# Cohomology foundations of one-loop amplitudes in pure spinor superspace 

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We describe a pure spinor BRST cohomology framework to compactly represent tendimensional one-loop amplitudes involving any number of massless open- and closed-string states. The method of previous work to construct scalar and vectorial BRST invariants in pure spinor superspace signals the appearance of the hexagon gauge anomaly when applied to tensors. We study the systematics of the underlying BRST anomaly by defining the notion of pseudo-cohomology. This leads to a rich network of pseudo-invariant superfields of arbitrary tensor rank whose behavior under traces and contractions with external momenta is determined from cohomology manipulations. Separate papers will illustrate the virtue of the superfields in this work to represent one-loop amplitudes of the superstring and of ten-dimensional super-Yang-Mills theory.

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## Contents

1 Introduction ..... 4
1.1. Amplitudes as expressions in pure spinor superspace ..... 4
1.2. Multiloop amplitude prescription and pure spinor superspace ..... 5
1.3. BRST cohomology considerations as a method to simplify computations ..... 6
1.4. The anatomy of one-loop amplitudes ..... 9
1.5. Outline ..... 10
2 Review and conventions ..... 11
2.1. Diagrammatic introduction of BRST blocks ..... 11
2.2. Recursive construction of BRST blocks ..... 12
2.3. Berends-Giele currents ..... 14
2.4. One-loop building blocks ..... 16
2.5. Scalar and vector one-loop cohomology ..... 18
3 Towards a BRST pseudo-cohomology ..... 20
3.1. Tensorial building blocks $M^{m n}$ ..... 21
3.2. Pseudo BRST cohomology ..... 22
3.3. Tools for constructing BRST pseudo-invariants ..... 22
3.4. Rank-two example of pseudo-cohomology ..... 24
4 Tensor pseudo-cohomology ..... 25
4.1. Higher-rank building blocks and anomaly blocks ..... 25
4.2. Anomalous BRST variations at higher rank ..... 26
4.3. Recursion for higher rank pseudoinvariants ..... 27
5 Towards a refined pseudo-cohomology ..... 28
5.1. Refined currents ..... 29
5.2. Berends-Giele version of refined currents ..... 30
5.3. Scalar pseudo cohomology ..... 33
5.4. Scalar pseudoinvariants versus tensor traces ..... 34
6 Generalizing the refined pseudo-cohomology ..... 35
6.1. Higher-rank refined currents and their anomaly ..... 35
6.2. Recursion for refined higher-rank pseudoinvariants ..... 36
6.3. Higher-refinement building blocks ..... 37
6.4. The general recursion for pseudoinvariants ..... 39
6.5. Trace relations among pseudoinvariants ..... 40
7 Anomalous BRST invariants ..... 42
7.1. BRST variation of unrefined anomaly blocks ..... 43
7.2. Unrefined anomaly invariants ..... 44
7.3. BRST variation of general anomaly blocks ..... 45
7.4. Refined anomaly invariants ..... 46
7.5. Anomaly trace relations ..... 47
8 Generalizing the recursion scheme ..... 47
8.1. The master recursion ..... 48
8.2. The $D$-superfields at ghost-number two ..... 50
8.3. The $L$-superfields at ghost-number two ..... 50
8.4. The $\Delta$-superfields at ghost-number three ..... 51
8.5. The $\Lambda$-superfields at ghost-number three ..... 51
9 Pseudoinvariant relations for $k_{B}$ momentum contractions ..... 52
9.1. BRST-exactness versus momentum phase space ..... 52
9.2. Momentum contractions of unrefined pseudoinvariants ..... 54
9.3. Momentum contractions of refined pseudoinvariants ..... 57
10 Pseudoinvariant relations for $k_{i}$ momentum contractions ..... 59
10.1. $k_{i}$ contractions of unrefined pseudoinvariants ..... 59
10.2. $k_{i}$ contractions of refined pseudoinvariants ..... 61
10.3. Trace relations and anomaly bookkeeping ..... 63
10.4. The web of relations between ghost-number two and four ..... 65
11 Canonicalizing pseudoinvariants ..... 66
11.1. Canonicalizing scalar invariants ..... 66
11.2. Canonicalizing unrefined pseudoinvariants ..... 68
11.3. Canonicalizing non-refined slots in refined pseudoinvariants ..... 69
11.4. Canonicalizing refined slots in refined pseudoinvariants ..... 70
12 Conclusion and outlook ..... 72
A Examples of BRST pseudoinvariants ..... 74
B Gauge transformations versus BRST transformations ..... 76
B.1. Gauge variations of multiparticle superfields ..... 77
B.2. Gauge variations of Berends-Giele currents $M_{A}$ ..... 78
B.3. Gauge variations of ghost number two building blocks ..... 78
B.4. Gauge variations of pseudoinvariants ..... 79
B.5. Parity odd nature of multiparticle anomaly tensors ..... 81
C BRST variations of miscellaneous superfields ..... 82
C.1. BRST variations before the Berends-Giele map ..... 82
C.2. BRST variations after the Berends-Giele map ..... 83
D The $H$ superfields in the redefinition of refined currents ..... 84
D.1. The $H_{[A, B]}$ tensors from BRST blocks $V_{C}$ ..... 84
D.2. The $H_{[A, B]}$ tensors from BRST blocks $A_{C}^{m}$ ..... 86
D.3. The $\mathcal{H}_{[A, B]}$ tensors from Berends-Giele currents $\mathcal{K}_{C}$ ..... 87
E On BRST exact relations among pseudoinvariants ..... 88
E.1. BRST generator of $C_{1 \mid A, B, C, D}^{m}$ ..... 88
E.2. Seven-point momentum contractions of $C_{1 \mid A, B, C, D}^{m}$ ..... 88
F Examples of the canonicalization procedure ..... 89

## 1. Introduction

Pure spinors are known to facilitate the superspace description of ten-dimensional super-Yang-Mills (SYM) [1,2] which descends from the pure spinor superstring [3]. As will be explained below, ten-dimensional pure spinor superspace allows to take advantage of BRST symmetry to provide valuable guidance for the construction of scattering amplitudes in both string- and field-theory.

### 1.1. Amplitudes as expressions in pure spinor superspace

The prescription to compute multiloop superstring amplitudes in the pure spinor formalism $[3,4,5]$ is considerably simpler than in the Ramond-Neveu-Schwarz (RNS) [6] and Green-Schwarz (GS) [7] formulations of superstring theory. Unlike in the RNS, spacetime supersymmetry is manifest and there is no need to sum over spin structures since there are no worldsheet spinors. And in contrast to the GS, the amplitudes are computed in a manifestly super-Poincaré covariant manner. These two features combined allow to bypass the technical challenges associated with amplitude computations in the RNS and GS. However, there is another feature in the pure spinor setup which is not as prominently stressed but is of equal importance: The result of amplitude computations belongs to the BRST cohomology of pure spinor superspace expressions at ghost number three.

Pure spinor superspace ${ }^{1}$ is defined in terms of the standard ten-dimensional superspace variables $\left(x^{m}, \theta^{\alpha}\right)$ and the pure spinor $\lambda^{\alpha}$ (of ghost number one) satisfying $\lambda^{\alpha} \gamma_{\alpha \beta}^{m} \lambda^{\beta}=0[8]$. As will be explained below, it turns out that the kinematic factors ${ }^{2}$ of multiloop amplitudes can be written as pure spinor superspace expressions of the form

$$
\begin{equation*}
K=\lambda^{\alpha} \lambda^{\beta} \lambda^{\gamma} f_{\alpha \beta \gamma}(x, \theta), \tag{1.1}
\end{equation*}
$$

where $f_{\alpha \beta \gamma}(x, \theta)$ represents a function of ten-dimensional superspace and includes the dependence on polarizations and momenta. This novel type of superspace was shown in [3] to encode the results of tree-level string amplitudes and proven to be supersymmetric and gauge invariant when $K$ is in the cohomology of the pure spinor BRST charge.

1 The superspace defined here is the minimal pure spinor superspace associated with the original formulation in [3]. The non-minimal superspace appropriate in the context of the non-minimal formalism of [5] also contains $\bar{\lambda}{ }_{\alpha}$ variables and is not the subject of the present paper.
${ }^{2}$ Kinematic factors are understood as the polarization-dependent parts of amplitudes accompanying a basis of worldsheet integrals.

Furthermore, in order to extract the precise contractions of polarizations and momenta from a superspace expression $K$, one computes its pure spinor bracket $\langle K\rangle$ defined by $\left\langle\left(\lambda \gamma^{m} \theta\right)\left(\lambda \gamma^{n} \theta\right)\left(\lambda \gamma^{p} \theta\right)\left(\theta \gamma_{m n p} \theta\right)\right\rangle=1$ [3]. Since component expansions are straightforward to evaluate and can be automated [9,10], the real challenge in computing string scattering amplitudes consists of obtaining their corresponding superspace expressions in the BRST cohomology, which motivates the studies presented in this paper.

### 1.2. Multiloop amplitude prescription and pure spinor superspace

The prescription to compute multiloop amplitudes in the pure spinor formalism was presented in $[4,5]$ and integrates out all eleven pure spinor components $\lambda^{\alpha}$ and all sixteen $\theta^{\alpha}$ variables. Performing those integrations leads to awkward expressions which are hard to manipulate. As observed at one-loop [11] and emphasized in the two-loop computations of [12], one can equivalently rewrite those complicated-looking expressions by reinstating three pure spinors $\lambda^{\alpha}$ in such a way as to obtain the same type of tree-level pure spinor superspace expressions (1.1) discussed above. To illustrate the above point, the two-loop kinematic factor of [13] after performing the path integral over all variables is written as

$$
\begin{equation*}
\left(T^{-1}\right)_{\rho_{1} \ldots \rho_{11}}^{\alpha \beta \gamma} \epsilon^{\rho_{1} \ldots \rho_{16}} \frac{\partial}{\partial \theta^{\rho_{12}}} \cdots \frac{\partial}{\partial \theta^{\rho_{16}}}\left(\gamma^{m n p q r}\right)_{\alpha \beta} \gamma_{\gamma \delta}^{s} F_{m n}^{1}(\theta) F_{p q}^{2}(\theta) F_{r s}^{3}(\theta) W^{4 \delta}(\theta) . \tag{1.2}
\end{equation*}
$$

The superfields $F_{m n}(\theta)$ and $W^{\delta}(\theta)$ represent the gauge multiplet of ten-dimensional SYM, and the tensor $\left(T^{-1}\right)_{\rho_{1} \ldots \rho_{11}}^{\alpha \beta \gamma}$ is proportional to a complicated combination of gamma matrices, $\epsilon_{\rho_{1} \ldots \rho_{16}}\left(\gamma^{m}\right)^{\kappa \rho_{12}}\left(\gamma^{n}\right)^{\sigma \rho_{13}}\left(\gamma^{p}\right)^{\tau \rho_{14}}\left(\gamma_{m n p}\right)^{\rho_{15} \rho_{16}}\left(\delta_{\kappa}^{(\alpha} \delta_{\sigma}^{\beta} \delta_{\tau}^{\gamma)}-\frac{1}{40} \gamma_{q}^{(\alpha \beta} \delta_{\kappa}^{\gamma)} \gamma_{\sigma \tau}^{q}\right)$. However, the kinematic factor (1.2) can be equivalently written as the tree-level pure spinor superspace expression $\langle K\rangle$ after reinstating three pure spinors, where

$$
\begin{equation*}
K=\left(\lambda \gamma^{m n p q r} \lambda\right)\left(\lambda \gamma^{s} W^{4}\right) F_{m n}^{1} F_{p q}^{2} F_{r s}^{3} \tag{1.3}
\end{equation*}
$$

In writing the kinematic factor (1.2) as (1.3), its BRST invariance becomes easier to prove; using standard manipulations of gamma matrices, the pure spinor constraint and SYM equations of motion for $D_{\alpha} W^{\delta}$ and $D_{\alpha} F^{m n}$, it follows that

$$
\begin{equation*}
Q\left[\left(\lambda \gamma^{m n p q r} \lambda\right)\left(\lambda \gamma^{s} W^{4}\right) F_{m n}^{1} F_{p q}^{2} F_{r s}^{3}\right]=0 \tag{1.4}
\end{equation*}
$$

Furthermore, one can also show that the kinematic factor (1.3) is in the cohomology of the BRST charge ${ }^{3}$. The compact nature of pure spinor superspace expressions as exemplified by (1.3) compared to (1.2) and the observation (1.4) constitute the central pillars in the study of multiloop string scattering amplitudes as objects in the BRST cohomology of tree-level pure spinor superspace.
${ }^{3}$ A superspace proof that (1.3) is not BRST exact requires a combination of the identities in [14] and section 9 of this paper.

### 1.3. BRST cohomology considerations as a method to simplify computations

Following observations based on the BRST structure of explicit lower-point results it was suggested in [15] that the field-theory amplitudes at tree-level could be uniquely obtained as pure spinor superspace expressions in the BRST cohomology. Indeed, the color-ordered $N$-point tree amplitude of SYM can be compactly written as [16]

$$
\begin{equation*}
A(1,2, \ldots, N)=\left\langle V_{1} E_{23 \ldots N}\right\rangle \tag{1.5}
\end{equation*}
$$

where $E_{23 \ldots N}$ is a superfield in the BRST cohomology. The pursuit of the general expression of the SYM tree amplitude as the solution of a cohomology problem in pure spinor superspace led to the discovery of interesting mathematical objects such as the BRST blocks and supersymmetric Berends-Giele currents reviewed in section 2. These BRSTcovariant objects also played an essential role in the derivation of the general $N$-point open superstring tree-level amplitude in [17]. And as a byproduct of the BRST-covariant organization of the string tree-level amplitudes, the worldsheet integrals conspire to a particularly symmetric form which was later exploited to find interesting patterns in their $\alpha^{\prime}$ expansion [18-23].

### 1.3.1. Challenges at one-loop

Studying the one-loop open superstring amplitudes as a BRST cohomology problem was firstly put forward in [24]. Using the multiloop prescription of [4] as a guide to obtain the patterns of zero-mode saturation, the kinematic factors could be expressed in pure spinor superspace. Furthermore, integration by parts identities among the worldsheet integrals built up BRST-closed linear combinations of those kinematic factors, denoted $C_{i \mid A, B, C}$ and reviewed in section 2.4.

This cohomology setup led to a general and manifestly BRST-invariant expression for the $N$-point amplitude. For example, the six-point amplitude of open superstrings was found to be a worldsheet integral over ${ }^{4}$

$$
\begin{equation*}
X_{23} X_{34}\left\langle C_{1 \mid 234,5,6}\right\rangle+X_{23} X_{45}\left\langle C_{1 \mid 23,45,6}\right\rangle+\text { permutations . } \tag{1.6}
\end{equation*}
$$

However, there is one subtlety in the one-loop BRST cohomology program outlined above, the bare one-loop amplitudes are in general anomalous and therefore not BRST invariant.

4 The objects $X_{i j}$ are related to the one-loop worldsheet Green function $G(z)$, and the KobaNielsen factor $\prod_{i<j} e^{\alpha^{\prime}\left(k_{i} \cdot k_{j}\right) G\left(z_{i}-z_{j}\right)}$ is suppressed [24].

The cancelation of the anomaly as described in [25] involves a sum over amplitudes with different worldsheet topologies, but the composing amplitudes are still anomalous when the number of external particles is six or higher. The BRST-invariant expression (1.6) could not be the whole story since it is non-anomalous ${ }^{5}$.

It is clear that in order to study the missing pieces of the one-loop amplitudes associated to the anomaly in a BRST cohomology setup one needs to relax the condition of BRST invariance. So in this work, among other things, we introduce the notion of a pseudo BRST cohomology which meets this criterium. The essential idea behind the pseudo BRST cohomology goes back to the pure spinor analysis of the gauge anomaly in [26]. It was shown that the gauge variation of the six-point amplitude w.r.t particle one is proportional to the pure spinor superspace expression,

$$
\begin{equation*}
\left\langle\left(\lambda \gamma^{m} W^{2}\right)\left(\lambda \gamma^{n} W^{3}\right)\left(\lambda \gamma^{p} W^{4}\right)\left(W^{5} \gamma_{m n p} W^{6}\right)\right\rangle \tag{1.7}
\end{equation*}
$$

whose component expansion correctly reproduces the known form of the anomaly, $\epsilon_{10} F^{5}$. As discussed in section 3, one can recursively construct objects whose BRST variation is proportional to the anomalous superfield (1.7). It will be shown in a subsequent work that these pseudo BRST invariants correctly capture the anomalous parts of the one-loop amplitudes which were not considered in [24].

So as the main focus of this work, we will study the (pseudo-)cohomology properties of various superfields expected to appear in one-loop amplitudes of open- and closed-strings. We introduce a grid of superspace kinematic factors which naturally describe the BRST cohomology properties of one-loop amplitudes. The axes of this grid are set by the number of free vector indices $m, n, p, \ldots$ and the number of multiparticle slots $A, B, \ldots, G$ which are interpreted as representing external tree-level subdiagrams. This leads to the arrangement in fig. 1, and we will derive recursion relations whose flow is indicated by the diagonal arrows. The tensorial superfields therein play an important role in two different contexts:

5 We thank Michael Green for insisting on a clarification of this point.


Fig. 1 Overview of (pseudo-)invariants. The arrows indicate whenever superfields of different type enter the recursion for the pseudoinvariants on their right.
(i) Closed string amplitudes involving five and more external legs allow for vector contractions between left- and right-moving degrees of freedom, see e.g. [27,28]. They originate from the zero modes of the worldsheet fields $\partial x^{m}$ and $\bar{\partial} x^{m}$. In a manifestly BRST-invariant representation of the five- and six-point torus amplitude, the left-right contractions enter in the form $C_{1 \mid 2,3,4,5}^{m} \tilde{C}_{1 \mid 2,3,4,5}^{m}$ and $C_{1 \mid 2,3,4,5,6}^{m n} \tilde{C}_{1 \mid 2,3,4,5,6}^{m n}$, details will be elaborated in [29]. Accordingly, scattering of $N$ closed strings requires tensors of rank $r \leq N-4$.
(ii) The field theory limit of open and closed string amplitudes reproduces $n$-gon Feynman integrals [30] where the loop momentum $\ell^{m}$ may contract kinematic factors. In a manifestly BRST invariant form of the five-point amplitude, this loop momentum dependence enters in the form $\ell_{m} C_{1 \mid 2,3,4,5}^{m}$. At six-points, the significance of the tensor hexagon $\ell_{m} \ell_{n} C_{1 \mid 2,3,4,5,6}^{m n}$ for the gauge anomaly of SYM will be clarified in [29]. More generally, the systematic association of tensorial Feynman integrals with the superfields in fig. 1 is discussed in [31].

The present work is devoted to the cohomology foundations of one-loop amplitudes in string- and field-theory. The key definitions and results are formulated in generality to describe any number of external legs. Applications to six-point string amplitudes and to field-theory amplitudes at multiplicity $\leq 7$ are given in upcoming work [29,31], and the generalization to arbitrary multiplicity is left for the future.

### 1.4. The anatomy of one-loop amplitudes

One can embed the pseudoinvariants listed in fig. 1 into a broader context. BRST cohomology methods are of crucial importance to decompose the computation of scattering amplitudes into smaller and more manageable problems. As we will see in various places of this work, pseudo-invariance of the superfields $C_{1 \mid \ldots}^{m \ldots}, P_{1 \mid \ldots}^{m \ldots}$ in fig. 1 requires BRST-covariant substructures, which in turn furnish systematic arrangements of smaller constituents. The different hierarchy levels of this decomposition are made more specific in fig.2. The figure applies universally to one-loop scattering amplitudes involving SYM or supergravity states in maximally supersymmetric string- and field-theory.

These classes of one-loop amplitudes are claimed to have a beautiful representation in terms of pseudoinvariants $C_{1 \mid \ldots}^{m \ldots}$ and $P_{1 \mid \ldots}^{m \ldots}$ (or their holomorphic squares in case of supergravity and closed-string amplitudes). Their composition rules in terms of integrals over a loop momentum or over worldsheet moduli are the subject of upcoming work [29,31]. As detailed in the first six sections, pseudoinvariants are built from Berends-Giele currents $M_{B}$ (whose trilinears exhaust tree amplitudes of SYM [16] and the open superstring [17]) and ghost-number-two superfields $\mathcal{J}$ which are specific to the one-loop order. These $\mathcal{J}$ superfields in turn encompass various numbers of further Berends-Giele superfields $\mathcal{K}_{B}$ with kinematic poles in external momenta [32]. Both $M_{B}$ and $\mathcal{K}_{B}$ represent external tree subamplitudes which can be expanded in terms of (prod-


Fig. 2.The seven hierarchy levels describing the anatomy of one-loop amplitudes in a BRST cohomology setup. For each step, details can be found in the references alongside the arrows. ucts of) external propagators. Their numerators are multiparticle superfields of SYM $K_{B} \in\left\{A_{\alpha}^{B}, A_{B}^{m}, W_{B}^{\alpha}, F_{B}^{m n}\right\}$ which have been recursively constructed in [32]. They encompass the degrees of freedom of several standard superfields $A_{\alpha}^{i}, A_{i}^{m}, W_{i}^{\alpha}, F_{i}^{m n}$ describing a single particle $i$. Finally, the components of the supersymmetry multiplet - a gluon with polarization vector $e_{i}$ and a gaugino with spinor
wavefunction $\chi_{i}$ - furnish the lowest hierarchy level. They are incorporated into the expansion of the superfields in terms of the Grassmann coordinate $\theta$ of pure spinor superspace [33], e.g.

$$
\begin{equation*}
A_{\alpha}(x, \theta)=\left(\frac{1}{2} e_{m}\left(\gamma^{m} \theta\right)_{\alpha}-\frac{1}{3}\left(\chi \gamma_{m} \theta\right)\left(\gamma^{m} \theta\right)_{\alpha}-\frac{1}{16} k_{m} e_{n}\left(\gamma_{p} \theta\right)_{\alpha}\left(\theta \gamma^{m n p} \theta\right)+\cdots\right) e^{i k \cdot x} \tag{1.8}
\end{equation*}
$$

As the number of external legs increases, every intermediate structure in fig. 2 reduces the complexity of amplitudes by more and more orders of magnitude. And it should be stressed that the four lower hierarchy levels in fig. 2 - from Berends-Giele superfields $M_{B}, \mathcal{K}_{B}$ to the components $e_{i}, \chi_{i}$ - are expected to play a universal role at any loop order.

### 1.5. Outline

The main body of this work begins with a review of multiparticle SYM superfields [32] in section 2. This sets the stage to define the notion of anomalous superfields and BRST pseudo-cohomology in section 3. The introductory examples are then generalized to arbitrary tensor rank in section 4 . The resulting tensor traces are shown to involve several constituents (indicated by $P$ in fig. 1) which are separately BRST pseudoinvariant, see sections 5 and 6 . This completes the construction of the pseudo cohomology at ghost number three which is visualized in fig. 1.

In section 7, we point out a close parallel between the superfields in anomalous BRST variations and the previously-constructed pseudoinvariants. The more abstract viewpoint on this connection is opened up in section 8 and rewarded by manifold relations between superfields at different rank, see sections 9 and 10 . The same approach leads to the proof in section 11 that - up to anomaly subtleties - the span of the pseudoinvariants in fig. 1 is independent on the choice of reference leg 1 which descends from the choice of unintegrated vertex operator $V_{1}$ in the one-loop string amplitude prescription [4].

Some appendices supplement the discussion by examples or serve to outsource technical aspects from the main body. For example, appendix A displays the expansions of superfields in fig. 1 at higher multiplicity, and appendix $B$ provides the prerequisites to extract anomalous gauge variations of pseudoinvariants from their BRST transformations given in section 7.

## 2. Review and conventions

This section provides a brief review of multiparticle SYM superfields introduced in [32] as well as their simplest applications to one-loop kinematic factors. It also introduces notation and conventions used in the rest of this work.

### 2.1. Diagrammatic introduction of BRST blocks

Linearized super-Yang-Mills (SYM) theory in ten dimensions can be described using the superfields ${ }^{6} A_{\alpha}^{i}(x, \theta), A_{m}^{i}(x, \theta), W_{i}^{\alpha}(x, \theta)$ and $F_{m n}^{i}(x, \theta)$ encoding the on-shell degrees of freedom of one external particle $i$.

They satisfy equations of motion $[34,35]$

$$
\begin{align*}
2 D_{(\alpha} A_{\beta)}^{i} & =\gamma_{\alpha \beta}^{m} A_{m}^{i} & & D_{\alpha} A_{m}^{i}=\left(\gamma_{m} W_{i}\right)_{\alpha}+k_{m} A_{\alpha}^{i} \\
D_{\alpha} F_{m n}^{i} & =2 k_{[m}^{i}\left(\gamma_{n]} W_{i}\right)_{\alpha} & & D_{\alpha} W_{i}^{\beta}=\frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}{ }^{\beta} F_{m n}^{i} \tag{2.1}
\end{align*}
$$

with light-like momentum $k_{i}$ and gauge transformations $\delta_{i} A_{\alpha}^{i}=D_{\alpha} \omega_{i}$ as well as $\delta_{i} A_{m}^{i}=$ $k_{m}^{i} \omega_{i}$ for some scalar superfield $\omega_{i}$. The fermionic operator

$$
\begin{equation*}
D_{\alpha} \equiv \frac{\partial}{\partial \theta^{\alpha}}+\frac{1}{2} k^{m}\left(\gamma_{m} \theta\right)_{\alpha} \tag{2.2}
\end{equation*}
$$

denotes the standard superspace covariant derivative. Note that if a superfield $K(x, \theta)$ depends only on the zero-modes of $\theta$, the action of the pure spinor BRST charge $Q$ is given by a covariant derivative:

$$
\begin{equation*}
Q \equiv \oint \lambda^{\alpha} d_{\alpha} \quad \Longrightarrow \quad Q K=\lambda^{\alpha} D_{\alpha} K \tag{2.3}
\end{equation*}
$$

This fact allows one to use the equations of motion (2.1) in combination with BRST cohomology manipulations to simplify expressions considerably.

In previous work [32], these superfields were promoted to multiparticle versions

$$
\begin{equation*}
K_{i} \in\left\{A_{\alpha}^{i}, A_{m}^{i}, W_{i}^{\alpha}, F_{m n}^{i}\right\} \rightarrow K_{B} \in\left\{A_{\alpha}^{B}, A_{m}^{B}, W_{B}^{\alpha}, F_{m n}^{B}\right\} \tag{2.4}
\end{equation*}
$$

where the multiparticle label $B=b_{1} b_{2} \ldots b_{p}$ describes $|B| \equiv p$ external particles attached to a tree subdiagram as shown in fig. 3. The off-shell leg indicated by the ... in the figure reflects that the overall momentum $k_{B}^{m} \equiv \sum_{i=1}^{p} k_{b_{i}}^{m}$ is no longer lightlike in general, $k_{B}^{2} \neq 0$.
${ }^{6}$ It is customary to use a calligraphic letter for the superfield field-strength. However in this paper calligraphic letters will denote the Berends-Giele currents associated to the superfields, see section 4.


Fig. 3 Four superfield realizations $K_{B} \in\left\{A_{\alpha}^{B}, A_{m}^{B}, W_{B}^{\alpha}, F_{m n}^{B}\right\}$ of cubic tree graphs $B=b_{1} b_{2} \ldots b_{p}$.

As detailed in [32], the diagrammatic interpretation is supported by the Lie symmetries of $K_{B}$ matching with the color representative of the diagram in fig. 3,

$$
\begin{equation*}
K_{1234 \ldots p} \leftrightarrow f^{12 a_{2}} f^{a_{2} 3 a_{3}} f^{a_{3} 4 a_{4}} \ldots f^{a_{p-1} p a_{p}} \tag{2.5}
\end{equation*}
$$

subject to antisymmetry $f^{a b c}=f^{[a b c]}$ and Jacobi identities $f^{e[a b} f^{c] d e}=0$. For example,

$$
\begin{align*}
& 0=K_{12}+K_{21}, \quad 0=K_{123}+K_{231}+K_{312}  \tag{2.6}\\
& 0=K_{1234}-K_{1243}+K_{3412}-K_{3421} \tag{2.7}
\end{align*}
$$

furnish the kinematic analogue of $f^{12 a}=-f^{21 a}$ and Jacobi identities among permutations of $f^{12 a} f^{a 3 b}$ and $f^{12 a} f^{a 3 b} f^{b 4 c}$.

### 2.2. Recursive construction of BRST blocks

A recursive procedure was described in [32] to construct BRST blocks at arbitrary multiplicity from the elementary SYM superfields. The definition of the multiparticle fields (2.4) is inspired by the OPE among integrated massless vertex operators in the pure spinor formalism [3]. For two particles, this directly leads to

$$
\begin{align*}
A_{\alpha}^{12} & =-\frac{1}{2}\left[A_{\alpha}^{1}\left(k^{1} \cdot A^{2}\right)+A_{m}^{1}\left(\gamma^{m} W^{2}\right)_{\alpha}-(1 \leftrightarrow 2)\right]  \tag{2.8}\\
A_{m}^{12} & =\frac{1}{2}\left[A_{p}^{1} F_{p m}^{2}-A_{m}^{1}\left(k^{1} \cdot A^{2}\right)+\left(W^{1} \gamma_{m} W^{2}\right)-(1 \leftrightarrow 2)\right] \\
W_{12}^{\alpha} & =\frac{1}{4}\left(\gamma^{m n} W^{2}\right)^{\alpha} F_{m n}^{1}+W_{2}^{\alpha}\left(k^{2} \cdot A^{1}\right)-(1 \leftrightarrow 2) \\
F_{m n}^{12} & =k_{m}^{12} A_{n}^{12}-k_{n}^{12} A_{m}^{12}-\left(k^{1} \cdot k^{2}\right)\left(A_{m}^{1} A_{n}^{2}-A_{n}^{1} A_{m}^{2}\right),
\end{align*}
$$

compatible with antisymmetry $K_{12}=-K_{21}$. Remarkably, the equations of motion for these two-particle superfields take the same form as their single-particle equations (2.1)
with the addition of contact terms,

$$
\begin{align*}
2 D_{(\alpha} A_{\beta)}^{12}= & \gamma_{\alpha \beta}^{m} A_{m}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} A_{\beta}^{2}+A_{\beta}^{1} A_{\alpha}^{2}\right)  \tag{2.9}\\
D_{\alpha} A_{m}^{12}= & \left(\gamma_{m} W^{12}\right)_{\alpha}+k_{m}^{12} A_{\alpha}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} A_{m}^{2}-A_{\alpha}^{2} A_{m}^{1}\right) \\
D_{\alpha} W_{12}^{\beta}= & \frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} F_{m n}^{12}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} W_{2}^{\beta}-A_{\alpha}^{2} W_{1}^{\beta}\right) \\
D_{\alpha} F_{m n}^{12}= & k_{m}^{12}\left(\gamma_{n} W^{12}\right)_{\alpha}-k_{n}^{12}\left(\gamma_{m} W^{12}\right)_{\alpha}+\left(k^{1} \cdot k^{2}\right)\left(A_{\alpha}^{1} F_{m n}^{2}-A_{\alpha}^{2} F_{m n}^{1}\right) \\
& +\left(k^{1} \cdot k^{2}\right)\left(A_{n}^{1}\left(\gamma_{m} W^{2}\right)_{\alpha}-A_{n}^{2}\left(\gamma_{m} W^{1}\right)_{\alpha}-A_{m}^{1}\left(\gamma_{n} W^{2}\right)_{\alpha}+A_{m}^{2}\left(\gamma_{n} W^{1}\right)_{\alpha}\right) .
\end{align*}
$$

Starting from multiplicity three, application of the recursion (2.8) yields superfields

$$
\begin{align*}
& \widehat{A}_{\alpha}^{123}=-\frac{1}{2}\left[A_{\alpha}^{12}\left(k^{12} \cdot A^{3}\right)+A_{m}^{12}\left(\gamma^{m} W^{3}\right)_{\alpha}-(12 \leftrightarrow 3)\right]  \tag{2.10}\\
& \widehat{A}_{m}^{123}=\frac{1}{2}\left[A_{12}^{p} F_{p m}^{3}-A_{m}^{12}\left(k^{12} \cdot A^{3}\right)+\left(W^{12} \gamma_{m} W^{3}\right)-(12 \leftrightarrow 3)\right]
\end{align*}
$$

which require redefinitions

$$
\begin{align*}
& A_{m}^{123}=\widehat{A}_{m}^{123}-k_{m}^{123} H^{123}, \quad A_{\alpha}^{123}=\widehat{A}_{\alpha}^{123}-D_{\alpha} H^{123}  \tag{2.11}\\
& H^{123}=\frac{1}{6}\left[\left(A^{1} \cdot A^{23}\right)-\left(k_{p}^{2}-k_{p}^{3}\right) A_{1}^{p}\left(A^{2} \cdot A^{3}\right)+\operatorname{cyclic}(123)\right] \tag{2.12}
\end{align*}
$$

by some scalar superfield $H_{i j k}$ [32] before they satisfy the Lie symmetries in (2.6) and qualify as BRST blocks. The three-particle set of BRST blocks $K_{123} \in\left\{A_{\alpha}^{123}, A_{m}^{123}, W_{123}^{\alpha}, F_{m n}^{123}\right\}$ is completed by field strengths

$$
\begin{align*}
W_{123}^{\alpha} & =\left[\frac{1}{4}\left(\gamma^{r s} W^{3}\right)^{\alpha} F_{r s}^{12}+W_{3}^{\alpha}\left(k^{3} \cdot A^{12}\right)-(12 \leftrightarrow 3)\right]+\frac{1}{2}\left(k^{1} \cdot k^{2}\right)\left[W_{2}^{\alpha}\left(A^{1} \cdot A^{3}\right)-(1 \leftrightarrow 2)\right] \\
F_{m n}^{123} & =k_{m}^{123} A_{n}^{123}-k_{n}^{123} A_{m}^{123}-\left(k^{1} \cdot k^{2}\right)\left[2 A_{[m}^{1} A_{n]}^{23}-(1 \leftrightarrow 2)\right]-\left(k^{12} \cdot k^{3}\right) 2 A_{[m}^{12} A_{n]}^{3} \cdot \tag{2.13}
\end{align*}
$$

As shown in [32], the equations of motion for the $K_{123}$ reproduce the universal structure of (2.1) and (2.9) and incorporate a richer set of contact terms $\sim\left(k^{1} \cdot k^{2}\right)$ and $\left(k^{12} \cdot k^{3}\right)$ :

$$
\begin{align*}
2 D_{(\alpha} A_{\beta)}^{123}= & \gamma_{\alpha \beta}^{m} A_{m}^{123}+\left(k^{12} \cdot k^{3}\right)\left[A_{\alpha}^{12} A_{\beta}^{3}-(12 \leftrightarrow 3)\right]  \tag{2.14}\\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} A_{\beta}^{23}+A_{\alpha}^{13} A_{\beta}^{2}-(1 \leftrightarrow 2)\right] \\
D_{\alpha} A_{m}^{123}= & \left(\gamma_{m} W^{123}\right)_{\alpha}+k_{m}^{123} A_{\alpha}^{123}+\left(k^{12} \cdot k^{3}\right)\left(A_{\alpha}^{12} A_{m}^{3}-A_{\alpha}^{3} A_{m}^{12}\right) \\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} A_{m}^{23}+A_{\alpha}^{13} A_{m}^{2}-A_{\alpha}^{23} A_{m}^{1}-A_{\alpha}^{2} A_{m}^{13}\right] \\
D_{\alpha} W_{123}^{\beta}= & \frac{1}{4}\left(\gamma^{m n}\right)_{\alpha}^{\beta} F_{m n}^{123}+\left(k^{12} \cdot k^{3}\right)\left[A_{\alpha}^{12} W_{3}^{\beta}-(12 \leftrightarrow 3)\right] \\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} W_{23}^{\beta}+A_{\alpha}^{13} W_{2}^{\beta}-(1 \leftrightarrow 2)\right] \\
D_{\alpha} F_{m n}^{123}= & 2 k_{[m}^{123}\left(\gamma_{n]} W^{123}\right)_{\alpha}+\left(k^{12} \cdot k^{3}\right)\left[A_{\alpha}^{12} F_{m n}^{3}-(12 \leftrightarrow 3)\right] \\
& +\left(k^{12} \cdot k^{3}\right)\left[2 A_{[n}^{12}\left(\gamma_{m]} W^{3}\right)_{\alpha}-(12 \leftrightarrow 3)\right] \\
& +\left(k^{1} \cdot k^{2}\right)\left[A_{\alpha}^{1} F_{m n}^{23}+A_{\alpha}^{13} F_{m n}^{2}-(1 \leftrightarrow 2)\right] \\
& +\left(k^{1} \cdot k^{2}\right)\left[2 A_{[n}^{1}\left(\gamma_{m]} W^{23}\right)_{\alpha}+2 A_{[n}^{13}\left(\gamma_{m]} W^{2}\right)_{\alpha}-(1 \leftrightarrow 2)\right] .
\end{align*}
$$



Fig. 4 From cubic diagrams $K_{A}$ to Berends-Giele currents $\mathcal{K}_{A}$.

The starting point towards BRST blocks at higher multiplicity $p$ is the recursion

$$
\begin{equation*}
\widehat{A}_{\alpha}^{12 \ldots p}=-\frac{1}{2}\left[A_{\alpha}^{12 \ldots p-1}\left(k^{12 \ldots p-1} \cdot A^{p}\right)+A_{m}^{12 \ldots p-1}\left(\gamma^{m} W^{p}\right)_{\alpha}-(12 \ldots p-1 \leftrightarrow p)\right] \tag{2.15}
\end{equation*}
$$

with manifest Lie symmetries (2.5) in the first $p-1$ labels. Then, an algorithmic redefinition along the lines of (2.11) enforces the remaining symmetry (2.5) involving the last label $p$, see [32] for details.

The equations of motion at any multiplicity combine the single-particle structure from (2.1) with a growing tail of contact terms $\sim\left(k^{12 \ldots j-1} \cdot k^{j}\right)$ generalizing the three-particle example (2.14). Their explicit forms can be found in [32].

### 2.3. Berends-Giele currents

BRST blocks $K_{B}$ of multiplicity $|B|$ are diagrammatically interpreted as off-shell cubic graphs shown in fig. 3. This suggests to assemble diagrams to a color-ordered SYM $(|B|+1)$ point tree amplitude where one of the legs is off-shell, as schematically depicted in fig. 4. The precise form of this diagrammatic construction was explained in the appendix A of [32], and the result is the promotion of BRST blocks $K_{B}$ to Berends-Giele currents $\mathcal{K}_{B}$,

$$
\begin{equation*}
K_{B} \in\left\{A_{\alpha}^{B}, A_{B}^{m}, W_{B}^{\alpha}, F_{B}^{m n}\right\} \rightarrow \mathcal{K}_{B} \in\left\{\mathcal{A}_{\alpha}^{B}, \mathcal{A}_{B}^{m}, \mathcal{W}_{B}^{\alpha}, \mathcal{F}_{B}^{m n}\right\} \tag{2.16}
\end{equation*}
$$

The name goes back to Berends and Giele who recursively constructed gluonic currents which were then used to compute tree-level amplitudes [36]. From now on the ordered subsets $B=b_{1} b_{2} \ldots b_{|B|}$ of external particle labels which appear along with Berends-Giele currents $\mathcal{K}_{B}$ will be denoted "words".

The first examples of Berends-Giele currents $\mathcal{K}_{B}$ are given by,

$$
\begin{align*}
\mathcal{K}_{12} & =\frac{K_{12}}{s_{12}}, \quad \mathcal{K}_{123}=\frac{K_{123}}{s_{12} s_{123}}+\frac{K_{321}}{s_{23} s_{123}}  \tag{2.17}\\
\mathcal{K}_{1234} & =\frac{1}{s_{1234}}\left(\frac{K_{1234}}{s_{12} s_{123}}+\frac{K_{3214}}{s_{23} s_{123}}+\frac{K_{3421}}{s_{34} s_{234}}+\frac{K_{3241}}{s_{23} s_{234}}+\frac{2 K_{12[34]}}{s_{12} s_{34}}\right), \tag{2.18}
\end{align*}
$$

where the conventions for the generalized Mandelstam invariants is

$$
\begin{equation*}
s_{12 \ldots p}=\sum_{1 \leq i<j}^{p}\left(k_{i} \cdot k_{j}\right)=\frac{1}{2} k_{12 \ldots p}^{2} . \tag{2.19}
\end{equation*}
$$

It turns out that they enjoy simplified BRST variations compared to their corresponding BRST blocks $K_{B}$. In particular, the Berends-Giele version of the unintegrated ${ }^{7}$ multiparticle vertex $V_{B} \equiv \lambda^{\alpha} A_{\alpha}^{B}$,

$$
\begin{equation*}
M_{B} \equiv \lambda^{\alpha} \mathcal{A}_{\alpha}^{B} \tag{2.20}
\end{equation*}
$$

satisfies

$$
\begin{equation*}
Q M_{B}=\sum_{X Y=B} M_{X} M_{Y}=\sum_{j=1}^{|B|-1} M_{b_{1} b_{2} \ldots b_{j}} M_{b_{j+1} \ldots b_{p}} \tag{2.21}
\end{equation*}
$$

This has been exploited in $[16,17]$ to construct tree amplitudes of ten-dimensional SYM and of the open superstring. Throughout this paper the notation $X Y=B$ (as in (2.21)) denotes a sum over all deconcatenations of the word $B$ into smaller (non-empty) words $X=b_{1} b_{2} \ldots b_{j}$ and $Y=b_{j+1} \ldots b_{p}$ with $j=1,2, \ldots,|B|-1$.

As partially used in [24] and generalized in [32], the equations of motion for the remaining $\mathcal{K}_{B}$ representatives are given by,

$$
\begin{align*}
Q \mathcal{A}_{B}^{m}= & \left(\lambda \gamma^{m} \mathcal{W}_{B}\right)+k_{B}^{m} M_{B}+\sum_{X Y=B}\left(M_{X} \mathcal{A}_{Y}^{m}-M_{Y} \mathcal{A}_{X}^{m}\right) \\
Q \mathcal{W}_{B}^{\alpha}= & \frac{1}{4}\left(\lambda \gamma_{m n}\right)^{\alpha} \mathcal{F}_{B}^{m n}+\sum_{X Y=B}\left(M_{X} \mathcal{W}_{Y}^{\alpha}-M_{Y} \mathcal{W}_{X}^{\alpha}\right)  \tag{2.22}\\
Q \mathcal{F}_{B}^{m n}= & 2 k_{B}^{[m}\left(\lambda \gamma^{n]} \mathcal{W}_{B}\right)+\sum_{X Y=B}\left(M_{X} \mathcal{F}_{Y}^{m n}-M_{Y} \mathcal{F}_{X}^{m n}\right) \\
& +\sum_{X Y=B} 2\left[\mathcal{A}_{X}^{[n}\left(\lambda \gamma^{m]} \mathcal{W}_{Y}\right)-\mathcal{A}_{Y}^{[n}\left(\lambda \gamma^{m]} \mathcal{W}_{X}\right)\right]
\end{align*}
$$

where the contact terms present in $Q K_{B}$ are traded by deconcatenations as the result of the Berends-Giele map (2.16). At general multiplicity, the transformation matrix between BRST blocks and their Berends-Giele currents was identified in [21] to be the momentum kernel [37,38], see [32] for further details.
${ }^{7}$ According to the calligraphic-letter convention of (2.16) the Berends-Giele current associated to $V_{B}$ would be denoted $\mathcal{V}_{B}$. However, the definition (2.20) is used for historic reasons.

According to their definition as off-shell SYM amplitudes, the Lie symmetries of the BRST-blocks $K_{B}$ translate into Kleiss-Kuijf relations [39] among their Berends-Giele counterparts $\mathcal{K}_{B}[16,17]$. Up to multiplicity four, these are

$$
\begin{align*}
& 0=\mathcal{K}_{12}+\mathcal{K}_{21}, \quad 0=\mathcal{K}_{123}-\mathcal{K}_{321}=\mathcal{K}_{123}+\mathcal{K}_{231}+\mathcal{K}_{312}  \tag{2.23}\\
& 0=\mathcal{K}_{1234}+\mathcal{K}_{4321}=\mathcal{K}_{1234}+\mathcal{K}_{2134}+\mathcal{K}_{2314}+\mathcal{K}_{2341}
\end{align*}
$$

and higher multiplicity generalizations are most compactly written as ${ }^{8}$

$$
\begin{equation*}
\mathcal{K}_{B 1 A}=(-1)^{|B|} \mathcal{K}_{1\left(A ш B^{T}\right)} \tag{2.24}
\end{equation*}
$$

The superscript along with $B^{T}$ denotes the reversal of the word $B$ in external particles $b_{j}$ such as $\left(b_{1} b_{2} \ldots b_{|B|}\right)^{T}=\left(b_{|B|} \ldots b_{2} b_{1}\right)$, and $\amalg$ denotes the shuffle product.

### 2.4. One-loop building blocks

The saturation of fermionic zero modes in the pure spinor formalism [3] imposes tight constraints on contributions to loop amplitudes, see e.g. [4,40]. As argued in [24], the oneloop prescription in the minimal version of the formalism requires zero modes $d_{\alpha} d_{\beta} N^{m n}$ from the external vertices, leaving behind $\left(\lambda \gamma^{[m}\right)_{\alpha}\left(\lambda \gamma^{n]}\right)_{\beta}$. This effective rule leads to the BRST-closed expression $\left(\lambda \gamma^{m} W^{i}\right)\left(\lambda \gamma^{n} W^{j}\right) F_{m n}^{k}$ in the four-point amplitude [4] and motivates the following higher-point generalization $[24,32]$

$$
\begin{equation*}
T_{A, B, C} \equiv \frac{1}{3}\left(\lambda \gamma_{m} W_{A}\right)\left(\lambda \gamma_{n} W_{B}\right) F_{C}^{m n}+(C \leftrightarrow A, B), \tag{2.25}
\end{equation*}
$$

as well as its associated Berends-Giele current

$$
\begin{equation*}
M_{A, B, C} \equiv \frac{1}{3}\left(\lambda \gamma_{m} \mathcal{W}_{A}\right)\left(\lambda \gamma_{n} \mathcal{W}_{B}\right) \mathcal{F}_{C}^{m n}+(C \leftrightarrow A, B) \tag{2.26}
\end{equation*}
$$

From now on, the Berends-Giele versions of various superfield combinations will be emphasized since explicit results for BRST (pseudo-)invariants and amplitudes simplify in this basis. Furthermore, their BRST-block counterparts such as $T_{A, B, C}$ can always be trivially recovered by using the superfields $A_{B}, W_{B}, F_{B}$ instead of $\mathcal{A}_{B}, \mathcal{W}_{B}, \mathcal{F}_{B}$.
${ }^{8}$ We follow the convention $\mathcal{K} \ldots A ш B \ldots \equiv \sum_{C \in A ш B} \mathcal{K} \ldots C \ldots$.

The universal form (2.22) of $Q \mathcal{W}_{B}^{\alpha}$ and $Q \mathcal{F}_{B}^{m n}$ gives rise to the BRST-covariant ${ }^{9}$ transformation [24,32]

$$
\begin{equation*}
Q M_{A, B, C}=\sum_{X Y=A}\left(M_{X} M_{Y, B, C}-M_{Y} M_{X, B, C}\right)+(A \leftrightarrow B, C), \tag{2.27}
\end{equation*}
$$

governed by deconcatenations of the multiparticle labels. Note that $Q M_{1,2,3}=0$ and that $M_{A, B, C}$ is totally symmetric in $A, B$ and $C$.

In closed-string amplitudes of multiplicity higher than four, additional zero-mode contributions can arise from the $\Pi^{m}$ fields in the external vertices. In the simplest case at five points [28], this leads to a single vector index contraction among left- and right-movers. The aforementioned $d_{\alpha} d_{\beta} N^{m n} \rightarrow\left(\lambda \gamma^{[m}\right)_{\alpha}\left(\lambda \gamma^{n]}\right)_{\beta}$ prescription identifies contributions of the form $\mathcal{A}_{A}^{m} M_{B, C, D}$ to the left/right-contracting part of the closed-string amplitude.

However, as pointed out in [28] and generalized in [32], a separate b-ghost contribution proportional to $\Pi^{m} d_{\alpha} d_{\beta}$ leads to an additional kinematic factor

$$
\begin{gather*}
\mathcal{W}_{A, B, C, D}^{m} \equiv \frac{1}{12}\left(\lambda \gamma_{n} \mathcal{W}_{A}\right)\left(\lambda \gamma_{p} \mathcal{W}_{B}\right)\left(\mathcal{W}_{C} \gamma^{m n p} \mathcal{W}_{D}\right)+(A, B \mid A, B, C, D)  \tag{2.28}\\
Q \mathcal{W}_{A, B, C, D}^{m}=\sum_{X Y=A}\left(M_{X} \mathcal{W}_{Y, B, C, D}^{m}-M_{Y} \mathcal{W}_{X, B, C, D}^{m}\right) \\
\quad-\left(\lambda \gamma^{m} \mathcal{W}_{A}\right) M_{B, C, D}+(A \leftrightarrow B, C, D) \tag{2.29}
\end{gather*}
$$

The notation $\left(A_{1}, \ldots, A_{p} \mid A_{1}, \ldots, A_{n}\right)$ will also be used in later sections and instructs to sum over all possible ways to choose $p$ elements $A_{1}, A_{2}, \ldots, A_{p}$ out of the set $\left\{A_{1}, \ldots, A_{n}\right\}$, for a total of $\binom{n}{p}$ terms. This yields the following vector building block,

$$
\begin{gather*}
M_{A, B, C, D}^{m} \equiv\left[\mathcal{A}_{A}^{m} M_{B, C, D}+(A \leftrightarrow B, C, D)\right]+\mathcal{W}_{A, B, C, D}^{m}  \tag{2.30}\\
Q M_{A, B, C, D}^{m}=\sum_{X Y=A}\left(M_{X} M_{Y, B, C, D}^{m}-M_{Y} M_{X, B, C, D}^{m}\right) \\
\quad+k_{A}^{m} M_{A} M_{B, C, D}+(A \leftrightarrow B, C, D), \tag{2.31}
\end{gather*}
$$

which is totally symmetric in $A, B, C, D$. Apart from the last line, the BRST covariant transformation (2.31) stems from deconcatenation terms in $Q \mathcal{K}_{B}$. This goes back to cancellations between (2.29) and the first term of $Q \mathcal{A}_{B}^{m}=\left(\lambda \gamma^{m} \mathcal{W}_{B}\right)+\ldots$, see (2.22).
${ }^{9}$ Due to the tensor $\left(\lambda \gamma^{[m}\right)_{\alpha}\left(\lambda \gamma^{n]}\right)_{\beta}$ in (2.26), the pure spinor constraint projects out all terms in (2.22) with an explicit appearance of $\lambda^{\alpha}$, regardless of the words $A, B$ and $C$.

### 2.5. Scalar and vector one-loop cohomology

The simplest kinematic expressions compatible with the one-loop amplitude prescription [4] are $M_{A} M_{B, C, D}$ and $M_{A} M_{B, C, D, E}^{m}$ where the Berends-Giele current $M_{A}$ stems from OPE contractions of the unintegrated vertex $V_{i}$ with its integrated counterparts. Their covariant BRST transformations motivate to combine them to BRST-closed expressions. Following the experience with scalars [24], BRST invariants are classified in [32] by a "leading term" where a reference leg $i$ is represented through a single-particle unintegrated vertex $M_{i}=V_{i}$

$$
\begin{align*}
C_{i \mid A, B, C} & \equiv M_{i} M_{A, B, C}+\sum_{E \neq \emptyset} M_{i E} \ldots  \tag{2.32}\\
C_{i \mid A, B, C, D}^{m} & \equiv M_{i} M_{A, B, C, D}^{m}+\sum_{E \neq \emptyset} M_{i E} \ldots . \tag{2.33}
\end{align*}
$$

Apart from the explicit leading term, the singled-out label $i$ always enters in a multiparticle Berends-Giele current. This is formally represented by a sum over (non-empty) words $E$ of external particles which join the reference leg $i$ in $M_{i E}$. The $\ldots$ in $C_{i \mid A, B, C}\left(C_{i \mid A, B, C, D}^{m}\right)$ represent linear combinations of $M_{A, B, C}\left(M_{A, B, C, D}^{m}\right.$ and $\left.M_{A, B, C} k_{D}^{m}\right)$ such that

$$
\begin{equation*}
Q C_{i \mid A, B, C}=Q C_{i \mid A, B, C, D}^{m}=0 . \tag{2.34}
\end{equation*}
$$

In later sections, we will encounter plenty of further examples where a leading term $M_{i} \ldots$ is combined with a BRST completion made of multiparticle currents $M_{i E}$ (with $E \neq \emptyset$ ) and ghost-number-two objects. Note that $C_{i \mid A, B, C}$ and $C_{i \mid A, B, C, D}^{m}$ are totally symmetric in $A, B, C$ and $A, B, C, D$ which follows a general convention used throughout this work: Whenever multiparticle slots $A, B$ in a subscript are separated by a comma rather than by a vertical bar as in $\ldots, A \mid B, \ldots$, then the parental object is understood to be symmetric in $A \leftrightarrow B$.

In [32], the following two observations were exploited to set up a recursive construction of BRST invariants in (2.32) and (2.33):
(i) Nilpotency $Q^{2}=0$ implies that also $Q M_{A, B, C}$ and $Q M_{A, B, C, D}^{m}$ as given by (2.27) and (2.31) are BRST closed. By promoting each $M_{i} M_{A, B, C}$ and $M_{i} M_{A, B, C, D}^{m}$ therein
to the corresponding invariant $C_{i \mid A, B, C}$ and $C_{i \mid A, B, C, D}^{m}$, one arrives at an alternative form of the BRST transformations ${ }^{10}$,

$$
\begin{align*}
Q M_{A, B, C}= & C_{a_{1} \mid a_{2} \ldots a_{|A|}, B, C}-C_{a_{|A|} \mid a_{1} \ldots a_{|A|-1}, B, C}+(A \leftrightarrow B, C)  \tag{2.35}\\
Q M_{A, B, C, D}^{m}= & C_{a_{1} \mid a_{2} \ldots a_{|A|}, B, C, D}^{m}-C_{a_{|A|} \mid a_{1} \ldots a_{|A|-1}, B, C, D}^{m} \\
& +\delta_{|A|, 1} k_{a_{1}}^{m} C_{a_{1} \mid B, C, D}+(A \leftrightarrow B, C, D) . \tag{2.36}
\end{align*}
$$

Note that $\delta_{|A|, 1}$ is equal to one when the word $A$ represents a single particle (i.e. $|A|=1)$ and zero otherwise.
(ii) We define a linear concatenation operator

$$
\begin{equation*}
M_{B} \otimes_{a_{1}} M_{a_{1} a_{2} \ldots a_{|A|}} \equiv M_{B a_{1} a_{2} \ldots a_{|A|}} \tag{2.37}
\end{equation*}
$$

acting on $M_{A}$ which does not interfere with ghost-number-two superfields such as $M_{A, B, C}$ or $M_{A, B, C, D}^{m}$. The deconcatenation formula (2.21) for $Q M_{A}$ and (2.34) imply

$$
\begin{align*}
Q\left(M_{i} \otimes_{j} C_{j \mid A, B, C}\right) & =M_{i} C_{j \mid A, B, C}  \tag{2.38}\\
Q\left(M_{i} \otimes_{j} C_{j \mid A, B, C, D}^{m}\right) & =M_{i} C_{j \mid A, B, C, D}^{m} \tag{2.39}
\end{align*}
$$

see subsection 3.3 for a more detailed and general derivation.
On these grounds, one can show that the following recursions generate BRST invariants for arbitrary multiplicity:

$$
\begin{align*}
C_{i \mid A, B, C}= & M_{i} M_{A, B, C}
\end{align*}+M_{i} \otimes\left[C_{a_{1} \mid a_{2} \ldots a_{|A|}, B, C}-C_{a_{|A|} \mid a_{1} \ldots a_{|A|-1}, B, C}+(A \leftrightarrow B, C)\right]
$$

Once the leading terms $Q\left(M_{i} M_{A, B, C}\right)$ and $Q\left(M_{i} M_{A, B, C, D}^{m}\right)$ are evaluated via (i), they are easily seen to cancel the BRST variations (ii) of the concatenated terms.

In (2.40) as well as later equations in this paper, the subscript $j$ of the concatenation $\otimes_{j}$ in (2.37) is suppressed and understood to match the reference leg $j$ of subsequent kinematic factor such as $C_{j \mid \ldots}$ or $C_{j \mid \ldots}^{m}$. In principle, $\otimes$ without further specification does

10 At this point, uniqueness of the BRST completions in (2.32) and (2.33) is assumed. We don't have a rigorous argument to prove this in full generality but rely on "experimental" evidence at finite multiplicities.
not preserve the Kleiss-Kuijf symmetries (2.24) of the Berends-Giele currents ${ }^{11}$. However, this slight ambiguity does not matter in the recursive formulas (2.40) as long as the objects generated by the recursion are directly used in the next steps without any prior symmetry manipulations.

The simplest instances of scalar and vector invariants following from the recursions in (2.40) are

$$
\begin{align*}
C_{1 \mid 2,3,4} & \equiv M_{1} M_{2,3,4}  \tag{2.41}\\
C_{1 \mid 23,4,5} & \equiv M_{1} M_{23,4,5}+M_{1} \otimes\left[C_{2 \mid 3,4,5}-C_{3 \mid 2,4,5}\right] \\
& =M_{1} M_{23,4,5}+M_{12} M_{3,4,5}-M_{13} M_{2,4,5} \\
C_{1 \mid 2,3,4,5}^{m} & \equiv M_{1} M_{2,3,4,5}^{m}+M_{1} \otimes\left[k_{2}^{m} C_{2 \mid 3,4,5}+(2 \leftrightarrow 3,4,5)\right] \\
& =M_{1} M_{2,3,4,5}^{m}+\left[k_{2}^{m} M_{12} M_{3,4,5}+(2 \leftrightarrow 3,4,5)\right] .
\end{align*}
$$

The corresponding six-point examples are expanded in appendix A, see (A.1). Seven- and eight-point examples of $C_{i \mid A, B, C}$ can be found in [24].

The families of scalar and vector invariants $C_{i \mid A, B, C}, C_{i \mid A, B, C, D}^{m}$ as well as their recursive construction in (2.40) furnish the first two cells from the left in fig. 1.

## 3. Towards a BRST pseudo-cohomology

In this section, we investigate the applicability of the BRST program in sections 2.4 and 2.5 to tensorial building blocks. The one-loop prescription of the pure spinor formalism [4] suggests a natural two-tensor generalization $M_{\ldots . .}^{m n}$ of the $M_{A, B, C}$ and $M_{A, B, C, D}^{m}$ above, but its BRST variation turns out to involve new classes of superfields. This obstruction is closely related to the pure spinor superspace description of the hexagon anomaly [26]. It leads us to define a pseudo-cohomology as an extension of the standard cohomology in order to systematically study the multiparticle superfields which play a role in the gauge anomaly of open superstring amplitudes and its cancellation [25].

[^1]
### 3.1. Tensorial building blocks $M^{m n}$

Higher-point loop amplitudes in the closed-string allow for an arbitrary number of $\Pi^{m}$ zero mode contractions between left- and right-movers. This motivates the study of higher-rank tensors generalizing (2.30) such as

$$
\begin{align*}
M_{A, B, C, D, E}^{m n} \equiv & 2\left[\mathcal{A}_{A}^{(m} \mathcal{A}_{B}^{n)} M_{C, D, E}+(A, B \mid A, B, C, D, E)\right] \\
& +2\left[\mathcal{A}_{A}^{(m} \mathcal{W}_{B, C, D, E}^{n)}+(A \leftrightarrow B, C, D, E)\right] \\
= & \mathcal{A}_{A}^{m} \mathcal{W}_{B, C, D, E}^{n}+\mathcal{A}_{A}^{n} M_{B, C, D, E}^{m}+(A \leftrightarrow B, C, D, E) \tag{3.1}
\end{align*}
$$

firstly relevant for the six-point amplitude. Its first term $\sim \mathcal{A}_{A}^{(m} \mathcal{A}_{B}^{n)}$ stems from the $\Pi^{m} \Pi^{n} d_{\alpha} d_{\beta} N_{p q}$ zero-mode coefficient and its second term $\sim \mathcal{A}_{A}^{(m} \mathcal{W}_{B, C, D, E}^{n)}$ originates from the $b$-ghost sector linear in $\Pi^{m}$. The BRST variations (2.22), (2.29) and (2.31) for its constituents imply that

$$
\begin{align*}
Q M_{A, B, C, D, E}^{m n}= & \delta^{m n} \mathcal{Y}_{A, B, C, D, E}+\left[\sum_{X Y=A}\left(M_{X} M_{Y, B, C, D, E}^{m n}-M_{Y} M_{X, B, C, D, E}^{m n}\right)\right. \\
& \left.+2 k_{A}^{(m} M_{A} M_{B, C, D, E}^{n)}+(A \leftrightarrow B, C, D, E)\right] \tag{3.2}
\end{align*}
$$

where the first term is a shorthand for

$$
\begin{equation*}
\mathcal{Y}_{A, B, C, D, E} \equiv \frac{1}{2}\left(\lambda \gamma^{m} \mathcal{W}_{A}\right)\left(\lambda \gamma^{n} \mathcal{W}_{B}\right)\left(\lambda \gamma^{p} \mathcal{W}_{C}\right)\left(\mathcal{W}_{D} \gamma_{m n p} \mathcal{W}_{E}\right) \tag{3.3}
\end{equation*}
$$

The superfield $\mathcal{Y}_{A, B, C, D, E}$ has ghost-number three and is totally symmetric in $A, B, C, D, E$ due to the pure spinor constraint. It stems from the term $\left(\lambda \gamma^{(m} \mathcal{W}_{A}\right) \mathcal{W}_{B, C, D, E}^{n)}$ in $Q\left(\mathcal{A}_{A}^{(m} \mathcal{W}_{B, C, D, E}^{n)}\right)$ where a group-theoretic analysis ${ }^{12}$ has been used to replace

$$
\begin{equation*}
\left(\lambda \gamma_{p}\right)_{\left[\alpha_{1}\right.}\left(\lambda \gamma_{q}\right)_{\alpha_{2}}\left(\lambda \gamma^{(m}\right)_{\alpha_{3}} \gamma_{\left.\alpha_{4} \alpha_{5}\right]}^{n) p q}=\frac{1}{10} \delta^{m n}\left(\lambda \gamma_{p}\right)_{\left[\alpha_{1}\right.}\left(\lambda \gamma_{q}\right)_{\alpha_{2}}\left(\lambda \gamma_{r}\right)_{\alpha_{3}} \gamma_{\left.\alpha_{4} \alpha_{5}\right]}^{p q r} \tag{3.4}
\end{equation*}
$$

Apart from the extra term $\mathcal{Y}_{A, B, C, D, E}$, the BRST variation (3.2) of $M_{A, B, C, D, E}^{m n}$ is a direct rank-two generalization of $Q M_{A, B, C, D}^{m}$ given in (2.31).

12 The spinors indices in $\left(\lambda \gamma_{p}\right)_{\left[\alpha_{1}\right.}\left(\lambda \gamma_{q}\right)_{\alpha_{2}}\left(\lambda \gamma^{(m}\right)_{\alpha_{3}} \gamma_{\left.\alpha_{4} \alpha_{5}\right]}^{n) p q}$ fall into the tensor product of $[0,0,0,0,3] \ni \lambda^{3}$ and $[0,0,0,0,1]^{\wedge 5}=[0,0,0,3,0] \oplus[1,1,0,1,0] \ni W^{5}$. The LiE program [41] identifies one scalar $[0,0,0,0,0]$ but no symmetric and traceless [2, $0,0,0,0]$ component in $[0,0,0,0,3] \otimes[0,0,0,0,1]^{\wedge 5}$. Hence, only the trace with respect to vector indices $m, n$ contributes. We are using standard Dynkin label notation $\left[a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right]$ for $S O(10)$ irreducibles and denote an antisymmetrized $k^{\text {th }}$ tensor power by $\left[a_{1}, a_{2}, a_{3}, a_{4}, a_{5}\right]^{\wedge k}$.

### 3.2. Pseudo BRST cohomology

The pure spinor analysis of the hexagon gauge anomaly [25] showed that the gauge variation of the open superstring six-point amplitude (w.r.t. leg one) is proportional to [26],

$$
\begin{equation*}
\mathcal{Y}_{2,3,4,5,6}=\frac{1}{2}\left(\lambda \gamma^{m} W_{2}\right)\left(\lambda \gamma^{n} W_{3}\right)\left(\lambda \gamma^{p} W_{4}\right)\left(W_{5} \gamma_{m n p} W_{6}\right) \tag{3.5}
\end{equation*}
$$

Since gauge invariance is related to BRST invariance of the kinematic factors in the amplitudes (see appendix B), terms of the form (3.3) are expected to describe the BRST anomaly of the amplitudes. They represent obstructions in finding elements in the cohomology.

Therefore it will be convenient to define a pseudo BRST cohomology in which the variation of pseudo BRST-closed elements vanish up to anomalous superfields such as (3.3). These objects give rise to gauge transformations with bosonic components proportional to the $\epsilon_{10}$ tensor, see appendix B.5, so they are suitable to describe the parity odd gauge anomaly of the open superstring. Indeed, it will be shown in [29] that the scalar pseudo BRST cohomology for six particles discussed in the next section correctly describes the anomaly terms of the six-point one-loop amplitude which were not included in the discussion of [24].

Definition 1. A superfield $\mathcal{Y}$ of ghost-number three or four is called anomalous if it contains a factor of $\mathcal{Y}_{A, B, C, D, E}$ as in (3.3) with some multiparticle labels $A, B \ldots, E$.

Definition 2. Superfields of ghost-number two and three are called pseudo-invariant if their BRST variation is entirely anomalous. The space of pseudo-invariants is referred to as the pseudo-cohomology.

### 3.3. Tools for constructing BRST pseudo-invariants

The subsequent sections are concerned with a systematic construction of BRST pseudoinvariants. As a driving force for this endeavor, we generalize the recursions of section 2.5 to situations with anomalous BRST variations.

Lemma 1. Let $\mathcal{J}$ denote any superfield unaffected by the operation $\otimes$ defined in (2.37). Then, concatenations of $M_{A} \mathcal{J}$ satisfy

$$
\begin{equation*}
Q\left(M_{i} \otimes M_{A} \mathcal{J}\right)=M_{i} M_{A} \mathcal{J}+M_{i} \otimes Q\left(M_{A} \mathcal{J}\right) \tag{3.6}
\end{equation*}
$$

Proof. By the deconcatenation formula (2.21) for $Q M_{A}$, one can identify

$$
Q\left(M_{A} \mathcal{J}\right)=\sum_{X Y=A} M_{X} M_{Y} \mathcal{J}-M_{A} Q \mathcal{J}
$$

in the third line of

$$
\begin{aligned}
Q\left(M_{i} \otimes M_{A} \mathcal{J}\right) & =Q\left(M_{i A} \mathcal{J}\right) \\
& =\left\{M_{i} M_{A}+\sum_{X Y=A} M_{i X} M_{Y}\right\} \mathcal{J}-M_{i A} Q \mathcal{J} \\
& =M_{i} M_{A} \mathcal{J}+M_{i} \otimes\left\{\sum_{X Y=A} M_{X} M_{Y} \mathcal{J}-M_{A} Q \mathcal{J}\right\} \\
& =M_{i} M_{A} \mathcal{J}+M_{i} \otimes Q\left(M_{A} \mathcal{J}\right)
\end{aligned}
$$

Corollary 1. Let $\mathcal{C}$ denote a BRST-invariant superspace expression

$$
\begin{equation*}
\mathcal{C}=\sum_{k} M_{A_{k}} \mathcal{J}_{k}, \quad Q \mathcal{C}=0 \tag{3.7}
\end{equation*}
$$

where $\otimes$ acts trivially on the $\mathcal{J}_{k}$, then

$$
\begin{equation*}
Q\left(M_{i} \otimes \mathcal{C}\right)=M_{i} \mathcal{C} \tag{3.8}
\end{equation*}
$$

Proof. Upon applying Lemma 1 to $M_{A_{k}} \mathcal{J}_{k}$, the second term $M_{i} \otimes Q\left(M_{A_{k}} \mathcal{J}_{k}\right)$ in (3.6) builds up $M_{i} \otimes Q \mathcal{C}$ by linearity of $\otimes$ which vanishes by the assumption (3.7).

Note that (2.38) and (2.39) are special cases of (3.8) with $\mathcal{C} \rightarrow C_{j \mid A, B, C}$ and $\mathcal{C} \rightarrow$ $C_{j \mid A, B, C, D}^{m}$, respectively.

Corollary 2. Let $\mathcal{P}$ denote a BRST-pseudoinvariant superspace expression

$$
\begin{equation*}
\mathcal{P}=\sum_{k} M_{A_{k}} \mathcal{J}_{k}, \quad Q \mathcal{P}=\sum_{l} M_{B_{l}} \mathcal{Y}_{l} \tag{3.9}
\end{equation*}
$$

where the $\mathcal{Y}_{l}$ are anomalous and $\otimes$ does not act on $\mathcal{J}_{k}$ or $\mathcal{Y}_{l}$, then the right-hand side of

$$
\begin{equation*}
Q\left(M_{i} \otimes \mathcal{P}\right)=M_{i} \mathcal{P}+M_{i} \otimes Q \mathcal{P} \tag{3.10}
\end{equation*}
$$

is anomalous up to the first term $M_{i} \mathcal{P}$.

Proof. Again, apply Lemma 1 to $M_{A_{k}} \mathcal{J}_{k}$, and the second term $M_{i} \otimes Q\left(M_{A_{k}} \mathcal{J}_{k}\right)$ in (3.6) builds up the expression $M_{i} \otimes Q \mathcal{P}$. The latter is anomalous by (3.9) since $M_{i} \otimes$ action does not alter the anomaly nature of the $M_{B_{l}} \mathcal{Y}_{l}$ in $Q \mathcal{P}$.

### 3.4. Rank-two example of pseudo-cohomology

As a first example of BRST pseudo-invariants, we derive a rank-two analogue of the recursions in (2.40) for scalar and vectorial BRST invariants. According to the anomalous transformation (3.2) of the tensorial building block $M_{\ldots n}^{m n}$ in (3.1), the resulting tensors $C_{i \mid A, B, C, D, E}^{m n}$ can at best be pseudo-invariant in the sense of Definition 2.

BRST pseudo-completions of rank-two tensors originate from an ansatz

$$
\begin{equation*}
C_{i \mid A, B, C, D, E}^{m n} \equiv M_{i} M_{A, B, C, D, E}^{m n}+\sum_{F \neq \emptyset} M_{i F} \cdots \tag{3.11}
\end{equation*}
$$

similar to (2.32) and (2.33). Apart from the leading term $M_{i} M_{A, B, C, D, E}^{m n}$, particle $i$ always appears in a multiparticle word $i F$, and the ellipsis represents tensor superfields of the form $M_{A, B, C, D, E}^{m n}, k_{A}^{(m} M_{B, C, D, E}^{n}$ or $k_{A}^{(m} k_{B}^{n)} M_{C, D, E}$.

Similar to the expressions (2.35) and (2.36) for $Q M_{A, B, C}$ and $Q M_{A, B, C, D}^{m}$, one can express $Q M_{A, B, C, D, E}^{m n}$ given in (3.2) in terms of pseudoinvariants: Each term containing a factor of $M_{i}$ in (3.2) signals the leading term of a (pseudo-)invariant, hence:

$$
\begin{align*}
Q M_{A, B, C, D, E}^{m n} & =\delta^{m n} \mathcal{Y}_{A, B, C, D, E}+\left[C_{a_{1} \mid a_{2} \ldots a_{|A|}, B, C, D, E}^{m n}-C_{a_{|A|} \mid a_{1} \ldots a_{|A|-1}, B, C, D, E}^{m n}\right. \\
& \left.+\delta_{|A|, 1}\left(k_{a_{1}}^{m} C_{a_{1} \mid B, C, D, E}^{n}+k_{a_{1}}^{n} C_{a_{1} \mid B, C, D, E}^{m}\right)+(A \leftrightarrow B, C, D, E)\right] . \tag{3.12}
\end{align*}
$$

This motivates the following recursion for pseudo-invariants $C_{i \mid A, B, C, D, E}^{m n}$,

$$
\begin{align*}
C_{i \mid A, B, C, D, E}^{m n} & =M_{i} M_{A, B, C, D, E}^{m n}+M_{i} \otimes\left[C_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B, C, D, E}^{m n}-C_{a_{|A|} \mid a_{1} \ldots a_{|A|-1}, B, C, D, E}^{m n}\right. \\
& \left.+\delta_{|A|, 1}\left(k_{a_{1}}^{m} C_{a_{1} \mid B, C, D, E}^{n}+k_{a_{1}}^{n} C_{a_{1} \mid B, C, D, E}^{m}\right)+(A \leftrightarrow B, C, D, E)\right] . \tag{3.13}
\end{align*}
$$

Their BRST variation is purely anomalous by (3.12) and (3.10) at $\mathcal{P} \rightarrow C_{\ldots}^{m n}$. The simplest example occurs at six points and uses the expression (2.41) for $C_{2 \mid 3,4,5,6}^{m}$,

$$
\begin{align*}
C_{1 \mid 2,3,4,5,6}^{m n} & =M_{1} M_{2,3,4,5,6}^{m n}+\left[M_{1} \otimes\left(k_{2}^{m} C_{2 \mid 3,4,5,6}^{n}+k_{2}^{n} C_{2 \mid 3,4,5,6}^{m}\right)+(2 \leftrightarrow 3,4,5,6)\right] \\
& =M_{1} M_{2,3,4,5,6}^{m n}+\left[k_{2}^{m} M_{12} M_{3,4,5,6}^{n}+k_{2}^{n} M_{12} M_{3,4,5,6}^{m}+(3 \leftrightarrow 4,5,6)\right]  \tag{3.14}\\
& +\left[\left(k_{2}^{m} k_{3}^{n}+k_{2}^{n} k_{3}^{m}\right)\left(M_{123}+M_{132}\right) M_{4,5,6}+(2,3 \mid 2,3,4,5,6)\right] .
\end{align*}
$$

The seven-point analogue $C_{1 \mid 23,4,5,6,7}^{m n}$ is displayed in appendix A, see (A.2).
One can explicitly check their BRST pseudo-invariant nature at low orders,

$$
\begin{align*}
Q C_{1 \mid 2,3,4,5,6}^{m n} & =-\delta^{m n} M_{1} \mathcal{Y}_{2,3,4,5,6}  \tag{3.15}\\
Q C_{1 \mid 23,4,5,6,7}^{m n} & =-\delta^{m n}\left(M_{1} \mathcal{Y}_{23,4,5,6,7}+M_{12} \mathcal{Y}_{3,4,5,6,7}-M_{13} \mathcal{Y}_{2,4,5,6,7}\right) \tag{3.16}
\end{align*}
$$

The general structure of $Q C_{i \mid A, B, C, D, E}^{m n}=-\delta^{m n}(\ldots)$ will be discussed in section 7.2. Note that traceless components are BRST-closed,

$$
\begin{equation*}
Q\left(C_{i \mid A, B, C, D, E}^{m n}-\frac{1}{10} \delta^{m n} C_{i \mid A, B, C, D, E}^{p p}\right)=0 \tag{3.17}
\end{equation*}
$$

The family of two-tensor pseudo-invariants constructed via (3.13) furnishes the third cell in the leading diagonal of the overview grid in fig. 1. As we will see in the next section, the tools of this section enable to address an infinite tower of higher-rank generalizations to complete this diagonal.

## 4. Tensor pseudo-cohomology

This section introduces a recursive method to construct higher-rank generalizations of the scalar, vector and two-tensor BRST (pseudo-)invariants discussed in the previous sections. As shown below, $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ for $r \geq 2$ is a pseudo-invariant according to Definition 2.

### 4.1. Higher-rank building blocks and anomaly blocks

Following the logic behind the two-tensor in (3.1), we define a building block of arbitrary rank $r$ by extracting zero modes of $\Pi^{m_{1}} \ldots \Pi^{m_{r}} d_{\alpha} d_{\beta} N_{p q}$ from $r+3$ integrated vertex operators in their multiparticle Berends-Giele version. Similarly, the $b$-ghost sector proportional to the zero-mode of $\Pi^{m}$ gives rise to a second sort of superfield with a factor of $\mathcal{W}_{A, B, C, D}^{m}$ :

$$
\begin{align*}
M_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} & \equiv r!\left[M_{B_{1}, B_{2}, B_{3}} \mathcal{A}_{B_{4}}^{\left(m_{1}\right.} \mathcal{A}_{B_{5}}^{m_{2}} \ldots \mathcal{A}_{B_{r+3}}^{\left.m_{r}\right)}+\left(B_{1}, B_{2}, B_{3} \mid B_{1}, B_{2}, \ldots, B_{r+3}\right)\right] \\
& +r!\left[\mathcal{W}_{B_{1}, B_{2}, B_{3}, B_{4}}^{\left(m_{1}\right.} \mathcal{A}_{B_{5}}^{m_{2}} \ldots \mathcal{A}_{\left.B_{r+3}\right)}^{m_{r}}+\left(B_{1}, \ldots, B_{4} \mid B_{1}, B_{2}, \ldots, B_{r+3}\right)\right] \tag{4.1}
\end{align*}
$$

In order to get a recursive handle on the combinatorics in (4.1), it is convenient to define higher-rank versions of $\mathcal{W}_{A, B, C, D}^{m}$ in (2.28),

$$
\begin{equation*}
\mathcal{W}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}} \equiv \mathcal{A}_{B_{1}}^{m_{1}} \mathcal{W}_{B_{2}, \ldots, B_{r+3}}^{m_{2} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right) . \tag{4.2}
\end{equation*}
$$

Then, the rank- $r$ building block $M_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ in (4.1) can be written recursively as

$$
\begin{align*}
M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} & =\mathcal{A}_{B_{1}}^{m_{1}} M_{B_{2}, \ldots, B_{r+3}}^{m_{2} \ldots m_{r}}+\mathcal{A}_{B_{1}}^{m_{r}} \mathcal{W}_{B_{2}, \ldots, B_{r+3}}^{m_{r-1} \ldots m_{2} \mid m_{1}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)  \tag{4.3}\\
& =\left[\mathcal{A}_{1}^{m_{1}} M_{B_{2}, \ldots, B_{r+3}}^{m_{2} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+3}\right)\right]+\mathcal{W}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{r} m_{r-1} \ldots m_{2} \mid m_{1}},
\end{align*}
$$

for example (3.1) at rank two and

$$
\begin{align*}
& M_{B_{1}, \ldots, B_{6}}^{m n p}=\mathcal{A}_{B_{1}}^{m} M_{B_{2}, \ldots, B_{6}}^{n p}+\mathcal{A}_{B_{1}}^{p} \mathcal{W}_{B_{2}, \ldots, B_{6}}^{n \mid m}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{6}\right)  \tag{4.4}\\
& M_{B_{1}, \ldots, B_{7}}^{m n n p q}=\mathcal{A}_{B_{1}}^{m} M_{B_{2}, \ldots, B_{7}}^{n p q}+\mathcal{A}_{B_{1}}^{q} \mathcal{W}_{B_{2}, \ldots, B_{7}}^{p n \mid m}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{7}\right)
\end{align*}
$$

at $r=3,4$. Note that $\mathcal{W}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}$ defined in (4.2) is symmetric in all its slots $B_{i}$ but only in its first $r-1$ vector indices $m_{i}$. That explains the notation $\ldots m_{r-1} \mid m_{r}$ in (4.2).

Also the scalar anomaly building block $\mathcal{Y}_{A, B, C, D, E}$ defined in (3.3) has a natural higher-rank generalization. It can be defined explicitly in analogy to (4.1),

$$
\begin{equation*}
\mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}} \equiv r!\mathcal{Y}_{B_{1}, \ldots, B_{5}} \mathcal{A}_{B_{6}}^{\left(m_{1}\right.} \mathcal{A}_{B_{7}}^{m_{2}} \ldots \mathcal{A}_{\left.B_{r+5} m_{r}\right)}^{m_{r}}+\left(B_{1}, \ldots, B_{5} \mid B_{1}, \ldots, B_{r+5}\right), \tag{4.5}
\end{equation*}
$$

or recursively like (4.3),

$$
\begin{equation*}
\mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}} \equiv \mathcal{A}_{B_{1}}^{m_{1}} \mathcal{Y}_{B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{2} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+5}\right) . \tag{4.6}
\end{equation*}
$$

Even though the recursion (4.6) for anomaly blocks resembles (4.2) for $\mathcal{W}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}$, the vector indices of the latter are not entirely carried by $\mathcal{A}_{B}^{m}$ superfields. That is why only $\mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$ is totally symmetric in both $B_{i}$ and $m_{i}$.

### 4.2. Anomalous BRST variations at higher rank

Both expressions (4.1) and (4.3) for higher-rank building blocks serve as a starting point to determine their BRST variation

$$
\begin{align*}
& Q M_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{\left.m_{3} \ldots m_{r}\right)}+\left[r M_{B_{1}} k_{B_{1}}^{\left(m_{1}\right.} M_{B_{2}, B_{3}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}\right.  \tag{4.7}\\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} M_{Y, B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}-M_{Y} M_{X, B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right] .
\end{align*}
$$

The $\delta^{m n}$ tensors in the anomalous part are due to the group-theory identity (3.4). The rank-two example has been given in (3.2), and ranks three and four give rise to

$$
\begin{align*}
& Q M_{B_{1}, B_{2}, \ldots, B_{6}}^{m n p}=3 \delta^{(m n} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{6}}^{p)}+\left[3 M_{B_{1}} k_{B_{1}}^{(m} M_{B_{2}, B_{3}, \ldots, B_{6}}^{n p)}\right.  \tag{4.8}\\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} M_{Y, B_{2}, B_{3}, \ldots, B_{6}}^{m n p}-M_{Y} M_{X, B_{2}, B_{3}, \ldots, B_{6}}^{m n p}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{6}\right)\right] \\
& Q M_{B_{1}, B_{2}, \ldots, B_{7}}^{m n n q}=6 \delta^{(m n} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{7}}^{p q)}+\left[4 M_{B_{1}} k_{B_{1}}^{(m} M_{B_{2}, B_{3}, \ldots, B_{7}}^{n p q)}\right.  \tag{4.9}\\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} M_{Y, B_{2}, B_{3}, \ldots, B_{7}}^{m n p q}-M_{Y} M_{X, B_{2}, B_{3}, \ldots, B_{7}}^{m n p q}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{7}\right)\right] .
\end{align*}
$$

The recursive approach makes use of the BRST variation

$$
\begin{gather*}
Q \mathcal{W}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}=(r-1) \delta^{m_{r}\left(m_{1}\right.} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r-1}\right)}  \tag{4.10}\\
+\left[(r-1) M_{B_{1}} k_{B_{1}}^{\left(m_{1}\right.} \mathcal{W}_{B_{2}, B_{3}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r-1}\right) \mid m_{r}}-\left(\lambda \gamma^{m_{r}} \mathcal{W}_{B_{1}}\right) M_{B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1}}\right. \\
\left.+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{W}_{Y, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}-M_{Y} \mathcal{W}_{X, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right],
\end{gather*}
$$

which generalizes the rank-one variation (2.29) and specializes as follows at rank $r \leq 3$,

$$
\begin{align*}
& Q \mathcal{W}_{B_{1}, B_{2}, \ldots, B_{5}}^{m \mid n}=\delta^{m n} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{5}}+\left[k_{B_{1}}^{m} M_{B_{1}} \mathcal{W}_{B_{2}, B_{3}, B_{4}, B_{5}}^{n}-\left(\lambda \gamma^{n} \mathcal{W}_{B_{1}}\right) M_{B_{2}, B_{3}, B_{4}, B_{5}}^{m}\right. \\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{W}_{Y, B_{2}, \ldots, B_{5}}^{m \mid n}-M_{Y} \mathcal{W}_{X, B_{2}, \ldots, B_{5}}^{m \mid n}\right)+\left(B_{1} \leftrightarrow B_{2}, B_{3}, B_{4}, B_{5}\right)\right]  \tag{4.11}\\
& Q \mathcal{W}_{B_{1}, B_{2}, \ldots, B_{6}}^{m n \mid p}=2 \delta^{p(m} \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{6}}^{n)}+\left[2 k_{B_{1}}^{(m} M_{B_{1}} \mathcal{W}_{B_{2}, \ldots, B_{6}}^{n) \mid p}-\left(\lambda \gamma^{p} \mathcal{W}_{B_{1}}\right) M_{B_{2}, B_{3}, B_{4}, B_{5}, B_{6}}^{m n}\right. \\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{W}_{Y, B_{2}, \ldots, B_{6}}^{m n \mid p}-M_{Y} \mathcal{W}_{X, B_{2}, \ldots, B_{6}}^{m n \mid p}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{6}\right)\right] . \tag{4.12}
\end{align*}
$$

### 4.3. Recursion for higher rank pseudoinvariants

The construction of general BRST pseudo-invariants

$$
\begin{equation*}
C_{i \mid A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}} \equiv M_{i} M_{A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}+\sum_{B \neq \emptyset} M_{i B} \ldots \tag{4.13}
\end{equation*}
$$

generalizes the scalars (2.32), vectors (2.33) and two-tensors (3.11) to arbitrary rank. As before, the leading term $M_{i} M_{A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ is the only instance where the reference leg $i$ enters through a single-particle vertex operator $M_{i}$. The ellipsis along with multiparticle $M_{i B}$ takes the form $k_{A_{1}}^{\left(m_{1}\right.} \ldots k_{A_{j}}^{m_{j}} M_{A_{j+1}, \ldots, A_{r+3}}^{\left.m_{j+1}, m_{r}\right)}$ with $j=0,1, \ldots, r$. The role of $M_{i}$ as defining the pseudoinvariant (4.13) leads to the following alternative form of (4.7):

$$
\begin{align*}
& Q M_{A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{A_{1}, A_{2}, \ldots, A_{r+3}}^{\left.m_{3} \ldots m_{r}\right)}+\left\{r \delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{\left(m_{1}\right.} C_{a_{1} \mid A_{2}, \ldots, A_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}\right.  \tag{4.14}\\
& \left.+C_{a_{1} \mid a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}-C_{a_{\left|A_{1}\right|} \mid a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+3}\right)\right\} .
\end{align*}
$$

This in turn gives rise to a recursion for the pseudo-invariants $C_{i \mid A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ in terms of lower-multiplicity representatives of rank $r$ and $r-1$,

$$
\begin{align*}
& \quad C_{i \mid A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}=M_{i} M_{A_{1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}+M_{i} \otimes\left\{r \delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{\left(m_{1}\right.} C_{a_{1} \mid A_{2}, \ldots, A_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}\right.  \tag{4.15}\\
& \left.+C_{a_{1} \mid a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{r+3}}^{m_{1}}-C_{a_{\left|A_{1}\right|} \mid a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+3}\right)\right\},
\end{align*}
$$

which reduces to (2.40) and (3.13) for $r \leq 2$. BRST pseudoinvariance follows from (3.10) at $\mathcal{P} \rightarrow C_{\ldots}^{m_{1} m_{2} \ldots m_{r}}$. The anomalous BRST variations entirely reside in trace components $\sim \delta^{m_{i} m_{j}}$ and will be systematically discussed in section 7.2. Similar as before, the traceless components are BRST invariant, e.g.

$$
\begin{equation*}
Q\left(C_{i \mid A, B, C, D, E, F}^{m n p}-\frac{1}{4} \delta^{(m n} C_{i \mid A, B, C, D, E, F}^{p) q q}\right)=0 . \tag{4.16}
\end{equation*}
$$

The simplest pseudoinvariant of rank greater than two is $C_{1 \mid 2,3,4,5,6,7}^{m n p}$, its expansion is displayed in appendix A, see (A.3).

The recursion (4.15) for pseudo-invariants of arbitrary rank completes the leading diagonal of the overview grid in fig. 1. In the next sections we explore the building blocks and recursions governing the subleading diagonals.

## 5. Towards a refined pseudo-cohomology

The discussion of BRST invariance of the closed-string five-point amplitude in [28] naturally led to consider the following combination of superfields ${ }^{13}$

$$
\begin{equation*}
k_{1}^{m} V_{1} T_{2,3,4,5}^{m}+\left[V_{12} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]+Y_{1,2,3,4,5} \tag{5.1}
\end{equation*}
$$

which was shown to be BRST exact in the appendix B of [28]. Given the appearance of the anomalous superfield $Y_{1,2,3,4,5}$, it will not be surprising to discover that this particular combination (5.1) signals a much deeper family of pseudo cohomology elements which will play an important role in the discussion of anomalous terms in the one-loop open superstring amplitudes [29].

[^2]
### 5.1. Refined currents

It turns out that to extend and generalize the discussion of (5.1) it is convenient to define the following superfield

$$
\begin{equation*}
\widehat{J}_{A \mid B, C, D, E} \equiv \frac{1}{2}\left(A_{A}^{m} T_{B, C, D, E}^{m}+A_{A}^{m} W_{B, C, D, E}^{m}\right) \tag{5.2}
\end{equation*}
$$

symmetric in $B, C, D, E$. In view of the special role of the first slot $A$, we refer to such objects as refined currents. Accordingly, any slot $A \mid \ldots$ on the left of the vertical bar of the subscript will be referred to as refined. It is not hard to check that the simplest case $\widehat{J}_{1 \mid 2,3,4,5}$ gives rise to (5.1) under $Q$ variation and that higher-multiplicity currents satisfy

$$
\begin{align*}
Q \widehat{J}_{1 \mid 2,3,4,5}= & k_{1}^{m} V_{1} T_{2,3,4,5}^{m}+\left[V_{12} T_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]+Y_{1,2,3,4,5}  \tag{5.3}\\
Q \widehat{J}_{12 \mid 3,4,5,6}= & k_{12}^{m} V_{12} T_{3,4,5,6}^{m}+\left[\widehat{V}_{123} T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right]+Y_{12,3,4,5,6} \\
& +s_{12}\left(V_{1} \widehat{J}_{2 \mid 3,4,5,6}-V_{2} \widehat{J}_{1 \mid 3,4,5,6}\right)  \tag{5.4}\\
Q \widehat{J}_{1 \mid 23,4,5,6}= & k_{1}^{m} V_{1} T_{23,4,5,6}^{m}-\widehat{V}_{231} T_{4,5,6}+\left[V_{14} T_{23,5,6}+(4 \leftrightarrow 5,6)\right] \\
& +Y_{1,23,4,5,6}+s_{23}\left(V_{2} \widehat{J}_{1 \mid 3,4,5,6}-V_{3} \widehat{J}_{1 \mid 2,4,5,6}\right) . \tag{5.5}
\end{align*}
$$

Both $V_{12}$ in (5.3) and the hatted superfields $\widehat{V}_{A}$ in (5.4) and (5.5) build up through the recursions (2.8) and (2.10) for $A_{12}^{\alpha}$ and $\widehat{A}_{123}^{\alpha}$. More generally, the recursion (2.15) relates the multiparticle spinor superpotential $A_{\alpha}^{B}$ to BRST blocks $K_{C}$ at lower multiplicity $|C|<|B|$ which are generated by $Q \widehat{J}_{A \mid B, C, D, E}$. However, the direct output $\widehat{A}_{\alpha}^{B}$ of the recursion requires redefinitions by BRST trivial components $H_{12 \ldots p} \equiv H_{[12 \ldots p-1, p]}$ in order to yield the BRST block $A_{\alpha}^{B}$ subject to Lie symmetries, see (2.11) and [32]. The appearance of $\widehat{V}_{B}=\lambda^{\alpha} \widehat{A}_{\alpha}^{B}$ in (5.4) and (5.5) suggests to redefine $\widehat{J}$ by the tensors $H_{i j k} \equiv H_{[i j, k]}$ in (2.12) such that their $Q$ variation can be expressed in terms of the BRST block $V_{B}=\lambda^{\alpha} A_{\alpha}^{B}$, e.g.

$$
\begin{align*}
J_{1 \mid 2,3,4,5} & \equiv \widehat{J}_{1 \mid 2,3,4,5} \\
J_{12 \mid 3,4,5,6} & \equiv \widehat{J}_{12 \mid 3,4,5,6}-\left[H_{[12,3]} T_{4,5,6}+(3 \leftrightarrow 4,5,6)\right],  \tag{5.6}\\
J_{1 \mid 23,4,5,6} & \equiv \widehat{J}_{1 \mid 23,4,5,6}+H_{[23,1]} T_{4,5,6}
\end{align*}
$$

Generalizations $H_{[A, B]}$ of the redefining superfields are explained in appendix D. They give rise to

$$
\begin{equation*}
J_{A \mid B, C, D, E} \equiv \widehat{J}_{A \mid B, C, D, E}-\left[H_{[A, B]} T_{C, D, E}+(A \leftrightarrow B, C, D, E)\right] \tag{5.7}
\end{equation*}
$$

with the understanding that $H_{[A, B]}=0$ for $|A|=|B|=1$ [32]. After the redefinition (5.7), the BRST transformation of $J_{A \mid B, C, D, E}$ contains BRST blocks $V_{X}$ rather than $\widehat{V}_{X}$,

$$
\begin{align*}
Q J_{A \mid B, C, D, E} & =k_{A}^{m} V_{A} T_{B, C, D, E}^{m}+V_{[A, B]} T_{C, D, E}+V_{[A, C]} T_{B, D, E} \\
& +V_{[A, D]} T_{B, C, E}+V_{[A, E]} T_{B, C, D}+Y_{A, B, C, D, E}+\mathcal{O}\left(k_{i} \cdot k_{j}\right) \tag{5.8}
\end{align*}
$$

Appendix C displays the four inequivalent seven-point examples of (5.8), including the contact terms $\mathcal{O}\left(k_{i} \cdot k_{j}\right)$, see (C.4). The latter represent the generalization of $s_{12}\left(V_{1} \widehat{J}_{2 \mid 3,4,5,6}-\right.$ $\left.V_{2} \widehat{J}_{1 \mid 3,4,5,6}\right)$ in (5.4) and $s_{23}\left(V_{2} \widehat{J}_{1 \mid 3,4,5,6}-V_{3} \widehat{J}_{1 \mid 2,4,5,6}\right)$ in (5.5) which simplifies once the $J_{A \mid B, C, D, E}$ in (5.7) are converted to Berends-Giele currents $\mathcal{J}_{A \mid B, C, D, E}$.

The bracket notation $V_{[A, B]}$ has been explained in the appendix A of [32] and can be diagrammatically understood from figure fig. 5. A few explicit examples are as follows ${ }^{14}$,

$$
\begin{gather*}
V_{[1,2]}=V_{12}, \quad V_{[12,3]}=V_{123}, \quad V_{[12 \ldots p-1, p]}=V_{12 \ldots p}  \tag{5.9}\\
V_{[1,23]}=V_{123}-V_{132}=-V_{231}, \quad V_{[12,34]}=V_{1234}-V_{1243}=-V_{3412}+V_{3421} .
\end{gather*}
$$

### 5.2. Berends-Giele version of refined currents

The Berends-Giele version $\mathcal{J}_{A \mid B, C, D, E}$ of refined currents $J_{A \mid B, C, D, E}$ in (5.7) can be obtained by applying the Berends-Giele map discussed in section 2.3 to each of its five slots. The resulting definition ${ }^{15}$

$$
\begin{equation*}
\mathcal{J}_{A \mid B, C, D, E} \equiv \frac{1}{2}\left(\mathcal{A}_{A}^{m} M_{B, C, D, E}^{m}+\mathcal{A}_{A}^{m} \mathcal{W}_{B, C, D, E}^{m}\right)-\left[\mathcal{H}_{[A, B]} M_{C, D, E}+(A \leftrightarrow B, C, D, E)\right] \tag{5.10}
\end{equation*}
$$

incorporates the Berends-Giele version $\mathcal{H}_{[A, B]}$ of the above superfields $H_{[A, B]}$, see appendix D and in particular (D.11) for examples. The contact terms in $Q J_{A \mid B, C, D, E}$ translate into deconcatenations in $Q \mathcal{J}_{A \mid B, C, D, E}$ in the same way as contact terms in $Q V_{B}$ are mapped into the deconcatenation formula (2.21) for $Q M_{B}$. Moreover, $k_{A}^{m} V_{A} T_{B, C, D, E}^{m}$ and $Y_{A, B, C, D, E}$ on the right-hand side of (5.8) can be straightforwardly replaced by $k_{A}^{m} M_{A} M_{B, C, D, E}^{m}$ and $\mathcal{Y}_{A, B, C, D, E}$, respectively. Only the four permutations of $V_{[A, B]} T_{C, D, E}$ require closer inspection since their expansion in terms of $M_{X} M_{C, D, E}$ will introduce explicit Mandelstam variables.
${ }^{14}$ As explained in [32], the multiparticle label $B=b_{1} b_{2} \ldots b_{|B|}$ in a BRST block $K_{B}$ satisfies Lie symmetries. They can be elegantly incorporated by writing $B=\left[\ldots\left[\left[\left[b_{1}, b_{2}\right], b_{3}\right], b_{4}\right], \ldots, b_{|B|}\right]$ and using Jacobi identities for iterated brackets. In particular $B=23$ translates into $B=[2,3]$. Furthermore, the translation from bracketed to non-bracketed labels is given by $K_{[\ldots[[[1,2], 3], 4], \ldots, \mid B]]}=$ $K_{123 \ldots|B|}$, for example $K_{[[1,2], 3]}=K_{123}$.
15 As will become clear in later sections, $\mathcal{J}_{A \mid B, C, D, E}$ should really be denoted $M_{A \mid B, C, D, E}$ since many formulas would acquire a more natural interpretation. However, we use the notation of (5.10) for hysterical reasons.

### 5.2.1. The $S[A, B]$ map

At the superfield level, the recursive definition of BRST blocks in [32] has the structure of a commutator; $V_{[A, B]} \rightarrow\left[V_{A}, V_{B}\right]$. At the level of diagrams, $V_{[A, B]}$ can be interpreted as connecting the off-shell legs in the subdiagrams represented by $V_{A}$ and $V_{B}$ through a cubic vertex, see [32] and fig. 5. Expanding any $V_{C}$ in terms of Berends-Giele currents $M_{C}$ gives rise to a similar diagrammatic interpretation shown in fig. 5, i.e. if two BerendsGiele currents $M_{A}$ and $M_{B}$ are attached to a cubic vertex, the resulting diagram is a linear combination of currents $M_{C}$ at overall multiplicity $|C|=|A|+|B|$. We denote this linear combination by $M_{S[A, B]}$, where the letter $S$ reminds of a factor of $s_{i j}$ which enters on dimensional grounds. In other words, the $S[A, B]$ map captures the difference of applying the Berends-Giele map as described in section 2.3 to the multiparticle label $C$ as a whole as compared to applying it simultaneously and individually to $A$ and $B$, where $|C|=|A|+|B|$,

$$
\begin{equation*}
V_{[A, B]}=\left[V_{A}, V_{B}\right] \quad \Longrightarrow \quad M_{S[A, B]}=\left[M_{A}, M_{B}\right] \tag{5.11}
\end{equation*}
$$

For example, converting both sides of $V_{12}=\left[V_{1}, V_{2}\right]$ to Berends-Giele currents leads to $s_{12} M_{12}=\left[M_{1}, M_{2}\right]$ and therefore $M_{S[1,2]}=s_{12} M_{12}$. Similarly, converting both sides of $V_{123}=\left[V_{12}, V_{3}\right]$ to Berends-Giele currents gives

$$
\begin{equation*}
s_{12}\left(s_{23} M_{123}-s_{13} M_{213}\right)=\left[s_{12} M_{12}, M_{3}\right] \quad \Longrightarrow \quad M_{S[12,3]}=s_{23} M_{123}-s_{13} M_{213} . \tag{5.12}
\end{equation*}
$$

To find $S[1,23]$ one repeats the analysis with $\left[V_{1}, V_{23}\right]=-\left[V_{23}, V_{1}\right]$ and uses the antisymmetry $M_{S[A, B]}=-M_{S[B, A]}$ due to (5.11). Following this procedure one obtains,

$$
\begin{align*}
M_{S[1,2]} & =s_{12} M_{12}  \tag{5.13}\\
M_{S[1,23]} & =s_{12} M_{123}-s_{13} M_{132} \\
M_{S[1,234]} & =s_{12} M_{1234}-s_{13}\left(M_{1324}+M_{1342}\right)+s_{14} M_{1432} \\
M_{S[12,34]} & =-s_{13} M_{2134}+s_{14} M_{2143}+s_{23} M_{1234}-s_{24} M_{1243} .
\end{align*}
$$



Fig. 5 Diagrammatic interpretation of $M_{S[A, B]}$.

It turns out that the general formula for $M_{S[A, B]}$ reads,
$M_{S[A, B]} \equiv \sum_{i=1}^{|A|} \sum_{j=1}^{|B|}(-1)^{i-j+|A|-1} s_{a_{i} b_{j}} M_{\left(a_{1} a_{2} \ldots a_{i-1} ш a_{|A|} a_{|A|-1} \ldots a_{i+1}\right) a_{i} b_{j}\left(b_{j-1} \ldots b_{2} b_{1} ш b_{j+1} \ldots b_{|B|}\right)}$.

The $S[A, B]$ map has been investigated in appendix B of [32] in a different context - it facilitates the expansion of $C_{i \mid A, B, C}$ in terms of SYM tree subamplitudes.

### 5.2.2. The BRST variation of refined currents

Let us compare the $M_{S[A, B]}$ in (5.13) with the BRST transformations of various $\mathcal{J}_{A \mid B, C, D, E}$, starting with the trivial five point case,

$$
\begin{equation*}
Q \mathcal{J}_{1 \mid 2,3,4,5}=k_{1}^{m} M_{1} M_{2,3,4,5}^{m}+\left[s_{12} M_{12} M_{3,4,5}+(2 \leftrightarrow 3,4,5)\right]+\mathcal{Y}_{1,2,3,4,5} \tag{5.15}
\end{equation*}
$$

At six points there are two inequivalent partitions of legs in $\mathcal{J}$ satisfying

$$
\begin{align*}
Q \mathcal{J}_{12 \mid 3,4,5,6}= & k_{12}^{m} M_{12} M_{3,4,5,6}^{m}+\left[\left(s_{23} M_{123}-s_{13} M_{213}\right) M_{4,5,6}+(3 \leftrightarrow 4,5,6)\right] \\
& +\mathcal{Y}_{12,3,4,5,6}+M_{1} \mathcal{J}_{2 \mid 3,4,5,6}-M_{2} \mathcal{J}_{1 \mid 3,4,5,6}  \tag{5.16}\\
Q \mathcal{J}_{1 \mid 23,4,5,6}= & k_{1}^{m} M_{1} M_{23,4,5,6}^{m}+\left(s_{12} M_{123}-s_{13} M_{132}\right) M_{4,5,6}+\left[s_{14} M_{14} M_{23,5,6}+(4 \leftrightarrow 5,6)\right] \\
& +\mathcal{Y}_{1,23,4,5,6}+M_{2} \mathcal{J}_{1 \mid 3,4,5,6}-M_{3} \mathcal{J}_{1 \mid 2,4,5,6} \tag{5.17}
\end{align*}
$$

The four inequivalent seven-point examples are displayed in appendix C, see (C.5).
Using the $S[A, B]$ map in (5.14), we can write down a general formula for the BRST variation of refined currents,

$$
\begin{align*}
& Q \mathcal{J}_{A \mid B, C, D, E}=\mathcal{Y}_{A, B, C, D, E}+k_{A}^{m} M_{A} M_{B, C, D, E}^{m}+\sum_{X Y=A}\left(M_{X} \mathcal{J}_{Y \mid B, C, D, E}-M_{Y} \mathcal{J}_{X \mid B, C, D, E}\right) \\
& \quad+\left[M_{S[A, B]} M_{C, D, E}+\sum_{X Y=B}\left(M_{X} \mathcal{J}_{A \mid Y, C, D, E}-M_{Y} \mathcal{J}_{A \mid X, C, D, E}\right)+(B \leftrightarrow C, D, E)\right] .(5.18) \tag{5.18}
\end{align*}
$$

It is amusing that the anomaly building block $\mathcal{Y}_{A, B, C, D, E}$ is completely symmetric in $A, B, C, D, E$ whereas $\mathcal{J}_{A \mid B, C, D, E}$ has a reduced symmetry in $B, C, D, E$ due to the refined slot $A$. Note that the non-anomalous part of the right-hand side contains the same kind of terms as they appear in $Q M_{A, B, C}$ and $Q M_{A, B, C, D}^{m}$, see (2.27) and (2.31). This allows to assemble superfields $\mathcal{J}_{A \mid B, C, D, E}, k_{F}^{m} M_{A} M_{B, C, D, E}^{m}$ and $s_{i j} M_{A} M_{B, C, D}$ into BRSTpseudoinvariants, the details are worked out in following section.

### 5.3. Scalar pseudo cohomology

Scalar BRST pseudoinvariants involving refined currents $\mathcal{J}_{A \mid B, C, D, E}$ are defined along the lines of the pseudoinvariants $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ in $(4.13)^{16}$,

$$
\begin{equation*}
P_{i|A| B, C, D, E} \equiv M_{i} \mathcal{J}_{A \mid B, C, D, E}+\sum_{F \neq \emptyset} M_{i F} \ldots . \tag{5.19}
\end{equation*}
$$

The leading term $M_{i} \mathcal{J}_{A \mid B, C, D, E}$ furnishes the only instance of a single-particle vertex $M_{i}$ and therefore defines the reference leg $i$ as well as the multiparticle labels of the pseudoinvariant $P_{i|A| B, C, D, E}$. The suppressed terms in the $\ldots$ along with a multiparticle $M_{i F}$ follow from demanding $Q P_{i|A| B, C, D, E}$ to be purely anomalous. We determine them by writing (5.18) in terms of vector invariants from section 2.5 and pseudoinvariants (5.19),

$$
\begin{align*}
Q \mathcal{J}_{A \mid B, C, D, E}= & \mathcal{Y}_{A, B, C, D, E}+\delta_{|A|, 1} k_{a_{1}}^{m} C_{a_{1} \mid B, C, D, E}^{m}  \tag{5.20}\\
& +P_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B, C, D, E}-P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}\right| B, C, D, E} \\
& +\left[P_{b_{1}|A| b_{2} \ldots b_{|B|}, C, D, E}-P_{b_{|B|}|A| b_{1} \ldots b_{|B|-1}, C, D, E}+(B \leftrightarrow C, D, E)\right] .
\end{align*}
$$

16 As will become clear in later sections, $P_{i|A| B, C, D, E}$ should really be denoted $C_{i|A| B, C, D, E}$ since many formulas would acquire a more natural interpretation. However, we use the notation of (5.19) for hysterical reasons.

As usual, any $M_{i}$ in (5.18) has been identified as a leading term of some $C_{i \mid A, B, C, D}^{m}$ or $P_{i|A| B, C, D, E}$, see (2.33) and (5.19). Furthermore, equation (5.20) can be verified once the explicit form of the pseudo-invariants $P_{i|A| B, C, D, E}$ to be determined below is plugged in and the result compared to (5.18). By (3.10), the non-anomalous terms in (5.20) drop out from the BRST variation of the following recursion:

$$
\begin{align*}
P_{i|A| B, C, D, E}= & M_{i} \mathcal{J}_{A \mid B, C, D, E}+M_{i} \otimes\left\{\delta_{|A|, 1} k_{a_{1}}^{m} C_{a_{1} \mid B, C, D, E}^{m}\right. \\
& +P_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B, C, D, E}-P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}\right| B, C, D, E}  \tag{5.21}\\
& \left.+\left[P_{b_{1}|A| b_{2} \ldots b_{|B|}, C, D, E}-P_{b_{|B|}|A| b_{1} \ldots b_{|B|-1}, C, D, E}+(B \leftrightarrow C, D, E)\right]\right\} .
\end{align*}
$$

When applied to the simplest six-point example, the recursion yields

$$
\begin{align*}
P_{1|2| 3,4,5,6} & =M_{1} \mathcal{J}_{2 \mid 3,4,5,6}+k_{2}^{m} M_{1} \otimes C_{2 \mid 3,4,5,6}^{m}  \tag{5.22}\\
& =M_{1} \mathcal{J}_{2 \mid 3,4,5,6}+M_{12} k_{2}^{m} M_{3,4,5,6}^{m}+\left[s_{23} M_{123} M_{4,5,6}+(3 \leftrightarrow 4,5,6)\right]
\end{align*}
$$

and the two inequivalent seven-point analogues are displayed in appendix A, see (A.4) and (A.5). For these simple cases, pseudoinvariance is still easy to check explicitly,

$$
\begin{align*}
Q P_{1|2| 3,4,5,6} & =-M_{1} \mathcal{Y}_{2,3,4,5,6}  \tag{5.23}\\
Q P_{1|23| 4,5,6,7} & =-M_{1} \mathcal{Y}_{23,4,5,6,7}-M_{12} \mathcal{Y}_{3,4,5,6,7}+M_{13} \mathcal{Y}_{2,4,5,6,7} \\
Q P_{1|2| 34,5,6,7} & =-M_{1} \mathcal{Y}_{2,34,5,6,7}-M_{13} \mathcal{Y}_{2,4,5,6,7}+M_{14} \mathcal{Y}_{2,3,5,6,7}
\end{align*}
$$

A general discussion of $Q P_{i|A| B, C, D, E}$ is given in the later section 7.4. Note that the $P_{i|A| B, C, D, E}$ furnish the first cell of the subleading diagonal in the overview grid in fig. 1.

### 5.4. Scalar pseudoinvariants versus tensor traces

The definition (5.10) of the refined current $\mathcal{J}_{A \mid B, C, D, E}$ exhibits a strong similarity to the trace of the two tensor $M_{A, B, C, D, E}^{m n}$ in (3.1). Only the redefinitions by $H_{[A, B]}=-H_{[B, A]}$ terms might pose an obstruction, but their antisymmetry implies that the difference between $\widehat{J}_{A \mid B, C, D, E}$ and $J_{A \mid B, C, D, E}$ in (5.7) drops out upon symmetrization in $A, B, C, D, E$,

$$
\begin{equation*}
\widehat{J}_{A \mid B, C, D, E}-J_{A \mid B, C, D, E}+(A \leftrightarrow B, C, D, E)=0 . \tag{5.24}
\end{equation*}
$$

Similarly, the $\mathcal{H}_{[A, B]}$ corrections in (5.10) cancel out when symmetrizing their corresponding Berends-Giele currents and one gets

$$
\begin{equation*}
\delta_{m n} M_{A, B, C, D, E}^{m n}=2 \mathcal{J}_{A \mid B, C, D, E}+(A \leftrightarrow B, C, D, E) \tag{5.25}
\end{equation*}
$$

After multiplication with $M_{i}$, (5.25) can be viewed as relating the leading terms of $\delta_{m n} C_{i \mid A, B, C, D, E}^{m n}$ and permutations of (5.19), leading to

$$
\begin{equation*}
\delta_{m n} C_{i \mid A, B, C, D, E}^{m n}=2 P_{i|A| B, C, D, E}+(A \leftrightarrow B, C, D, E) . \tag{5.26}
\end{equation*}
$$

In other words, scalar pseudoinvariants $P_{i|A| B, C, D, E}$ describe the tensor trace of $C_{i \mid A, B, C, D, E}^{m n}$ in terms of more fundamental objects. Similarly, it will be shown in the next section that traces of higher-rank pseudo-invariants $\delta_{m_{1} m_{2}} C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ decompose into tensorial generalizations of $P_{i|A| B, C, D, E}$. Starting from rank two, the latter give rise to traces by themselves (corresponding to double traces of $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ ), and one can anticipate an infinite number of all-rank families of pseudoinvariants. These are the different diagonals in the overview grid in fig. 1 where contractions with $\delta_{m n}$ move any tensorial object downwards to the next diagonal. Since an individual $P_{i|A| B, C, D, E}$ contains more information than the trace $\delta_{m n} C_{i \mid A, B, C, D, E}^{m n}$, we refer to the former as belonging to the refined pseudocohomology.

## 6. Generalizing the refined pseudo-cohomology

In this section, we generalize the refined currents (5.10) in two directions: firstly by defining tensorial counterparts and secondly by increasing the number of refined slots such as the distinguished word $A$ in $\mathcal{J}_{A \mid B, C, D, E}$. Each of these currents gives rise to a pseudoinvariant which can be recursively constructed along the lines of sections 4.3 and 5.3.

### 6.1. Higher-rank refined currents and their anomaly

We define the higher-rank version of the scalar refined current (5.10) by

$$
\begin{align*}
\mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}} \equiv & \frac{1}{2} \mathcal{A}_{A}^{p}\left[M_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}}+\mathcal{W}_{B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r} \mid p}\right] \\
& \quad-\left[\mathcal{H}_{\left[A, B_{1}\right]} M_{B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+4}\right)\right] \tag{6.1}
\end{align*}
$$

in terms of higher-rank building blocks $M_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}}$ and $\mathcal{W}_{B_{1}, \ldots, B_{r+4}}^{m_{1}, \ldots m_{r} \mid p}$ defined in (4.3) and (4.2). The redefinition by superfields $\mathcal{H}_{\left[A, B_{i}\right]}$ as in appendix D is necessary to trade the $\widehat{V}_{A}$ in its BRST variation for BRST blocks $V_{A}$. As before, it vanishes whenever both slots are of single-particle type, i.e. $|A|=\left|B_{i}\right|=1$.

At rank $r \geq 2$, the BRST variation of $\mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ turns out to involve anomalous traces in the same way as $Q M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ given by (4.7). They are accompanied by anomalous counterparts of the refined current (6.1),

$$
\begin{equation*}
\mathcal{Y}_{A \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}} \equiv \frac{1}{2} \mathcal{A}_{A}^{p} \mathcal{Y}_{B_{1}, \ldots, B_{r+6}}^{p m_{1} \ldots m_{r}}-\left[\mathcal{H}_{\left[A, B_{1}\right]} \mathcal{Y}_{B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right)\right], \tag{6.2}
\end{equation*}
$$

whose corrections $\sim \mathcal{H}_{\left[A, B_{i}\right]}$ are analogous to (6.1) and ensure a BRST variation in terms of $V_{A}$ rather than $\widehat{V}_{A}$, see section 5.1.

Equipped with the definitions above, we can write down the general BRST variation of higher-rank refined currents,

$$
\begin{align*}
& Q \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{A \mid B_{1}, \ldots, B_{r+4}}^{\left.m_{3} \ldots m_{r}\right)}+\mathcal{Y}_{A, B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+k_{A}^{p} M_{A} M_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}} \\
& \quad+\left[M_{S\left[A, B_{1}\right]} M_{B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+r k_{B_{1}}^{\left(m_{1}\right.} M_{B_{1}} \mathcal{J}_{A \mid B_{2}, \ldots, B_{r+4}}^{\left.m_{2} \ldots m_{r}\right)}\right. \\
& \left.\quad+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{J}_{A \mid Y, B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{J}_{A \mid X, B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right)\right] \\
& \quad+\sum_{X Y=A}\left(M_{X} \mathcal{J}_{Y \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{J}_{X \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}\right), \tag{6.3}
\end{align*}
$$

see (5.14) for the map $S\left[A, B_{i}\right]$. For example, in the case of vectors and two-tensors the general formula (6.3) yields

$$
\begin{align*}
& Q \mathcal{J}_{A \mid B_{1}, B_{2}, B_{3}, B_{4}, B_{5}}^{m}=\mathcal{Y}_{A, B_{1}, B_{2}, B_{3}, B_{4}, B_{5}}^{m}+k_{A}^{p} M_{A} M_{B_{1}, B_{2}, B_{3}, B_{4}, B_{5}}^{p m}+\left[M_{S\left[A, B_{1}\right]} M_{B_{2}, B_{3}, B_{4}, B_{5}}^{m}\right. \\
& \left.\quad+k_{B_{1}}^{m} M_{B_{1}} \mathcal{J}_{A \mid B_{2}, \ldots, B_{5}}+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{J}_{A \mid Y, B_{2}, \ldots, B_{5}}^{m}-M_{Y} \mathcal{J}_{A \mid X, B_{2}, \ldots, B_{5}}^{m}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{5}\right)\right] \\
& \quad+\sum_{X Y=A}\left(M_{X} \mathcal{J}_{Y \mid B_{1}, \ldots, B_{5}}^{m}-M_{Y} \mathcal{J}_{X \mid B_{1}, \ldots, B_{5}}^{m}\right)  \tag{6.4}\\
& Q \mathcal{J}_{A \mid B_{1}, B_{2}, B_{3}, B_{4}, B_{5}, B_{6}}^{m n}=\delta^{m n} \mathcal{Y}_{A \mid B_{1}, \ldots, B_{6}}+\mathcal{Y}_{A, B_{1}, \ldots, B_{6}}^{m n}+k_{A}^{p} M_{A} M_{B_{1}, \ldots, B_{6}}^{p m n}+\left[M_{S\left[A, B_{1}\right]} M_{B_{2}, \ldots, B_{6}}^{m n}\right. \\
& \left.\quad+2 k_{B_{1}}^{(m} M_{B_{1}} \mathcal{J}_{A \mid B_{2}, \ldots, B_{6}}^{n)}+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{J}_{A \mid Y, B_{2}, \ldots, B_{6}}^{m n}-M_{Y} \mathcal{J}_{A \mid X, B_{2}, \ldots, B_{6}}^{m n}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{6}\right)\right] \\
& \quad+\sum_{X Y=A}\left(M_{X} \mathcal{J}_{Y \mid B_{1}, \ldots, B_{6}}^{m n}-M_{Y} \mathcal{J}_{X \mid B_{1}, \ldots, B_{6}}^{m n}\right) .
\end{align*}
$$

### 6.2. Recursion for refined higher-rank pseudoinvariants

Each of the tensorial refined currents in (6.1) can be regarded as the leading term of a tensorial refined pseudoinvariant,

$$
\begin{equation*}
P_{i|A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}} \equiv M_{i} \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+\sum_{C \neq \emptyset} M_{i C} \ldots \tag{6.6}
\end{equation*}
$$

The BRST pseudo-completion through multiparticle $M_{i C}$ along with momenta and ghost number two objects $\mathcal{J}_{A \mid B_{1}, \ldots, B_{p+4}}^{m_{1} \ldots m_{p}}, M_{B_{1}, \ldots, B_{p+3}}^{m_{1} \ldots m_{p}}$ follows the same logic as explained below (4.13) and (5.19). The recursive construction of the $P_{i|A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ relies on an alternative form of the BRST variation (6.3),

$$
\begin{align*}
& Q \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{A \mid B_{1}, \ldots, B_{r+4}}^{\left.m_{3} \ldots m_{r}\right)}+\mathcal{Y}_{A, B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+\delta_{|A|, 1} k_{a_{1}}^{p} C_{a_{1} \mid B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}} \\
& +P_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}-P_{a_{\mid A A}\left|a_{1} \ldots a_{|A|-1}\right| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} P_{b_{1}|A| B_{2}, \ldots, B_{r+4}}^{\left.m_{2}, m_{r}\right)}\right. \\
& \left.+P_{b_{1}|A| b_{2} \ldots b_{\left|B_{1}\right|}, B_{2}, \ldots, B_{r+4}}^{m_{1}, m_{r}}-P_{b_{\left|B_{1}\right|}|A| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{r+4}}^{m_{1}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right)\right] \tag{6.7}
\end{align*}
$$

As usual, this follows from isolating the single-particle $M_{i}$ in (6.3) and promoting them to a (pseudo-)invariant $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ or $P_{i|A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$. By (3.10) and (6.7), the following recursion eliminates any non-anomalous contribution from $Q P_{i|A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$,

$$
\begin{align*}
& P_{i|A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}=M_{i} \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+M_{i} \otimes\left\{\delta_{|A|, 1} k_{a_{1}}^{p} C_{a_{1} \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r} p}+P_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}\right. \\
& -P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}\right| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} P_{b_{1}|A| B_{2}, \ldots, B_{r+4}}^{\left.m_{2} \ldots m_{r}\right)}+P_{b_{1}|A| b_{2} \ldots b_{\left|B_{1}\right|, B_{2}, \ldots, B_{r+4}}^{m_{1}},}\right. \\
& -P_{\left.\left.b_{\left|B_{1}\right||A| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{r+4}}^{m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+4}\right)\right]\right\} .} \tag{6.8}
\end{align*}
$$

This completes the subleading diagonal in the overview grid in fig. 1. The anomalous BRST variations of (6.8) are discussed in section 7.4.

At rank one, the general recursion (6.7) reduces to

$$
\begin{align*}
P_{i|A| B, C, D, E, F}^{m} & =M_{i} \mathcal{J}_{A \mid B, C, D, E, F}^{m}+M_{i} \otimes\left\{\delta_{|A|, 1} k_{a_{1}}^{p} C_{a_{1} \mid B, C, D, E, F}^{p m}+P_{a_{1}\left|a_{2} \ldots a_{|A|}\right| B, C, D, E, F}^{m}\right. \\
& -P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}\right| B, C, D, E, F}^{m}+\left[\delta_{|B|, 1} k_{b_{1}}^{m} P_{b_{1}|A| C, D, E, F}+P_{b_{1}|A| b_{2} \ldots b_{|B|}, C, D, E, F}^{m}\right. \\
& \left.\left.-P_{b_{|B|}|A| b_{1} \ldots b_{|B|-1}, C, D, E, F}^{m}+(B \leftrightarrow C, D, E, F)\right]\right\}, \tag{6.9}
\end{align*}
$$

and the simplest vectorial pseudoinvariant $P_{1|2| 3,4,5,6,7}^{m}$ is displayed in appendix A.

### 6.3. Higher-refinement building blocks

The definition (6.1) of refined building blocks can be endowed with a recursive structure which allows to successively increase the number $d$ of refined slots. In order to do that, first define the generalization of the tensor $\mathcal{W}_{B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r} \mid p}$ in (4.2) for any number of specialized legs,

$$
\begin{align*}
\mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r-} \mid m_{r}} & \equiv \frac{1}{2} \mathcal{A}_{A_{1}}^{p} \mathcal{W}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r-1} \mid m_{r}}  \tag{6.10}\\
& -\left[\mathcal{H}_{\left[A_{1}, B_{1}\right]} \mathcal{W}_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1}, m_{r} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] .
\end{align*}
$$

Then the recursion for refined currents $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ of arbitrary refinement can be immediately written down ${ }^{17}$,

$$
\begin{align*}
\mathcal{J}_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} & \equiv M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}  \tag{6.11}\\
\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}} & \equiv \frac{1}{2} \mathcal{A}_{A_{1}}^{p}\left[\mathcal{J}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r}}+\mathcal{W}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r} \mid p}\right] \\
& -\left[\mathcal{H}_{\left[A_{1}, B_{1}\right]} \mathcal{J}_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] .
\end{align*}
$$

Even though it is not manifest from their definitions (6.10) and (6.11), the objects $\mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r-} \mid m_{r}}$ and $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ are totally symmetric under exchange of refined slots $A_{i} \leftrightarrow A_{j}$. Moreover, symmetry in $B_{i} \leftrightarrow B_{j}$ is obviously inherited from $M_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}}$ and $\mathcal{W}_{B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r} \mid p}$ in the first step (6.1) of the recursion.

The BRST variation of (6.11) involves anomaly building blocks of higher refinement which are defined in analogy to (6.10),

$$
\begin{align*}
\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}} & \equiv \frac{1}{2} \mathcal{A}_{A_{1}}^{p} \mathcal{Y}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{p m_{1} \ldots m_{r}}  \tag{6.12}\\
& -\left[\mathcal{H}_{\left[A_{1}, B_{1}\right]} \mathcal{Y}_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right] .
\end{align*}
$$

These definitions give rise to the following formula for the most general case,

$$
\begin{align*}
& Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{\left.m_{3} \ldots m_{r}\right)} \\
+ & {\left[\mathcal{Y}_{A_{2}, \ldots, A_{d} \mid A_{1}, B_{1}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}}+k_{A_{1}}^{p} M_{A_{1}} \mathcal{J}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] } \\
+ & {\left[r k_{B_{1}}^{\left(m_{1}\right.} M_{B_{1}} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{2} \ldots m_{2}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] } \\
+ & {\left[M_{S\left[A_{1}, B_{1}\right]} \mathcal{J}_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}+\binom{A_{1} \leftrightarrow A_{2}, A_{3}, \ldots, A_{d}}{B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}}\right] }  \tag{6.13}\\
+ & {\left[\sum_{X Y=A_{1}}\left(M_{X} \mathcal{J}_{Y, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{J}_{X, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}\right)+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] } \\
+ & {\left[\sum_{X Y=B_{1}}\left(M_{X} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid Y, B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid X, B_{2}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}\right)\right.} \\
& \left.+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] .
\end{align*}
$$

Any term of $Q \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ as given by (6.3) has a counterpart in (6.13) at higher degree $d$ of refinement. Moreover, the three classes of terms $\mathcal{Y}_{A, B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}, k_{A}^{p} M_{A} M_{B_{1}, \ldots, B_{r+4}}^{p m_{1} \ldots m_{r}}$ and

17 We keep both notations for $M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}=\mathcal{J}_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ and $C_{i \mid B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}=P_{i \mid B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ to make the source of anomalous BRST transformations more transparent in the scattering amplitudes presented in [29,31].
$M_{S\left[A, B_{1}\right]} M_{B_{2}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ in (6.3) which release $A$ from the refined slot of $\mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ have multiple images in $Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ according to $A_{1} \leftrightarrow A_{2}, \ldots, A_{d}$.

For scalars of refinement $d=2$, for instance,

$$
\begin{align*}
& Q \mathcal{J}_{A, B \mid C, D, E, F, G}=\mathcal{Y}_{A \mid B, C, D, E, F, G}+\mathcal{Y}_{B \mid A, C, D, E, F, G}+M_{A} k_{A}^{m} \mathcal{J}_{B \mid C, D, E, F, G}^{m} \\
& \quad+M_{B} k_{B}^{m} \mathcal{J}_{A \mid C, D, E, F, G}^{m}+\left[M_{S[A, C]} \mathcal{J}_{B \mid D, E, F, G}+M_{S \mid B, C]} \mathcal{J}_{A \mid D, E, F, G}\right. \\
& \left.\quad+\sum_{X Y=C}\left(M_{X} \mathcal{J}_{A, B \mid Y, D, E, F, G}-M_{Y} \mathcal{J}_{A, B \mid X, D, E, F, G}\right)+(C \leftrightarrow D, E, F, G)\right] \\
& +\sum_{X Y=A}\left(M_{X} \mathcal{J}_{Y, B \mid C, D, E, F, G}-M_{Y} \mathcal{J}_{X, B \mid C, D, E, F, G}\right) \\
& \quad+\sum_{X Y=B}\left(M_{X} \mathcal{J}_{A, Y \mid C, D, E, F, G}-M_{Y} \mathcal{J}_{A, X \mid C, D, E, F, G}\right) \tag{6.14}
\end{align*}
$$

### 6.4. The general recursion for pseudoinvariants

The refined current (6.11) of arbitrary rank $r$ and refinement $d$ can be promoted to a pseudoinvariant via

$$
\begin{equation*}
P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}} \equiv M_{i} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}+\sum_{C \neq \emptyset} M_{i C} \ldots, \tag{6.15}
\end{equation*}
$$

which generalizes (6.6) to $d>1$ and by convention reduces to $P_{i \mid B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}} \equiv C_{i \mid B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ at $d=0$. The suppressed companions of the multiparticle $M_{i C}$ are further instances of momenta and $\mathcal{J}_{A_{1}, \ldots, A_{q} \mid B_{1}, \ldots, B_{p+q+3}}^{m_{1} \ldots m_{p}}$ which have to be chosen such that $Q P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ is purely anomalous. These contributions are determined by the following rewriting of (6.13):

$$
\begin{align*}
& Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r}}^{m_{3} \ldots m_{r}}+\left[\mathcal{Y}_{A_{2}, \ldots, A_{d} \mid A_{1}, B_{1}, \ldots, B_{d+r}}^{m_{1} \ldots m_{r}}\right. \\
& \quad+\delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{p} P_{a_{1}\left|A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{p m_{1} \ldots m_{r}}+P_{a_{1}\left|a_{2} \ldots m_{\left|A_{1}\right|}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}} \\
& \left.\quad-P_{a_{\left|A_{1}\right|}\left|a_{r} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
& +\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} P_{b_{1}\left|A_{1}, \ldots, A_{d}\right| B_{2}, \ldots, B_{d+r+3}}^{m_{2}, \ldots}+P_{b_{1}\left|A_{1}, \ldots, A_{d}\right| b_{2} \ldots b_{\left|B_{1}\right|}, B_{2}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}}\right. \\
& \left.\quad-P_{b_{\left|B_{1}\right|}\left|A_{1}, \ldots, A_{d}\right| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{d+r+3}}^{m_{1} m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right] . \tag{6.16}
\end{align*}
$$

As usual, (3.10) allows to derive a recursion from (6.16):

$$
\begin{align*}
& P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1}}=M_{i} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}} \\
& + \\
& M_{i} \otimes\left\{\left[\delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{p} P_{a_{1}\left|A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r}}^{p m_{1} \ldots m_{r}}+P_{a_{1}\left|a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}\right.\right. \\
& \left.\quad-P_{a_{\left|A_{1}\right|}\left|a_{1} \ldots a_{1} \ldots A_{1}\right|-1}^{m_{1}, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
& \quad+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} P_{b_{1}\left|A_{1}, \ldots, A_{d}\right| B_{2}, \ldots, B_{d+r+3}}^{m_{2}}+P_{b_{1}\left|A_{1}, \ldots, A_{d}\right| b_{2} \ldots b_{\left|B_{1}\right|}, B_{2}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}}\right.  \tag{6.17}\\
& \left.\left.\quad-P_{b_{\left|B_{1}\right|}\left|A_{1}, \ldots, A_{d}\right| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{d+r+3}}^{m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+3}\right)\right]\right\} .
\end{align*}
$$

This is the most general pseudoinvariant presented in this work, it completes the overview grid in fig. 1. Its anomalous BRST variation will be discussed in section 7.4.

In the $d=2$ example of $\mathcal{J}_{A, B \mid C, D, E, F, G}$, the variation in (6.14) can be rewritten as

$$
\begin{align*}
Q \mathcal{J}_{A, B \mid C, D, E, F, G} & =\mathcal{Y}_{A \mid B, C, \ldots, G}+\mathcal{Y}_{B \mid A, C, \ldots, G}+\delta_{|A|, 1} k_{a}^{m} P_{a|B| C, \ldots, G}^{m}+\delta_{|B|, 1} k_{b}^{m} P_{b|A| C, \ldots, G}^{m} \\
& +\left[P_{a_{1}\left|a_{2} \ldots a_{|A|}, B\right| C, \ldots, G}-P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}, B\right| C, \ldots, G}+(A \leftrightarrow B)\right]  \tag{6.18}\\
& +\left[P_{c_{1}|A, B| c_{2} \ldots c_{|C|}, D, \ldots, G}-P_{c_{|C|}|A, B| c_{1} \ldots c_{|C|-1}, D, \ldots, G}+(C \leftrightarrow D, \ldots, G)\right]
\end{align*}
$$

and converted to the recursion

$$
\begin{aligned}
P_{i|A, B| C, D, E, F, G} & =M_{i} \mathcal{J}_{A, B \mid C, D, E, F, G}+M_{i} \otimes\left\{\delta_{|A|, 1} k_{a}^{m} P_{a|B| C, \ldots, G}^{m}+\delta_{|B|, 1} k_{b}^{m} P_{b|A| C, \ldots, G}^{m}\right. \\
& +\left[P_{a_{1}\left|a_{2} \ldots a_{|A|}, B\right| C, \ldots, G}-P_{a_{|A|}\left|a_{1} \ldots a_{|A|-1}, B\right| C, \ldots, G}+(A \leftrightarrow B)\right] \\
& \left.+\left[P_{c_{1}|A, B| c_{2} \ldots c_{|C|}, D, \ldots, G}-P_{c_{|C|}|A, B| c_{1} \ldots c_{|C|-1}, D, \ldots, G}+(C \leftrightarrow D, \ldots, G)\right]\right\} .
\end{aligned}
$$

The simplest example $P_{1|2,3| 4,5,6,7,8}$ is displayed in appendix A, see (A.7).

### 6.5. Trace relations among pseudoinvariants

In section 5.4, we have discussed the relation between tensor traces $\delta_{m n} M_{A, B, C, D, E}^{m n}$, $\delta_{m n} C_{i \mid A, B, C, D, E}^{m n}$ and the refined objects, $\mathcal{J}_{A \mid B, C, D, E}, P_{i|A| B, C, D, E}$. The trace relations (5.25) and (5.26) are now generalized to higher rank $r$ and refinement $d$.

Lemma 2. The following is true,

$$
\begin{align*}
\delta_{n p} \mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{n p m_{1} \ldots m_{r-1} \mid m_{r}} & =2 \mathcal{W}_{A_{1}, \ldots, A_{d}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r-1} \mid B_{2}}+\left(B_{1}, \ldots, B_{d+r+5}\right)  \tag{6.20}\\
\delta_{n p} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{n p m_{1} \ldots m_{r}} & =2 \mathcal{J}_{A_{1}, \ldots, A_{d}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right) \tag{6.21}
\end{align*}
$$

Proof. To prove this inductively, first assume that (6.20) is true for $d-1$,

$$
\begin{equation*}
\delta_{n p} \mathcal{W}_{A_{1}, \ldots, A_{d-1} \mid B_{1}, \ldots, B_{d+r+4}}^{n p m_{1} \ldots m_{r-1} \mid m_{r}}=2 \mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{2}, \ldots, B_{d+r+4}}^{m_{1} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+4}\right) . \tag{6.22}
\end{equation*}
$$

From the definition (6.10) it follows that,

$$
\begin{aligned}
\delta_{n p} \mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{n p m_{1} \ldots m_{r-1} \mid m_{r}} & =\frac{1}{2} \mathcal{A}_{A_{d}}^{t} \mathcal{W}_{A_{1}, \ldots, A_{d-1} \mid B_{1}, \ldots, B_{d+r+5}}^{t p p m_{1} \ldots m_{r-1} \mid m_{r}} \\
& -\left[\mathcal{H}_{\left[A_{d}, B_{1}\right]} \mathcal{W}_{A_{1}, \ldots, A_{d-1} \mid B_{2}, \ldots, B_{d+r+5}}^{p m_{1} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right]
\end{aligned}
$$

and therefore (6.22) leads to,

$$
\begin{aligned}
\delta_{n p} \mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{n p m_{1} \ldots m_{r-1} \mid m_{r}}= & \frac{1}{2} \mathcal{A}_{A_{d}}^{t} 2 \mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{t m_{1} \ldots m_{r-1} \mid m_{r}} \\
- & 2\left[\mathcal{H}_{\left[A_{d}, B_{1}\right]}\left(\mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{2} \mid B_{3}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r-1} \mid m_{r}}+\left(B_{2} \leftrightarrow B_{3}, \ldots, B_{d+r+5}\right)\right)\right. \\
& \left.+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right] \\
= & 2 \mathcal{W}_{A_{1}, \ldots, A_{d}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right),
\end{aligned}
$$

where in the last line one uses that,

$$
\begin{align*}
& \mathcal{H}_{\left[A_{d}, B_{1}\right]}\left[\mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{2} \mid B_{3}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r-1} \mid m_{2}}+\left(B_{2} \leftrightarrow B_{3}, \ldots, B_{d+r+5}\right)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right) \\
& =\left[\mathcal{H}_{\left[A_{d}, B_{2}\right]} \mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{3}, \ldots, B_{d+r+5}}^{m_{1}, \ldots, A_{r}}+\left(B_{2} \leftrightarrow B_{3}, \ldots, B_{d+r+5}\right)\right]+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right) . \tag{6.23}
\end{align*}
$$

Furthermore, it is easy to show that when $d=0$,

$$
\begin{equation*}
\delta_{n p} \mathcal{W}_{B_{1}, B_{2} \ldots, B_{r+5}}^{n p m_{1} \ldots m_{r-1} \mid m_{r}}=2 \mathcal{W}_{B_{1} \mid B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right), \tag{6.24}
\end{equation*}
$$

finishing the proof of (6.20).
To show (6.21) one proceeds similarly by first assuming that it holds for $d-1$,

$$
\begin{equation*}
\delta_{n p} \mathcal{J}_{A_{1}, \ldots, A_{d-1} \mid B_{1}, \ldots, B_{d+r+4}}^{n p m_{1} \ldots m_{r}}=2 \mathcal{J}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{2}, \ldots, B_{d+r+4}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+4}\right) . \tag{6.25}
\end{equation*}
$$

A direct application of the definition (6.11) leads to

$$
\begin{gather*}
\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, B_{2}, \ldots, B_{d+r+5}}^{p p m_{1} \ldots m_{r}}=\frac{1}{2} \mathcal{A}_{A_{d}}^{q}\left[\mathcal{J}_{A_{1}, \ldots, A_{d-1} \mid B_{1}, B_{2}, \ldots, B_{d+r+5}}^{p m_{1} \ldots m_{r}}+\mathcal{W}_{A_{1}, \ldots, A_{d-1} \mid B_{1}, B_{2}, \ldots, B_{d+r+5}}^{p p m_{1} \ldots m_{r} \mid q}\right] \\
\quad-\left[\mathcal{H}_{\left[A_{d}, B_{1}\right]} \mathcal{J}_{A_{1}, \ldots, A_{d-1} \mid B_{2}, \ldots, B_{d+r+5}}^{p p m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5)}\right)\right] . \tag{6.26}
\end{gather*}
$$

Now one rewrites (6.26) using (6.20) together with the assumption (6.25),

$$
\begin{align*}
\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, B_{2}, \ldots, B_{d+r+5}}^{p p m_{1} \ldots m_{r}} & =\frac{1}{2} \mathcal{A}_{A_{d}}^{q}\left[2 \mathcal{J}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r} q}+2 \mathcal{W}_{A_{1}, \ldots, A_{d-1}, B_{1} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r} \mid q}\right] \\
& -2\left[\mathcal{H}_{\left[A_{d}, B_{1}\right]} \mathcal{J}_{A_{1}, \ldots, A_{d-1}, B_{2} \mid B_{3}, \ldots, B_{d+r+5}}^{m_{1}, m_{r}}+\left(B_{2} \leftrightarrow B_{3}, \ldots, B_{d+r+5}\right)\right] \\
& +\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right), \tag{6.27}
\end{align*}
$$

to finally obtain

$$
\begin{equation*}
\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, B_{2}, \ldots, B_{d+r+5}}^{p p m_{1} \ldots m_{r}}=2 \mathcal{J}_{A_{1}, \ldots, A_{d}, B_{1} \mid B_{2}, B_{3}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{d+r+5}\right), \tag{6.28}
\end{equation*}
$$

where a relation analogous to (6.23) has been used to arrive at (6.28). When $d=0$ (6.21) can be easily verified using the definitions (6.11) and (4.3) since the commutator drops out due to the sum over permutations,

$$
\begin{align*}
\mathcal{J}_{B_{1}, B_{2}, \ldots, B_{r+5}}^{p p m_{1} \ldots m_{r}} & =M_{B_{1}, B_{2}, \ldots, B_{r+5}}^{p m_{1} \ldots m_{r} p}  \tag{6.29}\\
& =\mathcal{A}_{B_{1}}^{p} M_{B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r} p}+\mathcal{A}_{B_{1}}^{p} \mathcal{W}_{B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r} \mid p}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+5}\right) \\
& =2 \mathcal{J}_{B_{1} \mid B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+5}\right) .
\end{align*}
$$

The above manipulations make use of the symmetry properties $M^{p p m_{1} \ldots m_{r}}=M^{p m_{1} \ldots m_{r} p}$ and $\mathcal{W}^{m_{r} m_{r-1} \ldots m_{1} \mid p}=\mathcal{W}^{m_{1} \ldots m_{r} \mid p}$.

After multiplication by $M_{i},(6.29)$ and (6.21) relate the leading terms of pseudoinvariants, so we can directly promote them to their BRST pseudo-completion:

$$
\begin{align*}
\delta_{n p} C_{i \mid B_{1}, \ldots, B_{r+5}}^{n p m_{1} \ldots m_{r}} & =2 P_{i\left|B_{1}\right| B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right)  \tag{6.30}\\
\delta_{n p} P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{n p m_{1} \ldots m_{r}} & =2 P_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+5}}^{m_{1}, m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right) .
\end{align*}
$$

This demonstrates that the family of pseudoinvariants defined in (6.15) and recursively constructed in (6.17) is closed under the trace operation. This is particularly relevant for their contractions with loop momenta in one-loop amplitudes, see [31].

## 7. Anomalous BRST invariants

This section is devoted to BRST variations of anomaly blocks such as $\mathcal{Y}_{A, B, C, D, E}$ given by (3.3) as well as its generalization to higher rank and refinement, see (4.6) and (6.12). We are led to BRST-invariant ghost-number-four objects built from $M_{C} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$ and momenta. They turn out to share the grid structure of pseudoinvariants in fig. 1, see fig. 6 for an overview and the subsequent sections for the notation therein.

These anomaly invariants capture the systematics of anomalous BRST variations of pseudoinvariants. Moreover, we point out close analogies between the $Q$ action on $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ and $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$. This firstly allows to recycle a lot of results from previous sections and secondly motivates a more abstract viewpoint on the recursion for pseudoinvariants which will prove essential for the subsequent sections.


Fig. 6 Overview of anomaly invariants. The arrows indicate whenever superfields of different type enter the recursion for the invariants on their right.

### 7.1. BRST variation of unrefined anomaly blocks

Let us firstly analyze the BRST variations of unrefined anomaly building blocks $\mathcal{Y}_{A_{1}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}$. In the scalar case (3.3),

$$
\begin{equation*}
Q \mathcal{Y}_{A, B, C, D, E}=\sum_{X Y=A}\left(M_{X} \mathcal{Y}_{Y, B, C, D, E}-M_{Y} \mathcal{Y}_{X, B, C, D, E}\right)+(A \leftrightarrow B, C, D, E) \tag{7.1}
\end{equation*}
$$

has the same structure as $Q M_{A, B, C}$ in (2.27), and in particular $\mathcal{Y}_{1,2,3,4,5}$ is BRST closed. The pure spinor constraint guarantees that the first term in $Q \mathcal{W}_{B}$ given by (2.22) does not contribute. Starting from the vector building block as in (4.6), we additionally need the group-theoretic identity ${ }^{18}$ [4],

$$
\begin{equation*}
\left(\lambda \gamma^{m}\right)_{\left[\alpha_{1}\right.}\left(\lambda \gamma_{p}\right)_{\alpha_{2}}\left(\lambda \gamma_{q}\right)_{\alpha_{3}}\left(\lambda \gamma_{r}\right)_{\alpha_{4}} \gamma_{\left.\alpha_{5} \alpha_{6}\right]}^{p q r}=0 \tag{7.2}
\end{equation*}
$$

to prove that

$$
\begin{align*}
Q \mathcal{Y}_{A, B, C, D, E, F}^{m}= & \sum_{X Y=A}\left(M_{X} \mathcal{Y}_{Y, B, C, D, E, F}^{m}-M_{Y} \mathcal{Y}_{X, B, C, D, E, F}^{m}\right) \\
& +k_{A}^{m} M_{A} \mathcal{Y}_{B, C, D, E, F}+(A \leftrightarrow B, C, D, E, F) \tag{7.3}
\end{align*}
$$

i.e. the first term of $Q \mathcal{A}_{B}^{m}=\left(\lambda \gamma^{m} \mathcal{W}_{B}\right)+\ldots$ drops out from $Q \mathcal{Y}_{A, B, C, D, E, F}^{m}$. Note the direct correspondence of (7.3) with $Q M_{A, B, C, D}^{m}$ given by (2.31). Accordingly, higher-tensor
${ }^{18}$ This is a consequence of having no vector representation $[1,0,0,0,0]$ in the decomposition $[0,0,0,0,4] \otimes[0,0,0,0,1]^{\wedge 6}$.
generalizations of $Q \mathcal{Y}_{A, B, C, D, E, F}^{m}$ can be almost literally borrowed from $Q M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ given by (4.7) except for one simplification: There is no anomaly analogue of the $\mathcal{W}_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}}$ tensor in (4.6) which prevents trace contributions in the following expression,

$$
\begin{align*}
Q \mathcal{Y}_{B_{1}, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}= & \sum_{X Y=B_{1}}\left(M_{X} \mathcal{Y}_{Y, B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{Y}_{X, B_{2}, B_{3}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}\right) \\
& +r M_{B_{1}} k_{B_{1}}^{\left(m_{1}\right.} \mathcal{Y}_{B_{2}, B_{3}, \ldots, B_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right) . \tag{7.4}
\end{align*}
$$

### 7.2. Unrefined anomaly invariants

We repeat the steps of section 4.3 to recursively construct tensorial BRST invariants $\Gamma_{i \mid \ldots}^{m_{1} \ldots}$ at ghost-number four from anomaly blocks. They are defined by a leading term $\sim M_{i}$,

$$
\begin{equation*}
\Gamma_{i \mid A_{1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}} \equiv M_{i} \mathcal{Y}_{A_{1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}+\sum_{B \neq \emptyset} M_{i B} \ldots \tag{7.5}
\end{equation*}
$$

The suppressed terms ... along with multiparticle $M_{i B}$ are also anomalous and can be found from a recursion relation. To see this, firstly rewrite (7.4) as follows

$$
\begin{gather*}
Q \mathcal{Y}_{A_{1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}=r \delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{\left(m_{1}\right.} \Gamma_{a_{1} \mid A_{2}, \ldots, A_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}  \tag{7.6}\\
+\Gamma_{a_{1} \mid a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}-\Gamma_{a_{\left|A_{1}\right|} \mid a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+5}\right),
\end{gather*}
$$

which resembles (4.14) for $Q M_{A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$. Together with (3.8), this implies BRST invariance of the recursively-generated objects

$$
\begin{align*}
& \Gamma_{i \mid A_{1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}=M_{i} \mathcal{Y}_{A_{1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}+M_{i} \otimes\left[r \delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{\left(m_{1}\right.} \Gamma_{a_{1} \mid A_{2}, \ldots, A_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}\right.  \tag{7.7}\\
+ & \left.\Gamma_{a_{1} \mid a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{r+5}}^{m_{1}}-\Gamma_{a_{\left|A_{1}\right|} \mid a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+5}\right)\right]
\end{align*}
$$

see (4.15) for the non-anomalous counterpart. For example,

$$
\begin{align*}
\Gamma_{1 \mid 2,3,4,5,6} & =M_{1} \mathcal{Y}_{2,3,4,5,6}  \tag{7.8}\\
\Gamma_{1 \mid 23,4,5,6,7} & =M_{1} \mathcal{Y}_{23,4,5,6,7}+M_{1} \otimes\left[\Gamma_{2 \mid 3,4,5,6,7}-\Gamma_{3 \mid 2,4,5,6,7}\right] \\
& =M_{1} \mathcal{Y}_{23,4,5,6,7}+M_{12} \mathcal{Y}_{3,4,5,6,7}-M_{13} \mathcal{Y}_{2,4,5,6,7} \\
\Gamma_{1 \mid 2,3,4,5,6,7}^{m} & =M_{1} \mathcal{Y}_{2,3,4,5,6,7}^{m}+M_{1} \otimes\left[k_{2}^{m} \Gamma_{2 \mid 3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right] \\
& =M_{1} \mathcal{Y}_{2,3,4,5,6,7}^{m}+\left[k_{2}^{m} M_{12} \mathcal{Y}_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right]
\end{align*}
$$

furnish the anomaly counterparts of $C_{1 \mid 2,3,4}, C_{1 \mid 23,4,5}$ and $C_{1 \mid 2,3,4,5}^{m}$ given in (2.41).

The anomaly invariants in (7.7) allow to concisely describe the anomalous BRST variations of pseudoinvariants at ghost-number three:

$$
\begin{equation*}
Q C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}=-\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Gamma_{i \mid A_{1}, \ldots, A_{r+3}}^{\left.m_{3} \ldots m_{r}\right)} \tag{7.9}
\end{equation*}
$$

The anomaly counterpart of this statement is simply

$$
\begin{equation*}
Q \Gamma_{i \mid A_{1}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}=0, \tag{7.10}
\end{equation*}
$$

which follows from $Q^{2}=0$. To justify (7.9), it is sufficient to study the anomalous BRST variation of the leading term $M_{i} M_{A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ in $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ and to promote the resulting $M_{i} \mathcal{Y}_{A_{1}, \ldots, A_{p+5}}^{m_{1} \ldots m_{p}}$ to their BRST invariant completion in (7.5).

### 7.3. BRST variation of general anomaly blocks

The close parallels between BRST manipulations of building blocks $M_{A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ and their anomaly counterparts $\mathcal{Y}_{A_{1}, \ldots, A_{r+5}}^{m_{1} \ldots m_{r}}$ propagate to their refined versions. This can be seen by comparing the recursions (6.11) and (6.12) for $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ and $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$, respectively. The absence of the $\mathcal{W}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r}+3}^{m_{1} \ldots m_{r-} \mid m_{r}}$ tensor in the anomalous case implies that there is no higher anomaly image of the $\mathcal{Y}$ contribution to $Q \mathcal{J}$. In particular, the expression (6.3) for $Q \mathcal{J}_{A \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ implies that

$$
\begin{gather*}
Q \mathcal{Y}_{A \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}=k_{A}^{p} M_{A} \mathcal{Y}_{B_{1}, \ldots, B_{r+6}}^{p m_{1} \ldots m_{r}}+\left[M_{S\left[A, B_{1}\right]} \mathcal{Y}_{B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}+r k_{B_{1}}^{\left(m_{1}\right.} M_{B_{1}} \mathcal{Y}_{A \mid B_{2}, \ldots, B_{r+6}}^{\left.m_{2} \ldots m_{r}\right)}\right. \\
\left.+\sum_{X Y=B_{1}}\left(M_{X} \mathcal{Y}_{A \mid Y, B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{Y}_{A \mid X, B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right)\right] \\
\quad+\sum_{X Y=A}\left(M_{X} \mathcal{Y}_{Y \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{Y}_{X \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}\right) \tag{7.11}
\end{gather*}
$$

More generally, (6.13) for $Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ leads to

$$
\begin{gather*}
Q \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}=\left[k_{A_{1}}^{p} M_{A_{1}} \mathcal{Y}_{A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{p_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
+\left[r k_{B_{1}}^{\left(m_{1}\right.} M_{B_{1}} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+5} m_{2},}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right]  \tag{7.12}\\
+\left[M_{S\left[A_{1}, B_{1}\right]} \mathcal{Y}_{A_{2}, \ldots, A_{d} \mid B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}+\binom{A_{1} \leftrightarrow A_{2}, A_{3}, \ldots, A_{d}}{B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}}\right] \\
+\left[\sum_{X Y=A_{1}}\left(M_{X} \mathcal{Y}_{Y, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{Y}_{X, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}\right)+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
+\left[\sum_{X Y=B_{1}}\left(M_{X} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid Y, B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}-M_{Y} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid X, B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right] .
\end{gather*}
$$

Recall that the $S[A, B]$ map entering $M_{S\left[A, B_{i}\right]}$ is explained in section 5.2 and defined in (5.14). The $\mathcal{H}_{\left[A_{i}, B_{j}\right]}$ corrections in the recursion (6.12) for $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1}, \ldots m_{r}}$ ensure that any $M_{S\left[A, B_{i}\right]}$ in (7.11) and (7.12) is built from BRST blocks $V_{C}$ rather than $\widehat{V}_{C}$.

### 7.4. Refined anomaly invariants

Similar to (7.5), we introduce refined anomaly invariants with a more general leading term,

$$
\begin{equation*}
\Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}} \equiv M_{i} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}+\sum_{C \neq \emptyset} M_{i C} \ldots \tag{7.13}
\end{equation*}
$$

The BRST completion ... along with multiparticle $M_{i C}$ is built from $\mathcal{Y}_{A_{1}, \ldots, A_{q} \mid B_{1}, \ldots, B_{p+q+5}}^{m_{1} \ldots m_{p}}$ and momenta to ensure that $Q \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}=0$. This is the anomaly counterpart of the pseudoinvariant $P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ given by (6.15). The definition (7.13) leads to the following rewriting of (7.12),

$$
\begin{gather*}
Q \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}=\left[\delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{p} \Gamma_{a_{1}\left|A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{p m_{1} \ldots m_{r}}\right. \\
\left.+\Gamma_{a_{1}\left|a_{2} \ldots a_{\left|A_{1}\right|}\right|, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}-\Gamma_{a_{1 A_{1} \mid}\left|a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1}, m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} \Gamma_{b_{1}\left|A_{1}, \ldots, A_{d}\right| B_{2}, \ldots, B_{d+r+5}}^{\left.m_{2}, m_{r}\right)}+\Gamma_{b_{1}\left|A_{1}, \ldots, A_{d}\right| b_{2} \ldots b_{\left|B_{1}\right|}, B_{2}, \ldots, B_{d+r+5}}^{m_{1}, m_{r}}\right. \\
\left.-\Gamma_{b_{\left|B_{1}\right|}\left|A_{1}, \ldots, A_{d}\right| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{d+r+5}}^{m_{1}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right] . \tag{7.14}
\end{gather*}
$$

This in turn suggests a recursion for the most general anomaly invariant in (7.13),

$$
\begin{gather*}
\Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}=M_{i} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}  \tag{7.15}\\
+M_{i} \otimes\left\{\left[\delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{p} \Gamma_{a_{1}\left|A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r}, 5}^{p m_{1} \ldots m_{r}}+\Gamma_{a_{1}\left|a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{2}}\right.\right. \\
-\Gamma_{a_{\left|A_{1}\right|}\left|a_{1} \ldots a_{\left|A_{1}\right|-1}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{\left.m_{1},\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right]} \\
+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} \Gamma_{b_{1}\left|A_{1}, \ldots, A_{d}\right| B_{2}, \ldots, B_{d+r+5}}^{\left.m_{2}\right)}+\Gamma_{b_{1}\left|A_{1}, \ldots, A_{d}\right| b_{2} \ldots b_{\left|B_{1}\right|, B_{2}, \ldots, B_{d+r+5}}^{m_{1}}}\right. \\
\left.\left.-\Gamma_{b_{\left|B_{1}\right|}\left|A_{1}, \ldots, A_{d}\right| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+5}\right)\right]\right\} .
\end{gather*}
$$

For example

$$
\begin{aligned}
\Gamma_{1|2| 3,4,5,6,7,8} & =M_{1} \mathcal{Y}_{2 \mid 3,4,5,6,7,8}+k_{2}^{m} M_{1} \otimes \Gamma_{2 \mid 3,4,5,6,7,8}^{m} \\
& =M_{1} \mathcal{Y}_{2 \mid 3,4,5,6,7,8}+M_{12} k_{2}^{m} \mathcal{Y}_{3,4,5,6,7,8}^{m}+\left[s_{23} M_{123} \mathcal{Y}_{4,5,6,7,8}+(3 \leftrightarrow 4, \ldots, 8)\right]
\end{aligned}
$$

Note that (7.14) and (7.15) resemble the derivation of pseudoinvariants $P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ via (6.16) and (6.17), and the example (7.16) is the anomaly counterpart of $P_{1|2| 3,4,5,6}$ given in (5.22).

The ghost-number-four invariants (7.15) describe the anomalous BRST transformation

$$
\begin{align*}
Q P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}} & =-\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{\left.m_{3} \ldots m_{r}\right)}  \tag{7.17}\\
& -\left[\Gamma_{i\left|A_{2}, \ldots, A_{d}\right| A_{1}, B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right]
\end{align*}
$$

with anomaly counterpart

$$
\begin{equation*}
Q \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}=0 \tag{7.18}
\end{equation*}
$$

The former can be see from the BRST variation of the leading term $M_{i} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ in $P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ where any $M_{i} \mathcal{Y}_{A_{1}, \ldots, A_{q} \mid B_{1}, \ldots, B_{p+q+5}}^{m_{1} \ldots m_{p}}$ is identified as a leading term in (7.13) and promoted to its completion $\Gamma_{i\left|A_{1}, \ldots, A_{q}\right| B_{1}, \ldots, B_{p+q+5}}^{m_{1} \ldots m_{p}}$.

### 7.5. Anomaly trace relations

The analysis of trace relations in section 6.5 straightforwardly carries over to anomalous building blocks. As before, the $\mathcal{H}_{[A, B]}$ corrections in the definition (6.12) of $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$ drop out in the combinations on the right-hand side of

$$
\begin{equation*}
\delta_{n p} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+7}}^{n p m_{1} \ldots m_{r}}=2 \mathcal{Y}_{A_{1}, \ldots, A_{d}, B_{1} \mid B_{2}, \ldots, B_{d+r+7}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+7}\right) . \tag{7.19}
\end{equation*}
$$

The inductive proof for the non-anomalous counterpart (6.21) fits to the present setting after trivial adjustments - adding two extra slots and suppressing the $\mathcal{W}_{\ldots}^{m_{1} \cdots}$ contribution. Similar to (6.30), one can uplift $M_{i}$ times (7.19) to the BRST completions,

$$
\begin{equation*}
\delta_{n p} \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+7}}^{n p m_{1} \ldots m_{r}}=2 \Gamma_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+7}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+7}\right), \tag{7.20}
\end{equation*}
$$

consistent with the BRST variations in (7.17).

## 8. Generalizing the recursion scheme

In sections 4 to 6 , we have built up a grid of BRST pseudo-invariant objects of ghost number three whose structure is summarized in fig. 1. As we have seen in section 7 and in particular fig. 6, the grid of pseudoinvariants has a straightforward extension to the anomaly sector at ghost-number four with two further slots. Given the almost identical recursion relations (6.17) and (7.15) for the BRST (pseudo-)invariants $P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ and $\Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$, it is natural to embed these two cases into a unified framework.

We will do so in section 8.1 by promoting (6.17) and (7.15) to a "master recursion". The latter points towards further special cases besides $P_{i \mid \ldots}^{m_{1} \ldots}$ and $\Gamma_{i \mid \ldots}^{m_{1}}$. In sections 8.2 and 8.3, we are led to two families of ghost-number-two objects, and their anomalous counterparts at ghost-number three are discussed in subsection 8.4 and 8.5. Each of these four cases exhibits a grid structure almost identical to fig. 1 and fig. 6. The superficial
disparity in the number of unrefined slots $B_{i}$ is taken care of as an integer parameter of the master recursion.

As a major benefit of the ghost-number-two families described in sections 8.2 and 8.3, their BRST variation generates a rich network of relations among pseudoinvariants, beyond the trace identities in section 6.5,

$$
\begin{equation*}
Q(\text { ghost-number-two object })=\text { ghost-number-three relation } . \tag{8.1}
\end{equation*}
$$

These BRST-exact relations turn out to connect momentum contractions $k_{B_{i}}^{p} P_{i \mid \ldots}^{p m_{1} \ldots}$ with pseudoinvariants of lower rank. For example, the five-point combinations

$$
\begin{equation*}
k_{1}^{m} C_{1 \mid 2,3,4,5}^{m}, \quad k_{2}^{m} C_{1 \mid 2,3,4,5}^{m}+s_{23} C_{1 \mid 23,4,5}+s_{24} C_{1 \mid 24,3,5}+s_{25} C_{1 \mid 25,3,4} \tag{8.2}
\end{equation*}
$$

will be identified as BRST-exact if momentum conservation $k_{12345}^{m}=0$ holds. As will be detailed in sections 9 and 10 , the master recursion in section 8.1 systematically constructs the required ghost-number-two superfields which generate meaningful relations via (8.1).

### 8.1. The master recursion

The purpose of this section is to unify the almost identical recursions (6.17) and (7.15) for the BRST (pseudo-)invariants $P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ and anomaly invariants $\Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$. The superficial difference set by the number three and five of unrefined slots in the simplest constituents $M_{A, B, C}$ and $\mathcal{Y}_{A, B, C, D, E}$ is described by an integer parameter. This amounts to replacing the superfields by an abstract symbol $\mathcal{U}_{A_{1}, A_{2}, \ldots, A_{N}}$ with a variable number $N$ of slots.

In the same way as $M_{A, B, C}$ and $\mathcal{Y}_{A, B, C, D, E}$ have been generalized to arbitrary rank $r$ and refinement $d$, we introduce formal symbols at all values of $d$ and $r$,

$$
\begin{equation*}
\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}, \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}} \rightarrow \mathcal{U}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+N}}^{m_{1} \ldots m_{r}} \tag{8.3}
\end{equation*}
$$

They are defined to be symmetric in $m_{i}, A_{i}, B_{i}$ but not under exchange of $A_{i} \leftrightarrow B_{j}$, so they may be identified with $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1}, \ldots m_{r}}$ and $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$ if $N=3$ and $N=5$, respectively. In terms of the standard Berends-Giele currents $M_{A}$ and the symbol in (8.3), we recursively define abstract tensors

$$
\begin{gather*}
R_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+N}}^{(N), m_{1} \ldots m_{r}} \equiv M_{i} \mathcal{U}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+N}}^{m_{1} \ldots m_{r}} \\
+M_{i} \otimes\left\{\left[\delta_{\left|A_{1}\right|, 1} k_{a_{1}}^{p} R_{a_{1}\left|A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+N}}^{(N), p m_{1} \ldots m_{r}}+R_{a_{1}\left|a_{2} \ldots a_{\left|A_{1}\right|}, A_{2}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+N}}^{(N), m_{1}}\right.\right. \\
\left.-R_{a_{\left|A_{1}\right|}\left|a_{1} \ldots m_{r} \ldots A_{1}\right|-1, A_{2}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+N}}^{(N), m_{1}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
+\left[r \delta_{\left|B_{1}\right|, 1} k_{b_{1}}^{\left(m_{1}\right.} R_{b_{1}\left|A_{1}, \ldots, A_{d}\right| B_{2}, \ldots, B_{d+r+N}}^{\left.(N), m_{2} \ldots m_{2}\right)}+R_{b_{1}\left|A_{1}, \ldots, A_{d}\right| b_{2} \ldots b_{\left|B_{1}\right|, B_{2}, \ldots, B_{d+r+N}}^{(N), m_{1}}}^{\left.\left.-R_{b_{\left|B_{1}\right|}\left|A_{1}, \ldots, A_{d}\right| b_{1} \ldots b_{\left|B_{1}\right|-1}, B_{2}, \ldots, B_{d+r+N}}^{(N), m_{1} \ldots m_{2}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+N}\right)\right]\right\} .}\right.
\end{gather*}
$$

The following two specializations reproduce the known recursions (6.17) and (7.15):

$$
\begin{align*}
& P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=R_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{(N=3), m_{1} \ldots m_{r}}\left[\mathcal{U} \cdots \rightarrow \mathcal{J}_{\ldots}\right]  \tag{8.5}\\
& \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{1}}=R_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+5}}^{(N=5), m_{1}, m_{r}}\left[\mathcal{U} \cdots \rightarrow \mathcal{Y}_{\ldots} \cdots .\right. \tag{8.6}
\end{align*}
$$

The simplest examples for the generalized $R_{i \mid \ldots}^{(N), \ldots}$ are

$$
\begin{align*}
R_{1 \mid 2,3, \ldots, N+1}^{(N)}= & M_{1} \mathcal{U}_{2,3, \ldots, N+1}  \tag{8.7}\\
R_{1 \mid 23,4, \ldots, N+2}^{(N)}= & M_{1} \mathcal{U}_{23,4, \ldots, N+2}+M_{12} \mathcal{U}_{3,4, \ldots, N+2}-M_{13} \mathcal{U}_{2,4, \ldots, N+2}  \tag{8.8}\\
R_{1 \mid 2,3, \ldots, N+2}^{(N)}= & M_{1} \mathcal{U}_{2,3, \ldots, N+2}^{m}+\left[k_{2}^{m} M_{12} \mathcal{U}_{3,4, \ldots, N+2}+(2 \leftrightarrow 3,4, \ldots, N+2)\right]  \tag{8.9}\\
R_{1|2| 3, \ldots, N+3}^{(N)}= & M_{1} \mathcal{U}_{2 \mid 3,4, \ldots, N+3}+M_{12} k_{2}^{m} \mathcal{U}_{3,4, \ldots, N+3}^{m} \\
& +\left[s_{23} M_{123} \mathcal{U}_{4, \ldots, N+3}+(3 \leftrightarrow 4,5, \ldots, N+3)\right] \tag{8.10}
\end{align*}
$$

The right-hand sides obviously specialize to familiar expressions such as

- (2.41) and (5.22) for $C_{1 \mid 2,3,4}, C_{1 \mid 23,4,5}, C_{1 \mid 2,3,4,5}^{m}$ and $P_{1|2| 3,4,5,6}$ under (8.5)
- (7.8) and (7.16) for $\Gamma_{1 \mid 2,3,4,5,6}, \Gamma_{1 \mid 23,4,5,6,7}, \Gamma_{1 \mid 2,3,4,5,6,7}^{m}$ and $\Gamma_{1|2| 3,4,5,6,7,8}$ under (8.6).

In the following sections, we consider the abstract tensors $R_{i \mid \ldots}^{(N), \ldots}$ in (8.4) at values $N=$ $2,4,6$. In order to accommodate this with the number of slots of $\mathcal{U} \in\{\mathcal{J}, \mathcal{Y}\}$, we have to eliminate the Berends-Giele currents $M_{A}$ and adjoin the word $A$ to the slots of the accompanying symbol. In the non-anomalous case $\mathcal{U}=\mathcal{J}$, this gives rise to ghost-numbertwo objects, and the anomalous choice $\mathcal{U}=\mathcal{Y}$ yields ghost number three. Moreover, the word $A$ from the eliminated $M_{A}$ can become either a refined or a non-refined slot of the symbol, leading to $\mathcal{U}_{A, \ldots \mid \ldots}$ or $\mathcal{U}_{\ldots \mid A, \ldots}$. These two independent choices yield a total of four new families of superfields whose notation and schematic form is summarized by

$$
\begin{align*}
& D_{i \mid \ldots} \equiv R_{i \mid \ldots}^{(N=2), \ldots\left[M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{J}_{\left\{\ddot{B}_{j}\right\} \mid A,\left\{C_{j}\right\}}\right]}  \tag{8.11}\\
& L_{i \ddot{ } \mid \ldots} \equiv R_{i \mid \ldots}^{(N=4), \ldots}\left[M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{J}_{\ddot{A},\left\{B_{j}\right\} \mid\left\{C_{j}\right\}}\right]  \tag{8.12}\\
& \Delta_{i \mid \ldots} \equiv R_{i \mid \ldots}^{(N=4)}, \ldots\left[M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{Y}_{\left\{\ddot{B}_{j}\right\} \mid A,\left\{C_{j}\right\}}\right]  \tag{8.13}\\
& \Lambda_{\ddot{i} \mid \ldots} \equiv R_{i \mid \ldots}^{(N=6), \ldots\left[M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{Y}_{\dddot{A},\left\{B_{j}\right\} \mid\left\{C_{j}\right\}}\right] .} \tag{8.14}
\end{align*}
$$

The precise definitions and simplest examples are given in the following subsections.

### 8.2. The D-superfields at ghost-number two

As a first avenue towards BRST generators at ghost number two, we consider the tensors $R_{i \mid \ldots}^{(N), \ldots}$ in (8.4) at $N=2$ and convert the word $A$ associated with $M_{A}$ to a non-refined slot of the associated symbol $\mathcal{U} \rightarrow \mathcal{J}$. As sketched in (8.11), this gives rise to the definition

$$
\begin{equation*}
D_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+2}}^{m_{1} \ldots m_{2}} \equiv R_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+2}}^{(N=2), m_{1} \ldots m_{r}}\left[M_{A} \mathcal{U}_{\left\{\dddot{F}_{j}\right\} \mid\left\{G_{j}\right\}} \rightarrow \mathcal{J}_{\left\{\ddot{F}_{j}\right\} \mid A,\left\{G_{j}\right\}}\right] \tag{8.15}
\end{equation*}
$$

The replacement rule in (8.15) converts the formal objects (8.7), (8.8), (8.9) and (8.10) to

$$
\begin{align*}
D_{1 \mid 2,3} & =M_{1,2,3}  \tag{8.16}\\
D_{1 \mid 23,4} & =M_{12,3,4}+M_{1,23,4}+M_{31,2,4} \\
D_{1 \mid 2,3,4}^{m} & =M_{1,2,3,4}^{m}+k_{2}^{m} M_{12,3,4}+k_{3}^{m} M_{13,2,4}+k_{4}^{m} M_{14,2,3}, \\
D_{1|2| 3,4,5} & =\mathcal{J}_{2 \mid 1,3,4,5}+k_{2}^{m} M_{12,3,4,5}^{m}+\left[s_{23} M_{123,4,5}+(3 \leftrightarrow 4,5)\right] .
\end{align*}
$$

In the next section 9, these ghost-number-two objects and their generalizations are shown to serve as powerful BRST generators in the sense of (8.1).

### 8.3. The L-superfields at ghost-number two

The $N=4$ version of the $R_{i \mid \ldots}^{(N), \ldots}$ in (8.4) allows to convert the word $A$ associated with $M_{A}$ to a refined slot of the associated symbol $\mathcal{U} \rightarrow \mathcal{J}$. The precise form of the definition sketched in (8.12) is

$$
\begin{equation*}
L_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+4}}^{m_{1} \ldots m_{r}} \equiv R_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+4}}^{(N=4), m_{1} \ldots m_{r}}\left[M_{A} \mathcal{U}_{\left\{\ddot{F}_{j}\right\} \mid\left\{G_{j}\right\}} \rightarrow \mathcal{J}_{\ddot{A},\left\{F_{j}\right\} \mid\left\{G_{j}\right\}}\right] . \tag{8.17}
\end{equation*}
$$

Starting from the examples in (8.7) to (8.10), the prescription (8.17) yields

$$
\begin{align*}
L_{1 \mid 2,3,4,5} & =\mathcal{J}_{1 \mid 2,3,4,5}  \tag{8.18}\\
L_{1 \mid 23,4,5,6} & =\mathcal{J}_{1 \mid 23,4,5,6}+\mathcal{J}_{12 \mid 3,4,5,6}-\mathcal{J}_{13 \mid 2,4,5,6} \\
L_{1 \mid 2,3,4,5,6}^{m} & =\mathcal{J}_{1 \mid 2,3,4,5,6}^{m}+\left[k_{2}^{m} \mathcal{J}_{12 \mid 3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right] \\
L_{1|2| 3,4,5,6,7} & =\mathcal{J}_{1,2 \mid 3,4,5,6,7}+k_{2}^{m} \mathcal{J}_{12 \mid 3,4,5,6,7}^{m}+\left[s_{23} \mathcal{J}_{123 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] .
\end{align*}
$$

These superfields of ghost-number two serve as another family of BRST generators, see section 10.

### 8.4. The $\Delta$-superfields at ghost-number three

Another specialization of the $R_{i \mid \ldots}^{(N), \ldots}$ in (8.4) to $N=4$ generates anomalous superfields. In this case, the word $A$ associated with $M_{A}$ is adjoined to the non-refined slots of the associated symbol $\mathcal{U} \rightarrow \mathcal{Y}$. As sketched in (8.13), this gives rise to the definition

$$
\begin{equation*}
\Delta_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+4}}^{m_{1} \ldots m_{r}} \equiv R_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+4}}^{(N=4), m_{1} \ldots m_{r}}\left[M_{A} \mathcal{U}_{\left\{\dddot{F}_{j}\right\} \mid\left\{G_{j}\right\}} \rightarrow \mathcal{Y}_{\left\{\dot{F}_{j}\right\} \mid A,\left\{G_{j}\right\}}\right] \tag{8.19}
\end{equation*}
$$

which can be viewed as the anomaly counterparts of $D_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+2}}^{m_{1} \ldots m_{r}}$ in (8.15).
Applying the replacement rule (8.19) to the examples in (8.7) to (8.10), one arrives at

$$
\begin{align*}
\Delta_{1 \mid 2,3,4,5} & =\mathcal{Y}_{1,2,3,4,5}  \tag{8.20}\\
\Delta_{1 \mid 23,4,5,6} & =\mathcal{Y}_{1,23,4,5,6}+\mathcal{Y}_{12,3,4,5,6}-\mathcal{Y}_{13,2,4,5,6}  \tag{8.21}\\
\Delta_{1 \mid 2,3,4,5,6}^{m} & =\mathcal{Y}_{1,2,3,4,5,6}^{m}+\left[k_{2}^{m} \mathcal{Y}_{12,3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right]  \tag{8.22}\\
\Delta_{1|2| 3,4,5,6,7} & =\mathcal{Y}_{2 \mid 1,3,4,5,6,7}+k_{2}^{m} \mathcal{Y}_{12,3,4,5,6,7}^{m}+\left[s_{23} \mathcal{Y}_{123,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \tag{8.23}
\end{align*}
$$

which can be easily recognized as the anomaly analogues of (8.16).

### 8.5. The $\Lambda$-superfields at ghost-number three

Finally, there is a $N=6$ version of the $R_{i \mid \ldots}^{(N), \ldots}$ in (8.4) where the word $A$ associated with $M_{A}$ becomes a refined slot of the symbol $\mathcal{U} \rightarrow \mathcal{Y}$. The resulting anomalous superfields were sketched in (8.14) and are more cleanly defined as

$$
\begin{equation*}
\Lambda_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+6}}^{m_{1} \ldots m_{r}} \equiv R_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+6}}^{(N=6), m_{1} \ldots m_{r}}\left[M_{A} \mathcal{U}_{\left\{\ddot{F}_{j}\right\} \mid\left\{G_{j}\right\}} \rightarrow \mathcal{Y}_{A,\left\{F_{j}\right\} \mid\left\{G_{j}\right\}}\right] \tag{8.24}
\end{equation*}
$$

This is the anomaly counterpart of the objects $L_{i\left|B_{1}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+4}}^{m_{1} \ldots m_{r}}$ in (8.17).
Under the prescription in (8.24), the examples in (8.7) to (8.10) are mapped to

$$
\begin{align*}
\Lambda_{1 \mid 2,3,4,5,6,7}= & \mathcal{Y}_{1 \mid 2,3,4,5,6,7}  \tag{8.25}\\
\Lambda_{1 \mid 23,4,5,6,7,8}= & \mathcal{Y}_{1 \mid 23,4,5,6,7,8}+\mathcal{Y}_{12 \mid 3,4,5,6,7,8}-\mathcal{Y}_{13 \mid 2,4,5,6,7,8}  \tag{8.26}\\
\Lambda_{1 \mid 2,3,4,5,6,7,8}^{m}= & \mathcal{Y}_{1 \mid 2,3,4,5,6,7,8}^{m}+\left[k_{2}^{m} \mathcal{Y}_{12 \mid 3,4,5,6,7,8}+(2 \leftrightarrow 3, \ldots, 8)\right]  \tag{8.27}\\
\Lambda_{1|2| 3,4,5,6,7,8,9}= & \mathcal{Y}_{1,2 \mid 3,4,5,6,7,8,9}+k_{2}^{m} \mathcal{Y}_{12 \mid 3,4,5,6,7,8,9}^{m} \\
& +\left[s_{23} \mathcal{Y}_{123 \mid 4,5,6,7,8,9}+(3 \leftrightarrow 4, \ldots, 9)\right] . \tag{8.28}
\end{align*}
$$

They can be quickly seen to furnish the anomaly counterparts of (8.18).

## 9. Pseudoinvariant relations for $k_{B}$ momentum contractions

The main concern in this paper is to systematically study the properties of and relations among the pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ which carry the polarization dependence of one-loop amplitudes. In the previous section, we have constructed two families of ghost-number-two superfields whose BRST variations will be demonstrated to generate relations among the $P_{i \mid \ldots}^{m_{1} \ldots}$.

Recall that pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ single out a reference leg $i$ which always enter through a Berends-Giele current of type $M_{i \ldots}$ and which is represented by an unintegrated vertex $V_{i}$ in the one-loop amplitude prescription [4]. The ghost-number-two generators of relations among pseudoinvariants in (8.1) must be carefully chosen in order to avoid admixtures of pseudoinvariants $P_{k \neq i \mid \ldots}^{m_{1} \ldots}$ with a different reference leg $k \neq i$.

It turns out that both the $D$ superfields from section 8.2 and the $L$ superfields from section 8.3 satisfy this criterion, see (8.15) and (8.17) for their precise definitions. BRST variations of type $Q D$ are systematically analyzed in the present section, and section 10 is devoted to $Q L$. We will see how the ghost-number-three expression for $Q D_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+2}}^{m_{1} \ldots m_{r}}$ relates momentum contractions $k_{B_{j}}^{n} P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{n m_{2} \ldots m_{r}}$ to pseudoinvariants at lower rank. These relations are crucial to translate the SYM one-loop amplitudes presented in [31] into worldline parametrization and to make contact with their string theory ancestors.

### 9.1. BRST-exactness versus momentum phase space

As a starting point, we investigate the $Q$ action on unrefined $D$ superfields $D_{i \mid A_{1}, \ldots, A_{r+2}}^{m_{1} \ldots m_{r}}$ such as the simplest examples given in (8.16). For scalars and vectors, we find

$$
\begin{align*}
Q D_{i \mid A, B} & =0  \tag{9.1}\\
Q D_{i \mid A, B, C}^{m} & =k_{i A B C}^{m} C_{i \mid A, B, C} \tag{9.2}
\end{align*}
$$

with overall momentum $k_{i A B C}^{m} \equiv k_{i}^{m}+k_{A}^{m}+k_{B}^{m}+k_{C}^{m}$. This can be verified case by case using the BRST variations (2.35) and (2.36) for each $M_{A, B, C}$ and $M_{A, B, C, D}^{m}$ occurring in $D_{i \mid A, B}$ and $D_{i \mid A, B, C}^{m}$, respectively. Analogous methods are used in all the subsequent cases when $Q$ variations are computed.

Contracting (9.2) with any momentum, one can solve for $C_{i \mid A, B, C}$,

$$
\begin{equation*}
C_{i \mid A, B, C}=Q\left[\frac{k_{m}^{i} D_{i \mid A, B, C}^{m}}{\left(k_{i} \cdot k_{i A B C}\right)}\right] \tag{9.3}
\end{equation*}
$$

i.e. $C_{i \mid A, B, C}$ is BRST exact unless $k_{i} \cdot k_{i A B C}=0$. Note, however, that momentum conservation $k_{i A B C}^{m}=0$ in an $n$-point amplitude (with $n=1+|A|+|B|+|C|$ ) implies $k_{i} \cdot k_{i A B C}=0$ and renders the right-hand side of (9.3) ill-defined. Hence, momentum phase space constraints for $n$ massless particles save $C_{i \mid A, B, C}$ from being BRST exact and preserve its cohomological nature in $n$ point amplitudes.

This is analogous to the superspace representation $\sum_{j=1}^{n-2} M_{12 \ldots j} M_{j+1 \ldots n-1} V_{n}$ of color ordered SYM tree amplitudes [17]. This expression can be rewritten as $Q\left(M_{12 \ldots n-1} V_{n}\right)$ as long as the overall propagator $M_{12 \ldots n-1} \sim s_{12 \ldots n-1}^{-1}$ does not diverge. Again, $n$ particle momentum conservation implying $s_{12 \ldots n-1}=0$ is essential to avoid BRST exactness of the tree amplitude.

In both cases, the cohomology nature of BRST-closed kinematic factors crucially depends on vanishing conformal weight $h \sim s_{12 \ldots n}$. Recall that in a topological conformal field theory where $Q b_{0}=L_{0}$, the cohomology at non-zero conformal weight is empty since every BRST-closed operator would also be BRST-exact [42],

$$
\begin{equation*}
Q \phi=0, \quad L_{0} \phi=h \phi, \quad h \neq 0 \Rightarrow \phi=Q\left(\frac{b_{0} \phi}{h}\right) . \tag{9.4}
\end{equation*}
$$

Starting from rank two, the $Q$ transformations of $D_{i \mid A_{1}, \ldots, A_{r+2}}^{m_{1} m_{2} \ldots m_{r}}$ additionally give rise to anomalous superfields $\Delta_{i \mid A_{1}, \ldots, A_{p+4}}^{m_{1} \ldots m_{p}}$ defined in (8.19), e.g.

$$
\begin{equation*}
Q D_{i \mid A, B, C, D}^{m n}=\delta^{m n} \Delta_{i \mid A, B, C, D}+2 k_{i A B C D}^{(m} C_{i \mid A, B, C, D}^{n)}, \tag{9.5}
\end{equation*}
$$

and more generally,

$$
\begin{equation*}
Q D_{i \mid A_{1}, \ldots, A_{r+2}}^{m_{1} m_{2} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Delta_{i \mid A_{1}, \ldots, A_{r+2}}^{\left.m_{3} m_{4} \ldots m_{r}\right)}+r k_{i A_{1} \ldots A_{r+2}}^{\left(m_{1}\right.} C_{i \mid A_{1}, \ldots, A_{r+2}}^{\left.m_{2} m_{3} \ldots m_{r}\right)} . \tag{9.6}
\end{equation*}
$$

As a consequence, the identification of BRST-exact quantities crucially depends on the momentum phase space. In case of momentum conservation $k_{i A_{1} \ldots A_{r+2}}^{m}=0,(9.6)$ implies $^{19}$ $Q$ exactness of the anomalous superfield $\Delta_{i \mid A_{1}, \ldots, A_{p+4}}^{m_{1} \ldots m_{p}}$, hence the latter does not contribute to physical amplitudes at multiplicity $1+\sum_{j=1}^{p-4}\left|A_{j}\right|$. However, it is important to stress that the hexagon gauge anomaly superfield $\Delta_{2 \mid 3,4,5,6}=\mathcal{Y}_{2,3,4,5,6}$ in the one-loop six-point

19 At rank $r=2$ and $r=3$, BRST exactness of $\Delta_{i \mid A, B, C, D}$ and $\Delta_{i \mid A, B, C, D, E}^{m}$ immediately follows from single traces of (9.6) at $k_{i A_{1} \ldots A_{r+2}}^{m}=0$. Higher rank $r \geq 4$ requires combinations of multiple $\delta_{m_{i} m_{j}}$ contractions in order to identify the BRST generator of $\Delta_{i \mid A_{1}, \ldots, A_{p+4}}^{m_{1} \ldots m_{p}}$ at any rank.
amplitude [26] is not BRST exact for the momentum phase space of six particles since $k_{23456}^{m}$ is not zero.

On the other hand, generic momentum configurations with $k_{i A_{1} \ldots A_{r+2}}^{m} \neq 0$ render the traceless components of pseudoinvariants $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} m_{2} \ldots m_{r}}$ BRST exact. This can be seen from the traceless projection of (9.6), see (E.2) in appendix E for the explicit form of the BRST generator for $C_{i \mid A, B, C, D}^{m}$.

The correspondence between the pseudoinvariants $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} m_{2} \ldots m_{r}}$ and the anomaly invariants $\Gamma_{i \mid A_{1}, \ldots, A_{r+5}}^{m_{1} m_{2}, \ldots m_{r}}$ described in section 7 and formalized in section 8.1 allows to immediately write down the anomaly correspondent of (9.6):

$$
\begin{equation*}
Q \Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} m_{2} \ldots m_{r}}=r k_{i A_{1} \ldots A_{r+4}}^{\left(m_{1}\right.} \Gamma_{i \mid A_{1}, \ldots, A_{r+4}}^{\left.m_{2} m_{3} \ldots m_{r}\right)} . \tag{9.7}
\end{equation*}
$$

We exploit that $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} m_{2} \ldots m_{r}}$ is the anomaly counterpart of $D_{i \mid A_{1}, \ldots, A_{r+2}}^{m_{1} m_{2} \ldots m_{r}}$ which, loosely speaking, does not have a higher anomaly image. Of course, (9.7) confirms that momentum conservation $k_{i A_{1} \ldots A_{r+4}}^{m}=0$ implies BRST closure of $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} m_{2} \ldots m_{r}}$, in lines with the discussion of BRST exactness along with (9.6).

### 9.2. Momentum contractions of unrefined pseudoinvariants

Refined versions of the $D$ superfields turn out to generate a much richer set of ghost number three relations than their unrefined counterparts studied in section 9.1. We start by exploring the case of minimal refinement $d=1$ and will find that $Q D_{i|A| B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ relates momentum contractions $\sim k_{A_{j}}$ of $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ to pseudoinvariants at lower rank.

As a first example, consider the inequivalent cases at five- and six-points,

$$
\begin{align*}
Q D_{1|2| 3,4,5} & =\Delta_{1 \mid 2,3,4,5}+k_{2}^{m} C_{1 \mid 2,3,4,5}^{m}+\left[s_{23} C_{1 \mid 23,4,5}+(3 \leftrightarrow 4,5)\right]  \tag{9.8}\\
Q D_{1|23| 4,5,6} & =\Delta_{1 \mid 23,4,5,6}+P_{1|3| 2,4,5,6}-P_{1|2| 3,4,5,6}+k_{23}^{m} C_{1 \mid 23,4,5,6}^{m} \\
& +\left[s_{34} C_{1 \mid 234,5,6}-s_{24} C_{1 \mid 324,5,6}+(4 \leftrightarrow 5,6)\right] \\
Q D_{1|4| 23,5,6} & =\Delta_{1 \mid 23,4,5,6}+k_{4}^{m} C_{1 \mid 23,4,5,6}^{m}+s_{24} C_{1 \mid 324,5,6}-s_{34} C_{1 \mid 234,5,6} \\
& +s_{45} C_{1 \mid 23,45,6}+s_{46} C_{1 \mid 23,46,5}
\end{align*}
$$

where (9.8) underpins the second example in (8.2) provided that momentum conservation $k_{12345}^{m}=0$ renders $\Delta_{1 \mid 2,3,4,5}$ BRST exact.

The combinations of $s_{i j} C_{1 \mid A, B, C}$ can be neatly described using the $S[A, B]$ map in (5.14), see in particular (5.13) for examples. The seven-point instances of $Q D_{1|A| B, C, D}$
displayed in (E.3) to (E.6) support this pattern and help to identify the appearance of $P_{1|A| B, C, D, E}$ as deconcatenations. These observations lead to the following generalization,

$$
\begin{gather*}
Q D_{i|A| B, C, D}=\Delta_{i \mid A, B, C, D}+k_{A}^{m} C_{i \mid A, B, C, D}^{m}+\sum_{X Y=A}\left(P_{i|Y| X, B, C, D}-P_{i|X| Y, B, C, D}\right) \\
+C_{i \mid S[A, B], C, D}+C_{i \mid S[A, C], B, D}+C_{i \mid S[A, D], B, C} \tag{9.9}
\end{gather*}
$$

Note that $Q D_{i|A| B, C, D}$ always generates relations for contractions of $C_{i \mid A, B, C, D}^{m}$ with the entire momentum $k_{A}^{m}=\sum_{j=1}^{|A|} k_{a_{j}}^{m}$ in the slot $A=a_{1} a_{2} \ldots a_{|A|}$. This method does not provide any information on contractions with partial slot momenta, e.g. $k_{a_{1}}^{m} C_{i \mid A, B, C, D}^{m}$ with $|A| \geq 2$.

It is natural to repeat the BRST manipulations for vectorial $D$ superfields such as

$$
\begin{align*}
Q D_{1|2| 3,4,5,6}^{m} & =\Delta_{1 \mid 2,3,4,5,6}^{m}+k_{p}^{2} C_{1 \mid 2,3,4,5,6}^{m p}+\left(k_{123456}^{m}-k_{2}^{m}\right) P_{1|2| 3,4,5,6}  \tag{9.10}\\
& +s_{23} C_{1 \mid 23,4,5,6}^{m}+s_{24} C_{1 \mid 24,3,5,6}^{m}+s_{25} C_{1 \mid 25,3,4,6}^{m}+s_{26} C_{1 \mid 26,3,4,5}^{m} \\
Q D_{1|23| 4,5,6,7}^{m} & =\Delta_{1 \mid 23,4,5,6,7}^{m}+k_{23}^{p} C_{1 \mid 23,4,5,6,7}^{m p}+\left(k_{1234567}^{m}-k_{23}^{m}\right) P_{1|23| 4,5,6,7} \\
& -P_{1|2| 3,4,5,6,7}^{m}+P_{1|3| 2,4,5,6,7}^{m}+\left[s_{34} C_{1 \mid 234,5,6,7}^{m}-s_{24} C_{1 \mid 324,5,6,7}^{m}+(4 \leftrightarrow 5,6,7)\right] \\
Q D_{1|4| 23,5,6,7}^{m} & =\Delta_{1 \mid 23,4,5,6,7}^{m}+k_{4}^{p} C_{1 \mid 23,4,5,6,7}^{m p}+\left(k_{1234567}^{m}-k_{4}^{m}\right) P_{1|4| 23,5,6,7} \\
& +s_{24} C_{1 \mid 324,5,6,7}^{m}-s_{34} C_{1 \mid 234,5,6,7}^{m}+\left[s_{45} C_{1 \mid 23,45,6,7}^{m}+(5 \leftrightarrow 6,7)\right]
\end{align*}
$$

which can be summarized by a general formula similar to the scalar case in (9.9),

$$
\begin{align*}
& Q D_{i|A| B, C, D, E}^{m}=\Delta_{i \mid A, B, C, D, E}^{m}+k_{A}^{p} C_{i \mid A, B, C, D, E}^{p m}+\left(k_{i A B C D E}^{m}-k_{A}^{m}\right) P_{i|A| B, C, D, E} \\
+ & {\left[C_{i|S| A, B], C, D, E}^{m}+(B \leftrightarrow C, D, E)\right]+\sum_{X Y=A}^{m}\left(P_{i|Y| X, B, C, D, E}^{m}-P_{i|X| Y, B, C, D, E}^{m}\right) . } \tag{9.11}
\end{align*}
$$

Tensorial generalizations at rank $r \geq 2$ additionally involve refined versions of the anomalous $\Delta$ superfields in (8.19). The simplest example occurs at seven points,

$$
\begin{align*}
Q D_{1|2| 3,4,5,6,7}^{m n} & =\Delta_{1 \mid 2,3,4,5,6,7}^{m n}+\delta^{m n} \Delta_{1|2| 3,4,5,6,7}+k_{2}^{p} C_{1 \mid 2,3,4,5,6,7}^{m n p}  \tag{9.12}\\
& +2\left(k_{1234567}^{(m}-k_{2}^{(m}\right) P_{1|2| 3,4,5,6,7}^{n)}+\left[s_{23} C_{1 \mid 23,4,5,6,7}^{m n}+(3 \leftrightarrow 4,5,6,7)\right]
\end{align*}
$$

where $\Delta_{1|2| 3,4,5,6,7}$ is given by (8.23). The structure of the scalar and vector cases (9.9) and (9.11) inspires the following generalization of (9.12) to multiparticle slots:

$$
\begin{align*}
& Q D_{i|A| B, C, D, E, F}^{m n}=\Delta_{i \mid A, B, C, D, E, F}^{m n}+\delta^{m n} \Delta_{i|A| B, C, D, E, F}+k_{A}^{p} C_{i \mid A, B, C, D, E, F}^{m n p} \\
& \quad+2\left(k_{i A B C D E F}^{(m}-k_{A}^{(m}\right) P_{i|A| B, C, D, E, F}^{n)}+\left[C_{i \mid S[A, B], C, D, E, F}^{m n}+(B \leftrightarrow C, D, E, F)\right] \\
& \quad+\sum_{X Y=A}\left(P_{i|Y| X, B, C, D, E, F}^{m n}-P_{i|X| Y, B, C, D, E, F}^{m n}\right) . \tag{9.13}
\end{align*}
$$

This in turn allows to infer the BRST variation of $D_{i|A| \ldots}^{m_{1} \ldots}$ superfields at generic rank:

$$
\begin{align*}
& Q D_{i|A| B_{1}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}=\Delta_{i \mid A, B_{1}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}+\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Delta_{i|A| B_{1}, \ldots, B_{r+3}}^{\left.m_{3} \ldots m_{r}\right)}+k_{A}^{p} C_{i \mid A, B_{1}, \ldots, B_{r+3}}^{p m_{1} \ldots m_{r}} \\
& \quad+r\left(k_{i A B_{1} \ldots B_{r+3}}^{\left(m_{1}\right.}-k_{A}^{\left(m_{1}\right.}\right) P_{i|A| B_{1}, \ldots, B_{r+3}}^{\left.m_{2} \ldots m_{r}\right)}+\left[C_{i \mid S\left[A, B_{1}\right], B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)\right] \\
& \quad+\sum_{X Y=A}\left(P_{i|Y| X, B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}-P_{i|X| Y, B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}\right) . \tag{9.14}
\end{align*}
$$

Recall that momentum conservation $k_{i A_{1} \ldots A_{r+4}}^{m}=0$ implies BRST exactness of the unrefined representatives $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ of the anomalous $\Delta$ superfields. Hence, the latter do not contribute when the relations (9.9), (9.11), (9.13) and (9.14) are applied to physical amplitudes. However, the situation is completely different for their refined counterparts $\Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$. As will be demonstrated in the following, refined $\Delta$ superfields are not BRST closed, regardless of momentum phase space constraints, so $Q$ exactness can be clearly ruled out. Starting from seven points, refined anomaly superfields $\Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$ at ghost-number three cannot be discarded in the discussion of one-loop amplitudes, see [31].

The ghost-number-four expression for $Q \Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$ can be inferred by analogy with (9.9) to (9.14). Since (8.19) identifies $\Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$ to be the anomaly counterpart of $D_{i|A| B_{1}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}$, the BRST transformation of the former follows from (9.9) and (9.14) upon discarding anomalous terms and converting $C_{i \mid \ldots}^{m_{1} \ldots}, P_{i \mid \ldots}^{m_{1} \ldots} \rightarrow \Gamma_{i \mid \ldots}^{m_{1} \ldots}$ :

$$
\begin{align*}
Q \Delta_{i|A| B, \ldots, F}= & k_{A}^{m} \Gamma_{i \mid A, B, \ldots, F}^{m}+\sum_{X Y=A}\left(\Gamma_{i|Y| X, B, \ldots, F}-\Gamma_{i|X| Y, B, \ldots, F}\right) \\
& +\left[\Gamma_{i \mid S[A, B], C, D, E, F}+(B \leftrightarrow C, D, E, F)\right]  \tag{9.15}\\
Q \Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} m_{2} \ldots m_{r}}= & k_{A}^{p} \Gamma_{i \mid A, B_{1}, \ldots, B_{r+5}}^{p m_{1} \ldots m_{r}}+r\left(k_{i A B_{1} \ldots B_{r+5}}^{\left(m_{1}\right.}-k_{A}^{\left(m_{1}\right.}\right) \Gamma_{i|A| B_{1}, \ldots, B_{r+5}}^{\left.m_{2} \ldots m_{r}\right)} \\
& +\left[\Gamma_{i \mid S\left[A, B_{1}\right], B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+5}\right)\right] \\
& +\sum_{X Y=A}\left(\Gamma_{i|Y| X, B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}-\Gamma_{i|X| Y, B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}\right) . \tag{9.16}
\end{align*}
$$

Together with expressions for $Q C_{i \mid \ldots}^{m_{1} \ldots}$ and $Q P_{i \mid \ldots}^{m_{1} \ldots}$ in (7.9) and (7.17), one can check BRST closure of the right-hand side of (9.9) and (9.14).

### 9.3. Momentum contractions of refined pseudoinvariants

The procedure from the previous section is now extended to higher refinement. In the simplest scalar cases at seven- and eight-points, we find

$$
\begin{align*}
Q D_{1|2,3| 4,5,6,7} & =\Delta_{1|2| 3,4,5,6,7}+\Delta_{1|3| 2,4,5,6,7}+k_{3}^{m} P_{1|2| 3,4,5,6,7}^{m}+k_{2}^{m} P_{1|3| 2,4,5,6,7}^{m} \\
& +\left[s_{34} P_{1|2| 34,5,6,7}+s_{24} P_{1|3| 24,5,6,7}+(4 \leftrightarrow 5,6,7)\right]  \tag{9.17}\\
Q D_{1|23,4| 5,6,7,8} & =\Delta_{1|23| 4,5,6,7,8}+\Delta_{1|4| 23,5,6,7,8}+k_{23}^{m} P_{1|4| 23,5,6,7,8}^{m}+k_{4}^{m} P_{1|23| 4,5,6,7,8}^{m} \\
& +\left[s_{35} P_{1|4| 235,6,7,8}-s_{25} P_{1|4| 325,6,7,8}+s_{45} P_{1|23| 45,6,7,8}+(5 \leftrightarrow 6,7,8)\right] \\
& -P_{1|2,4| 3,5,6,7,8}+P_{1|3,4| 2,5,6,7,8} \\
Q D_{1|4,5| 23,6,7,8} & =\Delta_{1|4| 23,5,6,7,8}+\Delta_{1|5| 23,4,6,7,8}+k_{4}^{m} P_{1|5| 23,4,6,7,8}^{m}+k_{5}^{m} P_{1|4| 23,5,6,7,8}^{m} \\
& +s_{24} P_{1|5| 324,6,7,8}-s_{34} P_{1|5| 234,6,7,8}+s_{25} P_{1|4| 325,6,7,8}-s_{35} P_{1|4| 235,6,7,8} \\
& +\left[s_{46} P_{1|5| 23,46,7,8}+s_{56} P_{1|4| 23,56,7,8}+(6 \leftrightarrow 7,8)\right]
\end{align*}
$$

signaling the general rule

$$
\begin{align*}
& \quad Q D_{i|A, B| C, D, E, F}=\Delta_{i|A| B, C, D, E, F}+\Delta_{i|B| A, C, D, E, F}+k_{A}^{m} P_{i|B| A, C, D, E, F}^{m} \\
&+k_{B}^{m} P_{i|A| B, C, D, E, F}^{m}+\left[P_{i|A| S[B, C], D, E, F}+P_{i|B| S[A, C], D, E, F}+(C \leftrightarrow D, E, F)\right]  \tag{9.18}\\
&+ \sum_{X Y=A}\left(P_{1|Y, B| X, C, D, E, F}-P_{1|X, B| Y, C, D, E, F}\right)+\sum_{X Y=B}\left(P_{1|Y, A| X, C, D, E, F}-P_{1|X, A| Y, C, D, E, F}\right) .
\end{align*}
$$

Given the appearance of two different momentum contractions $k_{A}^{m} P_{i|B| A, \ldots}^{m}$ and $k_{B}^{m} P_{i|A| B, \ldots,}^{m}$, (9.18) can be viewed as a weaker result in comparison to the relations in section 9.2 for a single $k_{A}^{p} C_{i \mid A, \ldots}^{p m_{1} \ldots}$.

Recall that tensorial superfields $D_{i|A| B_{1}, \ldots .}^{m_{1}{ }_{2}}$ give rise to additional terms $\sim k^{m}, \delta^{m n}$ absent in the scalar case, see (9.11), (9.13) and (9.14). The same kind of contributions appear in the vector and tensor generalization of (9.18), e.g.

$$
\begin{align*}
Q D_{1|2,3| 4, \ldots, 8}^{m} & =\Delta_{1|2| 3, \ldots, 8}^{m}+\Delta_{1|3| 2,4, \ldots, 8}^{m}+k_{3}^{p} P_{1|2| 3, \ldots, 8}^{m p}+k_{2}^{p} P_{1|3| 2,4, \ldots, 8}^{m p} \\
& +\left[s_{34} P_{1|2| 34,5, \ldots, 8}^{m}+s_{24} P_{1|3| 24,5, \ldots, 8}^{m}+(4 \leftrightarrow 5, \ldots, 8)\right] \\
& +\left(k_{12345678}^{m}-k_{23}^{m}\right) P_{1|2,3| 4, \ldots, 8},  \tag{9.19}\\
Q D_{1|2,3| 4, \ldots, 9}^{m n} & =\Delta_{1|2| 3, \ldots, 9}^{m n}+\Delta_{1|3| 2,4, \ldots, 9}^{m n}+\delta^{m n} \Delta_{1|2,3| 4, \ldots, 9}+k_{3}^{p} P_{1|2| 3, \ldots, 9}^{m n p} \\
& +k_{2}^{p} P_{1|3| 2,4, \ldots, 9}^{m n p}+\left[s_{34} P_{1|2| 34,5, \ldots, 9}^{m n}+s_{24} P_{1|3| 24,5, \ldots, 9}^{m n}+(4 \leftrightarrow 5, \ldots, 9)\right] \\
& +2\left(k_{123456789}^{(m}-k_{23}^{(m}\right) P_{1|2,3| 4, \ldots, 9}^{n)} . \tag{9.20}
\end{align*}
$$

This allows to anticipate the multiparticle version at general rank,

$$
\begin{align*}
Q D_{i|A, B| C_{1}, \ldots, C_{r+4}}^{m_{1} \ldots m_{r}} & =\Delta_{i|A| B, C_{1}, \ldots, C_{r+4}}^{m_{1} \ldots m_{r}}+\Delta_{i|B| A, C_{1}, \ldots, C_{r+4}}^{m_{1} \ldots m_{r}}+\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Delta_{i|A, B| C_{1}, \ldots, C_{r+4}}^{\left.m_{3} \ldots m_{r}\right)} \\
& +\left[P_{i|A| S\left[B, C_{1}\right], C_{2}, \ldots, C_{r+4}}^{m_{1} \ldots m_{r}}+P_{i|B| S\left[A, C_{1}\right], C_{2}, \ldots, C_{r+4}}^{m_{1} \ldots m_{r}}+\left(C_{1} \leftrightarrow C_{2}, \ldots, C_{r+4}\right)\right] \\
& +r\left(k_{i A B C_{1} \ldots C_{r+4}}^{\left(m_{1}\right.}-k_{A B}^{\left(m_{1}\right.}\right) P_{i|A, B| C_{1}, \ldots, C_{r+4}}^{\left.m_{2} \ldots m_{r}\right)} \\
& +k_{A}^{p} P_{i|B| A, C_{1}, \ldots, C_{r+4}}^{p m_{1} \ldots m_{r}}+k_{B}^{p} P_{i|A| B, C_{1}, \ldots, C_{r+4}}^{p m_{1} \ldots m_{r}} \\
& -\sum_{X=A}\left(P_{i|X, B| Y, C_{1} \ldots C_{r+4}}^{m_{1} \ldots m_{r}}-P_{i|Y, B| X, C_{1} \ldots C_{r+4}}^{m_{1} \ldots m_{r}}\right) \\
& -\sum_{X Y=B}\left(P_{i|A, X| Y, C_{1} \ldots C_{r+4}}^{m_{1} \ldots m_{r}}-P_{i|A, Y| X, C_{1} \ldots C_{r+4}}^{m_{1} \ldots m_{r}}\right), \tag{9.21}
\end{align*}
$$

where the deconcatenation terms $\sim \sum_{X Y=A, B}$ follow by analogy with (9.18). In comparison to the counterpart (9.14) of lower refinement, terms of the form $\Delta_{i|A| B, C_{1}, \ldots,}^{m_{1} \ldots m_{r}}$, $r k_{A}^{\left(m_{1}\right.} P_{i|A, B| C_{1}, \ldots .}^{\left.m_{2} \ldots m_{r}\right)}, k_{A}^{p} P_{i|B| A, C_{1}, \ldots,}^{p m_{1} \ldots m_{r}}, P_{i|A| S\left[B, C_{1}\right], C_{2}, \ldots}^{m_{1} \ldots m_{r}}$ and $\sum_{X Y=A} \ldots$ are doubled in (9.21). This suggests the following BRST variation for $D$ superfields at general refinement,

$$
\begin{align*}
& Q D_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+2}}^{m_{1} \ldots m_{r}}=\left[\Delta_{i\left|A_{2}, \ldots, A_{d}\right| A_{1}, B_{1}, \ldots, B_{r+d+2}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right] \\
& +\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Delta_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+2}}^{\left.m_{3} \ldots m_{r}\right)}+r k_{i B_{1} B_{2} \ldots B_{r+d+2}}^{\left(m_{1}\right.} P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+2}}^{\left.m_{2} \ldots m_{r}\right)}  \tag{9.22}\\
& +\left(k_{A_{1}}^{p} P_{i\left|A_{2}, \ldots, A_{d}\right| A_{1}, B_{1}, \ldots, B_{r+d+2}}^{p m_{1} \ldots m_{r}}+\left[P_{i\left|A_{2}, \ldots, A_{d}\right| S\left[A_{1}, B_{1}\right], B_{2}, \ldots, B_{r+d+2}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+d+2}\right)\right]\right. \\
& \left.-\sum_{X Y=A_{1}}\left(P_{i\left|X, A_{2}, \ldots, A_{d}\right| Y, B_{1}, \ldots, B_{r+d+2}}^{m_{1} \ldots m_{r}}-P_{i\left|Y, A_{2}, \ldots, A_{d}\right| X, B_{1}, \ldots, B_{r+d+2}}^{m_{1} \ldots m_{r}}\right)+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right) .
\end{align*}
$$

Again, we can directly infer the BRST variation of the anomalous counterparts $\Delta_{i \mid \ldots}^{m_{1} \ldots}$ by discarding their appearance in the right-hand side of (9.22) and replacing the remaining terms via $C_{i \mid \ldots}^{m_{1} \ldots}, P_{i \mid \ldots}^{m_{1} \ldots} \rightarrow \Gamma_{i \mid \ldots}^{m_{1} \ldots}$ :

$$
\begin{align*}
& Q \Delta_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}=r k_{1 B_{1} B_{2} \ldots B_{r+d+4}}^{\left(m_{1}\right.} \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{\left.m_{2} \ldots m_{r}\right)}  \tag{9.23}\\
& +\left(k_{A_{1}}^{p} \Gamma_{i\left|A_{2}, \ldots, A_{d}\right| A_{1}, B_{1}, \ldots, B_{r+d+4}}^{p m_{1} \ldots m_{r}}+\left[\Gamma_{i\left|A_{2}, \ldots, A_{d}\right| S\left[A_{1}, B_{1}\right], B_{2}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+d+4}\right)\right]\right. \\
& \left.-\sum_{X Y=A_{1}}\left(\Gamma_{i\left|X, A_{2}, \ldots, A_{d}\right| Y, B_{1}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}-\Gamma_{i\left|Y, A_{2}, \ldots, A_{d}\right| X, B_{1}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}\right)+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right) .
\end{align*}
$$

Using (9.23) and the $Q$ variation (7.17) of the pseudoinvariants, one can verify BRST closure of the right-hand side of (9.22). This is a strong consistency check since it requires every single term in (9.22) to conspire.

## 10. Pseudoinvariant relations for $k_{i}$ momentum contractions

In the previous section, $D$ superfields defined in (8.15) were shown to generate relations for $k_{B_{j}}$ contractions of pseudoinvariants. We shall now investigate the second family of ghost-number-two objects, the $L$ superfields defined in (8.17). It turns out that their BRST variations relate contractions of $P_{i \mid \ldots}^{m_{1} \ldots}$ with the momentum $k_{i}$ of the reference leg $i$ to pseudoinvariants of lower rank.

## 10.1. $k_{i}$ contractions of unrefined pseudoinvariants

This section is devoted to the unrefined superfields $L_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$, see (8.17). The BRST variations of the simplest scalars are given by

$$
\begin{align*}
Q L_{1 \mid 2,3,4,5}= & \Delta_{1 \mid 2,3,4,5}+k_{1}^{m} C_{1 \mid 2,3,4,5}^{m}  \tag{10.1}\\
Q L_{1 \mid 23,4,5,6}= & \Delta_{1 \mid 23,4,5,6}+k_{1}^{m} C_{1 \mid 23,4,5,6}^{m}+P_{1|2| 3,4,5,6}-P_{1|3| 2,4,5,6} \\
Q L_{1 \mid 234,5,6,7}= & \Delta_{1 \mid 234,5,6,7}+k_{1}^{m} C_{1 \mid 234,5,6,7}^{m} \\
& +P_{1|23| 4,5,6,7}+P_{1|2| 34,5,6,7}-P_{1|34| 2,5,6,7}-P_{1|4| 23,5,6,7} \\
Q L_{1 \mid 23,45,6,7}= & \Delta_{1 \mid 23,45,6,7}+k_{1}^{m} C_{1 \mid 23,45,6,7}^{m} \\
& +P_{1|2| 3,45,6,7}-P_{1|3| 2,45,6,7}+P_{1|4| 23,5,6,7}-P_{1|5| 23,4,6,7}
\end{align*}
$$

and thereby provide relations for $k_{i}^{m} C_{i \mid A, B, C, D}^{m}$. The explicit form of $L_{1 \mid 2,3,4,5}$ and $L_{1 \mid 23,4,5,6}$ can be found in (8.18), and the former underpins the first example in (8.2) provided that momentum conservation $k_{12345}^{m}=0$ renders $\Delta_{1 \mid 2,3,4,5}$ BRST exact.

The examples in (10.1) suggest the multiparticle pattern,

$$
\begin{align*}
Q L_{i \mid A, B, C, D}= & \Delta_{i \mid A, B, C, D}+k_{i}^{m} C_{i \mid A, B, C, D}^{m}  \tag{10.2}\\
& +\left[\sum_{X Y=A}\left(P_{i|X| Y, B, C, D}-P_{i|Y| X, B, C, D}\right)+(A \leftrightarrow B, C, D)\right]
\end{align*}
$$

where the anomalous $\Delta_{i \mid A, B, C, D}$ are defined by (8.19) and also appear in the relations (9.9) for different contractions $k_{A}^{m} C_{i \mid A, B, C, D}^{m}$.

The simplest $Q$ variations of ghost number two vectors $L_{i \mid A, B, C, D, E}^{m}$ read

$$
\begin{align*}
Q L_{1 \mid 2,3,4,5,6}^{m}= & \Delta_{1 \mid 2,3,4,5,6}^{m}+k_{1}^{n} C_{1 \mid 2,3,4,5,6}^{m n}+\left[k_{2}^{m} P_{1|2| 3,4,5,6}+(2 \leftrightarrow 3,4,5,6)\right]  \tag{10.3}\\
Q L_{1 \mid 23,4,5,6,7}^{m}= & \Delta_{1 \mid 23,4,5,6,7}^{m}+k_{1}^{n} C_{1 \mid 23,4,5,6,7}^{m n}+\left[k_{4}^{m} P_{1|4| 23,5,6,7}+(4 \leftrightarrow 5,6,7)\right] \\
& +k_{23}^{m} P_{1|23| 4,5,6,7}+P_{1|2| 3,4,5,6,7}^{m}-P_{1|3| 2,4,5,6,7}^{m}, \tag{10.4}
\end{align*}
$$

see (8.18) for the expansion of $L_{1 \mid 2,3,4,5,6}^{m}$. The novel class of terms $\sim k^{m}$ in (10.3) and (10.4) are reproduced by the general formula,

$$
\begin{align*}
Q L_{i \mid A, B, C, D, E}^{m}= & \Delta_{i \mid A, B, C, D, E}^{m}+k_{i}^{n} C_{i \mid A, B, C, D, E}^{m n}+\left[k_{A}^{m} P_{i|A| B, C, D, E}\right.  \tag{10.5}\\
& \left.+\sum_{X Y=A}\left(P_{i|X| Y, B, C, D, E}^{m}-P_{i|Y| X, B, C, D, E}^{m}\right)+(A \leftrightarrow B, C, D, E)\right] .
\end{align*}
$$

As the last explicit example in this section, consider the two-tensor relation,

$$
\begin{align*}
Q L_{1 \mid 2,3,4,5,6,7}^{m n}= & \Delta_{1 \mid 2,3,4,5,6,7}^{m n}+\delta^{m n} \Lambda_{1 \mid 2,3,4,5,6,7}+k_{1}^{p} C_{1 \mid 2,3,4,5,6,7}^{m n n} \\
& +2\left[k_{2}^{(m} P_{1|2| 3,4,5,6,7}^{n)}+(2 \leftrightarrow 3,4,5,6,7)\right], \tag{10.6}
\end{align*}
$$

subject to an anomalous trace with $\Lambda_{1 \mid 2,3,4,5,6,7}$ given by (8.25). Its generalization $\Lambda_{i \mid A_{1}, A_{2}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}$ to multiparticle slots and higher rank is defined in (8.24) and finds appearance in the general rank-two relation,

$$
\begin{align*}
& Q L_{i \mid A, B, C, D, E, F}^{m n}=\Delta_{i \mid A, B, C, D, E, F}^{m n}+\delta^{m n} \Lambda_{i \mid A, B, C, D, E, F}+k_{i}^{p} C_{i \mid A, B, C, D, E, F}^{m n p}  \tag{10.7}\\
+ & {\left[\sum_{X Y=A}\left(P_{i|X| Y, B, C, D, E, F}^{m n}-P_{i|Y| X, B, C, D, E, F}^{m n}\right)+2 k_{A}^{(m} P_{i|A| B, C, D, E, F}^{n)}+(A \leftrightarrow B, \ldots, F)\right] . }
\end{align*}
$$

The expressions (10.2), (10.5) and (10.7) for $Q L_{i \mid A_{1}, A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ at rank $r=0,1,2$ lead to a natural generalization for higher ranks,

$$
\begin{align*}
Q L_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}= & \Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}+\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Lambda_{i \mid A_{1}, \ldots, A_{r+4}}^{\left.m_{3} \ldots m_{r}\right)}+k_{i}^{p} C_{i \mid A_{1}, \ldots, A_{r+4}}^{p m_{1} \ldots m_{r}} \\
+ & {\left[\sum_{X Y=A_{1}}\left(P_{i|X| Y, A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}-P_{i|Y| X, A_{2}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}\right)\right.} \\
& \left.+r k_{A_{1}}^{\left(m_{1}\right.} P_{i\left|A_{1}\right| A_{2}, \ldots, A_{r+4}}^{\left.m_{2} \ldots m_{r}\right)}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+4}\right)\right] . \tag{10.8}
\end{align*}
$$

Similar to (9.14) for $k_{A_{1}}^{p} C_{i \mid A_{1}, \ldots, A_{r+4}}^{p m_{1} \ldots m_{r}}$ as derived from $Q D_{i \mid \ldots}^{m_{1} \ldots}$, two classes of anomalous terms appear in (10.8). The unrefined $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ common to both relations are BRST exact under momentum conservation, see (9.6), and can be discarded in the context of amplitudes. However, the second anomalous term $\Lambda_{i \mid A_{1}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}$ in the trace of (10.8) is not even BRST closed, regardless of the momentum phase space. Its non-vanishing $Q$ variation follows as the anomaly analogue of (10.2) and (10.8), i.e. by dropping the anomalous
contributions from the latter and replacing the rest according to $C_{i \mid \ldots}^{m_{1} \ldots}, P_{i \mid \ldots}^{m_{1} \ldots} \rightarrow \Gamma_{i \mid \ldots}^{m_{1} \ldots}$ :

$$
\begin{align*}
& Q \Lambda_{i \mid A, B, C, D, E, F}=k_{i}^{m} \Gamma_{i \mid A, B, C, D, E, F}^{m}  \tag{10.9}\\
& \quad+\left[\sum_{X Y=A}\left(\Gamma_{i|X| Y, B, C, D, E, F}-\Gamma_{i|Y| X, B, C, D, E, F}\right)+(A \leftrightarrow B, C, D, E, F)\right] \\
& Q \Lambda_{i \mid A_{1}, A_{2}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}=k_{i}^{p} \Gamma_{i \mid A_{1}, \ldots, A_{r+6}}^{p m_{1} \ldots m_{r}}+\left[r k_{A_{1}}^{\left(m_{1}\right.} \Gamma_{i\left|A_{1}\right| A_{2}, \ldots, A_{r+6}}^{\left.m_{2} \ldots m_{r}\right)}\right.  \tag{10.10}\\
& \left.\quad+\sum_{X Y=A_{1}}\left(\Gamma_{i|X| Y, A_{2}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}-\Gamma_{i|Y| X, A_{2}, \ldots, A_{r+6}}^{m_{1} \ldots m_{r}}\right)+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{r+6}\right)\right]
\end{align*}
$$

Together with the BRST transformations (7.9) and (7.17) of pseudoinvariants, (10.9) and (10.10) allow to check BRST closure of (10.8) and furnish a strong consistency check on the results in this section.

## 10.2. $k_{i}$ contractions of refined pseudoinvariants

We next proceed to refined versions $L_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots .}^{m_{1} \ldots}$ of the ghost-number-two objects under discussion. In the simplest scalar cases,

$$
\begin{align*}
Q L_{1|2| 3,4,5,6,7} & =\Delta_{1|2| 3,4,5,6,7}+\Lambda_{1 \mid 2,3,4,5,6,7}+k_{12}^{m} P_{1|2| 3,4,5,6,7}^{m} \\
& +\left[s_{23} P_{1|23| 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]  \tag{10.11}\\
Q L_{1|23| 4,5,6,7,8} & =\Delta_{1|23| 4,5,6,7,8}+\Lambda_{1 \mid 23,4,5,6,7,8}+k_{123}^{m} P_{1|23| 4,5,6,7,8}^{m} \\
& +\left[s_{34} P_{1|234| 5,6,7,8}-s_{24} P_{1|324| 5,6,7,8}+(4 \leftrightarrow 5,6,7,8)\right] \\
Q L_{1|4| 23,5,6,7,8} & =\Delta_{1|4| 23,5,6,7,8}+\Lambda_{1 \mid 23,4,5,6,7,8}+k_{14}^{m} P_{1|4| 23,5,6,7,8}^{m} \\
& +s_{24} P_{1|324| 5,6,7,8}-s_{34} P_{1|234| 5,6,7,8}+\left[s_{45} P_{1|45| 23,6,7,8}+(5 \leftrightarrow 6,7,8)\right] \\
& +P_{1|2,4| 3,5,6,7,8}-P_{1|3,4| 2,5,6,7,8},
\end{align*}
$$

where the expansion of $L_{1|2| 3,4,5,6,7}$ is given in (8.18). This generalizes to

$$
\begin{gather*}
Q L_{i|A| B, C, D, E, F}=\Delta_{i|A| B, C, D, E, F}+\Lambda_{i \mid A, B, C, D, E, F}+k_{i A}^{m} P_{i|A| B, C, D, E, F}^{m}  \tag{10.12}\\
+\left[P_{i|S[A, B]| C, D, E, F}+\sum_{X Y=B}\left(P_{i|A, X| Y, C, D, E, F}-P_{i|A, Y| X, C, D, E, F}\right)+(B \leftrightarrow C, D, E, F)\right] .
\end{gather*}
$$

Interestingly, (10.12) contains a contraction of $P_{i \mid \ldots}^{m}$ with the combined momentum $k_{i A}^{m}=$ $k_{i}^{m}+k_{A}^{m}$ including the refined slot $A$ and does not isolate its $k_{i}$ contraction. This is analogous to the shortcoming of the relation (9.18) to address the two-term combinations $k_{A}^{m} P_{i|B| A, C, D, E, F}^{m}+k_{B}^{m} P_{i|A| B, C, D, E, F}^{m}$ instead of the individual terms.

Vectorial and tensorial BRST variations exhibit novel terms $\sim k^{m}, \delta^{m n}$ which are absent for the scalars (10.12), e.g.

$$
\begin{align*}
Q L_{1|2| 3, \ldots, 8}^{m} & =\Delta_{1|2| 3, \ldots, 8}^{m}+\Lambda_{1 \mid 2,3, \ldots, 8}^{m}+k_{12}^{p} P_{1|2| 3, \ldots, 8}^{p m} \\
& +\left[s_{23} P_{1|23| 4, \ldots, 8}^{m}+k_{3}^{m} P_{1|2,3| 4, \ldots, 8}+(3 \leftrightarrow 4, \ldots 8)\right],  \tag{10.13}\\
Q L_{1|2| 3, \ldots, 9}^{m n} & =\Delta_{1|2| 3, \ldots, 9}^{m n}+\Lambda_{1 \mid 2,3, \ldots, 9}^{m n}+\delta^{m n} \Lambda_{1|2| 3, \ldots, 9}+k_{12}^{p} P_{1|2| 3, \ldots, 9}^{p m n} \\
& +\left[s_{23} P_{1|23| 4, \ldots, 9}^{m n}+2 k_{3}^{m} P_{1|2,3| 4, \ldots, 9}^{n)}+(3 \leftrightarrow 4, \ldots 9)\right] . \tag{10.14}
\end{align*}
$$

Experience with the scalar counterpart (10.12) suggests that only the non-refined multiparticle slots $B_{j}$ give rise to deconcatenation terms. This leads to the following all-rank generalization:

$$
\begin{align*}
Q L_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}} & =\Delta_{i|A| B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+\Lambda_{i \mid A, B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Lambda_{i|A| B_{1}, \ldots, B_{r+5}}^{\left.m_{3} \ldots m_{r}\right)}  \tag{10.15}\\
& +k_{i A}^{p} P_{i|A| B_{1}, \ldots, B_{r+5}}^{p m_{1} \ldots m_{r}}+\left[P_{i\left|S\left[A, B_{1}\right]\right| B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}+r k_{B_{1}}^{\left(m_{1}\right.} P_{i\left|A, B_{1}\right| B_{2}, \ldots, B_{r+5}}^{\left.m_{2} \ldots m_{r}\right)}\right. \\
& \left.+\sum_{X Y=B_{1}}\left(P_{i|A, X| Y, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}-P_{i|A, Y| X, B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots B_{r+5}\right)\right] .
\end{align*}
$$

For the extension of (10.15) to $L$ superfields of higher refinement $d$, one can expect that three classes of terms $\Lambda_{i \mid A, B_{1}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}, k_{A}^{p} P_{i|A| B_{1}, \ldots, B_{r+5}}^{p m_{1} \ldots m_{r}}$ and $P_{i\left|S\left[A, B_{1}\right]\right| B_{2}, \ldots, B_{r+5}}^{m_{1} \ldots m_{r}}$ have to be symmetrized in $A_{1} \leftrightarrow A_{2}, \ldots, A_{d}$. We therefore propose the following expression for the most general case:

$$
\begin{align*}
& Q L_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}=\Delta_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}+\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Lambda_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{\left.m_{3} \ldots m_{r}\right)}(10.16)  \tag{10.16}\\
& \quad+\left[\Lambda_{i\left|A_{2}, \ldots, A_{d}\right| A_{1}, B_{1}, \ldots, B_{r+d+4}^{m_{1}}+m_{r}}^{m_{1}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right]+k_{i A_{1} A_{2} \ldots A_{d}}^{p} P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+4}}^{p m_{1} \ldots m_{r}} \\
& \quad+\left[\left\{P_{i\left|A_{2}, \ldots, A_{d}, S\left[A_{1}, B_{1}\right]\right| B_{2}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right\}+r k_{B_{1}}^{\left(m_{1}\right.} P_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{r+d+4}}^{\left.m_{2}, m_{r}\right)}\right. \\
& \left.\quad+\sum_{X Y=B_{1}}\left(P_{i\left|A_{1}, \ldots, A_{d}, X\right| Y, B_{2}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}-P_{i\left|A_{1}, \ldots, A_{d}, Y\right| X, B_{2}, \ldots, B_{r+d+4}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots B_{r+d+4}\right)\right] .
\end{align*}
$$

Similar to the correspondence between (10.8) and (10.10), one can infer the BRST variation of $\Lambda_{i \mid \ldots}^{m_{1} \ldots}$ by trading the constituents of $(10.16)$ for their anomaly counterparts,

$$
\begin{align*}
& Q \Lambda_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+6}}^{m_{1} \ldots m_{r}}=k_{i A_{1} A_{2} \ldots A_{d}}^{p} \Gamma_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{r+d+6}}^{p m_{1} \ldots m_{r}}  \tag{10.17}\\
& +\left[\left\{\Gamma_{i\left|A_{2}, \ldots, A_{d}, S\left[A_{1}, B_{1}\right]\right| B_{2}, \ldots, B_{r+d+6}}^{m_{1} \ldots m_{r}}+\left(A_{1} \leftrightarrow A_{2}, \ldots, A_{d}\right)\right\}+r k_{B_{1}}^{\left(m_{1}\right.} \Gamma_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{r+d+6}}^{m_{2}},\right. \\
& \left.+\sum_{X Y=B_{1}}\left(\Gamma_{i\left|A_{1}, \ldots, A_{d}, X\right| Y, B_{2}, \ldots, B_{r+d+6}}^{m_{1} \ldots m_{r}}-\Gamma_{i\left|A_{1}, \ldots, A_{d}, Y\right| X, B_{2}, \ldots, B_{r+d+6}}^{m_{1} \ldots m_{r}}\right)+\left(B_{1} \leftrightarrow B_{2}, \ldots B_{r+d+6}\right)\right] .
\end{align*}
$$

Using (10.17) and the BRST transformations (7.9) and (7.17) of pseudoinvariants, one can verify that the right-hand side of (10.16) is BRST closed. Given the delicate interplay of every single term in (10.16), this is a highly nontrivial consistency check for the results in this section.

### 10.3. Trace relations and anomaly bookkeeping

The purpose of the above BRST-exact relations is to express momentum contractions of pseudoinvariants in terms of simpler pseudoinvariants at lower rank. However, two classes of obstructions arose, set by anomalous superfields $\Delta$ and $\Lambda$. Only the unrefined special case $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ was shown to be BRST trivial under momentum conservation, see (9.6), whereas refined $\Delta$ superfields and any $\Lambda$ superfield cannot be discarded in scattering amplitudes. Hence, it is desirable to identify relations among these anomalous admixtures.

In particular, one might wonder if the trace relations (6.30) and (7.20) found among pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ and anomaly invariants $\Gamma_{i \mid \ldots}^{m_{1} \ldots}$ carry over to their counterparts $D, L$ and $\Delta, \Lambda$ at different ghost-number. Even though they all originate from the same master recursion (8.4), there are subtleties under the slot rearrangements $M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow$ $\mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid A,\left\{C_{j}\right\}}$ and $M_{A} \mathcal{U}_{\left\{\dddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{U}_{\dddot{A},\left\{B_{j}\right\} \mid\left\{C_{j}\right\}}$ entering the definitions (8.11) to (8.14).

Since the formal symbols $\mathcal{U}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+N}}^{m_{1} \ldots m_{r}}$ are eventually identified with either $\mathcal{J}$ or $\mathcal{Y}$, it is safe to impose their trace relations (6.21) and (7.19) on the $\mathcal{U}$,

$$
\begin{equation*}
\delta_{n p} \mathcal{U}_{B_{1}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+N}}^{n p m_{1} \ldots m_{r-2}}=2 \mathcal{U}_{B_{1}, \ldots, B_{d}, C_{1} \mid C_{2}, \ldots, C_{d+r+N}}^{m_{1} \ldots m_{r-2}}+\left(C_{1} \leftrightarrow C_{2}, \ldots, C_{d+r+N}\right) . \tag{10.18}
\end{equation*}
$$

It turns out that this relation is not preserved under the slot rearrangement $M_{A} \mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow$ $\mathcal{U}_{\left\{\ddot{B}_{j}\right\} \mid A,\left\{C_{j}\right\}}$ relevant for $D$ and $\Delta$ (followed by multiplication with $M_{A}$ ) since

$$
\begin{equation*}
\delta_{n p} \mathcal{U}_{B_{1}, \ldots, B_{d} \mid A, C_{1}, \ldots, C_{d+r+N}}^{n p m_{1} \ldots m_{r-1}} \neq 2 \mathcal{U}_{B_{1}, \ldots, B_{d}, C_{1} \mid A, C_{2}, \ldots, C_{d+r+N}}^{m_{1} \ldots m_{r-1}}+\left(C_{1} \leftrightarrow C_{2}, \ldots, C_{d+r+N}\right) . \tag{10.19}
\end{equation*}
$$

The missing term to restore (10.18) is easily seen to be $2 \mathcal{U}_{B_{1}, \ldots, B_{d}, A \mid C_{1}, C_{2}, \ldots, C_{d+r+N}}^{m_{1} \ldots m_{r}}$ which in turn originates from the alternative slot rearrangement $M_{A} \mathcal{U}_{\left\{\dddot{B}_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{U}_{\ddot{A},\left\{B_{j}\right\} \mid\left\{C_{j}\right\}}$. Since this is the defining map for $L$ and $\Lambda$, we are led to the following trace relation:

$$
\begin{align*}
& \delta_{n p} D_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+4}}^{n p m_{1} \ldots m_{r}}=2 L_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+4}}^{m_{1} \ldots m_{r}}  \tag{10.20}\\
& \quad+2\left[D_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+4}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+4}\right)\right] .
\end{align*}
$$

The mixing between $L$ and $D$ superfields propagates to their anomalous counterparts $(D, L) \rightarrow(\Delta, \Lambda):$

$$
\begin{align*}
& \delta_{n p} \Delta_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+6}}^{n p m_{1} \ldots m_{r}}=2 \Lambda_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+6}}^{m_{1} \ldots m_{r}}  \tag{10.21}\\
& \quad+2\left[\Delta_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+6},}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+6}\right)\right]
\end{align*}
$$

Note that (10.21) and the BRST variation of (10.20) are consistent with the expressions (9.22) and (10.16) for $Q D_{i \mid \ldots}^{m_{1} \ldots}$ and $Q L_{i \mid \ldots}^{m_{1} \ldots}$.

The slot rearrangement $M_{A} \mathcal{U}_{\left\{B_{j}\right\} \mid\left\{C_{j}\right\}} \rightarrow \mathcal{U}_{\dddot{A},\left\{B_{j}\right\} \mid\left\{C_{j}\right\}}$ entering the definition of $L$ and $\Lambda$ preserves the trace relations (10.18) and bypasses the subtlety in (10.19). Hence, traces of $L$ and $\Lambda$ fall into the same pattern found for $P_{i \mid \ldots}^{m_{1} \ldots}$ and $\Gamma_{i \mid \ldots}^{m_{1} \ldots}$ in (6.30) and (7.20),

$$
\begin{align*}
& \delta_{n p} L_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+6}}^{n p m_{1} \ldots m_{r}}=2\left[L_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+6}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+6}\right)\right]  \tag{10.22}\\
& \delta_{n p} \Lambda_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+8}}^{n p m_{1} \ldots m_{r}}=2\left[\Lambda_{i\left|A_{1}, \ldots, A_{d}, B_{1}\right| B_{2}, \ldots, B_{d+r+8}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d+r+8}\right)\right] . \tag{10.23}
\end{align*}
$$

Again, one can verify consistency of (10.23) and the BRST variation of (10.22) by means of the expression (10.16) for $Q L_{i \mid \ldots}^{m_{1} \ldots}$.

As a main benefit of this discussion, the trace relations (10.21) and (10.23) are useful in manipulating anomalous ghost-number-three contributions to scattering amplitudes. In particular, we can take advantage of the decoupling of unrefined objects $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ as well as its traces and discard the following right-hand sides:

$$
\begin{align*}
\frac{1}{2} \delta_{n p} \Delta_{i \mid B_{1}, \ldots, B_{r+6}}^{n p m_{1} \ldots m_{r}} & =\Lambda_{i \mid B_{1}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}+\left[\Delta_{i\left|B_{1}\right| B_{2}, \ldots, B_{r+6}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+6}\right)\right] \\
\frac{1}{4} \delta_{n p} \delta_{q r} \Delta_{i \mid B_{1}, \ldots, B_{r+8}}^{n p q r m_{1} \ldots m_{r}} & =\left[\Lambda_{i\left|B_{1}\right| B_{2}, \ldots, B_{r+8}}^{m_{1} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+8}\right)\right]  \tag{10.24}\\
& +\left[\Delta_{i\left|B_{1}, B_{2}\right| B_{3}, \ldots, B_{r+8}}^{m_{1} \ldots m_{r}}+\left(B_{1}, B_{2} \mid B_{1}, B_{2}, \ldots, B_{r+8}\right)\right] .
\end{align*}
$$

Generalizations to multitraces of $\Delta_{i \mid A_{1}, \ldots, A_{r+4}}^{m_{1} \ldots m_{r}}$ are straightforward.


Fig. 7 Overview of superfield families. Diagonal lines point towards the images of BRST variations, and horizontal lines remind of trace relations.

### 10.4. The web of relations between ghost-number two and four

In the last sections we have constructed a variety of superfields and derived a rich set of relations among them. The recursions which led to pseudoinvariants $P$ in section 6 and to anomaly invariants $\Gamma$ in section 7 were unified to a master recursion (8.4) in section 8 . As detailed in sections 8.2 to 8.5 , the master recursion points towards four further replicae $D$, $L, \Delta$ and $\Lambda$ of the family of $P$ and $\Gamma$ which can be visualized in grids similar to fig. 1 and fig. 6.

The BRST-exact relations presented in this section and section 9 mediate between the six families of superfields as visualized in fig. 7. Since BRST action increases the ghost number by one, we arrange the superfields according to their ghost number. As a second coordinate for the roadmap of superfields, we take the number $N$ of multiparticle slots in the scalar and unrefined representatives, see section 8.1.

In fig. 7 , the BRST variations of all the six families are represented by solid lines, the arrows pointing towards the image of higher ghost number. The underlying expressions are given in

- (9.22), (10.16) and (7.17) for BRST action on the non-anomalous fields $D, L$ and $P$
- (9.23) and (10.17) for their anomalous counterparts $\Delta$ and $\Lambda$
whereas $Q \Gamma=0$. Horizontal dashed arrows additionally remind of the mixing of $D \leftrightarrow L$ as well as $\Delta \leftrightarrow \Lambda$ under the trace relations (10.20) and (10.21).


## 11. Canonicalizing pseudoinvariants

In the previous sections, we have systematically derived two families of relations among pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ at fixed reference leg $i$. This section is devoted to a different class of relations which mediates between different choices of the reference particle $i$. It will be demonstrated that any $P_{k \mid \ldots}^{m_{1} \ldots}$ with $k \neq i$ can be expressed in terms of $P_{i \mid \ldots}^{m_{1} \ldots}$, up to a constrained set of anomaly terms. This rearrangement will be referred to as canonicalization.

The anomalous admixtures in the canonicalization process reflect the property of the hexagon gauge anomaly in both field and string theory to break the permutation symmetry of one-loop amplitudes at multiplicity $n \geq 6$. Apart from this anomaly subtlety, however, we learn that the set of pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots m_{r}}$ at fixed reference leg $i$ spans the same space of kinematic factors as would be obtained by any other choice of reference leg $k \neq i$. This is a necessary condition for the independence of string amplitudes on the choice of the unintegrated vertex $i$.

The methods parallel the procedure in section 9 and 10, i.e. we identify suitable BRST generators at ghost-number two to derive relations among ghost-number-three objects. The $Q$ generators to trade different choices of the reference leg $i$ in $P_{i \mid \ldots}^{m_{1} \ldots}$ turn out to be minor modifications of the ghost-number-two superfields of types $D$ and $L$, see section 8 .

We start by discussing the canonicalization of scalar invariants in detail. This serves as a motivation of certain operations which capture the structure of the canonicalization process and easily carry over to more general pseudoinvariants.

### 11.1. Canonicalizing scalar invariants

As pointed out in [24] through a few examples at low multiplicity, any scalar invariant $C_{k \mid A, B, C}$ as given in (2.40) can be cast into a basis of $C_{i \mid D, E, F}$ with $i \neq k$. We shall now present the general solution for arbitrary $A, B, C$ and thereby develop the maps and notation to extend the procedure to higher tensors and to refined pseudoinvariants.

The trial and error method in [24] to canonicalize $C_{2 \mid A, B, C}$ towards $C_{1 \mid D, E, F}$ leads to the following expressions at multiplicity $n \leq 6$ (suppressing the laborious case $C_{2 \mid 34,56,1}$ ):

$$
\begin{align*}
C_{2 \mid 1,3,4} & =-Q\left(M_{12,3,4}\right)+C_{1 \mid 2,3,4}  \tag{11.1}\\
C_{2 \mid 13,4,5} & =-Q\left(M_{132,4,5}\right)+C_{1 \mid 32,4,5}  \tag{11.2}\\
C_{2 \mid 1,34,5} & =-Q\left(M_{12,34,5}+M_{123,4,5}-M_{124,3,5}\right)+C_{1 \mid 2,34,5}+C_{1 \mid 23,4,5}-C_{1 \mid 24,3,5}  \tag{11.3}\\
C_{2 \mid 134,5,6} & =-Q\left(M_{1342,5,6}\right)+C_{1 \mid 342,5,6} \tag{11.4}
\end{align*}
$$

$$
\begin{align*}
& C_{2 \mid 13,45,6}=-Q\left(M_{132,45,6}+M_{1324,5,6}-M_{1325,4,6}\right)+C_{1 \mid 32,45,6}+C_{1 \mid 324,5,6}-C_{1 \mid 325,4,6}  \tag{11.5}\\
& C_{2 \mid 1,345,6}=-Q\left(M_{12,345,6}+M_{1234,5,6}+M_{1254,3,6}-M_{1235,4,6}-M_{1253,4,6}+M_{123,45,6}+M_{125,43,6}\right) \\
& \quad+C_{1 \mid 2,345,6}+C_{1 \mid 234,5,6}+C_{1 \mid 254,3,6}-C_{1 \mid 235,4,6}-C_{1 \mid 253,4,6}+C_{1 \mid 23,45,6}+C_{1 \mid 25,43,6} \tag{11.6}
\end{align*}
$$

These examples can be easily verified using the form (2.35) of $Q M_{A, B, C}$. The following three observations on (11.1) to (11.6) guide the way towards a general solution for both the BRST ancestor and the set of $C_{1 \mid D, E, F}$ which appear in the canonicalization of $C_{2 \mid \ldots}$ :
(i) Each term of the form $-Q M_{1 D, E, F}$ in the BRST generator is accompanied by a corresponding invariant $C_{1 \mid D, E, F}$. By assuming this pattern to hold in general, knowledge of the BRST generator already determines the canonicalization in terms of $C_{1 \mid D, E, F}$.
(ii) Suppose particle 1 appears in the left hand side as $C_{2 \mid 1 A, B, C}$ (where $A$ can be empty), then each term of the BRST generator is of the form $M_{1 A D, E, F}$. In other words, the entire slot $1 A$ of the desired reference leg is concatenated with a pattern of $M_{\ldots D, E, F}$.
(iii) This pattern of $M_{\ldots D, E, F}$ must contain the information on the remaining labels $2, B, C$. The superfield $D_{2 \mid B, C}$ as defined in (8.15) is the natural object to do so, and indeed, its concatenation (2.37) with $M_{1 A}$ reproduces the above BRST generators, e.g.

$$
\begin{align*}
M_{1} \otimes D_{2 \mid 34,5}= & M_{12,34,5}+M_{123,4,5}-M_{124,3,5}  \tag{11.7}\\
M_{13} \otimes D_{2 \mid 45,6}= & M_{132,45,6}+M_{1324,5,6}-M_{1325,4,6}  \tag{11.8}\\
M_{1} \otimes D_{2 \mid 345,6}= & M_{12,345,6}+M_{1234,5,6}+M_{1254,3,6}-M_{1235,4,6} \\
& \quad-M_{1253,4,6}+M_{123,45,6}+M_{125,43,6} \tag{11.9}
\end{align*}
$$

for (11.3), (11.5) and (11.6), respectively. The $D$ superfields in the first two cases are given in (8.16) whereas $D_{2 \mid 345,6}$ can be inferred from $C_{1 \mid 234,5,6}$ in (A.1).

So one can promote the sample canonicalizations of $C_{2 \mid 1 A, B, C}$ in (11.1) to (11.6) to the following general formula ${ }^{20}$,

$$
\begin{equation*}
C_{k \mid i A, B, C}=\left(\wp_{i}-Q\right)\left(M_{i A} \otimes D_{k \mid B, C}\right) . \tag{11.10}
\end{equation*}
$$

${ }^{20}$ To support the plausibility of the canonicalization prescription in (11.10), note that any term in the concatenation product $M_{i A} \otimes D_{k \mid B, C}$ takes the form $M_{i A \ldots, D, E}$. BRST action then generates one term $C_{i \mid A \ldots, D, E}$ and up to five others, see (2.35). The map $\wp_{i}$ makes sure that the $C_{i \mid A \ldots, D, E}$ contribution in $-Q M_{i A \ldots, D, E}$ is compensated. Other invariants of the form $C_{l \neq i \mid F, G, H}$ largely cancel thanks to the fine-tuned arrangement of slots governed by the recursive origin (8.4) of $D_{k \mid B, C}$. Only one term $C_{k \mid i A, B, C}$ with reference leg $\neq i$ will emerge from the $Q$ action on the leading term $D_{k \mid B, C} \rightarrow M_{k, B, C}$, as required by the left-hand side of (11.10).

The "pseudoinvariantization" map $\wp_{i}$ in (11.10) is defined to transform $M_{i A, B, C} \rightarrow$ $C_{i \mid A, B, C}$ according to observation (i) and, more generally,

$$
\begin{align*}
& \wp_{i}\left(\mathcal{J}_{B_{1}, \ldots, B_{d} \mid i A, C_{2}, \ldots, C_{d+r}}^{m_{1}}\right) \equiv P_{i\left|B_{1}, \ldots, B_{d}\right| A, C_{2}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}  \tag{11.11}\\
& \wp_{i}\left(\mathcal{J}_{i A, B_{2}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}\right) \equiv P_{i\left|A, B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}} . \tag{11.12}
\end{align*}
$$

Loosely speaking, $\wp_{i}$ in (11.11) and (11.12) removes particle $i$ from the leading position of a word in $\mathcal{J}_{B_{1}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}$ and converts it into the reference leg of a pseudoinvariant. Its remaining labels are those of the above current with particle $i$ removed.

Further applications of (11.10) can be found in appendix F, see (F.1) to (F.6).

### 11.2. Canonicalizing unrefined pseudoinvariants

In order to canonicalize tensorial pseudoinvariants $C_{k \mid i A, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ to the form $C_{i \mid C_{1}, \ldots . C_{r+3}}^{m_{1}}$, it is tempting to simply replace $D_{k \mid B, C} \rightarrow D_{k \mid B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ in the scalar prescription (11.10). However, $D$ superfields at rank $r \geq 2$ additionally generate anomalous terms such as $\Delta_{i \mid A_{1}, \ldots, A_{4}}^{m_{1} \ldots m_{r}}$ in (9.5) and (9.6). In order to have a well-defined notion of $M_{i A} \otimes \Delta_{k \mid B_{1}, \ldots, B_{r+4}}^{m_{1}}$ and $M_{i A} \otimes D_{k \mid B_{1}, \ldots, B_{r+2}}^{m_{1} \ldots m_{r}}$ (needed in the subsequent), we extend the concatenation operation to anomaly building blocks and refined currents,

$$
\begin{align*}
& M_{i A} \otimes \mathcal{Y}_{k B, B_{2}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+5}}^{m_{1} \ldots m_{r}} \equiv \mathcal{Y}_{i A k B B_{2}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r}+5}^{m_{1} \ldots m_{r}}  \tag{11.13}\\
& M_{i A} \otimes \mathcal{Y}_{B_{1}, \ldots, B_{d} \mid k C, C_{2}, \ldots, C_{d+r+5}}^{m_{1}} \equiv \mathcal{Y}_{B_{1}, \ldots, B_{d} \mid i A k C, C_{2}, \ldots, C_{d+r+5}}^{m_{1} \ldots m_{r}}  \tag{11.14}\\
& M_{i A} \otimes \mathcal{J}_{k B, B_{2}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+3}}^{m_{1}} \equiv \mathcal{J}_{i A k B, B_{2}, \ldots, B_{d} \mid C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}  \tag{11.15}\\
& M_{i A} \otimes \mathcal{J}_{B_{1}, \ldots, B_{d} \mid k C, C_{2}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}} \equiv \mathcal{J}_{B_{1}, \ldots, B_{d} \mid i A k C, C_{2}, \ldots, C_{d+r+3}}^{m_{1}, m_{r}} \tag{11.16}
\end{align*}
$$

As before, the instruction to concatenate the word $i A$ with $k B$ and $k C$ is clear from the reference leg $k$ of the parental $D_{k \mid \ldots}^{m \omega_{1} \ldots}$. The anomaly concatenations $M_{i A} \otimes \Delta_{k \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}$ serve to compensate for the anomalous part of $Q\left(M_{i A} \otimes D_{k \mid \ldots}^{m_{1} \ldots}\right)$ :

$$
\begin{align*}
C_{k \mid i A, B_{2}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}= & \left(\wp{ }_{i}-Q\right)\left(M_{i A} \otimes D_{k \mid B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}\right)  \tag{11.17}\\
& +\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.}\left(M_{i A} \otimes \Delta_{k \mid B_{2}, B_{3}, \ldots, B_{r+3}}^{\left.m_{3} m_{4} \ldots m_{r}\right)}\right) .
\end{align*}
$$

The cancellation of anomalous superfields on the right-hand side can be understood as follows: The anomalous term in $Q M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}=\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \mathcal{Y}_{B_{1}, \ldots, B_{r+3}}^{\left.m_{3}, m_{r}\right)}+\ldots$ preserves the structure of the slots $B_{k}$. In a truncation to anomaly building blocks, BRST action and
the concatenation through $M_{i A} \otimes$ commute. Since $\delta^{\left(m_{1} m_{2}\right.} \Delta_{k \mid B_{2}, B_{3}, \ldots, B_{r+3}}^{\left.m_{3} m_{4} \ldots m_{r}\right)}$ in the second line of (11.17) can be traced back to $Q D_{k \mid B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}$, see (9.6), any anomalous contribution effectively originates from a "commutator" of the operations $M_{i A} \otimes$ and $Q$ acting on $D_{k \mid B_{2}, B_{3}, \ldots, B_{r+3}}^{m_{1} m_{2} \ldots m_{r}}$.

Two simple examples of the general procedure in (11.17) are given by,

$$
\begin{align*}
C_{2 \mid 1,3,4,5}^{m}= & \left(\wp_{1}-Q\right)\left(M_{1} \otimes D_{2 \mid 3,4,5}^{m}\right) \\
= & -Q\left(M_{12,3,4,5}^{m}+\left[k_{3}^{m} M_{123,4,5}+(3 \leftrightarrow 4,5)\right]\right) \\
& +C_{1 \mid 2,3,4,5}^{m}+\left[k_{3}^{m} C_{1 \mid 23,4,5}+(3 \leftrightarrow 4,5)\right]  \tag{11.18}\\
C_{2 \mid 1,3,4,5,6}^{m n}= & \left(\wp_{1}-Q\right)\left(M_{1} \otimes D_{2 \mid 3,4,5,6}^{m n}\right)+\delta^{m n}\left(M_{1} \otimes \Delta_{2 \mid 3,4,5,6}\right) \\
= & -Q\left(M_{12,3,4,5,6}^{m n}+\left[2 k_{3}^{(m} M_{123,4,5,6}^{n)}+(3 \leftrightarrow 4,5,6)\right]\right. \\
& \left.+\left[2 k_{3}^{(m} k_{4}^{n)}\left(M_{1234,5,6}+M_{1243,5,6}\right)+(3,4 \mid 3,4,5,6)\right]\right) \\
& +\delta^{m n} \mathcal{Y}_{12,3,4,5,6}+C_{1 \mid 2,3,4,5,6}^{m n}+\left[2 k_{3}^{(m} C_{1 \mid 23,4,5,6}^{n)}+(3 \leftrightarrow 4,5,6)\right] \\
& +\left[2 k_{3}^{(m} k_{4}^{n)}\left(C_{1 \mid 234,5,6}+C_{1 \mid 243,5,6}\right)+(3,4 \mid 3,4,5,6)\right] . \tag{11.19}
\end{align*}
$$

Further examples can be found in appendix F, see (F.7).

### 11.3. Canonicalizing non-refined slots in refined pseudoinvariants

Only minor modifications are required to generalize the above canonicalization rules to refined pseudoinvariants $P_{k\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}$, as long as the preferred reference leg $i$ resides in a unrefined slot $B_{j}$. By analogy with the first line of (11.17), it is natural to expect a BRST generator of the form $M_{i A} \otimes D_{k\left|B_{1}, \ldots, B_{d}\right| C_{2}, \ldots, C_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}$.

Similar to the unrefined tensors, the BRST variation of refined $D$ superfields incorporates anomalous $\Delta$ superfields, see (9.22). In order to address them, recall that the anomalous part of $Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}$ given in (6.16) has an unmodified slot structure. Hence, we can neglect the concatenation with $M_{i A}$ and compensate the anomalous part of $Q\left(M_{i A} \otimes D_{k\left|B_{1}, \ldots, B_{d}\right| C_{2}, \ldots}^{m_{1} m_{2} \ldots m_{r}}\right)$ using the corresponding terms of $M_{i A} \otimes Q D_{k\left|B_{1}, \ldots, B_{d}\right| C_{2}, \ldots}^{m_{1} m_{2} \ldots m_{r}}$. This reasoning motivates the last two lines of

$$
\begin{align*}
P_{k\left|B_{1}, \ldots, B_{d}\right| i A, C_{2}, \ldots, C_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}= & \left(\wp_{i}-Q\right)\left(M_{i A} \otimes D_{k\left|B_{1}, \ldots, B_{d}\right| C_{2}, \ldots, C_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}\right) \\
+ & M_{i A} \otimes\left\{\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Delta_{k\left|B_{1}, \ldots, B_{d}\right| C_{2}, C_{3}, \ldots, C_{d+r+3}}^{m_{3} m_{4} \ldots m_{r}}\right.  \tag{11.20}\\
& \left.+\left[\Delta_{k\left|B_{2}, \ldots, B_{d}\right| B_{1}, C_{2}, C_{3}, \ldots, C_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{d}\right)\right]\right\}
\end{align*}
$$

and guarantees that they cancel the anomalous contributions from the first line.
As the simplest application of (11.20),

$$
\begin{align*}
P_{2|3| 1,4,5,6}= & \left(\wp_{1}-Q\right)\left(M_{1} \otimes D_{2|3| 4,5,6}\right)+M_{1} \otimes \Delta_{2 \mid 3,4,5,6}  \tag{11.21}\\
= & -Q\left(\mathcal{J}_{3 \mid 12,4,5,6}+k_{3}^{m} M_{123,4,5,6}^{m}+\left[s_{34} M_{1234,5,6}+(4 \leftrightarrow 5,6)\right]\right) \\
& +\mathcal{Y}_{12,3,4,5,6}+P_{1|3| 2,4,5,6}+k_{3}^{m} C_{1 \mid 23,4,5,6}^{m}+\left[s_{34} C_{1 \mid 234,5,6}+(4 \leftrightarrow 5,6)\right],
\end{align*}
$$

and more involved cases are displayed in appendix F, see (F.8) to (F.11).

### 11.4. Canonicalizing refined slots in refined pseudoinvariants

A different canonicalization procedure is needed when the preferred reference label $i$ resides in a refined slot $A_{j}$ of a pseudoinvariant $P_{k\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} m_{2} \ldots m_{r}}$ at $d \neq 0$. In order to gain intuition for suitable BRST generators, consider the following examples:

$$
\begin{align*}
P_{2|1| 3,4,5,6}= & -Q \mathcal{J}_{12 \mid 3,4,5,6}+\mathcal{Y}_{12,3,4,5,6}+P_{1|2| 3,4,5,6}  \tag{11.22}\\
P_{2|13| 4,5,6,7}= & -Q \mathcal{J}_{132 \mid 4,5,6,7}+\mathcal{Y}_{132,4,5,6,7}+P_{1|32| 4,5,6,7} \\
P_{2|1| 34,5,6,7}= & -Q\left(\mathcal{J}_{12 \mid 34,5,6,7}+\mathcal{J}_{123 \mid 4,5,6,7}-\mathcal{J}_{124 \mid 3,5,6,7}\right) \\
& +\mathcal{Y}_{12,34,5,6,7}+\mathcal{Y}_{123,4,5,6,7}-\mathcal{Y}_{124,3,5,6,7} \\
& +P_{1|2| 34,5,6,7}+P_{1|23| 4,5,6,7}-P_{1|24| 3,5,6,7}
\end{align*}
$$

The appearance of anomalous contributions on the right-hand side is not surprising in view of the examples (11.19) and (11.21). In the present cases, however, the BRST generators are entirely built from refined building blocks $\mathcal{J}$ at $d \neq 0$. This is a defining property of the $L$ superfields defined in (8.17). Indeed, (11.22) is consistently described by

$$
\begin{equation*}
P_{k|i A| B, C, D, E}=\left(\wp_{i}-Q\right)\left(M_{i A} \otimes L_{k \mid B, C, D, E}\right)+M_{i A} \otimes \Delta_{k \mid B, C, D, E} . \tag{11.23}
\end{equation*}
$$

Similar to (11.17) and (11.20), the anomalous part of the BRST generator (i.e. the first term of $Q L_{k \mid B, C, D, E}=\Delta_{k \mid B, C, D, E}+\ldots$ in (10.2)) is manually compensated by the last term in (11.23), see the arguments in the previous sections 11.2 and 11.3.

In the tensorial generalization of (11.23), the anomalous traces in the expression (10.8) for $Q L_{\ddot{k} \mid \ldots}=\delta \cdots \Lambda_{\ddot{k} \mid \ldots}+\ldots$ have to be taken into account. This leads to the second line of

$$
\begin{align*}
P_{k|i A| B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}= & \left(\wp_{i}-Q\right)\left(M_{i A} \otimes L_{k \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}\right) \\
& +M_{i A} \otimes\left\{\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Lambda_{k \mid B_{1}, \ldots, B_{r+4}}^{\left.m_{3} \ldots m_{r}\right)}+\Delta_{k \mid B_{1}, \ldots, B_{r+4}}^{m_{1} \ldots m_{r}}\right\} . \tag{11.24}
\end{align*}
$$

The structure of the following vector and tensor examples resembles the canonicalization of $C_{2 \mid 1,3,4,5}^{m}$ and $C_{2 \mid 1,3,4,5,6}^{m n}$ performed in (11.18) and (11.19), respectively:

$$
\begin{align*}
P_{2|1| 3,4,5,6,7}^{m}= & \left(\wp_{1}-Q\right)\left(M_{1} \otimes L_{2 \mid 3,4,5,6,7}^{m}\right)+M_{1} \otimes \Delta_{2 \mid 3,4,5,6,7}^{m} \\
= & -Q\left(\mathcal{J}_{12 \mid 3,4,5,6,7}^{m}+\left[k_{3}^{m} \mathcal{J}_{123 \mid 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]\right) \\
& +\mathcal{Y}_{12,3,4,5,6,7}^{m}+\left[k_{3}^{m} \mathcal{Y}_{123,4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right] \\
& +P_{1|2| 3,4,5,6,7}^{m}+\left[k_{3}^{m} P_{1|23| 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]  \tag{11.25}\\
P_{2|1| 3,4,5,6,7,8}^{m n}= & \left(\wp_{1}-Q\right)\left(M_{1} \otimes L_{2 \mid 3,4,5,6,7,8}^{m n}\right)+M_{1} \otimes\left(\delta^{m n} \Lambda_{2 \mid 3,4, \ldots, 8}+\Delta_{2 \mid 3,4, \ldots, 8}^{m n}\right) \\
= & -Q\left(\mathcal{J}_{12 \mid 3,4,5,6,7,8}^{m n}+\left[2 k_{3}^{(m} \mathcal{J}_{123 \mid 4,5,6,7,8}^{n)}+(3 \leftrightarrow 4,5,6,7,8)\right]\right. \\
& \left.+\left[2 k_{3}^{(m} k_{4}^{n)}\left(\mathcal{J}_{1234 \mid 5,6,7,8}+\mathcal{J}_{1243 \mid 5,6,7,8}\right)+(3,4 \mid 3,4,5,6,7,8)\right]\right) \\
& +\delta^{m n} \mathcal{Y}_{12 \mid 3,4,5,6,7,8}+\mathcal{Y}_{12,3,4,5,6,7,8}^{m n}+\left[2 k_{3}^{(m} \mathcal{Y}_{123,4,5,6,7,8}^{n)}+(3 \leftrightarrow 4,5,6,7,8)\right] \\
& +\left[2 k_{3}^{(m} k_{4}^{n)}\left(\mathcal{Y}_{1234,5,6,7,8}+\mathcal{Y}_{1243,5,6,7,8}\right)+(3,4 \mid 3,4,5,6,7,8)\right] \\
& +P_{1|2| 3,4,5,6,7,8}^{m n}+\left[2 k_{3}^{m} P_{1|23| 4,5,6,7,8}^{n)}+(3 \leftrightarrow 4,5,6,7,8)\right] \\
& +\left[2 k_{3}^{(m} k_{4}^{n)}\left(P_{1|234| 5,6,7,8}+P_{1|243| 5,6,7,8}\right)+(3,4 \mid 3,4,5,6,7,8)\right] \tag{11.26}
\end{align*}
$$

Finally, the canonicalization prescription (11.24) can be generalized to higher refinement. We follow the usual logic and manually remove the anomalous contributions from the natural BRST generator, see (10.16) for $Q L_{k\left|B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}$ :

$$
\begin{gather*}
P_{k\left|i A, B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}=\left(\wp_{i}-Q\right)\left(M_{i A} \otimes L_{k\left|B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}\right) \\
+M_{i A} \otimes\left\{\binom{r}{2} \delta^{\left(m_{1} m_{2}\right.} \Lambda_{k\left|B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{\left.m_{3} \ldots m_{r}\right)}+\Delta_{k\left|B_{2}, \ldots, B_{d}\right| C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}\right. \\
\left.+\left[\Lambda_{k\left|B_{3}, \ldots, B_{d}\right| B_{2}, C_{1}, \ldots, C_{d+r+3}}^{m_{1} \ldots m_{r}}+\left(B_{2} \leftrightarrow B_{3}, \ldots, B_{d}\right)\right]\right\} . \tag{11.27}
\end{gather*}
$$

The simplest application occurs at eight-points,

$$
\begin{align*}
P_{2|1,3| 4,5,6,7,8}= & \left(\wp \wp_{1}-Q\right)\left(M_{1} \otimes L_{2|3| 4,5,6,7,8}\right)+M_{1} \otimes\left(\Delta_{2|3| 4,5,6,7,8}+\Lambda_{2 \mid 3,4,5,6,7,8}\right) \\
= & -Q\left(\mathcal{J}_{12,3 \mid 4,5,6,7,8}+k_{3}^{m} \mathcal{J}_{123 \mid 4,5,6,7,8}^{m}+\left[s_{34} \mathcal{J}_{1234 \mid 5,6,7,8}+(4 \leftrightarrow 5,6,7,8)\right]\right) \\
& +\mathcal{Y}_{12 \mid 3,4,5,6,7,8}+\mathcal{Y}_{3 \mid 12,4,5,6,7,8}+k_{3}^{m} \mathcal{Y}_{123,4,5,6,7,8}^{m} \\
& +\left[s_{34} \mathcal{Y}_{1234,5,6,7,8}+(4 \leftrightarrow 5,6,7,8)\right]  \tag{11.28}\\
& +P_{1|2,3| 4,5,6,7,8}+k_{3}^{m} P_{1|23| 4,5,6,7,8}^{m}+\left[s_{34} P_{1|234| 5,6,7,8}+(4 \leftrightarrow 5,6,7,8)\right] .
\end{align*}
$$

With the most general canonicalization prescriptions (11.20) and (11.27), any pseudoinvariant $P_{k \mid \ldots}^{m_{1} \ldots}$ with reference leg $k \neq i$ can be rewritten in terms of $P_{i \mid \ldots}^{m_{1} \ldots}$. The anomalous extra terms built from concatenations of superfields $\Delta$ and $\Lambda$ signal the breakdown of permutation symmetry in anomalous one-loop amplitudes, see [29,31].

## 12. Conclusion and outlook

As explained in the Introduction, the result of a multiloop superstring scattering amplitude in the pure spinor formalism can be written in terms of pure spinor superspace expressions in the cohomology of the BRST charge. This realization was the guiding principle which led us to consider the general structures for one-loop amplitudes presented in this work. The claim is that the kinematic factors considered in the previous sections form a convenient and complete set of building blocks which make manifest the BRST cohomology properties of one-loop amplitudes in pure spinor superspace.

Saturation of zero-modes in the pure spinor prescription implies that the external vertices in the four-point amplitude contribute through a term proportional to $d_{\alpha} d_{\beta} N^{m n}$ [4]. The measures defined in [4] summarize the net effect of zero-mode integrations by the following rule,

$$
\begin{equation*}
d_{\alpha} d_{\beta} N^{m n} \rightarrow\left(\lambda \gamma^{[m}\right)_{\alpha}\left(\lambda \gamma^{n]}\right)_{\beta} \tag{12.1}
\end{equation*}
$$

The resulting kinematic factor of the open superstring four-point amplitude [4] is written as $V_{1} T_{2,3,4}$ in the notation of equation (2.25) and can be checked to be in the BRST cohomology. Its multiparticle generalization found in [24] incorporates the contributions from OPE singularities through the basic structure $V_{A} T_{B, C, D}$ which is most elegantly described using the BRST blocks from [32]. The reduction of the associated worldsheet integrals [24] organizes these superfields into BRST-invariants such as the scalar

$$
\begin{equation*}
C_{1 \mid 23,4,5}=M_{1} M_{23,4,5}+M_{12} M_{3,4,5}-M_{13} M_{2,4,5} \tag{12.2}
\end{equation*}
$$

in the five-point amplitude. The trial-and-error construction of scalar BRST-invariants up to multiplicity eight [24] was improved to a systematic and recursive procedure in [32], see section 2 . Since their origin is ultimately related to the one-loop zero-mode pattern (12.1) from the one-loop amplitude prescription, these BRST invariants encode the manifestly gauge-invariant pieces of the $N$-point one-loop superstring amplitudes of [24].

However, each topology of open superstring amplitudes at genus one is anomalous for $N \geq 6$ external legs, and the cancellation of the anomaly relies on an interplay between the cylinder and the Möbius strip [25]. Therefore the manifestly gauge-invariant form of the amplitudes in [24] could not be the complete answer. Finding the missing anomalous terms from their BRST properties was one of the main goals of this paper and led to the concept of pseudo-cohomology introduced in section 3 .

As discussed in the main body, a more general class of superfields $\mathcal{J}$ extending the prescription (12.1) gives rise to a recursive procedure to construct anomalous superfields $P_{i \mid \ldots}^{m n p \ldots}$, called BRST pseudoinvariants. As a defining property, their BRST variation takes the form $V_{A}\left(\lambda \gamma^{m} W_{B}\right)\left(\lambda \gamma^{n} W_{C}\right)\left(\lambda \gamma^{p} W_{D}\right)\left(W_{E} \gamma_{m n p} W_{F}\right)$ which generalizes the hexagon anomaly $\epsilon_{10} F^{5}$ to superspace and to higher number of external particles. The bosonic components of several pseudoinvariants can be downloaded from the website [43].

Therefore, BRST cohomology considerations point towards superfields with the correct properties to describe the anomalous parts of one-loop open superstring amplitudes which were not considered in [24]. The methods to generate these pseudoinvariants are natural extensions of the well-tested recursion [32] for scalar BRST-invariants [24]. In an upcoming work [29] these pseudoinvariants will be assembled into six-point one-loop amplitudes of the open and closed superstring, the analogous treatment of higher multiplicity is left for the future.

The field theory limit of superstring amplitudes is composed of scalar and tensorial Feynman integrals. The underlying degeneration limit of the worldsheet reorganizes the scalar kinematic factors of the superstring such that loop momenta contract tensorial BRST pseudoinvariants. The recursive construction of this work naturally includes superfields of arbitrary tensor rank and motivates kinematic companions for loop momenta. Their precise appearance in one-loop amplitudes of SYM will be detailed in upcoming work [31]. The matching of worldsheet and momentum space representations of one-loop amplitudes requires a precise control of the momentum contractions of pseudoinvariants. As shown in sections 8 to 10 , this problem is addressed by cohomology considerations which will allow to identify the difference of the two representations as BRST exact [31].

Tensorial pseudoinvariants also play an essential role for closed string amplitudes and capture their contributions beyond the naive doubling of open string worldsheet correlators. As will be demonstrated in [29], the tensorial kinematic factors in this work provide a compact description of the interactions between left- and right-moving degrees of freedom. From a field-theory perspective, this points towards a squaring relation between the numerators of Feynman integrals in SYM and supergravity amplitudes. It would be interesting to realize the BCJ duality between color and kinematics [44] through the pseudoinvariants of this work.

After finding the recursive formulas for pseudoinvariants of arbitrary orders, the natural question to ask is how these pseudoinvariants can be derived from the pure spinor
multiloop amplitude prescription ${ }^{21}$ [4]. This is a challenge for the future. We suspect that the solution involves a careful treatment of OPE contractions between the b-ghost and the external vertices. The combinatorics must be such that spurious OPE singularities combine to local functions on the worldsheet (which are regular for all values of $z_{i}-z_{j}$ and denoted by $f_{i j}$ in [31]). This might bypass the subtleties related to b-ghost singularities pointed out in [45]. Given that the b-ghost is a source of technical difficulties in amplitude calculations with the pure spinor superstring, a first principles explanation for the pseudoinvariants constructed in this paper might shed new light into this difficult corner of the formalism.

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## Appendix A. Examples of BRST pseudoinvariants

This appendix gathers recursively generated expansions of various (pseudo-)invariants. Their component expansions can be found at the website [43].

At six points, the recursions (2.40) for scalar and vector invariants yield

$$
\begin{align*}
C_{1 \mid 234,5,6}= & M_{1} M_{234,5,6}+M_{1} \otimes\left[C_{2 \mid 34,5,6}-C_{4 \mid 23,5,6}\right] \\
= & M_{1} M_{234,5,6}+M_{12} M_{34,5,6}+M_{123} M_{4,5,6}-M_{124} M_{3,5,6}  \tag{A.1}\\
& -M_{14} M_{23,5,6}-M_{142} M_{3,5,6}+M_{143} M_{2,5,6} \\
C_{1 \mid 23,45,6}= & M_{1} M_{23,45,6}+M_{1} \otimes\left[C_{2 \mid 45,3,6}-C_{3 \mid 45,2,6}+C_{4 \mid 23,5,6}-C_{5 \mid 23,4,6}\right] \\
= & M_{1} M_{23,45,6}+M_{12} M_{45,3,6}-M_{13} M_{45,2,6}+M_{14} M_{23,5,6}-M_{15} M_{23,4,6} \\
& +M_{124} M_{3,5,6}-M_{134} M_{2,5,6}+M_{142} M_{3,5,6}-M_{152} M_{3,4,6}
\end{align*}
$$

${ }^{21}$ Some terms in the scalar $P_{i \mid \ldots}$ can be explained as the leftovers of the partial fraction manipulations described in [24]. This applies to both the single-pole contribution from iterated OPEs and to spurious double-pole singularities which can be removed via integration by parts.

$$
\begin{aligned}
& -M_{125} M_{3,4,6}+M_{135} M_{2,4,6}-M_{143} M_{2,5,6}+M_{153} M_{2,4,6} \\
C_{1 \mid 23,4,5,6}^{m}= & M_{1} M_{23,4,5,6}^{m}+M_{1} \otimes\left[C_{2 \mid 3,4,5,6}^{m}-C_{3 \mid 2,4,5,6}^{m}+\left\{k_{4}^{m} C_{4 \mid 23,5,6}+(4 \leftrightarrow 5,6)\right\}\right] \\
= & M_{1} M_{23,4,5,6}^{m}+M_{12} M_{3,4,5,6}^{m}-M_{13} M_{2,4,5,6}^{m}+k_{3}^{m} M_{123} M_{4,5,6}-k_{2}^{m} M_{132} M_{4,5,6} \\
& +\left[k_{4}^{m} M_{14} M_{23,5,6}+\left(M_{124}+M_{142}\right) M_{3,5,6}-\left(M_{134}+M_{143}\right) M_{2,5,6}+(4 \leftrightarrow 5,6)\right] .
\end{aligned}
$$

The five-point invariants $C_{1 \mid 23,4,5}$ and $C_{1 \mid 2,3,4,5}^{m}$ entering (A.1) are given by (2.41).
The recursions (3.13) and (4.15) for tensorial pseudoinvariants gives rise to

$$
\begin{align*}
C_{1 \mid 23,4,5,6,7}^{m n}= & M_{1} M_{23,4,5,6,7}^{m n}+M_{1} \otimes\left[C_{2 \mid 3,4,5,6,7}^{m n}-C_{3 \mid 2,4,5,6,7}^{m n}+2\left\{k_{4}^{(m} C_{4 \mid 23,5,6,7}^{n)}+(4 \leftrightarrow 5,6,7)\right\}\right] \\
= & M_{1} M_{23,4,5,6,7}^{m n}+M_{12} M_{3,4,5,6,7}^{m n}-M_{13} M_{2,4,5,6,7}^{m n}+2\left(k_{3}^{(m} M_{123}-k_{2}^{(m} M_{132}\right) M_{4,5,6,7}^{n)} \\
+ & 2\left[k _ { 4 } ^ { ( m } k _ { 5 } ^ { n ) } \left\{\left(M_{145}+M_{154}\right) M_{23,6,7}+\left(M_{1245}+\operatorname{symm}(2,4,5)\right) M_{3,6,7}\right.\right. \\
& \left.\left.-\left(M_{1345}+\operatorname{symm}(3,4,5)\right) M_{2,6,7}\right\}+(4,5 \mid 4,5,6,7)\right] \\
+ & {\left[2 k _ { 4 } ^ { ( m } \left\{M_{14} M_{23,5,6,7}^{n)}+\left(M_{124}+M_{142}\right) M_{3,5,6,7}^{n)}\right.\right.} \\
& -\left(M_{134}+M_{143}\right) M_{2,5,6,7}^{n)}-k_{2}^{n)}\left(M_{1432}+M_{1342}+M_{1324}\right) M_{5,6,7} \\
& \left.\left.+k_{3}^{n)}\left(M_{1423}+M_{1243}+M_{1234}\right) M_{5,6,7}\right\}+(4 \leftrightarrow 5,6,7)\right]  \tag{A.2}\\
C_{1 \mid 2,3,4,5,6,7}^{m n p}= & M_{1} M_{2,3,4,5,6,7}^{m n p}+M_{1} \otimes\left[3 k_{2}^{(m} C_{2 \mid 3,4,5,6,7}^{n p)}+(2 \leftrightarrow 3,4,5,6,7)\right] \\
= & M_{1} M_{2,3,4,5,6,7}^{m n p}+\left[3 k_{2}^{(m} M_{12} M_{3,4,5,6,7}^{n p}+(2 \leftrightarrow 3,4,5,7)\right] \\
+ & {\left[6 k_{2}^{(m} k_{3}^{n}\left(M_{123}+M_{132}\right) M_{4,5,6,7}^{p)}+(2,3 \mid 2,3,4,5,6,7)\right] } \\
+ & {\left[6 k_{2}^{(m} k_{3}^{n} k_{4}^{p)}\left(M_{1234}+\operatorname{symm}(2,3,4)\right) M_{5,6,7}+(2,3,4 \mid 2,3,4,5,6,7)\right], \quad \text { (A.3) } } \tag{A.3}
\end{align*}
$$

where $C_{1 \mid 2,3,4,5,6}^{m n}$ is given by (3.14).
One can extract the following seven-point pseudoinvariants from the recursion in (5.21) where the expansion of $P_{1|2| 3,4,5,6}$ is given by (5.22):

$$
\begin{align*}
P_{1|23| 4,5,6,7}= & M_{1} \mathcal{J}_{23 \mid 4,5,6,7}+M_{1} \otimes\left[P_{2|3| 4,5,6,7}-P_{3|2| 4,5,6,7}\right] \\
= & M_{1} \mathcal{J}_{23 \mid 4,5,6,7}+M_{12} \mathcal{J}_{3 \mid 4,5,6,7}-M_{13} \mathcal{J}_{2 \mid 4,5,6,7}+k_{3}^{m} M_{123} M_{4,5,6,7}^{m} \\
& -k_{2}^{m} M_{132} M_{4,5,6,7}^{m}+\left[\left(s_{34} M_{1234}-s_{24} M_{1324}\right) M_{5,6,7}+(4 \leftrightarrow 5,6,7)\right] \quad(\mathrm{A}  \tag{A.4}\\
P_{1|2| 34,5,6,7}= & M_{1} \mathcal{J}_{2 \mid 34,5,6,7}+M_{1} \otimes\left[P_{3|2| 4,5,6,7}-P_{4|2| 3,5,6,7}+k_{2}^{m} C_{2 \mid 34,5,6,7}^{m}\right] \\
= & M_{1} \mathcal{J}_{2 \mid 34,5,6,7}+M_{13} \mathcal{J}_{2 \mid 4,5,6,7}-M_{14} \mathcal{J}_{2 \mid 3,5,6,7} \\
& -s_{23}\left(M_{1243}+M_{1423}\right) M_{5,6,7}+s_{24}\left(M_{1234}+M_{1324}\right) M_{5,6,7} \\
+ & k_{2}^{m}\left(M_{12} M_{34,5,6,7}^{m}+\left(M_{123}+M_{132}\right) M_{4,5,6,7}^{m}-\left(M_{124}+M_{142}\right) M_{3,5,6,7}^{m}\right) \\
& +\left[s _ { 2 5 } \left(M_{125} M_{34,6,7}+\left(M_{1325}+M_{1235}+M_{1253}\right) M_{4,6,7}\right.\right. \\
& \left.\left.-\left(M_{1425}+M_{1245}+M_{1254}\right) M_{3,6,7}\right)+(5 \leftrightarrow 6,7)\right] . \tag{A.5}
\end{align*}
$$

The recursion (6.9) for vector pseudoinvariants yields the following seven-point example:

$$
\begin{align*}
P_{1|2| 3,4,5,6,7}^{m}= & M_{1} \mathcal{J}_{2 \mid 3,4,5,6,7}^{m}+M_{1} \otimes\left\{k_{2}^{p} C_{2 \mid 3,4,5,6,7}^{p m}+\left[k_{3}^{m} P_{3|2| 4,5,6,7}+(3 \leftrightarrow 4,5,6,7)\right]\right\} \\
= & M_{1} \mathcal{J}_{2 \mid 3,4,5,6,7}^{m}+\left[k_{3}^{m}\left\{M_{13} \mathcal{J}_{2 \mid 4,5,6,7}+\left(M_{123}+M_{132}\right) k_{2}^{p} M_{4,5,6,7}^{p}\right\}+(3 \leftrightarrow 4,5,6,7)\right] \\
& +k_{2}^{p} M_{12} M_{3,4,5,6,7}^{p m}+\left[s _ { 2 3 } \left\{M_{123} M_{4,5,6,7}^{m}+k_{4}^{m}\left(M_{1234}+M_{1243}+M_{1423}\right) M_{5,6,7}\right.\right. \\
& +k_{5}^{m}\left(M_{1235}+M_{1253}+M_{1523}\right) M_{4,6,7}+k_{6}^{m}\left(M_{1236}+M_{1263}+M_{1623}\right) M_{4,5,7} \\
& \left.\left.+k_{7}^{m}\left(M_{1237}+M_{1273}+M_{1723}\right) M_{4,5,6}\right\}+(3 \leftrightarrow 4,5,6,7)\right] . \tag{A.6}
\end{align*}
$$

The simplest pseudoinvariant of refinement $d>1$ is generated by the recursion (6.19):

$$
\begin{align*}
& P_{1|2,3| 4,5,6,7,8}= M_{1} \mathcal{J}_{2,3 \mid 4,5,6,7,8}+M_{1} \otimes\left[k_{2}^{m} P_{2|3| 4,5,6,7,8}^{m}+k_{3}^{m} P_{3|2| 4,5,6,7,8}^{m}\right] \\
&= M_{1} \mathcal{J}_{2,3 \mid 4,5, \ldots, 8}+M_{12} k_{m}^{2} \mathcal{J}_{3 \mid 4,5, \ldots, 8}^{m}+M_{13} k_{m}^{3} \mathcal{J}_{2 \mid 4,5, \ldots, 8}^{m}+\left(M_{123}+M_{132}\right) k_{m}^{2} k_{n}^{3} M_{4,5,6,7,8}^{m n} \\
&+ {\left[s_{24} M_{124} \mathcal{J}_{3 \mid 5,6,7,8}+s_{34} M_{134} \mathcal{J}_{2 \mid 5,6,7,8}+s_{34}\left(M_{1234}+M_{1324}+M_{1342}\right) k_{2}^{m} M_{5,6,7,8}^{m}\right.} \\
&\left.\quad+s_{24}\left(M_{1324}+M_{1234}+M_{1243}\right) k_{3}^{m} M_{5,6,7,8}^{m}+(4 \leftrightarrow 5,6,7,8)\right] \\
&+ {\left[s_{24} s_{35}\left(M_{12435}+M_{13245}+M_{13524}+M_{12345}+M_{12354}+M_{13254}\right) M_{6,7,8}\right.} \\
& \quad+s_{25} s_{34}\left(M_{12534}+M_{13254}+M_{13425}+M_{12354}+M_{12345}+M_{13245}\right) M_{6,7,8} \\
&\quad+(4,5 \mid 4,5,6,7,8)] \tag{A.7}
\end{align*}
$$

## Appendix B. Gauge transformations versus BRST transformations

The purpose of this appendix is to clarify the relation between gauge transformations and BRST variations. As mentioned below (2.1), the response of the superfields in tendimensional SYM to a gauge transformation $\delta_{i}$ in particle $i$ is given by

$$
\begin{equation*}
\delta_{i} A_{\alpha}^{i}=D_{\alpha} \omega_{i}, \quad \delta_{i} A_{m}^{i}=k_{m}^{i} \omega_{i}, \quad \delta_{i} W_{i}^{\alpha}=\delta_{i} F_{i}^{m n}=0, \tag{B.1}
\end{equation*}
$$

with some scalar superfield $\omega_{i}$. In the following, we infer the gauge transformation of multiparticle superfields from (B.1) using their recursive definition presented in [32] and reviewed in section 2.2. This in turn determines the action of $\delta_{i}$ on the complete set of building blocks for one-loop amplitudes as well as their (pseudo-)invariant combinations. In particular, we will arrive at a dictionary to translate anomalous BRST variations at ghostnumber four to the corresponding anomalous gauge variations at ghost-number three. This is a convenient approach to component expansions of the hexagon gauge anomaly in oneloop amplitudes of multiplicity $n \geq 6$.

## B.1. Gauge variations of multiparticle superfields

The recursive definitions of the rank-two superfields $K_{12} \in\left\{A_{\alpha}^{12}, A_{m}^{12}, W_{12}^{\alpha}, F_{m n}^{12}\right\}$ in (2.8) allows to infer their gauge variation from (B.1),

$$
\begin{array}{ll}
\delta_{1} A_{\alpha}^{12}=D_{\alpha} \omega_{1 \mid 2}+\left(k_{1} \cdot k_{2}\right) \omega_{1} A_{\alpha}^{2} & \delta_{1} W_{12}^{\alpha}=\left(k_{1} \cdot k_{2}\right) \omega_{1} W_{2}^{\alpha} \\
\delta_{1} A_{m}^{12}=k_{12}^{m} \omega_{1 \mid 2}+\left(k_{1} \cdot k_{2}\right) \omega_{1} A_{2}^{m} & \delta_{1} F_{m n}^{12}=\left(k_{1} \cdot k_{2}\right) \omega_{1} F_{2}^{m n} \tag{B.2}
\end{array}
$$

We have introduced a shorthand for the multiparticle gauge scalar,

$$
\begin{equation*}
\omega_{1 \mid 2} \equiv-\frac{1}{2} \omega_{1}\left(k_{1} \cdot A_{2}\right), \tag{B.3}
\end{equation*}
$$

to unify the two-particle expressions in $\delta_{1} A_{\alpha}^{12}$ and $\delta_{1} A_{m}^{12}$. The two-particle gauge transformations (B.2) reproduce the single particle pattern (B.1) with $\omega_{1} \rightarrow \omega_{1 \mid 2}$ and enrich it by contact terms $\sim\left(k_{1} \cdot k_{2}\right)$. This closely mimics the appearance of contact terms in the two-particle equations of motion (2.9).

It is straightforward to work out the multiparticle gauge transformation at $|B|>2$ using the recursion for $K_{B}$ as described in section 2.2. The structure of contact terms in gauge transformations up to multiplicity three is captured by the following variations for the unintegrated vertex $V_{B} \equiv \lambda^{\alpha} A_{\alpha}^{B}$,

$$
\begin{align*}
\delta_{1} V_{1} & =Q \omega_{1}, \quad \delta_{1} V_{12}=Q \omega_{1 \mid 2}+\left(k_{1} \cdot k_{2}\right) \omega_{1} V_{2}  \tag{B.4}\\
\delta_{1} V_{123} & =Q \omega_{1 \mid 23}+\left(k_{1} \cdot k_{2}\right)\left(\omega_{1} V_{23}+\omega_{1 \mid 3} V_{2}\right)+\left(k_{12} \cdot k_{3}\right) \omega_{1 \mid 2} V_{3}  \tag{B.5}\\
\delta_{1} V_{231} & =Q \omega_{23 \mid 1}+\left(k_{2} \cdot k_{3}\right)\left(\omega_{1 \mid 3} V_{2}-\omega_{1 \mid 2} V_{3}\right)-\left(k_{1} \cdot k_{23}\right) \omega_{1} V_{23} . \tag{B.6}
\end{align*}
$$

This parallels the contact terms in (2.14) and defines additional multiparticle gauge scalars

$$
\begin{align*}
& \omega_{1 \mid 23} \equiv-\frac{1}{2} \omega_{1 \mid 2}\left(k_{12} \cdot A_{3}\right)-\frac{1}{6}\left[\omega_{1}\left(k^{1} \cdot A^{23}\right)+\omega_{1 \mid 3}\left(k^{3} \cdot A^{2}\right)-\omega_{1 \mid 2}\left(k^{2} \cdot A^{3}\right)\right],  \tag{B.7}\\
& \omega_{23 \mid 1} \equiv \frac{1}{2} \omega_{1}\left(k_{1} \cdot A_{23}\right)-\frac{1}{6}\left[\omega_{1}\left(k^{1} \cdot A^{23}\right)+\omega_{1 \mid 3}\left(k^{3} \cdot A^{2}\right)-\omega_{1 \mid 2}\left(k^{2} \cdot A^{3}\right)\right],
\end{align*}
$$

where the terms proportional to $\frac{1}{6}$ come from the corrections $H_{i j k}$ of (2.11). As a consistency check of (B.7) one can show that $\delta_{1}\left(V_{123}+V_{231}+V_{312}\right)=0$ after using $\delta_{1} V_{312}=-\delta_{1} V_{132}$ due to the rank-two Lie symmetry in the first two labels.

## B.2. Gauge variations of Berends-Giele currents $M_{A}$

As detailed in section 2.3, a convenient basis of multiparticle fields $K_{B}$ is furnished by Berends-Giele currents $\mathcal{K}_{B}$, represented by calligraphic letters. In a cubic graph interpretation of multiparticle fields $K_{B}$ shown in fig. 3, Berends-Giele currents $\mathcal{K}_{B}$ assemble the diagrams of a color-ordered SYM tree including $|B|-1$ propagators. The dictionary up to multiplicity four is given in (2.17) and (2.18).

For the Berends-Giele current $M_{B}=\lambda^{\alpha} \mathcal{A}_{\alpha}^{B}$ associated with the unintegrated vertex $V_{B}$, (B.4) to (B.6) translate into

$$
\begin{align*}
\delta_{1} M_{1} & =Q \Omega_{1}, \quad \delta_{1} M_{12}=Q \Omega_{1 \mid 2}+\Omega_{1} M_{2}  \tag{B.8}\\
\delta_{1} M_{123} & =Q \Omega_{1 \mid 23}+\Omega_{1 \mid 2} M_{3}+\Omega_{1} M_{23} \tag{B.9}
\end{align*}
$$

with Berends-Giele gauge scalars

$$
\begin{equation*}
\Omega_{1} \equiv \omega_{1}, \quad \Omega_{1 \mid 2} \equiv \frac{\omega_{1 \mid 2}}{s_{12}}, \quad \Omega_{1 \mid 23} \equiv \frac{\omega_{1 \mid 23}}{s_{12} s_{123}}-\frac{\omega_{23 \mid 1}}{s_{23} s_{123}} . \tag{B.10}
\end{equation*}
$$

With suitable multiparticle generalizations $\Omega_{1 \mid 23 \ldots p}$ of (B.10), one can directly write down a closed formula for the gauge transformations of $M_{B}$,

$$
\begin{equation*}
\delta_{1} M_{12 \ldots p}=\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} M_{j+1 \ldots p}+Q \Omega_{1 \mid 23 \ldots p} \tag{B.11}
\end{equation*}
$$

With this form of $\delta_{1} M_{12 \ldots p}$, the superspace representation $\sum_{j=1}^{n-2} M_{12 \ldots j} M_{j+1 \ldots n-1} M_{n}$ of the SYM tree amplitude [17] can be easily checked to be gauge invariant up to BRST exact terms,

$$
\begin{equation*}
\delta_{1}\left(\sum_{j=1}^{n-2} M_{12 \ldots j} M_{j+1 \ldots n-1} M_{n}\right)=Q\left(\sum_{j=1}^{n-2} \Omega_{1 \mid 2 \ldots j} M_{j+1 \ldots n-1} M_{n}\right) . \tag{B.12}
\end{equation*}
$$

## B.3. Gauge variations of ghost number two building blocks

Similarly to $\delta_{1} M_{12 \ldots p}$ in (B.11), the Berends-Giele currents $\mathcal{A}_{B}^{m}, \mathcal{W}_{B}^{\alpha}$ and $\mathcal{F}_{B}^{m n}$ give rise to gauge transformations

$$
\begin{align*}
\delta_{1} \mathcal{A}_{12 \ldots p}^{m} & =k_{12 \ldots p}^{m} \Omega_{1 \mid 23 \ldots p}+\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} \mathcal{A}_{j+1 \ldots p}^{m}  \tag{B.13}\\
\delta_{1} \mathcal{W}_{12 \ldots p}^{\alpha} & =\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} \mathcal{W}_{j+1 \ldots p}^{\alpha}  \tag{B.14}\\
\delta_{1} \mathcal{F}_{12 \ldots p}^{m n} & =\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} \mathcal{F}_{j+1 \ldots p}^{m n} \tag{B.15}
\end{align*}
$$

which resemble their BRST variations (2.22). With (B.13) to (B.15), one can straightforwardly compute the gauge variations of all the building blocks $M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ and $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ introduced in (4.3) and (6.11), respectively. The simplest examples $M_{A, B, C}$ and $M_{A, B, C, D}^{m}$ are defined in (2.26) and (2.30), and their gauge variation

$$
\begin{align*}
\delta_{1} M_{12 \ldots p, B, C} & =\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} M_{j+1 \ldots p, B, C}  \tag{B.16}\\
\delta_{1} M_{12 \ldots p, B, C, D}^{m} & =k_{12 \ldots p}^{m} \Omega_{1 \mid 23 \ldots p} M_{B, C, D}+\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} M_{j+1 \ldots p, B, C, D}^{m} \tag{B.17}
\end{align*}
$$

resembles the contributions from a single slot to the BRST variations (2.27) and (2.31). The variation of tensors or refined building blocks

$$
\begin{align*}
\delta_{1} M_{12 \ldots p, B, C, D, E}^{m n}= & 2 k_{12 \ldots p}^{(m} \Omega_{1 \mid 23 \ldots p} M_{B, C, D, E}^{n)}+\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} M_{j+1 \ldots p, B, C, D, E}^{m n}  \tag{B.18}\\
\delta_{1} \mathcal{J}_{12 \ldots p \mid B, C, D, E}= & k_{12 \ldots p}^{m} \Omega_{1 \mid 23 \ldots p} M_{B, C, D, E}^{m}+\left[\Omega_{S[12 \ldots p, B]} M_{C, D, E}+(B \leftrightarrow C, D, E)\right] \\
& +\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} \mathcal{J}_{j+1 \ldots p \mid B, C, D, E}  \tag{B.19}\\
\delta_{1} \mathcal{J}_{B \mid 12 \ldots p, C, D, E}= & -\Omega_{S[12 \ldots p, B]} M_{C, D, E}+\sum_{j=1}^{p-1} \Omega_{1 \mid 23 \ldots j} \mathcal{J}_{B \mid j+1 \ldots p, C, D, E} \tag{B.20}
\end{align*}
$$

does not reproduce the anomalous terms $\mathcal{Y}_{A, B, C, D, E}$ present in the BRST variations (3.2) and (5.18). The $S[A, B]$ map is defined in (5.14) and $\Omega_{S[12 \ldots p, B]}$ is understood to be arranged in the form $\Omega_{1 \mid \ldots}$.

As general rule, $\delta_{i} \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ can be reconstructed from those terms in $Q \mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ given in (6.13) where particle $i$ appears in a current $M_{i C}$. The gauge variation follows by replacing $M_{i C} \rightarrow \Omega_{i \mid C}$ and discarding any other term in the BRST variation. The same prescription applies to anomaly building blocks $\mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}$ which are recursively defined by (6.12), see (3.3) for the simplest scalar $\mathcal{Y}_{A, B, C, D, E}$ and (7.12) for the general BRST transformation.

## B.4. Gauge variations of pseudoinvariants

The above gauge variations of $M_{B}$ and $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$ are sufficient to study the anomalous gauge transformations of pseudoinvariants. This in turn allows to probe the
hexagon anomaly in field theory and string theory, see [31,29] for details. All the pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ constructed by the recursion (6.17) are combinations of $M_{i A} \mathcal{J}$ where $\mathcal{J}$ represents any ghost number two building block $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$, possibly adjoined by momenta. This makes the gauge variation in the reference leg $i$ particularly convenient to study: By (B.11), we have

$$
\begin{equation*}
\delta_{i} M_{i A} \mathcal{J}=\sum_{j=0}^{|A|-1} \Omega_{i \mid a_{1} a_{2} \ldots a_{j}} M_{a_{j+1} \ldots a_{|A|}} \mathcal{J}-\Omega_{i \mid A} Q \mathcal{J}+Q\left(\Omega_{i \mid A} \mathcal{J}\right) \tag{B.21}
\end{equation*}
$$

Up to the last BRST-exact term, this is closely related to the BRST transformation

$$
\begin{equation*}
Q M_{i A} \mathcal{J}=\sum_{j=0}^{|A|-1} M_{i a_{1} a_{2} \ldots a_{j}} M_{a_{j+1} \ldots a_{|A|}} \mathcal{J}-M_{i A} Q \mathcal{J} \tag{B.22}
\end{equation*}
$$

upon interchanging $\Omega_{i \mid B} \leftrightarrow M_{i B}$ for any $|B|=0,1, \ldots,|A|$, i.e.

$$
\begin{equation*}
\delta_{i} M_{i A} \mathcal{J}=\left.\left(Q M_{i A} \mathcal{J}\right)\right|_{M_{i B} \rightarrow \Omega_{i \mid B}}+Q\left(\Omega_{i \mid A} \mathcal{J}\right) \tag{B.23}
\end{equation*}
$$

This can be applied term by term to the pseudoinvariants $P_{i \mid \ldots}^{m_{1} \ldots}$ obtained from (6.17),

$$
\begin{align*}
\delta_{i} P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}=( & \left.Q P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}\right)\left.\right|_{M_{i C} \rightarrow \Omega_{i \mid C}} \\
& +Q\left(\left.P_{i\left|A_{1}, \ldots, A_{d}\right| B_{1}, \ldots, B_{d+r+3} m_{1} \ldots m_{r}}\right|_{M_{i C} \rightarrow \Omega_{i \mid C}}\right), \tag{B.24}
\end{align*}
$$

where the BRST transformations are given by (7.17). For example, the anomalous $Q$ variations of the simplest pseudoinvariants at six and seven points

$$
\begin{align*}
Q P_{1|2| 3,4,5,6} & =-M_{1} \mathcal{Y}_{2,3,4,5,6}  \tag{B.25}\\
Q P_{1|23| 4,5,6,7} & =-M_{1} \mathcal{Y}_{23,4,5,6,7}-M_{12} \mathcal{Y}_{3,4,5,6,7}+M_{13} \mathcal{Y}_{2,4,5,6,7}  \tag{B.26}\\
Q P_{1|2| 3,4,5,6,7}^{m} & =-M_{1} \mathcal{Y}_{2,3,4,5,6,7}^{m}-\left[k_{2}^{m} M_{12} \mathcal{Y}_{3,4,5,6,7}+(2 \leftrightarrow 3,4,5,6,7)\right] \tag{B.27}
\end{align*}
$$

translate into gauge variations

$$
\begin{align*}
\delta_{1} P_{1|2| 3,4,5,6} & =-\Omega_{1} \mathcal{Y}_{2,3,4,5,6}+Q(\ldots)  \tag{B.28}\\
\delta_{1} P_{1|23| 4,5,6,7} & =-\Omega_{1} \mathcal{Y}_{23,4,5,6,7}-\Omega_{1 \mid 2} \mathcal{Y}_{3,4,5,6,7}+\Omega_{1 \mid 3} \mathcal{Y}_{2,4,5,6,7}+Q(\ldots)  \tag{B.29}\\
\delta_{1} P_{1|2| 3,4,5,6,7}^{m} & =-\Omega_{1} \mathcal{Y}_{2,3,4,5,6,7}^{m}-\left[k_{2}^{m} \Omega_{1 \mid 2} \mathcal{Y}_{3,4,5,6,7}+(2 \leftrightarrow 3, \ldots, 7)\right]+Q(\ldots) \tag{B.30}
\end{align*}
$$

Gauge variations $\delta_{k} P_{i \mid \ldots}^{m_{1} \cdots}$ beyond the reference leg $i$ can still be obtained by trading any $M_{k B}$ in the BRST variation $Q P_{i \mid \ldots}^{m_{1} \ldots}$ for $\Omega_{k \mid B}$, e.g.

$$
\begin{align*}
\delta_{2} P_{1|2| 3,4,5,6} & =Q(\ldots) \\
\delta_{2} P_{1|23| 4,5,6,7} & =\Omega_{2 \mid 1} \mathcal{Y}_{3,4,5,6,7}+Q(\ldots)  \tag{B.31}\\
\delta_{2} P_{1|2| 3,4,5,6,7}^{m} & =\Omega_{2 \mid 1} k_{2}^{m} \mathcal{Y}_{3,4,5,6,7}+Q(\ldots)
\end{align*}
$$

These expressions of ghost number three can be evaluated in components using the methods in [9]. As shown in [26], the bosonic components of (B.28) yield a Levi-Civita contraction of five gluon field strengths, i.e. $\sim \epsilon_{m_{1} n_{1} \ldots m_{5} n_{5}} k_{2}^{m_{1}} e_{2}^{n_{1}} \ldots k_{6}^{m_{5}} e_{6}^{n_{5}}$. We will argue in the next section that any anomalous superfield has parity odd bosonic components.

## B.5. Parity odd nature of multiparticle anomaly tensors

The component evaluation of $\left\langle\left(\lambda \gamma^{m} W_{2}\right)\left(\lambda \gamma^{n} W_{3}\right)\left(\lambda \gamma^{p} W_{4}\right)\left(W_{5} \gamma_{m n p} W_{6}\right)\right\rangle$ using the prescription $\left\langle\lambda^{3} \theta^{5}\right\rangle=1[3]$ is particularly simple for five external bosons: The lowest bosonic component in the superfields occurs at order $W_{i}^{\alpha} \rightarrow-\frac{1}{4}\left(\gamma_{m n} \theta\right)^{\alpha} f_{i}^{m n}$ (with $f_{i}^{m n}=2 k_{i}^{[m} e_{i}^{n]}$ in terms of the gluon polarization vector $e_{i}^{n}$ ), so the contribution from five factors of $W_{i}$ to the order $\theta^{5}$ is unique. The single-particle instance $\mathcal{Y}_{2,3,4,5,6}$ of the anomaly superfields in (3.3) therefore reduces to the correlator

$$
\begin{equation*}
\left\langle\left(\lambda \gamma^{m} \gamma^{a_{1} b_{1}} \theta\right)\left(\lambda \gamma^{n} \gamma^{a_{2} b_{2}} \theta\right)\left(\lambda \gamma^{p} \gamma^{a_{3} b_{3}} \theta\right)\left(\theta \gamma^{a_{4} b_{4}} \gamma_{m n p} \gamma^{a_{5} b_{5}} \theta\right)\right\rangle=\frac{1}{45} \epsilon^{a_{1} b_{1} a_{2} b_{2} \ldots a_{5} b_{5}} \tag{B.32}
\end{equation*}
$$

which has been evaluated in [26] and shown to flip sign under spacetime parity.
It turns out that the same correlator (B.32) governs the bosonic components of a generic $\mathcal{Y}_{A, B, C, D, E}$ built from multiparticle superfields $W_{A}^{\alpha}$. This follows from the twoform nature of the lowest bosonic component in the $\theta$-expansion of the BRST blocks,

$$
\begin{equation*}
W_{A}^{\alpha}=-\frac{1}{4}\left(\gamma_{m n} \theta\right)^{\alpha} f_{A}^{m n}+\mathcal{O}\left(\theta^{3}\right), \quad F_{A}^{m n}=f_{A}^{m n}+\mathcal{O}\left(\theta^{2}\right) \tag{B.33}
\end{equation*}
$$

where the two-particle instance of the bosonic field strength $f_{A}^{m n}$ is given by,

$$
\begin{equation*}
\frac{1}{2} f_{12}^{m n} \equiv k_{12}^{[m} e_{2}^{n]}\left(e_{1} \cdot k_{2}\right)-k_{12}^{[m} e_{1}^{n]}\left(e_{2} \cdot k_{1}\right)-k_{1}^{[m} k_{2}^{n]}\left(e_{1} \cdot e_{2}\right)-e_{1}^{[m} e_{2}^{n]}\left(k_{1} \cdot k_{2}\right) \tag{B.34}
\end{equation*}
$$

The appearance of $f_{A}^{m n}$ in both superfields in (B.33) is an inevitable consequence of the multiparticle equation of motion for $D_{\alpha} W_{A}^{\beta}$, see (2.9) and (2.14) for $|A|=2,3$ and (2.22) for the Berends-Giele version at general multiplicity. The contact terms $\sim A_{\alpha}^{B} W_{C}^{\beta}$ in the
multiparticle equations of motion do not contribute at zero'th order in $\theta$ since both factors are fermionic with lowest gluon contributions at order $\theta^{1}$.

The correlator (B.32) and the leading $\theta$ behavior in (B.33) for the bosonic part of $W_{A}^{\alpha}$ imply the gluon component

$$
\begin{equation*}
\left\langle\mathcal{Y}_{A, B, C, D, E}\right\rangle=\frac{1}{45}\left(-\frac{1}{4}\right)^{5} \epsilon_{a_{1} b_{1} a_{2} b_{2} \ldots a_{5} b_{5}} f_{A}^{a_{1} b_{1}} f_{B}^{a_{2} b_{2}} \ldots f_{E}^{a_{5} b_{5}} \tag{B.35}
\end{equation*}
$$

for the scalar and unrefined anomaly superfield $\mathcal{Y}_{A, B, C, D, E}$. Its generalization to higher rank or refinement simply adjoins superspace factors of $\mathcal{A}_{B}^{m}$, see (4.6) and (6.12). The latter can only contribute through their $\theta=0$ component since $\mathcal{Y}_{A, B, C, D, E}$ has a minimum contribution of five thetas for external bosons. The same is true for the $\Omega_{i \mid C}$ superfields due to gauge transformations in particle $i$. Hence, the gluon components of an anomalous gauge transformation $\left\langle\Omega_{i \mid C} \mathcal{Y}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+5}}^{m_{1} \ldots m_{r}}\right\rangle$ are proportional to the $\epsilon_{10}$ tensor generated by the correlator (B.32).

## Appendix C. BRST variations of miscellaneous superfields

In this appendix we display explicit BRST variations of various superfields that were omitted from the main text.

## C.1. BRST variations before the Berends-Giele map

Even though we emphasized the simpler BRST transformations of the Berends-Giele version of the various building blocks in the main body of this work, it is still convenient to know the explicit $Q$ variations of those building blocks prior to the application of the Berends-Giele map in (2.17) and (2.18).

The precursor of the Berends-Giele recursion (4.3) for $M_{B_{1}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r}}$ is based on the expression (2.25) for $T_{A, B, C}$ as well as

$$
\begin{equation*}
W_{A, B, C, D}^{m} \equiv \frac{1}{12}\left(\lambda \gamma_{n} W_{A}\right)\left(\lambda \gamma_{p} W_{B}\right)\left(W_{C} \gamma^{m n p} W_{D}\right)+(A, B \mid A, B, C, D) \tag{C.1}
\end{equation*}
$$

For the higher rank generalizations

$$
\begin{align*}
W_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1} \ldots m_{r-1} \mid m_{r}} & \equiv A_{B_{1}}^{m_{1}} W_{B_{2}, \ldots, B_{r+3}}^{m_{2} \ldots m_{r-1} \mid m_{r}}+\left(B_{1} \leftrightarrow B_{2}, \ldots, B_{r+3}\right)  \tag{C.2}\\
T_{B_{1}, B_{2}, \ldots, B_{r+3}}^{m_{1}, m_{r}} & \equiv A_{B_{1}}^{m_{1}} T_{B_{2}, \ldots, B_{r+3}}^{m_{2}, \ldots m_{r}}+A_{B_{1}}^{m_{r}} W_{B_{2}, \ldots, B_{r+3}}^{m_{r-1} \ldots m_{2} \mid m_{1}}+\left(B_{1} \leftrightarrow B_{2}, B_{3}, \ldots, B_{r+3}\right),
\end{align*}
$$

one can show that

$$
\begin{align*}
Q T_{1,2,3,4}^{m}= & k_{1}^{m} V_{1} T_{2,3,4}+(1 \leftrightarrow 2,3,4)  \tag{C.3}\\
Q T_{12,3,4,5}^{m}= & {\left[k_{12}^{m} V_{12} T_{3,4,5}+(12 \leftrightarrow 3,4,5)\right]+\left(k^{1} \cdot k^{2}\right)\left(V_{1} T_{2,3,4,5}^{m}-V_{2} T_{1,3,4,5}^{m}\right) } \\
Q T_{123,4,5,6}^{m}= & {\left[k_{123}^{m} V_{123} T_{4,5,6}+(123 \leftrightarrow 4,5,6)\right] } \\
& +\left(k^{1} \cdot k^{2}\right)\left[V_{1} T_{23,4,5,6}^{m}+V_{13} T_{2,4,5,6}^{m}-(1 \leftrightarrow 2)\right] \\
& +\left(k^{12} \cdot k^{3}\right)\left[V_{12} T_{3,4,5,6}^{m}-(12 \leftrightarrow 3)\right] \\
Q T_{1,2,3,4,5}^{m n}= & {\left[2 k_{1}^{(m} V_{1} T_{2,3,4,5}^{n)}+(1 \leftrightarrow 2,3,4,5)\right]+\delta^{m n} Y_{1,2,3,4,5} } \\
Q T_{12,3,4,5,6}^{m n}= & {\left[2 k_{12}^{(m} V_{12} T_{3,4,5,6}^{n)}+(12 \leftrightarrow 3,4,5,6)\right]+\delta^{m n} Y_{12,3,4,5,6} } \\
& +\left(k^{1} \cdot k^{2}\right)\left[V_{1} T_{2,3,4,5,6}^{m n}-(1 \leftrightarrow 2)\right] \\
Q T_{1,2,3,4,5,6}^{m n p}= & 3 \delta^{(m n} Y_{1,2,3,4,5,6}^{p)}+\left[3 V_{1} k_{1}^{(m} T_{2,3,4,5,6}^{n p)}+(1 \leftrightarrow 2,3,4,5,6)\right] .
\end{align*}
$$

Similarly, the BRST variations of refined currents (5.7) can be computed to be

$$
\begin{align*}
Q J_{1 \mid 23,45,6,7}= & k_{1}^{m} V_{1} T_{23,45,6,7}^{m}+\left[V_{[1,23]} T_{45,6,7}+(23 \leftrightarrow 45,6,7)\right]+Y_{1,23,45,6,7}  \tag{C.4}\\
& +\left(k^{2} \cdot k^{3}\right)\left[V_{2} J_{1 \mid 3,45,6,7}-(2 \leftrightarrow 3)\right]+\left(k^{4} \cdot k^{5}\right)\left[V_{4} J_{1 \mid 23,5,6,7}-(4 \leftrightarrow 5)\right] \\
Q J_{12 \mid 34,5,6,7}= & k_{12}^{m} V_{12} T_{34,5,6,7}^{m}+\left[V_{[12,34]} T_{5,6,7}+(34 \leftrightarrow 5,6,7)\right]+Y_{12,34,5,6,7} \\
& +\left(k^{1} \cdot k^{2}\right)\left[V_{1} J_{2 \mid 34,5,6,7}-(1 \leftrightarrow 2)\right]+\left(k^{3} \cdot k^{4}\right)\left[V_{3} J_{12 \mid 4,5,6,7}-(3 \leftrightarrow 4)\right] \\
Q J_{123 \mid 4,5,6,7}= & k_{123}^{m} V_{123} T_{4,5,6,7}^{m}+\left[V_{[123,4]} T_{5,6,7}+(4 \leftrightarrow 5,6,7)\right]+Y_{123,4,5,6,7} \\
& +\left(k^{1} \cdot k^{2}\right)\left[V_{1} J_{23 \mid 4,5,6,7}+V_{13} J_{2 \mid 4,5,6,7}-(1 \leftrightarrow 2)\right] \\
& +\left(k^{12} \cdot k^{3}\right)\left[V_{12} J_{3 \mid 4,5,6,7}-(12 \leftrightarrow 3)\right] \\
Q J_{1 \mid 234,5,6,7}= & k_{1}^{m} V_{1} T_{234,5,6,7}^{m}+\left[V_{[1,234]} T_{5,6,7}+(234 \leftrightarrow 5,6,7)\right]+Y_{1,234,5,6,7} \\
& +\left(k^{2} \cdot k^{3}\right)\left[V_{2} J_{1 \mid 34,5,6,7}+V_{24} J_{1 \mid 3,5,6,7}-(2 \leftrightarrow 3)\right] \\
& +\left(k^{23} \cdot k^{4}\right)\left[V_{23} J_{1 \mid 4,5,6,7}-(23 \leftrightarrow 4)\right] .
\end{align*}
$$

## C.2. BRST variations after the Berends-Giele map

After applying the Berends-Giele map to the refined currents from (C.4), their BRST variations become

$$
\begin{align*}
Q \mathcal{J}_{1 \mid 23,45,6,7}= & k_{1}^{m} M_{1} M_{23,45,6,7}^{m}  \tag{C.5}\\
& +\left(s_{12} M_{123}-s_{13} M_{132}\right) M_{45,6,7}+\left(s_{14} M_{145}-s_{15} M_{154}\right) M_{23,6,7}
\end{align*}
$$

$$
\begin{aligned}
& +s_{16} M_{16} M_{23,45,7}+s_{17} M_{17} M_{23,45,6}+\mathcal{Y}_{1,23,45,6,7} \\
& +M_{2} \mathcal{J}_{1 \mid 3,45,6,7}-M_{3} \mathcal{J}_{1 \mid 2,45,6,7}+M_{4} \mathcal{J}_{1 \mid 23,5,6,7}-M_{5} \mathcal{J}_{1 \mid 23,4,6,7} \\
Q \mathcal{J}_{12 \mid 34,5,6,7}= & k_{12}^{m} M_{12} M_{34,5,6,7}^{m} \\
& +\left(s_{23} M_{1234}-s_{13} M_{2134}-s_{24} M_{1243}+s_{14} M_{2143}\right) M_{5,6,7} \\
& +\left[\left(s_{25} M_{125}-s_{15} M_{215}\right) M_{34,6,7}+(5 \leftrightarrow 6,7)\right]+\mathcal{Y}_{12,34,5,6,7} \\
& +M_{1} \mathcal{J}_{2 \mid 34,5,6,7}-M_{2} \mathcal{J}_{1 \mid 34,5,6,7}+M_{3} \mathcal{J}_{12 \mid 4,5,6,7}-M_{4} \mathcal{J}_{12 \mid 3,5,6,7} \\
Q \mathcal{J}_{123 \mid 4,5,6,7}= & k_{123}^{m} M_{123} M_{4,5,6,7}^{m} \\
& +\left[\left(s_{34} M_{1234}-s_{24}\left(M_{1324}+M_{3124}\right)+s_{14} M_{3214}\right) M_{5,6,7}+(4 \leftrightarrow 5,6,7)\right] \\
& +\mathcal{Y}_{123,4,5,6,7}+M_{12} \mathcal{J}_{3 \mid 4,5,6,7}-M_{3} \mathcal{J}_{12 \mid 4,5,6,7}+M_{1} \mathcal{J}_{23 \mid 4,5,6,7}-M_{23} \mathcal{J}_{1 \mid 4,5,6,7} \\
Q \mathcal{J}_{1 \mid 234,5,6,7}= & k_{1}^{m} M_{1} M_{234,5,6,7}^{m} \\
& -\left(s_{12} M_{4321}-s_{13}\left(M_{2431}+M_{4231}\right)+s_{14} M_{2341}\right) M_{5,6,7} \\
& +\left[s_{15} M_{15} M_{234,6,7}+(5 \leftrightarrow 6,7)\right]+\mathcal{Y}_{1,234,5,6,7} \\
& +M_{23} \mathcal{J}_{1 \mid 4,5,6,7}-M_{4} \mathcal{J}_{1 \mid 23,5,6,7}+M_{2} \mathcal{J}_{1 \mid 34,5,6,7}-M_{34} \mathcal{J}_{1 \mid 2,5,6,7} .
\end{aligned}
$$

The following relations are useful to derive (C.5) from (C.4):

$$
\begin{align*}
T_{12}= & s_{12} M_{12}, \quad T_{123}=s_{12}\left(s_{23} M_{123}-s_{13} M_{213}\right)  \tag{C.6}\\
T_{1234}-T_{1243}= & s_{12} s_{34}\left(s_{23} M_{1234}-s_{13} M_{2134}-s_{24} M_{1243}+s_{14} M_{2143}\right)  \tag{C.7}\\
T_{1234}= & s_{12}\left[s_{23} s_{24}\left(M_{1234}+M_{1243}\right)-s_{13} s_{14}\left(M_{2134}+M_{2143}\right)+s_{23} s_{34} M_{1234}\right. \\
& \left.-s_{13} s_{34} M_{2134}+s_{14} s_{23} M_{3214}-s_{13} s_{24} M_{3124}\right] \tag{C.8}
\end{align*}
$$

## Appendix D. The $H$ superfields in the redefinition of refined currents

This appendix provides a general definition for the superfields $H_{[A, B]}$ and $\mathcal{H}_{[A, B]}$ relevant for the redefinition of refined currents $\mathcal{J}_{A_{1}, \ldots, A_{d} \mid B_{1}, \ldots, B_{d+r+3}}^{m_{1} \ldots m_{r}}$.
D.1. The $H_{[A, B]}$ tensors from BRST blocks $V_{C}$

As mentioned in section 2.2 and detailed in [32], the recursive construction of BRST blocks $K_{B} \in\left\{A_{\alpha}^{B}, A_{B}^{m}, W_{B}^{\alpha}, F_{B}^{m n}\right\}$ requires redefinitions by BRST trivial quantities to maintain the Lie symmetries. We define $\widehat{V}_{[A, B]}$ through a generalization of the recursion in (2.15) to situations where both $|A| \neq 1$ and $|B| \neq 1$ :

$$
\begin{equation*}
\widehat{V}_{[A, B]} \equiv-\frac{1}{2}\left[V_{A}\left(k_{A} \cdot A_{B}\right)+A_{m}^{A}\left(\lambda \gamma^{m} W_{B}\right)-(A \leftrightarrow B)\right] \tag{D.1}
\end{equation*}
$$

There are two obstructions to express $\widehat{V}_{[A, B]}$ as a linear combination of BRST blocks $V_{C}$ at multiplicity $|C|=|A|+|B|$ :
(i) Generic contributions to $Q \widehat{V}_{[A, B]}$ have the form $s_{i j} V_{C} \widehat{V}_{[D, E]}$. They can be corrected to $s_{i j} V_{C} V_{[D, E]}$ by subtracting combinations of $s_{i j} V_{C} H_{[D, E]}$ for some scalar superfields $H_{[D, E]}$ to be defined in the next step.
(ii) After the above subtraction, the modified $\widehat{V}_{[A, B]}$ must still be shifted by a BRST exact quantity $Q H_{[A, B]}$ before it can be expressed in a basis of BRST blocks $V_{C}$. If $B=b_{1}$, i.e. $|B|=1$, this amounts to enforcing the Lie symmetries at multiplicity $|A|+1$ by adding $Q H_{\left[A, b_{1}\right]}$. The latter was denoted by $H_{\left[A, b_{1}\right]} \equiv H_{a_{1} a_{2} \ldots a_{|A|} b_{1}}$ in [32].
Let us illustrate the recursive nature of these points through the simplest examples at multiplicity $|A|+|B| \leq 5$. Cases with $|B|=1$ have been discussed in [32],

$$
\begin{align*}
\widehat{V}_{[12,3]} & =V_{[12,3]}+Q H_{[12,3]}  \tag{D.2}\\
\widehat{V}_{[123,4]} & =V_{[123,4]}+\left(k_{12} \cdot k_{3}\right) H_{[12,4]} V_{3}+\left(k_{1} \cdot k_{2}\right)\left(H_{[13,4]} V_{2}-H_{[23,4]} V_{1}\right)+Q H_{[123,4]}  \tag{D.3}\\
\widehat{V}_{[1234,5]} & =V_{[1234,5]}+\left(k_{123} \cdot k_{4}\right) H_{[123,5]} V_{4}+\left(k_{12} \cdot k_{3}\right)\left(H_{[124,5]} V_{3}+H_{[12,5]} V_{34}-H_{[34,5]} V_{12}\right) \\
& +\left(k_{1} \cdot k_{2}\right)\left(H_{[134,5]} V_{2}+H_{[13,5]} V_{24}+H_{[14,5]} V_{23}-H_{[24,5]} V_{13}-H_{[23,5]} V_{14}-H_{[234,5]} V_{1}\right) \\
& +Q H_{[1234,5]}, \tag{D.4}
\end{align*}
$$

where the corrections by $Q H_{[12 \ldots p-1, p]}$ are explained in (ii) and the remaining terms are due to step (i), see [32] for a closed formula. Requiring $V_{[12 \ldots p-1, p]}=V_{12 \ldots p}$ to satisfy the Lie symmetries at rank $p$ turns (D.2) to (D.4) into a recursive procedure to determine $H_{[12 \ldots p-1, p]}=H_{12 \ldots p}$ and $V_{12 \ldots p}$ [32].

Cases with $|B| \neq 1$ introduce new classes of corrections $H_{[A, B]}$ :

$$
\begin{align*}
\widehat{V}_{[12,34]} & =V_{[12,34]}+\left(k_{1} \cdot k_{2}\right)\left(H_{[34,2]} V_{1}-H_{[34,1]} V_{2}\right) \\
& +\left(k_{3} \cdot k_{4}\right)\left(H_{[12,3]} V_{4}-H_{[12,4]} V_{3}\right)+Q H_{[12,34]}  \tag{D.5}\\
\widehat{V}_{[123,45]} & =V_{[123,45]}+\left(k_{12} \cdot k_{3}\right)\left(H_{[12,45]} V_{3}-H_{[3,45]} V_{12}\right) \\
& +\left(k_{1} \cdot k_{2}\right)\left(H_{[13,45]} V_{2}+H_{[1,45]} V_{23}-H_{[2,45]} V_{13}-H_{[23,45]} V_{1}\right) \\
& +\left(k_{4} \cdot k_{5}\right)\left(H_{[123,4]} V_{5}-H_{[123,5]} V_{4}\right)+Q H_{[123,45]}, \tag{D.6}
\end{align*}
$$

The bracket notation $V_{[A, B]}$ on the right-hand side represents linear combinations of BRST blocks such as $V_{[12,34]}=V_{1234}-V_{1243}$ or $V_{[123,45]}=V_{12345}-V_{12354}$, see appendix A of [32] for more details. They follow by identifying $V_{[A, B]}$ with the cubic diagram depicted in fig. 5 and expanding the latter in terms of a multiperipheral basis in fig. 3. The required


Fig. 8 Triplet of subdiagrams whose color representatives sum to zero by virtue of the Jacobi identity $f^{e[a b} f^{c] d e}=0$. The expansion of the above $V_{[A, B]}$ in a basis of BRST blocks $V_{C}$ can be understood using the same vanishing statement for triplets of diagrams. The latter allows to expand the diagrammatic representative for $V_{[A, B]}$ shown in fig. 5 in terms of multiperipheral trees depicted in fig. 3 and described by $V_{C}$.
diagram manipulations are depicted in fig. 8 and can be though of as the kinematic dual of the Jacobi identity $f^{e[a b} f^{c] d e}=0$ among color tensors along the lines of [44]. The $S[A, B]$ map defined in (5.14) will efficiently address the conversion of $V_{[A, B]} \rightarrow V_{C}$ once the participating superfields are transformed into a basis of Berends-Giele currents.

With this understanding of the $V_{[A, B]}$ on the right-hand side of (D.2) to (D.6), the redefining tensors $Q H_{[A, B]}$ with $|B| \neq 1$ can be obtained recursively. In order to bypass the inconvenience of "inverting" the BRST charge, we next present a setup to determine the $H_{[A, B]}$ directly.

## D.2. The $H_{[A, B]}$ tensors from BRST blocks $A_{C}^{m}$

Also the BRST block $\widehat{A}_{12 \ldots p}^{m}$ in its hatted version before $H_{[B, C]}$ modifications is defined recursively in [32]. The expression given for $\widehat{A}_{12 \ldots p}^{m} \equiv \widehat{A}_{[12 \ldots p-1, p]}^{m}$ can be straightforwardly generalized to $\widehat{A}_{[B, C]}^{m}$ with $|C| \neq 1$ :

$$
\begin{equation*}
\widehat{A}_{[B, C]}^{m} \equiv \frac{1}{2}\left[A_{B}^{p} F_{C}^{p m}+A_{C}^{m}\left(k_{C} \cdot A_{B}\right)-(B \leftrightarrow C)\right]+\left(W_{B} \gamma^{m} W_{C}\right) \tag{D.7}
\end{equation*}
$$

As shown in [32] for $|C|=1$, the redefinitions of $\widehat{A}_{[B, C]}^{m}$ and $\widehat{V}_{[B, C]}$ are mapped into each other by exchanging the relevant BRST blocks $V_{D} \leftrightarrow A_{D}^{m}$ and trading $Q \leftrightarrow k_{B C}^{m}=k_{B}^{m}+k_{C}^{m}$. Up to multiplicity four, this converts (D.2), (D.3) and (D.5) into

$$
\begin{align*}
\widehat{A}_{[12,3]}^{m} & =A_{[12,3]}^{m}+k_{123}^{m} H_{[12,3]}  \tag{D.8}\\
\widehat{A}_{[123,4]}^{m} & =A_{[123,4]}^{m}+\left(k_{12} \cdot k_{3}\right) H_{[12,4]} A_{3}^{m} \\
& +\left(k_{1} \cdot k_{2}\right)\left(H_{[13,4]} A_{2}^{m}-H_{[23,4]} A_{1}^{m}\right)+k_{1234}^{m} H_{[123,4]}  \tag{D.9}\\
\widehat{A}_{[12,34]}^{m} & =A_{[12,34]}^{m}+\left(k_{1} \cdot k_{2}\right)\left(H_{[34,2]} A_{1}^{m}-H_{[34,1]} A_{2}^{m}\right) \\
& +\left(k_{3} \cdot k_{4}\right)\left(H_{[12,3]} A_{4}^{m}-H_{[12,4]} A_{3}^{m}\right)+k_{1234}^{m} H_{[12,34]} . \tag{D.10}
\end{align*}
$$

As emphasized in [32], knowledge of $k_{B C}^{m} H_{[B, C]}$ is a more convenient starting point to solve for the scalar $H_{[B, C]}$ as compared to $Q H_{[B, C]}$.

## D.3. The $\mathcal{H}_{[A, B]}$ tensors from Berends-Giele currents $\mathcal{K}_{C}$

A convenient basis of multiparticle SYM fields $K_{C}$ to construct BRST (pseudo-)invariants is furnished by the Berends-Giele currents, see section 2.3. The contact terms in the above formulae turn out to simplify once we transform the superfields involved according to

$$
\begin{align*}
\mathcal{H}_{[12,3]} & =\frac{H_{[12,3]}}{s_{12}}, \quad \mathcal{H}_{[12,34]}=\frac{H_{[12,34]}}{s_{12} s_{34}}, \quad \mathcal{H}_{[123,4]}=\frac{H_{[123,4]}}{s_{12} s_{123}}+\frac{H_{[321,4]}}{s_{23} s_{123}} \\
\mathcal{H}_{[123,456]} & =\frac{1}{s_{123} s_{456}}\left(\frac{H_{[123,456]}}{s_{12} s_{45}}+\frac{H_{[321,456]}}{s_{23} s_{45}}+\frac{H_{[123,654]}}{s_{12} s_{56}}+\frac{H_{[321,654]}}{s_{23} s_{56}}\right) . \tag{D.11}
\end{align*}
$$

This amounts to applying the map in (2.17) and (2.18) separately to $B$ and $C$ in $H_{[B, C]}$. The resulting $\mathcal{H}_{[B, C]}$ are the natural superfields to describe the redefinitions of

$$
\begin{align*}
\widehat{\mathcal{V}}_{[B, C]} & \equiv-\frac{1}{2}\left[M_{B}\left(k_{B} \cdot \mathcal{A}_{C}\right)+\mathcal{A}_{m}^{B}\left(\lambda \gamma^{m} \mathcal{W}_{C}\right)-(B \leftrightarrow C)\right]  \tag{D.12}\\
\widehat{\mathcal{A}}_{[B, C]}^{m} & \equiv \frac{1}{2}\left[\mathcal{A}_{B}^{p} \mathcal{F}_{C}^{p m}+\mathcal{A}_{C}^{m}\left(k_{C} \cdot \mathcal{A}_{B}\right)-(B \leftrightarrow C)\right]+\left(\mathcal{W}_{B} \gamma^{m} \mathcal{W}_{C}\right) \tag{D.13}
\end{align*}
$$

In order to obtain the Berends-Giele images of the BRST blocks $V_{D}$ and $A_{D}^{m}$ with Lie symmetries, we have to modify $\widehat{\mathcal{V}}_{[A, B]}$ and $\widehat{\mathcal{A}}_{[B, C]}^{m}$ via

$$
\begin{align*}
\widehat{\mathcal{V}}_{[B, C]} & \equiv M_{S[B, C]}+\sum_{X Y=B}\left(\mathcal{H}_{[X, C]} M_{Y}-\mathcal{H}_{[Y, C]} M_{X}\right) \\
& +\sum_{X Y=C}\left(\mathcal{H}_{[B, X]} M_{Y}-\mathcal{H}_{[B, Y]} M_{X}\right)+Q \mathcal{H}_{[B, C]}  \tag{D.14}\\
\widehat{\mathcal{A}}_{[B, C]}^{m} & \equiv \mathcal{A}_{S[B, C]}^{m}+\sum_{X Y=B}\left(\mathcal{H}_{[X, C]} \mathcal{A}_{Y}^{m}-\mathcal{H}_{[Y, C]} \mathcal{A}_{X}^{m}\right) \\
& +\sum_{X Y=C}\left(\mathcal{H}_{[B, X]} \mathcal{A}_{Y}^{m}-\mathcal{H}_{[B, Y]} \mathcal{A}_{X}^{m}\right)+k_{B C}^{m} \mathcal{H}_{[B, C]} \tag{D.15}
\end{align*}
$$

Note that $\mathcal{H}_{[B, C]}=0$ whenever $|B|=|C|=1$. Comparison of (D.14) and (D.15) with the above examples (say (D.6) or (D.10)) reveals two benefits of the basis of BerendsGiele currents: Firstly, the pattern of BRST blocks on the right-hand side without an accompanying factor of $\mathcal{H}_{[B, C]}$ can be described by the $S[B, C]$ map defined in (5.14). Secondly, the contact terms in (D.6) or (D.10) are converted to simple deconcatenations. Since $M_{S[B, C]}$ and $\mathcal{A}_{S[B, C]}^{m}$ are known in terms of BRST blocks $V_{D}$ and $A_{D}^{m}$ of multiplicity $|D|=|B|+|C|[32]$, one can view (D.15) as a constructive definition of $\mathcal{H}_{[B, C]}$.

## Appendix E. On BRST exact relations among pseudoinvariants

## E.1. BRST generator of $C_{1 \mid A, B, C, D}^{m}$

According to the discussion in section 9.1 and in particular (9.6), the traceless components of $C_{i \mid A_{1}, \ldots, A_{r+3}}^{m_{1} \ldots m_{r}}$ are BRST exact. However, it is difficult to extract the BRST generators, so we will explicitly carry out the analysis for vectors $C_{i \mid A, B, C, D}^{m}$.

When contracting (9.5) with momenta $k_{r}^{m} k_{r}^{n}$ of any particle $r=i$ or $r \in A, B, C, D$, the on-shell condition $k_{r}^{2}$ decouples the anomalous term $\sim \delta^{m n}$, and we obtain the BRST generator for the corresponding momentum contraction,

$$
\begin{equation*}
Q\left[\frac{k_{r}^{m} k_{r}^{n} D_{i \mid A, B, C, D}^{m n}}{2\left(k_{r} \cdot k_{i A B C D}\right)}\right]=k_{r}^{m} C_{i \mid A, B, C, D}^{m} . \tag{E.1}
\end{equation*}
$$

Plugging this back into the $k_{i}^{n}$ contraction of (9.5) with its trace subtracted:

$$
\begin{gather*}
C_{i \mid A, B, C, D}^{m}=Q \frac{1}{\left(k_{i} \cdot k_{i A B C D}\right)}\left[k_{i}^{p} D_{i \mid A, B, C, D}^{p m}-\frac{1}{10} k_{i}^{m} \delta_{n p} D_{i \mid A, B, C, D}^{n p}\right. \\
\left.-\frac{k_{i A B C D}^{m} k_{i}^{n} k_{i}^{p} D_{i \mid A, B, C, D}^{n p}}{2\left(k_{i} \cdot k_{i A B C D}\right)}+\frac{1}{10} k_{i}^{m} \sum_{r \in i, A, B, C, D} \frac{k_{r}^{n} k_{r}^{p} D_{i \mid A, B, C, D}^{n p}}{\left(k_{r} \cdot k_{i A B C D}\right)}\right] . \tag{E.2}
\end{gather*}
$$

Similar to (9.3), the right-hand side is ill-defined if momentum conservation $k_{i A B C D}^{m}=0$ is imposed, so the vector invariant $C_{i \mid A, B, C, D}^{m}$ is not BRST-exact in the momentum phase space of $1+|A|+|B|+|C|+|D|$ massless particles. The BRST generator for traceless tensors of rank $r \geq 2$ can be found by the same method.

## E.2. Seven-point momentum contractions of $C_{1 \mid A, B, C, D}^{m}$

The general formula (9.9) for $Q D_{i|A| B, C, D}$ specializes to the following BRST exact relations at seven-points:

$$
\begin{align*}
& Q D_{1|234| 5,6,7}= \Delta_{1 \mid 234,5,6,7}+k_{234}^{m} C_{1 \mid 234,5,6,7}^{m}-P_{1|2| 34,5,6,7}-P_{1|23| 4,5,6,7} \\
&+ P_{1|34| 2,5,6,7}+P_{1|4| 23,5,6,7}+\left[-s_{25} C_{1 \mid 5234,6,7}-s_{45} C_{1 \mid 5432,6,7}\right. \\
&\left.+s_{35}\left(C_{1 \mid 5324,6,7}+C_{1 \mid 5342,6,7}\right)+(5 \leftrightarrow 6,7)\right]  \tag{E.3}\\
& Q D_{1|5| 234,6,7}=\Delta_{1 \mid 234,5,6,7}+k_{5}^{m} C_{1 \mid 234,5,6,7}^{m}+s_{56} C_{1 \mid 234,56,7}+s_{57} C_{1 \mid 234,57,6} \\
&+ {\left[s_{25} C_{1 \mid 5234,6,7}-s_{35}\left(C_{1 \mid 5324,6,7}+C_{1 \mid 5342,6,7}\right)+s_{45} C_{1 \mid 5432,6,7}\right] }  \tag{E.4}\\
& Q D_{1|23| 45,6,7}=\Delta_{1 \mid 23,45,6,7}+k_{23}^{m} C_{1 \mid 23,45,6,7}^{m}-P_{1|2| 3,45,6,7}+P_{1|3| 2,45,6,7} \\
&+ {\left[s_{25} C_{1 \mid 3254,6,7}-s_{24} C_{1 \mid 3245,6,7}-s_{35} C_{1 \mid 2354,6,7}+s_{34} C_{1 \mid 2345,6,7}\right] } \\
&+ {\left[s_{36} C_{1 \mid 236,45,7}-s_{26} C_{1 \mid 326,45,7}+(6 \leftrightarrow 7)\right] }  \tag{E.5}\\
& Q D_{1|6| 23,45,7}= \Delta_{1 \mid 23,45,6,7}+k_{6}^{m} C_{1 \mid 23,45,6,7}^{m}+\left(s_{26} C_{1 \mid 326,45,7}-s_{36} C_{1 \mid 236,45,7}\right) \\
&+\left(s_{46} C_{1 \mid 546,23,7}-s_{56} C_{1 \mid 456,23,7}\right)+s_{67} C_{1 \mid 23,45,67} . \tag{E.6}
\end{align*}
$$

## Appendix F. Examples of the canonicalization procedure

This appendix gathers further applications of the canonicalization procedure in section 11. We suppress the BRST generators since they can be reconstructed from the right-hand side and do not contribute to amplitudes.

The canonicalization prescription (11.10) for scalar invariants implies that

$$
\begin{align*}
C_{2 \mid 1,34,56} & =C_{1 \mid 2,34,56}+C_{1 \mid 23,56,4}-C_{1 \mid 24,56,3}+C_{1 \mid 25,34,6}-C_{1 \mid 26,34,5} \\
& +C_{1 \mid 235,6,4}-C_{1 \mid 236,5,4}-C_{1 \mid 245,6,3}+C_{1 \mid 246,5,3}+C_{1 \mid 253,4,6} \\
& -C_{1 \mid 254,3,6}-C_{1 \mid 263,4,5}+C_{1 \mid 264,3,5}+Q(\ldots)  \tag{F.1}\\
C_{2 \mid 1345,6,7} & =C_{1 \mid 3452,6,7}+Q(\ldots)  \tag{F.2}\\
C_{2 \mid 1,3456,7} & =C_{1 \mid 2,3456,7}+C_{1 \mid 23,456,7}-C_{1 \mid 26,345,7}+C_{1 \mid 234,56,7}-C_{1 \mid 236,45,7}-C_{1 \mid 263,45,7} \\
& +C_{1 \mid 265,34,7}+C_{1 \mid 2345,6,7}-C_{1 \mid 2346,5,7}-C_{1 \mid 2364,5,7}+C_{1 \mid 2365,4,7}-C_{1 \mid 2634,5,7} \\
& +C_{1 \mid 2635,4,7}+C_{1 \mid 2653,4,7}-C_{1 \mid 2654,3,7}+Q(\ldots)  \tag{F.3}\\
C_{2 \mid 134,56,7} & =C_{1 \mid 342,56,7}+C_{1 \mid 3425,6,7}-C_{1 \mid 3426,5,7}+Q(\ldots)  \tag{F.4}\\
C_{2 \mid 13,456,7} & =C_{1 \mid 32,456,7}+C_{1 \mid 324,56,7}-C_{1 \mid 326,45,7}+C_{1 \mid 3245,6,7} \\
& -C_{1 \mid 3246,5,7}-C_{1 \mid 3264,5,7}+C_{1 \mid 3265,4,7}+Q(\ldots)  \tag{F.5}\\
C_{2 \mid 13,45,67} & =C_{1 \mid 32,45,67}+C_{1 \mid 324,67,5}-C_{1 \mid 325,67,4}+C_{1 \mid 326,45,7}-C_{1 \mid 327,45,6} \\
& +C_{1 \mid 3246,7,5}-C_{1 \mid 3247,6,5}-C_{1 \mid 3256,7,4}+C_{1 \mid 3257,6,4}+C_{1 \mid 3264,5,7} \\
& -C_{1 \mid 3265,4,7}-C_{1 \mid 3274,5,6}+C_{1 \mid 3275,4,6}+Q(\ldots) . \tag{F.6}
\end{align*}
$$

Except for the more laborious $C_{2 \mid 345,67,1}$, (F.1) to (F.6) and the opening examples of section 11.1 cover all canonicalizations of scalars $C_{2 \mid A, B, C}$ up to multiplicity seven.

Next, we apply the canonicalization rule (11.17) to vectors and tensors:

$$
\begin{align*}
C_{2 \mid 13,4,5,6}^{m} & =C_{1 \mid 32,4,5,6}^{m}+\left[k_{4}^{m} C_{1 \mid 324,5,6}+(4 \leftrightarrow 5,6)\right]+Q(\ldots)  \tag{F.7}\\
C_{2 \mid 1,34,5,6}^{m} & =C_{1 \mid 2,34,5,6}^{m}+C_{1 \mid 23,4,5,6}^{m}-C_{1 \mid 24,3,5,6}^{m}+k_{4}^{m} C_{1 \mid 234,5,6}-k_{3}^{m} C_{1 \mid 243,5,6}+Q(\ldots) \\
& +\left[k_{5}^{m}\left(C_{1 \mid 25,34,6}+C_{1 \mid 235,4,6}+C_{1 \mid 253,4,6}-C_{1 \mid 245,3,6}-C_{1 \mid 254,3,6}\right)+(5 \leftrightarrow 6)\right] \\
C_{2 \mid 134,5,6,7}^{m} & =C_{1 \mid 342,5,6,7}^{m}+\left[k_{5}^{m} C_{1 \mid 3425,6,7}+(5 \leftrightarrow 6,7)\right]+Q(\ldots) \\
C_{2 \mid 13,45,6,7}^{m} & =C_{1 \mid 32,45,6,7}^{m}+C_{1 \mid 324,5,6,7}^{m}-C_{1 \mid 325,4,6,7}^{m}+k_{5}^{m} C_{1 \mid 3245,6,7}-k_{4}^{m} C_{1 \mid 3254,6,7}+Q(\ldots) \\
& +\left[k_{6}^{m}\left(C_{1 \mid 326,45,7}+C_{1 \mid 3264,5,7}+C_{1 \mid 3246,5,7}-C_{1 \mid 3265,4,7}-C_{1 \mid 3256,4,7}\right)+(6 \leftrightarrow 7)\right] \\
C_{2 \mid 13,4,5,6,7}^{m n} & =C_{1 \mid 32,4,5,6,7}^{m n}+\delta^{m n} \mathcal{Y}_{132,4,5,6,7}+2\left[k_{4}^{(m} C_{1 \mid 324,5,6,7}^{n)}+(4 \leftrightarrow 5,6,7)\right]
\end{align*}
$$

$$
\begin{aligned}
& +2\left[k_{4}^{(m} k_{5}^{n)}\left(C_{1 \mid 3245,6,7}+C_{1 \mid 3254,6,7}\right)+(4,5 \mid 4,5,6,7)\right]+Q(\ldots) \\
C_{2 \mid 1,34,5,6,7}^{m n}= & C_{1 \mid 2,34,5,6,7}^{m n}+C_{1 \mid 23,4,5,6,7}^{m n}-C_{1 \mid 24,3,5,6,7}^{m n}+\delta^{m n}\left(\mathcal{Y}_{12,34,5,6,7}+\mathcal{Y}_{123,4,5,6,7}-\mathcal{Y}_{124,3,5,6,7}\right) \\
+ & 2\left[k_{5}^{(m}\left(C_{1 \mid 25,34,6,7}^{n)}+C_{1 \mid 235,4,6,7}^{n)}+C_{1 \mid 253,4,6,7}^{n)}-C_{1 \mid 245,3,6,7}^{n)}-C_{1 \mid 254,3,6,7}^{n)}\right)+(5 \leftrightarrow 6,7)\right] \\
& +2 k_{4}^{(m} C_{1 \mid 234,5,6,7}^{n)}-2 k_{3}^{(m} C_{1 \mid 243,5,6,7}^{n)}+2\left[k _ { 5 } ^ { ( m } \left\{k_{4}^{n)}\left(C_{1 \mid 2345,6,7}+C_{1 \mid 2354,6,7}+C_{1 \mid 2534,6,7}\right)\right.\right. \\
& \left.-k_{3}^{n)}\left(C_{1 \mid 2435,6,7}+C_{1 \mid 2453,6,7}+C_{1 \mid 2543,6,7}\right)+(5 \leftrightarrow 6,7)\right] \\
+ & 2\left[k _ { 5 } ^ { ( m } k _ { 6 } ^ { n ) } \left\{C_{1 \mid 256,34,7}+C_{1 \mid 265,34,7}+\left(C_{1 \mid 2356,4,7}+\operatorname{symm}(3,5,6)\right)\right.\right. \\
& \left.\left.-\left(C_{1 \mid 2456,3,7}+\operatorname{symm}(3,5,6)\right)\right\}+(5,6 \mid 5,6,7)\right]+Q(\ldots) .
\end{aligned}
$$

The more laborious vectors $C_{2 \mid 1,345,6,7}^{m}$ and $C_{2 \mid 1,34,56,7}^{m}$ at multiplicity seven are omitted.
Finally, the following pseudoinvariants are canonicalized using (11.20):

$$
\begin{align*}
P_{2|3| 14,5,6,7}= & P_{1|3| 42,5,6,7}+\mathcal{Y}_{142,3,5,6,7}+k_{3}^{m} C_{1 \mid 423,5,6,7}^{m} \\
+ & {\left[s_{35} C_{1 \mid 4235,6,7}+(5 \leftrightarrow 6,7)\right]+Q(\ldots) }  \tag{F.8}\\
P_{2|34| 1,5,6,7}= & P_{1|34| 2,5,6,7}+P_{1|4| 23,5,6,7}-P_{1|3| 24,5,6,7}+\mathcal{Y}_{12,34,5,6,7}+\mathcal{Y}_{123,4,5,6,7} \\
- & \mathcal{Y}_{124,3,5,6,7}+\left[s_{45} C_{1 \mid 2345,6,7}-s_{35} C_{1 \mid 2435,6,7}+(5 \leftrightarrow 6,7)\right] \\
+ & k_{4}^{m} C_{1 \mid 234,5,6,7}^{m}-k_{3}^{m} C_{1 \mid 243,5,6,7}^{m}+Q(\ldots)  \tag{F.9}\\
P_{2|3| 1,45,6,7}= & P_{1|3| 2,45,6,7}+P_{1|3| 24,5,6,7}-P_{1|3| 25,4,6,7}+\mathcal{Y}_{12,3,45,6,7}+\mathcal{Y}_{124,3,5,6,7} \\
- & \mathcal{Y}_{125,3,4,6,7}+\left[s_{35}\left(C_{1 \mid 2345,6,7}+C_{1 \mid 2435,6,7}\right)-(4 \leftrightarrow 5)\right] \\
+ & k_{3}^{m}\left(C_{1 \mid 23,45,6,7}^{m}+C_{1 \mid 234,5,6,7}^{m}-C_{1 \mid 235,4,6,7}^{m}+C_{1 \mid 243,5,6,7}^{m}-C_{1 \mid 253,4,6,7}^{m}\right) \\
+ & {\left[s _ { 3 6 } \left(C_{1 \mid 236,45,7}+C_{1 \mid 2346,5,7}+C_{1 \mid 2364,5,7}+C_{1 \mid 2436,5,7}\right.\right.} \\
& \left.\left.-C_{1 \mid 2356,4,7}-C_{1 \mid 2365,4,7}-C_{1 \mid 2536,4,7}\right)+(6 \leftrightarrow 7)\right]+Q(\ldots)  \tag{F.10}\\
P_{2|3| 1,4,5,6,7}^{m}= & P_{1|3| 2,4,5,6,7}^{m}+\mathcal{Y}_{12,3,4,5,6,7}^{m}+\left[k_{4}^{m}\left(P_{1|3| 24,5,6,7}+\mathcal{Y}_{124,3,5,6,7}\right)+(4 \leftrightarrow 5,6,7)\right] \\
+ & {\left[s_{34} C_{1 \mid 234,5,6,7}^{m}+k_{4}^{m} k_{3}^{p}\left(C_{1 \mid 234,5,6,7}^{p}+C_{1 \mid 243,5,6,7}^{p}\right)+(4 \leftrightarrow 5,6,7)\right] } \\
+ & k_{3}^{p} C_{1 \mid 23,4,5,6,7}^{p m}+k_{3}^{m} \mathcal{Y}_{123,4,5,6,7}+\left[s _ { 3 4 } \left\{k_{5}^{m}\left(C_{1 \mid 2345,6,7}+C_{1 \mid 2354,6,7}+C_{1 \mid 2534,6,7}\right)\right.\right. \\
& +k_{6}^{m}\left(C_{1 \mid 2346,5,7}+C_{1 \mid 2364,5,7}+C_{1 \mid 2634,5,7}\right) \\
& \left.\left.+k_{7}^{m}\left(C_{1 \mid 2347,5,6}+C_{1 \mid 2374,5,6}+C_{1 \mid 2734,5,6}\right)\right\}+(4 \leftrightarrow 5,6,7)\right]+Q(\ldots) .(\mathrm{F} .11) \tag{F.11}
\end{align*}
$$

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[^1]:    11 For example, $M_{132} \neq-M_{123}$ implies that $M_{1} \otimes M_{32} \neq-M_{1} \otimes M_{23}$ even though $M_{32}=-M_{23}$.

[^2]:    ${ }^{13}$ In various places of this section, we will encounter the local representatives $V_{A}, T_{A, B, C}$, $W_{A, B, C, D}^{m}, T_{A, B, C, D}^{m}$ and $Y_{A, B, C, D, E}$ of the more frequently-used Berends-Giele superfields $M_{A}$, $M_{A, B, C}, W_{A, B, C, D}^{m}, M_{A, B, C, D}^{m}$ and $\mathcal{Y}_{A, B, C, D, E}$ as defined by (2.20), (2.26), (2.28), (2.30) and (3.3). They follow by trading any $\mathcal{K}_{B} \in\left\{\mathcal{A}_{\alpha}^{B}, \mathcal{A}_{B}^{m}, \mathcal{W}_{B}^{\alpha}, \mathcal{F}_{B}^{m n}\right\}$ in these definitions for the standard BRST blocks $K_{B} \in\left\{A_{\alpha}^{B}, A_{B}^{m}, W_{B}^{\alpha}, F_{B}^{m n}\right\}$, see section 2.3. Some of their BRST variations are displayed in appendix C .

