 1 \right) \left(\sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t) + 1 \right).$$

By the bootstrap assumption (BA1), we obtain $\mathcal{E}^{(1+\epsilon)}(u, v)(t) \lesssim \mathcal{E}^{(1+\epsilon)}(u, v)(t_0) + 1$ for $t_0 \leq t \leq t_0 + T$ which implies (3.33).

It remains to prove (3.34). From (3.33), (3.3) and Sobolev embedding, it follows that

$$(3.37)$$
$$\|\widehat{\nabla} Y, \widehat{\nabla} n, ne_0(n)\|_{L^\infty} + \|\Lambda^{1/2+\epsilon}(\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n, \widehat{\nabla}(ne_0(n)))\|_{L^2} \lesssim \|\widehat{\nabla}(\widehat{\nabla} g, k)(t_0)\|_{H^\epsilon} + 1.$$

Thus for any pair (u, v) satisfying (3.18) there holds $\mathcal{I}(\epsilon, u, v) \lesssim \|\widehat{\nabla} u, v\|_{H^{1+\epsilon}}$ on Σ_t. We will rely on Lemma 3.3 to treat $\|\mu^\epsilon \widehat{\nabla} F_{U_1^\mu}\|_{l^2_\mu L^2}$ and $\|\mu^\epsilon V_1^\mu\|_{l^2_\mu L^2}$ in (3.30). We then obtain from (3.30) that

$$\partial_t \left(\sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t) \right) \leq \left(\|k, \widehat{\nabla} Y, \widehat{\nabla} g\|_{L^\infty} + 1 \right) \sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t)$$
$$(3.38) \qquad\qquad + (\|\widehat{\nabla}^2 F_u\|_{H^\epsilon} + \|\widehat{\nabla} F_v\|_{H^\epsilon} + \|\widehat{\nabla} u, v\|_{H^1}) \sum_\mu \mu^{2\epsilon} \mathcal{E}^{(1)}_\mu(t).$$

Now we consider lower order energy $\mathcal{E}^{(1)}(u, v)$, by using (3.19) for the pair $(U_1, V_1) = (\widehat{\nabla} u, \widehat{\nabla} v)$ and

$$\|\widehat{\nabla} F_{U^1}, F_{V^1}\|_{L^2_x} \lesssim \|\widehat{\nabla} u, v\|_{H^1} + \|u\|_{L^2_x} + \|\widehat{\nabla}^2 F_u\|_{L^2_x} + \|\widehat{\nabla} F_v\|_{L^2_x}$$

which can be derived by using Sobolev embedding and (3.37). By (2.18), we derive

$$\partial_t \mathcal{E}^{(1)}(u, v)(t) \lesssim \left(\|k, \widehat{\nabla} Y\|_{L^\infty} + 1 \right) \mathcal{E}^{(1)}(u, v)(t)$$
$$(3.39) \qquad\qquad + \|\widehat{\nabla}^2 u, \widehat{\nabla} v\|_{L^2_x} (\|\widehat{\nabla}(\widehat{\nabla} F_u, F_v)\|_{L^2} + \|\widehat{\nabla} u, v\|_{H^1} + \|u\|_{L^2_x}).$$

For $\mathcal{E}^{(0)}(u, v)(t)$, we employ (2.18) again. Combining (3.38), (3.39), (2.18), Lemma 3.3 and the Gronwall inequality gives (3.34). $\qquad\square$

3.1.2. *Geometric wave operator.* For the pair (u, v) satisfying (3.18), we can show that u satisfies the geometric wave equation

$$(3.40) \qquad n^2 \Box_{\mathbf{g}} u = -(nF_v + ne_0(F_u)) + e_0(n)F_u - n^2 \pi_{0a} \nabla^a u + n^2 \mathrm{Tr}k \, e_0(u).$$

Indeed, relative to an orthonormal frame $e_0 := \mathbf{T}, e_j, j = 1, 2, 3$, by straightforward calculation we have

$$\mathbf{g}^{ij} \mathbf{D}_{ij}^2 u = \Delta u + \mathrm{Tr}k \mathbf{D}_{\mathbf{T}} u, \qquad \mathbf{D}_{\mathbf{T}} \mathbf{D}_{\mathbf{T}} u = e_0(e_0(u)) + \pi_{0j} \nabla^j u.$$

Since $\Box_{\mathbf{g}} u = -\mathbf{D}_{\mathbf{T}} \mathbf{D}_{\mathbf{T}} u + \mathbf{g}^{ij} \mathbf{D}_{ij}^2 u$, we obtain

$$(3.41) \qquad n^2 \Box_{\mathbf{g}} u = -n^2 e_0(e_0(u)) + n^2 \hat{\Delta} u - n^2 \pi_{0j} \nabla^j u + n^2 \mathrm{Tr}k \, \mathbf{D}_{\mathbf{T}} u.$$

In view of (3.18) we have

$$ne_0(ne_0(u)) = ne_0(n) \cdot v + n^2 \hat{\Delta} u + nF_v + ne_0(F_u).$$

Combining this with (3.41) and using (3.18) we obtain (3.40) as desired.

Therefore, if ψ is a solution of the geometric wave equation

$$(3.42) \qquad\qquad\qquad\qquad \Box_{\mathbf{g}} \psi = W,$$

we can check by (3.40) that $(u, v) := (\psi, e_0(\psi))$ satisfies the hyperbolic system

$$(3.43) \qquad \begin{array}{ll} e_0(u) = v, & ne_0(v) = n\hat{\Delta} u + F_v, \\ F_u = 0, & F_v = -nW - n\pi_{0j} \nabla^j u + nv \mathrm{Tr}k. \end{array}$$

Lemma 3.6. *Let ψ be a scalar function satisfying the geometric wave equation $\Box_{\mathbf{g}} \psi = 0$. Then for $0 < \epsilon \leq s - 2$ there holds*

$$(3.44) \qquad \|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}(\widehat{\nabla}\psi, e_0(\psi))\|_{H^\epsilon}.$$

Proof. Indeed, in view of (10.26) and $\mathrm{Tr}k = t$ we have

$$\|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+\epsilon}} \|\widehat{\nabla}^2 u\|_{L_x^2} + \|\widehat{\nabla} n\|_{L_x^\infty} \|\widehat{\nabla}^2 u\|_{H^\epsilon} + \|\widehat{\nabla} v\|_{H^\epsilon}.$$

Since $0 < \epsilon \leq s - 2$, we have from (3.37) that $\|\widehat{\nabla} F_v\|_{H^\epsilon} \lesssim \|\widehat{\nabla}^2 u\|_{L_x^2} + \|\widehat{\nabla}(\widehat{\nabla} u, v)\|_{H^\epsilon}$ which gives the estimate. $\qquad\qquad\square$

It is standard to derive for ψ satisfying $\Box_{\mathbf{g}} \psi = 0$ that

$$(3.45) \qquad \|\mathbf{D}\psi(t)\|_{L^2} \lesssim \|\mathbf{D}\psi(t_1)\|_{L^2}, \quad t_0 \leq t_1 \leq t \leq t_0 + T.$$

We now give the following energy estimate.

Proposition 3.7. *Let ψ be a scalar function satisfying the geometric wave equation $\Box_{\mathbf{g}} \psi = 0$. Then for any $0 < \epsilon < 1/2$ and $t_0 \leq t_1 \leq t \leq t_0 + T$ there hold the energy estimates*

$$(3.46) \qquad \mathcal{E}^{(i)}(\psi, e_0(\psi))(t) \lesssim \sum_{0 \leq j \leq i} \mathcal{E}^{(j)}(\psi, e_0(\psi))(t_1), \quad i = 0, 1,$$

and

$$\mathcal{E}^{(1+\epsilon)}(\psi, e_0(\psi))(t) \lesssim \mathcal{E}^{(1+\epsilon)}(\psi, e_0(\psi))(t_1)$$

Proof. Since $(u, v) = (\psi, e_0(\psi))$ satisfies (3.43) with $W = 0$, we can easily derive that

$$\|F_v\|_{L_x^2} \lesssim (\|\widehat{\nabla} n\|_{L_x^\infty} + \|\mathrm{Tr}k\|_{L_x^\infty}) \left(\mathcal{E}^{(0)}(u, v) \right)^{\frac{1}{2}}$$

Recall that $F_u = 0$, an application of (2.18) gives (3.46) with $i = 0$. The case $i = 1$ can be proved by employing (2.18) with $(U^1, V^1) := (\widehat{\nabla}\psi, \widehat{\nabla}(e_0(\psi)))$. Indeed, in view of (3.19) and (3.37), we have

$$\|F_{V^1}, \widehat{\nabla}F_{U^1}\|_{L_x^2} \lesssim \|\widehat{\nabla}F_v\|_{L_x^2} + (\mathcal{E}^{(1)}(u,v))^{\frac{1}{2}},$$

and

$$\|\widehat{\nabla}F_v\|_{L_x^2} \lesssim (\|\widehat{\nabla}\nabla n\|_{L_x^3} + \|\widehat{\nabla}n\|_{L_x^\infty})\|\widehat{\nabla}u\|_{H^1} + (\|\widehat{\nabla}v\|_{L_x^2} + \|\widehat{\nabla}n\|_{L_x^\infty}\|v\|_{L_x^2})\|\mathrm{Tr}k\|_{L_x^\infty}$$

$$\lesssim \left(\mathcal{E}^{(1)}(u,v)(t)\right)^{\frac{1}{2}} + \left(\mathcal{E}^{(0)}(u,v)(t)\right)^{\frac{1}{2}}.$$

By substituting to (2.18), we can complete the proof of (3.46). Using (3.34), (3.44) and the Gronwall inequality, we can complete the proof of Proposition 3.7. $\qquad\square$

Consider either $(u,v) = (g, -2k)$ in view of (1.13), or the solution of $\Box_{\mathbf{g}}\psi = 0$ in view of the reduction in (3.43) with $(u,v) = (\psi, e_0(\psi))$ and $W = 0$ therein. For $0 < \epsilon \le s - 2$ we have obtained Proposition 3.5 and Proposition 3.7, which can be rewritten as

$$(3.47) \qquad \mathcal{E}^{(1+\epsilon)}(u,v)(t) \lesssim \mathcal{E}^{(1+\epsilon)}(u,v)(t_0) + 1, \qquad t_0 \le t \le t_0 + T.$$

By the inequality (3.29), we have

$$\sup_{t_0 \le t \le t_0+T} \mu^{2\epsilon}\mathcal{E}_\mu^{(1)}(t) \le \mu^{2\epsilon}\mathcal{E}_\mu^{(1)}(t_0)$$

$$+ \int_{t_0}^{t_0+T} \left(\|\mu^\epsilon\widehat{\nabla}F_{U_1^\mu}\|_{L^2}\|\mu^\epsilon\widehat{\nabla}U_1^\mu\|_{L^2} + \|\mu^\epsilon V_1^\mu\|_{L^2}\|\mu^\epsilon F_{V_1^\mu}\|_{L^2}\right) dt.$$

Using Lemma 3.3 together with Young's inequality, we have

$$\left(\sum_\mu \sup_{t_0 \le t \le t_0+T} \mu^{2\epsilon}\mathcal{E}_\mu^{(1)}(t)\right)^{\frac{1}{2}} \lesssim \left(\mathcal{E}^{(1+\epsilon)}(t_0)\right)^{\frac{1}{2}}$$

$$+ \int_{t_0}^{t_0+T} \left(\|\mu^\epsilon P_\mu(\widehat{\nabla}^2 F_u)\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon\widehat{\nabla}P_\mu F_v\|_{l_\mu^2 L_x^2}\right).$$

We then conclude that

Proposition 3.8. *For $0 < \epsilon \le s - 2$ there holds*

$$\left(\sum_\mu \sup_{t_0 \le t \le t_0+T} \mu^{2\epsilon}\mathcal{E}_\mu^{(1)}(t)\right)^{\frac{1}{2}} \lesssim \|\widehat{\nabla}u, v\|_{H^{1+\epsilon}(t_0)}$$

$$+ \int_{t_0}^{t_0+T} \left(\|\mu^\epsilon P_\mu(\widehat{\nabla}^2 F_u)\|_{l_\mu^2 L_x^2} + \|\mu^\epsilon\widehat{\nabla}P_\mu F_v\|_{l_\mu^2 L_x^2}\right).$$

3.2. **Flux.** In view of (2.14) and (2.15), we can see that $(u,v) := (\widehat{\nabla}g, -2\widehat{\nabla}k)$ satisfies the hyperbolic system (3.18) with

$$(3.48) \quad \begin{cases} F_u = g\widehat{\nabla}^2 Y + \widehat{\nabla}Y \cdot \widehat{\nabla}g + Y \cdot \hat{R} \cdot g + \widehat{\nabla}n \cdot k, \\ F_v = \widehat{\nabla}(2\nabla^2 n + n * k * k + k * \widehat{\nabla}Y) + \widehat{\nabla}n\hat{\Delta}g + \widehat{\nabla}Y \cdot \widehat{\nabla}k + Y \cdot \hat{R} \cdot k \\ \qquad + n(g \cdot \widehat{\nabla}g \cdot \widehat{\nabla}^2 g + g \cdot \hat{R} \cdot \widehat{\nabla}g + g \cdot \widehat{\nabla}\hat{R} \cdot g). \end{cases}$$

By straightforward calculation we have

$$\begin{cases} \widehat{\nabla} F_u = g \cdot \widehat{\nabla}^3 Y + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}), \\ F_v = \widehat{\nabla}^3 n + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}). \end{cases}$$

Next we give the first order hyperbolic system for the pair (k, E). Recall that (see [2, (3.11a)])

$$n^{-1}(\partial_t - \mathcal{L}_Y)E_{ij} = \mathrm{curl}\, H_{ij} - n^{-1}(\nabla n \wedge H)_{ij} - \frac{5}{2}(E \times k)_{ij}$$

(3.49)
$$- \frac{2}{3}(E * k)g_{ij} - \frac{1}{2}\mathrm{Tr}k E_{ij},$$

where, for any symmetric 2-tensor F,

$$(3.50) \qquad \mathrm{curl}\, F_{ab} = \frac{1}{2}(\epsilon_a{}^{cd}\nabla_d F_{cb} + \epsilon_b{}^{cd}\nabla_d F_{ca}).$$

with $\epsilon_a{}^{cd}$ denoting the components of the volume form of (Σ_t, g). When $\mathrm{div}\, F = 0$ and $\mathrm{Tr} F = t$, symbolically we can obtain the identity

$$(3.51) \qquad \mathrm{curl}\,\mathrm{curl}\, F = -\hat{\Delta} F + \mathrm{Ric} * F + \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot g \cdot F$$

In view of $\mathrm{curl}\, k = -H$, we can use (3.51) with $F = k$ to treat the term $\mathrm{curl}\, H$ in (3.49). Consequently we obtain

$$(3.52) \quad n^{-1}(\partial_t - \mathcal{L}_Y)E_{ij} = \hat{\Delta} k + \mathrm{Ric} * k + \widehat{\nabla} g \cdot \widehat{\nabla} g \cdot k \cdot g + \nabla n * H + k * k * k.$$

By coupling (3.52) with (1.4), we can see that the pair $(u, v) := (k, E)$ satisfies the first order hyperbolic system (3.18) with

$$(3.53) \qquad \begin{cases} F_u = \nabla_{ij}^2 n + nk * k + k \cdot \widehat{\nabla} Y, \\ F_v = nR * k + n\widehat{\nabla} g \cdot \widehat{\nabla} g \cdot g \cdot k + \nabla n * H + nk * k * k + E \cdot \widehat{\nabla} Y. \end{cases}$$

Using the Gauss equation $E = Ric + k \star k$ and (2.12) to treat E, we have

$$\begin{cases} \widehat{\nabla} F_u = \widehat{\nabla}^3 n + \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}), \\ F_v = \mathrm{err}_1 + \mathrm{err}_2 + \mathrm{err}(\hat{R}). \end{cases}$$

In view of Proposition 3.1, we obtain

Proposition 3.9 (Remainder estimates). *Let (F_u, F_v) be defined by either (3.48) or (3.53). Then for any $0 < \epsilon < 1/2$ there hold*

$$(3.54) \qquad \begin{cases} \|\widehat{\nabla} F_u\|_{H^\epsilon} + \|F_v\|_{H^\epsilon} \lesssim (\|\widehat{\nabla} g, k\|_{L^\infty} + 1)\|\widehat{\nabla} g, k\|_{H^{1+\epsilon}}, \\ \|F_u\|_{H^{1/2+\epsilon}} \lesssim \|\Lambda^\epsilon \widehat{\nabla}(k, \widehat{\nabla} g)\|_{L^2} \end{cases}$$

and

$$(3.55) \qquad \|\widehat{\nabla} F_u\|_{L^2} + \|F_v\|_{L^2} \lesssim (\|\widehat{\nabla} g, k\|_{L^\infty} + 1)\|\widehat{\nabla} g, k\|_{H^1}.$$

We now fix a point p in $\Sigma \times I$ and use Γ^+ to denote the time axis passing through p which is defined to be the integral curve of the forward unit normal $\mathbf{T}$ with $\Gamma^+(t_p) = p$. We use Γ_t to denote the intersection point of Γ with Σ_t. Let u be the outgoing solution of the eikonal equation $\mathbf{g}^{\alpha\beta}\partial_\alpha u \partial_\beta u = 0$ satisfying the initial condition $u(\Gamma_t) = t - t_p$ on the time axis. We will call this u an optical function. We denote by C_u the level sets of u which are the outgoing null cones with vertex

on the time axis. Let $S_{t,u} = C_u \cap \Sigma_t$ and let $\{e_1, e_2\}$ be an orthonormal frame on $S_{t,u}$. Let N be the exterior unit normal, along Σ_t, to the surface $S_{t,u}$. We define

$$(3.56) \qquad \mathbf{b}^{-1} := \mathbf{T}(u), \quad L := \mathbf{T} + N, \quad \underline{L} := \mathbf{T} - N = 2\mathbf{T} - L.$$

We will call $L, \underline{L}$ the canonical null pair. Then $\{e_1, e_2, e_3 := \underline{L}, e_4 := L\}$ also forms a null frame.

For $0 < u \le t_0 + T - t_p$ and $t_p \le t_1 \le t_2 \le t_0 + T$, we introduce the null hypersurface $\mathcal{H} := \cup_{t_1 \le t \le t_2} S_{t,u}$. We will use D^+ to denote the region enclosed by $\mathcal{H}$, Σ_{t_1} and Σ_{t_2}. For any scalar function ψ we introduce the flux

$$\mathcal{F}[\psi] = \int_{\mathcal{H}} \left(|L\psi|^2 + \gamma^{AB} \slashed{\nabla}_A \psi \slashed{\nabla}_B \psi \right),$$

where γ is the induced metric on $S_{t,u}$ and $\slashed{\nabla}$ is the corresponding covariant differentiation. For any scalar functions ϕ and ψ we introduce the energy-momentum tensor

$$(3.57) \qquad Q[\phi, \psi]_{\mu\nu} = \frac{1}{2}(\mathbf{D}_\mu \phi \mathbf{D}_\nu \psi + \mathbf{D}_\nu \phi \mathbf{D}_\mu \psi) - \frac{1}{2}\mathbf{g}_{\mu\nu}(\mathbf{g}^{\alpha\beta}\mathbf{D}_\alpha \phi \mathbf{D}_\beta \psi).$$

Let $Q[\psi] := Q[\psi, \psi]$ and define the energy

$$Q(\psi)(t) := \int_{\Sigma_t} Q[\psi](\mathbf{T}, \mathbf{T}) d\mu_g.$$

It is straightforward to check $\mathcal{F}[\psi] = 2\int_{C_u} Q[\psi](\mathbf{T}, L)$.

For any Σ-tangent tensor field F_μ, we set

$$(3.58) \qquad \nabla_L F_i = e_i^\mu L^\nu \mathbf{D}_\nu F_\mu, \qquad \slashed{\nabla}_A F_i = e_i^\mu \nabla_A F_\mu$$

and introduce the norms

$$|\nabla_L F|_g^2 := g^{ij}\mathbf{D}_L F_i \mathbf{D}_L F_j, \qquad |\slashed{\nabla} F|_g^2 := \gamma^{AB} g^{ij} \slashed{\nabla}_A F_i \slashed{\nabla}_B F_j.$$

We will drop the subscript g in the definition of norms whenever there occurs no confusion.

Following the same proof in [25, Section 5], we can obtain the following result on tensorial k-flux.

Proposition 3.10. *Under the bootstrap assumption (BA1), for the tensorial k-flux there holds on the null cone C_u the estimate*

$$\int_{C_u} \left(|\slashed{\nabla} k|_g^2 + |\nabla_L k|_g^2 \right) \le C.$$

The following estimate is the main result of this subsection.

Proposition 3.11. *Let the bootstrap assumptions (BA1) and (BA2) hold. Let f be the scalar components of $\widehat{\nabla} g$ and k. Then for $0 < \epsilon \le s - 2$ there holds*

$$\mathcal{F}^{\frac{1}{2}}[f] + \|\mu^\epsilon \mathcal{F}^{\frac{1}{2}}[P_\mu f]\|_{l_\mu^2} \le C.$$

In the following we will give the proof of Proposition 3.11. By the standard energy estimate we have

$$\mathcal{F}[\psi] \le |Q(\psi)(t_2) - Q(\psi)(t_1)|$$

$$(3.59) \qquad + \int_{t_1}^{t_2} Q^{\frac{1}{2}}(\psi)(t')\|\Box_\mathbf{g}\psi\|_{L_x^2} + \int_{t_1}^{t_2} C(\|\pi\|_{L_x^\infty} + 1)Q(\psi)(t')dt'.$$

Recall that $(\widehat{\nabla} g, -2\widehat{\nabla} k)$ and (k, E) satisfy the first order hyperbolic system (3.18). Thus, for $\psi = \widehat{\nabla} g$ or k, the expression of $\Box_{\mathbf{g}} \psi$ derived from (3.40) contains time derivatives of the shift vector field Y since F_u contains the term $\psi \cdot \widehat{\nabla} Y$ and other terms involving Y. The lack of control on $\mathbf{D_T} Y$ makes it impractical to apply (3.59) directly to $\psi = \widehat{\nabla} g, k$.

To get around the difficulty, we consider the following modified energy momentum tensor

$$\tilde{P}_\mu = -n^{-1} F_u \mathbf{D}_\mu u + Q_{\mu\nu} \mathbf{T}^\nu + \frac{1}{2}(n^{-1} F_u)^2 n \mathbf{D}_\mu(t).$$

When (u, v) satisfies the system (3.18), u must satisfy (3.40). We claim that

(3.60)
$$\mathbf{D}^\mu \tilde{P}_\mu = \left(-\pi_{0a} \nabla^a u + \mathrm{Tr} k e_0(u) - n^{-1} F_v\right) v - \mathbf{D}^i(n^{-1} F_u) \mathbf{D}_i u + Q_{\mu\nu}{}^{(\mathbf{T})} \pi^{\mu\nu}.$$

Indeed, since (3.18) implies $-n^{-1} F_u + \mathbf{D_T} u = v$, we have from the definition of $\tilde{P}_\mu$ that

$$\begin{aligned}
\mathbf{D}^\mu \tilde{P}_\mu &= -\mathbf{D}^\mu(n^{-1} F_u) \mathbf{D}_\mu u - n^{-1} F_u \Box_{\mathbf{g}} u + \mathbf{D}^\mu Q_{\mu\nu} \mathbf{T}^\nu + Q_{\mu\nu}{}^{(\mathbf{T})} \pi^{\mu\nu} \\
&\quad + (n^{-1} F_u) \mathbf{D}^\mu(n^{-1} F_u) n \mathbf{D}_\mu t \\
&= \mathbf{D}_0(n^{-1} F_u) \mathbf{D}_0 u - \mathbf{D}^i(n^{-1} F_u) \mathbf{D}_i u - n^{-1} F_u \Box_{\mathbf{g}} u + \Box_{\mathbf{g}} u \mathbf{D_T} u + Q_{\mu\nu}{}^{(\mathbf{T})} \pi^{\mu\nu} \\
&\quad + (n^{-1} F_u) \mathbf{D}^\mu(n^{-1} F_u) n \mathbf{D}_\mu t \\
&= (-n^{-1} F_u + \mathbf{D_T} u)(\Box_{\mathbf{g}} u + \mathbf{D}_0(n^{-1} F_u)) - \mathbf{D}^i(n^{-1} F_u) \mathbf{D}_i u + Q_{\mu\nu}{}^{(\mathbf{T})} \pi^{\mu\nu} \\
&= \left(\Box_{\mathbf{g}} u + \mathbf{D}_0(n^{-1} F_u)\right) v - \mathbf{D}^i(n^{-1} F_u) \mathbf{D}_i u + Q_{\mu\nu}{}^{(\mathbf{T})} \pi^{\mu\nu}.
\end{aligned}$$

In view of (3.40), we obtain (3.60).

By the divergence theorem we have for $\tilde{P}_\mu$ that

(3.61)
$$\int_{\mathcal{H}} L^\mu \tilde{P}_\mu = \int_{\Sigma_{t_2} \cap \mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu - \int_{\Sigma_{t_1} \cap \mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu - \int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu.$$

Note that

$$\begin{aligned}
\int_{\mathcal{H}} L^\mu \tilde{P}_\mu &= \int_{\mathcal{H}} \frac{1}{2}(|\mathbf{D}_L u|^2 + |\slashed{\nabla} u|^2 + (n^{-1} F_u)^2) - n^{-1} F_u \mathbf{D}_L u \\
&= \int_{\mathcal{H}} \frac{1}{2}(\frac{1}{2}|\mathbf{D}_L u|^2 + |\slashed{\nabla} u|^2 - (n^{-1} F_u)^2) + (\frac{1}{2}\mathbf{D}_L u - n^{-1} F_u)^2.
\end{aligned}$$

Thus

$$\int_{\mathcal{H}} \frac{1}{2}(\frac{1}{2}|\mathbf{D}_L u|^2 + |\slashed{\nabla} u|^2) \leq \int_{\mathcal{H}} L^\mu \tilde{P}_\mu + \frac{1}{2}(n^{-1} F_u)^2.$$

Also using (3.61), we obtain
(3.62)
$$\mathcal{F}[u](\mathcal{H}) \lesssim \int_{\mathcal{H}} \frac{1}{2}(n^{-1} F_u)^2 + \left|\int_{\Sigma_{t_2} \cap \mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu - \int_{\Sigma_{t_1} \cap \mathcal{D}^+} \tilde{P}_\mu \mathbf{T}^\mu\right| + \left|\int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu\right|.$$

Now consider the terms on the right of (3.62). By trace inequality,

(3.63)
$$\int_{\mathcal{H}} (n^{-1} F_u)^2 \lesssim \int_{t_1}^{t_2} \|F_u\|_{H^1} \|F_u\|_{L_x^2}.$$

By definition of $\tilde{P}_\mu$ and $C^{-1} < n < C$, for any $0 < t' \le T$,

$$(3.64) \qquad \left| \int_{\Sigma_{t'} \cap \mathcal{D}^+} \mathbf{T}^\mu \tilde{P}_\mu \right| \lesssim \|\mathbf{D}u\|_{L_x^2}^2 + \|F_u\|_{L_x^2}^2.$$

For the third term, by (3.60), there holds

$$\left| \int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu \right| \le \int_{t_0}^t \|^{(\mathbf{T})}\pi\|_{L_x^\infty} \|\mathbf{D}u\|_{L_x^2} (\|v\|_{L_x^2} + \|\mathbf{D}u\|_{L_x^2})$$

$$(3.65) \qquad\qquad + \int_{\mathcal{D}^+} \left| n^{-1} v F_v + \mathbf{D}^i (n^{-1} F_u) \mathbf{D}_i u \right|.$$

Proof of Proposition 3.11. We first apply (3.65) to the modified energy momentum tensor $\tilde{P}_\mu$ corresponding to $(u, v) = (\widehat{\nabla}g, -2\widehat{\nabla}k)$ or (k, E). In view of (3.55), we obtain

$$\left| \int_{\mathcal{D}^+} \mathbf{D}^\mu \tilde{P}_\mu \right| \lesssim \|\widehat{\nabla}g, k\|_{L_t^1 L_x^\infty} \|\widehat{\nabla}g, k\|_{L_t^\infty H^1}^2.$$

By Proposition 2.5 and (3.55) we have

$$\left| \int_{\Sigma_{t'} \cap \mathcal{D}^+} \mathbf{T}^\mu \tilde{P}_\mu \right| \le C \qquad \text{and} \qquad \int_{\mathcal{H}} (n^{-1} F_u)^2 \le C.$$

Therefore we can conclude that $\mathcal{F}[\widehat{\nabla}g, k] \le C$.

Recall again that $(u, v) = (\widehat{\nabla}g, -2\widehat{\nabla}k)$ and (k, E) satisfy (3.18) with (F_u, F_v) given by (3.48) and (3.53) respectively. Then the pair $(U^\mu, V^\mu) = (P_\mu u, P_\mu v)$ satisfies (3.18) with (F_{U^μ}, F_{V^μ}) given by in (3.20). In view of (3.22)–(3.24), (3.33), and Proposition 3.9, we have

$$(3.66) \qquad \|\mu^{\frac{1}{2}+\epsilon} F_{U^\mu}\|_{l_\mu^2 L^2} + \|\mu^{-\frac{1}{2}+\epsilon} \widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L_x^2} \lesssim \|\widehat{\nabla}(\widehat{\nabla}g, k)\|_{H^\epsilon} \lesssim 1,$$

$$(3.67) \qquad \|\mu^\epsilon \widehat{\nabla} F_{U^\mu}\|_{l_\mu^2 L^2} \lesssim \left(\|k, \widehat{\nabla}g\|_{L_x^\infty} + 1 \right) \|\widehat{\nabla}(\widehat{\nabla}g, k)\|_{H^\epsilon} \lesssim \|k, \widehat{\nabla}g\|_{L_x^\infty} + 1,$$

$$(3.68) \qquad \|\mu^\epsilon F_{V^\mu}\|_{l_\mu^2 L^2} \lesssim \left(\|k, \widehat{\nabla}g\|_{L_x^\infty} + 1 \right) \|\widehat{\nabla}(\widehat{\nabla}g, k)\|_{H^\epsilon} \lesssim \|k, \widehat{\nabla}g\|_{L_x^\infty} + 1.$$

Define

$$(3.69) \qquad\qquad \mathcal{B}_\mu^{(1)} = \mu^{2\epsilon} \|F_{U^\mu}\|_{H^1} \|F_{U^\mu}\|_{L_x^2}.$$

Similar to (3.63), we have $\int_{\mathcal{H}} \mu^{2\epsilon} (n^{-1} F_{U^\mu})^2 \lesssim \int_{t_1}^{t_2} \mathcal{B}_\mu^{(1)}$. Using (3.66), it yields

$$(3.70) \qquad \sum_{\mu>1} \mathcal{B}_\mu^{(1)} \lesssim \|\mu^{\epsilon-\frac{1}{2}} F_{U^\mu}\|_{l_\mu^2 H^1} \|\mu^{\epsilon+\frac{1}{2}} F_{U^\mu}\|_{l_\mu^2 L_x^2} \le C$$

In view of (3.65), it follows that

$$\sum_{\mu>1} \left| \int_{\mathcal{D}^+} \mu^{2\epsilon} \mathbf{D}^\alpha \tilde{P}_\alpha \right|$$

$$\lesssim \int_{t_1}^{t_2} \left(\|^{(\mathbf{T})}\pi\|_{L_x^\infty} \sum_{\mu>1} \|\mu^\epsilon (P_\mu \partial u, P_\mu v)\|_{L_x^2}^2 + \sum_{\mu>1} \|\mu^\epsilon F_{V^\mu}\|_{L_x^2} \|\mu^\epsilon P_\mu v\|_{L_x^2} \right.$$

$$\left. + \sum_{\mu>1} \|\mu^\epsilon \widehat{\nabla} F_{U^\mu}\|_{L_x^2} \|\mu^\epsilon \mathbf{D}_i P_\mu u\|_{L_x^2} \right).$$

Using (3.66)-(3.68), we obtain

$$(3.71) \qquad \sum_{\mu>1}\left|\iint_{\mathcal{D}^+}\mu^{2\epsilon}\mathbf{D}^\alpha\tilde{P}_\alpha\right| \lesssim \|^{(\mathbf{T})}\pi, \widehat{\nabla}g, \widehat{\nabla}Y\|_{L^1_t L^\infty_x}\|\widehat{\nabla}(\widehat{\nabla}g, k)\|^2_{H^\epsilon} \le C.$$

In view of (3.64), (3.66) and (3.33),

$$(3.72) \qquad \left|\int_{\Sigma_{t'}\cap\mathcal{D}^+}\mu^{2\epsilon}\mathbf{T}^\alpha\tilde{P}_\alpha\right| \lesssim \sum_{\mu>1}\mu^{2\epsilon}(\|\mathbf{D}P_\mu u\|^2_{L^2_x} + \|F_{U^\mu}\|^2_{L^2_x}) \le C$$

We conclude in view of (3.70), (3.71) and (3.72) that

$$\sum_{\mu>1}\mu^{2\epsilon}(\mathcal{F}[P_\mu\widehat{\nabla}g] + \mathcal{F}[P_\mu k]) \le C.$$

The proof is thus complete. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

4. Strichartz estimate and Main estimates

In this section we will show that (BA1) and (BA2) can be improved. For ease of exposition, we shift the origin of time coordinate to t_0 and consider $[0, T] \times \Sigma$. Now we make the following additional bootstrap assumption: there is a constant B_0 such that

$$(\text{BA3}) \qquad \|\mu^\delta P_\mu\widehat{\nabla}g\|_{l^2_\mu L^2_{[0,T]}L^\infty_x} + \|\mu^\delta P_\mu k\|_{l^2_\mu L^2_{[0,T]}L^\infty_x} \le B_0,$$

where $0 < \delta < s - 2$ is a sufficiently small number. As an immediate consequence of (BA2), (BA3), and (10.27), there holds

$$\|\mu^\delta P_\mu(g\widehat{\nabla}g)\|_{l^2_\mu L^2_{[0,T]}L^\infty_x} \lesssim B_0.$$

This estimate will always be used together with (BA3). Our goal is to show that the estimates in (BA1), (BA2) and (BA3) can be improved by shrinking the time interval if necessary. We will achieve this by establishing the following main estimates.

Theorem 4.1 (Main Estimates). *Let (BA2) and (BA3) hold for some sufficiently small number $0 < \delta < s - 2$. Then for any number $q > 2$ that is sufficiently close to 2 there holds*

$$\|\widehat{\nabla}g, k\|_{L^2_{[0,T]}L^\infty_x} + \|\mu^\delta P_\mu(\widehat{\nabla}g, k)\|_{l^2_\mu L^2_{[0,T]}L^\infty_x} \lesssim T^{\frac{1}{2}-\frac{1}{q}}.$$

If ϕ is a function satisfying $\Box_{\mathbf{g}}\phi = 0$, then there holds

$$\|\partial\phi\|^2_{L^2_I L^\infty_x} + \|\mu^\delta P_\mu\partial\phi\|^2_{l^2_\mu L^2_I L^\infty_x} \le CT^{1-\frac{2}{q}}\|\widehat{\nabla}\phi, e_0\phi\|^2_{H^{1+\epsilon}(0)}.$$

4.1. Decay estimate $\Rightarrow$ Strichartz estimates. Let us rescale the coordinate $(t, x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$ for some positive constant λ. We first prove Strichartz estimate by assuming the following decay estimate.

Theorem 4.2 (Decay estimate). *Let $0 < \epsilon_0 < s - 2$ be a given number. There exists a large number Λ such that for any $\lambda \ge \Lambda$ and any solution ψ of the equation*

$$(4.1) \qquad\qquad\qquad\qquad \Box_{\mathbf{g}}\psi = 0$$

on the time interval $I_* = [0, t_*]$ with $t_* \le \lambda^{1-8\epsilon_0}T$ there is a function $d(t)$ satisfying

$$(4.2) \qquad\qquad \|d\|_{L^{\frac{q}{2}}} \lesssim 1, \text{ for } q > 2 \text{ sufficiently close to 2}$$

such that for any $0 \le t \le t_*$ there holds

(4.3) $$\|Pe_0\psi(t)\|_{L_x^\infty} \le \left(\frac{1}{(1+t)^{\frac{2}{q}}} + d(t)\right)(\|\psi[0]\|_{H^1} + \|\psi(0)\|_{L^2}),$$

where $\psi[0] := (\psi(0), \partial_t\psi(0))$ and $\|\psi[0]\|_{H^1} := \|\widehat{\nabla}\psi(0)\|_{L^2} + \|\partial_t\psi(0)\|_{L^2}$.

Using Theorem 4.2, we can prove the following result.

Theorem 4.3 (Dyadic Strichartz estiamte)**.** *There is a large universal constant C_0 such that if on the time interval $I_* := [0, t_*]$ there holds*

(4.4) $$C_0\|\pi, \widehat{\nabla}g, \widehat{\nabla}Y\|_{L_{I_*}^1 L_x^\infty} \le 1,$$

then for any ϕ satisfying the wave equation $\Box_{\mathbf{g}}\phi = 0$ and $q > 2$ sufficiently close to 2, there holds

(4.5) $$\|P\partial\phi\|_{L_{I_*}^q L_x^\infty} \lesssim \|\phi[0]\|_{H^1},$$

where P denote the Littlewood-Paley projection on the frequency domain $\{1/2 \le |\xi| \le 2\}$.

We will prove Theorem 4.3 by adapting a $\mathcal{T}\mathcal{T}^*$ argument from [11, 12]. Applying the $\mathcal{T}\mathcal{T}^*$ argument therein directly to our setting requires the control over $\partial\mathbf{g}$ including the undesired quantity $\mathbf{D_T}Y$. To get around this difficulty, we give a careful refinement.

Definition 4.4. *Let $\omega := (\omega_0, \omega_1) \in H^1(\Sigma) \times L^2(\Sigma)$. We denote by $\phi(t; s, \omega)$ the unique solution of the homogeneous geometric wave equation $\Box_{\mathbf{g}}\phi = 0$ satisfying the initial condition $\phi(s; s, \omega) = \omega_0$ and $\mathbf{D}_0\phi(s; s, \omega) = \omega_1$. We set $\Phi(t; s, \omega) := (\phi(t; s, \omega), \mathbf{D}_0\phi(t; s, \omega))$. By uniqueness we have $\Phi(t; s, \Phi(s; t_0, \omega)) = \Phi(t; t_0, \omega)$.*

We first show that

(4.6) $$\|P(e_0\phi)\|_{L_{I_*}^q L_x^\infty} \lesssim \|\phi[0]\|_{H^1}.$$

To this end, we let $\mathcal{H} := H^1(\Sigma) \times L^2(\Sigma)$ endowed with the inner product

$$\langle \omega, v \rangle = \int_\Sigma \left(\omega_1 \cdot v_1 + \delta^{ij}\mathbf{D}_i\omega_0 \cdot \mathbf{D}_j v_0\right)$$

relative to the orthornormal frame $\{e_0 = \mathbf{T}, e_i = 1, 2, 3\}$. Let $I = [t', t_*]$ with $0 \le t' \le t_*$ and let $X = L_I^q L_x^\infty$. Then the dual of X is $X' = L_I^{q'} L_x^1$, where $1/q' + 1/q = 1$. Let $\mathcal{T}(t') : \mathcal{H} \to X$ be the linear operator defined by

(4.7) $$\mathcal{T}(t')\omega := P\mathbf{D}_0\phi(t; t', \omega),$$

where $\phi := \phi(t; t', \omega)$ is the unique solution of $\Box_{\mathbf{g}}\phi = 0$ satisfying $\phi(t') = \omega_0$ and $D_0\phi(t') = \omega_1$ with $\omega := (\omega_0, \omega_1)$.

By using the Bernstein inequality for LP projections and the energy estimate it is easy to see that $\mathcal{T}(t') : \mathcal{H} \to X$ is a bounded linear operator, i.e.

(4.8) $$\|\mathcal{T}(t')\omega\|_X = \|P(e_0\phi)\|_{L_I^q L_x^\infty} \le C(\lambda)\|\mathbf{D}\phi(t')\|_{L_x^2}$$

for some constant $C(\lambda)$ possibly depending on λ. Let $M(t') := \|\mathcal{T}(t')\|_{\mathcal{H} \to X}$. Then $M(t') < \infty$, and for the adjoint $\mathcal{T}(t')^* : X' \to \mathcal{H}$ we have

$$\|\mathcal{T}(t')^*\|_{X' \to \mathcal{H}} = M(t'), \quad \|\mathcal{T}(t')\mathcal{T}(t')^*\|_{X' \to X} = M(t')^2.$$

Note that $M(\cdot)$ is a continuous function on I_*, whose maximum, denoted by M, is achieved at certain $t_0 \in [0, t_*)$. Our goal is to show that M is independent of λ. Our strategy is to show that

$$(4.9) \qquad M^2 \leq C + \frac{1}{2}M^2$$

for some universal positive constant C independent of λ. Let us set $I_0 = [t_0, t_*]$ and consider $X = L_{I_0}^q L_x^\infty$, $X' = L_{I_0}^{q'} L_x^1$, and the operators $\mathcal{T}(t_0)$ and $\mathcal{T}(t_0)^*$. For convenience, we drop the t_0 in the notation for operators.

We first calculate $\mathcal{T}^* : X' \to \mathcal{H}$. For any $f \in X'$ and $\omega \in \mathcal{H}$ we have

$$\langle \mathcal{T}^* f, \omega \rangle_\mathcal{H} = \langle f, \mathcal{T}\omega \rangle_{X', X} = \int_{I_0 \times \Sigma} f P \mathbf{D}_0 \phi = \int_{I_0 \times \Sigma} (Pf) \mathbf{D}_0 \phi(t, t_0, \omega).$$

We introduce the function ψ to be the solution of the initial value problem

$$(4.10) \qquad \begin{cases} \Box_\mathbf{g} \psi = -Pf, & \text{in } [t_0, t_*) \times \Sigma, \\ \psi(t_*) = \partial_t \psi(t_*) = 0. \end{cases}$$

Recall the energy momentum tensor $Q[\phi, \psi]$ introduced in (3.57). For any vector field Z we set $P_\mu := Q[\phi, \psi]_{\mu\nu} Z^\nu$. In view of $\Box_\mathbf{g} \phi = 0$, it is easy to check that

$$\mathbf{D}^\beta P_\beta = \frac{1}{2} \left((Z\phi) \Box_\mathbf{g} \psi + Q[\phi, \psi]_{\alpha\beta} {}^{(Z)}\pi^{\alpha\beta} \right).$$

By the divergence theorem we have

$$(4.11) \qquad \int_{\Sigma_{t_*}} Q[\phi, \psi]_{\mu\nu} Z^\mu \mathbf{T}^\nu - \int_{\Sigma_{t_0}} Q[\phi, \psi]_{\mu\nu} Z^\mu \mathbf{T}^\nu = \int_{I_0 \times \Sigma} \mathbf{D}^\beta P_\beta$$

which together with the initial conditions in (4.10) implies that

$$(4.12) \qquad \int_{I_0 \times \Sigma} -(Z\phi)\Box_\mathbf{g}\psi = 2 \int_{\Sigma_{t_0}} \mathbf{T}^\alpha P_\alpha + \int_{I_0 \times \Sigma} Q[\phi, \psi]_{\alpha\beta} {}^{(Z)}\pi^{\alpha\beta}.$$

Now we take $Z = \mathbf{T}$. Then it follows from (4.12) that

$$(4.13) \quad \int_{I_0 \times \Sigma} -\mathbf{D}_0\phi\Box_\mathbf{g}\psi = \int_{\Sigma_{t_0}} \mathbf{D}_0\phi\mathbf{D}_0\psi + \delta^{ij}\mathbf{D}_i\phi\mathbf{D}_j\psi + \int_{I_0 \times \Sigma} Q[\phi, \psi]_{\alpha\beta} {}^{(\mathbf{T})}\pi^{\alpha\beta}.$$

Therefore

$$(4.14) \qquad \langle \mathcal{T}^* f, \omega \rangle_\mathcal{H} = \langle \psi[t_0], \omega \rangle_\mathcal{H} + l(\omega),$$

where $l(\cdot)$ is a linear functional on $\mathcal{H}$ defined by

$$l(\omega) := \int_{I_0 \times \Sigma} Q[\phi, \psi]_{\alpha\beta} {}^{(\mathbf{T})}\pi^{\alpha\beta}.$$

We claim that $l(\cdot)$ is a bounded linear functional on $\mathcal{H}$. To see this, let $\omega \in \mathcal{H}$ with $\|\omega\|_\mathcal{H} \leq 1$. Then by the energy estimate we have $\|\mathbf{D}\phi\|_{L_t^\infty L_x^2} \leq \|\omega\|_\mathcal{H} \lesssim 1$. Thus

$$|l(\omega)| \leq \|\pi\|_{L_t^1 L_x^\infty} \|\mathbf{D}\phi\|_{L_{I_0}^\infty L_x^2} \|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2} \lesssim \|\pi\|_{L_t^1 L_x^\infty} \|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2}.$$

Hence, by the Riesz representation theorem we have $l(\omega) = \langle R(f), \omega \rangle_\mathcal{H}$ for some $R(f) \in \mathcal{H}$ and there is a universal constant C_1 such that

$$\|R(f)\|_\mathcal{H} \leq C_1 \|\pi\|_{L_t^1 L_x^\infty} \|\mathbf{D}\psi\|_{L_{I_0}^\infty L_x^2}.$$

Moreover, we have from (4.14) that $\mathcal{T}^* f = \psi[t_0] + R(f)$ and hence

$$(4.15) \qquad \mathcal{T}\mathcal{T}^* f = \mathcal{T}\psi[t_0] + \mathcal{T}R(f).$$

We claim that there is a universal constant C_2 such that

(4.16) $$\|\mathbf{D}\psi\|_{L^\infty_{I_0}L^2_x} \le C_2 M \|f\|_{L^{q'}_{I_0}L^1_x}.$$

Assuming this claim for a moment. Then it follows from the definition of M that

$$\|\mathcal{T}R(f)\|_{L^q_{I_0}L^\infty_x} \le C_1 C_2 M^2 \|\pi\|_{L^1_t L^\infty_x} \|f\|_{L^{q'}_{I_0}L^1_x}.$$

Thus, if (4.4) holds with $C_0 \ge 2C_1 C_2$, then

(4.17) $$\|\mathcal{T}R(f)\|_{L^q_{I_0}L^\infty_x} \le \frac{1}{2}M^2\|f\|_{L^{q'}_{I_0}L^1_x}.$$

Next we will estimate $\|\mathcal{T}\psi[t_0]\|_{L^q_{I_0}L^\infty_x}$. We set $F := (0, -n^2 Pf)$. By the Duhamel principle we have

$$\psi[t] = \int_{t_*}^t \Phi(t; s, F(s))ds.$$

Then $\psi[t_0] = -\int_{t_0}^{t_*} \Phi(t_0; s, F(s))ds$ and thus

$$\mathcal{T}\psi[t_0] = P\left[e_0\phi\left(t; t_0, -\int_{t_0}^{t_*}\Phi(t_0; s, F(s))\right)\right] = -P\left[e_0\left(\int_{t_0}^{t_*}\Phi(t, s, F(s))ds\right)\right]$$

$$= -\int_{t_0}^{t_*} P\left[e_0\Phi(t, s, F(s))\right]ds.$$

It follows from Theorem 4.2 that

$$\|P\left[e_0\Phi(t, s, F(s))\right]\|_{L^\infty_x} \le C\left((1 + |t-s|)^{-\frac{2}{q}} + d(|t-s|)\right)\|\widehat{\nabla}(n^2 Pf)(s)\|_{L^2_x}.$$

Observe that on Σ there holds

$$\|\widehat{\nabla}(n^2 Pf)\|_{L^2_x} \lesssim \|\widehat{\nabla}(n^2)Pf\|_{L^2_x} + \|n^2\widehat{\nabla}Pf\|_{L^2_x} \lesssim (\|\widehat{\nabla}n\|_{L^\infty_x} + 1)\|f\|_{L^1_x} \lesssim \|f\|_{L^1_x}.$$

Thus, in view of the Hardy-Littlewood-Sobolev inequality, (4.2) and Hausdorff Young inequality we obtain

(4.18) $$\|\mathcal{T}\psi[t_0]\|_{L^q_{I_0}L^\infty_x} \lesssim \|f\|_{L^{q'}_{I_0}L^1_x} + \left\|\int_{t_0}^{t_*} d(|t-s|)\|f(s)\|_{L^1_x}ds\right\|_{L^q_{I_0}} \lesssim \|f\|_{L^{q'}_{I_0}L^1_x}.$$

Combining (4.15), (4.17) and (4.18), we therefore obtain (4.9).

It remains to prove (4.16). Let $\tilde{\phi}$ be a solution of $\Box_{\mathbf{g}}\tilde{\phi} = 0$ in I_*. Then for any $t_0 \in [0, t_*]$ there holds the energy estimate $\|\mathbf{D}\tilde{\phi}(t)\|_{L^2(\Sigma)} \lesssim \|\mathbf{D}\tilde{\phi}(t_0)\|_{L^2(\Sigma)}$ for $t \in [t_0, t_*]$. Let $t_0 \le t' < t_*$. Similar to the derivation of (4.13), we have on $I = [t', t_*]$ that

(4.19) $$\int_{I\times\Sigma} -\mathbf{D}_0\tilde{\phi}\Box_{\mathbf{g}}\psi = \int_{\Sigma_{t'}} \mathbf{D}_0\phi\mathbf{D}_0\psi + \delta^{ij}\mathbf{D}_i\phi\mathbf{D}_j\psi + \int_{I\times\Sigma} Q[\phi, \psi]_{\alpha\beta}{}^{(\mathbf{T})}\pi^{\alpha\beta},$$

which together with $\Box_{\mathbf{g}}\psi = -Pf$ gives

$$\langle\mathbf{D}\psi, \mathbf{D}\tilde{\phi}\rangle(t') \le \|Pe_0\tilde{\phi}\|_{L^q_I L^\infty_x}\|f\|_{L^{q'}_I L^1_x} + \|^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_t L^2_x}\|\mathbf{D}\tilde{\phi}\|_{L^\infty_I L^2_x}$$

According to definition of M, we can obtain $\|Pe_0\tilde{\phi}\|_{L^q_I L^\infty_x} \le M\|\mathbf{D}\tilde{\phi}(t')\|_{L^2_x}$. Thus

$$\langle\mathbf{D}\psi, \mathbf{D}\tilde{\phi}\rangle(t') \lesssim \left(M\|f\|_{L^{q'}_I L^1_x} + \|^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_t L^2_x}\right)\|\mathbf{D}\tilde{\phi}(t')\|_{L^2_x}.$$

Since $\mathbf{D}\tilde{\phi}(t')$ can be arbitrary, there is a universal constant C_3 such that

$$\|\mathbf{D}\psi(t')\|_{L^2_x} \le C_3 M\|f\|_{L^{q'}_I L^1_x} + C_3\|^{(\mathbf{T})}\pi\|_{L^1_I L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_I L^2_x}.$$

Recall that $t' \in [t_0, t_*)$ is arbitrary. Thus, if (4.4) holds with $C_0 \geq 2C_3$ then

$$\|\mathbf{D}\psi\|_{L^\infty_{[t_0,t_*)}L^2_x} \leq C_3 M \|f\|_{L^{q'}_{[t_0,t_*)}L^1_x} + \frac{1}{2}\|\mathbf{D}\psi\|_{L^\infty_{[t_0,t_*)}L^2_x}.$$

This implies claim (4.16) with $C_2 = 2C_3$. The proof of (4.6) is thus completed. We also have proved for any $t \in I_*$

(4.20) $$\|\mathbf{D}\psi\|_{L^\infty_{[t,t_*)}L^2_x} \leq C_2 M \|f\|_{L^{q'}_{[t,t_*)}L^1_x}.$$

Now we consider $\|P(\partial_m\phi)\|_{L^q_{I_*}L^\infty_x}$. It suffices to estimate

$$\mathcal{I} = \int_{I_*\times\Sigma} fP(\partial_m\phi) = \int_{I_*\times\Sigma} \partial_m\phi Pf$$

for any function f satisfying $\|f\|_{L^{q'}_{I_*}L^1_x} \leq 1$. Let ψ be the solution of (4.10), then

$$\mathcal{I} = \int_{I_*\times\Sigma} -\partial_m\phi\Box_{\mathbf{g}}\psi.$$

In view of (4.12), we have with $Z = \partial_m$ that

$$\int_{I_*\times\Sigma} -Z\phi\Box_{\mathbf{g}}\psi = 2\int_{\Sigma_0} \mathbf{T}^\alpha P_\alpha + \int_{I_*\times\Sigma} Q[\phi,\psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}.$$

By direct calculation we can see that ${}^{(Z)}\pi = \mathbf{g}\cdot\tilde{\pi}$. Thus it follows from the energy estimate (4.20) and (4.4) that

$$\left|\int_{I_*\times\Sigma} Q[\phi,\psi]_{\alpha\beta}{}^{(Z)}\pi^{\alpha\beta}\right| \lesssim \|\tilde{\pi}\|_{L^1_{I_*}L^\infty_x}\|\mathbf{D}\psi\|_{L^\infty_{I_*}L^2_x}\|\mathbf{D}\phi\|_{L^\infty_{I_*}L^2_x}$$

$$\lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}\|f\|_{L^{q'}_{I_*}L^1_x}$$

and

$$\left|\int_{\Sigma_0} \mathbf{T}^\alpha P_\alpha\right| \lesssim \|\mathbf{D}\phi(0)\|_{L^2}\|\mathbf{D}\psi\|_{L^\infty_{I_*}L^2_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2}\|f\|_{L^{q'}_{I_*}L^1_x}.$$

Therefore $|\mathcal{I}| \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}\|f\|_{L^{q'}_{I_*}L^1_x}$. Hence we can conclude that

(4.21) $$\|P\partial_m\phi\|_{L^q_{I_*}L^\infty_x} \leq C\|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Finally we prove (4.5) for the case that $\partial = \partial_t$, i.e.

(4.22) $$\|P\partial_t\phi\|_{L^q_{I_*}L^\infty_x} \leq C\|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Note that $\partial_t f = ne_0(f) + Y^m\partial_m f$. We can write

$$P\partial_t\phi = nPe_0(\phi) + Y^m P\partial_m\phi + [P,n]e_0\phi + [P,Y^m]\partial_m\phi.$$

By using (10.12), the Bernstein inequality and the finite band property for the Littlewood-Paley projections, we obtain

$$\|[P,n]e_0\phi\|_{L^q_{I_*}L^\infty_x} \lesssim \|\widehat{\nabla}n\|_{L^q_{I_*}L^\infty_x}\|(e_0\phi)_{\leq 1}\|_{L^\infty} + \sum_{\ell>1}\|P(n_\ell\cdot(e_0\phi)_\ell)\|_{L^q_{I_*}L^\infty_x}$$

$$\lesssim \|\widehat{\nabla}n\|_{L^q_{I_*}L^\infty_x}\|e_0\phi\|_{L^\infty_{I_*}L^2_x}.$$

By using (2.4) and (BA2) under the rescaling coordinates $\|\widehat{\nabla}n\|_{L^q_{I_*}L^\infty_x} \lesssim \lambda^{-1+1/q}$, also using the energy estimate for ϕ, we can obtain

$$\|[P,n]e_0\phi\|_{L^q_{I_*}L^\infty_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Similarly, we have $\|\widehat{\nabla} Y\|_{L^q_{I_*} L^\infty_x} \lesssim \lambda^{-1 + \frac{1}{q}}$ and

$$\|[P, Y^m] \partial_m \phi\|_{L^q_{I_*} L^\infty_x} \lesssim \|\mathbf{D}\phi(0)\|_{L^2_x}.$$

Combining the above two estimates with (4.6) and (4.21) we therefore obtain (4.22). The proof is thus complete.

4.2. **Strichartz estimates $\Rightarrow$ Main estimates.** In this section we will use Theorem 4.3 to prove Theorem 4.1. According to the properties of Littlewood-Paley projections, it is easy to derive the desired estimates for the low frequency part. Therefore, to complete the proof of Theorem 4.1, it suffices to establish the following result.

Proposition 4.5. *There exists a large number $\Lambda \geq 1$ such that for any $q > 2$ sufficiently close to 2 and any $\delta > 0$ sufficiently close to 0 there holds on $I = [0, T]$*

$$\sum_{\mu > \Lambda} \left\{ \|\mu^\delta P_\mu \partial_m g\|^2_{L^2_I L^\infty_x} + \|\mu^\delta P_\mu \partial_t g\|^2_{L^2_I L^\infty_x} \right\} \lesssim T^{1 - \frac{2}{q}}.$$

Moreover for any solution of $\Box_\mathbf{g} \phi = 0$, there holds

$$\sum_{\mu > \Lambda} \|\mu^\delta P_\mu \partial \phi\|^2_{L^2_I L^\infty_x} \lesssim T^{1 - \frac{2}{q}} \|\widehat{\nabla}\phi, e_0 \phi\|^2_{H^{1+\epsilon}(0)}.$$

In order to carry out the proof of Proposition 4.5, we pick a sufficiently small $\epsilon_0 > 0$ and for each $\mu > 1$ we partition the interval $[0, T]$ into disjoint union of subintervals $I_k = [t_{k-1}, t_k)$ with the properties that

$$(4.23) \qquad |I_k| \lesssim \mu^{-8\epsilon_0} T \quad \text{and} \quad \|k, \widehat{\nabla} g, \widehat{\nabla} Y, \mathbf{D}n\|_{L^2_{I_k} L^\infty_x} \leq \mu^{-4\epsilon_0}.$$

Such partition is always possible. Let κ_μ denote the total number of subintervals in the partition. It is even possible to make $\kappa_\mu \lesssim \mu^{8\epsilon_0}$.

We first consider any pair (u, v) satisfying (3.18). Then $(U^\mu, V^\mu) := (P_\mu u, P_\mu v)$ also satisfies the system (3.18) with F_{U^μ} and F_{V^μ} given by (3.20), i.e.

$$(4.24) \qquad\qquad F_{U^\mu} = [P_\mu, Y^m]\partial_m u + [P_\mu, n]v + P_\mu F_u,$$

$$(4.25) \qquad\qquad F_{V^\mu} = [P_\mu, ng]\widehat{\nabla}^2 u + P_\mu F_v + [P_\mu, Y^m]\partial_m v.$$

Consequently, it follows from (3.40) that

$$n^2 \Box_\mathbf{g} P_\mu u = -n\mathbf{D_T} F_{U^\mu} + n(-F_{V^\mu} - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n)F_{U^\mu} + n\mathrm{Tr}ke_0 U^\mu).$$

Now we will use the Duhamel principle to represent $P_\mu u$. To simplify the notation, we use $W(t, s)$ to denote the operator defined on $\mathcal{H}$ such that, for each $\omega := (\omega_0, \omega_1) \in \mathcal{H}$, $\phi := W(t, s)(\omega)$ is the unique solution of the initial value problem

$$(4.26) \qquad\qquad \Box_\mathbf{g}\phi = 0, \quad \phi(t; s, x) = \omega_0, \quad \partial_t \phi(t; s, x) = \omega_1.$$

Then, by the Duhamel principle, we have for $t \in I_k = [t_{k-1}, t_k]$ that

$$P_\mu u(t) = W(t, t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1})\right)$$

$$(4.27) \qquad\qquad + \int_{t_{k-1}}^t W(t, s)(0, -R_\mu(s)) + W(t, s)(F_{U^\mu}(s), 0)ds.$$

Now we apply P_μ to the both sides and take the spacial derivative. Writing $P_\mu^2 = P_\mu$ by abuse of notation a little, we have

$$P_\mu \partial_m u(t) = \int_{t_{k-1}}^{t} \{\partial_m P_\mu W(t,s)(0,-R_\mu(s)) + \partial_m P_\mu W(t,s)(F_{U^\mu}(s),0)\} \, ds$$

(4.28)
$$+ \partial_m P_\mu W(t,t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1})\right).$$

where

$$R_\mu = n(-F_{V^\mu} - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n)F_{U^\mu} + n\mathrm{Tr}ke_0U^\mu) - Y^i\partial_i F_{U^\mu}.$$

By using (4.5) in Theorem 4.3 with suitable change of coordinates, we have for any one-parameter family of data $\omega(s) := (\omega_0(s), \omega_1(s)) \in \mathcal{H}$ with $s \in I_k := [t_{k-1}, t_k]$ that

$$\mu^{-1+\frac{1}{q}}\|P_\mu \partial W(t,s)(\omega(s))\|_{L^q_{[s,t_k]}L^\infty_x} \lesssim \mu^{\frac{1}{2}}\|\omega(s)\|_{\mathcal{H}}.$$

In view of the Minkowski inequality we then obtain

$$\left\|\int_{t_{k-1}}^{t} P_\mu \partial W(t,s)(\omega(s))ds\right\|_{L^2_{I_k}L^\infty_x} \lesssim \int_{t_{k-1}}^{t_k} \|P_\mu W(t,s)(\omega(s))\|_{L^2_{[s,t_k]}L^\infty_x} ds$$

$$\lesssim |I_k|^{\frac{1}{2}-\frac{1}{q}}\mu^{\frac{3}{2}-\frac{1}{q}}\int_{I_k}\|\omega(s)\|_{\mathcal{H}}ds.$$

Since $|I_k| \lesssim T\mu^{-8\epsilon_0}$, it follows that

$$\left\|\int_{t_{k-1}}^{t} P_\mu \partial W(t,s)(\omega(s))ds\right\|_{L^2_{I_k}L^\infty_x} \lesssim T^{\frac{1}{2}-\frac{1}{q}}\mu^{(\frac{1}{2}-\frac{1}{q})(1-8\epsilon_0)}\int_{I_k}\mu\|\omega(s)\|_{\mathcal{H}}ds.$$

Applying the above inequality to (4.28) gives, with $\delta_0 := (\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)$, that

$$\|P_\mu \partial_m u\|_{L^2_{I_k}L^\infty_x} \lesssim T^{\frac{1}{2}-\frac{1}{q}}\mu^{\delta_0}\left(\|\mu(0,R_\mu)\|_{L^1_{I_k}\mathcal{H}} + \|\mu(F_{U^\mu},0)\|_{L^1_{I_k}\mathcal{H}}\right)$$

(4.29)
$$+ T^{\frac{1}{2}-\frac{1}{q}}\left(B_\mu(t_{k-1}) + C_\mu(t_{k-1})\right),$$

where

$$B_\mu(t) := \mu^{\delta_0}\|\mu\left(P_\mu u(t), \partial_t P_\mu u(t)\right)\|_{\mathcal{H}}, \quad C_\mu(t) := \mu^{\delta_0}\|\mu(0,-F_{U^\mu}(t))\|_{\mathcal{H}}.$$

In the following we will give the estimates on R_μ, F_{U^μ}, $B_\mu(t_{k-1})$ and $C_\mu(t_{k-1})$ separately. Positive indices ϵ_0, q, δ are chosen such that $4\epsilon_0 + \delta_0 + \delta < s - 2$, and $\delta_0 + \delta < 4\epsilon_0$.

4.2.1. Estimates for R_μ, F_{U^μ}.

Lemma 4.6. *For any $\delta_1 > \delta > 0$ satisfying $b := \delta_0 + \delta_1 < 4\epsilon_0$, there holds*

$$\left(\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}R_\mu\|^2_{L^1_{I_k}L^2_x}\right)^{\frac{1}{2}} \leq T\|\widehat{\nabla}u,v\|_{L^\infty_t H^{1+b}} + \|\mu^{1+b}P_\mu F_u\|_{L^1_I l^2_\mu H^1}$$

(4.30)
$$+ \|\mu^{1+b}P_\mu F_v\|_{L^1_I l^2_\mu L^2_x} + \|\widehat{\nabla}^2 u\|_{L^\infty_I L^2_x}.$$

(4.31)
$$\sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}(F_{U^\mu}(s),0)\|^2_{L^1_{I_k}\mathcal{H}} \lesssim \left(\|\partial u,v\|_{L^1_I H^{1+b}} + \|\mu^{1+b}P_\mu \partial F_u\|_{L^1_I l^2_\mu L^2_x}\right)^2$$

Proof. We first write $R_\mu = -n[P_\mu, ng]\widehat{\nabla}^2 u - Y^i \partial_i F_{U^\mu} + \check{R}_\mu$, where

$$\check{R}_\mu := n\left(-P_\mu F_v - [P_\mu, Y^m]\partial_m v - n\pi_{0a}\nabla^a U^\mu + e_0(\ln n)F_{U^\mu} + n\mathrm{Tr}ke_0 U^\mu\right).$$

Let us set

$$\mathcal{I}_1 := \sum_{\mu > \Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}[P_\mu, ng]\widehat{\nabla}^2 u\|^2_{L^1_{I_k}L^2_x}, \quad \mathcal{I}_2 := \sum_{\mu > \Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}\check{R}_\mu\|^2_{L^1_{I_k}L^2_x}.$$

It suffices to show that

(4.32) $\quad \mathcal{I}_1^{\frac{1}{2}} \lesssim \|\widehat{\nabla}^2 u\|_{L^\infty_I L^2_x}$,

(4.33) $\quad \mathcal{I}_2^{\frac{1}{2}} \lesssim \|\widehat{\nabla}u, v\|_{L^\infty_t H^{1+b}}T + \|\mu^{1+b}P_\mu F_u\|_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+b}P_\mu F_v\|_{L^1_I l^2_\mu L^2_x}$.

(4.34)

$$\sum_{\mu > \Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta_0+\delta}\widehat{\nabla}F_{U^\mu}(s)\|^2_{L^1_{I_k}L^2_x} \lesssim \left(\|\partial u, v\|_{L^1_I H^{1+b}} + \|\mu^{1+b}P_\mu\partial F_u\|_{L^1_I l^2_\mu L^2_x}\right)^2.$$

By (4.23), we have $\|\widehat{\nabla}(ng)\|_{L^1_{I_k}L^\infty_x} \lesssim \mu^{-8\epsilon_0}$. We can apply Corollary 10.5 to obtain

$$\|\mu^{1+\delta+\delta_0}[P_\mu, ng]\widehat{\nabla}^2 u\|_{L^1_{I_k}L^2_x} \lesssim \mu^{\delta+\delta_0}\|\widehat{\nabla}(ng)\|_{L^1_{I_k}L^\infty_x}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x}$$

$$\lesssim \mu^{\delta+\delta_0-8\epsilon_0}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x}.$$

Recall also that $\kappa_\mu \lesssim \mu^{8\epsilon_0}$. We can obtain

$$\sum_{k=1}^{\kappa_\mu}\|\mu^{1+\delta+\delta_0}[P_\mu, ng]\widehat{\nabla}^2 u\|^2_{L^1_{I_k}L^2_x} \leq C\mu^{2(\delta+\delta_0-4\epsilon_0)}\|\widehat{\nabla}^2 u\|^2_{L^\infty_t L^2_x}.$$

Since $0 < \delta < \delta_1$ and $b := \delta_0 + \delta_1 < 4\epsilon_0$, we have $\mathcal{I}_1 \lesssim \Lambda^{2(b-4\epsilon_0)}\|\widehat{\nabla}^2 u\|_{L^\infty_t L^2_x} \lesssim \|\widehat{\nabla}^2 u\|^2_{L^\infty_t L^2_x}$ which gives (4.32).

Next we prove (4.33). Since $0 < \delta < \delta_1$, we observe that for any function a_μ there holds

$$\sum_{\mu > \Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^\delta a_\mu\|^2_{L^1_{I_k}L^2_x} \leq \sum_{\mu > \Lambda}\|\mu^\delta a_\mu\|^2_{L^1_I L^2_x} \leq \left(\int_I\sum_{\mu > \Lambda}\|\mu^\delta a_\mu\|_{L^2_x}\right)^2$$

(4.35) $$\lesssim \left(\int_I\|\mu^{\delta_1}a_\mu\|_{l^2_\mu L^2_x}\right)^2.$$

(4.34) and (4.31) can be derived immediately by using (4.35) and (3.25).

In view of (4.35), it suffices to estimate $\int_I\|\mu^{1+b}\check{R}_\mu\|_{l^2_\mu L^2_x}$. From (10.21) it follows that

$$\|\mu^{1+b}[P_\mu, Y^m]\partial_m v\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}Y\|_{L^\infty_x}\|\widehat{\nabla}v\|_{H^b} + \|\widehat{\nabla}^2 Y\|_{H^{\frac{1}{2}+b}}\|\widehat{\nabla}v\|_{L^2_x}$$

In view of (4.24), (3.23) and (3.37) we have

$$\|\mu^{1+b}e_0\ln nF_{U^\mu}\|_{l^2_\mu L^2_x} \lesssim \|e_0(\ln n)\|_{L^\infty}\left(\|\widehat{\nabla}Y, \widehat{\nabla}n\|_{L^\infty_x}\|\partial u, v\|_{H^b}\right.$$

$$+\|\widehat{\nabla}^2 Y, \widehat{\nabla}^2 n\|_{H^{\frac{1}{2}+b}}\|\partial u, v\|_{L^2_x} + \left.\|\mu^{1+b}P_\mu F_u\|_{l^2_\mu L^2_x}\right)$$

$$\lesssim \|\partial u, v\|_{H^b} + \|\mu^{1+b}P_\mu F_u\|_{l^2_\mu L^2_x}.$$

Recall that $\|\nabla n, \mathrm{Tr}k\|_{L_I^\infty L^\infty} \le C$, we can derive that

$$\|\mu^{1+b}(|\mathrm{Tr}k\,e_0(U^\mu)| + |\nabla_i n \mathbf{D}^i U^\mu|)\|_{l_\mu^2 L_x^2} \lesssim \|\nabla n, \mathrm{Tr}k\|_{L_x^\infty}\|\Lambda^b \widehat{\nabla}(\widehat{\nabla} u, v)\|_{L_x^2}$$
$$\lesssim \|\Lambda^b \widehat{\nabla}(\widehat{\nabla} u, v)\|_{L_x^2}$$

Combining the above three estimates we thus obtain (4.33). $\hfill\square$

4.2.2. *Estimate for $B_\mu(t_{k-1})$ and $C_\mu(t_{k-1})$.*

Lemma 4.7. *For any $\delta > 0$ satisfying $\alpha := 4\epsilon_0 + \delta_0 + \delta < s - 2$ there holds*

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \mu^{2\delta} B_\mu(t_{k-1})^2 \lesssim \sum_{\mu > \Lambda} \mu^{2\alpha} \sup_{t \in I} \mathcal{E}_\mu^{(1)}(t)$$
$$+ \sup_t \left(\|\widehat{\nabla} u, v\|_{H^{\frac{1}{2}+\alpha}}^2 + \|\mu^{\frac{3}{2}+\alpha} P_\mu F_u\|_{l_\mu^2 L_x^2}^2 \right).$$

Proof. Since $\kappa_\mu \lesssim \mu^{8\epsilon_0}$, we have from the expression of $B_\mu(t)$ that

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \mu^{2\delta} B_\mu(t_{k-1})^2$$
$$\lesssim \sum_{\mu > \Lambda} \sup_{t \in I}(\mu^{4\epsilon_0+\delta} B_\mu(t))^2 \lesssim \sum_{\mu > \Lambda} \mu^{2(1+\alpha)} \sup_{t \in I} \|(P_\mu u(t), \partial_t P_\mu u(t))\|_{\mathcal{H}}^2$$

According to the definition of $\mathcal{E}_\mu^{(1)}(t) := \mathcal{E}_\mu^{(1)}(u(t), v(t))$ and the equation for $\partial_t P_\mu u$ we obtain

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \mu^{2\delta} B_\mu(t_{k-1})^2 \lesssim \sum_{\mu > \Lambda} \mu^{2\alpha} \sup_{t \in I} \mathcal{E}_\mu^{(1)}(t) + \sum_{\mu > \Lambda} \sup_{t \in I} \|\mu^{1+\alpha} F_{U^\mu}(t)\|_{L_x^2}^2.$$

With the help of (10.22) we have

$$\|\mu^{1+\alpha} F_{U^\mu}\|_{L_x^2}^2 \le \mu^{-1} \sum_\lambda \|\lambda^{\frac{3}{2}+\alpha} F_{U^\lambda}(t)\|_{L_x^2}^2$$
$$\lesssim \mu^{-1}(\|\widehat{\nabla} u, v\|_{H^{\frac{1}{2}+\alpha}}^2 + \|\lambda^{\frac{3}{2}+\alpha} P_\lambda F_u\|_{l_\lambda^2 L_x^2}^2).$$

Plugging this into the above inequality and summing over $\mu > \Lambda$ gives the desired estimate. $\hfill\square$

Lemma 4.8. *For any $\delta_1 > \delta > 0$ satisfying $b := \delta_0 + \delta_1 < 4\epsilon_0$ there hold*

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \left(\mu^\delta C_\mu(t_{k-1}) \right)^2 \le \|\partial_m u, v\|_{L_t^\infty H^{4\epsilon_0+b}}^2 + \sup_{t \in I} \sum_{\mu > 1} \|\mu^{1+4\epsilon_0+b} P_\mu F_u\|_{L_x^2}^2.$$

Proof. Since

$$\sum_{k=1}^{\kappa_\mu} \left(\mu^\delta C_\mu(t_{k-1}) \right)^2 \le \sup_{t \in I} \|\mu^{4\epsilon_0+\delta_0+\delta+1} F_{U^\mu}(t)\|_{L_x^2}^2.$$

with $0 < \delta < \delta_1$, we have

$$\sum_{\mu > \Lambda} \sum_{k=1}^{\kappa_\mu} \left(\mu^\delta C_\mu(t_{k-1}) \right)^2 \le \sup_t \|\mu^{4\epsilon_0+\delta_0+\delta_1+1} F_{U^\mu}(t)\|_{l_\mu^2 L_x^2}^2.$$

In view of (10.22), we complete the proof of Lemma 4.8. $\hfill\square$

In view of (4.29), Lemma 4.6, Lemma 4.8, Lemma 4.7 and writing

$$\sum_{\mu>\Lambda}\|\mu^{\delta}P_{\mu}\partial_m u\|^2_{L^2_I L^\infty_x} = \sum_{\mu>\Lambda}\sum_{k=1}^{\kappa_\mu}\|\mu^{\delta}P_{\mu}\partial_m u\|^2_{L^2_{I_k}L^\infty_x},$$

we can obtain the following result.

Proposition 4.9. *For any $q > 2$ sufficiently close to 2 and any $\delta > 0$ sufficiently small such that $\alpha := 4\epsilon_0 + \delta_0 + \delta < s - 2$, where $\delta_0 := (\frac{1}{2} - \frac{1}{q})(1 - 8\epsilon_0)$. Then for any pair (u, v) satisfying (3.18) there holds*

$$\sum_{\mu>\Lambda}\|\mu^{\delta}P_{\mu}\partial_m u\|^2_{L^2_I L^\infty_x}$$

$$\lesssim T^{1-\frac{2}{q}}\left(\|\mu^{2+\alpha}P_{\mu}F_u\|^2_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+\alpha}P_{\mu}F_v\|^2_{L^1_I l^2_\mu L^2_x} + \|\mu^{\frac{1}{2}+\alpha}P_{\mu}\widehat{\nabla}F_u\|^2_{L^\infty_I l^2_\mu L^2_x}\right)$$

$$+ T^{1-\frac{2}{q}}\left(\sup_{t\in I}\mathcal{E}^{(1+\alpha)}(u,v)(t) + \sum_{\mu>\Lambda}\mu^{2\alpha}\sup_{t\in I}\mathcal{E}^{(1)}_\mu(t)\right).$$

Now we are ready to derive the estimates on the spacial derivative part in Proposition 4.5. Recall that $(u, v) := (g, -2k)$ satisfies (1.13). Recall also that Proposition 3.8 implies

$$\sum_{\mu>\Lambda}\sup_t \mu^{2\alpha}\mathcal{E}^{(1)}_\mu(t) \lesssim \mathcal{E}^{(1+\alpha)}(u,v)(0) + \|\mu^{2+\alpha}P_{\mu}F_u\|_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+\alpha}P_{\mu}F_v\|_{L^1_I l^2_\mu L^2_x}.$$

In view of $\mathcal{E}^{(1+\alpha)}(u,v)(t) \lesssim \mathcal{E}^{(1+\alpha)}(u,v)(0)$, we thus obtain from Proposition 4.9 that

$$\sum_{\mu>\Lambda}\|\mu^{\delta}P_{\mu}\partial_m u\|^2_{L^2_I L^\infty_x} \lesssim T^{1-\frac{2}{q}}\left(\|\mu^{2+\alpha}P_{\mu}F_u\|^2_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+\alpha}P_{\mu}F_v\|^2_{L^1_I l^2_\mu L^2_x}\right.$$

$$\left. + \|\mu^{\frac{1}{2}+\alpha}P_{\mu}\widehat{\nabla}F_u\|^2_{L^\infty_I l^2_\mu L^2_x} + \mathcal{E}^{(1+\alpha)}(u,v)(0)\right).$$

With the help of (3.31), (3.32), (3.33) and the bootstrap assumption (BA1), it follows that $\|\mu^{\frac{1}{2}+\alpha}P_{\mu}\widehat{\nabla}F_u\|_{L^\infty_I l^2_\mu L^2_x} \lesssim 1$ and

$$\|\mu^{1+\alpha}P_{\mu}F_u\|_{L^1_I l^2_\mu L^2_x} + \|\mu^{1+\alpha}P_{\mu}F_v\|_{L^1_I l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}g, k, \widehat{\nabla}Y, \widehat{\nabla}n\|_{L^1_I L^\infty_x} + 1 \lesssim 1.$$

Therefore

$$\|\partial_m g\|^2_{L^2_I L^\infty_x} + \sum_{\mu>\Lambda}\|\mu^{\delta}P_{\mu}\partial_m g\|^2_{L^2_I L^\infty_x} \le CT^{1-\frac{2}{q}}.$$

For a solution ϕ of the equation $\Box_{\mathbf{g}}\phi = 0$, we recall that $(u, v) = (\phi, e_0\phi)$ satisfies (3.43) with $W = 0$. Thus, in view of (3.44) and $F_u = 0$, we may use the same argument as above to conclude that

$$\|\partial_m \phi\|^2_{L^2_I L^\infty_x} + \sum_{\mu>\Lambda}\|\mu^{\delta}P_{\mu}\partial_m \phi\|^2_{L^2_I L^\infty_x} \le CT^{1-\frac{2}{q}}\|\widehat{\nabla}\phi, e_0\phi\|^2_{H^{1+\epsilon}(0)}.$$

4.2.3. *Estimate for $P_\mu \partial_t u$.* From (4.27) it follows that

$$P_\mu \partial_t u(t) = P_\mu F_{U^\mu}(t) + P_\mu \partial_t W(t, t_{k-1}) \left(P_\mu u(t_{k-1}), \partial_t P_\mu u(t_{k-1}) - F_{U^\mu}(t_{k-1}) \right)$$

$$(4.36) \qquad + \int_{t_{k-1}}^{t} P_\mu \{ \partial_t W(t, s)(0, -R_\mu(s)) + \partial_t W(t, s)(F_{U^\mu}(s), 0) \} ds.$$

We can use the same argument for dealing with $P_\mu \partial_m u$ to estimate the terms on the right hand side except the first term $P_\mu F_{U^\mu}(t)$.

Lemma 4.10. *For sufficiently small $\delta > 0$ there holds*

$$\sum_{\mu > \Lambda} \| \mu^\delta P_\mu F_{U^\mu} \|_{L_I^2 L_x^\infty}^2 \lesssim \sum_{\mu > \Lambda} \| \mu^\delta P_\mu F_u \|_{L_I^2 L_x^\infty}^2 + T \| \widehat{\nabla}^2 u, \widehat{\nabla} v \|_{L_t^\infty H^\delta}^2$$

Proof. From (10.28) we can find $0 < \eta < 1/2$ such that

$$\mu^\delta \| [P_\mu, Y^m] \partial_m u \|_{L_x^\infty} + \mu^\delta \| [P_\mu, n] v \|_{L_x^\infty}$$
$$\lesssim \mu^{-\eta} \left(\| \widehat{\nabla} Y, \widehat{\nabla} n \|_{L_x^\infty} \| \widehat{\nabla}^2 u, \widehat{\nabla} v \|_{H^\delta} + \| \widehat{\nabla} u, v \|_{H^1} \| \widehat{\nabla}^2 Y, \widehat{\nabla}^2 n \|_{H^{\delta + \frac{1}{2}}} \right).$$

This together with (4.24) implies that

$$\sum_{\mu > \Lambda} \| \mu^\delta P_\mu F_{U^\mu} \|_{L_I^2 L_x^\infty}^2 \lesssim \sum_{\mu > \Lambda} \| \mu^\delta P_\mu F_u \|_{L_I^2 L_x^\infty}^2 + T \| \widehat{\nabla} Y, \widehat{\nabla} n \|_{L_I^\infty L_x^\infty}^2 \| \widehat{\nabla}^2 u, \widehat{\nabla} v \|_{L_I^\infty H^\delta}^2$$
$$+ T \| \widehat{\nabla} u, v \|_{L_I^\infty H^1}^2 \| \widehat{\nabla}^2 Y, \widehat{\nabla}^2 n \|_{L_I^\infty H^{\frac{1}{2} + \delta}}^2.$$

In view of Proposition 3.1, we therefore obtain the desired estimate. $\qquad \square$

By using (4.36), Lemma 4.10 and Proposition 3.7 for the solution ϕ of $\Box_{\mathbf{g}} \phi = 0$, in view of $F_u = 0$ we derive that

$$\| \partial_t \phi \|_{L_I^2 L_x^\infty}^2 + \sum_{\mu > \Lambda} \| \mu^\delta P_\mu \partial_t \phi \|_{L_I^2 L_x^\infty}^2 \leq C T^{1 - \frac{2}{q}} \| \widehat{\nabla} \phi, e_0 \phi \|_{H^{1 + \epsilon}(0)}^2.$$

Next we consider $(u, v) = (g, -2k)$ which satisfies (1.13). Recall that $F_u = \widehat{\nabla} Y \cdot g$ in (1.13), we have

$$\| \mu^\delta P_\mu F_u \|_{L_I^2 L_x^\infty} \lesssim \| \mu^\delta [P_\mu, g] \widehat{\nabla} Y \|_{L_t^2 L_x^\infty} + \| \mu^\delta P_\mu \widehat{\nabla} Y \|_{L_t^2 L_x^\infty}.$$

By using (10.27) we have with $0 < \eta < 1/2$ that

$$\| \mu^\delta [P_\mu, g] \widehat{\nabla} Y \|_{L_x^\infty} \lesssim \mu^{-\eta} \| \widehat{\nabla} Y \|_{L_x^\infty} \| \widehat{\nabla}^2 g \|_{H^\delta}.$$

Therefore

$$\sum_{\mu > \Lambda} \| \mu^\delta P_\mu F_u \|_{L_I^2 L_x^\infty}^2 \lesssim T \left(\| \widehat{\nabla}^2 Y \|_{L_t^\infty H^{\frac{1}{2} + \delta}}^2 + \| \widehat{\nabla}^2 g, \widehat{\nabla} k \|_{L_t^\infty H^\delta}^2 \right)$$

From this, (4.36), (3.33) and (3.37), we conclude that $\sum_{\mu > \Lambda} \| \mu^\delta P_\mu \partial_t g \|_{L_I^2 L_x^\infty}^2 \leq C T^{1 - \frac{2}{q}}.$

4.3. Boundedness Theorem $\Rightarrow$ Decay estimates. In this subsection we give the proof of Theorem 4.2 under the rescaled coordinates. A time interval $I = [0, T]$ becomes $I_* = [0, \lambda T]$ after rescaling. Let τ_* denote a number such that $t_* \leq \tau_* \leq \lambda T$ and let t_0 be certain number satisfying $1 \approx t_0 < \tau_*$. We may take a sequence of balls $\{B_J\}$ of radius $1/2$ such that their union covers Σ_{t_0} and any ball in this collection intersect at most 10 other balls. Let $\{\chi_J\}$ be a partition of unity subordinate to the cover $\{B_J\}$. We may assume that $|\nabla \chi_J|_{L_x^\infty} \leq C_1$ uniformly in J. By using this partition of unity and a standard argument we can reduce the proof of Theorem 4.2 by establishing the following dispersive estimate result with initial data supported on a ball of radius $1/2$.

Proposition 4.11. *There exists a large constant Λ such that for any $\lambda \geq \Lambda$ and any solution ψ of*

$$\Box_{\mathbf{g}} \psi = 0$$

on the time interval $[0, \tau_*]$ with $\tau_* \leq \lambda T$, with certain $t_0 \in [1, C]$ and any initial data $\psi[t_0] = (\psi(t_0), \partial_t \psi(t_0))$ supported in the geodesic ball $B_{1/2}$ of radius $\frac{1}{2}$, there is a function $d(t)$ satisfying

$$(4.37) \qquad \|d\|_{L^{\frac{q}{2}}[0,\tau_*]} \lesssim 1, \text{ for } q > 2 \text{ sufficiently close to } 2$$

such that for all $t_0 \leq t \leq \tau_*$,

$$(4.38) \qquad \|Pe_0 \psi(t)\|_{L_x^\infty} \leq \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + d(t) \right) (\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2}).$$

Proof of Theorem 4.2. To derive Theorem 4.2, we apply the above result to ψ_I,

$$\Box_{\mathbf{g}} \psi_I = 0, \psi_I(t_0) = \chi_I \cdot \psi(t_0)$$

with ψ the solution of (4.1) with initial data $\psi[0] := (\psi(0), \partial_t \psi(0))$. It is easy to see that $\psi(t, x) = \sum_I \psi_I(t, x)$. By summing over I, we derive for $t_0 \leq t \leq \tau_*$

$$(4.39) \qquad \|Pe_0 \psi(t)\|_{L_x^\infty} \leq \left(\frac{1}{(1 + |t - t_0|)^{\frac{2}{q}}} + d(t) \right) (\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2}).$$

Then (4.3) follows by applying (3.46) to the solution ψ of (4.1)

$$\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2} \lesssim \|\psi[0]\|_{H^1} + \|\psi(0)\|_{L^2}.$$

For $0 < t < t_0$, it follows immediately from Bernstein inequality and (3.45) that

$$(4.40) \qquad \|Pe_0 \psi(t)\|_{L_x^\infty} \lesssim \|e_0 \psi(t)\|_{L_x^2} \lesssim \|\psi[0]\|_{H^1}.$$

Combining (4.39) with (4.40) implies Theorem 4.2. $\qquad\qquad \Box$

We will prove Proposition 4.11 by establishing boundedness theorem for conformal energy. For this purpose, we introduce the setup and notation. We denote by Γ^+ the portion in $[0, \lambda T]$ of the integral curve of $\mathbf{T}$ passing through the center of $B_{\frac{1}{2}}$. We define the optical function u to be the solution of eikonal equation $\mathbf{g}^{\alpha\beta} \partial_\alpha u \partial_\beta u = 0$ with $u = t$ on Γ^+. We denote the outgoing null cone initiating from Γ^+ by C_u with $0 \leq u \leq \lambda T$. Let $S_{t,u} = C_u \cap \Sigma_t$. Let us set $\mathcal{D}_0^+ = \cup_{\{t \in [t_0, \tau_*], 0 \leq u \leq t\}} S_{t,u}$ and $\mathcal{D}^+ = \cup_{\{t \in [0, \tau_*], 0 \leq u \leq t\}} S_{t,u}$. We denote the exterior region on Σ_t, $t \geq t_0$ by $\text{Ext}_t = \{0 \leq u \leq 3t/4\}$. By $C^{-1} < n < C$, we can always choose $t_0 \in [1, 2C]$ such that $B_{\frac{1}{2}} \subset (\mathcal{D}_0^+ \cap \Sigma_{t_0})$.

Next we extend the time axis $\Gamma^+ : u = t$ backward by following the integral curve of $\mathbf{T}$ to $t = -\lambda T$. Let us denote the extended portion of the integral curve of $\mathbf{T}$ by Γ^-. Let C_u be the outgoing null cone initiating from vertex $p(t) \in \Gamma^-$ with $u = t$. We also foliate the null hypersurfaces by time foliation, $C_u = \cup_{u \le t \le \tau_*} S_{t,u}$.

Let $\underline{\varpi}$ and ϖ be smooth cut-off functions depending only on two variables t, u. For $t > 0$, they are defined as follows

$$\underline{\varpi} = \begin{cases} 1 & \text{on } 0 \le u \le t \\ 0 & \text{off } -\frac{t}{4} \le u \le t, \end{cases} \qquad \varpi = \begin{cases} 1 & \text{on } 0 \le \frac{u}{t} \le \frac{1}{2} \\ 0 & \text{if } \frac{u}{t} \ge \frac{3}{4} \text{ or } u \le -\frac{t}{4} \end{cases} .$$

We also suppose ϖ and $\underline{\varpi}$ coincide in the region $\cup_{\{t \in [t_0, \tau_*], -\frac{t}{4} < u \le 0\}} S_{t,u}$.

Let us denote by N the outward unit normal of $S_{t,u} \in \Sigma_t$. Define $\theta_{AB} = \langle \mathbf{D}_A N, e_B \rangle$ and $\chi_{AB} = \langle \mathbf{D}_{e_A} L, e_B \rangle$. We decompose χ as $\chi_{AB} = \hat{\chi}_{AB} + \frac{1}{2} \text{tr}\chi \gamma_{AB}$.

(4.41) $\slashed{\nabla}_N N = -(\slashed{\nabla} \log \mathbf{b}) e_A, \qquad \slashed{\nabla}_A N_B = \theta_{AB} e_B, \qquad \chi_{AB} = \theta_{AB} - k_{AB}$

Under the rescaled coordinates we recall some useful results in Proposition 3.10, Proposition 3.11 and those established in [25, Sections 4 and 8]. Under (BA1) and (BA2) we have:

(i) there exists $\delta_* > 0$ depending only on B_1 and the norm of initial data $\|(g, k)\|_{H^2 \times H^1(\Sigma_0)}$ such that if $T \le \delta_*$ then the outgoing null radius of injectivity satisfies $i_*(p) > T - t(p)$ for any $p \in [-\lambda T, \lambda T] \times \Sigma$.

(ii) Let $\mathcal{N}^+(p)$ be an outgoing null cone initiating from $p \in [-\lambda T, \lambda T] \times \Sigma$ and contained therein. Then on every $\mathcal{N}^+(p)$ there holds $\mathcal{F}^{\frac{1}{2}}[\widehat{\nabla} g] \lesssim \lambda^{-\frac{1}{2}}$, the curvature flux $\mathcal{R}$ together with flux type norm of components of π on $\mathcal{N}^+(p)$ satisfies (for definition we refer to [25] and Section 6)

(4.42) $\mathcal{R} + \mathcal{N}_1[\slashed{\pi}] \lesssim \lambda^{-\frac{1}{2}}.$

(iii) For $0 \le T \le \delta_*$, consider $C_u \subset [0, \lambda T] \times \Sigma$, with $S_{t,u} = C_u \cap \Sigma_t$ and $r(t, u) = \sqrt{\frac{|S_{t,u}|}{4\pi}}$. As a consequence of (4.42) and $C^{-1} < n < C$ the metric $\gamma_{t,u}$ on $\mathbb{S}^2$, obtained by restricting the metric g on Σ_t to $S_{t,u}$ and then pulling it back to $\mathbb{S}^2$ by the exponential map $\mathcal{G}(t, u, \cdot)$, verifies with small quantity $0 < \epsilon < 1/2$ that

$$|r^{-2} \gamma_{t,u}(X, X) - \gamma_{\mathbb{S}^2}(X, X)| < \epsilon \gamma_{\mathbb{S}^2}(X, X), \quad \forall X \in T\mathbb{S}^2,$$

where $\gamma_{\mathbb{S}^2}$ is the standard metric on $\mathbb{S}^2$; there holds $(t - u) \approx r(t, u)$. There hold

(4.43) $$\left| \frac{\mathbf{b}}{n} - 1 \right| \le \frac{1}{2}, \qquad r \text{tr}\chi \approx 1, \qquad v_{t,u} := \sqrt{\frac{|\gamma_{t,u}|}{|\gamma_{\mathbb{S}^2}|}} \approx r^2$$

$$\|\tilde{\pi}, \hat{\chi}, \slashed{\nabla} \log \mathbf{b}\|_{L^4(S_{t,u})} + \|r^{-\frac{1}{2}}(\tilde{\pi}, \hat{\chi}, \slashed{\nabla} \log \mathbf{b})\|_{L^2(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}}.$$

We will constantly employ the following result, where all the constants suppressed in $\lesssim$ are independent of frequency λ.

Lemma 4.12. *For any Σ-tangent tensor field F there hold for $-\tau_*/2 \le u < t$*

$$\int_{S_{t,u}} |F|^2 \lesssim \|F\|_{H^1(\Sigma_t)} \|F\|_{L^2(\Sigma_t)}, \qquad \|F\|_{L^4(S_{t,u})} + \|r^{-\frac{1}{2}} F\|_{L^2(S_{t,u})} \lesssim \|F\|_{H^1(\Sigma_t)}.$$

Proof. This is [25, Proposition 7.5]. $\square$

More results that can be established under (BA1), (BA2) and the bounded H^2 norm of data will be revisited in Section 6. Now we prove a commutator estimate for P, the Littlewood Paley projection with frequency 1. This estimate is slightly more general than needed.

Lemma 4.13. *For scalar function f and G, with $3 < p \leq \infty$,*

$$\|[P,G]\partial_m f\|_{L_x^\infty} + \|[P,G]\partial_m f\|_{H^1} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\widehat{\nabla}f\|_{L_x^2}$$

Proof. We first establish L^∞ estimates in this lemma. Using (10.12), we have

$$(4.44) \qquad [P,G]\partial_m f = [P,G](\partial_m f)_{\leq 1} + \sum_{\ell > 1} P\left(G_\ell \cdot P_\ell(\partial_m f)\right).$$

By Sobolev embedding, with $b > 3$ sufficiently close to 3,

$$(4.45) \qquad \|[P,G](\partial_m f)_{\leq 1}\|_{L_x^\infty} \lesssim \|\widehat{\nabla}[P,G](\partial_m f)_{\leq 1}\|_{L_x^b} + \|[P,G](\partial_m f)_{\leq 1}\|_{L_x^2}.$$

Using Corollary 10.5, we can obtain for the second term with $\frac{1}{p} + \frac{1}{p'} = \frac{1}{2}$ that

$$(4.46) \qquad \|[P,G](\partial_m f)_{\leq 1}\|_{L_x^2} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\widehat{\nabla}f_{\leq 1}\|_{L_x^{p'}} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\widehat{\nabla}f\|_{L_x^2}.$$

Apply (10.16) to $(G, \partial_m f)$ and $\mu = 1$ we can obtain

$$\widehat{\nabla}[P,G](\partial_m f)_{\leq 1} = \int \widehat{\nabla}M_1(x-y)(x-y)^j \int_0^1 \partial_j G(\tau y + (1-\tau)x) d\tau (\partial_m f)_{\leq 1}(y) dy$$

$$(4.47) \qquad - \int M_1(x-y)\widehat{\nabla}G(x)(\partial_m f)_{\leq 1}(y) dy.$$

Then with $\frac{1}{b*} + \frac{1}{p} = \frac{1}{b}$,

$$(4.48) \qquad \|\widehat{\nabla}[P,G](\partial_m f)_{\leq 1}\|_{L_x^b} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\partial f_{\leq 1}\|_{L_x^{b*}}.$$

By (4.44), using (4.48) and (4.46), Bernstein inequality,

$$\|[P,G]\partial_m f\|_{L_x^\infty} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\partial_m f\|_{L_x^2} + \sum_{\ell > 1} \|P(G_\ell P_\ell(\partial_m f))\|_{L_x^\infty}.$$

Using Bernstein inequality and finite band property,

$$\sum_{\ell > 1} \|P\left(G_\ell P_\ell(\partial_m f)\right)\|_{L_x^\infty} \lesssim \sum_{\ell > 1} \ell^{-1}\|P_\ell \partial G\|_{L_x^p}\|P_\ell \partial_m f\|_{L_x^{p'}}$$

$$\lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\partial_m f\|_{L_x^2} \sum_\ell \ell^{-1+\frac{3}{p}} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\widehat{\nabla}f\|_{L_x^2}.$$

Thus the proof of the first part of the lemma is completed. Now we consider H^1 estimates in this lemma. Let $I = \widehat{\nabla}[P,G](\partial_m f)_{\leq 1}$. Similar to (4.48), it follows from (4.47) that

$$\|I\|_{L_x^2} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|(\partial_m f)_{\leq 1}\|_{L_x^{p'}}.$$

Now consider $J = \sum_{\ell > 1} \widehat{\nabla}P\left(G_\ell P_\ell(\partial_m f)\right)$, by finite band property,

$$\|J\|_{L_x^2} \lesssim \sum_{\ell > 1} \ell^{-1}\|\widehat{\nabla}G\|_{L_x^p}\|P_\ell(\partial_m f)\|_{L_x^{p'}}$$

$$\lesssim \sum_{\ell > 1} \ell^{-1+\frac{3}{p}}\|\partial_m f\|_{L_x^2}\|\widehat{\nabla}G\|_{L_x^p} \lesssim \|\widehat{\nabla}G\|_{L_x^p}\|\widehat{\nabla}f\|_{L_x^2}.$$

Combining the estimates for I and J, in view of (4.44), we can complete the proof. $\qquad\square$

For ease of exposition, let us introduce the first version of conformal energy and state its boundedness theorem whose proof occupies the rest of the paper.

Theorem 4.14 (Boundedness theorem). *Let ψ be a solution of $\Box_{\mathbf{g}}\psi = 0$ whose initial data is supported in $B_{\frac{1}{2}} \subset (\mathcal{D}_0^+ \cap \Sigma_{t_0})$. In the region $\mathcal{D}_0^+$,*

$$(4.49) \quad C[\psi](t) := \int_{\Sigma_t} \{t^2(|\nabla\!\!\!/\,\psi|^2 + |\nabla\!\!\!/_L\psi|^2) + u^2|\nabla\psi|^2 + (\frac{t^2}{(t-u)^2} + 1)\psi^2\}d\mu_g$$

there holds for $t \in [t_0, \tau_*]$, $C[\psi](t) \lesssim \|\psi[t_0]\|_{H^1}^2 + \|\psi(t_0)\|_{L^2(\Sigma)}^2$.

Lemma 4.15. *Let $q > 2$ and $0 < \delta \leq 1 - \frac{2}{q}$ be two numbers. Assuming (BA2) and*

$$(4.50) \qquad \|\varpi(\hat{\chi}, \nabla\!\!\!/\,\log\mathbf{b})\|_{L^2[0,\tau_*]L_x^\infty} \lesssim \lambda^{-\frac{1}{2}}, \quad with\ \tau_* \leq \lambda T,$$

for any solution ψ of $\Box_{\mathbf{g}}\psi = 0$, there holds

$$\|[P, \varpi N^m]\partial_m\psi\|_{L_x^\infty} + \|[\varpi\nabla\!\!\!/, P]\psi\|_{H^1} \lesssim \tilde{d}(t)(\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L_x^2}),$$

where

$$(1+t)^\delta \tilde{d}(t) \lesssim (1+t)^{-\frac{2}{q}} + d(t)$$

with $d(t)$ being a function satisfying (4.37).

The condition (4.50) is incorporated in (5.9) in Proposition 5.4 and is proved in Section 7.

Proof. We first claim for $t > t_0$ there hold

$$(4.51) \qquad \|\widehat{\nabla}(\varpi N)\|_{L_x^\infty} \lesssim \|\varpi(\hat{\chi}, \nabla\!\!\!/\,\log\mathbf{b}), k, \widehat{\nabla}g\|_{L^\infty} + (1+t)^{-1}$$

Indeed, for $t \geq t_0$, on the support of ϖ, i.e. $\cup_{-\frac{t}{4} \leq u \leq \frac{3t}{4}} S_{t,u}$, the radius r of $S_{t,u}$ within the support of ϖ satisfies $r \approx (1+t)$. (4.51) follows by using (4.41), (4.43) and the fact that $\|\widehat{\nabla}\varpi\|_{L_x^\infty} \lesssim (1+t)^{-1}$.

Let $\Pi_{ij} = g_{ij} - N_iN_j$ denote the projection tensor on Σ. Then for any scalar function f, we have $\nabla\!\!\!/_j f = \Pi_j^i \partial_i f$ and

$$(4.52) \qquad [P, \varpi\nabla\!\!\!/_j]f = -(\varpi N^iN_j)P\partial_i f + P((\varpi N_j N^i)\partial_i f) = [P, \varpi N_j N^i]\partial_i f.$$

Applying Lemma 4.13 to $(G, f) = (\varpi N^i, \psi), (\varpi N_j N^i, \psi)$, and using Proposition 3.7 for ψ, we have

$$\|[P, G]\partial_i f\|_{L_x^\infty} + \|[P, G]\partial_i f\|_{H^1} \lesssim \|\widehat{\nabla}G\|_{L_x^\infty}(\|\mathbf{D}\psi(t_0)\|_{L_x^2} + \|\psi(t_0)\|_{L_x^2}).$$

Now for $q > 2$, we set $\tilde{d}(t) = \|\widehat{\nabla}G\|_{L_x^\infty}$. We have from (4.51) that

$$\tilde{d}(t) \lesssim (1+t)^{-1} + \|\varpi(\hat{\chi}, \nabla\!\!\!/\,\log\mathbf{b}), \widehat{\nabla}g, k\|_{L_x^\infty} = (1+t)^{-1} + \tilde{d}^{(2)}(t).$$

By using (4.50) and Hölder inequality, we have

$$\|\tilde{d}^{(2)}(t)\|_{L^{\frac{q}{2}}} \lesssim \lambda^{\frac{2}{q}-1} T^{\frac{2}{q}-\frac{1}{2}}.$$

Thus, with $0 < \delta \leq 1 - \frac{2}{q}$ and $d(t) = (1+t)^\delta \tilde{d}^{(2)}(t)$, we can complete the proof. $\square$

Lemma 4.16. *(i) Let $S_t = \Sigma_t \cap \mathcal{N}^+(p)$, with $p \in [-\lambda T, \lambda T] \times \Sigma$. For S_t tangent tensor F, there holds*

$$(4.53) \qquad \|r^{1-2/q}F\|_{L^q(S_t)} \lesssim \|r\nabla\!\!\!/\,F\|_{L^2(S_t)}^{1-2/q}\|F\|_{L^2(S_t)}^{2/q} + \|F\|_{L^2(S_t)}. \quad 2 \leq q < \infty.$$

(ii) For any $\delta \in (0,1)$, any $q \in (2,\infty)$ and any scalar function f there hold

$$\sup_{S_{t,u}} |f| \lesssim r^{\frac{2\delta(q-2)}{2q+\delta(q-2)}} \left(\int_{S_{t,u}} \left(|\nabla\!\!\!\!/\, f|^2 + r^{-2}|f|^2 \right) \right)^{\frac{1}{2} - \frac{\delta q}{2q+\delta(q-2)}}$$

$$\times \left(\int_{S_{t,u}} \left(|\nabla\!\!\!\!/\, f|^q + r^{-q}|f|^q \right) \right)^{\frac{2\delta}{2q+\delta(q-2)}}.$$

Proof. This is [11, Theorem 5.2] $\qquad\square$

Now we are ready to complete the proof of Proposition 4.11.

Proof of Proposition 4.11. We first claim that

$$(4.54) \qquad \|\varpi P\psi\|_{L^\infty(\Sigma_t)} \lesssim \left(\frac{1}{(1+|t-t_0|)^{\frac{2}{q}}} + d(t) \right) (\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L^2}).$$

Since ϖ vanishes outside the region $-t/4 \leq u < 3t/4$, this claim is trivial there. Thus we may restrict our consideration to the region $-t/4 \leq u < 3t/4$. In view of $r \approx t - u$, we thus have $r \approx t$ for $t > 0$. Recall that ϖ is constant on each $S_{t,u}$, from Lemma 4.16 (ii), we can obtain

$$\sup_{S_{t,u}} |\varpi P\psi|^2 \lesssim r^\delta \left(\int_{S_{t,u}} \left(|\varpi \nabla\!\!\!\!/\, P\psi|^2 + r^{-2}|\varpi P\psi|^2 \right) \right)^{1-\delta}$$

$$\times \left(\int_{S_{t,u}} \left(|\varpi \nabla\!\!\!\!/\, P\psi|^4 + r^{-4}|\varpi P\psi|^4 \right) \right)^{\frac{1}{2}\delta}.$$

Applying Lemma 4.12 and the finite band property, we then obtain

$$\sup_{S_{t,u}} |\varpi P\psi|^2 \lesssim r^\delta \left(\int_{S_{t,u}} \left(|P(\varpi \nabla\!\!\!\!/\, \psi)|^2 + r^{-2}|\varpi P\psi|^2 + |[P,\varpi\nabla\!\!\!\!/\,]\psi|^2 \right) \right)^{1-\delta}$$

$$\times \left(\int_{S_{t,u}} \left(|P(\varpi \nabla\!\!\!\!/\, \psi)|^4 + r^{-4}|\varpi P\psi|^4 + |[P,\varpi\nabla\!\!\!\!/\,]\psi|^4 \right) \right)^{\frac{1}{2}\delta}$$

$$\lesssim r^\delta \left(t^{-2} C[\psi](t) + \|[P,\varpi\nabla\!\!\!\!/\,]\psi\|_{H^1}^2 \right)$$

By letting $0 < \delta \leq 2(1 - \frac{2}{q})$, (4.54) then follows from Theorem 4.14 and Lemma 4.15.

In order to derive the estimate on $P(e_0(\psi))$, we can write $\|P(e_0\psi)\|_{L_x^\infty} \leq \|P(\varpi e_0\psi)\|_{L_x^\infty} + \|P((\underline{\varpi} - \varpi)e_0\psi)\|_{L_x^\infty}$. By Bernstein inequality and Theorem 4.14, we have

$$\|P((\underline{\varpi} - \varpi)e_0\psi)\|_{L_x^\infty} \lesssim \|(\underline{\varpi} - \varpi)e_0\psi\|_{L_x^2} \lesssim (1+t)^{-1} C[\psi]^{\frac{1}{2}}(t)$$

$$\lesssim (1+t)^{-1} (\|\psi[t_0]\|_{H^1(\Sigma)} + \|\psi(t_0)\|_{L^2(\Sigma)})$$

and

$$\|P(\varpi e_0\psi)\|_{L_x^\infty} \lesssim \|P(\varpi L\psi)\|_{L_x^\infty} + \|P(\varpi N\psi)\|_{L_x^\infty} \lesssim \|\varpi L\psi\|_{L_x^2} + \|P(\varpi N\psi)\|_{L_x^\infty}.$$

From Theorem 4.14 we have

$$\|\varpi L\psi\|_{L_x^2} \lesssim (1+t)^{-1} C[\psi]^{\frac{1}{2}}(t) \lesssim (1+t)^{-1} (\|\psi[t_0]\|_{H^1(\Sigma)} + \|\psi(t_0)\|_{L_\Sigma^2}).$$

Moreover

(4.55) $\|P(\varpi N \psi)\|_{L_x^\infty} \le \|[P, \varpi N^l]\partial_l \psi\|_{L_x^\infty} + \|\varpi N^l P \partial_l \psi\|_{L_x^\infty}.$

The first term in (4.55) can be estimated by Lemma 4.15. By using (4.54), the second term in (4.55) can be estimated as

$$\|\varpi N^l P \partial_l \psi(t)\|_{L_x^\infty} \le \|\varpi \tilde{P} \psi(t)\|_{L_x^\infty} \lesssim \left((1+t)^{-\frac{2}{q}} + d(t)\right) \left(\|\psi[t_0]\|_{H^1} + \|\psi(t_0)\|_{L_\Sigma^2}\right),$$

where $\tilde{P}$ denotes a Littlewood Paley projection with frequency 1 associated to a different symbol. Putting the above estimates together completes the proof.   □

5. Boundedness theorem for Conformal energy

In this section we will present the proof of Theorem 4.14. We will work under the rescaled coordinates. Let $M_* = [0, \tau_*] \times \Sigma$, where $\tau_* \le \lambda T$, where $\lambda \ge \Lambda$ and Λ is a sufficiently large number.

Recall the definition of the optical function u. We will set $\underline{u} := 2t - u$ and introduce the Morawetz vector field $K := \frac{1}{2}n(u^2\underline{L} + \underline{u}^2 L)$. Associated to K we introduce the modified energy density

$$\bar{Q}(K, \mathbf{T}) = \bar{Q}[\psi](K, \mathbf{T}) = Q[\psi](K, \mathbf{T}) + 2t\psi \mathbf{T}\psi - \psi^2 \mathbf{T}(t),$$

and the total conformal energy

$$\bar{Q}[\psi](t) = \int_{\Sigma_t} \bar{Q}[\psi](K, \mathbf{T}).$$

Definition 5.1. *We define $\mathcal{C}[F]$ for a scalar function F with $suppF \subseteq \mathcal{D}_0^+$, by*

$$\mathcal{C}[F](t) = \mathcal{C}[F]^{(i)}(t) + \mathcal{C}[F]^{(e)}(t)$$

where $t_0 \le t \le \tau_*$, and

$$\mathcal{C}[F]^{(i)}(t) = \int_{\Sigma_t} (\underline{\varpi} - \varpi)\left(t^2|\boldsymbol{D}F|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)|F|^2\right)d\mu_g,$$

$$\mathcal{C}[F]^{(e)}(t) = \int_{\Sigma_t} \varpi\left(\underline{u}^2|\boldsymbol{D}_L F|^2 + u^2|\boldsymbol{D}_{\underline{L}} F|^2 + \underline{u}^2|\nabla\!\!\!\!/ F|^2 + |F|^2\right)d\mu_g.$$

From the definition it is easy to see that $\bar{Q}[\psi](t) \lesssim \mathcal{C}[\psi](t)$ and $C[\psi](t) \lesssim \mathcal{C}[\psi](t)$. We will prove the following results.

Theorem 5.2 (Comparison Theorem). *$T > 0$ can be chosen appropriately small but depending on universal constants, such that for any function ψ supported in $\mathcal{D}_0^+$ and any $1 \le t \le \tau_*$ there holds*

$$\mathcal{C}[\psi](t) \approx \bar{Q}[\psi](t).$$

Theorem 5.3 (Boundedness theorem). *There exists a large universal number Λ and a small universal number $T > 0$ such that for any $\lambda > \Lambda$, $\tau_* \le \lambda T$ and any function ψ satisfying the geometric wave equation*

(5.1) $\Box_{\mathbf{g}}\psi = \frac{1}{\sqrt{|\mathbf{g}|}}\partial_\alpha(\mathbf{g}^{\alpha\beta}\sqrt{|\mathbf{g}|}\partial_\beta \psi) = 0 \quad in \; [0, \tau_*] \times \mathbb{R}^3$

with initial data $\psi[t_0]$ supported on the ball $B_{1/2}(0)$ there holds

$$\bar{Q}[\psi](t) \lesssim \bar{Q}[\psi](t_0) \qquad \forall\, t_0 \le t \le \tau_*.$$

5.0.1. *Canonical null pair $L, \underline{L}$.* Recall that in (3.56) we have introduced along the null hypersurface C_u the canonical null frame $\{L, \underline{L}, e_1, e_2\}$, where $\{e_1, e_2\}$ is an orthonormal frame on $S_{t,u}$. Let $e_3 = \underline{L} = \mathbf{T} - N$ and $e_4 = L = \mathbf{T} + N$. Then from (3.56) it follows that

$$\mathbf{D}_3 u = 2\mathbf{b}^{-1}, \quad \mathbf{D}_3 \underline{u} = 2(n^{-1} - \mathbf{b}^{-1}), \quad \mathbf{D}_4 \underline{u} = 2n^{-1}, \quad \mathbf{D}_4 u = 0.$$

Associated to this canonical null frame, we define on each null cone $S_{t,u}$ the Ricci coefficients

$$\chi_{AB} = \langle \mathbf{D}_A e_4, e_B \rangle, \qquad \underline{\chi}_{AB} = \langle \mathbf{D}_A e_3, e_B \rangle$$

$$\zeta_A = \frac{1}{2} \langle \mathbf{D}_3 e_4, e_A \rangle, \qquad \underline{\zeta}_A = \frac{1}{2} \langle \mathbf{D}_4 e_3, e_A \rangle$$

$$\xi_A = \frac{1}{2} \langle \mathbf{D}_3 e_3, e_A \rangle.$$

It is well-known ([12]) that there hold the identities

$$\underline{\chi}_{AB} = -\chi_{AB} - 2k_{AB}, \qquad \underline{\zeta}_A = -k_{AN} + \slashed{\nabla}_A \log n,$$
$$\xi_A = k_{AN} - \zeta_A + \slashed{\nabla}_A \log n, \qquad \zeta_A = \slashed{\nabla}_A \log \mathbf{b} + k_{AN}.$$

and the frame equations

$$\mathbf{D}_4 e_4 = -(k_{NN} + \pi_{0N})e_4, \qquad \mathbf{D}_A e_4 = \chi_{AB}e_B - k_{AN}e_4,$$
$$\mathbf{D}_A e_3 = \underline{\chi}_{AB}e_B + k_{AN}e_3, \qquad \mathbf{D}_4 e_3 = 2\underline{\zeta}_A e_A + (k_{NN} + \pi_{0N})e_3,$$
$$\mathbf{D}_3 e_4 = 2\zeta_A e_A + (k_{NN} - \pi_{0N})e_4, \qquad \mathbf{D}_3 e_3 = 2\xi_A e_A - (k_{NN} - \pi_{0N})e_3,$$
$$\mathbf{D}_4 e_A = \slashed{\nabla}_4 e_A + \underline{\zeta}_A e_4, \qquad \mathbf{D}_3 e_A = \slashed{\nabla}_3 e_A + \zeta_A e_3 + \xi_A e_4.$$

Let

$$(5.2) \qquad z = \mathrm{tr}\chi - \frac{2}{n(t-u)}, \qquad \Omega = \frac{\mathbf{b}^{-1} - n^{-1}}{t-u}.$$

We rely on the following result to prove the boundedness theorem. (5.5)-(5.9) will be proved in Sections 6 and 7, and (5.3) is (4.43). (5.4) consists of (BA2) and the $L_t^2 L_x^\infty$ estimates on $e_0 n, \widehat{\nabla} n, \widehat{\nabla} Y$ that can be derived immediately by using (BA2), (2.4) and (2.25), under the rescaled coordinates.

Proposition 5.4. *Under the bootstrap assumptions (BA1), (BA2) and (BA3), on $\mathcal{D}^+ \subset [0, \tau_*] \times \Sigma$ there hold the estimates*

$$(5.3) \qquad (t-u)\mathrm{tr}\chi \approx 1$$

$$(5.4) \qquad \|\tilde{\pi}\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}$$

$$(5.5) \qquad \|\Omega\|_{L_t^2 L_x^\infty} + \|\Omega\|_{L^4(S_{t,u})} + \|(t-u)^{-1/2}\Omega\|_{L^2(S_{t,u})} \lesssim \lambda^{-1/2}$$

$$(5.6) \qquad \|z, \zeta\|_{L^4(S_{t,u})} + \|r^{-1/2}(z, \zeta, k)\|_{L^2(S_{t,u})} \lesssim \lambda^{-1/2}$$

$$(5.7) \qquad \|z\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2},$$

$$(5.8) \qquad \|r^{3/2}\slashed{\nabla} z, r^{3/2}\underline{L} z\|_{L_t^\infty L_u^\infty L_\omega^p} \lesssim \lambda^{-1/2}$$

$$(5.9) \qquad \|\hat{\chi}, \zeta\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2},$$

where $p > 2$ is such that $0 < 1 - 2/p < s - 2$.

Let $^{(K)}\pi$ denote the deformation tensor of K and let $^{(K)}\bar{\pi} := {}^{(K)}\pi - 4t\mathbf{g}$. Then we have

$$^{(K)}\bar{\pi}_{34} = -4un(\mathbf{b}^{-1} - n^{-1}) + nu^2(k_{NN} - \pi_{0N} - \mathbf{D}_3 \log n)$$
$$+ n\underline{u}^2(k_{NN} + \pi_{0N} - \mathbf{D}_4 \log n),$$
$$^{(K)}\bar{\pi}_{44} = -2u^2 n(\nabla_L \log n + k_{NN} + \pi_{0N}),$$
$$^{(K)}\bar{\pi}_{4A} = u^2 n(\underline{\zeta}_A - k_{AN} - \slashed{\nabla}_A \log n),$$
$$^{(K)}\bar{\pi}_{33} = -8\underline{u}n(n^{-1} - \mathbf{b}^{-1}) - 2n\underline{u}^2(k_{NN} - \pi_{0N} + \mathbf{D}_3 \log n),$$
$$^{(K)}\bar{\pi}_{3A} = n\underline{u}^2(\zeta_A + k_{AN} - \slashed{\nabla}_A \log n) + nu^2 \xi_A,$$
$$^{(K)}\bar{\pi}_{AB} = -2nu^2 \hat{k}_{AB} - nu^2 \mathrm{tr}k\delta_{AB} + 4tn(t-u)\hat{\chi}_{AB}$$
$$+ 2tn(t-u)\left(\mathrm{tr}\chi - \frac{2}{n(t-u)}\right)\delta_{AB}.$$

For simplicity of presentation, we will dropped the superscript K in $^{(K)}\bar{\pi}$. In view of (5.2), as an immediate consequence of Proposition 5.4, we have

Proposition 5.5. *Under the conditions in Proposition 5.4 we have on $[0, \tau_*] \times \Sigma$ that*

$$\|u^{-2} \bar{\pi}_{44}\|_{L_t^2 L_x^\infty} + \|(u\underline{u})^{-1} \bar{\pi}_{34}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2} \bar{\pi}_{33}\|_{L_t^2 L_x^\infty}$$
$$+ \|u^{-2} \bar{\pi}_{4A}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2} \bar{\pi}_{3A}\|_{L_t^2 L_x^\infty} + \|\underline{u}^{-2} \bar{\pi}_{AB}\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}.$$

Now we are ready to give the proof of Theorem 5.3 and Theorem 5.2.

5.1. **Proof of Theorem 5.3.** By calculating $\partial_t \bar{Q}[\psi]$ and integrating over the interval $[t_0, t]$, we have

$$\bar{Q}[\psi](t) = \bar{Q}[\psi](t_0) - \frac{1}{2}\mathcal{J}_1 + \mathcal{J}_2$$

where

$$\mathcal{J}_1 = \int_{[t_0,t] \times \Sigma} Q^{\alpha\beta}[\psi] \, {}^{(K)}\bar{\pi}_{\alpha\beta}, \qquad \mathcal{J}_2 = \int_{[t_0,t] \times \Sigma} \psi^2 \Box_{\mathbf{g}} t.$$

It is easy to see that

$$\mathcal{J}_1 = \int_{[t_0,t] \times \Sigma} \left(\frac{1}{4}\bar{\pi}_{33}(L\psi)^2 + \frac{1}{4}\bar{\pi}_{44}(\underline{L}\psi)^2 + \frac{1}{2}\bar{\pi}_{34}|\slashed{\nabla}\psi|^2 - \bar{\pi}_{4A}\underline{L}\psi\slashed{\nabla}_A\psi \right.$$
$$\left. - \bar{\pi}_{3A}L\psi\slashed{\nabla}_A\psi + \bar{\pi}_{AB}\slashed{\nabla}_A\psi\slashed{\nabla}_B\psi + \mathrm{tr}\bar{\pi}\left(\frac{1}{2}\underline{L}\psi L\psi - |\slashed{\nabla}\psi|^2\right)\right).$$

Observe that

$$\mathrm{tr}\bar{\pi} = \delta^{AB}\bar{\pi}_{AB} = 4tn(t-u)(\mathrm{tr}\chi - \frac{2}{n(t-u)}) - 2u^2 n\mathrm{tr}k.$$

It is easy to derive from (5.4) that

$$\int_{[t_0,t] \times \Sigma} |u^2 n\mathrm{tr}kL\psi\underline{L}\psi| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Thus, by letting

$$\mathcal{B} = \int_{[t_0,t] \times \Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right)\underline{L}\psi L\psi,$$

we have from Proposition 5.5 that

$$(5.10) \qquad |\mathcal{J}_1 - \mathcal{B}| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Since $\Box_{\mathbf{g}} t = -e_0(n^{-1}) + n^{-1}\mathrm{Tr}k$, we can conclude from (5.4) that

$$|\mathcal{J}_2| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t)$$

In the following we will show that

$$(5.11) \qquad |\mathcal{B}| \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t)$$

We can write $\mathcal{B} = \mathcal{B}^i + \mathcal{B}^e$, where

$$\mathcal{B}^e = \int_{[t_0,t] \times \Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right) L\psi \underline{L}\psi\varpi,$$

$$\mathcal{B}^i = \int_{[t_0,t] \times \Sigma} 2t'n(t'-u)\left(\mathrm{tr}\chi - \frac{2}{n(t'-u)}\right) L\psi \underline{L}\psi(\underline{\varpi} - \varpi).$$

In view of (5.7), we have

$$(5.12) \qquad \mathcal{B}^i \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

We still need to estimate $\mathcal{B}^e$. With the help of [12, p. 1162] and the integration by part we have

$$\frac{1}{2}\mathcal{B}^e = -I_1 + I_2 + I_3 - I_4,$$

where

$$I_1 = \int_{[t_0,t] \times \Sigma} \varpi nt'(t'-u)z(\underline{L}L\psi)\psi,$$

$$I_2 = \int_{[t_0,t] \times \Sigma} \left(-\underline{L}(\varpi nt'(t'-u)z) + (\mathrm{tr}\theta + N\log n - \mathrm{Tr}k - \operatorname{div}Y)\varpi nt'(t'-u)z\right)L\psi\psi,$$

$$I_3 = \int_{\Sigma_t} \varpi nt'(t'-u)zL\psi\psi,$$

$$I_4 = \int_{\Sigma_{t_0}} \varpi nt'(t'-u)zL\psi\psi.$$

Recall that in the exterior region $\{0 \le u \le 3t'/4\}$ we have $r(t',u) \approx t'$. Thus with the help of the Sobolev inequality (4.53) on $S_{t',u}$, for any function ψ there holds

$$(5.13) \qquad \int_{0 \le u \le 3t'/4} \|t'^{1-2/q}\psi\|^2_{L^q(S_{t',u})}du \lesssim \mathcal{C}[\psi](t'), \quad 2 \le q < \infty.$$

Therefore, by using (5.6) and (5.13), for the boundary term I_3 and I_4 we have the estimate

$$|I_3| + |I_4| \lesssim \|tL\psi\|_{L^\infty_t L^2_\Sigma}\|r^{\frac{1}{2}}\psi\|_{L^\infty_t L^2_u L^4_x} \sup_{t,0<u<\frac{3t}{4}} \|r^{\frac{1}{2}}z\|_{L^4(S_{t,u})} \lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

Now we consider I_2. We write $I_2 = I_2^{(1)} + I_2^{(2)}$, where

$$I_2^{(1)} = \int_{[t_0,t]\times\Sigma} \underline{L}(\varpi nt'(t'-u)z)L\psi\psi,$$

$$I_2^{(2)} = \int_{[t_0,t]\times\Sigma} (\mathrm{tr}\theta + N\log n - \mathrm{Tr}k - \mathrm{div}\,Y)\varpi nt'(t'-u)zL\psi\psi.$$

In view of (5.3), (5.4) and (5.7) in Proposition 5.4, by Hölder inequality we have

$$|I_2^{(2)}| \lesssim (\tau_*\|\pi,\nabla Y,z\|_{L_t^2 L_x^\infty} + \tau_*^{\frac{1}{2}})\|z\|_{L_t^2 L_x^\infty}\sup_t \mathcal{C}[\psi](t) \lesssim T^{\frac{1}{2}}\sup_t \mathcal{C}[\psi](t).$$

Observe that

(5.14) $$|\underline{L}\varpi| \lesssim r^{-1}, \quad \underline{L}t = n^{-1}, \quad \underline{L}u = 2\mathbf{b}^{-1}.$$

Let $p > 2$ be close to 2 such that $0 < 1 - 2/p < s - 2$ and let $1/p + 1/q = 1/2$. Then it follows from (5.8) that

$$|I_2^{(1)}| \lesssim \left(\|(|(t'-u)\underline{L}\varpi|+1)z\|_{L_t^1 L_x^\infty} + \tau_*\|\underline{L}n\|_{L_t^2 L_x^\infty}\|z\|_{L_t^2 L_x^\infty}\right)\|t'L\psi\|_{L_t^\infty L_\Sigma^2}\|\psi\|_{L_t^\infty L_\Sigma^2}$$

$$+ \|t'^{\frac{2}{q}}\varpi\underline{L}z\|_{L_t^1 L_u^\infty L_x^p}\sup_{t'}\left(\int_{0\le u\le\frac{3t'}{4}}\|t'^{1-\frac{2}{q}}\psi\|^2_{L^q(S_{t',u})}du\right)^{\frac{1}{2}}\|t'L\psi\|_{L_t^\infty L_\Sigma^2}.$$

In view of (5.13), (5.4), (5.7) and (5.8), we obtain

$$|I_2^{(1)}| \lesssim T^{\frac{1}{2}}\sup_t \mathcal{C}[\psi](t)$$

Finally we will use (5.1) to estimate I_1. We first rewrite (5.1) as

(5.15) $$\Box_{\mathbf{g}}\psi = -\underline{L}L\psi + \slashed{\Delta}\psi + 2\zeta_A\slashed{\nabla}_A\psi - \frac{1}{2}\mathrm{tr}\chi\underline{L}\psi - (\frac{1}{2}\mathrm{tr}\underline{\chi}+\nu)L\psi.$$

Then I_1 can be written as $I_1 = I_{11} + I_{12} + I_{13}$, where

$$I_{11} = \int_{[t_0,t]\times\Sigma} \varpi nt'(t'-u)z\slashed{\Delta}\psi\psi$$

$$I_{12} = -\frac{1}{2}\int_{[t_0,t]\times\Sigma} \varpi nt'(t'-u)z\mathrm{tr}\chi\psi\underline{L}\psi$$

$$I_{13} = \int_{[t_0,t]\times\Sigma} \varpi nt'(t'-u)z\left(2\zeta_A\slashed{\nabla}_A\psi - \left(\frac{1}{2}\mathrm{tr}\underline{\chi}+\nu\right)L\psi\right)\psi.$$

Recall that for any vector field X tangent to $S_{t,u}$ there holds

(5.16) $$\int_{\Sigma_t} F\,\mathrm{div}\,X = -\int_{\Sigma_t} \{\slashed{\nabla} + (\zeta+\underline{\zeta})\}F\cdot X.$$

In view of (5.16) we have

$$I_{11} = -\int_{[t_0,t]\times\Sigma} \slashed{\nabla}(\varpi nt'(t'-u)z\psi)\slashed{\nabla}\psi + (\zeta+\underline{\zeta})\varpi nt'(t'-u)z\psi\cdot\slashed{\nabla}\psi.$$

Now we introduce the following types of terms:

$$\mathrm{Er}_1 = nt'(t'-u)\left(\pi, z, \frac{\mathbf{b}^{-1}-n^{-1}}{(t-u)}\right)\cdot(L\psi,\slashed{\nabla}\psi)\cdot z,$$

$$\mathrm{Er}_2 = nzt'L\psi, \qquad\qquad \mathrm{Er}_3 = nt'(t'-u)\slashed{\nabla}z\cdot\slashed{\nabla}\psi,$$

$$\mathrm{Er}_4 = nt'z((t'-u)\slashed{\nabla}\psi,\psi), \qquad \mathrm{Er}_5 = nt'(t'-u)\zeta\cdot\slashed{\nabla}\psi\cdot z.$$

Then, symbolically we can write I_{11} and I_{13} as

$$|I_{11}| + |I_{13}| = \left| \int_{[t_0,t]\times\Sigma} \varpi(\mathrm{Er}_1 + \mathrm{Er}_2 + \mathrm{Er}_3 + \mathrm{Er}_5)\psi \right| + \left| \int_{[t_0,t]\times\Sigma} \varpi\mathrm{Er}_4 \cdot \slashed{\nabla}\psi \right|.$$

By using (5.4), (5.7), (5.5) and Hölder inequality, we can derive

$$\left| \int_{[t_0,t]\times\Sigma} |\varpi\mathrm{Er}_1 \cdot \psi| \right| \lesssim \|z\|_{L_t^2 L_x^\infty} \left\| z, \pi, \frac{\mathbf{b}^{-1} - n^{-1}}{n(t'-u)} \right\|_{L_t^2 L_x^\infty} \tau_* \sup_t \mathcal{C}[\psi](t)$$

$$\lesssim T \sup_t \mathcal{C}[\psi](t)$$

and

$$\left| \int_{[t_0,t]\times\Sigma} |\varpi\mathrm{Er}_2\psi| + |\varpi\mathrm{Er}_4\slashed{\nabla}\psi| \right|$$

$$\lesssim \|z\|_{L_t^1 L_x^\infty} \sup_t \left\{ \|t(|L\psi| + |\slashed{\nabla}\psi|)\|_{L_\Sigma^2} (\|\psi\|_{L_\Sigma^2} + \|t\slashed{\nabla}\psi\|_{L_\Sigma^2}) \right\}$$

$$\lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

By using (5.8) and (5.13) with $0 < 1 - 2/p < s - 2$ and $1/q + 1/p = \frac{1}{2}$, we have

$$\left| \int_{[t_0,t]\times\Sigma} |\varpi\mathrm{Er}_3 \cdot \psi| \right|$$

$$\lesssim \sup_t \|t^{\frac{2}{q}} \varpi\slashed{\nabla}z\|_{L_t^1 L_u^\infty L_x^p} \sup_{t'} \left\{ \mathcal{C}[\psi](t')^{\frac{1}{2}} \left(\int_{0 \leq u \leq \frac{3t'}{4}} \|t'^{1-\frac{2}{q}}\psi\|_{L^q(S_{t',u})}^2 du \right)^{\frac{1}{2}} \right\}$$

$$\lesssim T^{\frac{1}{2}} \sup_t \mathcal{C}[\psi](t).$$

With the help of (5.7), (5.13) with $q = 4$ and (5.6), we obtain

(5.17)

$$\left| \int_{[t_0,t]\times\Sigma} |\varpi\mathrm{Er}_5\psi| \right|$$

$$\lesssim \int_{t_0}^t \sup_u \|r^{\frac{1}{2}}\varpi z \cdot \zeta\|_{L_x^4} \cdot \sup_{t'} \left\{ \|t'\slashed{\nabla}\psi\|_{L_\Sigma^2} \left(\int_{0 \leq u \leq \frac{3t'}{4}} \|r'^{\frac{1}{2}}\psi\|_{L^4(S_{t',u})}^2 du \right)^{\frac{1}{2}} \right\}$$

$$\lesssim \|r^{\frac{1}{2}}\zeta\|_{L_t^\infty L_u^\infty L_x^4} \|z\|_{L_t^1 L_x^\infty} \sup_t \mathcal{C}[\psi](t) \lesssim T \sup_t \mathcal{C}[\psi](t).$$

Now we consider I_{12} by integration by part.

(5.18) $\qquad I_{12} = \dfrac{1}{4} \displaystyle\int_{[t_0,t]\times\Sigma} -\underline{L}(\varpi nt'(t'-u)z\mathrm{tr}\chi)\psi^2$

(5.19) $\qquad + \dfrac{1}{4} \displaystyle\int_{[t_0,t]\times\Sigma} (\mathrm{tr}\theta + N\log n - \mathrm{Tr}k - \mathrm{div}\,Y)\,\varpi nt'(t'-u)z\mathrm{tr}\chi\,\psi^2$

(5.20) $\qquad + \dfrac{1}{4} \displaystyle\int_{\Sigma_t} \varpi nt'(t'-u)z\mathrm{tr}\chi\psi^2 - \dfrac{1}{4} \displaystyle\int_{\Sigma_0} \varpi nt'(t'-u)z\mathrm{tr}\chi\psi^2.$

Using (5.3), (5.4) and (5.7), the terms in (5.19) is bounded by

$$\int_{[t_0,t]\times\Sigma} \left|\varpi(z+\pi+\nabla Y+\frac{1}{n(t-u)})z\right| t'n\psi^2$$

$$\lesssim \|z\|_{L_t^2 L_x^\infty}\||\psi|^2\|_{L_t^\infty L_\Sigma^1}(\tau_*^{1/2}+\tau_*\|\pi,\nabla Y,z\|_{L_t^2 L_x^\infty})$$

$$\lesssim T^{\frac{1}{2}}\sup_t \mathcal{C}[\psi](t).$$

For the terms in (5.20), using (5.3), (5.6) and (5.13) with $q=4$, they can be bounded by

$$\sup_{t,u}\|z\|_{L^2(S_{t,u})}\int_{0<u\le\frac{3t}{4}}\||t'|\psi|^2\|_{L^2(S_{t,u})}\lesssim T^{\frac{1}{2}}\sup_t\mathcal{C}[\psi](t).$$

It remains only to consider the terms in (5.18). We have

$$\left|\int_{[t_0,t]\times\Sigma}\underline{L}(\varpi nt'(t'-u)z\mathrm{tr}\chi)\psi^2\right|$$

$$\lesssim |\int_{[t_0,t]\times\Sigma}\{\underline{L}\varpi nt'(t'-u)\mathrm{tr}\chi+\varpi n\mathrm{tr}\chi\underline{L}(t'(t'-u))$$

(5.21) $$+\varpi\mathrm{tr}\chi\underline{L}nt'(t'-u)+\varpi\underline{L}(\frac{1}{n(t'-u)})nt'(t'-u)\}z\psi^2|$$

(5.22) $$+\int_{[t_0,t]\times\Sigma}|\varpi\underline{L}znt'(t'-u)|(|\mathrm{tr}\chi|+|z|)\psi^2.$$

Since $r\approx(t-u)$ and $t(|\mathrm{tr}\chi|+|z|)\approx 1$ on Ext_t, with $\frac{1}{p}+\frac{1}{q}=1$ and p is slightly greater than 2, also using Hölder inequality, we obtain

$$(5.22)\lesssim\|\varpi r^{1-\frac{2}{p}}\underline{L}z\|_{L_t^1 L_u^\infty L_x^p}\sup_{t'}(\int_{0\le u\le\frac{3t'}{4}}\|r^{\frac{2}{p}}|\psi|^2\|_{L^q(S_{t',u})}du)$$

$$\lesssim\|\varpi r^{1-\frac{2}{p}}\underline{L}z\|_{L_t^1 L_u^\infty L_x^p}\sup_{t'}(\int_{0\le u\le\frac{3t'}{4}}\|r^{1-\frac{2}{q_1}}\psi\|_{L^{q_1}(S_{t',u})}\|r^{1-\frac{2}{q_2}}\psi\|_{L^{q_2}(S_{t',u})}du)$$

where $\frac{1}{q_1}+\frac{1}{q_2}=\frac{1}{q}$. Using (5.13) and (5.8) we obtain

$$(5.22)\lesssim T^{\frac{1}{2}}\sup_{t'}\mathcal{C}[\psi](t').$$

To estimate (5.21), in view of (5.14) and

(5.23) $$\underline{L}(\frac{1}{n(t-u)})=\frac{n\mathbf{b}^{-1}}{n^2(t-u)^2}-\frac{\underline{L}\log n}{n(t-u)}-\frac{n(n^{-1}-\mathbf{b}^{-1})}{n^2(t-u)^2},$$

using (5.7), (5.4) and (5.5), we have

$$|(5.21)|\lesssim\|z\psi^2\|_{L_t^1 L_\Sigma^1}+\|\underline{L}n,\frac{\mathbf{b}^{-1}-n^{-1}}{n(t-u)}\|_{L_t^2 L_\Sigma^\infty}\|z\|_{L_t^2 L_\Sigma^\infty}\tau_*\sup_t\int_{0\le u\le\frac{3t}{4}}\|\psi^2\|_{L^1(S_{t,u})}$$

$$\lesssim T^{\frac{1}{2}}\sup_t\mathcal{C}[\psi](t).$$

The proof is therefore complete.

5.2. **Proof of Comparison theorem.** We will adapt the argument in [12] to prove Theorem 5.2. For simplicity, we use Θ to denote any term from the collection

$$\left\{ \text{tr}\chi - \frac{2}{n(t-u)}, \ \text{Tr}k, \ \frac{\mathbf{b}^{-1} - n^{-1}}{t-u}, \ \hat{k}_{NN} \right\}.$$

According to (5.6) and (5.5) in Proposition 5.4 we have

(5.24) $$\|r^{-\frac{1}{2}}\Theta\|_{L^2(S_{t,u})} \lesssim \lambda^{-\frac{1}{2}}.$$

By following the argument in [12, Section 6] we can derive

$$\bar{Q}[\psi](t) \gtrsim \int_{\Sigma} \left(\underline{u}^2 (L\psi)^2 + u^2 (\underline{L}\psi)^2 + (u^2 + \underline{u}^2)|\overline{\nabla}\psi|^2 + \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2 \right)$$

$$- \int_{\Sigma} \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2(t-u)\Theta$$

$$\gtrsim \mathcal{C}[\psi](t) - \int_{\Sigma} \left(1 + \frac{t^2}{(t-u)^2}\right)\psi^2(t-u)\Theta.$$

By using (5.24) and the inequality, which can be derived in view of (4.53),

$$\|t\psi\|_{L_u^2 L_\omega^4} \lesssim \|t\overline{\nabla}\psi\|_{L_x^2} + \|\frac{t}{t-u}\psi\|_{L_x^2} \lesssim \mathcal{C}[\psi](t),$$

we can obtain

$$\int_{\Sigma} \frac{t^2}{t-u}\Theta\psi^2 \lesssim \|r^{\frac{1}{2}}\Theta\|_{L_u^\infty L_\omega^2}\|r^{\frac{1}{2}}t^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim \lambda^{-\frac{1}{2}}\tau_*^{\frac{1}{2}}\|t^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

Similarly we have

$$\int_{\Sigma} \psi^2(t-u)\Theta \lesssim \lambda^{-\frac{1}{2}}\tau_*^{\frac{1}{2}}\|r^2\psi^2\|_{L_u^1 L_\omega^2} \lesssim T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

Therefore, there is a universal constant $C_0 > 0$ such that

$$\mathcal{C}[\psi](t) \leq C_0\bar{Q}[\psi](t) + C_0 T^{\frac{1}{2}}\mathcal{C}[\psi](t).$$

This implies the desired conclusion by taking T to be small universal constant.

6. **Estimates for Ricci coefficients**

In Sections 6 and 7, we will complete the proof of Proposition 5.4. We will consider only the case $\lambda = 1$ since the general case can be obtained by a simple rescaling. Thus, we will work on the spacetime $[0, T] \times \Sigma$.

Consider an outgoing null cone C_u in $[0, T] \times \Sigma$ initiating from a point p on Γ^+ with $u = t(p)$. Let $\mathcal{D}^+$ be the domain enclosed by C_u and Σ_T and let $S_{t,u} := C_u \cap \Sigma_t$. For simplicity, we will supress u and write $S_t := S_{t,u}$. Instead of the canonical null pair $\{L, \underline{L}\}$, we will work under a new pair of null vector fields $\{L', \underline{L}'\}$ which is defined as follows. The null vectors l'_ω in $T_p\mathbf{M}$ parametrized with $\omega \in \mathbb{S}^2$ are normalized by $\mathbf{g}(l'_\omega, \mathbf{T}_p) = -1$. We denote by $\Gamma_\omega(s)$ the outgoing null geodesic from p with $\Gamma_\omega(0) = p$ and $\frac{d}{ds}\Gamma_\omega(0) = l'_\omega$ and define the null vector field L' by

$$L'(\Gamma_\omega(s)) = \frac{d}{ds}\Gamma_\omega(s).$$

Then $\mathbf{D}_{L'}L' = 0$. The affine parameter s of null geodesic is chosen such that $s(p) = 0$ and $L'(s) = 1$. We will assume that the exponential map $\mathcal{G}_t : \omega \to \Gamma_\omega(s(t))$ is a global diffeomorphism from $\mathbb{S}^2$ to S_t for any $t \in (u, T]$. Along C_u we introduce

the null lapse function $a^{-1} = -\mathbf{g}(L', \mathbf{T})$. Then $a > 0$ on C_u with $a(p) = 1$. Moreover, along any null geodesic Γ_ω there holds

$$(6.1) \qquad \frac{dt}{ds} = n^{-1}a^{-1}, \quad t(p) = 0.$$

We can define a conjugate null vector $\underline{L}'$ on C_u with $\mathbf{g}(L', \underline{L}') = -2$ and such that $\underline{L}'$ is orthogonal to the leafs S_t. Let N be the outward unit normal of S_t in Σ_t. Then in terms of L', $\underline{L}'$ and a we have

$$(6.2) \qquad L' = a^{-1}(\mathbf{T} + N), \qquad \underline{L}' = a(\mathbf{T} - N).$$

Relative to the canonical null frame $L, \underline{L}$, we recall the null components of the Riemannian curvature tensor $\mathbf{R}$ as follows

$$
\begin{aligned}
&\alpha_{AB} = \mathbf{R}(L, e_A, L, e_B), && \beta_A = \frac{1}{2}\mathbf{R}(e_A, L, \underline{L}, L), \\
(6.3) \qquad &\rho = \frac{1}{4}\mathbf{R}(\underline{L}, L, \underline{L}, L), && \sigma = \frac{1}{4}{}^\star\mathbf{R}(\underline{L}, L, \underline{L}, L), \\
&\underline{\beta}_A = \frac{1}{2}\mathbf{R}(e_A, \underline{L}, \underline{L}, L), && \underline{\alpha}_{AB} = \mathbf{R}(\underline{L}, e_A, \underline{L}, e_B).
\end{aligned}
$$

and $\alpha', \beta', \rho', \sigma', \underline{\beta}', \underline{\alpha}'$ can be similarly defined with $L, \underline{L}$ replaced by $L', \underline{L}'$. We define

$$\mathcal{R}(C_u)^2 = \int_{C_u} \left(|\alpha|^2 + |\beta|^2 + |\underline{\beta}|^2 + |\rho|^2 + |\sigma|^2 \right).$$

It follows from Proposition 3.10 and the standard flux type estimate by using the Bel-Robinson tensor and Einstein vacuum equation,

$$(6.4) \qquad \|\,|\nabla_L k, \slashed{\nabla} k|_g\|_{L^2(C_u)} + \mathcal{R}(C_u) \leq C.$$

Let us denote by $\hat{k}$ the traceless part of k, $\hat{k} = k - \frac{1}{3}\mathrm{Tr}k \cdot g$. We decompose $\hat{k}$ on each S_t by introducing components

$$(6.5) \qquad \eta_{AB} = \hat{k}_{AB}, \qquad \epsilon_A = \hat{k}_{AN}, \qquad \delta = \hat{k}_{NN}$$

where $(e_A)_{A=1,2}$ is an orthonormal frame on S_t. Let $\hat{\eta}_{AB}$ be the traceless part of η. Since $\delta^{AB}\eta_{AB} = -\delta$, we have $\hat{\eta}_{AB} = \eta_{AB} + \frac{1}{2}\delta_{AB}\delta$. The Ricci coefficients $\chi', \underline{\chi}', \zeta', \underline{\zeta}', \xi', \mu'$ and null components of curvature associated with the null frame $\{e_1, e_2, L', \underline{L}'\}$ are related to their counterparts associated with $\{e_1, e_2, L, \underline{L}\}$ as follows:

$$\mathbf{b} = an_p, \quad \slashed{\nabla}\log \mathbf{b} = \slashed{\nabla}\log a, \quad \chi' = a^{-1}\chi, \quad \underline{\chi}' = a\underline{\chi}, \quad \zeta' = \zeta,$$

$$\underline{\zeta}' = \underline{\zeta}, \quad \xi' = a^2\xi, \quad -2\mu' = \mu := \mathbf{D}_3\mathrm{tr}\chi - \frac{1}{2}(\mathrm{tr}\chi)^2 - (k_{NN} + \nabla_N \log n)\mathrm{tr}\chi,$$

$$\alpha' = a^{-2}\alpha, \quad \beta' = a^{-1}\beta, \quad \rho' = \rho, \quad \sigma' = \sigma, \quad \underline{\beta}' = a\underline{\beta}$$

In the following we will consider the Ricci coefficients relative to the null frame $\{e_1, e_2, L', \underline{L}'\}$, and we will drop the prime for convenience. We will also fix the following conventions:

- $\slashed{\pi}$ denotes the collection of $\hat{\eta}$, ϵ, δ, $\nabla_N \log n$, $\slashed{\nabla}\log n$, $\mathrm{Tr}k$,
- $\check{\pi}$ denotes the collection of $\slashed{\pi}$, $\widehat{\nabla}g$,
- $\iota := \mathrm{tr}\chi - \frac{2}{r}, V := \mathrm{tr}\chi - \frac{2}{s}, \kappa := \mathrm{tr}\chi - (an)^{-1}\overline{an\mathrm{tr}\chi}$,
- A denotes the collection of $\hat{\chi}, \zeta, \underline{\zeta}, \nu = -\frac{da}{ds} = \pi_{0N} + \delta + \frac{\mathrm{Tr}k}{3}$,
- $\underline{A}$ denotes the collection of A and $\hat{\underline{\chi}}, \slashed{\nabla}\log a, \slashed{\pi}$,

- The pair of quantities $(M, \mathcal{D}_0 M)$ denotes either $(\nabla\!\!\!\!/\, \mathrm{tr}\chi, \nabla\!\!\!\!/\, \hat{\chi})$, or $(\mu, \nabla\!\!\!\!/\, \zeta)$,
- R_0 denotes the collection of $\alpha, \beta, \rho, \sigma, \underline{\beta}$,
- $\bar{R}$ denotes the collection of $R_0, \mathrm{tr}\chi\underline{A}, \bar{A} \cdot \underline{A}$,
- $S := S_t, \;\mathring{\gamma} := r^{-2}\gamma, \; \gamma^{(0)} := r^2 \gamma_{\mathbb{S}^2}$.

We define the norm of $\mathcal{N}_1[\cdot]$ on null cone C_u for S tangent tensor fields F by

$$(6.6) \qquad \mathcal{N}_1[F] = \|r^{-1}F\|_{L^2(C_u)} + \|\nabla\!\!\!\!/\, F\|_{L^2(C_u)} + \|\nabla\!\!\!\!/_L F\|_{L^2(C_u)}.$$

We first recall the following results that hold under the bounded H^2 data condition. The proofs can be found in can be found in [25, 26] and rely on the bounded flux (6.4) and (2.2)-(2.4).

Theorem 6.1. *There exists a universal constant $C_1 \geq 1$ such that on C_u there holds*

$$(6.7) \quad \mathcal{C}_0 := \mathcal{R}(C_u) + \mathcal{N}_1[\sharp] + \|r^{-\frac{1}{2}}\boldsymbol{D}n\|_{L^2(S_t)} + \|\boldsymbol{D}n\|_{L^4(S_t)} + \|\boldsymbol{D}n\|_{L^2_t L^\infty_\omega} \leq C_1.$$

Moreover, let $\mathcal{R}_0 := \mathcal{R}(C_u) + \mathcal{N}_1[\sharp]$, there exists a universal constant $\delta_0 > 0$ such that if $T - u \leq \delta_0$ then on C_u there hold

$$(6.8) \qquad \left\| \mathrm{tr}\chi - \frac{2}{s} \right\|_{L^\infty} \lesssim \mathcal{R}_0^2,$$

$$(6.9) \qquad |a - 1| \leq \frac{1}{2},$$

$$(6.10) \quad \left\| \int_0^1 (|\hat{\chi}|^2 + |\zeta|^2 + |\nu|^2 + |\underline{\zeta}|^2) na\, dt \right\|_{L^\infty_\omega} \lesssim \mathcal{R}_0^2,$$

$$(6.11) \quad \|\nabla\!\!\!\!/\, \mathrm{tr}\chi\|_{L^2(C_u)} + \|\mu\|_{L^2(C_u)} + \left\| \sup_{t \leq 1} |r^{3/2} \nabla\!\!\!\!/\, \mathrm{tr}\chi| \right\|_{L^2_\omega} + \left\| \sup_{t \leq 1} r^{\frac{3}{2}} |\mu| \right\|_{L^2_\omega} \lesssim \mathcal{R}_0$$

$$(6.12) \quad \mathcal{N}_1(\hat{\chi}) + \mathcal{N}_1(\zeta) + \mathcal{N}_1\left(\mathrm{tr}\chi - \frac{2}{r}\right) + \mathcal{N}_1\left(\mathrm{tr}\chi - (an)^{-1}\overline{antr\chi}\right) \lesssim \mathcal{R}_0,$$

$$(6.13) \quad \left\| \mathrm{tr}\chi - \frac{2}{r} \right\|_{L^2_t L^\infty_\omega} + \left\| \mathrm{tr}\chi - (an)^{-1}\overline{antr\chi} \right\|_{L^2_t L^\infty_\omega} \lesssim \mathcal{R}_0.$$

Proposition 6.2. *On each S_t we introduce the ratio of area elements*

$$(6.14) \qquad v_t(\omega) := \frac{\sqrt{|\gamma_t|}}{\sqrt{|\gamma_{\mathbb{S}^2}|}}, \qquad \omega \in \mathbb{S}^2.$$

Then there hold

$$(6.15) \qquad C^{-1} \leq v_t/s^2 \leq C, \qquad C^{-1} < r/s < C,$$

where C is a positive universal constant.

Finally we recall ([25, 26]) that for any Σ-tangent tensor field F there holds the trace inequality

$$(6.16) \qquad \|r^{-\frac{1}{2}}F\|_{L^2(S_{t,u})} + \|F\|_{L^4(S_{t,u})} \lesssim \|F\|_{H^1(\Sigma)},$$

and for any $-\frac{t}{4} \leq u \leq t$ and any Σ-tangent tensor field F, there holds

$$(6.17) \qquad \|F\|^2_{L^2(S_{t,u})} \lesssim (\|k\|_{L^3(\mathrm{Int}(S_{t,u}))} + 1)\|F\|_{H^1(\Sigma)}\|F\|_{L^2(\mathrm{Int}(S_{t,u}))}.$$

where $Int(S_{t,u})$ denotes the domain in Σ_t enclosed by $S_{t,u}$.

We recall also the transport Sobolev-type inequality for any S_t-tangent tensor field F,

$$(6.18) \qquad \|r^{\frac{1}{2}}F\|_{L^p_\omega L^\infty_t} \lesssim \|r^{-\frac{1}{2}}F\|_{L^2_x L^\infty_t(C_u)} + \|F\|_{L^4_x L^\infty_t(C_u)} \lesssim \mathcal{N}_1[F]$$

with $2 \leq p \leq 4$. Similar to [25, Proposition 8.1] and also using (6.16), we have for any Σ-tangent tensor field F that

$$\|s^{-1}F\|_{L^2(C_u)} \lesssim \|s^{-1/2}F\|_{L^2(S_T)} + \|\slashed{\nabla}_L F\|_{L^2(C_u)} + \|F\|_{L^2(C_u)}$$
$$(6.19) \qquad\qquad \lesssim \|F\|_{H^1(\Sigma_T)} + \|\slashed{\nabla}_L F\|_{L^2(C_u)},$$

As a immediate consequence of (6.18), (6.7) and (6.12), on C_u there holds for $F = \hat{\chi}, \zeta, \check{\pi}$ that

$$(6.20) \qquad\qquad \|r^{\frac{1}{2}}F\|_{L^p_\omega L^\infty_t} \lesssim \mathcal{N}_1[F] \leq C, \text{ with } 2 \leq p \leq 4.$$

Lemma 6.3. *For $q > 2$, there hold the following Sobolev inequalities on C_u:*

$$(6.21) \qquad \|F\|_{L^2_t L^q_\omega} \lesssim (\|\slashed{\nabla}F\|_{L^2(C_u)}^{1-\frac{2}{q}} + \|r^{-1}F\|_{L^2(C_u)}^{1-\frac{2}{q}})\|r^{-1}F\|_{L^2(C_u)}^{\frac{2}{q}}$$

$$(6.22) \qquad \|F\|_{L^\infty_\omega} \lesssim \|r\slashed{\nabla}F\|_{L^q_\omega} + \|F\|_{L^q_\omega}$$

Lemma 6.4. *Let $0 \leq 1 - \frac{2}{p} < s - 2$. For any Σ tangent tensor F there holds*

$$\|r^{1-\frac{2}{p}}|\nabla_L F, \slashed{\nabla}F|_g\|_{L^2_t L^p_x(C_u)} \lesssim \sum_i \|r^{1-\frac{2}{p}}(\slashed{\nabla}_L(F_i), \slashed{\nabla}(F_i))\|_{L^2_t L^p_x(C_u)} + \|F\|_{L^\infty_t H^1}.$$

Indeed, we can check for Σ tangent tensor F_{ij} the following symbolic identities,

$$\nabla_L F_i = \nabla_L(F_i) + \mathrm{err}(F), \qquad \slashed{\nabla}F_i = \slashed{\nabla}(F_i) + \mathrm{err}(F)$$

where $|\mathrm{err}(F)| \leq |\mathbf{g} \cdot (\widehat{\nabla}g, \widehat{\nabla}Y, k) \cdot F|$, which gives

$$\|r^{1-\frac{2}{p}}|\nabla_L F, \slashed{\nabla}F|_g\|_{L^2_t L^p_x(C_u)} \lesssim \sum_i \|r^{1-\frac{2}{p}}(\slashed{\nabla}_L(F_i), \slashed{\nabla}(F_i))\|_{L^2_t L^p_x(C_u)}$$
$$+ (T-u)^{\frac{1}{2}}\|\check{\pi}\|_{L^2_t L^\infty_x}\|r^{\frac{1}{2}-\frac{2}{p}}F\|_{L^\infty_t L^p(C_u)}$$

where to derive the last inequality, we employed (6.16).

Lemma 6.5. *Let f be the components of the Σ-tangent tensor fields $\nabla^2 n$, $\nabla(ne_0(n))$, and $\widehat{\nabla}^2 Y$. Then for $2 \leq q < 4$ there holds*

$$(6.23) \qquad\qquad \|r^{1-\frac{2}{q}}f\|_{L^2_t L^q_x(C_u)} \leq C.$$

For the Σ tangent tensor $\pi_0 = -\nabla \log n$, there holds [3]

$$(6.24) \qquad\qquad \|r^{1-\frac{2}{q}}(\nabla_L \pi_0, \nabla \pi_0)\|_{L^2_t L^q_x(C_u)} \leq C.$$

Proof. We decompose a scalar function f by $f = \sum_{\mu>1} P_\mu f + P_{\leq 1} f$. By the Sobolev inequality (4.53) on $S_{t,u}$ we have

$$r^{1-\frac{2}{q}}\|P_\mu f\|_{L^q_x} \lesssim \|r\slashed{\nabla}P_\mu f\|_{L^2_x}^{1-\frac{2}{q}}\|P_\mu f\|_{L^2_x}^{\frac{2}{q}} + \|P_\mu f\|_{L^2_x}.$$

---

[3]We can obtain stronger estimates than (6.24), but this estimate is sufficient for our purpose.

Using the finite band property of LP projections and (6.17), we derive that

$$\|r^{1-\frac{2}{q}}P_\mu f\|_{L_t^2 L_x^q} \lesssim (\mu^{1-\frac{2}{q}}+1)\|P_\mu f\|_{L^2(C_u)}$$
$$\lesssim (\mu^{1-\frac{2}{q}}+1)\|P_\mu f\|_{L_t^2 H^1([0,T]\times\Sigma)}^{\frac{1}{2}}\|P_\mu f\|_{L_t^2 L^2([0,T]\times\Sigma)}^{\frac{1}{2}}$$
$$\lesssim \mu^{-\frac{1}{2}}(\mu^{1-\frac{2}{q}}+1)\|f\|_{L_t^2 H^1([0,T]\times\Sigma)}.$$

The lower order part $P_{\leq 1}f$ can be treated similarly. Therefore summing over $\mu > 1$, we obtain

$$\|r^{1-\frac{2}{q}}f\|_{L_t^2 L_x^q(C_u)} \lesssim \|f\|_{L_t^2 H^1([0,T]\times\Sigma)}.$$

The estimate (6.23) then follows by using (2.3) and (2.24).

We next prove (6.24). It is straightforward to derive that $\nabla\pi_0 = n^{-1}\nabla^2 n + \pi_0\cdot\pi_0$. Then using (6.23) and (6.16)

$$\|r^{1-\frac{2}{q}}\nabla\pi_0\|_{L_t^2 L_x^q(C_u)} \lesssim \|r^{1-\frac{2}{q}}\nabla^2 n\|_{L_t^2 L_x^q(C_u)} + \|\pi_0\|_{L_t^2 L_x^\infty}\|r^{\frac{1}{2}-\frac{2}{q}}\pi_0\|_{L_t^\infty L_x^q} \lesssim 1.$$

Note that we have the following symbolic identity

$$\nabla_{n\mathbf{T}}\nabla_i \log n = \nabla_i \nabla_{n\mathbf{T}} \log n + nk\cdot\nabla\log n.$$

In view of $L = \mathbf{T} + N$, and using (6.23), we can derive

$$\|r^{1-\frac{2}{q}}\nabla_L\pi_0\|_{L_t^2 L_x^q(C_u)} \lesssim \|r^{1-\frac{2}{q}}(\nabla(ne_0(\log n)),\nabla\pi_0)\|_{L_t^2 L_x^q(C_u)}$$
$$+ \|\pi\|_{L_t^2 L_x^\infty}\|r^{\frac{1}{2}-\frac{2}{q}}\pi\|_{L_t^\infty L_x^q(C_u)} \lesssim 1.$$

Hence we complete the proof. $\qquad\square$

6.1. L^p flux type estimates.

In this subsection, we will use Proposition 3.11 to derive L^p flux type estimates on $\widehat{\nabla}g, k$ and null components of curvature $R_0 = (\alpha, \beta, \rho, \sigma, \underline{\beta})$. Let $0 < 1 - \frac{2}{p} < s - 2$. For any S_t-tangent tensor F along null hypersurface C_u we define the Sobolev norm

$$\mathcal{N}_{1,p}[F] = \|r^{-\frac{2}{p}}F\|_{L_t^2 L_x^p} + \|r^{1-\frac{2}{p}}\slashed{\nabla}F\|_{L_t^2 L_x^p} + \|r^{1-\frac{2}{p}}\nabla_L F\|_{L_t^2 L_x^p}.$$

We denote by

$$\mathcal{N}_{1,p}[\check{\pi}] = \mathcal{N}_{1,p}[\slashed{k}] + \mathcal{N}_{1,p}[\widehat{\nabla}g],$$

where $\widehat{\nabla}g$ in the last term denotes the scalar components of $\widehat{\nabla}g$ relative to arbitrary time independent coordinate frame in Σ. When $p = 2$, $\mathcal{N}_{1,2}[F] = \mathcal{N}_1[F]$. We have from Proposition 3.11 together with $\mathcal{N}_1[\slashed{k}] \leq C$ in Theorem 6.1 that

$$\mathcal{N}_1[\check{\pi}](C_u) \leq C. \tag{6.25}$$

We now state the main result of this subsection.

Proposition 6.6. *Let f be the components of k and $\widehat{\nabla}g$. Then on C_u there hold*

$$\|r^{1-\frac{2}{p}}(\slashed{\nabla}_L f, \slashed{\nabla}f)\|_{L_t^2 L_x^p(C_u)} \leq C, \tag{6.26}$$

$$\mathcal{N}_{1,p}[\check{\pi}] \leq C, \tag{6.27}$$

$$\|r^{1-\frac{2}{p}}(\alpha, \beta, \rho, \sigma, \beta, \underline{\beta})\|_{L_t^2 L_x^p(C_u)} \leq C, \tag{6.28}$$

where $0 < 1 - \frac{2}{p} < s - 2$.

As the first step, in what follows we build a connection between the dyadic flux in Proposition 3.11 and the ones that will be used to derive (6.26).

Proposition 6.7. *Let $\not{P}_\mu$ be the classical Littlewood-Paley decomposition on 2-dimensional slice $S_{t,u}$ on C_u. Let $\mathcal{D}_t = an\frac{d}{ds}$ along each null geodesic Γ_ω on C_u. Then for any $0 < \epsilon < 1$ and any scalar function f there hold*

$$\sum_{\mu>1} \|\mu^\epsilon \not{P}_\mu \not{\nabla} f\|_{L^2(C_u)}^2 \lesssim \sum_{\ell>1} \ell^{2\epsilon} \left(\mathcal{F}[P_\ell f](C_u) + \|P_\ell f\|_{H^1(\Sigma_T)}^2 \right)$$

(6.29)
$$+ \mathcal{F}[f] + \|f\|_{H^1(\Sigma_T)}^2,$$

$$\sum_{\mu>1} \|\mu^\epsilon \not{P}_\mu \mathcal{D}_t f\|_{L^2(C_u)}^2 \lesssim \sum_{\ell>1} \ell^{2\epsilon} \left(\mathcal{F}[P_\ell f](C_u) + \|P_\ell f\|_{H^1(\Sigma_T)}^2 \right)$$

(6.30)
$$+ \mathcal{F}[f] + \|f\|_{H^1(\Sigma_T)}^2.$$

Proof. We first prove (6.29) and (6.30) by assuming the following inequality, with $\ell > 1$ being dyadic

$$(6.31) \quad \|r\not{\nabla}\not{\nabla} P_\ell f\|_{L^2(C_u)} + \|r\not{\nabla}\mathcal{D}_t P_\ell f\|_{L^2(C_u)} \lesssim \ell \left(\mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f] + \|\tilde{P}_\ell f\|_{H^1(\Sigma_T)} \right).$$

We decompose $f = \sum_{l>1} P_\ell f + P_{\leq 1} f$. First we have

$$\mu^\epsilon \left\| \not{P}_\mu \not{\nabla} \sum_{\ell>\mu} P_\ell f \right\|_{L^2(C_u)} \lesssim \sum_{\ell>\mu} \left(\frac{\mu}{\ell} \right)^\epsilon \ell^\epsilon \|\not{\nabla} P_\ell f\|_{L^2(C_u)}.$$

Summing over $\mu > 1$, we obtain

$$(6.32) \quad \sum_{\mu>1} \mu^{2\epsilon} \left\| \not{P}_\mu \not{\nabla} \sum_{\ell>\mu} P_\ell f \right\|_{L^2(C_u)}^2 \lesssim \sum_{\ell} \ell^{2\epsilon} \|\not{\nabla} P_\ell f\|_{L^2(C_u)}^2.$$

Now we consider the part $\sum_{1<\ell<\mu} P_\ell f$ in the decomposition of f. The term $P_{\leq 1} f$ can be treated similarly. By the finite band property of $\not{P}_\mu$, we have

$$\|\not{P}_\mu \not{\nabla} P_\ell f\|_{L^2(C_u)} \lesssim \mu^{-1} \|r\not{\nabla}\not{\nabla} P_\ell f\|_{L^2(C_u)}.$$

In view of (6.31), it follows that

$$\|\not{P}_\mu \not{\nabla} P_\ell f\|_{L^2(C_u)} \lesssim \mu^{-1}\ell \left(\mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f] + \|\tilde{P}_\ell f\|_{H^1(\Sigma_{t_M})} \right).$$

Hence we have

$$\mu^\epsilon \left\| \not{P}_\mu \not{\nabla} \sum_{\ell\leq\mu} P_\ell f \right\|_{L^2(C_u)} \lesssim \sum_{\ell\leq\mu} \left(\frac{\ell}{\mu} \right)^{1-\epsilon} \ell^\epsilon \left(\mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f] + \|\tilde{P}_\ell f\|_{H^1(\Sigma_{t_M})} \right).$$

Taking l_μ^2 and also combining with (6.32) gives (6.29).

To see (6.30), we first derive by the basic property of $\not{P}_\mu$ that

$$\mu^\epsilon \left\| \not{P}_\mu \mathcal{D}_t \sum_{\ell>\mu} P_\ell f \right\|_{L^2(C_u)} \lesssim \sum_{\ell>1} \ell^\epsilon \left(\frac{\mu}{\ell} \right)^\epsilon \|\mathcal{D}_t P_\ell f\|_{L^2(C_u)}.$$

For the case $\ell \leq \mu$, we obtain by the finite band property of $\not{P}_\mu$ and (6.31) that

$$\mu^\epsilon \left\| \not{P}_\mu \mathcal{D}_t \sum_{1 < \ell \leq \mu} P_\ell f \right\|_{L^2(C_u)} \lesssim \sum_{1 < \ell \leq \mu} \mu^{\epsilon-1} \| r \not{\nabla} \mathcal{D}_t P_\ell f \|_{L^2(C_u)}$$

$$\lesssim \sum_{1 < \ell \leq \mu} \left(\frac{\ell}{\mu} \right)^{1-\epsilon} \ell^\epsilon \left(\mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f](C_u) + \|\tilde{P}_\ell f\|_{H^1(\Sigma_{t_M})} \right).$$

Taking l^2_μ for $\mu > 1$, we can obtain (6.30).

We next prove (6.31). Recall that $\Pi_{mn} = g_{mn} - N_m N_n$ and $\not{\nabla}_m = \Pi_m^{n'} \widehat{\nabla}_{n'}$. Note that for functions f on C_u we have

$$\not{\nabla} \not{\nabla} P_\ell f = \not{\nabla}(\Pi_m^n \widehat{\nabla}_n P_\ell f) = \not{\nabla}(\delta_m^n - N_m \cdot N^n) \widehat{\nabla} P_\ell f + \Pi_m^n \not{\nabla} \widehat{\nabla} P_\ell f$$

$$= -\not{\nabla} N_m \cdot N^n \widehat{\nabla}_n P_l f - N_m \not{\nabla} N^n \widehat{\nabla}_n P_l f + \Pi_m^n \not{\nabla} \widehat{\nabla} P_\ell f.$$

Therefore

$$|\not{\nabla} \not{\nabla} P_\ell f|_\gamma \leq \ell r^{-1} \left(|(\mathrm{tr}\theta + \hat{\theta}) \cdot \tilde{P}_\ell f|_\gamma + |\Pi_m^n \not{\nabla} \tilde{P}_\ell f|_\gamma \right),$$

where $\tilde{P}$ denote a classical Littlewood-Paley projection on $\mathbb{R}^3$ induced by a different symbol. By the commutation formula $[\not{\nabla}, \mathcal{D}_t]f = an\chi \cdot \not{\nabla} f$ for scalar functions f, we obtain

$$\not{\nabla} \mathcal{D}_t P_\ell f = [\not{\nabla}, \mathcal{D}_t] P_\ell f + \mathcal{D}_t \not{\nabla} P_\ell f = an\chi \not{\nabla} P_\ell f + \mathcal{D}_t \not{\nabla} P_\ell f.$$

For the last term on the right, we have

$$\mathcal{D}_t \not{\nabla} P_\ell f = \mathcal{D}_t(\Pi_m^n \widehat{\nabla}_n P_\ell f) = \mathcal{D}_t(\Pi_m^n) \widehat{\nabla}_n P_\ell f + \Pi_m^n \mathcal{D}_t \widehat{\nabla}_n P_\ell f$$

$$= \mathcal{D}_t(\delta_m^n - N_m N^n) \widehat{\nabla}_n P_\ell f + \ell \Pi_m^n \mathcal{D}_t \tilde{P}_\ell f.$$

Thus, in view of $\mathbf{D}_{L'} N^\rho = a^{-1} L'(a) \mathbf{T}^\rho - a^{-1} \underline{\zeta}_A e_A^\rho$ and $\mathcal{D}_t = an\mathbf{D}_{L'}$, we have

$$|\not{\nabla} \mathcal{D}_t P_\ell f|_\gamma \lesssim \ell r^{-1} \left(|\underline{\zeta} \tilde{P}_\ell f|_\gamma + |\mathcal{D}_t \tilde{P}_\ell f|_\gamma + |(\mathrm{tr}\chi + \hat{\chi}) \cdot \tilde{P}_\ell f|_\gamma \right).$$

With the help of the symbolic identity $\mathrm{tr}\theta = \mathrm{tr}\chi + k$ and $\hat{\theta} = \hat{\chi} + k$, we obtain

$$\|r \not{\nabla}^2 P_\ell f\|_{L^2(C_u)} + \|r \not{\nabla} \mathcal{D}_t P_\ell f\|_{L^2(C_u)}$$

$$\lesssim \ell \left(\|\not{\nabla} \tilde{P}_\ell f\|_{L^2(C_u)} + \|\mathcal{D}_t \tilde{P}_\ell f\|_{L^2(C_u)} + \|(\mathrm{tr}\chi + \hat{\chi} + \check{\pi}) \tilde{P}_\ell f\|_{L^2(C_u)} \right)$$

$$(6.33) \qquad \lesssim \ell \left(\mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f] + \|r^{-1} \tilde{P}_\ell f\|_{L^2(C_u)} + \|r^{-\frac{1}{2}} \tilde{P}_\ell f\|_{L^2_t L^4_x(C_u)} \right)$$

where we employed the Hölder inequality, (6.20) and $r\mathrm{tr}\chi \approx 1$. By the Sobolev embedding, we deduce that

$$\|r^{-\frac{1}{2}} \tilde{P}_\ell f\|_{L^2_t L^4_x(C_u)} \lesssim \|\not{\nabla} \tilde{P}_\ell f\|_{L^2(C_u)}^{\frac{1}{2}} \|r^{-1} \tilde{P}_\ell f\|_{L^2(C_u)}^{\frac{1}{2}} + \|r^{-1} \tilde{P}_\ell f\|_{L^2(C_u)}.$$

Applying (6.19) to $F = \tilde{P}_\ell f$ gives

$$(6.34) \qquad \|r^{-1} \tilde{P}_\ell f\|_{L^2(C_u)} \lesssim \|\not{\nabla}_L \tilde{P}_\ell f\|_{L^2(C_u)} + \|\tilde{P}_\ell f\|_{H^1(\Sigma_{t_M})},$$

Therefore

$$(6.35) \qquad \|r^{-\frac{1}{2}} \tilde{P}_\ell f\|_{L^2_t L^4_x(C_u)} \lesssim \mathcal{F}^{\frac{1}{2}}[\tilde{P}_\ell f](C_u) + \|\tilde{P}_\ell f\|_{H^1(\Sigma_{t_M})}.$$

In view of (6.33), (6.34) and (6.35), we thus obtain (6.31). $\qquad \square$

Note that the energy estimate on the Bel-Robinson tensor does not implies the $L^p, p > 2$, type curvature flux in (6.28). As the second step, we now build the connection between (6.27) and (6.28).

Proposition 6.8. *Let err denote the type of error terms taking the form $(\check{\pi} + F) \cdot \check{\pi} + tr\chi\check{\pi}$ with $F = \hat{\chi}, \slashed{\nabla} \log a, \hat{\theta}$. Relative to canonical null frame $\{e_1.e_2, L, \underline{L}\}$, there hold the following results for null components of curvature:*

(a) *There exist scalar functions $\overset{\circ}{\pi}$, S-tangent 2-tensor fields $\overset{\circ}{\pi}_{AB}$ such that the components $\beta, \underline{\beta}$ admit the decompositions*

$$\beta, \underline{\beta} = \slashed{\nabla}_A \overset{\circ}{\pi} + \slashed{\nabla}^B \overset{\circ}{\pi}_{AB} + c\slashed{u}rl \overset{\circ}{\pi} + err.$$

(b) *There exist two S-tangent vector fields $\overset{\circ}{\pi}_A^{(1)}$ and $\overset{\circ}{\pi}_A^{(2)}$ such that*

$$\rho = div \overset{\circ}{\pi}^{(1)} + err, \quad \sigma = curl \overset{\circ}{\pi}^{(2)} + err.$$

(c) *There holds the decomposition $\alpha = \slashed{\nabla}_L \hat{\eta} + \slashed{\nabla}\slashed{\eta} + err$.*

where $\overset{\circ}{\pi}, \overset{\circ}{\pi}^{(1)}$, and $\overset{\circ}{\pi}^{(2)}$ denote the terms formed by the sums of $N^\mu (g \cdot \widehat{\nabla} g, k)_{\mu \cdots}$ and $\Pi_\mu^{\mu'} (g \cdot \widehat{\nabla} g, k)_{\mu' \cdots}$.

As a key feature required by the proof of (5.9) (see Section 7.2), the higher order terms in the decompositions for β, ρ and σ can only contain $\slashed{\nabla}\check{\pi}$. Since time derivative of shift has to be excluded in our decompositions, Proposition 6.8 does not follow from the approach given in [13, Section 4]. We will derive in Appendix I the decompositions in a more invariant fashion.

Lemma 6.9. *Let err be the error term in Proposition 6.8. Let $0 \le 1 - 2/p < s - 2$. There holds*

$$\|r \cdot err\|_{L_t^2 L_\omega^p(C_u)} \lesssim (T - u)^{\frac{1}{2}} + 1.$$

Proof. It follows from (6.25), $rtr\chi \approx 1$ and (6.21) that $\|rtr\chi\check{\pi}\|_{L_t^2 L_\omega^p(C_u)} \lesssim \mathcal{N}_1[\check{\pi}] \le C$. In view of (6.16) and (6.18), we can obtain

$$\begin{cases} \|r\check{\pi} \cdot \check{\pi}\|_{L_\omega^p} \lesssim \|\check{\pi}\|_{L_x^\infty} \|r\check{\pi}\|_{L_\omega^p} \lesssim (t - u)^{\frac{1}{2}} \|\check{\pi}\|_{H^1(\Sigma)} \|\check{\pi}\|_{L_x^\infty} \\ \|rF \cdot \check{\pi}\|_{L_\omega^p} \lesssim (t - u)^{\frac{1}{2}} \mathcal{N}_1[F] \|\check{\pi}\|_{L_x^\infty} \end{cases}$$

where $F = \hat{\chi}, \slashed{\nabla} \log a, \hat{\theta}$. By integrating in t and using (BA2), $\|\check{\pi}\|_{H^1(\Sigma)} \le C$ and (6.20) we can complete the proof. $\qquad\square$

Proof of Proposition 6.6. First, applying (4.53) to $F = \slashed{P}_\mu \slashed{\nabla} f$ gives

$$\|r^{1 - \frac{2}{p}} \slashed{\nabla} f\|_{L_t^2 L_x^p(C_u)} \lesssim \sum_{\mu > 1} (\mu^{1 - \frac{2}{p}} + 1) \|\slashed{P}_\mu \slashed{\nabla} f\|_{L^2(C_u)} + \|\slashed{\nabla} f\|_{L^2(C_u)}$$

With the help of (6.29), we then obtain

$$\|r^{1 - \frac{2}{p}} \slashed{\nabla} f\|_{L_t^2 L_x^p(C_u)} \lesssim \mathcal{F}^{\frac{1}{2}}[f](C_u) + \sum_{\ell > 1} \ell^{1 - \frac{2}{p}} \left(\mathcal{F}^{\frac{1}{2}}[P_\ell f](C_u) + \|P_\ell f\|_{H^1(\Sigma_T)} \right).$$

On the other hand, applying (4.53) to $F = \slashed{P}_\mu \mathcal{D}_t f$ and using (6.30) we can obtain

$$\|r^{1 - \frac{2}{p}} \mathcal{D}_t f\|_{L_t^2 L_x^p(C_u)} \lesssim \mathcal{F}^{\frac{1}{2}}[f](C_u) + \sum_{\ell > 1} \ell^{1 - \frac{2}{p}} \left(\mathcal{F}^{\frac{1}{2}}[P_\ell f](C_u) + \|P_\ell f\|_{H^1(\Sigma_T)} \right).$$

Thus, applying the above two inequalities with $f = k_{ij}, \widehat{\nabla} g_{ij}$, and using

$$\|\mu^{1-\frac{2}{p}} P_\mu f\|_{l^1_\mu H^1_\Sigma} \le \|f\|_{H^{1+\epsilon}} \text{ with } 1 - \frac{2}{p} < \epsilon < s - 2$$

and Proposition 3.11, we have

$$\|\mu^{1-\frac{2}{p}} r(\slashed{\nabla}, \nabla_L) P_\mu f\|_{l^1_\mu L^2_t L^p_\omega(C_u)} \le C.$$

This completes the proof of (6.26).

We next prove (6.27). Let $\slashed{\nabla}_*$ denote either $\slashed{\nabla}_A$ or $\slashed{\nabla}_{L'}$ for S-tangent tensor, and ∇_* be either ∇_L or $\slashed{\nabla}_A$ for Σ tangent tensor as in (3.58). We recall from [25, (8.14), (8.15)] that with π being Σ tangent tensor π_0 or k,

(6.36) $\qquad \slashed{\nabla}_* \slashed{\pi} = \nabla_* \pi + \mathrm{tr}\theta \cdot \slashed{\pi} + (\hat{\theta}, \underline{\zeta}) \cdot \slashed{\pi} + (an)^{-1} = \nabla_* \pi + \mathrm{err}.$

By using (6.26) and (6.24), we have $\||r^{1-\frac{2}{p}} \nabla_* \pi|_g\|_{L^p(C_u)} \le C$, which together with (6.36) implies that,

$$\|r^{1-\frac{2}{p}} \slashed{\nabla}_* \slashed{\pi}\|_{L^2_t L^p(C_u)} \lesssim \||r^{1-\frac{2}{p}} \nabla_* \pi|_g\|_{L^p(C_u)} + 1 + (T-u)^{\frac{1}{2}} \le C,$$

where to derive the last inequality we employed Lemma 6.9. Hence we obtain (6.27).

In view of the decompositions in Proposition 6.8, we have $R_0 = \slashed{\nabla}_* \slashed{\pi} + g \cdot \slashed{\nabla}(\widehat{\nabla} g) + \mathrm{err}$. By using (6.27) and Lemma 6.9, we can obtain (6.28). $\qquad\square$

6.2. L^p estimates for Ricci coefficients.

We fix a null cone C_u contained in $\mathcal{D}^+$ with $t = u$ at its vertex on Γ^+. By rescaling the space-time coordinate as $(t, x) \to (\frac{t-u}{d}, \frac{x}{d})$ with $d = (T - u)$, we may restrict our consideration to the time interval $[0, 1]$. We denote by $\mathcal{H}$, the null cone C_u after change of coordinates. Then $\mathcal{H} = \cup_{0 < t \le 1} S_t$ with S_t being the intersection of $\mathcal{H}$ and Σ_t. We also denote $\mathcal{H}_t := \cup_{t' \in [0,t]} S_{t'}$ for $0 < t \le 1$.

We will rely on the quantities

(6.37) $$\mathcal{R}_{0,p} := \mathcal{N}_{1,p}[\check{\pi}] + \|r R_0\|_{L^2_t L^p_\omega(\mathcal{H})}$$

and

(6.38) $$\mathcal{R}_0 := \mathcal{R}(\mathcal{H}) + \mathcal{N}_1[\slashed{\pi}].$$

We will prove an L^p version of weakly spherical property on S_t in Proposition 6.20 and the following estimates

Proposition 6.10. *Let M denote either μ or $\slashed{\nabla} \mathrm{tr}\chi$. If $\mathcal{R}_{0,p} + \mathcal{R}_0$ is sufficiently small, there holds on $\mathcal{H}$*

(6.39) $\mathcal{N}_{1,p}(\underline{A}) + \|\underline{A}\|_{L^2_t L^\infty_\omega} + \|r^{\frac{1}{2}} \underline{A}\|_{L^{2p}_\omega L^\infty_t} + \|r^{\frac{3}{2}} M\|_{L^p_\omega L^\infty_t} + \|r M\|_{L^2_t L^p_\omega} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0.$

We remark the the smallness on $\mathcal{R}_0 + \mathcal{R}_{0,p}$ can be obtained by requiring $T - u$ sufficiently small but depends only on universal constants.

Let us record the structure equations for the Ricci coefficients relative to $\{e_1, e_2, \underline{L}', L'\}$:

(6.40) $\dfrac{d\mathrm{tr}\chi}{ds} + \dfrac{1}{2}(\mathrm{tr}\chi)^2 = -|\hat{\chi}|^2$

(6.41) $\dfrac{d}{ds}\hat{\chi} + \mathrm{tr}\chi\hat{\chi} = -\alpha$

(6.42) $\dfrac{d}{ds}\zeta = -\chi \cdot \zeta + \chi \cdot \underline{\zeta} - \beta$

(6.43) $\dfrac{d}{ds}\slashed{\nabla}\mathrm{tr}\chi + \dfrac{3}{2}\mathrm{tr}\chi\slashed{\nabla}\mathrm{tr}\chi = -\hat{\chi}\cdot\slashed{\nabla}\mathrm{tr}\chi - 2\hat{\chi}\cdot\slashed{\nabla}\hat{\chi} - (\zeta + \underline{\zeta})(|\hat{\chi}|^2 + \dfrac{1}{2}(\mathrm{tr}\chi)^2)$

(6.44) $\mathrm{div}\,\hat{\chi} = \dfrac{1}{2}\slashed{\nabla}\mathrm{tr}\chi + \dfrac{1}{2}\mathrm{tr}\chi\cdot\zeta - \hat{\chi}\cdot\zeta - \beta$

(6.45) $\mathrm{div}\,\zeta = -\mu - \rho + \dfrac{1}{2}\hat{\chi}\cdot\underline{\hat{\chi}} - |\zeta|^2 + \dfrac{1}{2}a\delta\mathrm{tr}\chi - \dfrac{1}{3}a\mathrm{Tr}k\mathrm{tr}\chi$

(6.46) $\mathrm{curl}\,\zeta = \sigma - \dfrac{1}{2}\hat{\chi}\wedge\underline{\hat{\chi}}.$

Let us record the following transport equation for μ' where $'$ will be dropped for convenience,

$$L\mu + \mathrm{tr}\chi\mu = 2\hat{\chi}\cdot\slashed{\nabla}\zeta + (\zeta - \underline{\zeta})\cdot(\slashed{\nabla}\mathrm{tr}\chi + \mathrm{tr}\chi\zeta) - \dfrac{1}{2}\mathrm{tr}\chi(\hat{\chi}\cdot\underline{\hat{\chi}} - 2\rho + 2\underline{\zeta}\cdot\zeta)$$

(6.47) $$+ 2\zeta\cdot\hat{\chi}\cdot\zeta + \left(\dfrac{1}{4}a^2\mathrm{tr}\chi + a(-\delta + \dfrac{2}{3}\mathrm{Tr}k)\right)|\hat{\chi}|^2 - \dfrac{1}{2}a\nu(\mathrm{tr}\chi)^2$$

Recall the following result on initial data (see [24]).

Lemma 6.11. *There holds*

$$V, \slashed{\nabla}a, r\slashed{\nabla}\mathrm{tr}\chi, r^2\mu \to 0 \ \text{as}\ t \to 0, \quad \lim_{t\to 0}\|\hat{\chi}, \zeta, \underline{\zeta}, \nu\|_{L^\infty(S_t)} < \infty.$$

Let us recall the transport lemma (see [13, 25]).

Lemma 6.12. *For an S tangent tensor field F verifying*

$$\slashed{\nabla}_L F + \dfrac{m}{2}\mathrm{tr}\chi F = G \cdot F + H$$

with $m \geq 1$ certain integer, if $\lim_{t\to 0} r(t)^m F = 0$ and $\|G\|_{L^\infty_\omega L^2_t} \leq C$, then there holds

(6.48) $$|F| \lesssim v_t^{-\frac{m}{2}} \int_0^t v_{t'}^{\frac{m}{2}} |H|na\,dt'.$$

We also need the following useful Sobolev inequality.

Lemma 6.13. *For any $q \geq 2$ and any S_t tangent tensor field F, there holds the estimate*

(6.49) $$\|r^{\frac{1}{2}-\frac{1}{q}}F\|^2_{L^{2q}_x L^\infty_t(\mathcal{H})} \lesssim \left(\|r\slashed{\nabla}_L F\|_{L^q_\omega L^2_t} + \|F\|_{L^q_\omega L^2_t}\right)\|F\|_{L^\infty_\omega L^2_t}.$$

Proof. The estimate with $q = 2$ is proved in [25, Lemma 8.2]; the same argument can be used to the result for any $q > 2$. $\square$

We now prove the following result on null cone $\mathcal{H}$,

Proposition 6.14. *For any $p \geq 2$ there hold*

(6.50) $$\|r\slashed{\nabla}_L(\hat{\chi}, \zeta)\|_{L^2_t L^p_\omega} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0$$

(6.51) $$\|\underline{A}\|_{L^2_t L^p_\omega} + \|r\nabla_L\underline{A}\|_{L^2_t L^p_\omega} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0$$

Proof. By the Sobolev embedding (6.21) we have for $p > 2$ that

$$\|\underline{A}\|_{L_t^2 L_\omega^p} \lesssim \mathcal{N}_1[\underline{A}] \lesssim \mathcal{R}_0.$$

The second inequality in (6.51) will follow from (6.50) and (6.37). We need only to prove (6.50). In view of (6.41) and (6.37), we have

$$\|r\nabla_L \hat{\chi}\|_{L_t^2 L_\omega^p} \lesssim \|\hat{\chi}\|_{L_t^2 L_\omega^p} + \|r\alpha\|_{L_t^2 L_\omega^p} \lesssim \mathcal{R}_{0,p}.$$

Let p'' be such that $\frac{1}{p} = \frac{1}{4} + \frac{1}{p''}$. By using (6.21) we have

(6.52) $$\|rA \cdot \underline{A}\|_{L_t^2 L_\omega^p} \lesssim \|r^{\frac{1}{2}} A\|_{L_t^\infty L_\omega^4} \|r^{\frac{1}{2}} \underline{A}\|_{L_t^2 L_\omega^{p''}} \lesssim \mathcal{N}_1(A)\mathcal{N}_1(r^{\frac{1}{2}} \underline{A}) \lesssim \mathcal{R}_0^2.$$

In view of (6.42), (6.52), (6.37) and the first part of (6.51), we obtain

$$\|r\nabla_L \zeta\|_{L_t^2 L_\omega^p} \lesssim \|\zeta\|_{L_t^2 L_\omega^p} + \|r\hat{\chi} \cdot \underline{\zeta}\|_{L_t^2 L_\omega^p} + \|\underline{\zeta}\|_{L_t^2 L_\omega^p} + \|r\beta\|_{L_t^2 L_\omega^p} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0^2.$$

The proof is thus complete. $\qquad\square$

6.2.1. *Weakly spherical condition.* Let γ be the induced metric on S_t, and define the rescaled metric $\overset{\circ}{\gamma}$ on S_t by $\overset{\circ}{\gamma} = r^{-2}\gamma$. Note that

(6.53) $$\lim_{t \to 0} \overset{\circ}{\gamma}_{ij} = \gamma_{\mathbb{S}^2 ij}, \qquad \lim_{t \to 0} \partial_k \overset{\circ}{\gamma}_{ij} = \partial_k \gamma_{\mathbb{S}^2 ij}$$

where $i, j, k = 1, 2$. Recall that we have proved in [26] that

(WS1) $$\|\overset{\circ}{\gamma}_{ij}(t) - \gamma_{\mathbb{S}^2 ij}\|_{L^\infty} \lesssim \mathcal{R}_0$$

(6.54) $$\|\partial_k \overset{\circ}{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\|_{L_\omega^2 L_t^\infty} \lesssim \mathcal{R}_0.$$

Now we make the following bootstrap assumption which will be improved at the end of this section.

Assumption 6.15. *For the transport local coordinates (t, ω), the following properties hold true for all surfaces S_t of the time foliation on the null cone $\mathcal{H}$: the metric $\overset{\circ}{\gamma}_{ij}(t)$ on each S_t verifies weakly spherical conditions i.e.*

$$\|\partial_k \overset{\circ}{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\|_{L_\omega^p L_t^\infty} \le \Delta_0 \text{ with } 0 < 1 - \frac{2}{p} < s - 2,$$

where we assume $0 < \Delta_0 < \frac{1}{2}$.

Let $\overset{\circ}{\nabla}^{(0)}$ be the covariant differentiation relative to $\gamma_{ij}^{(0)}$. We will improve (WS2) with the right side replaced by $\mathcal{R}_0 + \mathcal{R}_{0,p}$. By choosing $\Delta_0 < \mathcal{R}_{0,p} + \mathcal{R}_0$ sufficiently small, we can close the bootstrap argument.

Lemma 6.16. *Under the Assumption 6.15, for $2 < q \le p$,*

$$\|\overset{\circ}{\nabla} F\|_{L_x^q} + \|r^{-1} F\|_{L_x^q} \approx \|\overset{\circ}{\nabla}^{(0)} F\|_{L_x^q} + \|r^{-1} F\|_{L_x^q}.$$

Proof. It suffices to consider F to be 1-form. Relative to any coordinates on S, $\overset{\circ}{\nabla}_i F_j = \overset{\circ}{\nabla}_i^{(0)} F_j + (\Gamma_{ij}{}^l - \Gamma_{ij}^{(0)}{}^l) \cdot F_l$. Note that using (WS1) and Assumption 6.15 we have $\|r^{1-\frac{2}{q}}|\Gamma - \Gamma^{(0)}|_\gamma\|_{L_x^q} \lesssim \Delta_0$. Also in view of (6.22) and Hölder inequality, we obtain

$$\||(\Gamma - \Gamma^{(0)})_{ij}^l F_l|_\gamma\|_{L_x^q} \lesssim \|r^{1-\frac{2}{q}}|\Gamma - \Gamma^{(0)}|_\gamma\|_{L_x^q} \|r^{\frac{2}{q}-1} F\|_{L_\omega^\infty}$$

(6.55) $$\lesssim \Delta_0(\|\overset{\circ}{\nabla} F\|_{L_x^q} + \|r^{-1} F\|_{L_x^q}).$$

Note that we can choose $0 < \Delta_0 < \frac{1}{2}$ sufficiently small, then for $2 < q \le p$,

$$\|\nabla\!\!\!\!/\, F\|_{L_x^q} + \|r^{-1}F\|_{L_x^q} \lesssim \|\nabla\!\!\!\!/^{(0)}F\|_{L_x^q} + \|r^{-1}F\|_{L_x^q}.$$

The other direction can be proved in the same way. $\qquad\square$

We recall from [26] the following result which uses the smallness of $\mathcal{R}_0$ given in the remark immediately after Proposition 6.10.

Proposition 6.17. *On null cone $\mathcal{H}$ under the condition (6.38) and $C^{-1} < n < C$ that for S tangent tensor F, there holds*

(6.56) $$\|\nabla\!\!\!\!/\, F\|_{L^2(S)} + \|r^{-1}F\|_{L^2(S)} \lesssim \|\mathcal{D}F\|_{L^2(S)}.$$

By Assumption 6.15, let us prove the following L^p estimate for Hodge system.

Lemma 6.18. *Let $\mathcal{D}$ denote either $\mathcal{D}_1$ or $\mathcal{D}_2$ and F be S tangent tensor in the domain of $\mathcal{D}$, there holds for $2 < q \le p$,*

$$\|\nabla\!\!\!\!/\, F\|_{L^q(S)} + \|r^{-1}F\|_{L^q(S)} \lesssim \|\mathcal{D}F\|_{L^q(S)}$$

Proof. We only give the proof for $\mathcal{D} = \mathcal{D}_2$. The proof for the case that $\mathcal{D} = \mathcal{D}_1$ follows similarly. Relative to orthonormal frame associated to $\gamma^{(0)}$, for S-tangent, symmetric, traceless 2-tensor F, let us define $\tilde{F}_{AB} = \gamma_{AC}^{(0)}\gamma^{CD}F_{DB}$ and $2\bar{F}_{AB} = \tilde{F}_{AB} + \tilde{F}_{BA}$. Then we have

(6.57) $$\mathrm{div}\,^{(0)}\bar{F} = \mathrm{div}\, F + (\Gamma - \Gamma^{(0)})F \cdot \gamma, \qquad \mathrm{Tr}^{(0)}(\bar{F}) = 0.$$

Indeed, under the orthonormal frame associated to $\gamma^{(0)}$,

$$\nabla\!\!\!\!/_C(\gamma^{CD}F_{DA}) = \nabla\!\!\!\!/_C^{(0)}(\gamma^{CD}F_{DA}) + (\Gamma - \Gamma^{(0)})\gamma \cdot F,$$
$$\nabla\!\!\!\!/_C(F^{CD}\gamma_{DA}) = \nabla\!\!\!\!/_C^{(0)}(F^{CD}\gamma_{DA}) + (\Gamma - \Gamma^{(0)})F \cdot \gamma.$$

By Calderon Zygmund theory on 2-sphere with $\gamma^{(0)}$ and also using (6.57), we obtain

$$\int (|\nabla\!\!\!\!/^{(0)}\bar{F}|^q + |r^{-1}\bar{F}|^q) \lesssim \int |\mathcal{D}^{(0)}\bar{F}|^q$$
$$\le \int (|\mathcal{D}F|^q + |(\Gamma - \Gamma^{(0)})F \cdot \gamma|^q) \lesssim \int |\mathcal{D}F|^q$$

where to derive the last inequality, we employed (6.55), Lemma 6.16 and the smallness of Δ_0. By (WS1) and Assumption 6.15 we can show that

$$\|\nabla\!\!\!\!/^{(0)}F\|_{L_x^q} + \|r^{-1}F\|_{L_x^q} \lesssim \|\nabla\!\!\!\!/^{(0)}\bar{F}\|_{L_x^q} + \|r^{-1}\bar{F}\|_{L_x^q}.$$

Therefore, Lemma 6.18 follows by using Lemma 6.16. $\qquad\square$

Since
$$\mathcal{D}_2\hat{\chi} = \nabla\!\!\!\!/\,\mathrm{tr}\chi + \bar{R}, \quad \mathcal{D}_1\zeta = (\mu, 0) + \bar{R},$$
using Lemma 6.18, we can derive that for $2 \le q \le p$

$$\|\nabla\!\!\!\!/\,\hat{\chi}\|_{L^q(S)} + \|r^{-1}\hat{\chi}\|_{L^q(S)} \lesssim \|\nabla\!\!\!\!/\,\mathrm{tr}\chi\|_{L^q(S)} + \|\bar{R}\|_{L^q(S)},$$
$$\|\nabla\!\!\!\!/\,\zeta\|_{L^q(S)} + \|r^{-1}\zeta\|_{L^q(S)} \lesssim \|\mu\|_{L^q(S)} + \|\bar{R}\|_{L^q(S)}.$$

Since using (6.52), (6.51) and (6.37)

$$\|r\bar{R}\|_{L_t^2 L_\omega^p} \lesssim \|\underline{A}\|_{L_t^2 L_\omega^p} + \|rR_0\|_{L_t^2 L_\omega^p} + \|rA \cdot \underline{A}\|_{L_t^2 L_\omega^p} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0^2,$$

then on $\mathcal{H}_t$

(6.58) $$\|r\mathcal{D}_0 M\|_{L^2_t L^p_\omega} \lesssim \|rM\|_{L^2_t L^p_\omega} + \mathcal{R}_{0,p} + \mathcal{R}_0.$$

Recall that $M = \slashed{\nabla}\mathrm{tr}\chi$ or μ. (6.43) and (6.47) can be symbolically recast as

(6.59) $$\slashed{\nabla}_L M + \frac{m}{2}\mathrm{tr}\chi M = \hat{\chi}\cdot M + H_1 + H_2 + H_3$$

where [4]

$$H_1 = \hat{\chi}\cdot\mathcal{D}_0 M + F\cdot A, \quad H_2 = \underline{A}\cdot A\cdot A, \quad \text{and} \quad H_3 = r^{-1}\bar{R}, \iota\cdot\bar{R},$$

with

$$(m, F) = \left\{ \begin{array}{ll} (2, \slashed{\nabla}\mathrm{tr}\chi), & \text{if } M = \mu, \\ (3, 0), & \text{if } M = \slashed{\nabla}\mathrm{tr}\chi. \end{array} \right.$$

We now establish the following estimates, which, in view of (6.22) and (6.49), imply Proposition 6.10.

Proposition 6.19. *Let M denote either $\slashed{\nabla}\mathrm{tr}\chi$ or μ, there hold*

(6.60) $$\|r^{\frac{3}{2}}M\|_{L^p_\omega L^\infty_t} + \|rM\|_{L^2_t L^p_\omega} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0,$$

(6.61) $$\|r\slashed{\nabla}\zeta, r\slashed{\nabla}\hat{\chi}\|_{L^2_t L^p_\omega} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0.$$

Proof. (6.61) can be obtained immediately by combining the second estimate in (6.60) with (6.58), we consider the first norm in (6.60). By (6.59), Lemma 6.12 and 6.11, integrating along the null geodesic Γ_ω initiating from the vertex, we obtain

$$|M| \leq \sum_{i=1}^{3}\left|v_t^{-\frac{m}{2}}\int_0^t v_{t'}^{\frac{m}{2}}|H_i|na\,dt'\right| = I_1(t) + I_2(t) + I_3(t).$$

We have

(6.62) $$v_t^{\frac{3}{4}}I_3(t) \leq v_t^{\frac{3}{4}-\frac{m}{2}}\int_0^{s(t)}v_{t'}^{\frac{m}{2}-\frac{1}{2}}|\bar{R}|ds(t') \lesssim s^{\frac{3}{2}-m}\int_0^{s(t)}s^{m-2}|\bar{R}r|ds(t')$$

$$\lesssim \left(\int_0^{s(t)}|\bar{R}r|^2 ds(t')\right)^{\frac{1}{2}}$$

where we used $s^2 \approx v_t$, $r \approx s$ in Proposition 6.2 to derive the second inequality. Thus

(6.63) $$\left\|\sup_{t\in(0,1]}|v_t^{\frac{3}{4}}I_3(t)|\right\|_{L^p_\omega} \lesssim \|r\bar{R}\|_{L^p_\omega L^2_t} \lesssim \mathcal{R}_{0,p} + \mathcal{R}_0.$$

We can proceed in a similar fashion for the other two terms,

(6.64) $$v_t^{\frac{3}{4}}I_2(t) \leq v_t^{\frac{3}{4}-\frac{m}{2}}\int_0^{s(t)}v_{t'}^{\frac{m}{2}}|A|^2|\underline{A}|ds(t') \lesssim s^{\frac{3}{2}-m}\int_0^{s(t)}s^m|A|^2|\underline{A}|ds(t')$$

$$\lesssim \|A\|_{L^2_t}^2\|r^{3/2}\underline{A}\|_{L^\infty_t},$$

then

(6.65) $$\left\|\sup_{0<t\leq 1}|v_t^{\frac{3}{4}}I_2(t)|\right\|_{L^p_\omega} \lesssim \|A\|_{L^\infty_\omega L^2_t}^2\|r^{3/2}\underline{A}\|_{L^p_\omega L^\infty_t} \lesssim \mathcal{R}_0^3.$$

---

[4] In view of $|a-1| \leq 1/2$ in (6.9), the factors a^m, $m \in \mathbb{N}$ can be ignored in H_i when we employ (6.59) to prove Proposition 6.19.

Similarly,

$$(6.66) \qquad v_t^{\frac{3}{4}} I_1(t) \lesssim r(t)^{\frac{3}{2}-m} \int_0^t s^m |(\mathcal{D}_0 M + F) \cdot A| dt'.$$

Taking L_ω^p, using (6.58) for $\mathcal{D}_0 M$,

$$\left\| \sup_{0<t\le 1} |v_t^{\frac{3}{4}} I_1(t)| \right\|_{L_\omega^p} \lesssim (\|v_t^{\frac{3}{4}} \mathcal{D}_0 M\|_{L_\omega^p L_t^2} + \|v_t^{\frac{3}{4}} F\|_{L_\omega^p L_t^2}) \|A\|_{L_\omega^\infty L_t^2}$$

$$(6.67) \qquad\qquad \lesssim (\|v_t^{\frac{3}{4}} M\|_{L_t^2 L_\omega^p} + \|v_t^{\frac{3}{4}} F\|_{L_t^2 L_\omega^p} + \mathcal{R}_{0,p}) \|A\|_{L_\omega^\infty L_t^2}.$$

Note that $\|A\|_{L_\omega^\infty L_t^2} \lesssim \mathcal{R}_0$ can be sufficiently small, then we combine (6.63), (6.65) and (6.67) to obtain

$$\|r^{3/2} M\|_{L_\omega^p L_t^\infty(\mathcal{H})} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p}.$$

Using Minkowski inequality, we can obtain in view of (6.62)

$$r(t)^{-\frac{1}{2}} \|v_t^{\frac{3}{4}} I_3(t)\|_{L_\omega^p} \lesssim r(t)^{-1} \int_0^t r(t') \|\bar{R}\|_{L_\omega^p} dt'$$

then

$$\|r I_3(t)\|_{L_t^2 L_\omega^p} \lesssim \|r\bar{R}\|_{L_t^2[0,1]L_\omega^p} \lesssim \mathcal{R}_{0,p}.$$

We can also obtain in view of (6.64) that

$$v_t^{\frac{1}{2}} I_2(t) \lesssim \|A\|_{L_t^2}^2 \|r\underline{A}\|_{L^\infty}$$

by taking $L_t^2 L_\omega^p$ norm and using (6.18), we have

$$\|v_t^{\frac{1}{2}} I_2\|_{L_t^2 L_\omega^p} \lesssim \|A\|_{L_\omega^\infty L_t^2}^2 \|r\underline{A}\|_{L_\omega^p L_t^\infty} \lesssim \mathcal{R}_0^3.$$

For $I_1(t)$, in view of (6.66)

$$v_t^{\frac{1}{2}} I_1(t) \lesssim r(t)^{1-m} \int_0^t s^m |(\mathcal{D}_0 M + F) \cdot A| dt'.$$

Now taking $L_t^2([0,1])L_\omega^p$ by Hölder inequality,

$$\|v_t^{\frac{1}{2}} I_1\|_{L_t^2 L_\omega^p} \lesssim \|A\|_{L_\omega^\infty L_t^2} \|r(|\mathcal{D}_0 M| + |F|)\|_{L_\omega^p L_t^2}$$
$$\lesssim \mathcal{R}_0(\|rM\|_{L_t^2 L_\omega^p} + \|rF\|_{L_t^2 L_\omega^p} + \mathcal{R}_{0,p})$$

where for the last step we employed Minkowski inequality and (6.58). Since $\mathcal{R}_0$ can be sufficiently small, the second inequality in (6.60) can be proved. $\qquad\square$

Now we prove Assumption 6.15.

Proposition 6.20. *For the transport local coordinates (t, ω), the following properties hold true for all surfaces S_t of the time foliation on the null cone $\mathcal{H}$: the metric $\overset{\circ}{\gamma}_{ij}(t)$ on each S_t verifies weakly spherical conditions i.e.*

$$(6.68) \qquad \|\partial_k \overset{\circ}{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\|_{L_\omega^p L_t^\infty} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p},$$

where $i, j, k = 1, 2$.

Proof. Since relative to the transport coordinate on $\mathcal{H}$, $\frac{d}{ds}\gamma_{ij} = 2\chi_{ij}$,

$$(6.69) \qquad \frac{d}{dt}(\mathring{\gamma}_{ij}) = an(\kappa \cdot \gamma_{ij} + 2\hat{\chi}_{ij})r^{-2}.$$

Differentiate the above equation again, we obtain,

$$\frac{d}{dt}\partial_k \mathring{\gamma}_{ij} = an\left(\partial_k \log(an)\kappa\, \mathring{\gamma}_{ij} + \partial_k \mathrm{tr}\chi\, \mathring{\gamma}_{ij}\right.$$
$$\left. + \kappa\partial_k \mathring{\gamma}_{ij} + 2\partial_k \hat{\chi}_{ij} r^{-2} + 2\partial_k \log(an)\hat{\chi}_{ij} r^{-2}\right)$$

where $i, j, k = 1, 2$, with the initial condition given by (6.53). Integrating the following transport equation along a null geodesic initiating from vertex, by $\|\kappa\|_{L^2_t L^\infty_\omega} \lesssim \mathcal{R}_0$ in (6.13) and a similar argument to Lemma 6.12, we can obtain

$$\left|\partial_k \mathring{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\right| \lesssim \left|\int_0^{s(t)} \left(\partial_k \mathrm{tr}\chi \cdot \mathring{\gamma}_{ij} + r^{-2}(\mathring{\nabla}_k \hat{\chi}_{ij} - \Gamma \cdot \hat{\chi})\right.\right.$$
$$\left.\left. + \partial_k \log(an)\, \mathring{\gamma}_{ij}\, \kappa + 2\partial_k \log(an)\hat{\chi}_{ij} r^{-2} + \kappa\partial_k \gamma_{\mathbb{S}^2 ij}\right) ds(t')\right|,$$

where Γ represents Christoffel symbols, and $\Gamma \cdot \hat{\chi}$ stands for the terms $\sum_{l=1}^2 \Gamma^l_{ki}\hat{\chi}_{lj}$ with $l = 1, 2$. Then with the help of $\|\kappa\|_{L^2_t L^\infty_\omega(\mathcal{H})} \lesssim \mathcal{R}_0$ in (6.13) and Lemma 6.12,

$$\left\|\sup_{0<t\leq 1}\left|\partial_k \mathring{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\right|\right\|_{L^p_\omega}$$
$$\lesssim \|\Gamma\|_{L^p_\omega L^2_t}\||\hat{\chi}|_\gamma\|_{L^\infty_\omega L^2_t} + \|r|\mathring{\nabla}\mathrm{tr}\chi|_\gamma\|_{L^p_\omega L^1_t} + \|r|\mathring{\nabla}\hat{\chi}|_\gamma\|_{L^p_\omega L^1_t}$$
$$(6.70) \qquad + \|r|\zeta + \underline{\zeta}|_\gamma \cdot \kappa\|_{L^p_\omega L^1_t} + \|r|(\zeta + \underline{\zeta}) \cdot \hat{\chi}|_\gamma\|_{L^p_\omega L^1_t} + \|\kappa\|_{L^p_\omega L^1_t}.$$

We estimate the terms in the line of (6.70) by (6.10) and (6.52)

$$(6.70) \lesssim \|\kappa\|_{L^p_\omega L^2_t}(\|A\|_{L^\infty_\omega L^2_t} + 1) + \|rA \cdot A\|_{L^p_\omega L^1_t} \lesssim \mathcal{R}_0^2 + \mathcal{R}_0$$

Using Proposition 6.19,

$$\left\|\sup_{0<t\leq 1}\left|\partial_k \mathring{\gamma}_{ij}(t) - \partial_k \gamma_{\mathbb{S}^2 ij}\right|\right\|_{L^p_\omega} \lesssim \mathcal{R}_{0,p} + \|\hat{\chi}\|_{L^\infty_\omega L^2_t}\|\Gamma\|_{L^p_\omega L^2_t}.$$

Summing over all $i, j, k = 1, 2$, we have for the component of Christoffel Symbol

$$\|\Gamma\|_{L^p_\omega} \lesssim \sum_{i,j,k=1,2} \|\partial_k(\mathring{\gamma}_{ij} - \gamma_{\mathbb{S}^2 ij})\|_{L^p_\omega} + C,$$

where C is the constant such that the Christoffel symbol of $\gamma_{\mathbb{S}^2}$ satisfies $|\partial\gamma_{\mathbb{S}^2}| \leq C$, (6.68) then follows by $\|\hat{\chi}\|_{L^\infty_\omega L^2_t} \leq C\mathcal{R}_0$ which can be sufficiently small. $\qquad\square$

7. Proof of Proposition 5.4

In this section, we give the proofs of (5.4)–(5.9) in Proposition 5.4. Recall the definition of the region $\mathcal{D}^+$ from Section 4.3. We consider $\mathcal{D}^+$ under the rescaling coordinates according to $(t, x) \to (\frac{t}{\lambda}, \frac{x}{\lambda})$. The quantities $\mathcal{C}_0$ and $\mathcal{R}_0$ introduced in Theorem 6.1 in the frequency dependent coordinates then satisfy $\mathcal{C}_0 \lesssim \lambda^{-\frac{1}{2}}$, $\mathcal{R}_0 \lesssim \lambda^{-\frac{1}{2}}$ and the quantity $\mathcal{R}_{0,p} \lesssim \lambda^{-\frac{1}{2}}$. Estimates in Theorem 6.1 hold with $\mathcal{R}_0$ replaced by $\lambda^{-\frac{1}{2}}$ and Proposition 6.10 holds with $\mathcal{R}_0 + \mathcal{R}_{0,p}$ replaced by $\lambda^{-\frac{1}{2}}$. The estimates for ζ, k in (5.6) were derived by combining (6.12), (6.7) and (6.18).

7.1. **Estimates for z and Ω.** We first derive the estimates on z and Ω given in Proposition 5.4, where z and Ω are defined by (5.2), i.e. $z = a\mathrm{tr}\chi' - \frac{2}{n(t-u)}$ and $\Omega = \frac{\mathbf{b}^{-1}-n^{-1}}{t-u}$.

Proposition 7.1. *Let $\mathcal{U} = \frac{1}{s} - \frac{1}{an(t-u)}$, and let z and Ω be defined by (5.2). then there holds on $S := S_{t,u}$ that*

$$(7.1) \qquad \|r^{-\frac{1}{2}}(\mathcal{U}, z, \Omega)\|_{L^2(S)} + \|\mathcal{U}, z, \Omega\|_{L^4(S)} \lesssim \lambda^{-\frac{1}{2}}.$$

Proof. Observe that $z = aV + 2a\mathcal{U}$ and $\|V\|_{L^4(S)} + \|r^{-\frac{1}{2}}V\|_{L^2(S)} \lesssim \tau_*^{\frac{1}{2}}\mathcal{R}_0^2 \lesssim \lambda^{-\frac{1}{2}}$. Thus the estimates for z follow immediately from those estimates on $\mathcal{U}$.

In order to derive the estimates on $\mathcal{U}$, we write $\mathcal{U} = z_1 - z_2$, where

$$(7.2) \qquad z_1 = \frac{1}{s} - \frac{1}{an_p(t-u)}, \quad z_2 = \frac{1}{a(t-u)}\left(\frac{1}{n} - \frac{1}{n_p}\right)$$

with n_p being the value of the lapse function n at the vertex p of the null cone C_u. Integrating along any null geodesic initiating from vertex, we obtain,

$$(7.3) \qquad |z_2| \le \frac{1}{a(t-u)}\left|\int_u^t \frac{d}{ds}(\frac{1}{n})nadt'\right|.$$

Therefore

$$(7.4) \qquad \|z_2\|_{L^\infty_\omega L^2_t} \lesssim \|\nabla_L n\|_{L^\infty_\omega L^2_t} \lesssim \mathcal{C}_0.$$

Recall that $|a-1| \le 1/2$, $C^{-1} < n < C$ and $t-u \approx s$. We can derive from (7.3) that

$$(7.5)$$
$$\|r^{-\frac{1}{2}}z_2\|_{L^2(S)} \lesssim \|r^{-\frac{1}{2}}\nabla_L n\|_{L^2(S)} \lesssim \mathcal{C}_0 \lesssim \lambda^{-\frac{1}{2}}, \quad \|z_2\|_{L^4(S)} \lesssim \|\nabla_L n\|_{L^4(S)} \lesssim \mathcal{C}_0 \lesssim \lambda^{-\frac{1}{2}}.$$

We next consider z_1. On the outgoing null cone C_u with vertex p, we have the transport equation

$$(7.6) \qquad \frac{d}{ds}z_1 + \frac{2}{s}z_1 = z_1^2 + \frac{1}{an_p(t-u)}z_2 + \frac{\frac{da}{ds}}{a^2 n_p(t-u)}$$

Integrating along any null geodesic initiating from the vertex, we obtain,

$$(7.7) \qquad |z_1| \lesssim \left|\frac{1}{s^2}\int_u^t s^2\left(z_1^2 + \frac{1}{an_p(t-u)}z_2 + \frac{\frac{da}{ds}}{a^2 n_p(t-u)}\right)\right|.$$

In view of (7.4), it follows that

$$\|z_1\|_{L^\infty_\omega L^2_t} \lesssim \tau_*^{\frac{1}{2}}\|z_1\|^2_{L^\infty_\omega L^2_t} + \|z_2\|_{L^\infty_\omega L^2_t} + \|\nabla_L a\|_{L^\infty_\omega L^2_t}$$
$$(7.8) \qquad\qquad \le C_2\left(\tau_*^{\frac{1}{2}}\|z_1\|^2_{L^\infty_\omega L^2_t} + \mathcal{C}_0\right).$$

Recall that $\mathcal{C}_0 \le C_1\lambda^{-1/2}$ for some universal constant $C_1 > 0$. We claim that $\|z_1\|_{L^\infty_\omega L^2_t} \le 2C_2\mathcal{C}_0$ on any time interval $[0,T]$ with $T < (4C_1 C_2^2)^{-2}$. Observe that $\|z_1\|_{L^\infty_\omega L^2_t} \to 0$ as $T \to 0$, the estimate holds on sufficiently small interval $[0,T]$. Thus, it follows from (7.8) and $\tau_* \le \lambda T$ that on the same time interval $[0,T]$ with $T < (4C_1 C_2^2)^{-2}$ there holds $\|z_1\|_{L^\infty_\omega L^2_t} < 2C_2\mathcal{C}_0$. Consequently the estimate holds on larger interval and thus we must have the claim on any interval $[0,T]$ with $T < (4C_1 C_2^2)^{-2}$. Therefore

$$(7.9) \qquad \|z_1\|_{L^\infty_\omega L^2_t} \lesssim \mathcal{C}_0 \lesssim \lambda^{-1/2}.$$

With the help of (7.6), (7.9) and Lemma 6.12 we thus obtain

$$\|r^{-\frac{1}{2}}z_1\|_{L^2(S)} \lesssim \|r^{-\frac{1}{2}}z_2\|_{L^2(S)} + \|r^{-\frac{1}{2}}\nabla_L a\|_{L^2(S)}$$

which together with (7.5) gives

$$\|r^{-\frac{1}{2}}z_1\|_{L^2(S)} \lesssim \mathcal{C}_0 + \mathcal{N}_1[\nu] \lesssim \mathcal{C}_0 \lesssim \lambda^{-1/2}.$$

In the same way, we can also derive that

$$\|z_1\|_{L^4(S)} \lesssim \|z_2\|_{L_t^\infty L_x^4(C_u)} + \|\nu\|_{L_t^\infty L_x^4(C_u)} \lesssim \mathcal{N}_1[\nu] + \mathcal{C}_0 \lesssim \mathcal{C}_0 \lesssim \lambda^{-1/2}.$$

Combining this with (7.5), we obtain the estimate for $\mathcal{U}$ in (7.1).

Finally we consider Ω. Observe that $\Omega = \frac{a^{-1}-1}{n_p(t-u)} + \frac{n_p^{-1}-n^{-1}}{t-u}$, we have

$$|\Omega| \leq \frac{1}{n_p(t-u)}\left|\int_u^t L(a^{-1})ds\right| + \frac{1}{(t-u)}\left|\int_u^t L(n^{-1})ds\right|.$$

Similar to (7.5) we can derive that

$$\|r^{-\frac{1}{2}}\Omega\|_{L^2(S)} \lesssim \|r^{-\frac{1}{2}}(|\nabla_L n| + |\nabla_L a|)\|_{L^2(S)} \lesssim \mathcal{C}_0 \lesssim \lambda^{-1/2},$$

$$\|\Omega\|_{L^4(S)} \lesssim \|\nabla_L n\|_{L^4(S)} + \|\nabla_L a\|_{L^4(S)} \lesssim \mathcal{C}_0 \lesssim \lambda^{-1/2}.$$

Thus we complete the proof of (7.1). $\qquad\square$

Proposition 7.2. *Relative to the canonical null frame, there holds on every $S_{t,u}$ that*

(7.10) $$\|r^{\frac{3}{2}}\underline{L}z\|_{L_\omega^p} + \|r^{\frac{3}{2}}\slashed{\nabla}z\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

Proof. We consider $\underline{L}z$ first. By the definition of μ and using (5.23), we derive that

$$\underline{L}z - (\mu + \mathrm{tr}\chi(k_{NN} + \nabla_N \log n))$$

$$= \underline{L}z - \underline{L}\mathrm{tr}\chi + \frac{1}{2}(\mathrm{tr}\chi)^2 = \frac{1}{2}(\mathrm{tr}\chi)^2 - 2\underline{L}\left(\frac{1}{n(t-u)}\right)$$

$$= \frac{1}{2}\left(z + \frac{2}{n(t-u)}\right)^2 - 2\left(\frac{n\mathbf{b}^{-1}}{n^2(t-u)^2} - \frac{\underline{L}\log n}{n(t-u)} - \frac{n(n^{-1}-\mathbf{b}^{-1})}{n^2(t-u)^2}\right)$$

$$= \frac{1}{2}\left(z^2 + \frac{4z}{n(t-u)}\right) + 4\frac{1-n\mathbf{b}^{-1}}{n^2(t-u)^2} + 2\frac{\underline{L}\log n}{n(t-u)}.$$

In view of $r\mathrm{tr}\chi \approx 1$ and (7.1), we have $\|r^{\frac{3}{2}}z^2\|_{L_\omega^p} \lesssim \mathcal{R}_0 \lesssim \lambda^{-1/2}$. Therefore, by using (6.39) and applying

(7.11) $$\|r^{\frac{1}{2}}F\|_{L_\omega^p} \lesssim \|r^{-\frac{1}{2}}F\|_{L^2(S)} + \|F\|_{L^4(S)},$$

to $F = \Omega$, $\underline{L}\log n$, z, we can deduce in view of (7.1), (6.7) and $\|r^{\frac{3}{2}}\mu\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}$ in Proposition 6.10 that

$$\|r^{\frac{3}{2}}\underline{L}z\|_{L_\omega^p} \lesssim \|r^{\frac{3}{2}}\mu\|_{L_\omega^p} + \|r^{\frac{3}{2}}z^2\|_{L_\omega^p} + \|r^{\frac{1}{2}}(z, \underline{L}\log n, \Omega)\|_{L_\omega^p} \lesssim \lambda^{-\frac{1}{2}}.$$

In order to obtain the estimate for $\slashed{\nabla}z$, we apply (7.11) to $\slashed{\nabla}a$ and $\slashed{\nabla}n$ and use (6.39) for $\slashed{\nabla}\mathrm{tr}\chi'$. It then follows that

$$\|r^{\frac{3}{2}}\slashed{\nabla}z\|_{L_\omega^p} \lesssim \|r^{\frac{3}{2}}\mathrm{tr}\chi'\slashed{\nabla}a\|_{L_\omega^p} + \|r^{\frac{1}{2}}\slashed{\nabla}(n^{-1})\|_{L_\omega^p} + \|r^{\frac{3}{2}}a\slashed{\nabla}\mathrm{tr}\chi'\|_{L_\omega^p} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p}.$$

This completes the proof. $\qquad\square$

We define Hardy-Littlewood maximal function for scalar function $f(t)$ by

$$\mathcal{M}(f)(t) = \sup_{t'} \frac{1}{|t - t'|} \int_{t'}^{t} |f(\tau)| d\tau.$$

It is well-known that for any $1 < q < \infty$ there holds

$$\|\mathcal{M}(f)\|_{L_t^q} \lesssim \|f\|_{L_t^q}. \tag{7.12}$$

Proposition 7.3. *There holds on* $\mathcal{D}^+$ *that* $\|z\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}}$.

Proof. Recall the estimate (6.8) in Theorem 6.1. We then have

$$\|z\|_{L_t^2 L_x^\infty} \lesssim \|aV\|_{L_t^\infty L_x^\infty} \tau_*^{\frac{1}{2}} + \left\|\frac{1}{s} - \frac{1}{an(t - u)}\right\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-\frac{1}{2}} T^{\frac{1}{2}} + \|\mathcal{U}\|_{L_t^2 L_x^\infty}.$$

Assuming the following estimate on $\mathcal{D}^+$

$$\|\mathcal{U}\|_{L_t^2 L_x^\infty} \lesssim \|\nabla_L n, \nabla_L a\|_{L_t^2 L_x^\infty}, \tag{7.13}$$

noting that (5.4) implies $\|\nu, \nabla_L n\|_{L_t^2 L_x^\infty} \lesssim \lambda^{-1/2}$, therefore we can obtain (5.7).

To see (7.13), due to $\mathcal{U} = z_1 - z_2$, let us consider z_2 first. In view of (7.3), $|a - 1| \leq 1/2$, $C^{-1} < n < C$ and $t - u \approx s$, we can obtain

$$\sup_{\omega \in \mathbb{S}^2, u} |z_2|(t) \lesssim \mathcal{M}(\|\nabla_L n\|_{L_x^\infty}). \tag{7.14}$$

In view of (7.7), we also have

$$|z_1| \lesssim \frac{1}{s(t)^2} \int_u^t s(t')^2 \left|\frac{1}{an_p(t - u)} z_2 + \frac{1}{a^2 n_p(t - u)} \frac{da}{ds}\right| dt'.$$

Therefore

$$\sup_{\omega \in \mathbb{S}^2, u} |z_1|(t) \lesssim \mathcal{M}(\|z_2\|_{L_x^\infty}) + \mathcal{M}(\|\nabla_L a\|_{L_x^\infty}). \tag{7.15}$$

Combining (7.14) and (7.15), taking L_t^2 norm, and using (7.12) implies (7.13). □

7.2. $L_t^2 L_x^\infty$ estimate for $\hat{\chi}$ and ζ. We will use the equations (6.44)-(6.46) to derive the estimates for $\hat{\chi}$ and ζ. We first recall from [13] the following version of Calderon-Zygmund-type result for Hodge systems.

Proposition 7.4. *Let F be a covariant traceless symmetric 2-tensor satisfying the Hodge system*

$$div\, F = \slashed{\nabla} G + e \qquad on\ S_{t,u} \tag{7.16}$$

for some scalar function G and 1-form e. Then for $2 < p < \infty$ and $\frac{1}{q} = \frac{1}{2} + \frac{1}{p}$ there holds

$$\|F\|_{L^p(S_{t,u})} \lesssim \|G\|_{L^p(S_{t,u})} + \|e\|_{L^q(S_{t,u})} \tag{7.17}$$

and for $p > 2$ there holds
$$\tag{7.18}$$
$$\|F\|_{L^\infty(S_{t,u})} \lesssim \|G\|_{L^\infty(S_{t,u})} \ln\left(2 + r^{\frac{3}{2} - \frac{2}{p}} \lambda^{\frac{1}{2}} \|\slashed{\nabla} G\|_{L^p(S_{t,u})}\right) + r^{1 - \frac{2}{p}} \|e\|_{L^p(S_{t,u})}.$$

Similarly, for the Hodge system

$$\begin{cases} div\, F = \slashed{\nabla} \cdot G_1 + e_1, \\ curl\, F = \slashed{\nabla} \cdot G_2 + e_2, \end{cases} \tag{7.19}$$

where $G = (G_1, G_2)$ are 1-forms and $e = (e_1, e_2)$ are scalar functions, we have (7.17) and for any $p > 2$ there holds

(7.20)

$$\|F\|_{L^\infty(S_{t,u})} \lesssim \|G\|_{L^\infty(S_{t,u})} \ln\left(2 + r^{\frac{3}{2} - \frac{2}{p}} \lambda^{\frac{1}{2}} \|\slashed{\nabla} G, r^{-1} G\|_{L^p(S_{t,u})}\right) + r^{1 - \frac{2}{p}} \|e\|_{L^p(S_{t,u})}.$$

Relative to rough metric, the proof of Proposition 7.4 can follow from the standard Calderon-Zygmund theory with the help of weakly spherical condition (6.68) and (WS1). Consider the application of (7.18) to $\operatorname{div} \hat{\chi} = \slashed{\nabla} \overset{\circ}{\pi} + \slashed{\nabla} \operatorname{tr}\chi + \cdots$ which follows from the Hodge system (6.44) and Proposition 6.8 (a), where $\overset{\circ}{\pi}$ is the major part of G. In principle we only have the bounds of $\|r\slashed{\nabla}\overset{\circ}{\pi}\|_{L_t^2 L_\omega^p}$ on null cones, which needs one-half more derivative to give the estimate of $\sup_u r^{\frac{3}{2}} \|\slashed{\nabla} G\|_{L_\omega^p}$ for $p > 2$ in (7.18). This poses an issue of lack of half-derivative. Note that, by Strichartz estimate, we may expect to control $\|\mu^{0+} P_\mu(\widehat{\nabla} g, k)\|_{l_\mu^1 L_t^2 L_x^\infty}$, which is slightly stronger than $\|\overset{\circ}{\pi}\|_{L_t^2 L_x^\infty}$. On the other hand, since $\overset{\circ}{\pi}$ takes the form of $N^\mu(g\widehat{\nabla}g, k)_{\mu\dots}$ or $\Pi_\mu^{\mu'}(g\widehat{\nabla}g, k)_{\mu'\dots}$ relative to spacial harmonic coordinates, to control $\|\mu^{0+} P_\mu \overset{\circ}{\pi}\|_{l_\mu^1 L_t^2 L_x^\infty}$ causes a serious technical baggage due to the factor N^μ involved in the paradifferential calculation. To solve all the potential issues, our strategy is to establish (7.21) so as to take advantage of the extra differentiability of $k, \widehat{\nabla}g$.

Proposition 7.5. *Let F and G be S-tangent tensor fields of suitable type satisfying (7.16) or (7.19) with certain term e. Suppose G is a projection of Σ tangent tensor $\tilde{G}$ to tangent space of S by $\Pi_\mu^{\mu'} \tilde{G}_{\mu'\dots}$ or takes the form of $N^\mu \tilde{G}_{\mu\dots}$. Then for $p > 2$, $\delta > 0$ sufficiently close to 0, and $1 \le c < \infty$ there holds*

(7.21) $$\|F\|_{L^\infty(S_{t,u})} \lesssim \|\mu^\delta P_\mu \tilde{G}\|_{l_\mu^c L^\infty(S_{t,u})} + \|\tilde{G}\|_{L^\infty(S_{t,u})} + r^{1 - \frac{2}{p}} \|e\|_{L^p(S_{t,u})}.$$

The proof of (7.21) is presented in the Appendix II.

We slightly extend the definition of err in Proposition 6.8 to

(7.22) $$\operatorname{err} := \underline{A} \cdot \underline{A} + \operatorname{tr}\chi \cdot \check{\pi}.$$

where $\underline{A}$ was defined in Section 6. By using (6.39) we have $\|r^{1 - \frac{2}{p}} \underline{A} \cdot \underline{A}\|_{L^p(S_{t,u})} \lesssim (\mathcal{R}_{0,p} + \mathcal{R}_0)^2$. Also using the fact $r\operatorname{tr}\chi \approx 1$, we can obtain the following estimate on err.

Lemma 7.6. *For any $p > 2$ there holds*

(7.23) $$\|r^{1 - \frac{2}{p}} \operatorname{err}\|_{L^p(S_{t,u})} \lesssim \|\check{\pi}\|_{L_\omega^\infty} + (\mathcal{R}_{0,p} + \mathcal{R}_0)^2$$

7.2.1. *Estimate of $\hat{\chi}$.* In view of the equation (6.44) and Proposition 6.8 (a), we have

$$\operatorname{div} \hat{\chi} = \frac{1}{2} \slashed{\nabla} \operatorname{tr}\chi + \slashed{\nabla} \overset{\circ}{\pi} + \operatorname{err}.$$

We define $\check{\chi}$, a symmetric traceless 2-tensor, to be the solution of

(7.24) $$\operatorname{div} \check{\chi} = \frac{1}{2} \slashed{\nabla}\left(\operatorname{tr}\chi - \frac{2}{n(t-u)}\right).$$

It is straightforward to see that

(7.25) $$\operatorname{div}(\hat{\chi} - \check{\chi}) = \slashed{\nabla} \overset{\circ}{\pi} + \operatorname{err}.$$

Applying (7.18) in Proposition 7.4 to (7.24) and also using (7.10), we obtain

$$(7.26) \qquad \|\check{\chi}\|_{L^\infty(S_{t,u})} \lesssim \|z\|_{L^\infty(S_{t,u})} \ln(2 + \|r^{\frac{3}{2}-\frac{2}{p}}\lambda^{\frac{1}{2}}\not{\nabla}z\|_{L^p(S_{t,u})}) \lesssim \|z\|_{L^\infty(S_{t,u})}$$

Note that with $2 \le q \le 4$,

$$(7.27) \qquad \|r^{\frac{1}{2}-\frac{2}{q}}\check{\pi}\|_{L^q(S_{t,u})} \lesssim \mathcal{N}_1[\check{\pi}]\lambda^{-\frac{1}{2}}.$$

Applying (7.21) in Proposition 7.5 to (7.25) and using (7.27) give

$$(7.28) \quad \|\hat{\chi} - \check{\chi}\|_{L^\infty(S_{t,u})} \lesssim \|\mu^{0+}P_\mu \overset{\circ}{\pi}\|_{l^2_\mu L^\infty(S_{t,u})} + \|\overset{\circ}{\pi}\|_{L^\infty(S_{t,u})} + \|r^{1-\frac{2}{p}}\mathrm{err}\|_{L^p(S_{t,u})}$$

where $p > 2$ sufficiently close to 2.

Combining (7.26) and (7.28), and using (7.23) and (5.8), we can obtain [5]

$$\|\hat{\chi}\|_{L^2_t L^\infty_x} \lesssim \|\mu^{0+}P_\mu \overset{\circ}{\pi}\|_{L^2_t l^2_\mu L^\infty_x} + \|\check{\pi}\|_{L^2_t L^\infty_x} + (\mathcal{R}_{0,p} + \mathcal{R}_0)^2 \tau_*^{\frac{1}{2}}.$$

By using (BA3), we conclude $\|\hat{\chi}\|_{L^2_t L^\infty_x(\mathcal{D}^+)} \lesssim \lambda^{-1/2}$, which is the first part of (5.9).

7.2.2. *Estimate for ζ.* In view of (6.45), (6.46) and Proposition 6.8, we can write

$$\mathcal{D}_1\zeta = \not{\nabla} \overset{\circ}{\pi} + \mathrm{err} + (\mu, 0)$$

where $\overset{\circ}{\pi} = (\overset{\circ}{\pi}^{(1)}, \overset{\circ}{\pi}^{(2)})$.

Let $\not{\mu}_A$ be an S-tangent co-vector which is defined as a solution of the Hodge system

$$(7.29) \qquad \mathrm{div}\,\not{\mu} = \mu - \bar{\mu} \qquad \mathrm{curl}\,\not{\mu} = 0.$$

It follows from (6.56) and (6.11) that

$$(7.30) \qquad \|\not{\nabla}\not{\mu}\|_{L^2(C_u)} + \|r^{-1}\not{\mu}\|_{L^2(C_u)} \lesssim \|\mu\|_{L^2(C_u)} \lesssim \mathcal{R}_0.$$

Observe that by (6.45), Proposition 6.8 (b) and the divergence theorem, we can treat $\bar{\mu}$ as $\overline{nA \cdot A + n\mathrm{tr}\chi\pi}$. By ignoring the average sign, in view of (7.22), $\bar{\mu}$ can also be incorporated to err. With the help of (7.29), we have

$$(7.31) \qquad \mathcal{D}_1(\zeta - \not{\mu}) = \not{\nabla} \overset{\circ}{\pi} + \mathrm{err}.$$

In view of (7.21) in Proposition 7.5 and (7.27), we obtain
$$(7.32)$$
$$\|\zeta\|_{L^\infty(S_{t,u})} \lesssim \|\not{\mu}\|_{L^\infty(S_{t,u})} + \|\mu^{0+}P_\mu \overset{\circ}{\pi}\|_{l^2_\mu L^\infty_x} + \|\overset{\circ}{\pi}\|_{L^\infty(S_{t,u})} + \|r^{1-\frac{2}{p}}\mathrm{err}\|_{L^p(S_{t,u})}.$$

Lemma 7.7. *In the region $\mathcal{D}^+$, there holds*

$$\left(\int_0^{\tau_*} \sup_u \|\not{\mu}\|_{L^\infty(S_{t,u})}^2 dt'\right)^{1/2} \lesssim \|\mu^{0+}P_\mu \overset{\circ}{\pi}\|_{L^2_t l^2_\mu L^\infty_x} + \|\check{\pi}\|_{L^2_t L^\infty_x} + T^{\frac{1}{2}}\lambda^{-\frac{1}{2}}.$$

Assuming this result, we can obtain from (7.32) and (7.23) that

$$(7.33) \qquad \left(\int_0^{\tau_*} \sup_u \|\zeta\|_{L^\infty(S_{t,u})}^2\right)^{\frac{1}{2}} \lesssim \lambda^{-\frac{1}{2}}$$

which, in view of (BA3), gives the second part in (5.9).

In order to prove Lemma 7.7, we first derive an equation for

$$(7.34) \qquad \mathbf{D}_4\not{\mu} := \mathcal{D}_t\not{\mu} + \frac{1}{2}n\mathrm{tr}\chi\not{\mu} - n\hat{\chi}\cdot\not{\mu},$$

---

[5]We denote by $0+$ a constant $b > 0$ arbitrarily close to 0

where $\mathcal{D}_t F := n\mathbf{D}_L F$ for any tensor field F. We need the commutation formula ([7, Chapter 13], [26])

(7.35) $\qquad [\mathcal{D}_t, \mathbf{\nabla}\!\!\!/_B] F_A = -n\chi_{BC}\mathbf{\nabla}\!\!\!/_C F_A + n(\chi_{AB}\underline{\zeta}_C - \chi_{BC}\underline{\zeta}_A + \epsilon_{AC}{}^\star\beta_B) F_C.$

Lemma 7.8. $\mathbf{D}_4\mu\!\!\!/$ *satisfies the following Hodge system.*

(7.36) $\qquad \begin{cases} div\,\mathbf{D}_4\mu\!\!\!/ = \frac{1}{t-u}\mathbf{\nabla}\!\!\!/\,\overset{\circ}{\pi} + e_1' + n(tr\chi)^2\pi \\ curl\,\mathbf{D}_4\mu\!\!\!/ = n(\chi\cdot\pi + R_0)\mu\!\!\!/ + n\underline{A}\cdot(A+tr\chi)\mu\!\!\!/ + n\mathbf{\nabla}\!\!\!/ A\cdot\mu\!\!\!/, \end{cases}$

where $e_1' = n\left(A\cdot\mathbf{\nabla}\!\!\!/ A + (A+\mu\!\!\!/)\cdot(A\cdot\underline{A} + \underline{A}\cdot tr\chi) + (A+z)\cdot R_0\right).$

Proof. Let $G := L\mu + tr\chi\mu$. We can derive

(7.37) $\qquad\qquad \mathcal{D}_t(\mu - \bar\mu) + ntr\chi(\mu - \bar\mu) = nG - \overline{nG} - n\kappa\bar\mu$

In view of (7.35) we then obtain

$$\operatorname{div}\mathcal{D}_t\mu\!\!\!/ = \mathcal{D}_t\operatorname{div}\mu\!\!\!/ + \delta^{AB}n\chi_{BC}\mathbf{\nabla}\!\!\!/_C\mu\!\!\!/_A - \delta^{AB}n(\chi_{AB}\underline{\zeta}_C - \chi_{BC}\underline{\zeta}_A + \epsilon_{AC}{}^\star\beta_B)\mu\!\!\!/_C$$

$$= \mathcal{D}_t\operatorname{div}\mu\!\!\!/ + \frac{1}{2}ntr\chi\operatorname{div}\mu\!\!\!/ + n\hat\chi\cdot\mathbf{\nabla}\!\!\!/\mu\!\!\!/ - \frac{1}{2}ntr\chi\underline{\zeta}\cdot\mu\!\!\!/ + n\hat\chi\cdot\underline{\zeta}\cdot\mu\!\!\!/ - n\beta\cdot\mu\!\!\!/$$

$$= nG - \overline{nG} - n\kappa\bar\mu - \frac{1}{2}ntr\chi\operatorname{div}\mu\!\!\!/ + n\hat\chi\cdot\mathbf{\nabla}\!\!\!/\mu\!\!\!/ - \frac{1}{2}ntr\chi\underline{\zeta}\cdot\mu\!\!\!/$$

$$+ n\hat\chi\cdot\underline{\zeta}\cdot\mu\!\!\!/ - n\beta\cdot\mu\!\!\!/.$$

Consequently

$$\operatorname{div}\mathbf{D}_4\mu\!\!\!/ = nG - \overline{nG} - n\kappa\bar\mu + \frac{1}{2}\mathbf{\nabla}\!\!\!/(ntr\chi)\mu\!\!\!/ - \operatorname{div}(n\hat\chi)\mu\!\!\!/$$

$$- \frac{1}{2}ntr\chi\underline{\zeta}\cdot\mu\!\!\!/ + n\hat\chi\cdot\underline{\zeta}\cdot\mu\!\!\!/ - n\beta\cdot\mu\!\!\!/.$$

As explained before (7.31), $\kappa\bar\mu$ can be treated as $\kappa\cdot(anA\cdot A + antr\chi\pi)$, where the average sign has been ignored. We also will not distinguish $\overline{nG}$ with nG. Then, in view of (6.44) and (6.47), symbolically there holds

$$\operatorname{div}\mathbf{D}_4\mu\!\!\!/ = n\left(A\cdot\mathbf{\nabla}\!\!\!/ A + (A+\mu\!\!\!/)\cdot(A\cdot\underline{A} + \underline{A}\cdot tr\chi) + (tr\chi)^2\pi + tr\chi\cdot\rho\right).$$

In view of (7.35) and (7.29), we also have

$$\operatorname{curl}\mathbf{D}_4\mu\!\!\!/ = n(\chi\cdot{}^\star\underline{\zeta} + {}^\star\beta)\cdot\mu\!\!\!/ + \mathbf{\nabla}\!\!\!/(n\hat\chi)\mu\!\!\!/ + \mathbf{\nabla}\!\!\!/(ntr\chi)\mu\!\!\!/.$$

Thus, with the help of Proposition 6.8 (b), we obtain (7.36). $\qquad\qquad\square$

Proof of Lemma 7.7. In view of (6.21) and (7.30), there holds $\|\mu\!\!\!/\|_{L_t^2 L_\omega^p} \lesssim \mathcal{R}_0$. By Lemma 6.18, (7.29) and (6.39), we can obtain

(7.38) $\qquad\qquad \|r\mathbf{\nabla}\!\!\!/\mu\!\!\!/\|_{L_t^2 L_\omega^p} \lesssim \|r\mu\|_{L_t^2 L_\omega^p} + \|\mu\!\!\!/\|_{L_t^2 L_\omega^p} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p}.$

Let

$$\mathcal{E} := \|r\mathbf{\nabla}\!\!\!/(\underline{A},\mu\!\!\!/)\|_{L_t^2 L_\omega^p} + \|rR_0\|_{L_t^2 L_\omega^p} + \|rtr\chi(\underline{A},\mu\!\!\!/)\|_{L_t^2 L_\omega^p} + \|r\underline{A}\cdot\underline{A}\|_{L_t^2 L_\omega^p},$$

we have $\mathcal{E} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p}$. By (6.22), we also have

(7.39) $\qquad\qquad \left(\int_u^{\tau_*}\|\mu\!\!\!/\|_{L^\infty(S_{t,u})}^2\right)^{\frac{1}{2}} \lesssim \mathcal{R}_0 + \mathcal{R}_{0,p}.$

By (7.34) and Lemma 6.12,

$$\|\mu\!\!\!/\|_{L^\infty(S_{t,u})} \lesssim \left\|\frac{1}{t-u}\int_u^t s\mathbf{D}_4\mu\!\!\!/\right\|_{L^\infty(S_{t,u})}.$$

Using (7.21) in Proposition 7.5 , (7.27), (7.36) and $C^{-1} < \mathrm{str}\chi < C$ on C_u, we can obtain

$$\left\| \frac{1}{t-u} \int_u^t s\mathbf{D}_4 \not\!\!\mu \right\|_{L^\infty(S_{t,u})} \lesssim \int_u^t r^{1-\frac{2}{p}} \left(\|e_1'\|_{L^p(S_{t,u})} + \|\operatorname{curl}\mathbf{D}_4\not\!\!\mu\|_{L^p(S_{t,u})} \right)$$

$$(7.40) \qquad\qquad + \frac{1}{t-u} \int_u^t \|\mu^{0+} P_\mu \overset{\circ}{\pi}\|_{l_\mu^2 L_\omega^\infty} + \frac{1}{t-u} \int_u^t \|\pi\|_{L_\omega^p}$$

Thus, by using (7.39), we can derive that

$$\int_u^t \left\| r^{1-\frac{2}{p}}(|e_1'| + |\operatorname{curl}\mathbf{D}_4\not\!\!\mu|) \right\|_{L^p(S_{t,u})} \lesssim \||A| + |z| + |\not\!\!\mu|\|_{L_t^2 L_\omega^\infty(C_u)} \cdot \mathcal{E}$$

$$\lesssim (\mathcal{R}_{0,p} + \mathcal{R}_0)\lambda^{-\frac{1}{2}}.$$

In view of (5.4) and (7.12), we have

$$\left(\int_0^{\tau_*} \sup_u \|\not\!\!\mu\|_{L^\infty(s_{t,u})}^2 dt' \right)^{\frac{1}{2}} \lesssim \|\mu^{0+} P_\mu \overset{\circ}{\pi}\|_{L_t^2 l_\mu^2 L_x^\infty} + \|\check{\pi}\|_{L_t^2 L_x^\infty}$$

$$+ (\mathcal{R}_0 + \mathcal{R}_{0,p})(\lambda^{-1}\tau_*)^{\frac{1}{2}}$$

This completes the proof.   □

8. Appendix I: Proof of Proposition 6.8

8.1. **Decomposition for $\beta, \underline{\beta}$.** We need the following decomposition result on R_{AN}.

Lemma 8.1. *On $S_{t,u} \subset \Sigma_t$, relative to orthonormal frame $\{N, e_1, e_2\}$, where N denotes the unit normal of $S_{t,u}$ in Σ_t, there holds* [6]

$$(8.1) \qquad R_{AN} = c\not\!url \overset{\circ}{\pi}_1 + \not\!\nabla_A \overset{\circ}{\pi}_2 + g \cdot \theta \cdot \widehat{\nabla} g + g \cdot \widehat{\nabla} g \cdot \widehat{\nabla} g + \hat{R} \cdot g$$

where the 2-tensor $\overset{\circ}{\pi}_1$ and the scalar $\overset{\circ}{\pi}_2$ are terms formed by the sums of $N^\mu(g \cdot \widehat{\nabla} g, k)_{\mu\cdots}$ and $\Pi_\mu^{\mu'}(g \cdot \widehat{\nabla} g, k)_{\mu'\cdots}$.

By the Gauss equation $E_{AN} = R_{AN} + k \cdot k$, the identity $E_{AN} = \frac{1}{2}(\beta + \underline{\beta})$ (see [7, (7.3.3e)]), and

$$\operatorname{div}\hat\eta = \frac{1}{2}(\beta - \underline{\beta}) - \frac{1}{2}\not\!\nabla\delta + \hat\theta \cdot \epsilon - \frac{1}{2}\mathrm{tr}\theta \cdot \epsilon$$

(see [7, (11.1.2d)]), we have symbolically that

$$\beta = \operatorname{div}\hat\eta + \not\!\nabla\delta + R_{AN} + \theta \cdot \epsilon + k \cdot k.$$

Hence Proposition 6.8 (a) can be proved by using (8.1).

Proof of Lemma 8.1. We write $R_{AN} = N^k e_A^j R_{jk}$. Recall that

$$R_{jk} = \hat{R}_{jk} + \widehat{\nabla}_m U_{jk}^m - \widehat{\nabla}_j U_{mk}^m - U_{jk}^n U_{nm}^m - U_{mk}^n U_{nj}^m.$$

Then we have

$$R_{AN} = N^k e_A^j \widehat{\nabla}_m U_{jk}^m - \left(e_A^j \widehat{\nabla}_j (N^k U_{mk}^m) - e_A^j \widehat{\nabla}_j N^k U_{mk}^m \right) + N^k e_A^j \hat{R}_{jk} - U \cdot U.$$

---

[6]We ignored the frame coefficients e_A^j, N^k in the last three terms, since they are harmless for the purpose of application.

For simplicity of exposition, we set $U_{kjl} = U_{kj}{}^p \cdot g_{pl}$ and $A_{klm} := \widehat{\nabla}_m g_{lk} - \widehat{\nabla}_l g_{mk}$. Then we have

$$\widehat{\nabla}_m U_{jk}{}^m = \frac{1}{2}\widehat{\nabla}_m\{g^{ml}(\widehat{\nabla}_k g_{jl} + \widehat{\nabla}_j g_{kl} - \widehat{\nabla}_l g_{jk})\}$$

$$= \widehat{\nabla}_m g^{ml} U_{kjl} + \frac{1}{2}g^{ml}(\widehat{\nabla}_m\widehat{\nabla}_k g_{jl} + \widehat{\nabla}_m\widehat{\nabla}_j g_{kl} - \widehat{\nabla}_m\widehat{\nabla}_l g_{jk})$$

$$= \widehat{\nabla}_m g^{ml} U_{kjl} + \frac{1}{2}g^{ml}(\widehat{\nabla}_j\widehat{\nabla}_m g_{kl} + \hat{R}) + \frac{1}{2}g^{ml}\widehat{\nabla}_m A_{jlk}.$$

We now define

$$\Omega_{ij} = \frac{1}{2}(\epsilon_i{}^{lm}\widehat{\nabla}_l g_{mj} + \epsilon_j{}^{lm}\widehat{\nabla}_l g_{mi}), \qquad \varrho_k = g^{mn}\widehat{\nabla}_k g_{mn} - g^{mn}\widehat{\nabla}_m g_{kn}.$$

Since the spacial harmonic condition implies that $g^{ij}\widehat{\nabla}_i g_{jl} = \frac{1}{2}g^{ij}\widehat{\nabla}_l g_{ij}$, we can derive that $\varrho_k = g^{mn}\widehat{\nabla}_m g_{nk}$. We claim that

$$(8.2) \qquad A_{iab} = \left(\Omega_{ij} + \frac{1}{2}\varrho_n\epsilon_{ij}{}^n\right)\epsilon_{ab}{}^j.$$

Using (8.2), we can deduce that

$$g^{ml}\widehat{\nabla}_m A_{jlk} = \epsilon_{lk}^s g^{ml}\widehat{\nabla}_m\left(\Omega_{js} + \frac{1}{2}\varrho_n\epsilon_{js}{}^n\right) + g^{ml}\widehat{\nabla}_m\epsilon_{lk}^s\left(\Omega_{js} + \frac{1}{2}\varrho_n\epsilon_{js}{}^n\right).$$

Hence, with $\dot{\pi}_{jC} = \frac{1}{2}(\Omega_{jC} + \frac{1}{2}\varrho_n\epsilon_{jC}{}^n)$, we have

$$e_A^j N^k \widehat{\nabla}_m U_{jk}{}^m$$

$$= e_A^j N^k \epsilon^m{}_k{}^s\widehat{\nabla}_m(\Omega_{js} + \frac{1}{2}\varrho_n\epsilon_{js}{}^n) + \frac{1}{2}e_A^j N^k g^{ml}\widehat{\nabla}_j\widehat{\nabla}_m g_{kl} + g\widehat{\nabla}g \cdot \widehat{\nabla}g + g \cdot \hat{R}$$

$$= -e_A^j \epsilon^{BC}\slashed{\nabla}_B \dot{\pi}_{jC} + \frac{1}{2}e_A^j N^k g^{ml}\widehat{\nabla}_j\widehat{\nabla}_m g_{kl} + \theta \cdot g \cdot \widehat{\nabla}g + g\widehat{\nabla}g \cdot \widehat{\nabla}g + g \cdot \hat{R}$$

$$= -(\text{curl}\,\dot{\pi})_A + \frac{1}{2}e_A^j\widehat{\nabla}_j(N^k g^{ml}\widehat{\nabla}_m g_{kl}) + \theta \cdot g \cdot \widehat{\nabla}g + g\widehat{\nabla}g \cdot \widehat{\nabla}g + g \cdot \hat{R}.$$

We then conclude symbolically that

$$R_{AN} = \slashed{\nabla}_A\left(\frac{1}{2}g^{ml}\widehat{\nabla}_m g_{kl}N^k + N^k U_{mk}{}^m\right) + \text{curl}\,\dot{\pi} + g \cdot \theta \cdot \widehat{\nabla}g + g\widehat{\nabla}g \cdot \widehat{\nabla}g + g \cdot \hat{R}$$

as desired.

It remains to verify (8.2). By the definition of Ω_{ij} we have

$$2\Omega_{ij}\epsilon_{ab}{}^j = \epsilon_i{}^{lm}\epsilon_{ab}{}^j\widehat{\nabla}_l g_{mj} + \epsilon_j{}^{lm}\epsilon_{ab}{}^j\widehat{\nabla}_l g_{mi}.$$

From the properties of the volume form components $\epsilon_i{}^{jl}$, we have

$$\epsilon_j{}^{lm}\epsilon_{ab}{}^j\widehat{\nabla}_l g_{mi} = \widehat{\nabla}_a g_{bi} - \widehat{\nabla}_b g_{ai}, \quad \epsilon_i{}^{lm}\epsilon_{ab}{}^j\widehat{\nabla}_l g_{mj} = \widehat{\nabla}_a g_{bi} - \widehat{\nabla}_b g_{ai} + g_{ia}\varrho_b - g_{ib}\varrho_a.$$

It is easy to check that $g_{ia}\varrho_b - g_{ib}\varrho_a = -\varrho_n\epsilon_{ij}{}^n\epsilon_{ab}{}^j$. Hence we have

$$\Omega_{ij}\epsilon_{ab}{}^j = \widehat{\nabla}_a g_{bi} - \widehat{\nabla}_b g_{ai} - \frac{1}{2}\varrho_n\epsilon_{ij}{}^n\epsilon_{ab}{}^j.$$

This completes the proof of (8.2). $\qquad\square$

8.2. **Decompositions of α and σ.** The decompositions of α and σ follow immediately from the structure equations

$$\frac{1}{2}\alpha = \mathbf{D}_4\hat{\eta} + \frac{1}{2}\mathrm{tr}\chi\hat{\eta} + n^{-1}\hat{\slashed{\nabla}}^2 n - \frac{1}{2}\slashed{\nabla}\widehat{\otimes}\epsilon + \frac{1}{2}\delta\hat{\eta} - a^{-1}\slashed{\nabla}a\widehat{\otimes}\epsilon - \frac{3}{2}\hat{\theta}\delta$$
$$- \epsilon\widehat{\otimes}\epsilon + (\zeta - \slashed{\nabla}\log n)\widehat{\otimes}\epsilon$$

and $\sigma = \mathrm{c\slashed{u}rl}\,\epsilon - \hat{\theta} \wedge \hat{\eta}$ which can be found in [7, Chapter 11].

8.3. **Decomposition for ρ.** Let $(e_A)_{A=1,2}$ be an orthonormal frame on S, and let $\overset{\circ}{\pi}{}^C_{AB}$ be the projection of U^l_{ij} onto S, i.e.

$$\overset{\circ}{\pi}{}^C_{AB} = e^i_A e^j_B (e_C)_k U_{i'j'}{}^{k'} \Pi^{i'}_i \Pi^{j'}_j \Pi^k_{k'}.$$

where U^l_{ij} is the tensor defined by (1.8) and $\Pi^j_i = \delta^j_i - N_i N^j$ is the projection operator. Recall that

$$R^i{}_{jkl} = \widehat{\nabla}_k U_{lj}{}^i - \widehat{\nabla}_l U_{kj}{}^i + U_{km}{}^i U_{lj}{}^m - U_{lm}{}^i U_{kj}{}^m + \hat{R}^i{}_{jkl}$$

By using $\widehat{\nabla}_k U_{lj}{}^i = \nabla_k U_{lj}{}^i + U \cdot U$, we have

$$(8.3) \qquad R_{ABCD} = e_{Ai}e^j_B e^k_C e^l_D R^i{}_{jkl} = \slashed{\nabla}_C \overset{\circ}{\pi}_{DBA} - \slashed{\nabla}_D \overset{\circ}{\pi}_{CBA} + \tilde{E}_{ABCD}.$$

where $\tilde{E}$ consists of the projections $U \cdot U$ to S and the terms $\theta \cdot (N^l U_{l\dots}, \Pi^l_m U_{l\dots})$ and for $|\tilde{E}| := \max_{A,B,C,D} |\tilde{E}_{ABCD}|$ we have $|\tilde{E}| \lesssim |\widehat{\nabla}g \cdot \widehat{\nabla}g| + |\theta \cdot \widehat{\nabla}g|$. Recall from [7, P.167] that $\gamma^{AC}\gamma^{BD}\mathbf{R}_{ADCB} = -2\rho$. By the Gauss equation, we then have $-2\rho = \gamma^{AC}\gamma^{BD}(R_{ABCD} + k \cdot k)$. This together with (8.3) gives the decomposition for ρ in Proposition 6.8.

9. **Appendix II: Proof of Proposition 7.5**

In this section we will complete the proof of Proposition 7.5 by employing the geometric Littlewood-Paley theory on the surfaces S_t developed in [16]. In Section 6 we introduced the metric $\gamma^{(0)} := r^2\gamma_{\mathbb{S}^2}$ on $S_{t,u}$, where r denotes the the radius of $S_{t,u}$ with respect to γ, i.e $r := \sqrt{\frac{1}{4\pi}\int_{S_{t,u}} d\mu_\gamma}$. Let m be a function in the Schwartz class defined on $[0,\infty)$ having finite number of vanishing moments and set $m_\mu(\tau) := \mu^2 m(\mu^2\tau)$ for any dyadic numbers $\mu > 0$. The geometric Littlewood-Paley projection $P^{(0)}_\mu$ associated with $\gamma^{(0)}$ is defined by

$$P^{(0)}_\mu H = \int_0^\infty m_\mu(\tau)U(\tau)H d\tau$$

for any S-tangent tensor H, where $U(\tau')H$ denotes the solution of the heat flow

$$\frac{d}{d\tau}U(\tau)H = r^2\slashed{\Delta}^{(0)}U(\tau)H, \qquad U(0)H = H.$$

One can refer to [16] for various properties of $U(\tau)$ and $P^{(0)}_\mu$. In particular, it has been shown that one can always find an m such that the associated $P^{(0)}_\mu$ satisfies $\sum_{\mu>1} P^{(0)2}_\mu + P^{(0)2}_{\leq 1} = Id$.

Lemma 9.1. *For $2 \le p < \infty$, and any S-tangent tensor H there hold on S_t that*

(9.1)
$$\|[P_\mu^{(0)}, \slashed{\nabla}^{(0)^2}]H\|_{L^p} \lesssim r^{-2}\|H\|_{L^p},$$

(9.2)
$$[P_\mu^{(0)}, \slashed{\nabla}^{(0)}]H\|_{L^p} \lesssim r^{-2+\frac{2}{p}}\mu^{-\frac{2}{p}}\|H\|_{L^2}.$$

Proof. Symbolically we have $[\slashed{\Delta}^{(0)}, \slashed{\nabla}^{(0)^2}]H = r^{-2}\gamma^{(0)} \cdot \slashed{\nabla}^{(0)^2}H$. By the definition of $P_\mu^{(0)}H$ and the Duhumal principle for heat flow, we have

$$[P_\mu^{(0)}, \slashed{\nabla}^{(0)^2}]H = r^2 \int_0^\infty m_\mu(\tau) \int_0^\tau U(\tau - \tau')[\slashed{\Delta}^{(0)}, \slashed{\nabla}^{(0)^2}]U(\tau')H d\tau'$$

$$= \int_0^\infty m_\mu(\tau) \int_0^\tau U(\tau - \tau')\slashed{\nabla}^{(0)^2}U(\tau')H d\tau'.$$

Taking the L^p norm gives

$$\|[P_\mu^{(0)}, \slashed{\nabla}^{(0)^2}]H\|_{L^p} \lesssim r^{-2}\|H\|_{L^p} \int_0^\infty m_\mu(\tau) \int_0^\tau (\tau - \tau')^{-\frac{1}{2}}\tau'^{-\frac{1}{2}}d\tau' \lesssim r^{-2}\|H\|_{L^p}.$$

We next prove (9.2). In view of

$$[P_\mu^{(0)}, \slashed{\nabla}^{(0)}]H = r^2 \int_0^\infty m_\mu(\tau) \int_0^\tau U(\tau - \tau')[\slashed{\Delta}^{(0)}, \slashed{\nabla}^{(0)}]U(\tau')H$$

$$= \int_0^\infty m_\mu(\tau) \int_0^\tau U(\tau - \tau')\slashed{\nabla}^{(0)}U(\tau')H.$$

We then derive by the Sobolev inequality and the L^2 estimate of heat operator $U(\tau)$ that

$$\|[P_\mu^{(0)}, \slashed{\nabla}^{(0)}]H\|_{L^p} \lesssim r^{-2+\frac{2}{p}}\|H\|_{L^2} \int_0^\infty m_\mu(\tau) \int_0^\tau \tau'^{-\frac{1}{2}}(\tau - \tau')^{-\frac{1}{2}+\frac{1}{p}}d\tau'$$

$$\lesssim r^{-2+\frac{2}{p}}\mu^{-\frac{2}{p}}\|H\|_{L^2}$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Now we divide the proof of (7.21) into two steps. The first step is to prove

Lemma 9.2. *For S-tangent tensor fields F and G satisfying the Hodge system in Proposition 7.4, there holds with $\delta > 0$ and $p > 2$ that*

$$\|F\|_{L^\infty(S_{t,u})} \lesssim \|\mu^\delta P_\mu^{(0)}G\|_{l_\mu^1 L^\infty(S_{t,u})} + \|G\|_{L^\infty(S_{t,u})} + r^{1-\frac{2}{p}}\|e\|_{L^p(S_{t,u})}$$

Proof. Let $\bar{F}$ be defined by (6.57). Using the following standard identity for Hodge operators (see [7, Page 38])

$$-\frac{1}{2}\slashed{\Delta}^{(0)}\bar{F} = {}^*\mathcal{D}_2^{(0)}\mathcal{D}_2^{(0)}\bar{F} - \frac{1}{r^2}\bar{F},$$

we can obtain

$$-\frac{1}{2}\slashed{\Delta}^{(0)}\bar{F} = {}^*\mathcal{D}_2^{(0)}(\mathcal{D}_2 F + (\Gamma - \Gamma^{(0)})F \cdot \gamma) - \frac{1}{r^2}\bar{F}.$$

We can always find the Littlewood Paley projection $\tilde{P}_\mu^{(0)}$ associated to a different symbol $\tilde{m}$ such that

(9.3)
$$\mu^2 P_\mu^{(0)} = r^2 \slashed{\Delta}^{(0)} \tilde{P}_\mu^{(0)}$$

we can derive symbolically that

(9.4)
$$P_\mu^{(0)}\bar{F} = \mu^{-2}r^2\left(\tilde{P}_\mu^{(0)*}\mathcal{D}_2^{(0)}(\nabla\!\!\!/^{(0)}G + (\Gamma - \Gamma^{(0)})(F\cdot\gamma + G) + e)\right) + \mu^{-2}\tilde{P}_\mu^{(0)}\bar{F}.$$

In what follows, we will not distinguish $\tilde{P}_\mu^{(0)}$ with $P_\mu^{(0)}$. By the Bernstein inequality and the finite band property, for $p > 2$ satisfying $0 < 1 - 2/p < s - 2$, we can derive on S that

$$I_\mu =: \mu^{-2}r^2\|P_\mu^{(0)}\nabla\!\!\!/^{(0)}\left((\Gamma - \Gamma^{(0)})(F\cdot\gamma + G) + e\right)\|_{L^\infty} + \mu^{-2}\|P_\mu^{(0)}\bar{F}\|_{L^\infty}$$

$$\lesssim \mu^{-2+\frac{2}{p}+1}r^{1-\frac{2}{p}}\|(\Gamma - \Gamma^{(0)})(F\cdot\gamma + G) + e\|_{L^p} + \mu^{-2}\|P_\mu^{(0)}\bar{F}\|_{L^\infty}$$

$$\lesssim \mu^{\frac{2}{p}-1}\left(r^{1-\frac{2}{p}}(\|e\|_{L^p} + \|F\|_{L_x^\infty}\|\Gamma - \Gamma^{(0)}\|_{L^p}) + \|G\|_{L^\infty}\right)$$

where, with the help of the Bernstein inequality and (7.17), the lowest order term $\mu^{-2}\|P_\mu^{(0)}\bar{F}\|_{L_x^\infty}$ has been treated as

$$\|P_\mu^{(0)}\bar{F}\|_{L_x^\infty} \lesssim \mu^{\frac{2}{p''}}r^{-\frac{2}{p''}}\|F\|_{L_x^{p''}} \lesssim \mu^{\frac{2}{p''}}(\|G\|_{L_\omega^{p''}} + \|re\|_{L_\omega^q}),$$

with $p'' > 2$ being a large number and $1/q = 1/2 + 1/p''$. Therefore, we obtain that

$$\sum_{\mu>1}I_\mu \lesssim \|r^{1-\frac{2}{p}}|\Gamma - \Gamma^{(0)}|_\gamma\|_{L^p}\|F\|_{L_x^\infty} + \|G\|_{L_x^\infty} + r^{1-\frac{2}{p}}\|e\|_{L^p}.$$

It remains to consider the first term in (9.4). Using the Bernstein inequality with $q > 2$, we have $\mu^{-2}r^2\|P_\mu^{(0)*}\mathcal{D}_2^{(0)}\nabla\!\!\!/^{(0)}G\|_{L^\infty} \lesssim a_\mu + b_\mu$ where

$$a_\mu = (\mu^{-1}r)^{2-\frac{2}{q}}\|^*\mathcal{D}_2^{(0)}\nabla\!\!\!/^{(0)}P_\mu^{(0)}G\|_{L^q} \quad \text{and} \quad b_\mu = (\mu^{-1}r)^{2-\frac{2}{q}}\|[P_\mu^{(0)}, \nabla\!\!\!/^{(0)^2}]G\|_{L^q}.$$

By the L^q theory for elliptic equations, the finite band property, and (9.1), we have

(9.5) $$a_\mu \lesssim \mu^{\frac{2}{q}}r^{-\frac{2}{q}}\|P_\mu^{(0)}G\|_{L^q}, \qquad b_\mu \lesssim \mu^{-2+\frac{2}{q}}r^{-\frac{2}{q}}\|G\|_{L^q},$$

which together with the estimate on I_μ imply

$$\sum_{\mu>1}\|P_\mu^{(0)}\bar{F}\|_{L^\infty} \lesssim \sum_{\mu>1}\mu^{\frac{2}{q}}\|P_\mu^{(0)}G\|_{L_\omega^q} + \|r^{1-\frac{2}{p}}|\Gamma - \Gamma^{(0)}|_{\gamma^{(0)}}\|_{L^p}\|F\|_{L_x^\infty}$$

$$+ \|G\|_{L_x^\infty} + r^{1-\frac{2}{p}}\|e\|_{L^p}.$$

Using Proposition 6.20 and (WS1), we therefore complete the proof. $\square$

The second step is to prove Lemma 9.4, for which we need the following result.

Lemma 9.3. *For any $p > 2$ and any S-tangent tensor F, there holds on $S := S_{t,u}$ that*

(9.6) $$\|P_\mu^{(0)}F\|_{L^\infty} \lesssim \mu^{-1+\frac{2}{p}}r^{1-\frac{2}{p}}\|P_\mu^{(0)}\nabla\!\!\!/^{(0)}F\|_{L^p} + \mu^{-2}\|\nabla\!\!\!/^{(0)}F\|_{L^2}.$$

Proof. With the help of (9.3) and the Bernstein inequality, we have

$$\|P_\mu^{(0)}F\|_{L^\infty} = \mu^{-2}r^2\|\Delta\!\!\!/\tilde{P}_\mu^{(0)}F\|_{L^\infty} \lesssim \mu^{-2+\frac{2}{p}}r^{2-\frac{2}{p}}\|\tilde{P}_\mu^{(0)}\Delta\!\!\!/^{(0)}F\|_{L^p}$$

$$\lesssim \mu^{-2+\frac{2}{p}}r^{2-\frac{2}{p}}\left(\|[\tilde{P}_\mu^{(0)}, \nabla\!\!\!/^{(0)}]\nabla\!\!\!/^{(0)}F\|_{L^p} + \|\nabla\!\!\!/^{(0)}\tilde{P}_\mu^{(0)}\nabla\!\!\!/^{(0)}F\|_{L^p}\right)$$

$$:= \mathcal{A}_\mu^{(1)} + \mathcal{A}_\mu^{(2)}.$$

From (9.2) it follows that

$$r^{2-\frac{2}{p}}\|[\tilde{P}_\mu^{(0)}, \nabla\!\!\!/^{(0)}]\nabla\!\!\!/^{(0)}F\|_{L^p} \lesssim \mu^{-\frac{2}{p}}\|\nabla\!\!\!/^{(0)}F\|_{L^2},$$

which implies $\mathcal{A}_\mu^{(1)} \lesssim \mu^{-2}\|\nabla^{(0)}F\|_{L^2}$. In view of the finite band property, we also have

$$\mathcal{A}_\mu^{(2)} \lesssim \mu^{-1+\frac{2}{p}}r^{1-\frac{2}{p}}\|\tilde{P}_\mu^{(0)}\nabla^{(0)}F\|_{L^p}.$$

Thus, we complete the proof of (9.6). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Lemma 9.4. *(i) For $0 < \epsilon < 1$, $\mu > 1$ and any scalar function f, there holds on $S := S_{t,u}$ that*

$$(9.7) \quad \mu^\epsilon\|P_\mu^{(0)}f\|_{L^\infty} \lesssim \left(\sum_{\ell\geq\mu}\left(\frac{\mu}{\ell}\right)^\epsilon + \sum_{\ell<\mu}\left(\frac{\ell}{\mu}\right)^{1-\epsilon}\right)\|\ell^\epsilon P_\ell f\|_{L^\infty} + \mu^{-1+\epsilon}\|f\|_{L^\infty}.$$

(ii) Let $\delta, \delta' > 0$ be such that $\delta + \delta' < 1$ and let $q > 2$ be such that $2/q < 1 - \delta$. Then for any scalar functions f and H and any $\mu > 1$ there holds

$$\mu^\delta\|P_\mu^{(0)}(f\cdot H)\|_{L^\infty} \lesssim \|H\|_{L^\infty}\left(\sum_{\ell\geq\mu}\left(\frac{\mu}{\ell}\right)^\delta + \sum_{\ell<\mu}\left(\frac{\ell}{\mu}\right)^{1-\delta'-\delta}\ell^{\delta'}\right)\|\ell^\delta P_\ell f\|_{L_x^\infty}$$

$$(9.8) \qquad\qquad + \mu^{\frac{2}{q}-1+\delta}\|f\|_{L^\infty}\|r\nabla^{(0)}H\|_{L_\omega^q}.$$

Proof. We first prove (i). We can write

$$P_\mu^{(0)}f = P_\mu^{(0)}\sum_{\ell\geq\mu}P_\ell f + P_\mu^{(0)}\sum_{1\leq\ell<\mu}P_\ell f + P_\mu^{(0)}P_{\leq 1}f = I_\mu^{(1)} + I_\mu^{(2)} + I_\mu^{(3)}.$$

By using the properties of the geometric Littlewood-Paley projections, we have on $S_{t,u}$ that

$$\mu^\epsilon\|I_\mu^{(1)}\|_{L^\infty} \lesssim \sum_{\ell\geq\mu}\left(\frac{\mu}{\ell}\right)^\epsilon\|\ell^\epsilon P_\ell f\|_{L^\infty},$$

$$\mu^\epsilon\|I_\mu^{(2)}\|_{L^\infty} \lesssim \sum_{1\leq\ell<\mu}\mu^{-2+\epsilon}r^2\|\Delta^{(0)}\tilde{P}_\ell f\|_{L^\infty} \lesssim \sum_{1\leq\ell<\mu}\left(\frac{\ell}{\mu}\right)^{2-\epsilon}\|\ell^\epsilon\tilde{P}_\ell f\|_{L^\infty},$$

$$\|I_\mu^{(3)}\|_{L^\infty} \lesssim \mu^{-2}r^2\|\Delta^{(0)}P_{\leq 1}f\|_{L^\infty} \lesssim \mu^{-2}\|f\|_{L^\infty}.$$

Combining the above estimates gives (9.7).

Now we prove (ii). In view of $\sum_\ell P_\ell f = f$ and set $f_\ell := P_\ell f$, we can write

$$P_\mu^{(0)}(f\cdot H) = J_\mu^{(1)} + J_\mu^{(2)} + J_\mu^{(3)}$$

$$= \sum_{1<\ell<\mu}P_\mu^{(0)}(f_\ell\cdot H) + \sum_{\ell\geq\mu}P_\mu^{(0)}(f_\ell\cdot H) + P_\mu^{(0)}(f_{\leq 1}\cdot H).$$

It is straightforward to derive that

$$\mu^\delta\|J_\mu^{(2)}\|_{L^\infty} \lesssim \|H\|_{L^\infty}\sum_{\ell\geq\mu}\left(\frac{\mu}{\ell}\right)^\delta\ell^\delta\|f_\ell\|_{L^\infty}.$$

We now estimate $J_\mu^{(1)}$ by using (9.6). Let $f_{1<\cdot<\mu} := \sum_{1<\ell<\mu} f_\ell$. Then for sufficiently large p we have

$$\mu^\delta \|J_\mu^{(1)}\|_{L^\infty} \lesssim \mu^{\delta-1+\frac{2}{p}} r^{1-\frac{2}{p}} \|P_\mu^{(0)} \slashed{\nabla}^{(0)}(f_{1<\cdot<\mu} \cdot H)\|_{L^p} + \mu^{-2+\delta} \|\slashed{\nabla}^{(0)}(f_{1<\cdot<\mu} \cdot H)\|_{L^2}$$

$$\lesssim \mu^{\delta-1+\frac{2}{p}} r^{1-\frac{2}{p}} \|P_\mu^{(0)}(\slashed{\nabla}^{(0)} f_{1<\cdot<\mu} \cdot H + f_{1<\cdot<\mu} \cdot \slashed{\nabla}^{(0)} H)\|_{L^p}$$

$$+ \mu^{-2+\delta} \|\slashed{\nabla}^{(0)}(f_{1<\cdot<\mu} \cdot H)\|_{L^2}$$

$$\lesssim \mu^{\delta-1+\frac{2}{p}} \|H\|_{L^\infty} \sum_{1<\ell<\mu} \ell \|f_\ell\|_{L^p_\omega} + \mu^{\delta-2} \|f\|_{L^\infty} \|\slashed{\nabla}^{(0)} H\|_{L^2}$$

$$+ \mu^{\delta-1+\frac{2}{p}} r^{1-\frac{2}{p}} \|P_\mu^{(0)}(f_{1<\cdot<\mu} \cdot \slashed{\nabla}^{(0)} H)\|_{L^p}.$$

For the last term we proceed with $q > 2$ sufficiently close to 2 and the Bernstein inequality to obtain

$$\mu^{\delta-1+\frac{2}{p}} r^{1-\frac{2}{p}} \|P_\mu^{(0)}(f_{1<\cdot<\mu} \cdot \slashed{\nabla}^{(0)} H)\|_{L^p} \lesssim \mu^{\delta-1+\frac{2}{q}} r^{1-\frac{2}{q}} \|f\|_{L^\infty} \|\slashed{\nabla}^{(0)} H\|_{L^q}.$$

Thus

$$\mu^\delta \|J_\mu^{(1)}\|_{L^\infty} \lesssim \|H\|_{L^\infty} \sum_{1<\ell<\mu} \left(\frac{\ell}{\mu}\right)^{1-\delta-\frac{2}{p}} \|\ell^{\delta+\frac{2}{p}} f_\ell\|_{L^p_\omega} + \mu^{\delta-1+\frac{2}{q}} \|f\|_{L^\infty} \|r\slashed{\nabla}^{(0)} H\|_{L^q_\omega}$$

The low frequency part $J_\mu^{(3)}$ can be treated similarly. Hence we obtain (9.8). □

Now we prove (7.21). By abuse of notation, we can write G as the sum of two types of terms $\tilde{G}_\mu N^\mu$ and $\tilde{G}_\mu$. We may apply (9.7) to the components of $\tilde{G}_\mu$ to treat the second type of terms. To treat the first type of terms, we may apply (9.8) to $f = \tilde{G}_\mu$ and $H = N^\mu$. Note that $|\slashed{\nabla}^{(0)} N| \leq |\text{tr}\theta, \hat{\theta}, g \cdot \widehat{\nabla} g|$ for which we have

$$r^{1-\frac{2}{p}} \|\slashed{\nabla}^{(0)} N\|_{L^p(S_{t,u})} \lesssim 1, \quad \text{with } 2 \leq p \leq 4.$$

This together with Lemma 9.4 and Lemma 9.2 implies (7.21).

10. Appendix III: Commutator estimates

In this section we derive various commutator estimates involving the LP projections P_λ in fractional Sobolev spaces that are extensively used in this paper, where P_λ is defined by (1.11). One can refer to [20, 16] for various properties of LP projections. In view of the LP decomposition, the norm in the Sobolev space H^ϵ with $0 \leq \epsilon < 1$ is defined by

$$\|F\|_{H^\epsilon} := \|F\|_{L^2} + \left(\sum_{\lambda>1} \lambda^{2\epsilon} \|P_\lambda F\|_{L^2}^2\right)^{1/2}$$

for any scalar function F. For any nonnegative integer m and $0 \leq \epsilon < 1$, we define $\|F\|_{H^{m+\epsilon}} := \|F\|_{H^m} + \|\widehat{\nabla}^m F\|_{H^\epsilon}$. For simplicity of exposition, we will write $F_\lambda := P_\lambda F$, $F_{\leq \lambda} := \sum_{\mu \leq \lambda} P_\mu F$, and $\|\Lambda^r F\|_{L^2} := \left(\sum_{\lambda>1} \lambda^{2r} \|P_\lambda F\|_{L^2}^2\right)^{1/2}$. For any sequence (a_λ) we will use $\|a_\lambda\|_{L^2_\lambda}^2$ to denote $\sum_{\lambda \geq 1} |a_\lambda|^2$.

10.1. **Product estimates.** We first derive some useful product estimates. Let P_μ denote the LP projection. According to the Littlewood-Paley (LP) decomposition, one has the trichotomy law which schematically says that for any scalar functions F and G there holds

$$(10.1) \qquad P_\mu(F \cdot G) = P_\mu(F_{\leq \mu} \cdot G_\mu) + P_\mu(F_\mu \cdot G_{\leq \mu}) + \sum_{\lambda > \mu} P_\mu(F_\lambda \cdot G_\lambda).$$

We will use this decomposition repeatedly.

Lemma 10.1. *For any $0 < \epsilon < 1$ and any scalar functions F and G there hold*

$$(10.2) \qquad \|\Lambda^\epsilon(F \cdot G)\|_{L^2} \lesssim \|F\|_{H^{1/2+\epsilon}} \|G\|_{H^1} + \|G\|_{H^{1/2+\epsilon}} \|F\|_{H^1},$$

$$(10.3) \qquad \|\mu^{-1/2+\epsilon} P_\mu(F \cdot G)\|_{l^2_\mu L^2} \lesssim \|G\|_{H^\epsilon} \|F\|_{H^1}.$$

Proof. We first prove (10.2). By using the Bernstein inequality and the finite band property of the LP projections, we have

$$\mu^\epsilon \|P_\mu(F_{\leq \mu} \cdot G_\mu)\|_{L^2}$$

$$\lesssim \mu^\epsilon \sum_{\lambda \leq \mu} \|F_\lambda\|_{L^\infty} \|G_\mu\|_{L^2} \lesssim \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \|\lambda^{1/2+\epsilon} F_\lambda\|_{L^2} \|\widehat{\nabla} G_\mu\|_{L^2}.$$

Therefore

$$(10.4) \qquad \|\mu^\epsilon P_\mu(F_{\leq \mu} \cdot G_\mu)\|_{l^2_\mu L^2_x} \lesssim \|F\|_{H^{1/2+\epsilon}} \|\widehat{\nabla} G\|_{L^2}.$$

Similarly we have

$$(10.5) \qquad \|\mu^\epsilon P_\mu(F_\mu \cdot G_{\leq \mu})\|_{l^2_\mu L^2} \lesssim \|G\|_{H^{1/2+\epsilon}} \|\widehat{\nabla} F\|_{L^2}.$$

Moreover, we have

$$\mu^\epsilon \|P_\mu(F_\lambda \cdot G_\lambda)\|_{L^2} \lesssim \mu^{1/2+\epsilon} \|F_\lambda \cdot G_\lambda\|_{L^{\frac{3}{2}}} \lesssim \mu^{1/2+\epsilon} \|F_\lambda\|_{L^2} \|G_\lambda\|_{L^6}$$

$$\lesssim \left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon} \|\lambda^{1/2+\epsilon} F_\lambda\|_{L^2} \|\widehat{\nabla} G_\lambda\|_{L^2}.$$

This implies that

$$(10.6) \qquad \left\| \mu^\epsilon \sum_{\lambda > \mu} P_\mu(F_\lambda \cdot G_\lambda) \right\|_{l^2_\mu L^2} \lesssim \|\Lambda^{1/2+\epsilon} F\|_{L^2} \|\widehat{\nabla} G\|_{L^2}.$$

Combining the above estimates and using the trichotomy law we obtain (10.2).

Next we prove (10.3) by using again the trichotomy. We have

$$\|\mu^{-1/2+\epsilon} P_\mu(F_{\leq \mu} \cdot G_\mu)\|_{l^2_\mu L^2} \lesssim \left\| \mu^{-1/2+\epsilon} \|G_\mu\|_{L^2} \|F_{\leq \mu}\|_{L^\infty} \right\|_{l^2_\mu}$$

$$\lesssim \left\| \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1/2} \|\mu^\epsilon G_\mu\|_{L^2} \|\widehat{\nabla} F_\lambda\|_{L^2} \right\|_{l^2_\mu} \lesssim \|F\|_{H^1} \|G\|_{H^\epsilon},$$

$$\|\mu^{-1/2+\epsilon} P_\mu(F_\mu \cdot G_{\leq \mu})\|_{l^2_\mu L^2} \lesssim \left\| \mu^{-1/2+\epsilon} \|F_\mu\|_{L^2} \|G_{\leq \mu}\|_{L^\infty} \right\|_{l^2_\mu}$$

$$\lesssim \left\| \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{3/2-\epsilon} \|\widehat{\nabla} F\|_{L^2} \|\lambda^\epsilon G_\lambda\|_{L^2} \right\|_{l^2_\mu} \lesssim \|\widehat{\nabla} F\|_{L^2} \|G\|_{H^\epsilon}$$

and

$$\left\| \sum_{\lambda > \mu} \mu^{-1/2+\epsilon} P_\mu(F_\lambda \cdot G_\lambda) \right\|_{l^2_\mu L^2}$$

$$\lesssim \left\| \mu^{1/2+\epsilon} \sum_{\lambda > \mu} \|F_\lambda \cdot G_\lambda\|_{L^{6/5}} \right\|_{l^2_\mu} \lesssim \left\| \mu^{1/2+\epsilon} \sum_{\lambda \geq \mu} \|F_\lambda\|_{L^3} \|G_\lambda\|_{L^2} \right\|_{l^2_\mu}$$

$$\lesssim \left\| \sum_{\lambda \geq \mu} \left(\frac{\mu}{\lambda} \right)^{1/2+\epsilon} \|\widehat{\nabla} F_\lambda\|_{L^2} \|\lambda^\epsilon G_\lambda\|_{L^2} \right\|_{l^2_\mu} \lesssim \|F\|_{H^1} \|G\|_{H^\epsilon}.$$

Combining the above estimates yields (10.3).   □

Lemma 10.2. *For any $\epsilon > 0$ and any scalar functions G_1, G_2 and G_3 there holds*

$$\|\Lambda^\epsilon(G_1 G_2 G_3)\|_{L^2} \lesssim \sum_{j=1}^3 \left(\|G_j\|_{H^{1+\epsilon}} \prod_{l \neq j} \|G_l\|_{H^1} \right).$$

Proof. By using the properties of LP projections and the estimate (10.3) in Lemma 10.1 we have

$$\|\mu^\epsilon P_\mu\left((G_1)_{\leq \mu}(G_2 G_3)_\mu\right)\|_{l^2_\mu L^2} \lesssim \left\| \sum_{\lambda \leq \mu} \mu^\epsilon \|(G_1)_\lambda\|_{L^\infty} \|(G_2 G_3)_\mu\|_{L^2} \right\|_{l^2_\mu}$$

$$\lesssim \left\| \sum_{\lambda \leq \mu} \mu^\epsilon \lambda^{1/2} \|\widehat{\nabla}(G_1)_\lambda\|_{L^2} \|(G_2 G_3)_\mu\|_{L^2} \right\|_{l^2_\mu} \lesssim \|\widehat{\nabla} G_1\|_{L^2} \left\| \mu^{-1/2+\epsilon} P_\mu \widehat{\nabla}(G_2 G_3) \right\|_{l^2_\mu L^2}$$

$$\lesssim \|G_1\|_{H^1} \left(\|G_2\|_{H^{1+\epsilon}} \|G_3\|_{H^1} + \|G_3\|_{H^{1+\epsilon}} \|G_2\|_{H^1} \right),$$

$$\|\mu^\epsilon P_\mu\left((G_1)_\mu(G_2 G_3)_{\leq \mu}\right)\|_{l^2_\mu L^2} \lesssim \|\mu^\epsilon \|(G_1)_\mu\|_{L^3} \|(G_2 G_3)_{\leq \mu}\|_{L^6} \|_{l^2_\mu}$$

$$\lesssim \left\| \mu^{\epsilon-1/2} \|\widehat{\nabla}(G_1)_\mu\|_{L^3} \right\|_{l^2_\mu} \|G_2 G_3\|_{L^3}$$

$$\lesssim \|\mu^\epsilon \widehat{\nabla}(G_1)_\mu\|_{l^2_\mu L^2} \|G_2\|_{L^6} \|G_3\|_{L^6}$$

$$\lesssim \|G_1\|_{H^{1+\epsilon}} \|G_2\|_{H^1} \|G_3\|_{H^1}$$

and

$$\left\| \sum_{\lambda > \mu} \mu^\epsilon P_\mu\left((G_1)_\lambda(G_2 G_3)_\lambda\right) \right\|_{l^2_\mu L^2} \lesssim \left\| \sum_{\lambda > \mu} \mu^\epsilon \|(G_1)_\lambda\|_{L^6} \|G_2 G_3\|_{L^3} \right\|_{l^2_\mu}$$

$$\lesssim \left\| \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda} \right)^\epsilon \|\lambda^\epsilon \widehat{\nabla}(G_1)_\lambda\|_{L^2} \right\|_{l^2_\mu} \|G_2\|_{H^1} \|G_3\|_{H^1}$$

$$\lesssim \|G_1\|_{H^{1+\epsilon}} \|G_2\|_{H^1} \|G_3\|_{H^1}.$$

In view of the trichotomy and the above estimates, we thus complete the proof.   □

Lemma 10.3. *For any $0 < \epsilon < 1$ there hold*

$$(10.7) \qquad \|\Lambda^\epsilon(F \cdot \widehat{\nabla} G)\|_{L^2} \lesssim \|F\|_{L^\infty} \|G\|_{H^{1+\epsilon}} + \|G\|_{L^\infty} \|F\|_{H^{1+\epsilon}},$$

Proof. We will use the trichotomy law to derive the estimates. To derive (10.7), we use the properties of the LP projections to obtain

$$\|\mu^\epsilon P_\mu(F_{\leq \mu} \cdot \widehat{\nabla} G_\mu)\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty} \|\mu^\epsilon(\widehat{\nabla} G)_\mu\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty} \|\Lambda^\epsilon \widehat{\nabla} G\|_{L^2},$$

$$\|\mu^\epsilon P_\mu(F_\mu \cdot \widehat{\nabla} G_{\leq \mu})\|_{l^2_\mu L^2} \lesssim \left\| \mu^\epsilon \|F_\mu\|_{L^2} \sum_{\lambda < \mu} \lambda \|G_\lambda\|_{L^\infty} \right\|_{l^2_\mu} \lesssim \|\mu^{1+\epsilon} F_\mu\|_{l^2_\mu L^2} \|G\|_{L^\infty}$$

$$\lesssim \|\Lambda^\epsilon \widehat{\nabla} F\|_{L^2} \|G\|_{L^\infty}$$

and

$$\left\| \mu^\epsilon \sum_{\lambda > \mu} P_\mu(F_\lambda \cdot \widehat{\nabla} G_\lambda) \right\|_{l^2_\mu L^2} \lesssim \|F\|_{L^\infty} \left\| \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\epsilon \|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2} \right\|_{l^2_\mu}$$

$$\lesssim \|F\|_{L^\infty} \|\Lambda^\epsilon \widehat{\nabla} G\|_{L^2}.$$

Combining the above three estimates we therefore obtain (10.7). □

10.2. **Commutator estimates.** In this subsection we will derive various estimates related to the commutators $[P_\mu, F]G$. We first consider the general setting. Let $m(\xi)$ define a multiplier

$$(10.8) \qquad \mathcal{P}f(x) = \int e^{ix\xi} m(\xi) \hat{f}(\xi) d\xi.$$

By introducing the function $P(x)$ defined by

$$(10.9) \qquad M(x) = \int e^{ix \cdot \xi} m(\xi) d\xi = \hat{m}(-x),$$

then for any scalar functions F and G we can write

$$[\mathcal{P}, F]G(x) = \int M(x - y)(F(y) - F(x))G(y) dy$$

$$= \int M(x - y)(x - y)^j \int_0^1 \partial_j F(\tau y + (1 - \tau)x) d\tau G(y) dy$$

$$(10.10) \qquad = \int M(h) h^j \int_0^1 \partial_j F(x - \tau h) d\tau G(x - h) dh.$$

By taking the L^q-norm with $1 \leq q \leq \infty$ and using the Minkowski inequality we obtain

$$(10.11) \qquad \|[\mathcal{P}, F]G\|_{L^q} \leq \int |M(h)||h| \int_0^1 \|\partial F(\cdot - \tau h)G(\cdot - h)\|_{L^q} d\tau dh$$

An application of the Hölder inequality gives the following result whose special case with $p = \infty$ and $q = r = 2$ is [11, Lemma 8.2].

Lemma 10.4. *Let $\mathcal{P}$ be the multiplier operator defined by (10.8) and let M be the function given by (10.9). Then, for any $1 \leq p, q, r \leq \infty$ satisfying $1/p + 1/r = 1/q$ and any scalar functions F and G, there holds*

$$\|[\mathcal{P}, F]G\|_{L^q} \leq \|\partial F\|_{L^p} \|G\|_{L^r} \int |x| |M(x)| dx.$$

Recall that the LP projection P_μ is a multiplier operator with $m(\xi) = \Psi(\mu^{-1}\xi)$, where Ψ is a mollifier with support on $\{1/2 < |\xi| < 2\}$. Observe that $M(x) = \mu^3 \widehat{\Psi}(-\mu x)$. We have $\int |x| |M(x)| dx \lesssim \mu^{-1}$. Therefore, from Lemma 10.4 we obtain the following commutator estimate.

Corollary 10.5. *For any $1 \leq p, q, r \leq \infty$ satisfying $1/p + 1/r = 1/q$ and any scalar functions F and G there holds*

$$\|[P_\mu, F]G\|_{L^q} \lesssim \mu^{-1} \|\widehat{\nabla} F\|_{L^p} \|G\|_{L^r}.$$

In the following we will give further estimates related to the commutator $[P_\mu, F]G$ for any scalar functions F and G. We can write

$$[P_\mu, F]G = [P_\mu, F]G_{\leq 2\mu} + [P_\mu, F]G_{>2\mu}.$$

By the orthogonality of the LP projections, we have

$$[P_\mu, F]G_{>2\mu} = P_\mu(F \cdot G_{>2\mu}) - F P_\mu G_{>2\mu} = \sum_{\mu_1 > 2\mu} P_\mu(F \cdot G_{\mu_1})$$

$$= \sum_{\mu_1 > 2\mu} \sum_{\frac{\mu_1}{2} \leq \mu_2 \leq 2\mu_1} P_\mu(F_{\mu_2} \cdot G_{\mu_1}).$$

Thus, schematically we can write

(10.12) $$[P_\mu, F]G = [P_\mu, F]G_{\leq \mu} + \sum_{\lambda > \mu} P_\mu(F_\lambda G_\lambda)$$

which is not quite accurate but harmless to derive estimates.

Lemma 10.6. *For $0 < \epsilon < 1$, there holds*

$$\mu^\epsilon \|[P_\mu, F]\widehat{\nabla} G\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}$$

(10.13) $$+ \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^\epsilon \|\lambda^\epsilon G_\lambda\|_{L^2}.$$

Proof. By using Corollary 10.5 we have

$$\|[P_\mu, F]\widehat{\nabla} G_{\leq \mu}\|_{L^2} \lesssim \mu^{-1} \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda \leq \mu} \|\widehat{\nabla} G_\lambda\|_{L^2} \lesssim \mu^{-1} \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda \leq \mu} \lambda \|G_\lambda\|_{L^2}.$$

On the other hand, by using the properties of the LP projections we have

$$\sum_{\lambda > \mu} \|P_\mu(F_\lambda \cdot \widehat{\nabla} G_\lambda)\|_{L^2} \lesssim \sum_{\lambda > \mu} \|F_\lambda\|_{L^\infty} \|\widehat{\nabla} G_\lambda\|_{L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda > \mu} \|G_\lambda\|_{L^2}.$$

Combining these two estimates and using the decomposition (10.12) we therefore complete the proof. $\square$

Lemma 10.7. *For $0 < \epsilon < 1/2$ and $\mu \geq 1$ there hold*

$$\mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla}[P_\mu, F]G\|_{L^2} \lesssim \|\widehat{\nabla}F\|_{L^6} \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1/2-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}$$

(10.14)
$$+ \|\widehat{\nabla}F\|_{L^6} \sum_{\lambda > \mu} \left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}$$

and

$$\mu^{\frac{1}{2}+\epsilon}\|[P_\mu, F]G\|_{L^2} \lesssim \|\widehat{\nabla}F\|_{L^6} \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1/2-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}$$

(10.15)
$$+ \|\widehat{\nabla}F\|_{L^6} \sum_{\lambda \geq \mu} \left(\frac{\mu}{\lambda}\right)^{1+\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}.$$

Proof. From the expression (10.10) we can obtain

$$\widehat{\nabla}[P_\mu, F]G_{\leq\mu} = \int \widehat{\nabla}M_\mu(x-y)(x-y)^j \int_0^1 \partial_j F(\tau y + (1-\tau)x)d\tau G_{\leq\mu}(y)dy$$

(10.16)
$$+ \int M_\mu(x-y)\widehat{\nabla}F(x)G_{\leq\mu}(y)dy,$$

where M_μ is given by (10.9) with $m(\xi) = \Psi(\mu^{-1}\xi)$. One can check $\int |M_\mu(x)|dx \lesssim 1$ and $\int |x||\widehat{\nabla}M_\mu(x)|dx \lesssim 1$. Thus, it follows from the Minkowski inequality that

$$\|\widehat{\nabla}[P_\mu, F]G_{\leq\mu}\|_{L^2} \lesssim \|\widehat{\nabla}F\|_{L^6}\|G_{\leq\mu}\|_{L^3} \lesssim \|\widehat{\nabla}F\|_{L^6} \sum_{\lambda \leq \mu} \lambda^{1/2}\|G_\lambda\|_{L^2}.$$

On the other hand, by the finite band property and the Bernstein inequality of LP projections, we have

$$\sum_{\lambda > \mu} \|\widehat{\nabla}P_\mu(F_\lambda \cdot G_\lambda)\|_{L^2} \lesssim \mu \sum_{\lambda > \mu} \|F_\lambda\|_{L^6}\|G_\lambda\|_{L^3} \lesssim \mu\|\widehat{\nabla}F\|_{L^6} \sum_{\lambda > \mu} \lambda^{-1/2}\|G_\lambda\|_{L^2}.$$

With the help of the decomposition (10.12) we thus complete the proof of (10.14).

Next we prove (10.15). By using Corollary 10.5 we have

$$\|\mu^{\frac{1}{2}+\epsilon}[P_\mu, F]G_{\leq\mu}\|_{L^2} \lesssim \mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla}F\|_{L^6}\|G_{\leq\mu}\|_{L^3} \lesssim \mu^{-\frac{1}{2}+\epsilon}\|\widehat{\nabla}F\|_{L^6} \sum_{\lambda \leq \mu} \lambda^{\frac{1}{2}}\|G_\lambda\|_{L^2},$$

while by using the properties of the LP projections we have

$$\mu^{\frac{1}{2}+\epsilon} \sum_{\lambda > \mu} \|P_\mu(F_\lambda \cdot G_\lambda)\|_{L^2} \lesssim \mu^{1+\epsilon} \sum_{\lambda > \mu} \|F_\lambda \cdot G_\lambda\|_{L^{\frac{3}{2}}} \lesssim \mu^{1+\epsilon}\|\widehat{\nabla}F\|_{L^6} \sum_{\lambda > \mu} \lambda^{-1}\|G_\lambda\|_{L^2}.$$

The estimate (10.15) thus follows from an application of (10.12). $\qquad\square$

Lemma 10.8. *For $0 < \epsilon < 3/2$ there holds*

(10.17)
$$\|\mu^{-1/2+\epsilon}[P_\mu, F]\widehat{\nabla}G\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}F\|_{H^1}\|G\|_{H^\epsilon}.$$

Proof. We will use the decomposition (10.12). We first consider the term

$$a_\mu := \sum_{\lambda > \mu} \mu^{-1/2+\epsilon}P_\mu(F_\lambda \cdot \widehat{\nabla}G_\lambda).$$

By the Bernstein inequality and the finite band property of LP projections, we have

$$\mu^{-1/2+\epsilon}\|P_\mu(F_\lambda\cdot\widehat{\nabla}G_\lambda)\|_{L^2}\lesssim\mu^\epsilon\|F_\lambda\cdot\widehat{\nabla}G_\lambda\|_{L^{3/2}}\lesssim\mu^\epsilon\|F_\lambda\|_{L^6}\|\widehat{\nabla}G_\lambda\|_{L^2}$$
$$\lesssim\mu^\epsilon\|\widehat{\nabla}F_\lambda\|_{L^6}\|G_\lambda\|_{L^2}\lesssim\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon G_\lambda\|_{L^2}\|\widehat{\nabla}F\|_{H^1}.$$

Therefore

$$(10.18)\qquad\qquad\|a_\mu\|_{l_\mu^2 L^2}\lesssim\|\widehat{\nabla}F\|_{H^1}\|\Lambda^\epsilon G\|_{L^2}.$$

Next we consider the term $\mu^{-1/2+\epsilon}[P_\mu,F]\widehat{\nabla}G_{\leq\mu}$. By using (10.11) and setting $F_{h,\tau}(x)=F(x-\tau h)$, we obtain

$$\|[P_\mu,F]\widehat{\nabla}G_{\leq\mu}\|_{L^2}\lesssim\mu^{-1}\sup_{h,\tau}\|\widehat{\nabla}F_{h,\tau}\cdot\widehat{\nabla}G_{\leq\mu}\|_{L^2}.$$

By using the orthogonality of the LP projections, we can write $\widehat{\nabla}F_{h,\tau}\cdot\widehat{\nabla}G_{\leq\mu}=b_\mu+c_\mu$, where

$$b_\mu=\sum_{\lambda>\mu}P_\lambda((\widehat{\nabla}F_{h,\tau})_\lambda\cdot\widehat{\nabla}G_{\leq\mu})\quad\text{and}\quad c_\mu=\sum_{\lambda\leq\mu}P_\lambda(\widehat{\nabla}F_{h,\tau}\cdot\widehat{\nabla}G_{\leq\mu}).$$

We first have from the Hölder inequality, the finite band property and the Bernstein inequality that

$$\|b_\mu\|_{L^2}\lesssim\sum_{\lambda>\mu}\|(\widehat{\nabla}F_{h,\tau})_\lambda\|_{L^2}\|\widehat{\nabla}G_{\leq\mu}\|_{L^\infty}\lesssim\sum_{\lambda>\mu,\lambda'\leq\mu}\lambda^{-1}\lambda'^{5/2}\|\widehat{\nabla}F_{h,\tau}\|_{H^1}\|G_{\lambda'}\|_{L^2}$$
$$\lesssim\|\widehat{\nabla}F\|_{H^1}\sum_{\lambda'\leq\mu}\mu^{-1}\lambda'^{5/2}\|G_{\lambda'}\|_{L^2}$$

Therefore

$$(10.19)\qquad\qquad\|\mu^{-3/2+\epsilon}b_\mu\|_{l_\mu^2 L^2}\lesssim\|\widehat{\nabla}F\|_{H^1}\|G\|_{H^\epsilon}.$$

Next we consider c_μ. By using the orthogonality of the LP projections we can write

$$c_\mu=\sum_{\lambda\leq\mu}\sum_{\lambda'\leq\lambda}P_\lambda\left((\widehat{\nabla}F_{h,\tau})_\lambda\cdot\widehat{\nabla}G_{\lambda'}+(\widehat{\nabla}F_{h,\tau})_{\lambda'}\cdot\widehat{\nabla}G_\lambda\right)$$
$$+\sum_{\lambda\leq\mu}\sum_{\lambda<\lambda'\leq\mu}P_\lambda\left((\widehat{\nabla}F_{h,\tau})_{\lambda'}\cdot\widehat{\nabla}G_{\lambda'}\right).$$

By using the Bernstein inequality and the finite band property, we obtain

$$\|c_\mu\|_{L^2}\lesssim\sum_{\lambda\leq\mu}\sum_{\lambda'\leq\lambda}\left(\lambda^{-1}\|(\widehat{\nabla}F_{h,\tau})_\lambda\|_{H^1}\|\widehat{\nabla}G_{\lambda'}\|_{L^\infty}+\lambda'^{\frac{1}{2}}\|\widehat{\nabla}(\widehat{\nabla}F_{h,\tau})_{\lambda'}\|_{L_x^2}\|\widehat{\nabla}G_\lambda\|_{L_x^2}\right)$$
$$+\sum_{\lambda\leq\mu}\sum_{\lambda<\lambda'\leq\mu}\lambda\|(\widehat{\nabla}F_{h,\tau})_{\lambda'}\|_{L^2}\|\widehat{\nabla}G_{\lambda'}\|_{L^3}$$
$$\lesssim\|\widehat{\nabla}F\|_{H^1}\sum_{\lambda\leq\mu}\sum_{\lambda'\leq\lambda\leq\mu}(\lambda^{-1}\lambda'^{5/2}+\lambda'^{\frac{1}{2}}\lambda)\|G_{\lambda'}\|_{L^2}$$
$$+\|\widehat{\nabla}F\|_{H^1}\sum_{\lambda\leq\mu}\sum_{\lambda<\lambda'\leq\mu}\lambda\lambda'^{1/2}\|G_{\lambda'}\|_{L^2}$$
$$\lesssim\|\widehat{\nabla}F\|_{H^1}\cdot\left(\sum_{\lambda\leq\mu}\sum_{\lambda'\leq\mu}\lambda'^{1/2}\lambda\|G_{\lambda'}\|_{L^2}\right).$$

Therefore

(10.20) $$\|\mu^{-3/2+\epsilon}c_\mu\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}F\|_{H^1}\|G\|_{H^\epsilon}.$$

Combining (10.19) and (10.20) yields

$$\|\mu^{-1/2+\epsilon}[P_\mu, F]\widehat{\nabla}G_{\leq\mu}\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}F\|_{H^1}\|G\|_{H^\epsilon}$$

which together with (10.18) gives the desired estimate. $\qquad\square$

Lemma 10.9. *For* $0 < \epsilon < 1$ *there holds*

(10.21)
$$\|\mu^{1+\epsilon}[P_\mu, F]G\|_{l^2_\mu L^2} + \|\mu^\epsilon\widehat{\nabla}[P_\mu, F]G\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}^2 F\|_{H^{\frac12+\epsilon}}\|G\|_{L^2} + \|\widehat{\nabla}F\|_{L^\infty}\|G\|_{H^\epsilon}$$

(10.22)
$$\|\mu^{1+\epsilon}[P_\mu, F]G\|_{l^2_\mu L^2} \lesssim (\|\widehat{\nabla}^2 F\|_{H^{\frac12}} + \|\widehat{\nabla}F\|_{L^\infty})\|G\|_{H^\epsilon}$$

Proof. We first write $\mu^{1+\epsilon}[P_\mu, F]G = a_\mu + b_\mu + c_\mu$, where

$$a_\mu = \mu^{1+\epsilon}[P_\mu, F]G_{>\mu}, \quad b_\mu = \mu^{1+\epsilon}\sum_{\lambda\leq\mu}[P_\mu, F_{\leq\lambda}]G_\lambda, \quad c_\mu = \mu^{1+\epsilon}\sum_{\lambda\leq\mu}[P_\mu, F_{>\lambda}]G_\lambda.$$

By Corollary 10.5, we obtain

$$\|a_\mu\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\left\|\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon G_\lambda\|_{L^2}\right\|_{l^2_\mu} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\|\Lambda^\epsilon G\|_{L^2}.$$

In order to estimate b_μ and c_μ, we introduce the function $M_\mu(x) = \int e^{ix\cdot\xi}m_\mu(\xi)d\xi$ with $m_\mu(\xi) = \Psi(\mu^{-1}\xi)$. It is easy to see that $\int |x|^q|M_\mu(x)|dx \lesssim \mu^{-q}$ for any $q > -3$. It follows from (10.10) that

$$[P_\mu, F_{\leq\lambda}]G_\lambda(x) = A_{\mu,\lambda}(x) + B_{\mu,\lambda}(x),$$

where

$$A_{\mu,\lambda}(x) = \partial_j F_{\leq\lambda}(x)\int M_\mu(x-y)(x-y)^j G_\lambda(y)dy,$$

$$B_{\mu,\lambda}(x) = \int M_\mu(x-y)(x-y)^j\int_0^1 [\partial_j F_{\leq\lambda}(x-\tau(x-y)) - \partial_j F_{\leq\lambda}(x)]\,d\tau G_\lambda(y)dy.$$

For the term $A_{\mu,\lambda}(x)$, it is nonzero only if μ and λ are at the same magnitude since both M_μ and G_λ are frequency localized at the level μ and λ respectively. Thus

$$\sum_{\lambda\leq\mu}\|A_{\mu,\lambda}\|_{L^2} \lesssim \mu^{-1}\|\widehat{\nabla}F_{\leq\mu}\|_{L^\infty}\|G_\mu\|_{L^2} \lesssim \mu^{-1}\|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}.$$

For the term $B_{\mu,\lambda}$ we write

$$B_{\mu,\lambda}(x) = \int M_\mu(x-y)(x-y)^j(x-y)^l$$

$$\times \int_0^1\int_0^1 -\tau\partial_l\partial_j F_{\leq\lambda}\left(x - \tau\tau'(x-y)\right)d\tau d\tau' G_\lambda(y)dy.$$

Thus, by the Minkowski inequality we obtain

$$\|B_{\mu,\lambda}\|_{L^2} \lesssim \mu^{-2}\|\widehat{\nabla}^2 F_{\leq\lambda}\|_{L^\infty}\|G_\lambda\|_{L^2} \lesssim \mu^{-2}\sum_{\lambda'\leq\lambda}\lambda'\|\widehat{\nabla}F_{\lambda'}\|_{L^\infty}\|G_\lambda\|_{L^2}$$

$$\lesssim \mu^{-2}\lambda\|G_\lambda\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}.$$

We therefore obtain

$$\|b_\mu\|_{L^2} \lesssim \mu^\epsilon \|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty} + \sum_{\lambda \le \mu}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\|\lambda^\epsilon G_\lambda\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}$$

which implies that

$$\|b_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\|G\|_{H^\epsilon}.$$

In order to estimate the term c_μ, we write

$$[P_\mu, F_{>\lambda}]G_\lambda(x)$$

$$= \int M_\mu(x-y)(x-y)^j \int_0^1 \{\partial_j F_{>\lambda}(x - \tau(x-y)) - \partial_j F_{>\lambda}(x)\}d\tau G_\lambda(y)dy$$

$$+ \partial_j F_{>\lambda}(x)\int M_\mu(x-y)(x-y)^j G_\lambda(y)dy.$$

By using the similar argument as above, we can obtain

$$\sum_{\lambda \le \mu}\|[P_\mu, F_{>\lambda}]G_\lambda\|_{L^2}$$

$$\lesssim \mu^{-2}\sum_{\lambda \le \mu}\|\widehat{\nabla}^2 F_{>\lambda}\|_{L^2}\|G_\lambda\|_{L^\infty} + \mu^{-1}\|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}$$

$$\lesssim \sum_{\lambda \le \mu}\sum_{\lambda' > \lambda}\mu^{-2}\lambda^{3/2}\|\widehat{\nabla}^2 F_{\lambda'}\|_{L^2}\|G_\lambda\|_{L^2} + \mu^{-1}\|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}.$$

Therefore

$$\|c_\mu\|_{L^2} \lesssim \sum_{\lambda \le \mu}\sum_{\lambda' > \lambda}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\left(\frac{\lambda}{\lambda'}\right)^{1/2+\epsilon}\|\lambda'^{1/2+\epsilon}\widehat{\nabla}^2 F_{\lambda'}\|_{L^2}\|G_\lambda\|_{L^2}$$

$$+ \mu^\epsilon\|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}$$

which implies that $\|c_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\|G\|_{H^\epsilon} + \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}$. This together with the estimates on a_μ and b_μ gives the first inequality of (10.21). Note that c_μ can also be estimated as

$$\|c_\mu\|_{L^2} \lesssim \sum_{\lambda' > \lambda}\left(\frac{\lambda}{\mu}\right)^{1-\epsilon}\left(\frac{\lambda}{\lambda'}\right)^{1/2}\|\lambda'^{1/2}\widehat{\nabla}^2 F_{\lambda'}\|_{L^2}\|\lambda^\epsilon G_\lambda\|_{L^2}$$

$$+ \mu^\epsilon\|G_\mu\|_{L^2}\|\widehat{\nabla}F\|_{L^\infty}$$

which implies $\|c_\mu\|_{l_\mu^2 L^2} \lesssim (\|\widehat{\nabla}F\|_{L^\infty} + \|\widehat{\nabla}^2 F\|_{H^{\frac{1}{2}}})\|\Lambda^\epsilon G\|_{L^2}$. This together with the estimates for a_μ and b_μ gives (10.22).

Next we prove the second part of (10.21). By using (10.12) we can write $\mu^\epsilon\widehat{\nabla}[P_\mu, F]G = I_\mu + J_\mu$, where

$$I_\mu := \mu^\epsilon \sum_{\lambda > \mu}\widehat{\nabla}P_\mu(F_\lambda \cdot G_\lambda) \quad \text{and} \quad J_\mu := \mu^\epsilon\widehat{\nabla}[P_\mu, F]G_{\le \mu}.$$

We first have from the finite band property of the LP projections that

$$\|I_\mu\|_{L^2} \lesssim \mu^{1+\epsilon}\sum_{\lambda > \mu}\|F_\lambda \cdot G_\lambda\|_{L^2} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\sum_{\lambda > \mu}\left(\frac{\mu}{\lambda}\right)^{1+\epsilon}\|\lambda^\epsilon G_\lambda\|_{L^2}.$$

This gives

$$\|I_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla}F\|_{L^\infty}\|\Lambda^\epsilon G\|_{L^2}.$$

In order to estimate J_μ, we write

$$J_\mu(x) = \mu^\epsilon \int \widehat{\nabla}_x \left(M_\mu(x-y)(F(x)-F(y)) \right) G_{\leq\mu}(y)dy$$

$$= \mu^\epsilon \widehat{\nabla} F(x) \int M_\mu(x-y) G_{\leq\mu}(y)dy$$

$$+ \mu^\epsilon \int \widehat{\nabla} M_\mu(x-y)(F(x)-F(y)) G_{\leq\mu}(y)dy.$$

By writing $F(x) - F(y) = (x-y)^j \int_0^1 \partial_j F(x - \tau(x-y))d\tau$, we can decompose J_μ as $J_\mu = J_\mu^{(1)} + J_\mu^{(2)} + J_\mu^{(3)}$, where

$$J_\mu^{(1)} = \mu^\epsilon \widehat{\nabla} F(x) \int M_\mu(x-y) G_{\leq\mu}(y)dy,$$

$$J_\mu^{(2)} = \mu^\epsilon \partial_j F(x) \int \widehat{\nabla} M_\mu(x-y)(x-y)^j G_{\leq\mu}(y)dy,$$

$$J_\mu^{(3)} = \mu^\epsilon \int \widehat{\nabla} M_\mu(x-y)(x-y)^j \int_0^1 \left[\partial_j F(x - \tau(x-y)) - \partial_j F(x) \right] d\tau G_{\leq\mu}(y)dy.$$

By using $\int |x||\widehat{\nabla} M_\mu(x)|dx \lesssim 1$ and the frequency localization of M_μ and G_λ we can obtain

$$\|J_\mu^{(1)}\|_{L^2} + \|J_\mu^{(2)}\|_{L^2} \lesssim \mu^\epsilon \|G_\mu\|_{L^2} \|\widehat{\nabla} F\|_{L^\infty},$$

which gives

$$\|J_\mu^{(1)}\|_{l^2_\mu L^2} + \|J_\mu^{(2)}\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla} F\|_{L^\infty} \|\Lambda^\epsilon G\|_{L^2}.$$

For the term $J_\mu^{(3)}$, we can write

$$J_\mu^{(3)}(x) = \mu^\epsilon \sum_{\lambda \leq \mu} \int \widehat{\nabla} M_\mu(x-y)(x-y)^j$$

$$\times \int_0^1 \left[\partial_j F_{\leq\lambda}(x - \tau(x-y)) - \partial_j F_{\leq\lambda}(x) \right] d\tau G_\lambda(y)dy$$

$$+ \mu^\epsilon \sum_{\lambda \leq \mu} \int \widehat{\nabla} M_\mu(x-y)(x-y)^j$$

$$\times \int_0^1 \left[\partial_j F_{>\lambda}(x - \tau(x-y)) - \partial_j F_{>\lambda}(x) \right] d\tau G_\lambda(y)dy.$$

Now we can use the similar arguments in the proof of the first part of this lemma to obtain

$$\|J_\mu^{(3)}\|_{L^2} \lesssim \mu^{-1+\epsilon} \sum_{\lambda \leq \mu} \|\widehat{\nabla}^2 F_{\leq\lambda}\|_{L^\infty} \|G_\lambda\|_{L^2} + \mu^{-1+\epsilon} \sum_{\lambda \leq \mu} \|\widehat{\nabla}^2 F_{>\lambda}\|_{L^2} \|G_\lambda\|_{L^\infty}$$

$$\lesssim \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu} \right)^{1-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2}$$

$$+ \sum_{\lambda \leq \mu} \sum_{\lambda \leq \lambda'} \left(\frac{\lambda}{\mu} \right)^{1-\epsilon} \left(\frac{\lambda}{\lambda'} \right)^{1/2+\epsilon} \|\lambda'^{1/2+\epsilon} \widehat{\nabla}^2 F_{\lambda'}\|_{L^2} \|G_\lambda\|_{L^2}.$$

By taking the l^2_μ-norm we obtain

$$\|J_\mu^{(3)}\|_{l^2_\mu L^2} \lesssim \|\widehat{\nabla}^2 F\|_{H^{1/2+\epsilon}} \|G\|_{L^2} + \|\widehat{\nabla} F\|_{L^\infty} \|G\|_{H^\epsilon}.$$

This together with the estimates on $J_\mu^{(1)}$ and $J_\mu^{(2)}$ gives

$$\|J_\mu\|_{l_\mu^2 L^2} \lesssim \|\widehat\nabla F\|_{L^\infty}\|G\|_{H^\epsilon} + \|\widehat\nabla^2 F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}.$$

Combining this with the estimate on I_μ completes the proof of the second part of (10.21). $\qquad\square$

Proposition 10.10. *For $0 < \epsilon < 1/2$ there holds*

(10.23) $\qquad \|\mu^{1+\epsilon}\widehat\nabla[P_\mu, F]G\|_{l_\mu^2 L^2} \lesssim \|\widehat\nabla F\|_{L^\infty}\|G\|_{H^{1+\epsilon}} + \|\widehat\nabla^2 F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1}.$

Proof. As can be seen from the proof of the second part of (10.21), it suffices to estimate the term

$$a_\mu := \mu J_\mu^{(3)} = \mu^{1+\epsilon}\int \widehat\nabla M_\mu(x-y)(x-y)^j(x-y)^l$$
$$\times \int_0^1\int_0^1 -\tau\partial_{jl}^2 F(x-\tau\tau'(x-y))d\tau d\tau' G_{\leq\mu}(y)dy$$

which can be written as $a_\mu = a_\mu^{(1)} + a_\mu^{(2)} + a_\mu^{(3)}$, where

$$a_\mu^{(1)} = \mu^{1+\epsilon}\int \widehat\nabla M_\mu(x-y)(x-y)^j(x-y)^l$$
$$\times \int_0^1\int_0^1 -\tau\partial_{jl}^2 F_{\geq\mu}\left(x-\tau\tau'(x-y)\right) d\tau d\tau' G_{\leq\mu}(y)dy$$

$$a_\mu^{(2)} = \mu^{1+\epsilon}\partial_{jl}^2 F_{<\mu}(x)\int \widehat\nabla M_\mu(x-y)(x-y)^j(x-y)^l G_{\leq\mu}(y)dy$$

$$a_\mu^{(3)} = \mu^{1+\epsilon}\int \widehat\nabla M_\mu(x-y)(x-y)^j(x-y)^l$$
$$\times \int_0^1\int_0^1 -\tau\left[\partial_{jl}^2 F_{<\mu}(x-\tau\tau'(x-y)) - \partial_{jl}^2 F_{<\mu}(x)\right] d\tau d\tau' G_{\leq\mu}(y)dy.$$

By using the properties of the LP projections, it is easy to derive that

$$\|a_\mu^{(1)}\|_{L^2} \lesssim \mu^\epsilon \sum_{\lambda\geq\mu}\|\widehat\nabla^2 F_\lambda\|_{L^2}\|G_{\leq\mu}\|_{L^\infty} \lesssim \sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{1/2+\epsilon}\|\lambda^{1/2+\epsilon}\widehat\nabla^2 F_\lambda\|_{L^2}\|\widehat\nabla G\|_{L^2},$$

$$\|a_\mu^{(2)}\|_{L^2} \lesssim \mu^\epsilon\|\widehat\nabla^2 F_{<\mu}\|_{L^\infty}\|G_\mu\|_{L^2} \lesssim \mu^{1+\epsilon}\|G_\mu\|_{L^2}\|\widehat\nabla F\|_{L^\infty},$$

$$\|a_\mu^{(3)}\|_{L^2} \lesssim \mu^{-1+\epsilon}\|\widehat\nabla^3 F_{<\mu}\|_{L^2}\|G_{\leq\mu}\|_{L^\infty} \lesssim \sum_{\lambda<\mu}\left(\frac{\lambda}{\mu}\right)^{1/2-\epsilon}\|\lambda^{1/2+\epsilon}\widehat\nabla^2 F_\lambda\|_{L^2}\|\widehat\nabla G\|_{L^2}.$$

Therefore, by taking the l_μ^2-norm, we obtain

$$\|a_\mu^{(1)}\|_{l_\mu^2 L^2} + \|a_\mu^{(3)}\|_{l_\mu^2 L^2} \lesssim \|\widehat\nabla^2 F\|_{H^{1/2+\epsilon}}\|G\|_{H^1}, \quad \|a_\mu^{(2)}\|_{l_\mu^2 L^2} \lesssim \|\widehat\nabla F\|_{L^\infty}\|G\|_{H^{1+\epsilon}}.$$

The proof is thus complete. $\qquad\square$

Lemma 10.11. *For $0 < \epsilon < 1$ there holds*

$$\|\mu^\epsilon[P_\mu, F]G\|_{l_\mu^2 L^2} \lesssim \|F\|_{L^\infty}\|G\|_{H^\epsilon} + \|\widehat\nabla F\|_{H^{1/2+\epsilon}}\|G\|_{L^2}.$$

Proof. First we have

$$\mu^\epsilon \sum_{\lambda>\mu}\|P_\mu(F_\lambda\cdot G_\lambda)\|_{L^2} \lesssim \mu^\epsilon \sum_{\lambda>\mu}\|F_\lambda\|_{L^\infty}\|G_\lambda\|_{L^2} \lesssim \|F\|_{L^\infty}\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^\epsilon\|\lambda^\epsilon G_\lambda\|_{L^2}$$

which implies that

$$\left\|\mu^\epsilon \sum_{\lambda > \mu} P_\mu(F_\lambda \cdot G_\lambda)\right\|_{l_\mu^2 L^2} \lesssim \|F\|_{L^\infty} \|G\|_{H^\epsilon}.$$

Next we consider the term

$$[P_\mu, F]G_{\leq \mu} = \int M_\mu(x-y)[F(x) - F(y)]G_{\leq \mu}(y)dy$$

which can be decomposed as follows

$$[P_\mu, F]G_{\leq \mu}(x) = \sum_{\lambda \leq \mu} \int M_\mu(x-y)\left(F_{\leq \lambda}(x) - F_{\leq \lambda}(y)\right) G_\lambda(y)dy$$

$$+ \sum_{\lambda \leq \mu} \int M_\mu(x-y)\left(F_{>\lambda}(x) - F_{>\lambda}(y)\right) G_\lambda(y)dy.$$

We can estimate as before to obtain

$$\mu^\epsilon \|[P_\mu, F]G_{\leq \mu}\|_{L^2}$$

$$\lesssim \mu^{-1+\epsilon} \sum_{\lambda \leq \mu} \|\widehat{\nabla} F_{\leq \lambda}\|_{L^\infty} \|G_\lambda\|_{L^2} + \mu^{-1+\epsilon} \sum_{\lambda \leq \mu} \|\widehat{\nabla} F_{>\lambda}\|_{L^2} \|G_\lambda\|_{L^\infty}$$

$$\lesssim \sum_{\lambda \leq \mu} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \|\lambda^\epsilon G_\lambda\|_{L^2} \|F\|_{L^\infty}$$

$$+ \sum_{\lambda \leq \mu} \sum_{\lambda' > \lambda} \left(\frac{\lambda}{\mu}\right)^{1-\epsilon} \left(\frac{\lambda}{\lambda'}\right)^{1/2+\epsilon} \|\lambda'^{1/2+\epsilon} \widehat{\nabla} F_{\lambda'}\|_{L^2} \|G_\lambda\|_{L^2}.$$

Taking the l_μ^2-norm gives

$$\|\mu^\epsilon [P_\mu, F]G_{\leq \mu}\|_{l_\mu^2 L^2} \lesssim \|F\|_{L^\infty} \|G\|_{H^\epsilon} + \|\widehat{\nabla} F\|_{H^{1/2+\epsilon}} \|G\|_{L^2}.$$

The proof is therefore complete. $\qquad\square$

Lemma 10.12. *For any $\epsilon > 0$ and any scalar functions F and G, there holds*

$$(10.24) \qquad \|\mu^\epsilon P_\mu(\widehat{\nabla} F \cdot G)\|_{l_\mu^2 L^2} \lesssim \|\widehat{\nabla} F\|_{H^{\frac{1}{2}+\epsilon}} \|G\|_{H^1} + \|F\|_{L^\infty} \|G\|_{H^{1+\epsilon}}.$$

Proof. By the trichotomy law, we can write

$$P_\mu(\widehat{\nabla} F \cdot G) = a_\mu + b_\mu + c_\mu$$

$$= P_\mu\left((\widehat{\nabla} F)_\mu \cdot G_{\leq \mu}\right) + \sum_{\lambda > \mu} P_\mu\left(\widehat{\nabla} F_\lambda \cdot G_\lambda\right) + P_\mu\left(\widehat{\nabla} F_{\leq \mu} \cdot G_\mu\right).$$

For the terms b_μ and c_μ, it is easy to derive that

$$\|\mu^\epsilon b_\mu\|_{l_\mu^2 L^2} \lesssim \|F\|_{L^\infty} \|G\|_{H^{1+\epsilon}}, \qquad \|\mu^\epsilon c_\mu\|_{l_\mu^2 L^2} \lesssim \|F\|_{L^\infty} \|G\|_{H^{1+\epsilon}}.$$

For the term a_μ, we can write

$$(10.25) \qquad\qquad a_\mu = [P_\mu, G_{\leq \mu}]\widehat{\nabla} F_\mu + G_{\leq \mu} P_\mu(\widehat{\nabla} F)_\mu.$$

By the Bernstein inequality for LP projections, it is easy to obtain

$$\|\mu^\epsilon G_{\leq \mu} P_\mu(\widehat{\nabla} F)_\mu\|_{L^2} \lesssim \mu^\epsilon \|G_{\leq \mu}\|_{L^6} \|P_\mu(\widehat{\nabla} F)_\mu\|_{L^3} \lesssim \|G\|_{L^6} \|\mu^{\frac{1}{2}+\epsilon}(\widehat{\nabla} F)_\mu\|_{L^2},$$

while by using Corollary 10.5 we have

$$\|\mu^\epsilon [P_\mu, G_{\leq\mu}](\widehat{\nabla}F)_\mu\|_{L^2} \lesssim \mu^{\epsilon-1} \sum_{\lambda\leq\mu} \|\widehat{\nabla}G_\lambda\|_{L^\infty}\|(\widehat{\nabla}F)_\mu\|_{L^2}$$

$$\lesssim \sum_{\lambda\leq\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{3}{2}} \|\widehat{\nabla}G_\lambda\|_{L^2}\|\mu^{\frac{1}{2}+\epsilon}(\widehat{\nabla}F)_\mu\|_{L^2}.$$

Therefore

$$\|\mu^\epsilon a_\mu\|_{l^2_\mu L^2_x} \lesssim \|\widehat{\nabla}F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1}.$$

The combination of the estimates for a_μ, b_μ and c_μ gives (10.24). $\qquad\square$

By using Lemma 10.11 and (10.24), we can derive the following product estimate.

Lemma 10.13. *For $0 < \epsilon < 1$ there holds*

(10.26) $$\|\Lambda^\epsilon\widehat{\nabla}(F\cdot G)\|_{L^2} \lesssim \|\widehat{\nabla}F\|_{H^{\frac{1}{2}+\epsilon}}\|G\|_{H^1} + \|F\|_{L^\infty}\|G\|_{H^{1+\epsilon}}$$

Proof. Observe that

$$\|\Lambda^\epsilon\widehat{\nabla}(F\cdot G)\|_{L^2} \lesssim \|\mu^\epsilon P_\mu(\widehat{\nabla}F\cdot G)\|_{L^2_\mu L^2} + \|\mu^\epsilon P_\mu(F\cdot\widehat{\nabla}G)\|_{l^2_\mu L^2}.$$

From Lemma 10.11 it follows that

$$\|\mu^\epsilon P_\mu(F\cdot\widehat{\nabla}G)\|_{l^2_\mu L^2} \leq \|\mu^\epsilon[P_\mu, F]\widehat{\nabla}G\|_{l^2_\mu L^2} + \|F\|_{L^\infty_x}\|\mu^\epsilon P_\mu\widehat{\nabla}G\|_{l^2_\mu L^2}$$

$$\lesssim \|F\|_{L^\infty}\|\widehat{\nabla}G\|_{H^\epsilon} + \|\widehat{\nabla}F\|_{H^{\frac{1}{2}+\epsilon}}\|\widehat{\nabla}G\|_{L^2}.$$

This together with (10.24) then completes the proof. $\qquad\square$

Lemma 10.14. *Let $0 < \epsilon < 1/2$ and $p > 3$ be sufficiently close to 3. Then for any $\mu \geq 1$ and any scalar functions F and G, there hold*

$$\|[P_\mu, F]G\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|G\|_{L^\infty}\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{\frac{3}{p}-\frac{1}{2}+\epsilon}\|\lambda^\epsilon\widehat{\nabla}^2 F_\lambda\|_{L^2}$$

(10.27) $$+ \mu^{-\frac{1}{2}-\epsilon}\|G\|_{L^\infty}\sum_{\lambda\leq\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon}\|\lambda^\epsilon\widehat{\nabla}^2 F_\lambda\|_{L^2}$$

and

$$\|[P_\mu, F]G\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon}\|\widehat{\nabla}F\|_{L^\infty}\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{2+\epsilon}\|\lambda^\epsilon\widehat{\nabla}G_\lambda\|_{L^2}$$

$$+ \mu^{-\frac{1}{2}-\epsilon}\|\widehat{\nabla}F\|_{L^\infty}\sum_{\lambda\leq\mu}\left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon}\|\lambda^\epsilon\widehat{\nabla}G_\lambda\|_{L^2}$$

(10.28) $$+ \mu^{-\frac{3}{p}-\frac{1}{2}-\epsilon}\|G\|_{H^1}\sum_{\lambda>\mu}\left(\frac{\mu}{\lambda}\right)^{\frac{3}{p}+\epsilon}\|\lambda^{\frac{1}{2}+\epsilon}\widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

where $C > 0$ is a constant depending only on ϵ and p.

Proof. In view of (10.12), we can write $[P_\mu F]G = a_\mu + b_\mu + c_\mu$, where

$$a_\mu = \sum_{\lambda>\mu} P_\mu(F_\lambda G_\lambda), \quad b_\mu = \sum_{\lambda>\mu}[P_\mu, F_\lambda]G_{\leq\mu}, \quad c_\mu = \sum_{\lambda\leq\mu}[P_\mu, F_\lambda]G_{\leq\mu}.$$

It is easy to derive that

$$\|a_\mu\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{2+\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2_x} \|G_\lambda\|_{L^\infty}.$$

By using Corollary 10.5 and the Bernstein inequality, we also have

$$\|c_\mu\|_{L^\infty} \lesssim \|G\|_{L^\infty} \sum_{\lambda\leq\mu} \mu^{-1} \|\widehat{\nabla} F_\lambda\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon} \|G\|_{L^\infty} \sum_{\lambda\leq\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

Since $p > 3$, we have the Sobolev embedding

$$\|b_\mu\|_{L^\infty} \lesssim \|\widehat{\nabla} b_\mu\|_{L^p} + \|b_\mu\|_{L^2}.$$

From (10.16) and the properties of LP projections it follows that

$$\|\widehat{\nabla} b_\mu\|_{L^p} \lesssim \|G\|_{L^\infty} \sum_{\lambda>\mu} \|\widehat{\nabla} F_\lambda\|_{L^p} \lesssim \|G\|_{L^\infty} \sum_{\lambda>\mu} \lambda^{-\frac{3}{p}+\frac{1}{2}} \|\widehat{\nabla}^2 F_\lambda\|_{L^2}$$

$$\lesssim \mu^{-\frac{3}{p}-\epsilon-\frac{1}{2}} \|G\|_{L^\infty} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{\frac{3}{p}-\frac{1}{2}+\epsilon} \|\lambda^\epsilon \widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

The term $\|b_\mu\|$ can be estimated similarly but much easier. We therefore complete the proof of (10.27).

Similarly, in order to obtain (10.28), we can use the properties of LP projections and Corollary 10.5 to derive that

$$\|a_\mu\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon} \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda>\mu} \left(\frac{\mu}{\lambda}\right)^{2+\epsilon} \|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2},$$

$$\|c_\mu\|_{L^\infty} \lesssim \mu^{-1} \sum_{\lambda\leq\mu} \|\widehat{\nabla} F_{\leq\mu}\|_{L^\infty} \|G_\lambda\|_{L^\infty} \lesssim \mu^{-\frac{1}{2}-\epsilon} \|\widehat{\nabla} F\|_{L^\infty} \sum_{\lambda\leq\mu} \left(\frac{\lambda}{\mu}\right)^{\frac{1}{2}-\epsilon} \|\lambda^\epsilon \widehat{\nabla} G_\lambda\|_{L^2}.$$

$$\|\widehat{\nabla} b_\mu\|_{L^p} \lesssim \sum_{\lambda\leq\mu} \sum_{\ell>\mu} \|G_\lambda\|_{L^\infty} \|\widehat{\nabla} F_\ell\|_{L^p} \lesssim \sum_{\ell\leq\mu} \sum_{\lambda>\mu} \ell^{\frac{1}{2}} \|\widehat{\nabla} G_\ell\|_{L^2} \lambda^{-\frac{3}{p}-\epsilon} \|\lambda^{\frac{1}{2}+\epsilon} \widehat{\nabla}^2 F_\lambda\|_{L^2}$$

$$\lesssim \mu^{-\frac{3}{p}-\epsilon+\frac{1}{2}} \|G\|_{H^1} \sum_{\ell\leq\mu} \sum_{\lambda>\mu} \left(\frac{\ell}{\mu}\right)^{\frac{1}{2}} \left(\frac{\mu}{\lambda}\right)^{\frac{3}{p}+\epsilon} \|\lambda^{\epsilon+\frac{1}{2}} \widehat{\nabla}^2 F_\lambda\|_{L^2}.$$

Hence the proof of (10.28) is completed. $\qquad\square$

10.3. H^ϵ elliptic estimates.

Lemma 10.15. *For $0 < \epsilon < 1/2$ and any Σ-tangent tensor field F there hold*

(10.29) $$\|\widehat{\nabla}^2 F\|_{\dot{H}^\epsilon} \lesssim \|\hat{\Delta} F\|_{\dot{H}^\epsilon} + \|F\|_{H^1},$$

(10.30) $$\|\widehat{\nabla} F\|_{\dot{H}^{1/2+\epsilon}} \lesssim \|\Lambda^{\epsilon-1/2} \hat{\Delta} F\|_{L^2} + \|F\|_{H^1}.$$

Proof. Consider (10.29) first. We will use err_μ to denote any error term satisfying

$$\|\text{err}_\mu\|_{L^2} \lesssim \mu^{-1} \|F\|_{H^1}.$$

Using Corollary 10.5, we can obtain

(10.31) $$P_\mu \widehat{\nabla}^2 F = \widehat{\nabla}^2 P_\mu F + \text{err}_\mu.$$

Recall from Lemma 2.2 that

(10.32) $$\|\widehat{\nabla}^2 P_\mu F\|_{L^2} \lesssim \|\hat{\Delta} P_\mu F\|_{L^2} + \|\widehat{\nabla} P_\mu F\|_{L^2} + \|P_\mu F\|_{L^2}$$

Recall also that $\hat{\Delta} = g^{ij}\widehat{\nabla}_i\widehat{\nabla}_j$, we have

(10.33) $$\hat{\Delta} P_\mu F = P_\mu\hat{\Delta} F + [P_\mu, g^{ij}](\partial_i\partial_j F - \hat{\Gamma}_{ij}^k\partial_k F) + \mathrm{err}_\mu.$$

By Lemma 10.8 and 10.11, we have

(10.34) $$\|\mu^\epsilon[P_\mu, g^{ij}]\partial_i\partial_j F\|_{l_\mu^2 L^2} \lesssim \|\partial g\|_{H^1}\|\Lambda^{\epsilon+1/2}\partial F\|_{L^2},$$

(10.35) $$\|\mu^\epsilon[P_\mu, g^{ij}](\hat{\Gamma}_{ij}^k\partial_k F)\|_{l_\mu^2 L^2} \lesssim \|\partial g\|_{H^{\frac{1}{2}+\epsilon}}\|\partial F\|_{L^2}.$$

By the Sobolev inequality and (10.33), we obtain $\hat{\Delta} P_\mu F = P_\mu\hat{\Delta} F + \mathrm{err}_\mu$. This together with (10.31) and (10.32) gives (10.29).

To prove (10.30), we first use the equivalence between g and $\hat{g}$ and the integration by parts to obtain

(10.36) $$\|P_\mu\widehat{\nabla} F\|_{L^2}^2 \approx \int_\Sigma g^{ij}P_\mu\partial_i F P_\mu\partial_j F d\mu_g = \int_\Sigma P_\mu F\hat{\Delta} P_\mu F d\mu_g.$$

In view of (10.17) and (10.35), we can obtain

$$\sum_\mu \mu^{1+2\epsilon}\left|\int_\Sigma P_\mu F[P_\mu, g^{ij}]\partial_i\partial_j F d\mu_g\right|$$

$$\lesssim \sum_\mu \mu^{3/2+\epsilon}\|P_\mu F\|_{L^2}\|\mu^{-1/2+\epsilon}[P_\mu, g^{ij}]\partial_i\partial_j F\|_{L^2}$$

$$\lesssim \|\widehat{\nabla} F\|_{H^{1/2+\epsilon}}\|\widehat{\nabla} g\|_{H^1}\|\Lambda^\epsilon\partial F\|_{L^2},$$

and in view of (10.15)

$$\sum_\mu \mu^{1+2\epsilon}\left|\int P_\mu F \cdot [P_\mu, g\hat{\Gamma}]\partial F\right| \lesssim \|\mu^{\frac{1}{2}+\epsilon}P_\mu F\|_{l_\mu^2 L^2}\|\widehat{\nabla}(g\cdot\hat{\Gamma})\|_{L^6}\|\partial F\|_{H^\epsilon}$$

$$\lesssim \|F\|_{H^{\frac{1}{2}+\epsilon}}\|\partial F\|_{H^\epsilon}\|g\|_{H^2}.$$

In view of (10.33) and (10.36), we have with $p = \epsilon/(1/2 + \epsilon)$ that

$$\sum_\mu \mu^{1+2\epsilon}\|P_\mu\widehat{\nabla} F\|_{L^2}^2 \lesssim (\|F\|_{\dot{H}^{\frac{1}{2}+\epsilon}} + \|\partial F\|_{H^{1/2+\epsilon}})\|\partial F\|_{H^{1/2+\epsilon}}^p\|\partial F\|_{L^2}^{1-p}\|g\|_{H^2}$$

$$+ \sum_\mu \|\mu^{3/2+\epsilon}P_\mu F\|_{L^2}\|\mu^{-1/2+\epsilon}P_\mu\hat{\Delta} F\|_{L^2}.$$

By the fact $\|g\|_{H^2} \lesssim 1$ and the Young's inequality, we obtain (10.30).

To prove (10.30) for the vector field case, we note that

$$P_\mu\widehat{\nabla}_i F^m = P_\mu\widehat{\nabla}_i(F^m) + [P_\mu, \hat{\Gamma}]F + \hat{\Gamma}\cdot P_\mu F.$$

Using Corollary 10.5, then there holds $P_\mu\widehat{\nabla}_i F^m = P_\mu\widehat{\nabla}_i(F^m) + \mathrm{err}_\mu$, hence we can obtain

$$\|\widehat{\nabla} F^m\|_{H^{1/2+\epsilon}} \lesssim \|\widehat{\nabla}(F^m)\|_{H^{1/2+\epsilon}} + \|F\|_{H^1}.$$

Now we can use (10.30) for the scalar function case to derive

(10.37) $$\|\widehat{\nabla} F^m\|_{H^{1/2+\epsilon}} \lesssim \|\Lambda^{\epsilon-1/2}\hat{\Delta}(F^m)\|_{L^2} + \|F\|_{H^1}.$$

In view of (10.33), we can obtain

$$\|\Lambda^{\epsilon-1/2}(\hat{\Delta} F^m)\|_{L^2} \lesssim \|\Lambda^{\epsilon-1/2}(\hat{\Delta} F)^m\|_{L^2} + \|\widehat{\nabla} F\|_{L^2}.$$

combining this with (10.37) completes the proof. $\square$

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