Metric- and frame-like higher-spin gauge theories in three dimensions

Stefan Fredenhagen and Pan Kessel

Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Golm, Germany

E-mail: stefan.fredenhagen@aei.mpg.de and pan.kessel@aei.mpg.de

Abstract

We study the relation between the frame-like and metric-like formulation of higher-spin gauge theories in three space–time dimensions. We concentrate on the theory that is described by an $SL(3) \times SL(3)$ Chern–Simons theory in the frame-like formulation. The metric-like theory is obtained by eliminating the generalized spin connection by its equation of motion, and by expressing everything in terms of the metric and a spin-3 Fronsdal field. We give an exact map between fields and gauge parameters in both formulations. To work out the gauge transformations explicitly in terms of metric-like variables, we have to make a perturbative expansion in the spin-3 field. We describe an algorithm for how to do this systematically, and we work out the gauge transformations to cubic order in the spin-3 field. We use these results to determine the gauge algebra to this order, and explain why the commutator of two spin-3 transformations only closes on-shell.

Keywords: gauge theory, higher-spin field theory, Chern–Simons theory, low-dimensional field theory

1. Introduction

Higher-spin gauge theories have gained a lot of attention in recent years, in particular because of the proposed higher-spin AdS/CFT correspondence in four and three dimensions (see [1, 2] for reviews). Higher-spin gauge fields can either be described by extending the vielbein formalism of gravity to higher-spins [3], or by extending the metric formulation [4]. Although the metric-like description might be the more intuitive ansatz, because one needs less auxiliary fields, it is the frame-like formulation that allowed Vasiliev to construct a consistent nonlinear theory of interacting higher-spin gauge fields [5, 6]. In the metric-like formulation, on the other hand, one only knows how to construct interactions in a perturbative expansion, e.g. one has obtained a classification of consistent cubic terms [7–18].

It would be desirable to understand the theory also in the metric-like formulation. In particular one would hope that one could get a better geometric understanding of the higher-spin gauge symmetry as generalized diffeomorphisms. This might also improve our understanding of particular solutions of higher-spin theories like higher-spin analogues of black holes [19–23]. In [24] it was shown how one could use the Wald formula in a metric-like higher-spin formulation to compute the entropy of higher-spin black holes (for other approaches see e.g. [19, 25–34]).

Higher-spin gauge theories in three dimensions are considerably simpler than in higher dimensions, because they do not contain propagating degrees of freedom and can be written as a Chern–Simons theory [35, 36]. Also, in contrast to higher dimensions, it is possible to truncate the tower of typically infinitely many higher-spin gauge fields to a finite selection—the simplest theory only contains gravity and one spin-3 field. In this case the generalized vielbein $e = e_\mu dx^\mu$ and the generalized spin connection take values in the Lie algebra $sl(3, \mathbb{R})$,

$$e_\mu = e_\mu^{\mathcal{A}} J_{\mathcal{A}}, \qquad \omega_\mu = \omega_\mu^{\mathcal{A}} J_{\mathcal{A}}, \tag{1.1}$$

where $J_{\mathcal{A}}$ form a basis of $sl(3, \mathbb{R})$,

$$\left[J_{\mathcal{A}}, J_B \right] = f_{\mathcal{A}B}{}^C J_C. \tag{1.2}$$

The gauge sector of this theory is described by the action

$$S = \frac{1}{16\pi G} \int \mathrm{tr}\left(e \wedge R + \frac{1}{3l^2} e \wedge e \wedge e \right), \tag{1.3}$$

where

$$R = \mathrm{d}\omega + \omega \wedge \omega \quad \Leftrightarrow \quad R^{\mathcal{A}} = \mathrm{d}\omega^{\mathcal{A}} + \frac{1}{2} f^{\mathcal{A}}{}_{BC}\, \omega^B \wedge \omega^C \tag{1.4}$$

is the curvature of the generalized spin connection, G is the gravitational constant and tr is the trace in the fundamental representation of $sl(3, \mathbb{R})$. The parameter l is related to the cosmological constant—a real and positive l coincides with the radius of the AdS solution. This action can be rewritten as a Chern–Simons theory whose gauge group depends on the cosmological constant: e.g. for a negative constant (positive l^2) the gauge group is $SL(3, \mathbb{R}) \times SL(3, \mathbb{R})$.

The frame-like formulation being so simple, it is tempting to try to reformulate it in terms of metric-like fields. First one has to eliminate the spin connection by its equation of motion,

$$D_{[\mu} e_{\nu]} = 0, \tag{1.5}$$

where D_μ is the covariant derivative including spin connection and the Levi-Civita Christoffel symbols,

$$D_\mu e_\nu^{\mathcal{A}} = \partial_\mu e_\nu^{\mathcal{A}} + f^{\mathcal{A}}{}_{BC}\, \omega_\mu^B\, e_\nu^C - \Gamma^\lambda{}_{\mu\nu}\, e_\lambda^{\mathcal{A}}. \tag{1.6}$$

Then one has to express everything in terms of metric-like fields, which have to be expressions in the vielbeins where all $sl(3)$ indices are contracted with invariant tensors. In [37] it was proposed to define the metric and the spin-3 field as

$$g_{\mu\nu} = \kappa_{\mathcal{A}B}\, e_\mu^{\mathcal{A}}\, e_\nu^B \tag{1.7}$$

and

$$\phi_{\mu\nu\rho} = \frac{1}{3!} \, d_{ABC} \, e_\mu^A \, e_\nu^B \, e_\rho^C, \tag{1.8}$$

where κ (Killing form) and d are invariant symmetric tensors defined in (A.7) and (A.11). The remaining task is then to rewrite the action (after eliminating ω) in terms of these fields.

Because the vielbein is not invertible (it is not a square matrix), this is rather complicated. In [24] the action was worked out to quadratic order in the spin-3 field by making a general ansatz and then demanding that explicit solutions of the frame-like theory should map to solutions in the metric-like theory. We will follow here a different approach which was started also in [24] [1].

Instead of considering the action and its solutions we concentrate on the gauge transformations. We formulate an exact map of the gauge parameters in the frame- and the metric-like formulation (section 2). Notice that one always has the freedom to reparameterize the gauge transformations, therefore this map is not unique. We then formulate an algorithm that can be used to map any given contraction of frame-like quantities to metric-like quantities in a perturbative expansion in the spin-3 field (section 3). We use this algorithm to explicitly compute the gauge transformations in the metric-like theory to cubic order. We are then in the position to compute commutators of these transformations to better understand the gauge algebra in the metric-like theory (section 4). We also discuss there why the commutator of two spin-3 transformations only closes on-shell.

Our work clarifies a few issues that were left unanswered in [24]. First of all we could show that the perturbative ansatz for the map between gauge parameters in [24] can be used as an exact map without any corrections, such that one has an exact dictionary of fields and gauge parameters. Secondly we can explain why the gauge transformations only close on-shell in the metric-like theory (whereas they close off-shell on the frame-like side). Thirdly we worked out a systematic approach to obtain the explicit expressions on the metric-like side that does not require the use of specific solutions of the theory. Last but not least we worked out the gauge transformations and gauge algebra to one order higher than in [24] in the hope to understand the metric-like theory better.

The expressions that we obtain for the gauge transformations to the order we consider are already quite large, they fill two pages of the appendix. In principle one could now go on and determine the corresponding action (which is a fairly easy task if one uses a powerful computer algebra program), and the result will be of similar size. We have not found any pattern in our expressions that could help to organize them—but without such a pattern it does not make sense to work out the metric-like theory to even higher orders. On the other hand, one might hope that there is a clever redefinition of fields and gauge parameters which makes the theory more manageable.

2. Relating frame- and metric-like gauge transformations

In this section we relate the gauge transformations in the frame- and in the metric-like description. In the frame-like theory there are two types of gauge transformations; the generalized local Lorentz transformations,

$$\delta_\Lambda^L e_\mu = \left[\Lambda, e_\mu \right], \tag{2.1}$$

[1] For an alternative ansatz for a metric-like description see [38, 39].

$$\delta^L_\Lambda \omega_\mu = D_\mu \Lambda, \tag{2.2}$$

and the generalized local translations,

$$\delta_\Xi e_\mu = D_\mu \Xi, \tag{2.3}$$

$$\delta_\Xi \omega_\mu = \frac{1}{l^2} \big[e_\mu, \Xi \big]. \tag{2.4}$$

The local Lorentz transformations act trivially on all metric-like fields built from the vielbeins e_μ. The generalized local translations, on the other hand, induce non-trivial transformations on them, and they can be interpreted as diffeomorphisms and higher-spin generalizations thereof.

Let us first consider pure diffeomorphisms. It is well-known (see e.g. [40]) that a generalized translation, where the parameter Ξ is of the form

$$\Xi^{\mathcal{A}} = e_\mu^{\mathcal{A}} \, \xi^\mu, \tag{2.5}$$

induces a diffeomorphism generated by the vector-field ξ^μ (up to a local Lorentz rotation) if one imposes the torsion constraint (1.5). The action of such a diffeomorphism (spin-2 gauge transformation) on any metric-like field ϕ built from the vielbeins e is given by

$$\delta^{(2)}_\xi \phi = \mathcal{L}_\xi \phi, \tag{2.6}$$

where $\mathcal{L}_\xi$ denotes the Lie derivative.

For the higher-spin transformations we do not know how they act in general, but only in the linearized theory where they should reproduce the transformations of free Fronsdal fields [4]. The spin-3 transformation should act as

$$\delta^{(3)}_\xi g_{\mu\nu} = 0 + \cdots, \tag{2.7}$$

$$\delta^{(3)}_\xi \phi_{\mu\nu\rho} = \nabla_{(\mu} \left(\xi_{\nu\rho)} - \frac{1}{3} g_{\nu\rho)} \xi_\lambda{}^\lambda \right) + \cdots, \tag{2.8}$$

where $\xi^{\mu\nu}$ is a symmetric tensor that labels the spin-3 gauge transformations, and the dots indicate terms that are at least linear in the spin-3 field. The covariant derivative ∇_μ is defined with respect to the Levi-Civita connection. Note that instead of choosing the spin-3 gauge parameter to be traceless, we decided to put an explicit projector to the traceless part and work with an unconstrained gauge parameter $\xi_{\nu\rho}$, which turns out to be more convenient in the higher-order computations.

We combine the gauge parameters for spin-2 and spin-3 transformations into a single object $\xi = (\xi^\mu, \xi^{\nu\rho})$. We are looking for a map

$$\xi = (\xi^\mu, \xi^{\nu\rho}) \mapsto \Xi(\xi), \tag{2.9}$$

such that

$$\delta_{\Xi(\xi)} \phi = \delta^{(2)}_\xi \phi + \delta^{(3)}_\xi \phi. \tag{2.10}$$

Note that such a map is not unique, even if we have fixed the expression of the metric-like fields in terms of frame-like ones such that no field redefinitions are possible. We can still redefine the higher-spin gauge parameters by terms that are at least linear in the higher-spin fields, such that the linearized gauge transformations are untouched. In the following we will construct one such map that is valid at all orders in the spin-3 field.

2.1. A proposal for the map

The map $\xi \mapsto \Xi(\xi)$ is linear, so we can write it as

$$\Xi^{\mathcal{A}}(\xi) = \mathcal{S}^{\mathcal{A}}{}_{\mu}\, \xi^{\mu} + \mathcal{S}^{\mathcal{A}}{}_{\nu\rho}\, \xi^{\nu\rho}, \tag{2.11}$$

with possibly field-dependent matrices $\mathcal{S}$. The implementation of pure diffeomorphisms is given by (2.5), this fixes the coefficients $\mathcal{S}^{\mathcal{A}}{}_{\mu}$ to

$$\mathcal{S}^{\mathcal{A}}{}_{\mu} = e_{\mu}^{\mathcal{A}}. \tag{2.12}$$

An arbitrary frame-like gauge transformation $\Xi^{\mathcal{A}}$ will induce both a diffeomorphism and a spin-3 transformation, therefore there will be projections P and $(1 - P)$ such that $P\Xi$ induces a pure diffeomorphism, and $(1 - P)\Xi$ a pure spin-3 transformation. Instead of fixing $\mathcal{S}^{\mathcal{A}}{}_{\nu\rho}$ directly, we will rather first attempt to fix the projection P. It should project an arbitrary gauge transformation to a pure diffeomorphism, therefore we demand that

$$\text{for every } \Xi^{\mathcal{A}} \text{ there is a } \xi^{\mu} \text{ such that } P^{\mathcal{A}}{}_{\mathcal{B}}\, \Xi^{\mathcal{B}} = \mathcal{S}^{\mathcal{A}}{}_{\mu}\, \xi^{\mu}, \tag{2.13}$$

and $P^2 = P$. A natural requirement for the projector is that it is orthogonal w.r.t. the Killing form, in other words that

$$P^{\mathcal{A}\mathcal{B}} = P^{\mathcal{B}\mathcal{A}}, \tag{2.14}$$

where we have raised the indices with the Killing form. This then fixes the projector uniquely to be

$$P^{\mathcal{A}\mathcal{B}} = e_{\mu}^{\mathcal{A}}\, g^{\mu\nu}\, e_{\nu}^{\mathcal{B}}. \tag{2.15}$$

Indeed we can easily check that

$$P^{\mathcal{A}}{}_{\mathcal{B}}\, P^{\mathcal{B}}{}_{\mathcal{C}} = e_{\mu}^{\mathcal{A}}\, g^{\mu\nu}\, e_{\nu}^{\mathcal{D}}\, \kappa_{\mathcal{B}\mathcal{D}}\, e_{\rho}^{\mathcal{B}}\, g^{\rho\sigma}\, e_{\sigma}^{\mathcal{E}}\, \kappa_{\mathcal{E}\mathcal{C}} \tag{2.16}$$

$$= e_{\mu}^{\mathcal{A}}\, g^{\mu\nu}\, g_{\nu\rho}\, g^{\rho\sigma}\, e_{\sigma}^{\mathcal{E}}\, \kappa_{\mathcal{E}\mathcal{C}} \tag{2.17}$$

$$= e_{\mu}^{\mathcal{A}}\, g^{\mu\sigma}\, e_{\sigma}^{\mathcal{E}}\, \kappa_{\mathcal{E}\mathcal{C}} = P^{\mathcal{A}}{}_{\mathcal{C}}, \tag{2.18}$$

where we used the definition of the metric (1.7) to go to the second line. Furthermore, for an arbitrary $\Xi^{\mathcal{A}}$ we have

$$P^{\mathcal{A}}{}_{\mathcal{B}}\, \Xi^{\mathcal{B}} = e_{\mu}^{\mathcal{A}}\left(g^{\mu\nu}\, e_{\nu}^{\mathcal{C}}\, \kappa_{\mathcal{C}\mathcal{B}}\, \Xi^{\mathcal{B}} \right), \tag{2.19}$$

therefore P indeed projects onto pure diffeomorphisms (where we interpret the term in the parentheses as the corresponding vector field). Notice that defining P to be an orthogonal projector was a choice we made, but we will see in the next section that by redefining the gauge parameters it is always possible to bring the projector to the form above.

Having fixed P we can now look for an $\mathcal{S}^{\mathcal{A}}{}_{\nu\rho}$ that satisfies

$$P^{\mathcal{A}}{}_{\mathcal{B}}\, \mathcal{S}^{\mathcal{B}}{}_{\nu\rho} = 0. \tag{2.20}$$

In addition we want that $\mathcal{S}^{\mathcal{A}}{}_{\nu\rho}$ coincides with the free field expression when we set the higher-spin fields to zero, i.e.[2]

$$\mathcal{S}^{\mathcal{A}}{}_{\nu\rho} = 3\, d^{\mathcal{A}}{}_{bc}\, e_{\nu}^{b}\, e_{\rho}^{c} + \cdots. \tag{2.21}$$

---

[2] That this ansatz reproduces the correct free field transformation (2.8) is checked in appendix A, see (A.17).

Here the lower case Latin indices run over the principally embedded $sl(2)$-subalgebra as will be explained in more detail in section 3.1. A natural ansatz for a covariant expression that is consistent with the linearization and with the projection P is then

$$S^{\mathcal{A}}{}_{\nu\rho} = \left(\delta^{\mathcal{A}}_D - P^{\mathcal{A}}_D\right)3\, d^D{}_{BC}\, e^B_\nu\, e^C_\rho + \cdots. \tag{2.22}$$

The linearized gauge transformation does only depend on the traceless part of the gauge parameter $\xi^{\mu\nu}$. If we want this property to hold also at the nonlinear level, we have to add the projection to the traceless part,

$$\xi^{\mu\sigma} \mapsto \left(\delta^\mu_\nu\, \delta^\sigma_\rho - \frac{1}{3} g^{\mu\sigma}\, g_{\nu\rho}\right)\xi^{\nu\rho}. \tag{2.23}$$

Our final ansatz for $S^{\mathcal{A}}{}_{\nu\rho}$ then reads

$$S^{\mathcal{A}}{}_{\nu\rho} = \left(\delta^{\mathcal{A}}_D - P^{\mathcal{A}}_D\right)3\, d^D{}_{BC}\, e^B_\mu\, e^C_\sigma\left(\delta^\mu_\nu\, \delta^\sigma_\rho - \frac{1}{3} g^{\mu\sigma}\, g_{\nu\rho}\right) \tag{2.24}$$

$$= 3\left(d^{\mathcal{A}}{}_{BC}\, e^B_\mu\, e^C_\sigma - 6\, e^{\mathcal{A}}_\kappa\, g^{\kappa\lambda}\, \phi_{\lambda\mu\sigma}\right)\left(\delta^\mu_\nu\, \delta^\sigma_\rho - \frac{1}{3} g^{\mu\sigma}\, g_{\nu\rho}\right), \tag{2.25}$$

where we used our definition for the spin-3 field ϕ in (1.8).

To summarize we propose the following map for the gauge parameters,

$$\Xi^{\mathcal{A}} = S^{\mathcal{A}}{}_\mu\, \xi^\mu + S^{\mathcal{A}}{}_{\nu\rho}\, \xi^{\nu\rho} \tag{2.26}$$

$$= e^{\mathcal{A}}_\mu\, \xi^\mu + 3\left(\delta^{\mathcal{A}}_D - P^{\mathcal{A}}_D\right)d^D{}_{BC}\, e^B_\mu\, e^C_\sigma\left(\delta^\mu_\nu\, \delta^\sigma_\rho - \frac{1}{3} g^{\mu\sigma}\, g_{\nu\rho}\right)\xi^{\nu\rho}. \tag{2.27}$$

In the following section we will argue that this is a consistent choice to all orders in the higher-spin field.

2.2. The proposed map is exact

Our goal is to obtain an exact map between the gauge parameters on the metric-like side and on the frame-like side,

$$\Xi^{\mathcal{A}} = S^{\mathcal{A}}{}_M\, \xi^M. \tag{2.28}$$

Here, M is a collective label for the metric-like labels, e.g. in the $sl(3)$ case $\{M\} = \{\mu, (\nu\rho)\}$, where $(\nu\rho)$ denote symmetric pairs of space–time labels without any trace constraints. The matrix S is then not a square matrix, and it can depend on the fields and on the vielbein. In the last section we have made a proposal for such a map (see (2.27)). In this section we want to show that there will always be a redefinition of the gauge parameters such that the proposal (2.27) provides the exact map.

Given a frame-like gauge parameter $\Xi^{\mathcal{A}}$ we may ask what the corresponding diffeomorphism and spin-3 transformation are that it induces, in other words we want to have an inverse relation of the form

$$\xi^M = \mathcal{T}^M{}_{\mathcal{A}}\, \Xi^{\mathcal{A}}, \tag{2.29}$$

such that

$$S^{\mathcal{A}}{}_M \, \mathcal{T}^M{}_B = \delta^{\mathcal{A}}_B \quad , \quad \mathcal{T}^M{}_{\mathcal{A}} \, S^{\mathcal{A}}{}_N = \mathcal{K}^M{}_N = \begin{pmatrix} \delta^{\mu}_{\nu} & 0 \\ 0 & \mathcal{K}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \end{pmatrix}. \qquad (2.30)$$

Here, $\mathcal{K}$ is a projector: not all components of $\xi^{\mu\nu}$ give rise to independent gauge transformations, and those ξ that are annihilated by $\mathcal{K}$ do not contribute. Therefore $\mathcal{K}$ projects $\xi^{\mu\nu}$ to the part that contributes to non-trivial gauge transformations. In the linearized approximation, $\mathcal{K}$ projects onto traceless tensors, in the full theory $\mathcal{K}$ could act differently. Of course this structure generalizes straightforwardly to the situation with more higher-spin fields.

Such a map between frame- and metric-like gauge parameters, specified by S and $\mathcal{T}$, is not unique, because we can redefine the gauge parameter on the metric-like side. Suppose we are given S and $\mathcal{T}$, and the associated projector $\mathcal{K}$. Then we can parameterize ξ^M by a new gauge parameter $\tilde{\xi}^M$,

$$\xi^M = \Phi^M{}_N \, \tilde{\xi}^N, \qquad (2.31)$$

with a possibly field-dependent matrix Φ. Because some combinations of gauge parameters ξ^M label trivial gauge transformations (in the linearized theory this is the trace part of $\xi^{\mu\nu}$), the matrix Φ does not need to be invertible, but we have to ensure that $\tilde{\xi}$ still parameterizes the full set of gauge transformations. Therefore we request that

$$\mathrm{Im}(\mathcal{K}\Phi) = \mathrm{Im}(\mathcal{K}). \qquad (2.32)$$

Then there is a map Ψ in the opposite direction,

$$\tilde{\xi}^M = \Psi^M{}_N \, \xi^N, \qquad (2.33)$$

such that

$$\mathcal{K}\Phi\Psi = \mathcal{K}. \qquad (2.34)$$

Ψ acts as an inverse after projection by $\mathcal{K}$.

With such a redefinition of the gauge parameters, the map (2.28) between frame-like and metric-like gauge parameters is changed into

$$\Xi^{\mathcal{A}} = \tilde{S}^{\mathcal{A}}{}_M \, \tilde{\xi}^M, \qquad \tilde{S} = S\Phi. \qquad (2.35)$$

Similarly we can introduce a new inverse map $\tilde{\mathcal{T}}$,

$$\tilde{\xi}^M = \tilde{\mathcal{T}}^M{}_{\mathcal{A}} \, \Xi^{\mathcal{A}}, \qquad \tilde{\mathcal{T}} = \Psi\mathcal{T}, \qquad (2.36)$$

such that

$$\tilde{S}\tilde{\mathcal{T}} = \mathbf{1}, \qquad \tilde{\mathcal{T}}\tilde{S} = \Psi\mathcal{K}\Phi =: \tilde{\mathcal{K}}. \qquad (2.37)$$

Notice that the prescription is symmetric in the sense that we can also view ξ^M to provide a new parameterization of $\tilde{\xi}^M$ via Ψ, and the maps Φ and Ψ satisfy

$$\tilde{\mathcal{K}}\Psi\Phi = \tilde{\mathcal{K}}, \qquad (2.38)$$

in analogy to (2.34).

Let us now apply this general discussion to the situation we are interested in. We assume that there is an exact map relating the gauge parameters as above with corresponding matrices S and $\mathcal{T}$ which are *a priori* unknown. We then show that there is a redefinition of gauge parameters such that the transformed S coincides with our proposal.

We already know how a pure diffeomorphism is implemented on the frame-like side, therefore $S^{\mathcal{A}}{}_\mu = e_\mu^{\mathcal{A}}$ is fixed, and should not be altered by a reparameterization. We can then restrict to matrices Φ and Ψ of the form

$$\Phi = \begin{pmatrix} \mathbf{1} & * \\ 0 & * \end{pmatrix}, \qquad \Psi = \begin{pmatrix} \mathbf{1} & * \\ 0 & * \end{pmatrix}. \tag{2.39}$$

Suppose now that we have found a $\hat{\mathcal{T}}^\mu{}_{\mathcal{A}}$ such that

$$\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} S^{\mathcal{A}}{}_\nu = \delta_\nu^\mu. \tag{2.40}$$

In our case this will be given by

$$\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} = g^{\mu\nu} e_\nu^{\mathcal{B}} \kappa_{\mathcal{B}\mathcal{A}}, \tag{2.41}$$

such that we recover our projector on diffeomorphisms (see (2.15)) as

$$S^{\mathcal{A}}{}_\mu \hat{\mathcal{T}}^\mu{}_{\mathcal{B}} = P^{\mathcal{A}}{}_{\mathcal{B}}. \tag{2.42}$$

Then we set

$$\Psi = \begin{pmatrix} \mathbf{1} & \Psi^\mu{}_{\rho\sigma} \\ 0 & \mathbf{1} \end{pmatrix}, \qquad \Psi^\mu{}_{\rho\sigma} = \left(\hat{\mathcal{T}}^\mu{}_{\mathcal{A}} - \mathcal{T}^\mu{}_{\mathcal{A}} \right) S^{\mathcal{A}}{}_{\rho\sigma}. \tag{2.43}$$

With this transformation one finds

$$\tilde{\mathcal{T}}^\mu{}_{\mathcal{A}} = \mathcal{T}^\mu{}_{\mathcal{A}} + \Psi^\mu{}_{\rho\sigma} \mathcal{T}^{\rho\sigma}{}_{\mathcal{A}} \tag{2.44}$$

$$= \mathcal{T}^\mu{}_{\mathcal{A}} + \left(\hat{\mathcal{T}}^\mu{}_{\mathcal{B}} - \mathcal{T}^\mu{}_{\mathcal{B}} \right) S^{\mathcal{B}}{}_{\rho\sigma} \mathcal{T}^{\rho\sigma}{}_{\mathcal{A}} \tag{2.45}$$

$$= \mathcal{T}^\mu{}_{\mathcal{A}} + \left(\hat{\mathcal{T}}^\mu{}_{\mathcal{B}} - \mathcal{T}^\mu{}_{\mathcal{B}} \right) \left(\delta_{\mathcal{A}}^{\mathcal{B}} - S^{\mathcal{B}}{}_\nu \mathcal{T}^\nu{}_{\mathcal{A}} \right) \tag{2.46}$$

$$= \hat{\mathcal{T}}^\mu{}_{\mathcal{A}}, \tag{2.47}$$

so it is possible to transform $\mathcal{T}$ such that the new $\tilde{\mathcal{T}}$ coincides in its μ-components with $\hat{\mathcal{T}}$. This means that it is always possible to redefine the gauge parameters such that the projection on pure diffeomorphisms is indeed given by P as defined in (2.15).

Assume now that we have fixed $S^{\mathcal{A}}{}_\mu = e_\mu^{\mathcal{A}}$ as well as $\mathcal{T}^\mu{}_{\mathcal{A}}$ as in (2.41). Then we are left with block-diagonal transformation matrices Φ and Ψ with the identity matrix in the $\mu - \nu$-block. Suppose now that we have found a matrix $\hat{S}^{\mathcal{A}}{}_{\mu\nu}$ such that

$$\mathcal{T}^\rho{}_{\mathcal{A}} \hat{S}^{\mathcal{A}}{}_{\mu\nu} = 0. \tag{2.48}$$

In our case such a $\hat{S}$ is given by the expression in (2.25). Now we set

$$\Phi = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & * \end{pmatrix}, \qquad \Phi^{\mu\nu}{}_{\rho\sigma} = \mathcal{T}^{\mu\nu}{}_{\mathcal{A}} \hat{S}^{\mathcal{A}}{}_{\rho\sigma}. \tag{2.49}$$

With this map Φ, the matrix S is transformed to

$$\tilde{S}^{\mathcal{A}}{}_{\rho\sigma} = S^{\mathcal{A}}{}_{\mu\nu} \Phi^{\mu\nu}{}_{\rho\sigma} \tag{2.50}$$

$$= S^{\mathcal{A}}{}_{\mu\nu} \mathcal{T}^{\mu\nu}{}_{\mathcal{B}} \hat{S}^{\mathcal{B}}{}_{\rho\sigma} \tag{2.51}$$

$$= \left(\delta_{\mathcal{B}}^{\mathcal{A}} - S^{\mathcal{A}}{}_\mu \mathcal{T}^\mu{}_{\mathcal{B}} \right) \hat{S}^{\mathcal{B}}{}_{\rho\sigma} \tag{2.52}$$

$$= \hat{S}^{\mathcal{A}}{}_{\rho\sigma}. \tag{2.53}$$

We have to make sure that the transformation Φ that we defined is an allowed one, i.e. that it does not reduce the set of gauge transformations. In the case at hand this is clear at least in a perturbative expansion in the higher-spin fields, where we only have to check that the transformation is regular at leading order. The leading terms of $\hat{S}$ and S coincide and are given by the linearized expression (2.21), therefore the transformation is regular.

In conclusion we have shown that indeed there is a parameterization of the metric-like gauge transformations such that the proposed map (2.27) gives an exact relation between metric-like and frame-like gauge parameters.

3. Translating frame- to metric-like quantities

In this section we will discuss an algorithm to translate frame-like to metric-like quantities. We will first outline this algorithm for quantities which do not contain any covariant derivatives and illustrate it by explicitly calculating the cosmological constant term in the metric-like formulation up to quartic order in the spin-3 field. We will then generalize the algorithm appropriately for quantities containing covariant derivatives. This will allow us to explicitly calculate the gauge transformations of the metric and spin-3 field to cubic order. Finally we will discuss why in these cases the mapping between metric-like and frame-like quantities is unique despite the appearance of seemingly free parameters in the metric-like expressions.

3.1. Restricting to vielbeins only

The aim of this section is to describe an algorithm which allows us to rewrite a frame-like expression in terms of metric-like fields. This algorithm is based on a perturbative expansion of all quantities in the spin-3 field ϕ. To this end we split the $sl(3)$ generators into $sl(2)$ generators $\{J_a\}$, labelled by small Latin indices, and the remaining generators $\{J_A\}$, labelled by capital Latin indices and chosen to be orthogonal to the J_a with respect to the Killing form. Using this notation we decompose the vielbein into the following components

$$e^{\mathcal{A}} = (e^a, E^A). \tag{3.1}$$

We first note that a given order in the spin-3 field ϕ corresponds to the same order of vielbeins E^A,

$$\mathcal{O}(\phi) = \mathcal{O}(E). \tag{3.2}$$

This can be seen by expanding (1.8) and (1.7),

$$\phi_{\mu\nu\rho} = \frac{1}{2} d_{Abc} E^A_{(\mu} e^b_\nu e^c_{\rho)} + \frac{1}{6} d_{ABC} E^A_\mu E^B_\nu E^C_\rho , \tag{3.3}$$

$$g_{\mu\nu} = \kappa_{ab} e^a_\mu e^b_\nu + \kappa_{AB} E^A_\mu E^B_\nu =: \overset{(0)}{g}_{\mu\nu} + \overset{(2)}{g}_{\mu\nu}, \tag{3.4}$$

where we have used (A.9) and (A.14).

Given any frame-like expression[3], with space–time indices $\mu_1 \ldots \mu_m$ and all frame indices contracted, we can perturbatively find the metric-like equivalent by making an ansatz

---

[3] For the moment we will not consider terms containing covariant derivatives. But, as discussed in section 3.3, by slightly modifying our algorithm these kind of terms can be dealt with as well.

consisting of all possible contractions of metric-like fields up to a certain order n in the spin-3 field ϕ. We then proceed in five steps:

Step 1: Expand both sides in terms of E^A using (3.3) and (3.4) up to order n and subtract them from each other.

Step 2: Isolate different orders in E^A. For each order we obtain an equation of the following form

$$\sum_i c^{(i)} \; t^{(i)}_{a_1 \ldots a_{p_i} B_1 \ldots B_{l_i}}\bigg(\{e\}, \{E\}, \{\epsilon\}, \Big\{ \overset{(0)}{g}{}^{-1} \Big\} \bigg)^{a_1 \ldots a_{p_i} B_1 \ldots B_{l_i}}_{\mu_1 \ldots \mu_n} = 0, \tag{3.5}$$

where $(\{e\}, \{E\}, \{\epsilon\}, \{\overset{(0)}{g}{}^{-1}\})$ denotes a contraction of vielbeins of the given index structure containing the inverse zero-order metric, the vielbeins and the invariant space–time tensor $\epsilon^{\mu\nu\rho}$. We will assume that each term has the same number of $\epsilon^{\mu\nu\rho}$, which carries only upper indices. The $t^{(i)}$ are $sl(2)$-invariant tensors. Furthermore some of the $c^{(i)}$ are understood to be the coefficients of the terms arising from the expansion of the frame-like side and are therefore equal to 1.

The next steps are to be performed for each order separately.

Step 3: Replace the E^A by the equivalent expression

$$E^A_\mu \to E^A_\sigma \; \Delta^\sigma_\mu, \tag{3.6}$$

where we use Δ to denote the expression

$$\Delta^\sigma_\mu = \overset{(0)}{g}{}^{\sigma\rho} \; e^b_\rho \; e^c_\mu \; \kappa_{bc} , \tag{3.7}$$

which—when performing the contraction—is equivalent to the identity matrix. This formal replacement ensures that the space–time index of E is contracted (via the inverse metric) with an $sl(2)$-vielbein e.

Step 4: Impose

$$\overset{(0)}{g}{}^{\mu\nu} \; e^a_\mu \; e^b_\nu = \kappa^{ab} \tag{3.8}$$

for all contractions of this type. After this replacement all terms in the sum of (3.5) are of the same form and can therefore be written as

$$\tilde{t}_{a_1 \ldots a_p B_1 \ldots B_l}\Big(\big\{ c^{(i)} \big\}\Big) \; \bigg(\{e\}, \{E\}, \{\epsilon\}, \Big\{ \overset{(0)}{g}{}^{-1} \Big\} \bigg)^{a_1 \ldots a_p B_1 \ldots B_l}_{\mu_1 \ldots \mu_n} = 0. \tag{3.9}$$

This is because the replacement (3.6) will transfer the space–time index of the vielbein E to a $sl(2)$-vielbein e. If the vielbein E carries a free space–time index it is therefore ensured that the free index is now carried by an $sl(2)$-vielbein. If however the space–time indices of two vielbeins E are contracted with each other they will be contracted with a $sl(2)$-vielbein after the substitution (3.6) and imposing (3.8). Finally a vielbein E contracted with an $sl(2)$-vielbein e will stay invariant under performing (3.6) and (3.8). Note that the number of $sl(2)$ frame indices in (3.9) might have changed during this step.

Step 5: Solve (3.9) by stripping off the vielbeins. This leads to

$$\mathcal{P}\tilde{t}_{a_1 \ldots a_p B_1 \ldots B_l}\Big(\big\{ c^{(i)} \big\}\Big) = 0, \tag{3.10}$$

where $\mathcal{P}$ is a projector imposing the symmetry inherent in the tensor $(\{e\}, \{E\}, \{\epsilon\}, \{\overset{(0)}{g}{}^{-1}\})$. We will explain this aspect in more detail in the next

section. But (3.10) is a linear equation in the coefficients $c^{(i)}$ and can therefore easily be solved using a computer algebra program.

We stress again that steps 3–5 have to be performed for all orders from 0 to n separately.

3.2. Example: cosmological constant term

Let us illustrate the algorithm described in the previous section by an example. We will consider the higher-spin cosmological constant term which is given by

$$\frac{1}{3l^2} \,\mathrm{tr}(e \wedge e \wedge e) = \frac{1}{3l^2} f_{ABC} \, \epsilon^{\mu\nu\rho} \, e_\mu^A \, e_\nu^B \, e_\rho^C \, d^3x. \qquad (3.11)$$

We make an ansatz for the metric-like equivalent of this term by writing down all the possible contractions of the spin-3 field with the metric such that the resulting expression is a space–time scalar, i.e.

$$\frac{1}{3l^2} f_{ABC} \, \epsilon^{\mu\nu\rho} \, e_\mu^A \, e_\nu^B \, e_\rho^C = \frac{2}{l^2} \sqrt{-g} \left(1 + \sum_{n=2}^{\infty} \mathcal{L}_n \right). \qquad (3.12)$$

Here $\mathcal{L}_n$ denotes all possible contractions compatible with the symmetries of the equation's lhs containing n of the ϕ fields and an arbitrary number of metric tensors. In the case of $n = 2$ this is given by

$$\mathcal{L}_2 = c_1 \, \phi^{\mu\nu\rho} \, \phi_{\mu\nu\rho} + c_2 \, \phi^\mu \, \phi_\mu, \qquad (3.13)$$

where ϕ_μ denotes the trace of the spin-3 field. We will now explain how the algorithm described in the previous section allows us to fix the coefficients c_1 and c_2.

Step 1 and 2: We expand (3.12) up to second order in E^A which corresponds to second order in ϕ as explained in the previous section. For this we have to expand the determinant of the metric which will also depend on E^A. This yields

$$\sqrt{-g} = \frac{1}{3!} f_{uvw} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \left(1 + \frac{1}{2} \overset{(0)}{g}^{\mu\nu} \kappa_{AB} \, E_\mu^A \, E_\nu^B \right) + \mathcal{O}\left(E^4 \right). \qquad (3.14)$$

Subtracting the lhs from the rhs of equation (3.12) and considering only terms of quadratic order we obtain up to an overall factor

$$c_1 f_{uvw} \, d_{Acd} \, d_{Bef} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, e_\rho^c \, e_{\rho'}^e \, \overset{(0)}{g}^{\rho\rho'} e_\sigma^d \, e_{\sigma'}^f \, \overset{(0)}{g}^{\sigma\sigma'} E_\mu^A \, E_\nu^B \, \overset{(0)}{g}^{\mu\nu}$$

$$+ \; 2c_1 f_{uvw} \, d_{Acd} \, d_{Bef} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, e_\rho^c \, e_{\rho'}^e \, \overset{(0)}{g}^{\rho\rho'} E_\sigma^A \, e_{\sigma'}^f \, \overset{(0)}{g}^{\sigma\sigma'} e_\mu^d \, E_\nu^B \, \overset{(0)}{g}^{\mu\nu}$$

$$+ \; \dots$$

$$+ \; \mathcal{O}\left(E^4 \right) = 0. \qquad (3.15)$$

Here we have only written out two terms explicitly and we will now show how the algorithm transforms them into the same form.

Step 3: Performing the substitution (3.6) leads to

$$
c_1 \, f_{uvw} \, d_{Acd} \, d_{Bef} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, e_\rho^c \, e_{\rho'}^e \, \overset{(0)}{g}{}^{\rho\rho'} \, e_\sigma^d \, e_{\sigma'}^f \, \overset{(0)}{g}{}^{\sigma\sigma'}
$$
$$
\times \, E_\kappa^A \, \overset{(0)}{g}{}^{\kappa\gamma} \, e_\gamma^g \, e_\mu^h \, \kappa_{gh} \, E_\lambda^B \, \overset{(0)}{g}{}^{\lambda\tau} \, e_\tau^l \, e_\nu^j \, \kappa_{lj} \, \overset{(0)}{g}{}^{\mu\nu}
$$
$$
+ \, 2c_1 \, f_{uvw} \, d_{Acd} \, d_{Bef} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, e_\rho^c \, e_{\rho'}^e \, \overset{(0)}{g}{}^{\rho\rho'}
$$
$$
\times \, E_\kappa^A \, \overset{(0)}{g}{}^{\kappa\gamma} \, e_\gamma^g \, e_\sigma^h \, \kappa_{gh} \, e_{\sigma'}^f \, \overset{(0)}{g}{}^{\sigma\sigma'} \, e_\mu^d \, E_\lambda^B \, \overset{(0)}{g}{}^{\lambda\tau} \, e_\tau^l \, e_\nu^j \, \kappa_{lj} \, \overset{(0)}{g}{}^{\mu\nu}
$$
$$
+ \, \ldots
$$
$$
+ \, \mathcal{O}\!\left(E^4\right) = 0. \tag{3.16}
$$

Step 4: Imposing the relation (3.8) we obtain

$$
c_1 \, f_{uvw} \, d_{Acd} \, d_B{}^{cd} \, \kappa_{lg} \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, E_\kappa^A \, \overset{(0)}{g}{}^{\kappa\gamma} \, e_\gamma^g \, E_\lambda^B \, \overset{(0)}{g}{}^{\lambda\tau} \, e_\tau^l
$$
$$
+ \, 2c_1 \, f_{uvw} \, d_{Adc} \, d_{Bf}{}^c \, \epsilon^{\chi\delta\epsilon} \, e_\chi^u \, e_\delta^v \, e_\epsilon^w \, E_\kappa^A \, \overset{(0)}{g}{}^{\kappa\gamma} \, e_\gamma^f \, E_\lambda^B \, \overset{(0)}{g}{}^{\lambda\tau} \, e_\tau^d
$$
$$
+ \, \ldots
$$
$$
+ \, \mathcal{O}\!\left(E^4\right) = 0. \tag{3.17}
$$

Having written all terms in the same form we find up to a global factor

$$
\Big(-108 f_{ABc} \, \kappa_{gd} \, \kappa_{he} + f_{cde}\big(6c_1 \, d_{Ahf} \, d_{Bg}{}^f + 4c_2 \, d_{Agf} \, d_{Bh}{}^f + 3c_1 \, d_{Asf} \, d_B{}^{sf} \, \kappa_{gh} \big)
$$
$$
+ \, 18 f_{cde} \, \kappa_{AB} \, \kappa_{gh} \Big) \epsilon^{\chi\delta\epsilon} \, e_\chi^c \, e_\delta^d \, e_\epsilon^e \, E_\gamma^A \, \overset{(0)}{g}{}^{\alpha\gamma} \, e_\alpha^g \, E_\sigma^B \, \overset{(0)}{g}{}^{\beta\sigma} \, e_\beta^h = 0. \tag{3.18}
$$

Step 5: We can solve this equation by stripping off the vielbeins. The remaining term has to be antisymmetrized in c, d, e and symmetrized with respect to exchange of the pair g, A with h, B. This operation was denoted by $\mathcal{P}$ in (3.10). The resulting equation is linear in c_1, c_2 and can be easily solved,

$$
c_1 = -3 \, , \quad c_2 = \frac{9}{2}. \tag{3.19}
$$

This is most conveniently done by choosing an explicit representation for the invariant tensors of $sl(3)$ and solving the resulting equation using a computer algebra program.

Therefore by using the algorithm described in the previous section we have found the metric-like equivalent of the cosmological constant term to quadratic order in the spin-3 field.

By applying the algorithm also to the quartic order we obtain the result

$$
\frac{1}{3l^2} \int f_{ABC} \, e^{\mathcal{A}} \wedge e^{\mathcal{B}} \wedge e^{\mathcal{C}} = \frac{2}{l^2} \int d^3x \, \sqrt{-g} \left(1 + \mathcal{L}_2 + \mathcal{L}_4 \right) + \mathcal{O}\!\left(\phi^6\right), \tag{3.20}
$$

where the quadratic terms are given by

$$\mathcal{L}_2 = -3\,\phi^{\mu\nu\rho}\,\phi_{\mu\nu\rho} + \frac{9}{2}\,\phi^{\mu}\,\phi_{\mu}. \tag{3.21}$$

This result was already obtained in [24]. The quartic contribution is

$$
\begin{aligned}
\mathcal{L}_4 = {}& (9+c)\,\phi_{\mu}{}^{\sigma\kappa}\,\phi^{\mu\nu\rho}\,\phi_{\nu\sigma}{}^{\tau}\,\phi_{\rho\kappa\tau} + c\,\phi_{\mu\nu}{}^{\sigma}\,\phi^{\mu\nu\rho}\,\phi_{\rho}{}^{\kappa\tau}\,\phi_{\sigma\kappa\tau} \\
&- (54+4c)\,\phi^{\nu}\,\phi_{\nu}{}^{\rho\sigma}\,\phi_{\rho}{}^{\kappa\tau}\,\phi_{\sigma\kappa\tau} - 9\,\phi^{\nu}\,\phi_{\nu}{}^{\rho\sigma}\,\phi_{\rho}\,\phi_{\sigma} \\
&- \left(6+\tfrac{1}{2}c\right)\phi_{\mu\nu\rho}\,\phi^{\mu\nu\rho}\,\phi_{\sigma\kappa\tau}\,\phi^{\sigma\kappa\tau} + \left(\tfrac{9}{2}+c\right)\phi^{\nu}\,\phi_{\nu}\,\phi_{\sigma\kappa\tau}\,\phi^{\sigma\kappa\tau} \\
&+ (81+2c)\,\phi^{\nu}\,\phi_{\nu}{}^{\rho\sigma}\,\phi_{\rho\sigma}{}^{\kappa}\,\phi_{\kappa} - \left(\tfrac{81}{8}+\tfrac{1}{2}c\right)\phi^{\nu}\,\phi_{\nu}\,\phi^{\kappa}\,\phi_{\kappa}. \tag{3.22}
\end{aligned}
$$

The sum of all terms term proportional to c is zero due to a dimension dependent identity as will be explained in section 3.5.

　　Note that we cannot build a scalar by contracting an odd number of spin-3 fields and therefore there are no such contributions in (3.20).

3.3. Including covariant derivatives

In the last two sections we did not include terms involving covariant derivatives in our discussion. In principle we can apply our algorithm also to these types of terms, but there is an additional complication. In the frame-like approach covariant derivatives can act both on E^A and e^a. The algorithm described in section 3.1 crucially relies on the fact that we can bring our expressions into the form (3.9). For this to work for quantities involving covariant derivatives we need to be able to express $D_\mu e_\nu^a$ in terms of $D_\mu E_\nu^A$. This can be achieved as follows. The metric is covariantly constant,

$$\nabla_\rho g_{\mu\nu} = D_\rho g_{\mu\nu} = \kappa_{AB}\,e_\mu^A\,D_\rho e_\nu^B + \kappa_{AB}\,e_\nu^A\,D_\rho e_\mu^B = 0, \tag{3.23}$$

where we used (1.7). The covariant derivative D_μ was defined in (1.6), and ∇_ρ is the covariant derivative with respect to the Levi-Civita connection.

　　By summing three permutations of equation (3.23),

$$
\begin{aligned}
& \kappa_{AB}\,e_\mu^A\,D_\rho e_\nu^B + \kappa_{AB}\,e_\nu^A\,D_\rho e_\mu^B \\
& - \kappa_{AB}\,e_\rho^A\,D_\mu e_\nu^B - \kappa_{AB}\,e_\nu^A\,D_\mu e_\rho^B \\
& + \kappa_{AB}\,e_\rho^A\,D_\nu e_\mu^B + \kappa_{AB}\,e_\mu^A\,D_\nu e_\rho^B = 0\,, \tag{3.24}
\end{aligned}
$$

and using the torsion constraint (1.5) we conclude

$$\kappa_{AB}\,e_\mu^A\,D_\rho e_\nu^B = 0. \tag{3.25}$$

Expanding this we obtain

$$D_\mu e_\nu^c = -\kappa_{AB}\,\overset{(0)}{g}{}^{\sigma\rho}\,e_\sigma^c\,E_\rho^A\,D_\mu E_\nu^B. \tag{3.26}$$

Using this result we can reformulate the algorithm described in section 3.1 such that it is also applicable to expressions involving covariant derivatives. We only need to modify the prescriptions for steps 1 and 3.

　　Step 1′: Expand both sides in terms of E^A using (3.4) and (3.3) up to order n and subtract them from each other. Perform the following substitution

$$D_\mu e_\nu^c = -\kappa_{AB} \overset{(0)}{g}\vphantom{g}^{\sigma\rho} e_\sigma^c E_\rho^A D_\mu E_\nu^B. \tag{3.27}$$

This ensures that all the covariant derivatives act on E^A.

Step 3′: Replace covariant derivatives of E^A by

$$D_\mu E_\nu^A \to \Delta_\mu^{\ \sigma} \Delta_\nu^{\ \rho} D_\sigma E_\rho^A \tag{3.28}$$

and the E^A without a derivative by

$$E_\mu^A \to E_\sigma^A \Delta_\mu^{\ \sigma}. \tag{3.29}$$

The notation $\Delta_\mu^{\ \rho}$ was defined in (3.7).

All other steps are unchanged.

3.4. Spin-3 transformations

In this section we determine the spin-3 transformations of both the metric and the spin-3 field perturbatively. For this we again make the most general ansatz for the gauge transformations of the metric-like fields and fix its coefficients by applying the modified algorithm described in the last section. The gauge transformation of the spin-3 field is then given by

$$\delta_\Xi^{(3)} \phi_{\alpha\beta\chi} = 3 \nabla_{(\alpha} \hat\xi_{\beta\chi)} + \left(\hat\xi \phi \nabla\phi\right)_{\alpha\beta\chi} + \left(\nabla\hat\xi \phi\phi\right)_{\alpha\beta\chi} + \mathcal{O}\left(\phi^4\right). \tag{3.30}$$

Here $\hat\xi$ denotes the traceless component of ξ, see (A.3). The explicit expressions for $(\hat\xi \phi \nabla\phi)_{\alpha\beta\chi}$ and $(\nabla\hat\xi \phi\phi)_{\alpha\beta\chi}$ are quite involved and are given in appendix D.

The metric transforms as follows

$$\begin{aligned}
\delta_\Xi^{(3)} g_{\alpha\beta} = {}& 6 \Big(2 \hat\xi^{\chi\delta} \nabla_\delta \phi_{\alpha\beta\chi} + 4 \hat\xi^{\chi\delta} g_{\alpha\beta} \nabla_\delta \phi_\chi + \hat\xi_{\alpha\beta} \nabla_\delta \phi^\delta - 2 \hat\xi^{\chi\delta} g_{\alpha\beta} \nabla_\epsilon \phi_{\chi\delta}^{\ \ \epsilon} \\
& - 4 \hat\xi_{(\alpha}^{\ \chi} \nabla_{\beta)} \phi_\chi - 4 \hat\xi_{(\alpha}^{\ \chi} \nabla_{|\chi|} \phi_{\beta)} + 4 \hat\xi_{(\alpha}^{\ \chi} \nabla^\delta \phi_{\beta)\chi\delta} \\
& - 4 \hat\xi^{\chi\delta} \nabla_{(\alpha} \phi_{\beta)\chi\delta} \Big) + \left(\hat\xi \phi\phi \nabla\phi\right)_{\alpha\beta} + \mathcal{O}\left(\phi^5\right).
\end{aligned} \tag{3.31}$$

The explicit expression for $(\hat\xi \phi\phi \nabla\phi)_{\alpha\beta}$ can be found in appendix D.

3.5. Ambiguities

Equation (3.22) contains a free parameter c, seemingly suggesting that the frame-like cosmological constant term does not have a unique metric-like counterpart. However this is not the case. The parameter c is due to a dimensional dependent identity (DDI), which arises by over-antisymmetrization. An example for a DDI is

$$\delta_{[\sigma}^\mu \delta_\rho^\nu \delta_\sigma^\gamma \delta_{\tau]}^\lambda = 0, \tag{3.32}$$

which obviously vanishes in three dimensions. A systematic way to construct all DDIs of a set of tensors is described in [41]. For a certain tensor all possible contractions with (3.32) are determined. All identities which arise by over-antisymmetrization can be constructed in such a way as we can always pull out deltas on the over-antisymmetrized indices. Using the Mathematica package xTras, described in [41], these identities can automatically be constructed by this method. For the case of the cosmological constant term at quartic order there is the following relevant DDI,

$$\phi_\mu{}^{\delta\epsilon}\,\phi^{\mu\nu\chi}\,\phi_{\nu\delta}{}^\phi\,\phi_{\chi\epsilon\phi} + \phi_{\mu\nu}{}^\delta\,\phi^{\mu\nu\chi}\,\phi_\chi{}^{\epsilon\phi}\,\phi_{\delta\epsilon\phi}$$

$$- 4\,\phi^\nu\,\phi_\nu{}^{\chi\delta}\,\phi_\chi{}^{\epsilon\phi}\,\phi_{\delta\epsilon\phi} - \frac{1}{2}\,\phi_{\mu\nu\chi}\,\phi^{\mu\nu\chi}\,\phi_{\delta\epsilon\phi}\,\phi^{\delta\epsilon\phi}$$

$$+ \phi^\nu\,\phi_\nu\,\phi_{\delta\epsilon\phi}\,\phi^{\delta\epsilon\phi} + 2\,\phi^\nu\,\phi_\nu{}^{\chi\delta}\,\phi_{\chi\delta}{}^\epsilon\,\phi_\epsilon - \frac{1}{2}\,\phi^\nu\,\phi_\nu\,\phi^\epsilon\,\phi_\epsilon \equiv 0. \qquad (3.33)$$

But the terms proportional to c in (3.22) are exactly given by this DDI and therefore vanish. Thus the cosmological constant term to quartic order is uniquely determined by our calculation.

4. Gauge algebra

We will now discuss the algebra of the gauge transformations for the metric-like fields. While the algebra of the frame-like transformations closes off-shell, in the metric-like formulation the algebra only closes on-shell (in addition to the torsion constraint that is imposed anyway when going to the metric-like description, we also have to satisfy the remaining equations of motion). We start by explaining this phenomenon and discuss also why the commutators with spin-2 transformations (diffeomorphisms) still close off-shell (i.e. without imposing the generalized Einstein equation corresponding to the vielbein's equation of motion). We then determine explicitly the gauge algebra to linear order in the spin-3 field ϕ.

4.1. On-shell gauge algebra

Recall from section 2.1 that in the frame-like theory general local translations induce pure diffeomorphisms and spin-3 transformations. General translations are parameterized by $\Xi^{\mathcal{A}}$. Obviously this corresponds to having eight degrees of freedom which nicely matches the 3 + 5 degrees of freedom of the parameter v^μ corresponding to pure diffeomorphisms and $\hat{\xi}^{\mu\nu} = \xi^{\mu\nu} - \frac{1}{3}\,g^{\mu\nu}\,\xi^\lambda{}_\lambda$ parameterizing the spin-3 transformations. According to (2.3) the fields of the frame-like formalism transform as follows

$$\delta_\Xi e_\mu^{\mathcal{A}} = D_\mu \Xi^{\mathcal{A}}, \qquad (4.1)$$

$$\delta_\Xi \omega_\mu^{\mathcal{A}} = \frac{1}{l^2}\Big[e_\mu, \Xi^{\mathcal{A}}\Big], \qquad (4.2)$$

and the gauge algebra closes off-shell.

When we translate the frame-like theory to the metric-like formulation we have to use the torsion constraint (1.5) to express the spin connection in terms of vielbeins, $\omega = \omega(e)$. This implicit dependence induces a gauge transformation of the spin connection that differs from the transformation (4.2), and only coincides with it on-shell, i.e. after using the equation of motion. This can be seen as follows. The induced transformation of the spin connection can be calculated by varying the torsion constraint (1.5),

$$\delta_\Xi\Big(D_{[\mu} e_{\nu]}^{\mathcal{A}}\Big) = \delta_\Xi D_{[\mu} e_{\nu]}^{\mathcal{A}} + D_{[\mu} D_{\nu]} \Xi^{\mathcal{A}} = 0. \qquad (4.3)$$

The Christoffel symbol is symmetric in μ and ν and therefore the variation of the covariant derivative in the equation above is given by the transformation of the spin connection. We thus obtain

$$f^{\mathcal{A}}{}_{BC}\,\delta_\Xi\omega^B_{[\mu}\,e^C_{\nu]} + f^{\mathcal{A}}{}_{BC}\,R^B_{\mu\nu}\,\Xi^C = 0 \qquad (4.4)$$

with $R_{\mu\nu}^{\mathcal{A}}$ from (1.4). The equation of motion for the vielbein is

$$R_{\mu\nu}^{\mathcal{A}} = -\frac{1}{2l^2} f^{\mathcal{A}}{}_{BC}\, e_\mu^B\, e_\nu^C. \tag{4.5}$$

Using this equation of motion in (4.4), we find that the induced transformation reduces on-shell to (we assume that the vielbein is non-degenerate)

$$\delta_\Xi \omega_\mu^{\mathcal{A}} = \frac{1}{l^2} f^{\mathcal{A}}{}_{BC}\, e_\mu^B\, \Xi^C, \tag{4.6}$$

which coincides with the transformation (4.2) in the frame-like theory. Therefore we expect that the metric-like gauge algebra only closes on-shell.

Let us explicitly consider the commutator of two gauge transformations on a vielbein (all metric-like fields are built out of the vielbein). Using (4.1) we obtain

$$\begin{aligned}
\left[\delta_\Xi, \delta_\Pi\right] e_\mu^{\mathcal{A}} &= D_\mu\left(\delta_\Xi \Pi^{\mathcal{A}} - \delta_\Pi \Xi^{\mathcal{A}}\right) + f^{\mathcal{A}}{}_{BC}\left(\delta_\Xi \omega_\mu^B\, \Pi^C - \delta_\Pi \omega_\mu^B\, \Xi^C\right) \\
&= \delta_{(\delta_\Xi \Pi - \delta_\Pi \Xi)} e_\mu^{\mathcal{A}} + f^{\mathcal{A}}{}_{BC}\left(\delta_\Xi \omega_\mu^B\, \Pi^C - \delta_\Pi \omega_\mu^B\, \Xi^C\right).
\end{aligned} \tag{4.7}$$

The first term is a local translation of the vielbein and therefore can again be interpreted as a gauge transformation in the metric-like formulation. For the second term it might in general not be possible to rewrite it as a gauge transformation on the vielbein. On the other hand, on-shell the last term is a generalized local Lorentz transformation of the vielbein as can be checked by using the Jacobi identity,

$$\begin{aligned}
f^{\mathcal{A}}{}_{BC}\left(\delta_\Xi \omega_\mu^B\, \Pi^C - \delta_\Pi \omega_\mu^B\, \Xi^C\right) &= -\frac{1}{l^2} f^{\mathcal{A}}{}_{BC}\left(f^B{}_{D\mathcal{E}}\, e_\mu^D\, \Xi^{\mathcal{E}}\, \Pi^C - f^B{}_{D\mathcal{E}}\, e_\mu^D\, \Pi^{\mathcal{E}}\, \Xi^C\right) \\
&= \frac{1}{l^2} f^{\mathcal{A}}{}_{BD}\left(f^B{}_{\mathcal{E}C}\, \Xi^{\mathcal{E}}\, \Pi^C\right) e_\mu^D.
\end{aligned} \tag{4.8}$$

In the metric-like fields all frame indices are contracted with invariant tensors, and the local Lorentz transformations do not have any effect. Hence we find that on-shell the gauge algebra in the metric-like formulation is obtained by translating

$$\left[\delta_\Xi, \delta_\Pi\right] = \delta_{(\delta_\Xi \Pi - \delta_\Pi \Xi)} \tag{4.9}$$

into metric-like quantities.

4.2. Off-shell closure for spin-2 transformations

In the last subsection we have shown that after imposing the vielbein's equation of motion the second term in (4.7) can be written as a local Lorentz transformation. In this section we will show that in the special case in which at least one of the parameters describes a spin-2 transformation, i.e. $\Xi^{\mathcal{A}} = e_\mu^{\mathcal{A}} \xi^\mu$, the last term of (4.7) is a local Lorentz transformation even off-shell. Firstly, using (4.4) we calculate the variation of the spin connection in this special case.

$$\begin{aligned}
f^{\mathcal{A}}{}_{BC}\, \delta_\Xi \omega_{[\mu}^B\, e_{\nu]}^C &= -f^{\mathcal{A}}{}_{BC}\, R_{\mu\nu}^B\, e_\sigma^C\, \xi^\sigma \\
&= -f^{\mathcal{A}}{}_{BC}\left(R_{\mu\sigma}^B\, e_\nu^C + R_{\sigma\nu}^B\, e_\mu^C\right)\xi^\sigma \\
&= 2 f^{\mathcal{A}}{}_{BC}\, R_{\sigma[\mu}^B\, e_{\nu]}^C\, \xi^\sigma,
\end{aligned} \tag{4.10}$$

where we have used a Bianchi-like identity $f^{\mathcal{A}}{}_{BC}\, R_{[\mu\nu}^B\, e_{\sigma]}^C = 0$ (see appendix B). As the vielbein is non-degenerate we conclude from (4.10) that

$$\delta_\Xi \omega_\mu^B = 2\,\xi^\sigma\,R_{\sigma\mu}^B \tag{4.11}$$

is the induced transformation of the spin connection under a spin-2 transformation.

Plugging this result into the second term of (4.7) and using (4.4) we obtain

$$
\begin{aligned}
f^A{}_{BC}\left(\delta_\Xi \omega_\mu^B\,\Pi^C - \delta_\Pi \omega_\mu^B\,\Xi^C\right) &= f^A{}_{BC}\left(2\,\xi^\nu\,R_{\nu\mu}^B\,\Pi^C - \delta_\Pi \omega_\mu^B\,e_\nu^C\,\xi^\nu\right) \\
&= f^A{}_{BC}\left(2\,\xi^\nu\,R_{\nu\mu}^B\,\Pi^C - 2\,\delta_\Pi \omega_{[\mu}^B\,e_{\nu]}^C\,\xi^\nu - \delta_\Pi \omega_\nu^B\,e_\mu^C\,\xi^\nu\right) \\
&= f^A{}_{BC}\left(2\,\xi^\nu\,R_{\nu\mu}^B\,\Pi^C + 2\,\xi^\nu\,R_{\mu\nu}^B\,\Pi^C - \delta_\Pi \omega_\nu^B\,e_\mu^C\,\xi^\nu\right) \\
&= -f^A{}_{BC}\,\delta_\Pi \omega_\nu^B\,e_\mu^C\,\xi^\nu.
\end{aligned}
\tag{4.12}
$$

But the final expression is just a generalized local Lorentz transformation and we have therefore shown that the commutator of a spin-2 transformation with any other transformation can be expressed as a gauge transformation also off-shell.

In the following we will compute the various commutators that arise in the algebra of metric-like gauge transformations explicitly.

4.3. Spin-2 spin-2 commutator

Here we will consider the case of both transformations being diffeomorphisms, i.e. $\Pi^A = e_\mu^A \pi^\mu$ and $\Xi^A = e_\mu^A \xi^\mu$. As shown in the previous section this commutator closes off-shell in the metric-like theory (i.e. by only imposing the torsion constraint), and using (4.7) we can calculate the resulting transformation

$$
\begin{aligned}
\delta_\Pi\left(e_\mu^A\,\xi^\mu\right) - \delta_\Xi\left(e_\mu^A\,\pi^\mu\right) &= D_\mu\left(e_\nu^A\,\pi^\nu\right)\xi^\mu - \xi \leftrightarrow \pi \\
&= -e_\nu^A\,\mathcal{L}_\pi \xi^\nu + 2\,\xi^\mu\,\pi^\nu\,D_{[\mu}e_{\nu]}^A,
\end{aligned}
\tag{4.13}
$$

where $\mathcal{L}_\pi \xi^\nu = \pi^\mu \partial_\mu \xi^\nu - \xi^\mu \partial_\mu \pi^\nu$ is the Lie derivative. But the last term in the last line vanishes as we impose the torsion constraint (1.5). By (2.5) the result of this commutator therefore induces a diffeomorphism with vector field $-\mathcal{L}_\pi \xi^\nu$.

4.4. Spin-3 spin-2 commutator

We now want to discuss the commutator of a spin-3 and a spin-2 transformation. The spin-3 transformation is parameterized by

$$\Xi^A = S^A{}_{\mu_1\mu_2}\,\xi^{\mu_1\mu_2}, \tag{4.14}$$

where S is given in (2.25). The result for the commutator will not depend on the precise form of S, but only on the property that it is built from the vielbeins. In fact we can also consider the more general case of the commutator of a spin-$(s+1)$ and a spin-2 transformation without any additional complication, where the spin-2 and the spin-$(s+1)$ transformations are parameterized by

$$\Pi^A = e_\sigma^A \pi^\sigma \quad \text{and} \quad \Xi^A = S^A{}_{\mu_1\dots\mu_s}(e)\xi^{\mu_1\dots\mu_s}. \tag{4.15}$$

Here, $S^A{}_{\mu_1\dots\mu_s}(e)$ is built by contracting vielbeins and it is completely symmetric in all space–time indices. For a result that we need later we consider the following space–time tensor,

$$\mathcal{O}_{\nu\mu_1\dots\mu_s} = \kappa_{AB}\,e_\nu^A\,S^B{}_{\mu_1\dots\mu_s}(e). \tag{4.16}$$

Because it is constructed from the vielbeins, under the spin-2 transformation the tensor $\mathcal{O}_{\nu\mu_1\dots\mu_s}$ changes by the Lie derivative along π,

$$\delta_\Pi \mathcal{O}_{\nu\mu_1\ldots\mu_s} = \pi^\sigma \nabla_\sigma \mathcal{O}_{\nu\mu_1\ldots\mu_s} + s\ \nabla_{(\mu_1}\pi^\sigma \mathcal{O}_{|\nu\sigma|\mu_2\ldots\mu_s)} + \nabla_\nu \pi^\sigma\ \mathcal{O}_{\sigma\mu_1\ldots\mu_s}. \tag{4.17}$$

The lhs of this equation can be calculated by explicitly evaluating the variation of the vielbein, i.e.

$$
\begin{aligned}
\delta_\Pi \mathcal{O}_{\nu\mu_1\ldots\mu_s} &= \kappa_{AB}\, D_\nu \left(e_\sigma^{\mathcal{A}} \pi^\sigma \right) S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} + \kappa_{AB}\, e_\nu^{\mathcal{A}} \left(\delta_\Pi S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} \right) \\
&= \kappa_{AB}\, \pi^\sigma \left(D_\sigma e_\nu^{\mathcal{A}} \right) S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} + \kappa_{AB}\, e_\nu^{\mathcal{A}} \left(\delta_\Pi S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} \right) + \nabla_\nu \pi^\sigma \mathcal{O}_{\sigma\mu_1\ldots\mu_s},
\end{aligned} \tag{4.18}
$$

where we used (1.5) and suppressed the dependency of $S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s}$ on the vielbeins to simplify notation. Combining (4.17) with (4.18) yields

$$\kappa_{AB}\, e_\nu^{\mathcal{A}} \left(\delta_\Pi S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} \right) = \kappa_{AB}\, e_\nu^{\mathcal{A}}\, \pi^\sigma D_\sigma S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} + s\,\kappa_{AB}\, e_\nu^{\mathcal{A}}\ \nabla_{(\mu_1}\pi^\sigma\ S^{\mathcal{B}}{}_{|\sigma|\mu_2\ldots\mu_s)}. \tag{4.19}$$

We therefore conclude that

$$\delta_\Pi S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} = \pi^\sigma D_\sigma S^{\mathcal{B}}{}_{\mu_1\ldots\mu_s} + s\,\nabla_{(\mu_1}\pi^\sigma\ S^{\mathcal{B}}{}_{|\sigma|\mu_2\ldots\mu_s)}. \tag{4.20}$$

We are now in the position to determine the commutator of the spin-2 transformation Π and the spin-$(s+1)$ transformation Ξ given in (4.15), and we find

$$
\begin{aligned}
\delta_\Xi \Pi^{\mathcal{A}} - \delta_\Pi \Xi^{\mathcal{A}} &= \pi^\sigma D_\sigma \left(S^{\mathcal{A}}{}_{\mu_1\ldots\mu_s} \xi^{\mu_1\ldots\mu_s} \right) - \xi^{\mu_1\ldots\mu_s}\, \delta_\Pi S^{\mathcal{A}}{}_{\mu_1\ldots\mu_s} \\
&= S^{\mathcal{A}}{}_{\mu_1\ldots\mu_s} \left(\pi^\sigma \nabla_\sigma \xi^{\mu_1\ldots\mu_s} - s\ \xi^{\sigma(\mu_1\ldots\mu_{s-1}} \nabla_\sigma \pi^{\mu_s)} \right) \\
&= S^{\mathcal{A}}{}_{\mu_1\ldots\mu_s} \left(\mathcal{L}_\pi \xi^{\mu_1\ldots\mu_s} \right).
\end{aligned} \tag{4.21}
$$

Thus the commutator is a spin-$(s+1)$ transformation whose parameter is given by the Lie derivative of the original spin-$(s+1)$-parameter. In particular in our case we find

$$\left[\delta_\Xi^{(3)}, \delta_\pi^{(2)} \right] = \delta_{\mathcal{L}_\pi\xi}^{(3)}. \tag{4.22}$$

4.5. Spin-3 spin-3 commutator

In contrast to the commutation relation involving at least one spin-2 transformation we currently do not have an all order result for the commutator of two spin-3 transformations. The commutator is specified by traceless parameters $\hat{\xi}^{\mu\nu}$ and $\hat{\pi}^{\mu\nu}$, and generically it will lead to a combination of a spin-2 transformation and a spin-3 transformation, i.e.

$$\left[\delta_\Pi, \delta_\Xi \right] e_\mu^{\mathcal{A}} = \delta_{S(u,v)}\, e_\mu^{\mathcal{A}}, \tag{4.23}$$

where

$$S^{\mathcal{A}}(u, v) = S^{\mathcal{A}}{}_\mu v^\mu + S^{\mathcal{A}}{}_{\rho\sigma} u^{\rho\sigma} \tag{4.24}$$

denotes the map defined in (2.11). In the following we will determine the parameters $u^{\mu\nu}$ and v^μ perturbatively in the spin-3 field. First we will calculate explicitly the spin-2 parameter v^μ by only considering zeroth order contributions. Then we will use the algorithm discussed in section 3.1 to determine these parameters at linear order.

4.5.1. Spin-2 parameter v^μ.
This contribution was already calculated in [24] using a different method. We need to evaluate

$$\delta^{(3)}_\Pi \Xi^{\mathcal{A}} = 3\, \delta^{(3)}_\Pi \left\{ \delta^{\mathcal{A}}_{\mathcal{B}} - P^{\mathcal{A}}_{\mathcal{B}} \right\} d^{\mathcal{B}}{}_{CD}\, e^C_\mu\, e^D_\nu\, \hat{\xi}^{\mu\nu} + 3\left(\delta^{\mathcal{A}}_{\mathcal{B}} - P^{\mathcal{A}}_{\mathcal{B}} \right) \delta^{(3)}_\Pi \left\{ d^{\mathcal{B}}{}_{CD}\, e^C_\mu\, e^D_\nu\, \hat{\xi}^{\mu\nu} \right\}$$

$$= -3\left(\delta^{(3)}_\Pi P^{\mathcal{A}}_{\mathcal{B}} \right) d^{\mathcal{B}}{}_{CD}\, e^C_\mu\, e^D_\nu\, \hat{\xi}^{\mu\nu}$$

$$+ 6\left(\delta^{\mathcal{A}}_{\mathcal{B}} - P^{\mathcal{A}}_{\mathcal{B}} \right) d^{\mathcal{B}}{}_{CD}\left(D_\mu \Pi^C \right) e^D_\nu\, \hat{\xi}^{\mu\nu}$$

$$+ 3\left(\delta^{\mathcal{A}}_{\mathcal{B}} - P^{\mathcal{A}}_{\mathcal{B}} \right) d^{\mathcal{B}}{}_{CD}\, e^C_\mu\, e^D_\nu\left(\delta^{(3)}_\Pi \hat{\xi}^{\mu\nu} \right). \tag{4.25}$$

The variation of the projector $P^{\mathcal{A}}_{\mathcal{B}}$ is given by

$$\delta^{(3)}_\Pi P^{\mathcal{A}}_{\mathcal{B}} = \left(D_\mu \Pi^{\mathcal{A}} \right) g^{\mu\nu}\, e^C_\nu\, \kappa_{BC} + e^{\mathcal{A}}_\mu\, g^{\mu\nu} \kappa_{BC}\left(D_\nu \Pi^C \right)$$

$$- 2 e^{\mathcal{A}}_\mu\, \kappa_{\mathcal{E}F}\, e^C_\nu\, e^{\mathcal{E}}_{(\sigma}\, D_{\rho)} \Pi^{\mathcal{F}}\, g^{\sigma\nu}\, g^{\rho\mu}\, \kappa_{BC}, \tag{4.26}$$

where the last term arises due to the variation of the inverse metric in the projector. We will now evaluate (4.25) at leading order. Let us focus on the last term in (4.25) first. By using (3.31) it can be checked easily that this term is of higher order as

$$\delta^{(3)}_\Pi \hat{\xi}^{\mu\nu} = \mathcal{O}(E). \tag{4.27}$$

Note that if we choose $\mathcal{A} = A$, all terms in (4.25) will be at least of linear order. For $\mathcal{A} = a$ we can easily deduce that the second term in (4.25) does not contribute as

$$\left(\delta^a_b - P^a_b \right) = 0 \quad \text{and} \quad P^a_B = \mathcal{O}(E). \tag{4.28}$$

So only the first term in (4.25) will contribute to leading order. From $\mathcal{A} = a$ it follows that to leading order we have to choose $\mathcal{B} = B$. The variation of the projector is then given by

$$\delta^{(3)}_\Pi P^a_B = 3\, e^a_\mu\, g^{\mu\nu}\, \kappa_{BC}\, d^C{}_{ef}\, e^e_\rho\, e^f_\tau\, \nabla_\nu \hat{\pi}^{\rho\tau} + \mathcal{O}(E). \tag{4.29}$$

Plugging this in the only non-vanishing term of (4.25) we find at leading order

$$\delta^{(3)}_\Pi \Xi^a - \delta^{(3)}_\Xi \Pi^a = -9\, d_{Bef}\, d^B{}_{cd}\, e^a_\sigma\, g^{\sigma\tau}\, e^e_\rho\, e^f_\gamma\, e^c_\mu\, e^d_\nu\left(\hat{\xi}^{\mu\nu}\, \nabla_\tau \hat{\pi}^{\rho\gamma} - \hat{\xi} \leftrightarrow \hat{\pi} \right)$$

$$= -18\, e^a_\sigma\, g^{\sigma\tau}\left(\hat{\xi}^{\mu\nu}\, \nabla_\tau \hat{\pi}_{\mu\nu} - \hat{\xi} \leftrightarrow \hat{\pi} \right), \tag{4.30}$$

where we have used the identity (A.15b) in the last step.

But by (2.5) the result in (4.30) corresponds to a spin-2 transformation with the parameter

$$v^\mu = -18\, g^{\mu\nu}\left(\xi^{\rho\sigma}\, \nabla_\nu \pi_{\rho\sigma} - \frac{1}{3}\, \xi^\rho{}_\rho\, \nabla_\nu \pi^\sigma{}_\sigma - \xi \leftrightarrow \pi \right). \tag{4.31}$$

4.5.2. Spin-3 parameter $u^{\mu\nu}$. To determine the spin-3 parameter $u^{\mu\nu}$ we make the following ansatz

$$\delta^{(3)}_\Pi \Xi^{\mathcal{A}} - \delta^{(3)}_\Xi \Pi^{\mathcal{A}} = S^{\mathcal{A}}\left(u^{\mu\nu}, v^\mu \right), \tag{4.32}$$

where $S^{\mathcal{A}}$ is defined as in (4.24). The parameter v^μ cannot be corrected by terms linear in the spin-3 field as we cannot build a vector by contracting a spin-3 field, a covariant derivative and the parameter $u^{\mu\nu}$. In order to solve this equation we make an ansatz for the linear order of $u^{\mu\nu}$ by considering all possible contractions of

$$\xi^{\rho\sigma}, \quad \pi^{\rho\sigma} \quad \text{and} \quad \phi_{\sigma\tau\rho} \tag{4.33}$$

with two symmetric free indices, μ and ν, and antisymmetric with respect to the exchange of ξ and π. We use the algorithm described in section 3.3 to determine the coefficients of the ansatz. The result for $u^{\alpha\beta}$ contains three different contributions denoted by

$$u^{\alpha\beta} = u_1^{\alpha\beta} + u_2^{\alpha\beta} + u_3^{\alpha\beta}. \tag{4.34}$$

Firstly terms with a derivative acting on the spin-3 field

$$
\begin{aligned}
u_1^{\alpha\beta} = 6\Bigg(&-5\left(\hat{\xi}\hat{\pi}\right)^{\alpha\beta\chi\delta}\nabla_\delta\phi_\chi + \left(\hat{\xi}\hat{\pi}\right)^{\chi\delta}{}_\chi{}^\epsilon\left(g^{\alpha\beta}\nabla_\epsilon\phi_\delta - \nabla_\epsilon\phi^{\alpha\beta}{}_\delta\right) + 3\left(\hat{\xi}\hat{\pi}\right)^{\alpha\beta\chi\delta}\nabla_\epsilon\phi_{\chi\delta}{}^\epsilon \\
&- 6\left(\hat{\xi}\hat{\pi}\right)^{\chi\delta\xi}g^{\alpha\beta}\nabla_\xi\phi_{\chi\delta\epsilon} + 3\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi}{}_\chi{}^{\delta|}\nabla^{\beta)}\phi_\delta + 2\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi}{}_\chi{}^{\delta|}\nabla_\delta\phi^{\beta)} \\
&- 3\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi}{}_\chi{}^\delta\nabla^{\epsilon|}\phi^{\beta)}{}_{\delta\epsilon} - 5\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta\epsilon|}\nabla_\chi\phi^{\beta)}{}_{\delta\epsilon} \\
&+ 13\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta\epsilon|}\nabla_\delta\phi^{\beta)}{}_{\chi\epsilon}\Bigg) + k_1 D_1^{\alpha\beta},
\end{aligned}
\tag{4.35}
$$

where we have used the following notation

$$\left(\hat{\xi}\hat{\pi}\right)^{\mu\nu\rho\sigma} = \hat{\xi}^{\mu\nu}\hat{\pi}^{\rho\sigma} - \hat{\xi} \leftrightarrow \hat{\pi}. \tag{4.36}$$

The hatted tensors again denote the traceless components of the parameters, see (A.3). The term $D_1^{\alpha\beta}$ is given in appendix C and vanishes due to DDIs.

Secondly there are contributions with a derivative acting on one of the parameters,

$$
\begin{aligned}
u_2^{\alpha\beta} = 12\Bigg(&\left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta\alpha\beta}{}_{\chi\delta}\phi^\chi + \left(\nabla\hat{\xi}\hat{\pi}\right)^\delta{}_\chi{}^\epsilon{}_{\delta\epsilon}\phi^\chi g^{\alpha\beta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\chi\delta}\phi^{\alpha\beta}{}_\epsilon - \frac{3}{2}\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\alpha\beta\delta\epsilon}\phi_{\chi\delta\epsilon} \\
&+ 12\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_\chi{}^\xi g^{\alpha\beta}\phi_{\delta\epsilon\xi} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)}{}_\chi\phi_\delta - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|}{}_{\chi\delta}\phi^{\beta)} \\
&- 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta}{}_\chi{}^{\epsilon|}\phi^{\beta)}{}_{\delta\epsilon} - 34\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon(\alpha}{}_\chi\phi^{\beta)}{}_{\delta\epsilon}\Bigg) \\
&+ k_2 D_2^{\alpha\beta} + k_3 D_3^{\alpha\beta} + k_4 D_4^{\alpha\beta} + k_5 D_5^{\alpha\beta}.
\end{aligned}
\tag{4.37}
$$

Here we used the notation

$$\left(\nabla\hat{\xi}\hat{\pi}\right)_\mu{}^{\nu\rho\sigma\epsilon} = \hat{\pi}^{\nu\rho}\nabla_\mu\hat{\xi}^{\sigma\epsilon} - \hat{\pi} \leftrightarrow \hat{\xi}. \tag{4.38}$$

The terms $D_i^{\alpha\beta}$, $i = 2\ldots 5$, are given in appendix C and are identically zero due to DDIs.

Finally there are contributions containing the trace of the parameters of the gauge transformations

$$
\begin{aligned}
u_3^{\alpha\beta} = &4\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{\lambda\rho}\nabla_\rho\phi^{\alpha\beta}{}_\lambda + 4\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{\lambda\rho}g^{\alpha\beta}\nabla_\rho\phi_\lambda + 2\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{\alpha\beta}\nabla_\rho\phi^\rho \\
&- 8\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{(\alpha|\lambda|}\nabla^{\beta)}\phi_\lambda - 8\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{(\alpha|\lambda|}\nabla_\lambda\phi^{\beta)} + 8\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{(\alpha|\lambda}\nabla^{\rho|}\phi^{\beta)}{}_{\lambda\rho} \\
&- 8\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{\lambda\rho}\nabla^{(\alpha}\phi^{\beta)}{}_{\lambda\rho},
\end{aligned}
\tag{4.39}
$$

where we denoted

$$\left(\hat{\xi}\pi' - \hat{\pi}\xi'\right)^{\mu\nu} = \hat{\xi}^{\mu\nu}\pi^\sigma{}_\sigma - \hat{\pi}^{\mu\nu}\xi^\sigma{}_\sigma. \tag{4.40}$$

It might at first seem surprising that the commutator contains traces of the gauge parameters, whereas in a single gauge transformation only their traceless part contributes. This is due to the fact that the notion of the trace is field-dependent (it depends on the metric), and that the field changes under the gauge transformation.

Let us briefly explain this phenomenon in a very simple example. Consider an infinitesimal rotation of a vector $\vec{x} \in \mathbb{R}^3$ parameterized by a vector $\vec{v} \in \mathbb{R}^3$,

$$\delta_v \vec{x} = \vec{v} \times \vec{x}. \tag{4.41}$$

Obviously the component of $\vec{v}$ parallel to $\vec{x}$, i.e. $\vec{v}_\parallel = \frac{(\vec{v} \cdot \vec{x})}{\|x\|^2} \vec{x}$, does not contribute to the rotation. However the commutator of two rotations is given by

$$\left[\delta_v, \delta_w \right] \vec{x} = \left(\vec{w} \times \vec{v} \right) \times \vec{x} = \vec{v} \left(\vec{w} \cdot \vec{x} \right) - \vec{w} \left(\vec{v} \cdot \vec{x} \right). \tag{4.42}$$

Therefore the components parallel to $\vec{x}$ contribute in the commutator although an individual rotation only depends on the component orthogonal to $\vec{x}$. This is completely analogous to the observation above that the traces of the spin-3 parameter contribute to the commutator.

This concludes the computation of the commutator of two spin-3 transformations at linear order in the spin-3 field. Together with the expression derived for the commutator of a spin-2 with either a spin-2 or spin-3 transformation, which are exact results, we have therefore determined the gauge algebra to leading order.

5. Conclusion

In this article we have studied the metric-like formulation of the higher-spin gauge theory that in the frame-like formulation is described by an $SL(3) \times SL(3)$ Chern–Simons theory. Starting point is the identification of the metric and the spin-3 gauge field by contracting the generalized vielbeins with the invariant tensors corresponding to the quadratic and cubic Casimir, respectively.

There does not seem to be a simple way of explicitly solving for the spin connection, therefore we can only study the metric-like reformulation in a perturbative expansion in the spin-3 field. Instead of directly rewriting the action, we decided to analyse the structure of the gauge transformations in the metric-like description that is induced from the gauge transformations in Chern–Simons theory. We made a proposal for the relation between the metric-like and frame-like gauge parameters in section 2, and we showed that this proposal is consistent to all orders in the perturbative expansion. Having the dictionary fixed, we could then work out the gauge transformations in the metric-like theory order by order.

To systematically do this, we formulated an algorithm in section 3, which we implemented on a computer. Using this algorithm we worked out the gauge transformations to cubic order in the spin-3 field. In section 4 we analysed the gauge algebra and explained why the algebra only closes on-shell, i.e. after imposing the equations of motion in the metric-like theory.

The ultimate goal would be to find a complete nonlinear metric-like formulation of this higher-spin gauge theory. The simplicity of the Chern–Simons action led us to the suspicion that one could also find a simple metric-like formulation. On the other hand, our analysis shows that the expressions rapidly become very complicated, and we could not identify any pattern that would help to reorganize the results in a way that could be generalized to higher or even all orders.

There are now different possibilities. It might be that there is a clever field redefinition (and/or gauge parameter redefinition) that leads to a simpler answer (of which we did not see any sign), or it could just be that there is no simple metric-like formulation of this theory, and the frame-like approach is the natural description.

Some signs, however, point towards an alternative scenario. The on-shell closure of the gauge algebra is reminiscent of the situation in supersymmetric field theories, and it suggests to add additional auxiliary fields. As gauge transformations mix spin-2 and spin-3 gauge fields, one might try to reorganize gauge fields and auxiliary fields in a unified higher-spin field, similarly to the formulation of supersymmetric theories in terms of superfields. It would

be very interesting if one could find a higher-spin generalization of geometry, in which the metric-like formulation becomes simple and natural.

Acknowledgments

Appendix A. Conventions

We denote symmetrization by a pair of parentheses,

$$A_{(\mu} B_{\nu)} = \frac{1}{2}\Big(A_\mu B_\nu + A_\nu B_\mu\Big). \tag{A.1}$$

Likewise square brackets denote antisymmetrization. We often omit contracted indices of a tensor to simplify notation, for example

$$\phi_\mu \equiv \phi_{\mu\lambda}{}^\lambda. \tag{A.2}$$

Furthermore we will use hats to denote the traceless projection of a contravariant rank 2 tensor, i.e.

$$\hat{\xi}^{\mu\nu} = \Big(\delta^\mu_\sigma \, \delta^\nu_\kappa - \frac{1}{3}\, g^{\mu\nu}\, g_{\sigma\kappa}\Big)\xi^{\sigma\kappa}. \tag{A.3}$$

The algebra $sl(3, \mathbb{R})$ can be given in terms of generators J_a and T_{ab} with the commutation relations

$$\big[J_a, J_b\big] = \epsilon_{abc}\, J^c, \tag{A.4a}$$

$$\big[J_a, T_{bc}\big] = 2\, \epsilon^d{}_{a(b}T_{c)d}, \tag{A.4b}$$

$$\big[T_{ab}, T_{cd}\big] = -2\Big(\eta_{a(c}\epsilon_{d)be} + \eta_{b(c}\epsilon_{d)ae}\Big)J^e, \tag{A.4c}$$

and $T_{[ab]} = \eta^{ab}\, T_{ab} = 0$. Here the Levi-Civita symbol is given by

$$\epsilon^{012} = -\epsilon_{012} = 1, \tag{A.5}$$

and indices can be raised and lowered by $\eta_{ab} = \mathrm{diag}(-1, 1, 1)$. A 3×3 matrix representation for the T_{ab} is given by

$$T_{ab} = \Big(J_a J_b + J_b J_a - \frac{2}{3}\, \eta_{ab}\, J_c J^c\Big), \tag{A.6}$$

where J_a is in the three-dimensional representation of $sl(2, \mathbb{R}) \hookrightarrow sl(3, \mathbb{R})$. Furthermore $\{J_A\}$ denote a set of five independent generators built from the matrix representation T_{ab}. We use the notation $\{J_{\mathcal{A}}\}$ for the set of all generators $\{J_a, J_A\}$.

The Killing form is defined to be one half of the matrix trace in the fundamental representation of $sl(3, \mathbb{R})$,

$$\kappa_{\mathcal{A}\mathcal{B}} = \frac{1}{2}\, \mathrm{tr}\big(J_{\mathcal{A}}\, J_{\mathcal{B}}\big), \tag{A.7}$$

and therefore

$$\kappa_{ab} = \eta_{ab}, \tag{A.8}$$

$$\kappa_{aB} = 0. \tag{A.9}$$

The anti-symmetric and symmetric structure constants are given by

$$f_{ABC} = \frac{1}{2} \operatorname{tr}\left([J_A, J_B] J_C \right), \tag{A.10}$$

$$d_{ABC} = \frac{1}{2} \operatorname{tr}\left(\{J_A, J_B\} J_C \right), \tag{A.11}$$

such that

$$f_{Abc} = f_{ABC} = 0, \tag{A.12}$$

$$f_{abc} = \epsilon_{abc}, \tag{A.13}$$

$$d_{abc} = d_{ABc} = 0. \tag{A.14}$$

The structure constants satisfy a number of identities of which we used

$$d_{Abc}\, \kappa^{bc} = 0, \tag{A.15a}$$

$$d_{Abc}\, d^A{}_{de} = -\frac{2}{3}\,\kappa_{bc}\,\kappa_{de} + 2\,\kappa_{d(b}\,\kappa_{c)e}. \tag{A.15b}$$

The latter identity is for example useful to verify that at the linearized level the metric-like spin-3 field transforms as a free Fronsdal field. Applying a spin-3 transformation to $\phi_{\mu\nu\rho}$ (as given in (3.3)) yields

$$\begin{aligned}
\delta\phi_{\mu\nu\rho} &= \frac{1}{2}\, d_{Abc}\, e^c_{(\mu}\, e^b_{\nu}\, \delta E^A_{\rho)} + \mathcal{O}(E) \\
&= \frac{1}{2}\, d_{Abc}\, d^A{}_{ef}\, e^b_{(\mu}\, e^c_{\nu}\, D_{\rho)}\left(e^e_{\sigma}\, e^f_{\lambda}\, \xi^{\sigma\lambda} \right) + \mathcal{O}(E) ,
\end{aligned} \tag{A.16}$$

where the transformation $\delta E^A_\mu = D_\mu \Xi^A$ is determined from (2.11) and (2.21). Using the identity (A.15b) then leads to the expected gauge transformation at leading order (see (2.8)),

$$\delta\phi_{\mu\nu\rho} = \nabla_{(\rho}\left(\xi_{\mu\nu)} - \frac{1}{3} g_{\mu\nu)} \xi^\lambda{}_\lambda \right) + \cdots . \tag{A.17}$$

Appendix B. Bianchi-like identity

For the curvature of the spin connection we have the following Bianchi-like identity,

$$f^A{}_{BC}\, R^B_{[\mu\nu}\, e^C_{\rho]} = 0. \tag{B.1}$$

For convenience we display its proof here. We evaluate

$$
\begin{aligned}
f^{\mathcal{A}}{}_{BC} R^{B}_{[\mu\nu} e^{C}_{\rho]} &= f^{\mathcal{A}}{}_{BC}\left(\partial_{[\mu}\omega^{B}_{\nu} e^{C}_{\rho]} + \tfrac{1}{2} f^{B}{}_{\mathcal{E}\mathcal{F}}\omega^{\mathcal{E}}_{[\mu}\omega^{\mathcal{F}}_{\nu} e^{C}_{\rho]}\right) \\
&= f^{\mathcal{A}}{}_{BC}\partial_{[\mu}\omega^{B}_{\nu} e^{C}_{\rho]} - \tfrac{1}{2}\left(f^{\mathcal{A}}{}_{B\mathcal{E}}f^{B}{}_{\mathcal{F}C} + f^{\mathcal{A}}{}_{B\mathcal{F}}f^{B}{}_{C\mathcal{E}}\right)\omega^{\mathcal{E}}_{[\mu}\omega^{\mathcal{F}}_{\nu} e^{C}_{\rho]} \\
&= f^{\mathcal{A}}{}_{BC}\partial_{[\mu}\omega^{B}_{\nu} e^{C}_{\rho]} - \tfrac{1}{2}f^{\mathcal{A}}{}_{B\mathcal{E}}\omega^{\mathcal{E}}_{[\mu}\partial_{\nu} e^{B}_{\rho]} - \tfrac{1}{2}f^{\mathcal{A}}{}_{B\mathcal{F}}\omega^{\mathcal{F}}_{[\nu}\partial_{\mu} e^{B}_{\rho]} \\
&= f^{\mathcal{A}}{}_{BC}\partial_{[\mu}\omega^{B}_{\nu} e^{C}_{\rho]} + f^{\mathcal{A}}{}_{B\mathcal{E}}\partial_{[\mu}\omega^{\mathcal{E}}_{\nu} e^{B}_{\rho]} \\
&= f^{\mathcal{A}}{}_{BC}\partial_{[\mu}\left(\omega^{B}_{\nu} e^{C}_{\rho]}\right).
\end{aligned}
\tag{B.2}
$$

Here we have used (1.5) to obtain the third line. By using the torsion constraint (1.5) again we yield

$$
f^{\mathcal{A}}{}_{BC}\partial_{[\mu}\left(\omega^{B}_{\nu} e^{C}_{\rho]}\right) = -\partial_{[\mu}\partial_{\nu} e^{\mathcal{A}}_{\rho]} = 0,
\tag{B.3}
$$

which concludes the proof of (B.1).

Appendix C. DDI contributions to gauge algebra

In the following we will summarize the contributions to the parameter $u^{\alpha\beta}$ of the gauge algebra, given in (4.34), which vanish due to dimensional dependent identities. These might be helpful in comparing with our results.

First we give the term with a derivative acting on the spin-3 field.

$$
\begin{aligned}
D_1^{\alpha\beta} = \tfrac{1}{2}\Big(& \left(\hat{\xi}\hat{\pi}\right)^{\alpha\beta\chi\delta}\nabla_{\delta}\phi_{\chi} - \left(\hat{\xi}\hat{\pi}\right)^{\chi\delta}{}_{\chi}{}^{\epsilon}g^{\alpha\beta}\nabla_{\epsilon}\phi_{\delta} \\
& + \left(\hat{\xi}\hat{\pi}\right)^{\chi\delta}{}_{\chi}{}^{\epsilon}\nabla_{\epsilon}\phi^{\alpha\beta}{}_{\delta} - \left(\hat{\xi}\hat{\pi}\right)^{\alpha\beta\chi\delta}\nabla_{\epsilon}\phi_{\chi\delta}{}^{\epsilon} \\
& + 2\left(\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon\xi}g^{\alpha\beta}\nabla_{\xi}\phi_{\chi\delta\epsilon} - \left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi}{}_{\chi}{}^{\delta|}\nabla^{\beta)}\phi_{\delta} \\
& + 2\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta\epsilon|}\nabla^{\beta)}\phi_{\chi\delta\epsilon} + \left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi}{}_{\chi}{}^{\delta}\nabla^{\epsilon|}\phi^{\beta)}{}_{\delta\epsilon} \\
& + \left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta\epsilon|}\nabla_{\chi}\phi^{\beta)}{}_{\delta\epsilon} - 3\left(\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta\epsilon|}\nabla_{\delta}\phi^{\beta)}{}_{\chi\epsilon}\Big).
\end{aligned}
\tag{C.1}
$$

Furthermore there are four more quantities with a derivative acting on the parameters.

$$
\begin{aligned}
D_2^{\alpha\beta} = & \left(\nabla\hat{\xi}\hat{\pi}\right)_{\chi}{}^{\delta\epsilon}{}_{\delta\epsilon}\phi^{\chi}g^{\alpha\beta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\delta\epsilon}\phi^{\alpha\beta}{}_{\chi} \\
& - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\alpha\beta\delta\epsilon}\phi_{\chi\delta\epsilon} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon\alpha\beta}\phi_{\chi\delta\epsilon} - 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\delta}{}^{\xi}g^{\alpha\beta}\phi_{\chi\epsilon\xi} \\
& - 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)}{}_{\delta}\phi_{\chi} + 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)\epsilon}\phi_{\chi\delta\epsilon} \\
& + 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta}{}_{\delta}{}^{\epsilon|}\phi^{\beta)}{}_{\chi\epsilon} + 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon(\alpha}{}_{\delta}\phi^{\beta)}{}_{\chi\epsilon},
\end{aligned}
\tag{C.2}
$$

$$
\begin{aligned}
D_3^{\alpha\beta} =&\ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\chi\delta}{}^{\alpha\beta}\phi^{\chi} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\chi}{}^{\epsilon}{}_{\delta\epsilon}\phi^{\chi}g^{\alpha\beta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\chi\delta}\phi_{\epsilon}^{\alpha\beta} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon\alpha\beta}\phi_{\chi\delta\epsilon} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\delta}{}^{\xi}g^{\alpha\beta}\phi_{\chi\epsilon\xi} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta\epsilon\xi}g^{\alpha\beta}\phi_{\delta\epsilon\xi} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\chi}{}^{\xi}g^{\alpha\beta}\phi_{\delta\epsilon\xi} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta|\beta)}{}_{\chi}\phi_{\delta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)}{}_{\chi}\phi_{\delta} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha\beta)\chi\delta\epsilon}\phi_{\chi\delta\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta|\beta)\epsilon}\phi_{\chi\delta\epsilon} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta}{}^{\epsilon|}{}_{\chi}\phi^{\beta)}{}_{\delta\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)\epsilon}\phi_{\chi\delta\epsilon} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta}{}^{\epsilon|}{}_{\chi}\phi^{\beta)}{}_{\delta\epsilon} - 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta(\alpha|\epsilon|}\phi^{\beta)}{}_{\delta\epsilon} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon(\alpha}{}_{\chi}\phi^{\beta)}{}_{\delta\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon(\alpha}{}_{\delta}\phi^{\beta)}{}_{\chi\epsilon},
\end{aligned} \tag{C.3}
$$

$$
\begin{aligned}
D_4^{\alpha\beta} =&\ -\left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta\alpha\beta}{}_{\chi\delta}\phi^{\chi} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\delta}{}^{\epsilon}{}_{\chi\epsilon}\phi^{\chi}g^{\alpha\beta} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta}{}_{\delta}{}^{\epsilon}\phi^{\alpha\beta}{}_{\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\alpha\beta\delta\epsilon}\phi_{\chi\delta\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\delta}{}^{\xi}g^{\alpha\beta}\phi_{\chi\epsilon\xi} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta\epsilon\xi}g^{\alpha\beta}\phi_{\delta\epsilon\xi} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi\delta\epsilon}{}_{\chi}{}^{\xi}g^{\alpha\beta}\phi_{\delta\epsilon\xi} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha\beta)\chi}{}_{\chi}{}^{\delta}\phi_{\delta} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha}{}_{\chi}{}^{\beta)\delta}\phi_{\delta} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha\beta)\chi\delta\epsilon}\phi_{\chi\delta\epsilon} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta|\beta)\epsilon}\phi_{\chi\delta\epsilon} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta}{}^{\epsilon|}{}_{\chi}\phi^{\beta)}{}_{\delta\epsilon} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)\epsilon}\phi_{\chi\delta\epsilon} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha}{}_{\chi}{}^{|\delta\epsilon|}\phi^{\beta)}{}_{\delta\epsilon} + 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta}{}^{\epsilon|}{}_{\chi}\phi^{\beta)}{}_{\delta\epsilon} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta}{}^{\epsilon|}{}_{\delta}\phi^{\beta)}{}_{\chi\epsilon} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta(\alpha|\epsilon|}\phi^{\beta)}{}_{\delta\epsilon},
\end{aligned} \tag{C.4}
$$

$$
\begin{aligned}
D_5^{\alpha\beta} =&\ -\left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta\alpha\beta}{}_{\chi\delta}\phi^{\chi} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\chi\delta}{}^{\alpha\beta}\phi^{\chi} + \left(\nabla\hat{\xi}\hat{\pi}\right)_{\chi}{}^{\delta\epsilon}{}_{\delta\epsilon}\phi^{\chi}g^{\alpha\beta} \\
&- \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\chi}{}^{\epsilon}{}_{\delta\epsilon}\phi^{\chi}g^{\alpha\beta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{\delta}{}_{\delta}{}^{\epsilon}{}_{\chi\epsilon}\phi^{\chi}g^{\alpha\beta} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha\beta)\chi}{}_{\chi}{}^{\delta}\phi_{\delta} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta|\beta)}{}_{\chi}\phi_{\delta} - \left(\nabla\hat{\xi}\hat{\pi}\right)^{(\alpha|\chi\delta|}{}_{\chi\delta}\phi^{\beta)} \\
&+ \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha}{}_{\chi}{}^{\beta)\delta}\phi_{\delta} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)}{}_{\chi}\phi_{\delta} \\
&- 2\left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|\beta)}{}_{\delta}\phi_{\chi} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi(\alpha|\delta|}{}_{\chi\delta}\phi^{\beta)} + \left(\nabla\hat{\xi}\hat{\pi}\right)^{\chi}{}_{\chi}{}^{\delta(\alpha}{}_{\delta}\phi^{\beta)}.
\end{aligned} \tag{C.5}
$$

The quantities $(\nabla\hat{\xi}\hat{\pi})$ and $(\hat{\xi}\hat{\pi})$ are defined in (4.38) and (4.36).

Appendix D. Higher order corrections to spin-3 transformations

In this section we list the higher order corrections to the gauge transformations given in section 3.4. By making a particular choice of the undetermined constants due to dimensional dependent identities we reduced the size of the expressions considerably. For the gauge

transformation of the spin-3 field there are corrections with a derivative acting on the ϕ field,

$$
\begin{aligned}
\left(\hat{\xi}\phi\,\nabla\phi\right)_{\alpha\beta\chi} = 18&\Big(2\phi^{\delta}\hat{\xi}_{\delta}{}^{\epsilon}\nabla_{\epsilon}\phi_{\alpha\beta\chi} + \hat{\xi}^{\delta\epsilon}\Big(3\phi_{\delta\epsilon}{}^{\xi}\nabla_{\xi}\phi_{\alpha\beta\chi} + \phi_{\alpha\beta\chi}\Big(7\nabla_{\epsilon}\phi_{\delta} - 3\nabla_{\xi}\phi_{\delta\epsilon}{}^{\xi}\Big)\Big)\\
&- 4\phi_{(\alpha}\hat{\xi}_{\beta\chi)}\nabla^{\delta}\phi_{\delta} + 8\phi_{(\alpha}\hat{\xi}_{\beta}{}^{\delta}\nabla_{\chi)}\phi_{\delta} + 9\phi_{(\alpha}\hat{\xi}_{\beta}{}^{\delta}\nabla_{|\delta|}\phi_{\chi)} - 5\phi^{\delta}\hat{\xi}_{(\alpha\beta}\nabla_{|\delta|}\phi_{\chi)}\\
&- 2\phi^{\delta}\hat{\xi}_{(\alpha|\delta|}\nabla_{\beta}\phi_{\chi)} - 8\phi_{(\alpha}\hat{\xi}_{\beta}{}^{\delta}\nabla^{\epsilon}\phi_{\chi)\delta\epsilon} - 2\phi_{(\alpha}\hat{\xi}^{\delta\epsilon}\nabla_{|\delta|}\phi_{\beta\chi)\epsilon} + \phi^{\delta}\hat{\xi}_{(\alpha|\delta|}\nabla^{\epsilon}\phi_{\beta\chi)\epsilon}\\
&+ 13\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}\nabla_{|\delta|}\phi_{\beta\chi)\epsilon} - 3\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}\nabla_{|\epsilon|}\phi_{\beta\chi)\delta} + 12\hat{\xi}_{(\alpha\beta}\phi_{\chi)}{}^{\delta\epsilon}\nabla_{\delta}\phi_{\epsilon}\\
&+ 5\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta\chi)\delta}\nabla^{\epsilon}\phi_{\epsilon} - 12\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta\chi)}{}^{\epsilon}\nabla_{\delta}\phi_{\epsilon} - 7\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta\chi)}{}^{\epsilon}\nabla_{\epsilon}\phi_{\delta} - 6\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta|\delta|}{}^{\epsilon}\nabla_{\chi)}\phi_{\epsilon}\\
&+ 8\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta|\delta}{}^{\epsilon}\nabla_{\epsilon|}\phi_{\chi)} - 3\hat{\xi}^{\delta\epsilon}\phi_{(\alpha\beta|\delta|}\nabla_{\chi)}\phi_{\epsilon} - 4\hat{\xi}^{\delta\epsilon}\phi_{(\alpha\beta|\delta}\nabla_{\epsilon|}\phi_{\chi)}\\
&- 8\hat{\xi}_{(\alpha\beta}\phi_{\chi)}{}^{\delta\epsilon}\nabla^{\xi}\phi_{\delta\epsilon\xi} + \hat{\xi}_{(\alpha\beta}\phi^{\delta\epsilon\xi}\nabla_{\chi)}\phi_{\delta\epsilon\xi}\\
&+ 5\hat{\xi}_{(\alpha\beta}\phi^{\delta\epsilon\xi}\nabla_{|\delta|}\phi_{\chi)\epsilon\xi} + 11\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta\chi)}{}^{\epsilon}\nabla^{\xi}\phi_{\delta\epsilon\xi} + \hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta}{}^{\epsilon\xi}\nabla_{\chi)}\phi_{\delta\epsilon\xi} + 6\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta|\delta|}{}^{\epsilon}\nabla^{\xi}\phi_{\chi)\epsilon\xi}\\
&+ 3\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta}{}^{\epsilon\xi}\nabla_{|\delta|}\phi_{\chi)\epsilon\xi} - 18\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta}{}^{\epsilon\xi}\nabla_{|\epsilon|}\phi_{\chi)\delta\xi} - \hat{\xi}_{(\alpha}{}^{\delta}\phi_{|\delta|}{}^{\epsilon\xi}\nabla_{\beta}\phi_{\chi)\epsilon\xi} - 11\hat{\xi}_{(\alpha}{}^{\delta}\phi_{|\delta}{}^{\epsilon\xi}\nabla_{\epsilon|}\phi_{\beta\chi)\xi}\\
&- 9\xi^{\delta\epsilon}\phi_{(\alpha\beta}{}^{\xi}\nabla_{\chi)}\phi_{\delta\epsilon\xi} + 3\xi^{\delta\epsilon}\phi_{(\alpha\beta|\delta|}\nabla^{\xi}\phi_{\chi)\epsilon\xi} + 7\xi^{\delta\epsilon}\phi_{(\alpha\beta}{}^{\xi}\nabla_{|\delta|}\phi_{\chi)\epsilon\xi} + 3\xi^{\delta\epsilon}\phi_{(\alpha\beta}{}^{\xi}\nabla_{|\xi|}\phi_{\chi)\delta\epsilon}\\
&- 7\xi^{\delta\epsilon}\phi_{(\alpha|\delta}{}^{\xi}\nabla_{\epsilon|}\phi_{\beta\chi)\xi} - 9\xi^{\delta\epsilon}\phi_{(\alpha|\delta}{}^{\xi}\nabla_{\xi|}\phi_{\beta\chi)\epsilon} - 3\phi_{(\alpha}\hat{\xi}^{\delta\epsilon}g_{\beta\chi)}\nabla_{\delta}\phi_{\epsilon} + 2\phi^{\delta}\hat{\xi}_{(\alpha|\delta|}g_{\beta\chi)}\nabla^{\epsilon}\phi_{\epsilon}\\
&- 2\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}g_{\beta\chi)}\nabla_{\delta}\phi_{\epsilon} - 3\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}g_{\beta\chi)}\nabla_{\epsilon}\phi_{\delta} - 3\phi^{\delta}\hat{\xi}_{\delta}{}^{\epsilon}g_{(\alpha\beta}\nabla_{\chi)}\phi_{\epsilon} - 3\phi^{\delta}\hat{\xi}_{\delta}{}^{\epsilon}g_{(\alpha\beta}\nabla_{|\epsilon|}\phi_{\chi)}\\
&+ \phi_{(\alpha}\hat{\xi}^{\delta\epsilon}g_{\beta\chi)}\nabla^{\xi}\phi_{\delta\epsilon\xi} + 3\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}g_{\beta\chi)}\nabla^{\xi}\phi_{\delta\epsilon\xi} + 3\phi^{\delta}\hat{\xi}_{\delta}{}^{\epsilon}g_{(\alpha\beta}\nabla^{\xi}\phi_{\chi)\epsilon\xi} - 7\phi^{\delta}\hat{\xi}^{\epsilon\xi}g_{(\alpha\beta}\nabla_{|\delta|}\phi_{\chi)\epsilon\xi}\\
&+ 3\phi^{\delta}\hat{\xi}^{\epsilon\xi}g_{(\alpha\beta}\nabla_{|\epsilon|}\phi_{\chi)\delta\xi} - 3\hat{\xi}_{(\alpha}{}^{\delta}g_{\beta\chi)}\phi_{\delta}{}^{\epsilon\xi}\nabla_{\epsilon}\phi_{\xi} - 2\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)\delta\epsilon}\nabla^{\xi}\phi_{\xi} + 9\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)\delta}{}^{\xi}\nabla_{\epsilon}\phi_{\xi}\\
&+ 3\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)\delta}{}^{\xi}\nabla_{\xi}\phi_{\epsilon} - 4\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{|\delta\epsilon}{}^{\xi}\nabla_{\xi|}\phi_{\chi)} + \hat{\xi}_{(\alpha}{}^{\delta}g_{\beta\chi)}\phi_{\delta}{}^{\epsilon\xi}\nabla^{\gamma}\phi_{\epsilon\xi\gamma} + \hat{\xi}_{(\alpha}{}^{\delta}g_{\beta\chi)}\phi^{\epsilon\xi\gamma}\nabla_{\delta}\phi_{\epsilon\xi\gamma}\\
&- \hat{\xi}_{(\alpha}{}^{\delta}g_{\beta\chi)}\phi^{\epsilon\xi\gamma}\nabla_{\epsilon}\phi_{\delta\xi\gamma} - 8\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)\delta}{}^{\xi}\nabla^{\gamma}\phi_{\epsilon\xi\gamma} - 6\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)}{}^{\xi\gamma}\nabla_{\delta}\phi_{\epsilon\xi\gamma}\\
&+ 6\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{\chi)}{}^{\xi\gamma}\nabla_{\xi}\phi_{\delta\epsilon\gamma} + 6\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{|\delta|}{}^{\xi\gamma}\nabla_{\chi)}\phi_{\epsilon\xi\gamma} + 8\hat{\xi}^{\delta\epsilon}g_{(\alpha\beta}\phi_{|\delta}{}^{\xi\gamma}\nabla_{\xi|}\phi_{\chi)\epsilon\gamma}\Big).
\end{aligned}
\tag{D.1}
$$

Then there are contributions with a derivative acting on the parameter.

$$
\begin{aligned}
\left(\nabla\hat{\xi}\phi\phi\right)_{\alpha\beta\chi} = -9&\Big(12\phi^{\delta}\phi_{\alpha\beta\chi}\nabla_{\epsilon}\hat{\xi}_{\delta}{}^{\epsilon} + 4\phi_{\alpha\beta\chi}\phi^{\delta\epsilon\xi}\nabla_{\xi}\hat{\xi}_{\delta\epsilon} + 14\phi_{(\alpha}\phi_{\beta}\nabla^{\delta}\hat{\xi}_{\chi)\delta}\\
&+ 6\phi_{(\alpha}\phi^{\delta}\nabla_{\beta}\hat{\xi}_{\chi)\delta} + \phi^{\delta}\phi_{\delta}\nabla_{(\alpha}\hat{\xi}_{\beta\chi)} - 16\phi_{(\alpha}\phi_{\beta\chi)}{}^{\delta}\nabla^{\epsilon}\hat{\xi}_{\delta\epsilon} + 16\phi_{(\alpha}\phi_{\beta}{}^{\delta\epsilon}\nabla_{\chi)}\hat{\xi}_{\delta\epsilon}\\
&- 8\phi_{(\alpha}\phi_{\beta}{}^{\delta\epsilon}\nabla_{|\delta|}\hat{\xi}_{\chi)\epsilon} - 12\phi^{\delta}\phi_{(\alpha\beta}{}^{\epsilon}\nabla_{\chi)}\hat{\xi}_{\delta\epsilon}\\
&- 8\phi^{\delta}\phi_{(\alpha\beta|\delta|}\nabla^{\epsilon}\hat{\xi}_{\chi)\epsilon} + 32\phi^{\delta}\phi_{(\alpha\beta}{}^{\epsilon}\nabla_{|\delta|}\hat{\xi}_{\chi)\epsilon} - 12\phi^{\delta}\phi_{(\alpha\beta}{}^{\epsilon}\nabla_{|\epsilon|}\hat{\xi}_{\chi)\delta} + 8\phi^{\delta}\phi_{(\alpha|\delta}{}^{\epsilon}\nabla_{\epsilon|}\hat{\xi}_{\beta\chi)}\\
&+ 4\phi_{(\alpha\beta}{}^{\delta}\phi_{\chi)\delta}{}^{\epsilon}\nabla^{\xi}\hat{\xi}_{\epsilon\xi} - 4\phi_{(\alpha\beta}{}^{\delta}\phi_{\chi)}{}^{\epsilon\xi}\nabla_{\epsilon}\hat{\xi}_{\delta\xi} + 14\phi_{(\alpha\beta}{}^{\delta}\phi_{|\delta|}{}^{\epsilon\xi}\nabla_{\chi)}\hat{\xi}_{\epsilon\xi} - 12\phi_{(\alpha\beta}{}^{\delta}\phi_{|\delta}{}^{\epsilon\xi}\nabla_{\epsilon|}\hat{\xi}_{\chi)\xi}\\
&- 6\phi_{(\alpha}{}^{\delta\epsilon}\phi_{\beta|\delta\epsilon|}\nabla^{\xi}\hat{\xi}_{\chi)\xi} + 8\phi_{(\alpha}{}^{\delta\epsilon}\phi_{|\delta\epsilon}{}^{\xi}\nabla_{\xi|}\hat{\xi}_{\beta\chi)} - 12\phi_{(\alpha}\phi^{\delta}g_{\beta\chi)}\nabla^{\epsilon}\hat{\xi}_{\delta\epsilon} + 3\phi^{\delta}\phi^{\epsilon}g_{(\alpha\beta}\nabla_{\chi)}\hat{\xi}_{\delta\epsilon}\\
&- \phi^{\delta}\phi_{\delta}g_{(\alpha\beta}\nabla^{\epsilon}\hat{\xi}_{\chi)\epsilon} - 4\phi_{(\alpha}g_{\beta\chi)}\phi^{\delta\epsilon\xi}\nabla_{\delta}\hat{\xi}_{\epsilon\xi} + 10\phi^{\delta}g_{(\alpha\beta}\phi_{\chi)\delta}{}^{\epsilon}\nabla^{\xi}\hat{\xi}_{\epsilon\xi} - 16\phi^{\delta}g_{(\alpha\beta}\phi_{\chi)}{}^{\epsilon\xi}\nabla_{\delta}\hat{\xi}_{\epsilon\xi}\\
&+ 12\phi^{\delta}g_{(\alpha\beta}\phi_{\chi)}{}^{\epsilon\xi}\nabla_{\epsilon}\hat{\xi}_{\delta\xi} - 6\phi^{\delta}g_{(\alpha\beta}\phi_{|\delta|}{}^{\epsilon\xi}\nabla_{\chi)}\hat{\xi}_{\epsilon\xi} + 2\phi^{\delta}g_{(\alpha\beta}\phi_{|\delta}{}^{\epsilon\xi}\nabla_{\epsilon|}\hat{\xi}_{\chi)\xi}
\end{aligned}
$$

$$+ 2g_{(\alpha\beta}\phi_{\chi)}{}^{\delta\epsilon}\phi_{\delta\epsilon}{}^{\xi}\nabla^{\gamma}\hat{\xi}_{\xi\gamma} - 2g_{(\alpha\beta}\phi_{\chi)}{}^{\delta\epsilon}\phi_{\delta}{}^{\xi\gamma}\nabla_{\epsilon}\hat{\xi}_{\xi\gamma} + 10g_{(\alpha\beta}\phi_{\chi)}{}^{\delta\epsilon}\phi_{\delta}{}^{\xi\gamma}\nabla_{\xi}\hat{\xi}_{\epsilon\gamma}$$

$$+ g_{(\alpha\beta}\phi^{\delta\epsilon\xi}\phi_{|\delta\epsilon\xi|}\nabla^{\gamma}\hat{\xi}_{\chi)\gamma} - 2g_{(\alpha\beta}\phi^{\delta\epsilon\xi}\phi_{|\delta\epsilon}{}^{\gamma}\nabla_{\xi|}\hat{\xi}_{\chi)\gamma} - 16\phi_{(\alpha}\phi^{\delta}\nabla_{|\delta|}\hat{\xi}_{\beta\chi)}\Big). \tag{D.2}$$

Finally the transformation of the metric to cubic order is given by

$$\left(\hat{\xi}\phi\phi\nabla\phi\right)_{\alpha\beta} = 18\Big(16\phi^{\chi}\hat{\xi}^{\delta\epsilon}\phi_{\alpha\beta\delta}\nabla_{\chi}\phi_{\epsilon} - 8\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\delta\epsilon}{}^{\xi}\nabla_{\chi}\phi_{\xi}$$

$$+ 16\phi^{\chi}\hat{\xi}^{\delta\epsilon}\phi_{\delta\epsilon}{}^{\xi}\nabla_{\chi}\phi_{\alpha\beta\xi} + 16\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\delta}{}^{\xi\gamma}\nabla_{\chi}\phi_{\epsilon\xi\gamma} + 5\phi^{\chi}\phi^{\delta}\hat{\xi}_{\alpha\beta}\nabla_{\delta}\phi_{\chi}$$

$$- 14\phi_{\chi}\phi^{\chi}\hat{\xi}_{\alpha\beta}\nabla_{\delta}\phi^{\delta} + 4\phi^{\chi}\phi^{\delta}\hat{\xi}_{\chi}{}^{\epsilon}g_{\alpha\beta}\nabla_{\delta}\phi_{\epsilon} + 12\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\chi\epsilon}{}^{\xi}\nabla_{\delta}\phi_{\xi}$$

$$+ 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\epsilon\xi}{}^{\gamma}\nabla_{\delta}\phi_{\gamma} - 10\phi^{\chi}\phi^{\delta}\hat{\xi}^{\epsilon\xi}g_{\alpha\beta}\nabla_{\delta}\phi_{\chi\epsilon\xi} - 8\phi^{\chi}\hat{\xi}_{\chi}{}^{\delta}g_{\alpha\beta}\phi^{\epsilon\xi\gamma}\nabla_{\delta}\phi_{\epsilon\xi\gamma}$$

$$+ 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\epsilon}{}^{\gamma\eta}\nabla_{\delta}\phi_{\xi\gamma\eta} + 12\phi^{\chi}\phi^{\delta}\hat{\xi}_{\chi}{}^{\epsilon}g_{\alpha\beta}\nabla_{\epsilon}\phi_{\delta} - 20\phi_{\chi}\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\nabla_{\epsilon}\phi_{\delta}$$

$$+ 14\phi^{\chi}\hat{\xi}_{\alpha\beta}\phi_{\chi}{}^{\delta\epsilon}\nabla_{\epsilon}\phi_{\delta} - \phi^{\chi}\phi^{\delta}\hat{\xi}_{\chi\delta}g_{\alpha\beta}\nabla_{\epsilon}\phi^{\epsilon} - 9\phi^{\chi}\phi^{\delta}\hat{\xi}_{\alpha\beta}\nabla_{\epsilon}\phi_{\chi\delta}{}^{\epsilon}$$

$$+ 20\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi}{}^{\xi\gamma}\nabla_{\epsilon}\phi_{\delta\xi\gamma} + 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\chi\epsilon}{}^{\xi}\nabla_{\xi}\phi_{\delta} - 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\chi\delta}{}^{\xi}\nabla_{\xi}\phi_{\epsilon}$$

$$+ 18\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}{}^{\xi}\phi^{\chi\delta\epsilon}\nabla_{\xi}\phi_{\epsilon} - 10\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi\delta\epsilon}\nabla_{\xi}\phi^{\xi} + 4\hat{\xi}_{\alpha\beta}\phi_{\chi\delta\epsilon}\phi^{\chi\delta\epsilon}\nabla_{\xi}\phi^{\xi}$$

$$- 8\phi^{\chi}\hat{\xi}_{\chi}{}^{\delta}\phi_{\delta}{}^{\epsilon\xi}\nabla_{\xi}\phi_{\alpha\beta\epsilon} - 12\phi^{\chi}\phi^{\delta}\hat{\xi}_{\chi}{}^{\epsilon}g_{\alpha\beta}\nabla_{\xi}\phi_{\delta\epsilon}{}^{\xi}$$

$$+ 10\phi_{\chi}\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\nabla_{\xi}\phi_{\delta\epsilon}{}^{\xi} + 18\phi^{\chi}\hat{\xi}_{\alpha\beta}\phi_{\chi}{}^{\delta\epsilon}\nabla_{\xi}\phi_{\delta\epsilon}{}^{\xi}$$

$$+ 16\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\epsilon\xi}{}^{\gamma}\nabla_{\gamma}\phi_{\delta} + 20\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\delta\epsilon}{}^{\gamma}\nabla_{\gamma}\phi_{\xi} + 6\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\delta\epsilon\xi}\nabla_{\gamma}\phi^{\gamma}$$

$$+ 4\hat{\xi}^{\chi\delta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\delta\epsilon}{}^{\gamma}\nabla_{\gamma}\phi_{\alpha\beta\xi} - 24\hat{\xi}^{\chi\delta}\phi_{\chi\delta}{}^{\epsilon}\phi_{\epsilon}{}^{\xi\gamma}\nabla_{\gamma}\phi_{\alpha\beta\xi} - 8\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\epsilon}{}^{\xi\gamma}\nabla_{\gamma}\phi_{\chi\delta\xi}$$

$$+ 16\phi^{\chi}\hat{\xi}^{\delta\epsilon}g_{\alpha\beta}\phi_{\chi}{}^{\xi\gamma}\nabla_{\gamma}\phi_{\delta\epsilon\xi} + 24\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\chi}{}^{\xi\gamma}\nabla_{\gamma}\phi_{\delta\epsilon\xi} - 12\hat{\xi}_{\alpha\beta}\phi_{\chi}{}^{\xi\gamma}\phi^{\chi\delta\epsilon}\nabla_{\gamma}\phi_{\delta\epsilon\xi}$$

$$- 16\hat{\xi}^{\chi\delta}\phi_{\alpha\beta\chi}\phi^{\epsilon\xi\gamma}\nabla_{\gamma}\phi_{\delta\epsilon\xi} - 12\hat{\xi}^{\chi\delta}\phi_{\alpha\beta}{}^{\epsilon}\phi_{\chi\epsilon}{}^{\xi}\nabla_{\gamma}\phi_{\delta\xi}{}^{\gamma}$$

$$- 14\hat{\xi}_{\alpha\beta}\phi_{\chi\delta}{}^{\xi}\phi^{\chi\delta\epsilon}\nabla_{\gamma}\phi_{\epsilon\xi}{}^{\gamma} + 8\phi^{\chi}\hat{\xi}_{\chi}{}^{\delta}g_{\alpha\beta}\phi_{\delta}{}^{\epsilon\xi}\nabla_{\gamma}\phi_{\epsilon\xi}{}^{\gamma} - 4\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\epsilon\xi}{}^{\eta}\phi^{\epsilon\xi\gamma}\nabla_{\eta}\phi_{\chi\delta\gamma}$$

$$- 40\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\epsilon}{}^{\gamma\eta}\nabla_{\eta}\phi_{\delta\xi\gamma} - 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\epsilon\xi}{}^{\gamma}\nabla_{\eta}\phi_{\delta\gamma}{}^{\eta} - 16\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi}{}^{\epsilon\xi}\phi_{\delta\epsilon}{}^{\gamma}\nabla_{\eta}\phi_{\xi\gamma}{}^{\eta}$$

$$+ 8\hat{\xi}^{\chi\delta}g_{\alpha\beta}\phi_{\chi\delta}{}^{\epsilon}\phi_{\epsilon}{}^{\xi\gamma}\nabla_{\eta}\phi_{\xi\gamma}{}^{\eta} + 2\phi_{(\alpha}\phi^{\chi}\hat{\xi}_{\beta)\chi}\nabla^{\delta}\phi_{\delta} - 20\phi_{(\alpha}\phi^{\chi}\hat{\xi}_{\beta)}{}^{\delta}\nabla_{\chi}\phi_{\delta}$$

$$- 12\phi_{(\alpha}\phi^{\chi}\hat{\xi}_{\beta)}{}^{\delta}\nabla_{\delta}\phi_{\chi} + 20\phi^{\chi}\phi_{\chi}\hat{\xi}_{(\alpha}{}^{\delta}\nabla_{\beta)}\phi_{\delta} - 12\phi^{\chi}\phi^{\delta}\hat{\xi}_{(\alpha|\chi|}\nabla_{\beta)}\phi_{\delta}$$

$$+ 20\phi^{\chi}\phi_{\chi}\hat{\xi}_{(\alpha}{}^{\delta}\nabla_{|\delta|}\phi_{\beta)} - 4\phi^{\chi}\phi^{\delta}\hat{\xi}_{(\alpha|\chi}\nabla_{\delta|}\phi_{\beta)} + 12\phi_{(\alpha}\phi^{\chi}\hat{\xi}_{\beta)}{}^{\delta}\nabla^{\epsilon}\phi_{\chi\delta\epsilon}$$

$$+ 16\phi_{(\alpha}\phi^{\chi}\hat{\xi}^{\delta\epsilon}\nabla_{|\chi|}\phi_{\beta)\delta\epsilon} - 20\phi^{\chi}\phi_{\chi}\hat{\xi}_{(\alpha}{}^{\delta}\nabla^{\epsilon}\phi_{\beta)\delta\epsilon} + 12\phi^{\chi}\phi^{\delta}\hat{\xi}_{(\alpha|\chi|}\nabla^{\epsilon}\phi_{\beta)\delta\epsilon}$$

$$+ 20\phi^{\chi}\phi^{\delta}\hat{\xi}_{(\alpha}{}^{\epsilon}\nabla_{|\chi|}\phi_{\beta)\delta\epsilon} + 16\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{\beta)\chi}{}^{\epsilon}\nabla_{\epsilon}\phi_{\delta} + 20\phi\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta)\chi\delta}\nabla^{\epsilon}\phi_{\epsilon}$$

$$+ 8\phi^{\chi}\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta)\delta}{}^{\epsilon}\nabla_{\chi}\phi_{\epsilon} - 8\phi_{(\alpha}\hat{\xi}_{\beta)}{}^{\chi}\phi_{\chi}{}^{\delta\epsilon}\nabla^{\xi}\phi_{\delta\epsilon\xi} + 8\phi_{(\alpha}\hat{\xi}_{\beta)}{}^{\chi}\phi^{\delta\epsilon\xi}\nabla_{\chi}\phi_{\delta\epsilon\xi}$$

$$+ 16\phi_{(\alpha}\hat{\xi}_{\beta)}{}^{\chi}\phi^{\delta\epsilon\xi}\nabla_{\delta}\phi_{\chi\epsilon\xi} - 24\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{\beta)}{}^{\epsilon\xi}\nabla_{\epsilon}\phi_{\chi\delta\xi} - 8\phi_{(\alpha}\hat{\xi}^{\chi\delta}\phi_{|\chi}{}^{\epsilon\xi}\nabla_{\epsilon|}\phi_{\beta)\delta\xi}$$

$$- 8\phi^{\chi}\hat{\xi}_{(\alpha|\chi|}\phi_{\beta)}{}^{\delta\epsilon}\nabla^{\xi}\phi_{\delta\epsilon\xi} + 8\phi^{\chi}\hat{\xi}_{(\alpha}{}^{\delta}\phi_{\beta)}{}^{\epsilon\xi}\nabla_{\chi}\phi_{\delta\epsilon\xi} + 8\phi^{\chi}\hat{\xi}_{(\alpha|\chi|}\phi^{\delta\epsilon\xi}\nabla_{\beta)}\phi_{\delta\epsilon\xi}$$

$$
\begin{aligned}
&- 20\phi^\chi \hat{\xi}_{(\alpha}{}^\delta \phi_{|\chi|}{}^{\epsilon\xi} \nabla_{\beta)} \phi_{\delta\epsilon\xi} - 20\phi^\chi \hat{\xi}_{(\alpha}{}^\delta \phi_{|\chi}{}^{\epsilon\xi} \nabla_{\delta|} \phi_{\beta)\epsilon\xi} \\
&- 32\phi^\chi \hat{\xi}_{(\alpha}{}^\delta \phi_{|\chi}{}^{\epsilon\xi} \nabla_{\epsilon|} \phi_{\beta)\delta\xi} - 16\phi^\chi \hat{\xi}_{(\alpha}{}^\delta \phi_{|\delta}{}^{\epsilon\xi} \nabla_{\chi|} \phi_{\beta)\epsilon\xi} + 8\phi^\chi \hat{\xi}_{\chi}{}^\delta \phi_{(\alpha}{}^{\epsilon\xi} \nabla_{|\epsilon|} \phi_{\beta)\delta\xi} \\
&- 40\phi^\chi \xi^{\delta\epsilon} \phi_{(\alpha|\delta}{}^\xi \nabla_{\chi|} \phi_{\beta)\epsilon\xi} - 12\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\chi\delta\epsilon} \nabla^\xi \phi_\xi - 20\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\chi\delta}{}^\xi \nabla_\epsilon \phi_\xi \\
&- 12\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\chi\delta}{}^\xi \nabla_\xi \phi_\epsilon - 8\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\delta\epsilon}{}^\xi \nabla_\chi \phi_\xi - 8\hat{\xi}_{(\alpha}{}^\chi \phi_{|\chi}{}^{\delta\epsilon} \phi_{\delta\epsilon|}{}^\xi \nabla_{\beta)} \phi_\xi \\
&- 16\hat{\xi}_{(\alpha}{}^\chi \phi_{|\chi}{}^{\delta\epsilon} \phi_{\delta\epsilon}{}^\xi \nabla_{\xi|} \phi_{\beta)} + 8\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)\delta}{}^\xi \nabla_\epsilon \phi_\xi - 12\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)\epsilon}{}^\xi \nabla_\delta \phi_\xi \\
&- 40\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)\epsilon}{}^\xi \nabla_\xi \phi_\delta - 12\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi}{}^\epsilon \phi_{\delta\epsilon|}{}^\xi \nabla_{\beta)} \phi_\xi \\
&+ 12\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\delta\epsilon|} \nabla_{\beta)} \phi_\xi - 24\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi}{}^\epsilon \phi_{\delta\epsilon}{}^\xi \nabla_{\xi|} \phi_{\beta)} + 24\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\delta\epsilon} \nabla_{\xi|} \phi_{\beta)} \\
&- 16\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)\chi}{}^\delta \phi_\delta{}^{\epsilon\xi} \nabla^\gamma \phi_{\epsilon\xi\gamma} + 8\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)\chi}{}^\delta \phi^{\epsilon\xi\gamma} \nabla_\epsilon \phi_{\delta\xi\gamma} + 32\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\chi\delta}{}^\gamma \nabla^\gamma \phi_{\epsilon\xi\gamma} \\
&- 8\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_\chi{}^{\xi\gamma} \nabla_\xi \phi_{\delta\epsilon\gamma} + 8\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_{\delta\epsilon}{}^\xi \nabla^\gamma \phi_{\chi\xi\gamma} - 8\hat{\xi}_{(\alpha}{}^\chi \phi_{\beta)}{}^{\delta\epsilon} \phi_\delta{}^{\xi\gamma} \nabla_\chi \phi_{\epsilon\xi\gamma} \\
&- 8\hat{\xi}_{(\alpha}{}^\chi \phi_{|\chi}{}^{\delta\epsilon} \phi_{\delta|}{}^{\xi\gamma} \nabla_{\beta)} \phi_{\epsilon\xi\gamma} + 8\hat{\xi}_{(\alpha}{}^\chi \phi_{|\chi}{}^{\delta\epsilon} \phi_{\delta\epsilon|}{}^\xi \nabla^\gamma \phi_{\beta)\xi\gamma} + 40\hat{\xi}_{(\alpha}{}^\chi \phi_{|\chi}{}^{\delta\epsilon} \phi_\delta{}^{\xi\gamma} \nabla_{\xi|} \phi_{\beta)\epsilon\gamma} \\
&+ 8\hat{\xi}_{(\alpha}{}^\chi \phi^{\delta\epsilon\xi} \phi_{|\delta\epsilon}{}^\gamma \nabla_{\xi|} \phi_{\beta)\chi\gamma} + 12\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)\epsilon}{}^\xi \nabla^\gamma \phi_{\delta\xi\gamma} - 24\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)}{}^{\xi\gamma} \nabla_\epsilon \phi_{\delta\xi\gamma} \\
&+ 16\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi|}{}^\epsilon \phi_{\beta)}{}^{\xi\gamma} \nabla_\xi \phi_{\delta\epsilon\gamma} + 28\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{\beta)\epsilon}{}^\gamma \nabla_\xi \phi_{\chi\delta\gamma} + 12\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi}{}^\epsilon \phi_{\delta\epsilon|}{}^\xi \nabla^\gamma \phi_{\beta)\xi\gamma} \\
&+ 64\hat{\xi}^{\chi\delta} \phi_{(\alpha|\chi}{}^\epsilon \phi_{\epsilon}{}^{\xi\gamma} \nabla_{\xi|} \phi_{\beta)\delta\gamma} - 12\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\delta\epsilon|} \nabla^\gamma \phi_{\beta)\xi\gamma} - 8\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\delta}{}^\gamma \nabla_{\epsilon|} \phi_{\beta)\xi\gamma} \\
&+ 8\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\delta}{}^\gamma \nabla_{\gamma|} \phi_{\beta)\epsilon\xi} - 4\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\epsilon}{}^\gamma \nabla_{\xi|} \phi_{\beta)\delta\gamma} + 4\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\chi\epsilon}{}^\gamma \nabla_{\gamma|} \phi_{\beta)\delta\xi} \\
&- 16\hat{\xi}^{\chi\delta} \phi_{(\alpha}{}^{\epsilon\xi} \phi_{|\epsilon\xi}{}^\gamma \nabla_{\gamma|} \phi_{\beta)\chi\delta} - 8\phi^\chi \hat{\xi}^{\delta\epsilon} \phi_{\alpha\beta}{}^\xi \nabla_\chi \phi_{\delta\epsilon\xi} \Bigg).
\end{aligned}
\tag{D.3}
$$

References

[1] Giombi S and Yin X 2013 The higher spin/vector model duality *J. Phys. A: Math. Theor.* **46** 214003

[2] Gaberdiel M R and Gopakumar R 2013 Minimal model holography *J. Phys. A: Math. Theor.* **46** 214002

[3] Vasiliev M A 1980 'Gauge' form description of massless fields with arbitrary spin *Yad. Fiz.* **32** 855 (in Russian)

[4] Fronsdal C 1978 Massless fields with integer spin *Phys. Rev.* D **18** 3624

[5] Vasiliev M A 1990 Consistent equation for interacting gauge fields of all spins in (3+1)-dimensions *Phys. Lett.* B **243** 378

[6] Vasiliev M 2003 Nonlinear equations for symmetric massless higher spin fields in (A)dS(d) *Phys. Lett.* B **567** 139

[7] Bengtsson A K, Bengtsson I and Brink L 1983 Cubic interaction terms for arbitrary spin *Nucl. Phys.* B **227** 31

[8] Metsaev R 2006 Cubic interaction vertices of massive and massless higher spin fields *Nucl. Phys.* B **759** 147

[9] Buchbinder I, Fotopoulos A, Petkou A C and Tsulaia M 2006 Constructing the cubic interaction vertex of higher spin gauge fields *Phys. Rev.* D **74** 105018

[10] Fotopoulos A, Irges N, Petkou A C and Tsulaia M 2007 Higher-spin gauge fields interacting with scalars: the lagrangian cubic vertex *J. High Energy Phys.* JHEP10(2007)021

[11] Bekaert X, Boulanger N and Leclercq S 2010 Strong obstruction of the Berends–Burgers–van Dam spin-3 vertex *J. Phys. A: Math. Theor.* **43** 185401

[12] Manvelyan R, Mkrtchyan K and Rühl W 2010 General trilinear interaction for arbitrary even higher spin gauge fields *Nucl. Phys.* B **836** 204
[13] Sagnotti A and Taronna M 2011 String lessons for higher-spin interactions *Nucl. Phys.* B **842** 299
[14] Fotopoulos A and Tsulaia M 2010 On the tensionless limit of string theory, off-shell higher spin interaction vertices and BCFW recursion relations *J. High Energy Phys.* JHEP11(2010)086
[15] Joung E and Taronna M 2012 Cubic interactions of massless higher spins in (A)dS: metric-like approach *Nucl. Phys.* B **861** 145
[16] Joung E, Lopez L and Taronna M 2013 Solving the Noether procedure for cubic interactions of higher spins in (A)dS *J. Phys. A: Math. Theor.* **46** 214020
[17] Joung E, Lopez L and Taronna M 2013 Generating functions of (partially-)massless higher-spin cubic interactions *J. High Energy Phys.* JHEP01(2013)168
[18] Joung E and Taronna M 2014 Cubic-interaction-induced deformations of higher-spin symmetries *J. High Energy Phys.* JHEP03(2014)103
[19] Gutperle M and Kraus P 2011 Higher spin black holes *J. High Energy Phys.* JHEP05(2011)022
[20] Ammon M, Gutperle M, Kraus P and Perlmutter E 2011 Space–time geometry in higher spin gravity *J. High Energy Phys.* JHEP10(2011)053
[21] Castro A, Hijano E, Lepage-Jutier A and Maloney A 2012 Black holes and singularity resolution in higher spin gravity *J. High Energy Phys.* JHEP01(2012)031
[22] Bañados M, Canto R and Theisen S 2012 The action for higher spin black holes in three dimensions *J. High Energy Phys.* JHEP07(2012)147
[23] Bunster C, Henneaux M, Pérez A, Tempo D and Troncoso R 2014 Generalized black holes in three-dimensional space–time *J. High Energy Phys.* JHEP05(2014)031
[24] Campoleoni A, Fredenhagen S, Pfenninger S and Theisen S 2013 Towards metric-like higher-spin gauge theories in three dimensions *J. Phys. A: Math. Theor.* **46** 214017
[25] Kraus P and Perlmutter E 2011 Partition functions of higher spin black holes and their CFT duals *J. High Energy Phys.* JHEP11(2011)061
[26] Gaberdiel M R, Hartman T and Jin K 2012 Higher spin black holes from CFT *J. High Energy Phys.* JHEP04(2012)103
[27] Pérez A, Tempo D and Troncoso R 2013 Higher spin gravity in 3D: black holes, global charges and thermodynamics *Phys. Lett.* B **726** 444
[28] Pérez A, Tempo D and Troncoso R 2013 Higher spin black hole entropy in three dimensions *J. High Energy Phys.* JHEP04(2013)143
[29] de Boer J and Jottar J I 2014 Thermodynamics of higher spin black holes in AdS$_3$ *J. High Energy Phys.* JHEP01(2014)023
[30] Kraus P and Ugajin T 2013 An entropy formula for higher spin black holes via conical singularities *J. High Energy Phys.* JHEP05(2013)160
[31] Ammon M, Castro A and Iqbal N 2013 Wilson lines and entanglement entropy in higher spin gravity *J. High Energy Phys.* JHEP10(2013)110
[32] de Boer J and Jottar J I 2014 Entanglement entropy and higher spin holography in AdS$_3$ *J. High Energy Phys.* JHEP04(2014)089
[33] Compère G, Jottar J I and Song W 2013 Observables and microscopic entropy of higher spin black holes *J. High Energy Phys.* JHEP11(2013)054
[34] Datta S, David J R, Ferlaino M and Kumar S P 2014 Higher spin entanglement entropy from CFT *J. High Energy Phys.* JHEP06(2014)096
[35] Blencowe M 1989 A consistent interacting massless higher spin field theory in $D = (2+1)$ *Class. Quantum Grav.* **6** 443
[36] Bergshoeff E, Blencowe M and Stelle K 1990 Area preserving diffeomorphisms and higher spin algebra *Commun. Math. Phys.* **128** 213
[37] Campoleoni A, Fredenhagen S, Pfenninger S and Theisen S 2010 Asymptotic symmetries of three-dimensional gravity coupled to higher-spin fields *J. High Energy Phys.* JHEP11(2010)007
[38] Fujisawa I and Nakayama R 2013 Second-order formalism for 3D spin-3 gravity *Class. Quantum Grav.* **30** 035003
[39] Fujisawa I and Nakayama R 2014 Metric-like formalism for matter fields coupled to 3D higher spin gravity *Class. Quantum Grav.* **31** 015003
[40] Witten E 1988 (2+1)-dimensional gravity as an exactly soluble system *Nucl. Phys.* B **311** 46
[41] Nutma T 2014 xTras: a field-theory inspired xAct package for Mathematica *Comput. Phys. Commun.* **185** 1719
[42] Martin-Garcia J M xAct: efficient tensor computer algebra for Mathematica (http://xact.es/)