Canonical structure of the E_{10} model and supersymmetry

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A coset model based on the hyperbolic Kac-Moody algebra E_{10} has been conjectured to underlie 11-dimensional supergravity and M theory. In this note we study the canonical structure of the bosonic model for finite- and infinite-dimensional groups. In the case of finite-dimensional groups like $GL(n)$ we exhibit a convenient set of variables with Borel-type canonical brackets. The generalization to the Kac-Moody case requires a proper treatment of the imaginary roots that remains elusive. As a second result, we show that the supersymmetry constraint of $D = 11$ supergravity can be rewritten in a suggestive way using E_{10} algebra data. Combined with the canonical structure, this rewriting explains the previously observed association of the canonical constraints with null roots of E_{10}. We also exhibit a basic incompatibility between local supersymmetry and the $K(E_{10})$ "R symmetry" that can be traced back to the presence of imaginary roots and to the unfaithfulness of the spinor representations occurring in the present formulation of the E_{10} worldline model, and that may require a novel type of bosonization/fermionization for its resolution. This appears to be a key challenge for future progress with E_{10}.

I. INTRODUCTION

The conjectured E_{10} symmetry of the M theory completion of $D = 11$ supergravity [1] applies to both the bosonic and the fermionic sectors. The one-dimensional spinning E_{10} model constructed and analyzed in [2–5] has manifest symmetry under the hyperbolic Kac-Moody group E_{10}, and its dynamics have been shown to match exactly the $D = 11$ dynamics at the nonlinear level, when both are suitably truncated. However, it has so far proved impossible to remove the truncation of this correspondence, one central obstacle being a dichotomy between the bosonic and fermionic variables on the E_{10} side. Whereas the bosonic variables are described in terms of *infinitely* many coordinates of the infinite-dimensional coset space $E_{10}/K(E_{10})$, the fermionic variables are described by *finitely* many components of a finite-dimensional (unfaithful) spinor representation of $K(E_{10})$ [2]. This dichotomy is also reflected in the fact that the one-dimensional E_{10} model cannot be fully supersymmetric on its worldline, since in its presently known form it pairs an infinite number of bosonic with a finite number of fermionic degrees of freedom.

In view of the fact that a detailed understanding of supersymmetry has often been central in advances regarding the structure of hidden symmetries, we initiate in this note a more detailed study of the worldline supersymmetry in the E_{10} context. Though we will not be able to present a new supersymmetric E_{10} model, our results bring the obstacles in the current formulation to the front, and we hope they can serve as a first step to resolving the issues both in the physics and the mathematics associated with constructing a model that fully accommodates both supersymmetry and $K(E_{10})$ symmetry. In fact, progress toward solving the outstanding problems may well require some novel kind of bosonization/fermionization, and thus also involve quantization in a crucial way. This is not only because the distinction between bosons and fermions becomes fluid in low dimensions and thus also in the (one-dimensional) worldline model, but also because the very meaning of what is a space-time boson and what is a space-time fermion, and hence also the ultimate relevance of *space-time* supersymmetry (as opposed to worldline supersymmetry), must be questioned in the context of emergent space-time scenarios. The present results can be viewed as a first step in this direction; in particular we identify the proper canonical variables on the bosonic side that couple naturally to the fermions, and hence will be an essential ingredient in approaching quantization of the worldline model. We note that in the context of string theory the emergence of space-time fermions from bosonic fields was already suggested long ago in [6], and the relation of this construction to Kac-Moody algebras was discussed more recently in [7]. In the context of maximal supergravity in two dimensions [where $K(E_{10})$ is replaced by $K(E_9)$], it was already pointed out in [8] that the associated linear system effectively constitutes a bosonization of the supergravity fermions, especially in view of previous work in [9,10].

Our main tool is the detailed analysis of the canonical structure of one-dimensional coset models, starting with purely bosonic systems based on a coset G/K. We will exhibit explicitly a set of variables that makes the algebraic structure completely manifest, and we propose that these

variables are therefore also an appropriate starting point for quantum considerations extending the reduced quantum cosmological billiards of [11] that should eventually lead to an implementation of the Wheeler-DeWitt equation for the full theory. For the case $E_{10}/K(E_{10})$ our arguments remain somewhat formal since an explicit parametrization of the group E_{10} similar to the one used in the proof for finite-dimensional G is not available. Denoting the velocity type variables as P_α^r, where $\alpha > 0$ is a positive root of the E_{10} Borel algebra and r labels an orthonormal basis of elements in the root space associated with the root α (this extra label is only required for imaginary roots), we will in particular argue, and *prove* for finite-dimensional G, that the canonical commutation relations of the P_α^r are exactly those of the E_{10} algebra itself.

The bosonic expressions have to be completed by fermionic ones, and in Sec. III we then look at $D = 11$ supergravity [12]. A rewriting of the supersymmetry constraint, inspired by recent studies in quantum super-symmetric cosmology in relation to Kac-Moody symmetries [13,14], suggests a very simple underlying algebraic formulation. We will here restrict attention to terms *linear in the fermions*, as the consideration of higher order fermionic terms does not affect our main conclusions.[1] With every root α of E_{10} one can associate an element $\tilde{\Gamma}(\alpha)$ of the $SO(10)$ Clifford algebra and a polarization of the fermionic field $\phi(\alpha)$. In [3] the super-symmetry constraint was analyzed *to linear order* in fermions and shown to take the schematic form

$$\mathcal{S} = P \odot \Psi, \tag{1.1}$$

where P stands for the infinite component coset velocity of the E_{10} coset space model, and Ψ for the *finite-dimensional* unfaithful spinor representation. The symbol $\odot$ is short-hand for the particular combination of the fermions and the bosonic coset velocities identified from the canonical supersymmetry constraint in [3]. In this paper, we will show how the above expression can (again schematically) be transformed into a sum

$$\mathcal{S} = \cdots + \sum_\alpha P_\alpha \tilde{\Gamma}(\alpha)\phi(\alpha) + \cdots. \tag{1.2}$$

One main goal of this paper will be to explore the validity, and more specifically the limit of validity, of this expression, and thereby attach a more concrete representation theoretic meaning to the symbol $\odot$. Indeed, already from the form of (1.2) one may anticipate problems when trying to combine supersymmetry with the "R symmetry" $K(E_{10})$:

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[1]By contrast, the supersymmetric Bianchi-type model recently (and impressively) analyzed in [13,14] does retain all higher order fermionic terms and is thus fully supersymmetric also at the quantum level, but with only partial manifestations of $K(AE_3)$ symmetry.

supersymmetry requires an *equal* number of bosons and fermions, whereas in (1.2) an infinite number of bosonic degrees of freedom is to be paired with a finite number of fermionic degrees of freedom. To be sure, in the actual expression obtained from supergravity the above sum contains only finitely many bosonic contributions, as a result of "cutting off" the sum over roots α at level $\ell = 3$. Therefore the supersymmetry constraint $\mathcal{S}$ cannot, in its presently known form, be assigned to any known representation of $K(E_{10})$, even though separately, both P and ϕ do transform properly [although it is not known whether P transforms in an *irreducible* representation of $K(E_{10})$]. The novel techniques introduced in this paper will allow us to analyze in considerable detail the terms by which the supersymmetry constraint fails to transform properly, and to highlight the differences between the finite-dimensional and infinite-dimensional cases. Our analysis thus identifies the terms that have to be dealt with differently in the construction of a supersymmetric E_{10} model, and we offer more comments in the concluding section. There we also explain that the failure to transform covariantly under $K(E_{10})$ cannot be cured by higher order fermionic terms.

While the exact $D = 11$ supersymmetry constraint can be transformed into a *truncated* expression of the type above, we thus encounter obstacles when trying to remove the truncation and to explore what the dots in the above formula could stand for. The expression above does provide a sensible object for $GL(n, \mathbb{R})$ and other finite-dimensional groups in the sense that it transforms covariantly as spinor as the supersymmetry should, but a similar result is no longer true for E_{10}. From a more physical perspective, the mismatch between bosons and fermions in the latter case is also reflected in the fact that no fermionic analog of the gradient representations has been found so far, thus (so far, at least) precluding an expansion for the fermions à la Belinski-Khalatnikov-Lifshitz (BKL).

II. CANONICAL STRUCTURE OF BOSONIC WORLDLINE COSET MODELS

In this section, we study the canonical structure of a coset model describing the motion of a point particle on a symmetric space G/K, with G a split real simple Lie group and $K \equiv K(G)$ its maximal compact subgroup. To set the basic notations and conventions, we first discuss the case of finite-dimensional G where everything is well defined, and subsequently write down the corresponding expressions for Kac-Moody algebras and groups. In the latter case, of course, many expressions will remain formal. For previous work on the canonical structure of nonlinear σ models, see for example [15].

A. Setup in the finite-dimensional case

To begin with, we restrict attention to finite-dimensional and simply laced Lie group G. Then the Lie algebra

$\mathfrak{g} = \mathrm{Lie}(G)$ is finite dimensional and has a system of roots $\alpha \in \Delta = \Delta_- \cup \Delta_+$. The positive roots will also be written as $\alpha > 0$, and we designate the Cartan subalgebra by $\mathfrak{h}$. We assume a Cartan-Weyl basis with basis vectors H_a and E_α, where $a = 1, …, \dim \mathfrak{h}$. The commutation relations are [16]

$$[E_\alpha, E_\beta] = \begin{cases} c_{\alpha,\beta} E_{\alpha+\beta} & \text{if } \alpha + \beta \in \Delta, \\ \alpha^a H_a & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \quad (2.1a)$$

$$[H_a, E_\alpha] = \alpha_a E_\alpha, \quad (2.1b)$$

with $c_{\alpha,\beta} = \pm 1$ or $= 0$ for simply laced finite-dimensional Lie algebras. There is a nondegenerate invariant bilinear form on $\mathfrak{g}$ that satisfies[2]

$$\langle E_\alpha | E_\beta \rangle = \begin{cases} 1 & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \quad (2.2a)$$

$$\langle H_a | H_b \rangle = G_{ab} \quad (2.2b)$$

the metric G_{ab} is positive definite for any simple finite-dimensional Lie algebra $\mathfrak{g}$ but need not be positive definite for *nonsimple* $\mathfrak{g}$. The inverse G^{ab} of G_{ab} has been used to raise the index in (2.1a) according to $\alpha^a = G^{ab}\alpha_b$ and $\alpha_a = \alpha(H_a)$. The compact subalgebra $K(\mathfrak{g}) \equiv \mathfrak{k} \subset \mathfrak{g}$ is generated by $k_\alpha \equiv E_\alpha - E_{-\alpha}$ with $\alpha > 0$ and will be discussed in more detail in Sec. III B. The structure constants $c_{\alpha,\beta}$ are antisymmetric and satisfy standard identities [16], in particular

$$c_{\alpha,\beta} = -c_{\beta,\alpha} = -c_{-\alpha,-\beta}, \qquad c_{\alpha+\beta,-\beta} = c_{\alpha,\beta}. \quad (2.3)$$

The coset $G/K(G) \equiv G/K$ can be parametrized in a Borel gauge fixed form according to the Iwasawa decomposition $G = KAN$. For finite-dimensional G any element of the coset G/K can thus be written in the form[3]

$$V(q^a, A_\alpha) = \exp(q^a H_a) \exp\left(\sum_{\alpha>0} A_\alpha E_\alpha\right). \quad (2.4)$$

The worldline model describing the motion of a point particle on the coset manifold G/K is then parametrized by a map $V: \mathbb{R} \to G/K$, where $t \in \mathbb{R}$ is the time coordinate. The Cartan derivative is (with $\partial \equiv d/dt$)

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[2]For finite-dimensional simple $\mathfrak{g}$ this is just the appropriately normalized matrix trace.

[3]Here, we deviate from the standard notation in the cosmological billiards literature (see e.g. [17]) where the diagonal degrees of freedom are denoted by $-\beta^a \equiv +q^a$. This is done in order to avoid confusion with the labeling of the components of the root β below.

$$\partial V V^{-1} = P + Q = \partial q^a H_a + \sum_{\alpha>0} e^{q^a \alpha_a} DA_\alpha E_\alpha, \quad (2.5)$$

where $Q \in \mathfrak{k}, P \in \mathfrak{k}^\perp$, and, schematically and due to $\partial e^X e^{-X} = \partial X + \frac{1}{2}[X, \partial X] + \cdots$,

$$DA_\alpha = \partial A_\alpha + \frac{1}{2} \sum_{\substack{\beta>0 \\ \alpha-\beta>0}} c_{\alpha-\beta,\beta} A_{\alpha-\beta} \partial A_\beta + \cdots. \quad (2.6)$$

Importantly, the Borel gauge implies a triangular expansion of DA_α where the factors contributing to the terms quadratic in A_γ on the right-hand side (r.h.s.) are of *lower* height, whence the sum on the r.h.s. of (2.6) has only finitely many terms even for infinite-dimensional Kac-Moody algebras (a crucial fact for the calculability of the model). The invariant Lagrangian is given by

$$L = \frac{1}{2}\langle P|P \rangle = \frac{1}{2}\partial q^a G_{ab} \partial q^b + \sum_{\alpha>0} P_\alpha P_\alpha, \quad (2.7)$$

where we have defined

$$P = \partial q^a H_a + \sum_\alpha P_\alpha(E_\alpha + E_{-\alpha}), \qquad P_\alpha = \frac{1}{2}e^{q^a \alpha_a} DA_\alpha. \quad (2.8)$$

The compact part is then given by

$$Q = \sum_{\alpha>0} Q_\alpha(E_\alpha - E_{-\alpha}) = \sum_{\alpha>0} P_\alpha(E_\alpha - E_{-\alpha}), \quad (2.9)$$

where the equality $Q_\alpha = P_\alpha$ is a consequence of the triangular gauge choice.

The model has global G symmetry and local K symmetry that we use to fix the triangular gauge (2.4) everywhere. The symmetries act by

$$V(t) \to k(t)V(t)g^{-1} \Rightarrow P \to kPk^{-1},$$
$$Q \to kQk^{-1} + \partial kk^{-1}. \quad (2.10)$$

When the triangular gauge (2.4) is fixed, a local compensating K transformation is required to restore the gauge for every G transformation that throws V out of the triangular gauge.

The equations of motion of the coset model are

$$DP \equiv \partial P - [Q, P] = 0. \quad (2.11)$$

(We note that this of course implies that the equations of the original coordinates q^a and A_α are second order differential equations.) For a given root component P_α this means

$$\partial P_\alpha = -\partial q^{\mathrm{a}}\alpha_{\mathrm{a}}P_\alpha + 2\sum_{\beta>0}c_{\alpha+\beta,-\beta}P_\beta P_{\alpha+\beta}$$

$$\equiv -\partial q^{\mathrm{a}}\alpha_{\mathrm{a}}P_\alpha + 2\sum_{\beta>0}c_{\alpha,\beta}P_\beta P_{\alpha+\beta}. \qquad (2.12)$$

Note that [in contrast to (2.6)] the sum on the r.h.s. contains terms of *ascending* height.

B. Changes in the Kac-Moody case

When the Lie algebra $\mathfrak{g}$ is an infinite-dimensional Kac-Moody algebra [18], the definition of the corresponding group G requires more care; see for example [19,20]. Again we restrict to simply laced algebras, and more specifically to symmetric generalized Cartan matrices with at most one line linking any two nodes. Of course, our primary interest here will be with E_{10} and its maximal compact subgroup $K(E_{10})$.

There are now two types of roots of the algebra, called real and imaginary, and they are distinguished by their Cartan-Killing norm: Real roots α satisfy $\alpha^2 = 2$ and imaginary roots $\alpha^2 \leq 0$. The generators corresponding to real roots are unique up to normalization and can be denoted by E_α as above, but the generators corresponding to imaginary roots can have nontrivial multiplicities and are more appropriately denoted by E_α^r, where $r = 1, \ldots, \mathrm{mult}(\alpha)$ labels an orthonormal basis (with respect to the Cartan-Killing metric) in the root space. We will write all generators in this way, keeping in mind that for real roots r can take only one value. The commutation relations in the Cartan-Weyl basis [cf. (2.1)] then have to account also for the multiplicities and become

$$[E_\alpha^r, E_\beta^s] = \begin{cases} \sum_{t=1}^{\mathrm{mult}(\alpha+\beta)} c_{\alpha,\beta}^{rst} E_{\alpha+\beta}^t & \text{if } \alpha+\beta \in \Delta, \\ \delta^{rs}\alpha^{\mathrm{a}}H_{\mathrm{a}} & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \qquad (2.13a)$$

$$[H_{\mathrm{a}}, E_\alpha^r] = \alpha_{\mathrm{a}}E_\alpha^r. \qquad (2.13b)$$

We note that we still have $c_{\alpha,\beta} = \pm 1$ if α, β and $\alpha+\beta$ are all real, but this need no longer be true when any of these roots is imaginary. The bilinear form (2.2) generalizes to

$$\langle E_\alpha^r | E_\beta^s \rangle = \begin{cases} \delta^{rs} & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \qquad (2.14a)$$

$$\langle H_{\mathrm{a}} | H_{\mathrm{b}} \rangle = G_{\mathrm{ab}}, \qquad (2.14b)$$

where the metric G_{ab} is now indefinite (and Lorentzian for hyperbolic Kac-Moody algebras).

The other important modification concerns the parametrization of the elements of the formal coset space G/K that, using the Iwasawa decomposition, could be given in the finite-dimensional case as in (2.4). Even at a purely

formal level, and even if the sum in the exponent is truncated to a finite number of terms, it is not directly meaningful to parametrize a given element of the Kac-Moody group in the form

$$V(q^{\mathrm{a}}, A_\alpha^r) \text{ `` = ''} \exp(q^{\mathrm{a}}H_{\mathrm{a}}) \exp\left(\sum_{\alpha>0}\sum_{r=1}^{\mathrm{mult}(\alpha)} A_\alpha^r E_\alpha^r\right), \qquad (2.15)$$

where $\{q^{\mathrm{a}}, A_\alpha^r\}$ are local coordinates on the (infinite-dimensional) coset manifold, one coordinate for each Lie algebra element E_α^r. The reason is that the step operators E_α^r associated with imaginary roots are not (locally) nilpotent in any standard representation, and therefore the exponential is *a priori* ill defined.[4] For this reason, standard approaches to Kac-Moody groups involve writing down only exponentials of *real* root generators (that are nilpotent) and then defining the Kac-Moody group as the group generated by the products of these real root exponentials [19]. Although such a treatment is mathematically well defined, it does not solve by any means the problem of finding a manageable realization of the Kac-Moody group, because different orderings of exponentials of a given set of real root generators will yield new group elements. Organizing these differently ordered exponentials is thus directly associated with the (unsolved) problem of classifying the independent elements of the associated root space (where the problem is to count and classify the inequivalent ways in which a given set of Chevalley generators can be "distributed" over a multicommutator). In particular, a parametrization in terms of fields associated only with real roots of E_{10}, besides being incomplete, would also obscure the relation to the fields $\{A_\alpha^r\}$, and therefore does not appear to lead to a convenient parametrization of the coordinates on the coset space G/K.[5] (Let us emphasize that the Iwasawa decomposition $G = KAN$ is nevertheless still applicable [21].)

Irrespective of an explicit description of the coordinates on G/K we can still generalize the worldline σ model to infinite-dimensional cosets and consider the Cartan form

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[4]The notion of local nilpotency is defined as follows: an operator E_α^r is locally nilpotent in a representation V of $\mathfrak{g}$ if for all $x \in V$ there exists an $n_0 = n_0(x)$ such that

$$(E_\alpha^r)^n(x) = 0 \qquad \text{for all } n > n_0.$$

(In the adjoint representation, the action is simply by commutators: $E_\alpha^r(x) = \mathrm{ad}E_\alpha^r(x) \equiv [E_\alpha^r, x]$.) Clearly this holds for any real root α in the adjoint or any standard representation, but is no longer true for imaginary roots. The intuitive picture for this statement in the adjoint representation is that all roots lie in a solid hyperboloid $\{\alpha^2 \leq 2\}$ in a Lorentzian space. Imaginary roots point *into* the light-cone where infinitely many roots of $\mathfrak{g}$ lie whereas real roots point outside the light-cone and eventually will leave the solid hyperboloid.

[5]But let us note that the highest weights associated with the *gradient representations* of [1] are, in fact, real roots.

$$\partial VV^{-1} = P + Q = \partial q^{\mathrm{a}} H_{\mathrm{a}} + \sum_{\alpha>0} \sum_{r=1}^{\mathrm{mult}(\alpha)} P_{\alpha}^{r} E_{\alpha}^{r} \quad (2.16)$$

without spelling out the explicit parametrization of V and P_{α}^{r} in terms of coordinates and their time derivatives. The triangular structure on N implies, however, that the P_{α}^{r} are all *finite* combinations of coordinates and their derivatives, as we explained after (2.6).

The coset equations (2.11) take the same form if the Lagrangian is the formal extension of (2.7) to the infinite-dimensional Kac-Moody algebra, using the invariant bilinear form (2.14) on the Kac-Moody algebra. Therefore (2.12) becomes

$$\partial P_{\alpha}^{t} = -\partial q^{\mathrm{a}} \alpha_{\mathrm{a}} P_{\alpha}^{t} + 2 \sum_{\beta>0} \sum_{r,s} c_{\alpha+\beta,-\beta}^{rst} P_{\beta}^{r} P_{\alpha+\beta}^{s}. \quad (2.17)$$

As we noted after (2.12) the r.h.s. is a sum over terms of *ascending* height, and hence an infinite sum for infinite-dimensional $\mathfrak{g}$. This sum can be rendered finite and calculable only by consistently truncating the P_{α}^{r} to vanish beyond a given height, as is necessary for the comparison between supergravity and the E_{10} σ model. More concretely, this can be done for example by choosing a grading on the root lattice and cutting off P_{α}^{r} after a certain degree [22].

C. Canonical treatment

We now analyze the canonical structure, by again considering the *finite-dimensional* coset model (2.7) first. The canonical momenta from (2.7) are

$$\pi_{\mathrm{a}} = \frac{\partial L}{\partial \partial q^{\mathrm{a}}} = G_{\mathrm{ab}} \partial q^{\mathrm{b}},$$

$$\Pi_{\alpha} = \frac{\partial L}{\partial \partial A_{\alpha}}$$

$$= \frac{1}{2} \left(e^{2q^{\mathrm{a}}\alpha_{\mathrm{a}}} DA_{\alpha} + \frac{1}{2} \sum_{\beta>0} c_{\beta,\alpha} e^{2q^{\mathrm{a}}(\alpha+\beta)_{\mathrm{a}}} A_{\beta} DA_{\alpha+\beta} + \cdots \right),$$

$$(2.18)$$

displaying again a triangular structure. This can be inverted to write the P_{α} in triangular form in terms of canonical coordinates and momenta,

$$P_{\alpha} = e^{-q^{\mathrm{a}}\alpha_{\mathrm{a}}} \left(\Pi_{\alpha} - \frac{1}{2} \sum_{\beta>0} c_{\beta,\alpha} A_{\beta} \Pi_{\alpha+\beta} + \cdots \right). \quad (2.19)$$

From this and the standard relations

$$\{q^{\mathrm{a}}, \pi_{\mathrm{b}}\} = \delta_{\mathrm{b}}^{\mathrm{a}}, \qquad \{A_{\alpha}, \Pi_{\beta}\} = \delta_{\alpha,\beta}, \quad (2.20)$$

one can derive the canonical brackets among the π^{a} and P_{α},

$$\{\pi_{\mathrm{a}}, \pi_{\mathrm{b}}\} = 0, \quad (2.21\mathrm{a})$$

$$\{\pi_{\mathrm{a}}, P_{\alpha}\} = \alpha_{\mathrm{a}} P_{\alpha}, \quad (2.21\mathrm{b})$$

$$\{P_{\alpha}, P_{\beta}\} = c_{\alpha,\beta} P_{\alpha+\beta}. \quad (2.21\mathrm{c})$$

Only the first two of these relations are evident, while the third one is not and will be proven below. Of course, to the order given one can check the last relation easily from the expressions above, but the important point is that all the higher nonlinear terms combine in the right way to produce such a simple result. Our main point here is that the "composite" variables P_{α} are "good" canonical variables because the canonical brackets between them assume a very simple form, and furthermore display a graded structure which is nothing but the Borel subalgebra. Equally important, the P_{α}, being objects associated with the maximal compact subgroup K, couple naturally to the fermions. Let us note the relations

$$\{\pi_{\mathrm{a}}, V\} = -H_{\mathrm{a}} V, \qquad \{P_{\alpha}, V\} = -E_{\alpha} V,$$
$$\{P_{\alpha}, q^{\mathrm{a}}\} = 0, \qquad \{V, V\} = 0. \quad (2.22)$$

For any coset space σ model the canonical conserved Noether current (or more properly, conserved *charge*) is given by general formula

$$J = V^{-1} PV \equiv J^{\mathrm{a}} H_{\mathrm{a}} + \sum_{\alpha>0} (J_{-\alpha} E_{\alpha} + J_{\alpha} E_{-\alpha}) \quad (2.23)$$

such that $\partial J = 0$ by the equations of motion (2.12). Although the canonical commutation relations for this current reproduce the $GL(n)$ algebra (see below), we will see that the structure of its components is considerably more complicated, not least because J has both upper *and* lower triangular pieces. For finite-dimensional $\mathfrak{g}$ the lower triangular half of the matrix J takes a relatively simple form when one expresses the associated conserved components in terms of the momenta Π_{α}. By contrast the components of the upper triangular half involve all canonical variables and become increasingly more complicated with growing n; see also [17]. We will illustrate this explicitly with the example of the $GL(3)/SO(3)$ model in Appendix B.

The conserved current J of (2.23) generates the global G transformations in (2.10) and, since we are working in fixed triangular gauge, the "lower triangular" G transformations induce a compensating K transformation. That is, we expect the infinitesimal transformation of V to be

$$\{J, V\} = \delta V = -V\delta g + \delta k V, \quad (2.24)$$

where δg and δk are the infinitesimal versions of the group transformations in (2.10) and δk is determined by δg and V such that the resulting δV is in triangular gauge.

This can be worked out in terms of the basis components in (2.23) and the canonical brackets (2.21), with the result (for $\alpha > 0$)

$$\{J_\alpha, V\} = -VE_\alpha, \tag{2.25a}$$

$$\{J_\mathrm{a}, V\} = -VH_\mathrm{a}, \tag{2.25b}$$

$$\{J_{-\alpha}, V\} = -VE_{-\alpha} - \sum_{\beta>0}\langle VE_{-\alpha}V^{-1}|E_\beta\rangle(E_\beta - E_{-\beta})V, \tag{2.25c}$$

where the extra term on the r.h.s. in the last line corresponds to the compensating transformation in $K(G)$ required to bring V back into triangular gauge, so that

$$\langle V^{-1}\{J_{-\alpha}, V\}|E_\beta\rangle = 0 \tag{2.26}$$

and the explicit compensating element in $\mathfrak{k}$ is

$$\delta k_\alpha = -\sum_{\beta>0}\langle VE_{-\alpha}V^{-1}|E_\beta\rangle(E_\beta - E_{-\beta}). \tag{2.27}$$

In deriving the above brackets we made repeated use of the invariance of the invariant bilinear form (trace) and the orthonormality relations (2.2).[6] Relation (2.10) then implies directly the transformation of the velocity components P under J. For $\alpha > 0$ and $\beta > 0$ one has

$$\{J_\mathrm{a}, P_\beta\} = 0, \tag{2.28a}$$

$$\{J_\alpha, P_\beta\} = 0, \tag{2.28b}$$

$$\begin{aligned}\{J_{-\alpha}, P_\beta\} &= \pi_\mathrm{a}\beta^\mathrm{a}\langle VE_{-\alpha}V^{-1}|E_\beta\rangle \\ &\quad - \sum_{\gamma>0}c_{\beta-\gamma,\gamma}P_\gamma\langle VE_{-\alpha}V^{-1}|E_{\beta-\gamma}\rangle \\ &\quad - \sum_{\gamma>0}c_{\beta+\gamma,-\gamma}P_\gamma\langle VE_{-\alpha}V^{-1}|E_{\beta+\gamma}\rangle. \end{aligned} \tag{2.28c}$$

Because (2.25) equivalently expresses the standard non-linear realization of global symmetries in nonlinear σ models we can immediately infer the closure relations

$$\{J_\alpha, J_\beta\} = \begin{cases} c_{\alpha,\beta}J_{\alpha+\beta} & \text{if } \alpha+\beta \in \Delta, \\ \alpha^\mathrm{a}J_\mathrm{a} & \text{if } \alpha = -\beta, \\ 0 & \text{otherwise,} \end{cases} \tag{2.29a}$$

$$\{J_\mathrm{a}, J_\alpha\} = \alpha_\mathrm{a}J_\alpha, \tag{2.29b}$$

in the nonlinear realization of the G symmetry acting on V. An explicit verification of the above relations for the $GL(3)/SO(3)$ model can be found in Appendix B.

An important aspect of the variables P_α concerns quantization. When quantizing a nonlinear model there is always the question for which canonical variables one should perform the replacement of Poisson or Dirac brackets by quantum commutators (which in quantum field theory may yield *inequivalent* quantizations). Obviously, the variables P_α are ideally suited for this purpose; in particular, such a quantisation prescription eliminates all operator ordering ambiguities. Furthermore, we emphasise once again that the P_α are the natural variables coupling to fermions, as will be seen in more detail below.

D. Canonical structure for $GL(n,\mathbb{R})/SO(n)$

We now prove (2.21) and in particular the crucial third relation, for $G = GL(n, \mathbb{R})$. This is a slight generalization of the setup of the preceding sections since $GL(n, \mathbb{R})$ is not simple, but it is the case of direct interest for cosmological billiards [17].

Let us fix some notation. We denote the generators of $GL(n, \mathbb{R})$ by $K^a{}_b$ with $a, b = 1, \ldots, n$ and commutation relations

$$[K^a{}_b, K^c{}_d] = \delta^c_b K^a{}_d - \delta^a_d K^c{}_b. \tag{2.30}$$

The symmetric and antisymmetric combinations are defined as $S^{ab} = K^a{}_b + K^b{}_a$ and $J^{ab} = K^a{}_b - K^b{}_a$. The positive roots of $GL(n, \mathbb{R})$ are denoted by α_{ab} with $a < b$ and will be written as tuples $\alpha_{ab} = (0\cdots010\cdots0-10\cdots)$ with $(+1)$ in the ath and (-1) in the bth place. The generator corresponding to such a positive α_{ab} is then $E_{\alpha_{ab}} = K^a{}_b$ and the above commutation relations translate into (recall that $a < b$ and $c < d$)

$$[E_{\alpha_{ab}}, E_{\alpha_{cd}}] = c_{\alpha_{ab},\alpha_{cd}}E_{\alpha_{ab}+\alpha_{cd}} = \begin{cases} E_{\alpha_{ad}} & \text{if } b = c \\ -E_{\alpha_{cb}} & \text{if } a = d, \\ 0 & \text{otherwise.} \end{cases} \tag{2.31}$$

So we read off the general formula $c_{\alpha_{ab},\alpha_{cd}} = \delta_{bc} - \delta_{ad}$.

We write the coset element of $GL(n, \mathbb{R})/SO(n)$ in Borel gauge by an upper triangular $(n \times n)$ matrix through (as for notation; cf. footnote 1)

$$V = AN \quad \text{with} \quad A = \mathrm{diag}(e^{q^1}, \ldots, e^{q^n}), \qquad N = N^a{}_i. \tag{2.32}$$

Here, a is the local (row) index, and i a global (column) index. The matrix $N^a{}_i$ is equal to 1 on the diagonal and has

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[6]The invariance relation $\langle V^{-1}AV|B\rangle = \langle A|VBV^{-1}\rangle$ for $A, B \in \mathfrak{g}$ and $V \in G$ are also valid for infinite-dimensional $\mathfrak{g}$. In the finite-dimensional case, the relation can be viewed as the standard cyclic property of the matrix trace.

vanishing entries for $a > i$. The inverse matrix N^{-1} is also upper triangular and has components $N^i{}_a$ which vanish for $i > a$. The Borel gauge implies that some of the summations below are restricted in the way indicated. For the coset velocities one finds

$$(\partial V V^{-1})^a{}_b = \partial q^a \delta^a_b + \sum_i e^{q^a - q^b} \partial N^a{}_i N^i{}_b. \quad (2.33)$$

For a positive root α_{ab}, i.e. $a < b$, we define the quantity

$$P_{\alpha_{ab}} = \frac{1}{2} e^{q^a - q^b} \sum_{a < i \leq b} \partial N^a{}_i N^i{}_b. \quad (2.34)$$

This is just the variable P_α introduced in the previous section, except that we are now labeling the roots by indices a, b. In a convenient normalization the Lagrangian can be written as

$$L = \frac{1}{2} \mathrm{Tr}(P^2) - \frac{1}{2} (\mathrm{Tr}P)^2$$
$$= \frac{1}{2} \partial q^\mathrm{a} \partial q^\mathrm{b} G_{\mathrm{ab}} + \sum_{a < b} P_{\alpha_{ab}} P_{\alpha_{ab}}, \quad (2.35)$$

where G_{ab} is the DeWitt metric

$$\sum_{\mathrm{a,b}} \partial q^\mathrm{a} \partial q^\mathrm{b} G_{\mathrm{ab}} = \sum_\mathrm{a} (\partial q^\mathrm{a})^2 - \left(\sum_\mathrm{a} \partial q^\mathrm{a} \right)^2. \quad (2.36)$$

The canonical momenta conjugate to $N^a{}_i$ are

$$\Pi^i{}_a = \frac{\partial L}{\partial \partial N^a{}_i} = \sum_b e^{q^a - q^b} P_{\alpha_{ab}} N^i{}_b \quad (2.37)$$

and vanish for $a \geq i$. In other words, for $a < b$,

$$P_{\alpha_{ab}} = \sum_i e^{q^b - q^a} N^a{}_i \Pi^i{}_b. \quad (2.38)$$

The advantage of using the variables P_α is that they obey very simple commutation relations, *to wit*,

$$\{P_{\alpha_{ab}}, P_{\alpha_{cd}}\} = c_{\alpha_{ab}, \alpha_{cd}} P_{\alpha_{ab} + \alpha_{cd}} \quad (2.39)$$

whenever $\alpha_{ab} + \alpha_{cd}$ is a root, and we recall $c_{\alpha_{ab}, \alpha_{cd}} = \delta_{bc} - \delta_{ad}$. This can be verified by straightforward computation using $\{N^a{}_i, \Pi^j{}_b\} = \delta^a_b \delta^j_i$. It is equally easy to see that if $\alpha_{ab} + \alpha_{cd}$ is not a root, the canonical bracket vanishes, as does the structure constant. We thus have demonstrated the relation (2.21c) for $GL(n, \mathbb{R})$. The other relations (2.21a) and (2.21b) are also evident for $GL(n, \mathbb{R})$, given the explicit form of the Lagrangian (2.35) and (2.38).

The proof of (2.21c) can be extended to all other simple finite-dimensional Lie algebras, either by direct

computation, or more simply by looking at differently embedded $GL(n)$ subalgebras, and by observing that these commutation relations must be compatible with the action of the Weyl group, because all roots can be reached by Weyl transformations from the simple roots.

As a simple example we discuss the case of $GL(3)/SO(3)$ in Appendix B.

The extension of the above results to *infinite-dimensional* Kac-Moody algebras, and more specifically to the E$_{10}$ algebra, is more subtle, and here we do not have a complete picture. In particular, we do not have a proof that (2.21b) and (2.21c) remain valid for all roots. For instance, in the presence of imaginary roots (2.21c) would have to generalize to

$$\{P^r_\alpha, P^s_\beta\} = \sum_{t=1}^{\mathrm{mult}(\alpha)} c^{rst}_{\alpha,\beta} P^t_{\alpha+\beta}, \quad (2.40)$$

where α and β are any roots, and where the sum on t ranges over the multiplicity of the root $(\alpha + \beta)$ if this is an imaginary root. While a general derivation by the above methods seems beyond reach, we can extend the argument at least to those roots α and β for which $\alpha + \beta$ is also a real root, because the above commutation relation should respect the Weyl group, and because all real roots can be reached by E$_{10}$ Weyl transformations. Hence at least for this special case, the above relation should also hold for E$_{10}$.

E. Hamiltonian analysis

The canonical Hamiltonian is

$$H = \pi_\mathrm{a} \partial q^\mathrm{a} + \sum_{\alpha > 0} \Pi_\alpha \partial A_\alpha - L$$
$$= \frac{1}{2} \pi_\mathrm{a} G^{\mathrm{ab}} \pi_\mathrm{b} + \sum_{\alpha > 0} e^{-2q^\mathrm{a} \alpha_\mathrm{a}} \Pi_\alpha^2 + \cdots, \quad (2.41)$$

where the dots denote important nonlinear terms. In terms of the coset velocities they can be summarized as (see also [15] for a derivation in the finite-dimensional case)

$$H = \frac{1}{2} \pi_\mathrm{a} G^{\mathrm{ab}} \pi_\mathrm{b} + \sum_{\alpha > 0} P_\alpha^2. \quad (2.42)$$

Again, it is important that the nonlinear terms combine in the right way to yield such a simple expression. We note that (2.23) implies that we can rewrite the Hamiltonian alternatively as

$$H = \frac{1}{2} \langle J | J \rangle, \quad (2.43)$$

which we recognize as the standard bilinear form on the corresponding Lie algebra.

Let us verify the consistency expression (2.42) with the coset equations of motion

$$\partial P_\alpha = \{P_\alpha, H\} = -\alpha_a G^{ab} \pi_b P_\alpha + 2 \sum_{\beta>0} c_{\alpha,\beta} P_\beta P_{\alpha+\beta}.$$

$$(2.44)$$

Comparing the above relation with the general result (2.12) shows agreement. This shows that the Borel structure is correct, at least for all finite-dimensional algebras: any other algebra would not correctly reproduce the equations of motion.

Staying at the formal level, an analogous argument also works for the infinite-dimensional case. Namely, we can similarly deduce a statement of the canonical brackets of the P_α^r in the Kac-Moody case. Starting from the same Lagrangian

$$L = \frac{1}{2}\langle P|P \rangle \qquad (2.45)$$

as in the finite-dimensional case, but where $\langle \cdot | \cdot \rangle$ is now the standard invariant bilinear form on the Kac-Moody algebra, the Hamiltonian is given by the straightforward formal extension of (2.42), using the arguments of [15],

$$H = \frac{1}{2}\pi_a G^{ab} \pi_b + \sum_{\alpha>0}\sum_r P_\alpha^r P_\alpha^r. \qquad (2.46)$$

Because, formally, the conserved Noether current is still given by

$$J = V^{-1}PV \equiv J^a H_a + \sum_{\alpha>0}\sum_{r=1}^{\text{mult}(\alpha)} (J_{-\alpha}^r E_\alpha^r + J_\alpha^r E_{-\alpha}^r),$$

$$(2.47)$$

the Hamiltonian can again be cast into the form (2.43) with the bilinear form on the Kac-Moody algebra. When considered as a function of the phase space variables $\{J^a, J_{\pm\alpha}^r\}$ this is just the (unique) E_{10} invariant bilinear form. Let us mention, however, that in contrast to the finite-dimensional case a simple form of the lower triangular half can only be achieved by truncating the current components to $J_{-\alpha}^r = 0$ for α's exceeding a given height. A related discussion can be found in [17].

Compatibility of the canonical structure with the equations of motion (2.17) is then ensured by the canonical brackets

$$\{P_\alpha^r, P_\beta^s\} = \sum_t c_{\alpha,\beta}^{rst} P_{\alpha+\beta}^t. \qquad (2.48)$$

We thus see that *if* the Hamiltonian is given by the restriction of the E_{10} Casimir operator to the coset $E_{10}/K(E_{10})$, the compatibility of the canonical structure with the equations of motion *implies* the extension of the Borel-like structure found in (2.21) to the full Borel subalgebra of E_{10}. However, it is known that beyond level $\ell = 3$ the canonical supergravity Hamiltonian starts to

deviate from the Casimir operator, and therefore we will also have to eventually allow for modifications in the canonical algebra (2.48).

III. FERMIONS AND SUPERSYMMETRY

The extension of the E_{10} coset model to include fermions was discussed in [2–4]. We briefly review the salient features of the resulting model and its relation to maximal $D = 11$ supergravity in order to provide a self-contained presentation.

A. E_{10}, its level decomposition and the bosonic sector

The description of E_{10} that is most commonly used in connection with $D = 11$ supergravity is that where the Lie algebra is presented in $GL(10)$ level decomposition [1]. In this presentation, the infinitely many generators of E_{10} are organized into $\mathfrak{gl}(10,\mathbb{R})$ tensor representations and graded by a level ℓ such that each level only contains finitely many $\mathfrak{gl}(10,\mathbb{R})$ representations. The Lie bracket is compatible with the level. At low non-negative levels one finds the following $\mathfrak{gl}(10,\mathbb{R})$ representations corresponding to the (spatial) components of the $D = 11$ fields and their magnetic duals:

Level ℓ	Generator	Representation of $\mathfrak{gl}(10,\mathbb{R})$
0	$K^a{}_b$	**100** (adjoint; graviton)
1	$E^{abc} = E^{[abc]}$	**120** (three form)
2	$E^{a_1\cdots a_6} = E^{[a_1\cdots a_6]}$	**210** (six form)
3	$E^{a_0|a_1\cdots a_8} = E^{a_0|[a_1\cdots a_8]}$ with $E^{[a_0|a_1\cdots a_8]} = 0$	**440** [(8,1) hook; dual graviton]

The "coset velocity" P of (2.5) can be similarly decomposed by level

$$P = \sum_{\alpha>0}\sum_{r=1}^{\text{mult}(\alpha)} P_\alpha^r (E_\alpha^r + E_{-\alpha}^r) \equiv \sum_{\ell \geq 0} P^{(\ell)} * E^{(\ell)}$$

$$\equiv \frac{1}{2} P_{ab}^{(0)} S^{ab} + \frac{1}{3!} P_{abc}^{(1)} S^{abc} + \frac{1}{6!} P_{abcdef}^{(2)} S^{abcdef}$$

$$+ \frac{1}{9!} P_{a_0|a_1\cdots a_8}^{(3)} S^{a_0|a_1\cdots a_8} + \cdots. \qquad (3.1)$$

Here, the $P^{(\ell)}$ transform in the representation from the table branched to $SO(10)$ level [since P transforms covariantly under the "compact" subgroup $K(E_{10})$]. The generators are defined by

$$S^{ab} = K^a{}_b + K^b{}_a,$$
$$S^{abc} = E^{abc} + F_{abc},$$
$$S^{a_1\cdots a_6} = E^{a_1\cdots a_6} + F_{a_1\cdots a_6},$$
$$S^{a_0|a_1\cdots a_8} = E^{a_0|a_1\cdots a_8} + F_{a_0|a_1\cdots a_8}, \qquad (3.2)$$

where F_{abc}, etc., are the Chevalley transposed generators on the negative levels and correspond to the $E^r_{-\alpha}$ part in the general expression.

As was shown in [1,2], the bosonic coset model with Lagrangian $L = \frac{1}{2}\langle P|P \rangle$, when restricted to levels $\ell \leq 3$, is equivalent to $D = 11$ supergravity expanded about a fixed spatial point, $\mathbf{x}_0$ with the bosonic dictionary[7]

$$P^{(0)}_{ab}(t) = -N\omega_{ab0}(t, \mathbf{x}_0), \qquad P^{(1)}_{abc}(t) = NF_{0abc}(t, \mathbf{x}_0), \tag{3.3}$$

$$P^{(2)}_{a_1 \cdots a_6}(t) = \frac{1}{4!}N\epsilon_{a_1 \cdots a_6 b_1 \cdots b_4}F^{b_1 \cdots b_4}(t, \mathbf{x}_0), \qquad P^{(3)}_{a_0|a_1 \cdots a_8}(t) = \frac{1}{2}N\epsilon_{a_1 \cdots a_8 b_1 b_2}\omega_{b_1 b_2 a_0}(t, \mathbf{x}_0), \tag{3.4}$$

when all higher order spatial gradients are neglected and the $SO(10)$ connection is traceless $\omega_{bba} = 0$ (corresponding to the irreducibility condition of the $\ell = 3$ generator in the table) and t is the coordinate along the worldline that is identified with the physical time coordinate. The index 0 is a flat index in the time direction and N is the lapse function in the Arnowitt-Deser-Misner (ADM) gauge with zero shift. With the said truncations it can then be shown that the bosonic equations of motion of $D = 11$ supergravity coincide with those of the worldline E$_{10}$ sigma model.

In order to reexpress these $SO(10)$ objects in terms of E$_{10}$ root data and P^r_α we need to explain how the roots at the various levels are related to the components. As in Sec. II, we work in the so-called "wall basis" [13,23]. This means that we write a root α as $\alpha = \sum_a \alpha_a e^a$ where e^a are the basis of the $\mathfrak{h}^*$ dual to the Cartan generators H_a such that $e^a(H_b) = \delta^a_b$ and hence $\alpha(H_a) = \alpha_a$. In the wall basis, the inner product is given by

$$\langle e^a | e^b \rangle = G^{ab} = \delta^{ab} - \frac{1}{9} \tag{3.5}$$

and agrees with the (inverse) DeWitt metric for diagonal metrics. In order to avoid confusion with the labeling of the simple roots it will sometimes be convenient to also use the notation $p_a \equiv \alpha_a$ interchangeably for the component of α in the wall basis and to also write $\alpha = (p_1, \ldots, p_{10})$ as a row vector. The ten simple roots of E$_{10}$ are explicitly given by

$$\alpha_1 = (1, -1, 0, 0, \ldots, 0),$$

$$\alpha_2 = (0, 1, -1, 0, \ldots, 0),$$

$$\vdots$$

$$\alpha_9 = (0, \ldots, 0, 0, 1, -1),$$

$$\alpha_{10} = (0, 0, \ldots, 0, 1, 1, 1). \tag{3.6}$$

The $\mathfrak{gl}(10)$ level of an arbitrary root α expanded on the simple roots as $\alpha = \sum_{j=1}^{10} m^j \alpha_j$ is $\ell \equiv \ell(\alpha) = m_{10}$.

Roots on level $\ell = 0$ are roots of $\mathfrak{gl}(10)$ and can be written as α_{ab} as above in Sec. II D. The components P_α for these roots are identified with $P^{(0)}_{ab}$, and we let $a < b$ for positive roots as before. The components of the Cartan subalgebra are identified via $\pi^a = P^{(0)}_{aa}$ (no sum). Roots on level $\ell = 1$ have three entries 1 in the wall basis and the other entries p_a are zero, as for example in α_{10} above. Calling the three nonvanishing components a, b, and c with $a < b < c$, we identify P_α with $P^{(1)}_{abc}$. Roots $\alpha_{a_1 \cdots a_6}$ on level $\ell = 2$ have six entries $p_a = 1$ and four vanishing entries. We assume $a_1 < \cdots < a_6$ and then identify $P_{\alpha_{a_1 \cdots a_6}}$ with the corresponding level $\ell = 2$ coset velocity. Roots on level $\ell = 3$ come in two varieties: They either have one entry $p_a = 2$, seven entries $p_a = 1$, and two $p_a = 0$ or they have nine $p_a = 1$ and one $p_a = 0$. In the first case, we let a_0 be the component with entry $p_{a_0} = 0$ and assume again that the $p_a = 1$ components are ordered as $a_1 < \cdots < a_7$. Then we identify P_α with $P^{(3)}_{a_0|a_0 a_1 \cdots a_7}$. The second case corresponds to null roots (of multiplicity 8), and schematically we distribute the ordered nine $p_a = 1$ components as $P_{a_0|a_1 \cdots a_8}$. The multiplicity requires extra care and will be discussed in detail in Sec. III D.

In summary, we find that we can associate

$$P_{\alpha_{ab}} = P^{(0)}_{ab}, \qquad P_{\alpha_{abc}} = P^{(1)}_{abc}, \qquad P_{\alpha_{a_1 \cdots a_6}} = P^{(2)}_{a_1 \cdots a_6},$$

$$P_{\alpha_{a_0|a_1 \cdots a_8}} = P^{(3)}_{a_0|a_1 \cdots a_8}. \tag{3.7}$$

with root labels on the left-hand side (l.h.s.), and with the associated $SO(10)$ tensors on the r.h.s. Up to $\ell \leq 3$, this correspondence rule allows us to rewrite any expression involving $P^{(\ell)}$ in terms of P_α.

B. Unfaithful spinor representations of $K(E_{10})$

Fermions are associated with the compact subalgebra $K(E_{10})$ of E$_{10}$. This algebra is generated by the compact combinations ($\alpha > 0$)

$$k^r_\alpha = E^r_\alpha - E^r_{-\alpha}. \tag{3.8}$$

We have chosen the letter k^r_α for the $K(E_{10})$ generators, rather than using $J(\alpha)^r$ as in [24] in order to avoid confusion with the components of the conserved current J in (2.23). From (2.13) the $K(E_{10})$ elements satisfy

$$[\mathrm{k}_\alpha^r, \mathrm{k}_\beta^s] = \sum_{t=1}^{\mathrm{mult}(\alpha+\beta)} c_{\alpha,\beta}^{rst} \mathrm{k}_{\alpha+\beta}^t - \sum_{t=1}^{\mathrm{mult}(\alpha-\beta)} c_{\alpha,-\beta}^{rst} \mathrm{k}_{\alpha-\beta}^t. \quad (3.9)$$

In order to make sense of the above relation in general, and because $\alpha - \beta$ can be < 0 for $\alpha, \beta > 0$, one also requires a definition of k_α^r for $\alpha < 0$; from (3.8) we directly get

$$\mathrm{k}_\alpha^r := -\mathrm{k}_{-\alpha}^r \quad \text{for } \alpha < 0, \quad (3.10)$$

which is also consistent with (2.3).

$K(\mathrm{E}_{10})$ admits unfaithful finite-dimensional spinor representations [2–5,25], but unfortunately no faithful spinor representations are known up to now. The unfaithful representations relevant to supergravity involve the vector-spinor (gravitino) and Dirac-spinor (supersymmetry parameter). The representations can be represented conveniently using the wall basis [13,23], and we use the same formalism as in [24]. For the Dirac representation it is enough to restrict attention to real roots $\alpha, \beta, \ldots$, and we will thus drop the multiplicity labels in the remainder of this section. Then with every element v of the E_{10} root lattice $v = \sum n^j \alpha_j \equiv \sum_a v_a e^a$ (which need not be a root for arbitrary $n^j \in \mathbb{Z}$) we associate an element of the $SO(10)$ Clifford algebra through

$$\Gamma(v) = (\Gamma_1)^{v_1} \cdots (\Gamma_{10})^{v_{10}}, \quad (3.11)$$

where, of course, $\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}$ are the usual $SO(10)$ Γ matrices. The product of two such matrices is given by

$$\Gamma(u)\Gamma(v) = \varepsilon_{u,v}\Gamma(u \pm v), \quad (3.12)$$

where we have defined the cocycle

$$\varepsilon_{u,v} = (-1)^{\sum_{a<b} v_a u_b}, \quad (3.13)$$

which obeys

$$\varepsilon_{u,v}\varepsilon_{v,u} = (-1)^{u \cdot v}, \qquad \varepsilon_{u,v}\varepsilon_{u+v,w} = \varepsilon_{u,v+w}\varepsilon_{v,w}, \quad (3.14)$$

where $v \cdot w \equiv G^{ab} v_a w_b$. The cocycle $\varepsilon_{u,v}$ is defined only up to a coboundary; that is, we can modify the above definition (3.11) by

$$\Gamma(v) \to \tilde{\Gamma}(v) = \sigma_v \Gamma(v) \quad (3.15)$$

with $\sigma_v = \pm 1$ an (in principle) arbitrary sign factor; then

$$\tilde{\varepsilon}_{u,v} = \sigma_u \sigma_v \sigma_{u+v} \varepsilon_{u,v} \quad (3.16)$$

also obeys the cocycle relations (3.14). Next we specialize to elements $v = \alpha, \beta \in \Delta$ which *are* roots and choose the coboundary such that

$$\tilde{\Gamma}(\alpha) := \sigma_\alpha \Gamma(\alpha) := \begin{cases} \Gamma(\alpha) & \text{if } \alpha > 0, \\ -\Gamma(\alpha) \equiv -\Gamma(-\alpha) & \text{if } \alpha < 0; \end{cases} \quad (3.17)$$

that is, $\sigma_\alpha = \pm 1$ according to whether α is positive or negative, whence $\sigma_\alpha \sigma_{-\alpha} = -1$. The sign switch between positive and negative roots in (3.17) is necessary to remain consistent with (3.10). This definition can be extended to the whole root lattice by choosing $\sigma_v = \pm 1$ arbitrarily for nonroots v, but subject to the condition $\sigma_v \sigma_{-v} = -1$ (for $v \neq 0$). Indeed, in the relevant expressions in the supersymmetry constraint the matrix $\tilde{\Gamma}(\alpha)$ always comes with a factor P_α^r which vanishes when α is not a root. The extra sign in (3.17) leads to an important modification in the multiplication rule (3.12), viz.

$$\tilde{\Gamma}(\alpha)\tilde{\Gamma}(\beta) = \tilde{\varepsilon}_{\alpha,\beta}\tilde{\Gamma}(\alpha+\beta) = -\tilde{\varepsilon}_{\alpha,-\beta}\tilde{\Gamma}(\alpha-\beta). \quad (3.18)$$

With these definitions one can check that the map

$$\mathrm{k}_\alpha \mapsto \frac{1}{2}\tilde{\Gamma}(\alpha) \quad (3.19)$$

for all real α provides a representation of $K(\mathrm{E}_{10})$, when extended consistently by commutators. For consistency of this representation with (3.10) the sign in (3.17) is crucial. This representation has a large kernel; for instance, for null roots δ one has $\mathrm{k}_\delta^r \mapsto 0$, and for timelike imaginary roots all elements of the corresponding root space either vanish or are represented by the *same* element of the Clifford algebra. The quotient algebra of $K(\mathrm{E}_{10})$ by the kernel is isomorphic to $\mathfrak{so}(32)$ [2]. We will refer to this representation of $K(\mathrm{E}_{10})$ as the Dirac-spinor representation, or just "Dirac representation." (This type of representation can be straightforwardly generalized to other simply laced Kac-Moody algebras and also to arbitrary Kac-Moody algebras [25].)

Because (3.19) works for all real roots, the comparison of (3.9) with (3.18) shows that, for real roots α and β,

$$c_{\alpha,\beta} = -\tilde{\varepsilon}_{\alpha,\beta} \quad (3.20)$$

whenever $\alpha + \beta$ or $\alpha - \beta$ is also a real root. This is furthermore consistent with the fact that for real α and β only one of the terms on the r.h.s. of (3.9) can be nonzero. The minus sign in the above relation arises because below we will act on the components of the spinor rather than on the basis vectors.

While the Dirac representation corresponds to the supersymmetry transformation parameter, the vector spinor representation derives from the $D = 11$ gravitino and is "less unfaithful" than the Dirac representation. It was first obtained in [2] in terms of an $SO(10)$ covariant vector spinor Ψ_A^a with an $SO(10)$ vector index $a = 1, \ldots, 10$ and spinor indices $A, B, \ldots = 1, \ldots, 32$. This vector spinor is directly related via a fermionic dictionary to the spatial components of the $D = 11$ gravitino ψ_a through [[2] Eq. (5.1)],

$$\Psi_a(t) = g^{1/4}\psi_a(t, \mathbf{x}_0), \qquad (3.21)$$

where $g = \det(g_{mn})$ is the determinant of the spatial part of the metric. For the time component of the gravitino (the Lagrange multiplier for the supersymmetry constraint), we adopt the gauge $\psi_0 = \Gamma_0\Gamma^a\psi_a$, as in [2].

For the present purposes it is, however, advantageous to switch to a different description of the vector spinor in terms of fermions ϕ^{a} which are related to the $SO(10)$ covariant vector spinor Ψ^a of [2] above by the following crucial redefinition [23]:

$$\phi^{\mathrm{a}} = \Gamma^a\Psi^a \qquad \text{(no sum on } a\text{).} \qquad (3.22)$$

This relation clearly breaks $SO(10)$ covariance, but has an important advantage: in this way the Lorentz group $SO(10)$ gets replaced by the $SO(1,9)$ symmetry acting on the space of diagonal scale factors $\{q^{\mathrm{a}}\}$, which is also the invariance group of the DeWitt metric G_{ab}. It is for this reason that we adopt a different font (a, b, …), as we already did in [24]; the latter indices are then covariant under the (Lorentzian) invariance group of the DeWitt metric G_{ab}. We will also use the notation

$$\phi(\alpha) \equiv \alpha_{\mathrm{a}}\phi^{\mathrm{a}}. \qquad (3.23)$$

Like the Dirac representation the vector-spinor representation, now modeled by spinors ϕ^{a}_A, is obviously finite dimensional (we will often suppress explicitly writing out the spinor indices). The vector spinor ϕ^{a}_A satisfies the canonical (Dirac) brackets [2,23],

$$\{\Psi^a_A, \Psi^b_B\} = \delta^{ab}\delta_{AB} - \frac{1}{9}(\Gamma^a\Gamma^b)_{AB} \Rightarrow \{\phi^{\mathrm{a}}_A, \phi^{\mathrm{b}}_B\} = G^{\mathrm{ab}}\delta_{AB}$$
$$(3.24)$$

[recall the definition of G^{ab} in (3.5)]. A canonical representation of $K(E_{10})$ is then obtained by defining for any real root α

$$\mathrm{k}_\alpha = X_{\mathrm{ab}}(\alpha)\phi^{\mathrm{a}}\tilde{\Gamma}(\alpha)\phi^{\mathrm{b}},$$
$$X_{\mathrm{ab}} \equiv X_{\mathrm{ab}}(\alpha) = -\frac{1}{2}\alpha_{\mathrm{a}}\alpha_{\mathrm{b}} + \frac{1}{4}G_{\mathrm{ab}}, \qquad (3.25)$$

and this construction yields an unfaithful representation of $K(E_{10})$ [24]. Note that we again have to employ the $\tilde{\Gamma}$ matrices from (3.17) in order to extend this definition to both positive and negative real roots. We also note that the unfaithful spinor representation can be used to deduce partial information about the unknown structure constants of $K(E_{10})$, and thus E_{10}.

With the bosonic dictionary (3.3) and the fermionic dictionary (3.21) one can now convert any supergravity expression into the E_{10} variables $P^{(\ell)}$ and Ψ_a. With the relations (3.7) and (3.22) we can then rewrite in the next step everything into P_α and ϕ^{a} variables. This is the procedure we now apply to the supersymmetry constraint of $D = 11$ supergravity.

C. Supersymmetry constraint

In terms of the original canonical variables of $D = 11$ supergravity [12], the canonical supersymmetry constraint is given by [[2] Eq. (3.12)]

$$\tilde{\mathcal{S}} = \Gamma^{ab}\left[\partial_a\psi_b + \frac{1}{4}\omega_{acd}\Gamma^{cd}\psi_b + \omega_{abc}\psi_c + \frac{1}{2}\omega_{ac0}\Gamma^c\Gamma^0\psi_b\right]$$
$$+ \frac{1}{4}F_{0abc}\Gamma^0\Gamma^{ab}\psi^c + \frac{1}{48}F_{abcd}\Gamma^{abcde}\psi_e, \qquad (3.26)$$

where ω_{ABC} are the components of the $D = 11$ spin connection and F_{ABCD} the components of the four form (with flat indices $A, B, \ldots = 0, 1, \ldots, 10$). Using the dictionaries (3.3) and (3.21) one can rewrite this expression in terms of E_{10} coset variables. The translation between the coset model and $D = 11$ supergravity furthermore involves neglecting spatial gradients $\partial_a\psi_b$ on the fermions, terms of the form $\partial_a g \propto \omega_{bba}$, and all spatial gradients of second or higher order on the bosonic fields. It was then shown in [[2] Eq. (5.14)] that the supersymmetry constraint can be reexpressed in terms of the coset quantities $P^{(\ell)}$ and in an $SO(10)$ covariant manner as

$$\mathcal{S} = (P^{(0)}_{ab}\Gamma^a - P^{(0)}_{cc}\Gamma_b)\Psi^b + \frac{1}{2}P^{(1)}_{abc}\Gamma^{ab}\Psi^c$$
$$+ \frac{1}{5!}P^{(2)}_{abcdef}\Gamma^{abcde}\Psi^f$$
$$+ \frac{1}{6!}\left(P^{(3)}_{a|ac_1\cdots c_7}\Gamma^{c_1\cdots c_6}\Psi^{c_7} - \frac{1}{28}P^{(3)}_{a|c_1\cdots c_8}\Gamma^{c_1\cdots c_8}\Psi^a\right).$$
$$(3.27)$$

Compared to [2], we have rescaled the supersymmetry constraint by an overall factor of 2, and we also recall the normalization changes that we explained in footnote 7.

The notation $\mathcal{S}$ in place of $\tilde{\mathcal{S}}$ of (3.26) indicates that we have rescaled $\tilde{\mathcal{S}}$ and multiplied it by Γ_0. In this $SO(10)$ covariant form, repeated indices are summed over and indices are raised and lowered with the Euclidean metric δ_{ab}. We will now rewrite this expression once more, in order to bring it into a form that conforms more closely with the new variables introduced in the foregoing section. A key fact here is that by so doing we will give up manifest spatial Lorentz covariance and trade it for the Lorentzian $SO(1,9)$ symmetry on the space of scale factors exhibited above. In other words, the simplest form of the constraint is attained by trading a space-time symmetry for a symmetry in (a truncated version of) DeWitt superspace.

To convert the expression (3.27) to the E_{10} covariant notation above, we change fermionic variables according to (3.22) and analyze the various terms. For the contributions from $\ell = 0, 1, 2$, and now writing out the sums, we find

$$\sum_a P_{aa}^{(0)} \Gamma^a \Psi^a - \sum_c P_{cc}^{(0)} \sum_a \Gamma_a \Psi^a = G_{\mathrm{ab}} \pi^{\mathrm{a}} \phi^{\mathrm{b}},$$

$$\sum_{a<b} P_{ab}^{(0)} \Gamma^a \Psi^b + \sum_{a>b} P_{ab}^{(0)} \Gamma^a \Psi^b = \sum_{a<b} P_{ab}^{(0)} \Gamma^{ab} (\phi^b - \phi^a),$$

$$\sum_{a,b,c} P_{abc}^{(1)} \Gamma^{ab} \Psi^c = 2 \sum_{a<b<c} P_{abc}^{(1)} \Gamma^{abc} (\phi^a + \phi^b + \phi^c),$$

$$\sum_{a,b,c,d,e,f} P_{abcdef}^{(2)} \Gamma^{abcde} \Psi^f = 5! \sum_{a<b<c<d<e<f} P_{abcdef}^{(2)} \Gamma^{abcdef} (\phi^a + \cdots + \phi^f), \tag{3.28}$$

where we identified $\pi^{\mathrm{a}} = P_{aa}^{(0)}$. We now see that the expressions on the r.h.s. are already in the desired form; for instance,

$$\sum_{a<b<c} P_{abc}^{(1)} \Gamma^{abc} (\phi^a + \phi^b + \phi^c) \equiv \sum_{\alpha_{abc}} P_{\alpha_{abc}} \Gamma(\alpha_{abc}) \phi(\alpha_{abc})$$

$$= \frac{1}{2} \sum_{\ell(\alpha)=\pm 1} P_\alpha \tilde{\Gamma}(\alpha) \phi(\alpha), \tag{3.29}$$

where the middle sum on the r.h.s. runs over all level $\ell = 1$ roots α_{abc} (which are positive), while the last sum includes positive and negative roots. The level $\ell = 0, 2$ contributions work in an analogous manner.

At level $\ell = 3$ we encounter not only real roots, but for the first time also null roots. To see this distinction one has to separately analyze those terms in $P_{a|c_1 \cdots c_8}^{(3)}$ for which the index a coincides with one of the c_i (yielding real roots), and those terms for which all indices are different, i.e. $a \notin \{c_1, \ldots, c_8\}$ (yielding null roots). In order to analyze these terms we thus have to split up the various sums. We start with

$$\sum_{a,c_1,\ldots,c_7} P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{c_1 \cdots c_6} \Psi^{c_7} \equiv \sum_{c_1,\ldots,c_7} \sum_{a \neq c_i} P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{[c_1 \cdots c_6} \Psi^{c_7]}$$

$$= 6! \sum_{c_1 < \cdots < c_7} \sum_{a \neq c_i} P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{c_1 \cdots c_7} (\phi^{c_1} + \cdots + \phi^{c_7}), \tag{3.30}$$

where the c_i have been ordered in the second expression. The other contribution to the supersymmetry constraint (3.27) becomes

$$-\frac{1}{28} \sum_a \sum_{c_1,\ldots,c_8} P_{a|c_1 \cdots c_8}^{(3)} \Gamma^{c_1 \cdots c_8} \Psi^a = -2 \times 6! \sum_a \sum_{c_1 < \cdots < c_8} P_{a|c_1 \cdots c_8}^{(3)} \Gamma^{c_1 \cdots c_8} \Psi^a$$

$$= 2 \times 6! \sum_{c_1 < \cdots < c_7} \sum_{a \neq c_i} P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{c_1 \cdots c_7} \phi^a - 2 \times 6! \sum_{c_1 < \cdots < c_8} \sum_{a \neq c_i} P_{a|c_1 \cdots c_8}^{(3)} \Gamma^{ac_1 \cdots c_8} \phi^a. \tag{3.31}$$

Combining the two parts one finds

$$\frac{1}{6!} \left(P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{c_1 \cdots c_6} \Psi^{c_7} - \frac{1}{28} P_{a|c_1 \cdots c_8}^{(3)} \Gamma^{c_1 \cdots c_8} \Psi^a \right)$$

$$= \sum_{c_1 < \cdots < c_7} \sum_{a \neq c_i} P_{a|ac_1 \cdots c_7}^{(3)} \Gamma^{c_1 \cdots c_7} (2\phi^a + \phi^{c_1} + \cdots + \phi^{c_7}) - 2 \sum_{c_1 < \cdots < c_8} \sum_{a \neq c_i} P_{a|c_1 \cdots c_8}^{(3)} \Gamma^{ac_1 \cdots c_8} \phi^a. \tag{3.32}$$

The first term is exactly the contribution from the 360 (gravitational) real roots on level $\ell = 3$, viz.

$$\alpha = (2 1 1 1 1 1 1 0 0) \quad \text{and} \quad \text{permutations}, \tag{3.33}$$

where the root shown is associated with the component $P_{1|12345678}^{(3)}$. The normalization is different from the one used previously since the level $\ell = 3$ generators were normalized to 9 rather than 1 in [2]; cf. also footnote 7. The second term is a sum over the (gravitational) null roots

$$\delta = (1111111110) \quad \text{and} \quad \text{permutations}, \qquad (3.34)$$

where the first root is now associated with the component $P^{(3)}_{1|23456789}$. Note that the constraint as written above is overcounting them since there are $\binom{10}{8} \times 2 = 90$ instead of the required 80. The reason is a new type of gauge invariance related to the irreducibility of the $\ell = 3$ representation and that will be discussed in more detail in Sec. III D.

Let us summarize: altogether, the rewriting of the supersymmetry constraint (3.27) so far has led to the following expression up to and including all roots of $\ell \leq 3$:

$$
\mathcal{S} = \pi \cdot \phi + \sum_{\substack{\alpha^2=2 \\ \ell=0,\alpha>0}} P_\alpha \tilde{\Gamma}(\alpha)\phi(\alpha) + \sum_{\substack{\alpha^2=2 \\ \ell=1}} P_\alpha \tilde{\Gamma}(\alpha)\phi(\alpha)
$$

$$
+ \sum_{\substack{\alpha^2=2 \\ \ell=2}} P_\alpha \tilde{\Gamma}(\alpha)\phi(\alpha) + \sum_{\substack{\alpha^2=2 \\ \ell=3}} P_\alpha \tilde{\Gamma}(\alpha)\phi(\alpha)
$$

$$
+ \sum_{\substack{\delta^2=0 \\ \ell=3}} \sum_{r=1}^{8} P^r_\delta \tilde{\Gamma}(\delta)\phi(\epsilon^r), \qquad (3.35)
$$

where we have replaced Γ by $\tilde{\Gamma}$ to underline that the sum can also be extended to run over negative roots as in (3.29). The "polarization vectors" ϵ^r appearing for the null roots will be discussed in detail in the next section.

D. Null roots and gauge equivalences

We now return to the counting issue mentioned after (3.34). The association of a particular index set $(a_1 c_1 \cdots c_8)$ with all indices different with a null root component $P^{(3)}_{a_1|c_1 \cdots c_8}$ is subject to the irreducibility constraint (Young symmetry)

$$P^{(3)}_{[a_1|c_1 \cdots c_8]} = 0. \qquad (3.36)$$

This provides one linear relation between *a priori* nine different ways of distributing the nine indices on the hook tableau, bringing down the number of independent components to eight, in agreement with the multiplicity of null roots in E$_{10}$. Let us discuss in more detail how this is implemented in the supersymmetry constraint.

Gauge fixed form: To see this in more detail let us pick the particular null root corresponding to the indices $\{a, c_1, \ldots, c_8\} = \{1, \ldots, 9\}$. This root has contributions proportional to $\Gamma(\delta)$ through (reordering some of the indices)

$$-2(P_{1|2\cdots9}\phi^1 + P_{2|3\cdots91}\phi^2 + \cdots + P_{9|1\cdots8}\phi^9). \qquad (3.37)$$

Here, one could now trade the first term for a combination of the other terms by virtue of (3.36). This leads to

$$-2(P_{2|3\cdots91}(\phi^2 - \phi^1) + \cdots + P_{9|1\cdots8}(\phi^9 - \phi^1)); \qquad (3.38)$$

that is, it can be written in the form

$$\sum_{r=1}^{8} P^r_\delta \Gamma(\delta)\phi(\epsilon^r) \qquad (3.39)$$

with

$$P^1_\delta = P_{2|3\ldots91}, \ldots, P^8_\delta = P_{9|1\cdots8} \qquad (3.40)$$

and polarization vectors

$$\epsilon^1 = (2 - 200000000), \ldots, \epsilon^8 = (20000000 - 20). \qquad (3.41)$$

These polarization vectors are orthogonal to δ (as required) and correspond to positive $\ell = 0$ roots associated with generators $K^1_{\ r+1}$ (or their negatives).

Gauge unfixed form: We can avoid choosing a particular set of polarization vectors by instead letting the "multiplicity sum" run over an enlarged set

$$\sum_{r=1}^{9} P^r_\delta \Gamma(\delta)\phi(\epsilon^r). \qquad (3.42)$$

Here, P^r_δ denote the nine index arrangements and ϵ^r are nine independent polarization vectors that are orthogonal to δ. Shifting $\epsilon^r \to \epsilon^r + \delta$ leads to

$$\sum_{r=1}^{9} P^r_\delta \Gamma(\delta)\phi(\epsilon^r + \delta) = \sum_{r=1}^{9} P^r_\delta \Gamma(\delta)\phi(\epsilon^r) + \sum_{r=1}^{9} P^r_\delta \Gamma(\delta)\phi(\delta)$$

$$= \sum_{r=1}^{9} P^r_\delta \Gamma(\delta)\phi(\epsilon^r), \qquad (3.43)$$

since $\sum_{r=1}^{9} P^r_\delta = 0$ by virtue of (3.36). Therefore, we have a gauge invariance in the expression that we could use to fix the gauge in the way we have done above. This gauge invariance no longer "lives" in ordinary space time, but rather in the DeWitt superspace of (diagonal) metrics.

IV. PROPERTIES OF SUPERSYMMETRY CONSTRAINT

Having rewritten the supersymmetry constraint in terms of $K(E_{10})$ variables we will now reinvestigate the canonical algebra of supersymmetry constraints and its $K(E_{10})$ covariance. As for the algebra we will recover the previously derived results according to which the canonical constraints of $D = 11$ supergravity in the appropriate truncation are all associated with null roots of E$_{10}$. As for the transformation properties of the superconstraint, we will exhibit its noncovariance under the full $K(E_{10})$—a clear indication that the present construction is incomplete.

A. Supersymmetry constraint algebra

The above calculations led to the following expression for the supersymmetry constraint:

$$\mathcal{S}_A = \pi_a \phi_A^a + \sum_{\substack{\alpha^2=2 \\ \ell \le 3, \alpha > 0}} P_\alpha (\Gamma(\alpha)\phi(\alpha))_A + \sum_{\substack{\delta^2=0 \\ \ell=3}} P_\delta^r (\Gamma(\delta)\phi(\epsilon^r))_A$$

(4.1)

[recall that $\phi(\alpha)_A \equiv \alpha_a \phi_A^a$]. As shown above, the terms written out in the above formula agree precisely with the supersymmetry constraint derived from supergravity by dropping terms containing spatial gradients as well as cubic terms in the fermions. In other words, apart from these omitted contributions, the full content of the supersymmetry constraint is captured by the $\ell \le 3$ sector of the E_{10} model with fermions. However, from this restriction it is already clear that this expression cannot be the whole story, and we will make this point more explicit in the following section by showing that, contrary to first expectations, $\mathcal{S}$ does not transform in the Dirac representation, nor in any other known representation of $K(E_{10})$.

Nevertheless, under the canonical brackets, the supersymmetry constraint in the above form should yield the Hamiltonian and all other supergravity constraints in the gradient truncation (and ignoring higher order fermionic terms). Schematically, we see that

$$\{\mathcal{S}_A, \mathcal{S}_B\} = 2\mathcal{H}\delta_{AB} + \sum_{\delta^2=0} \mathcal{C}(\delta)\Gamma_{AB}(\delta) + \cdots . \quad (4.2)$$

Here, we have introduced the calligraphic letter $\mathcal{H}$ for the "Hamiltonian" arising from the commutator of two supersymmetry constraints, to distinguish it notationally from the coset Hamiltonian H discussed in the previous sections, since it is not clear *a priori* whether the two agree. Indeed, we will explain below that they do differ.

The anticommutator (4.2) contains many terms, but let us first concentrate on the ones containing no fermions (the ones bilinear in the fermions would also receive contributions from cubic fermionic terms, which are not included in the above formula for $\mathcal{S}$). Here we use [for roots α and β that are real and hence have antisymmetric $\Gamma(\alpha)$ and $\Gamma(\beta)$]

$$\{\pi \cdot \phi_A, \pi \cdot \phi_B\} = G_{ab}\pi^a\pi^b\delta_{AB}, \quad (4.3a)$$

$$\{\pi \cdot \phi_A, P_\alpha(\Gamma(\alpha)\phi(\alpha))_B\} = (\alpha \cdot \pi)P_\alpha\Gamma(\alpha)_{BA} + \cdots, \quad (4.3b)$$

$$\{\pi \cdot \phi_A, P_\delta^r(\Gamma(\delta)\phi(\epsilon^r))_B\} = (\epsilon^r \cdot \pi)P_\delta^r\Gamma(\delta)_{BA} + \cdots, \quad (4.3c)$$

$$\{P_\alpha(\Gamma(\alpha)\phi(\alpha))_A, P_\beta(\Gamma(\beta)\phi(\beta))_B\} = -(\alpha \cdot \beta)\varepsilon_{\alpha,\beta}P_\alpha P_\beta \Gamma(\alpha+\beta)_{AB} + \cdots, \quad (4.3d)$$

where dots stand for terms quadratic in the fermions. Now the anticommutator (4.2) is *symmetric* in A, B; hence the terms in the second line do not contribute because $\Gamma(\alpha)$ is antisymmetric for real roots α.[8] Consequently, the result will then contain only terms proportional to δ_{AB} (the Hamiltonian), and terms where $\alpha + \beta$ is lightlike (the constraints), and more generally, for which $(\alpha + \beta)^2$ is a multiple of four. This is indeed the structure displayed in (4.2).

Let us first look at the Hamiltonian. The first kind of contribution will come from those terms with $\beta = \alpha$; in this case we use $\varepsilon_{\alpha,\alpha} = -1$ to get

$$-(\alpha \cdot \alpha)\varepsilon_{\alpha,\alpha}P_\alpha P_\alpha \Gamma(2\alpha)_{AB} = +2P_\alpha P_\alpha \delta_{AB}, \quad (4.4)$$

which is positive, and agrees with what we get from the E_{10} Casimir (see below). For the second kind we have $\alpha \ne \beta$, but such that $(\alpha + \beta)$ has only even components, such that again $\Gamma(\alpha + \beta) = \mathbf{1}$; for example $\alpha + \beta = 2\delta = (22222\,22220)$ with

$$\alpha = (21111\,11100) \quad \text{and} \quad \beta = (01111\,11120).$$

In this case we still have $\varepsilon_{\alpha,\beta} = -1$ but $\alpha \cdot \beta = -2$; hence

$$-(\alpha \cdot \beta)\varepsilon_{\alpha,\beta}P_\alpha P_\beta \Gamma(\alpha+\beta)_{AB} = -2P_\alpha P_\beta \delta_{AB}. \quad (4.5)$$

As one can easily check these are indeed associated with the *negative definite* terms in the bosonic part of the supergravity Hamiltonian. To see this more explicitly, we recall from [[2] Eq. (6.6)] the $SO(10)$ covariant expressions for $\mathcal{H}$ arising from the supersymmetry commutator,

$$\begin{aligned}
\mathcal{H} &= \frac{1}{2}P_{ab}^{(0)}P_{ab}^{(0)} - \frac{1}{2}P_{aa}^{(0)}P_{bb}^{(0)} + \frac{1}{3!}P_{abc}^{(1)}P_{abc}^{(1)} \\
&\quad + \frac{1}{6!}P_{a_1\cdots a_6}^{(2)}P_{a_1\cdots a_6}^{(2)} \\
&\quad + \frac{2}{8!}(P_{a_0|a_1\cdots a_8}^{(3)}P_{a_0|a_1\cdots a_8}^{(3)} - 4P_{b|ba_1\cdots a_7}^{(3)}P_{c|ca_1\cdots a_7}^{(3)}) \\
&= \frac{1}{2}\pi_a G^{ab}\pi_b + \sum_{\substack{\alpha>0 \\ \alpha^2=2,\ell\le3}} P_\alpha P_\alpha - \sum_{\substack{\alpha,\beta>0,\alpha+\beta=2\delta \\ \alpha^2=\beta^2=2,\ell=3}} P_\alpha P_\beta
\end{aligned} \quad (4.6)$$

(see footnote 7 for the normalizations of the level-2 and level-3 terms). Writing out the sums in the last two terms we get exactly the two contributions (4.4) and (4.5) (plus the contribution from the null root). This result is to be contrasted with the coset Hamiltonian H

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[8]This is in agreement with the structure of the diagonal components $\pi_a = P_{aa}^{(0)}$ from the constraint $C_{[a_1\cdots a_9]}^{(3)} = P_{ca_1}^{(0)}P_{c|a_2\cdots a_9}^{(3)} + \cdots$ discussed in [26,27] that only couple to the null root components P_δ^r as determined by (4.3c).

$$H = \frac{1}{2}\langle P|P\rangle = \frac{1}{2}P_{ab}^{(0)}P_{ab}^{(0)} - \frac{1}{2}P_{aa}^{(0)}P_{bb}^{(0)} + \frac{1}{3!}P_{abc}^{(1)}P_{abc}^{(1)}$$
$$+ \frac{1}{6!}P_{a_1\cdots a_6}^{(2)}P_{a_1\cdots a_6}^{(2)}$$
$$+ \frac{1}{8!}P_{a_0|a_1\cdots a_8}^{(3)}P_{a_0|a_1\cdots a_8}^{(3)} + \cdots$$
$$= \frac{1}{2}\pi_a G^{ab}\pi_b + \sum_{\substack{\alpha>0 \\ \alpha^2=2,\ell\leq 3}} P_\alpha P_\alpha + \cdots, \qquad (4.7)$$

where the dots stand for higher level real roots, as well as imaginary roots. In $SO(10)$ form, this latter expression differs from the previous one not only by the appearance of the negative $\ell = 3$ term in (4.6) but also by the factor in front of the $\ell = 3$ term with the correct sign. However, when writing out the Hamiltonian (4.6) in terms of the $K(E_{10})$ variables in a manner completely analogous to the derivation in the foregoing section, we see that the terms at level $\ell = 3$ are in one-to-one correspondence with the two types of terms exhibited in (4.4) and (4.5).[9] We have thus isolated the source of the disagreement between the canonical Hamiltonian and the E$_{10}$ Casimir that appears from level $\ell = 3$ onwards, in terms of the $K(E_{10})$ covariant looking supersymmetry constraint (4.1). This disagreement is seen not only in the negativity, but also in the fact that the E$_{10}$ Casimir does not pair P_α with P_β for $\beta \neq \pm\alpha$. Clearly the source of the trouble resides in the unfaithfulness of the $K(E_{10})$ representation in terms of the $\Gamma(\alpha)$ matrices that we are dealing with here, and seems to require a generalization of the usual Clifford algebra. We also see that these troubles multiply when we extend the sum from roots with $\ell \leq 3$ to all real roots, as we will then have many more contributions proportional to δ_{AB}, which would ruin the agreement with the supergravity Hamiltonian found above.

The bosonic constraints identified in [2] and associated there with lightlike roots are also recovered from those combinations where $\alpha + \beta$ is a null root; as both α and β can go up to level $\ell = 3$, the resulting null roots go up to level $\ell = 6$, in agreement with [2]. So the constraints are generically of the form

$$\mathcal{C}(\delta) = \sum_r \varepsilon^r \cdot P_\delta^r + \sum_{\alpha+\beta=\delta} P_\alpha P_\beta + \cdots, \qquad (4.8)$$

---

[9]We stress that the coefficients of the $P_\alpha P_\alpha$ terms for real roots do agree in the two expressions. For $\ell = 3$, this might seem surprising in view of the *different* coefficients in the $SO(10)$ covariant expressions. A simple way of seeing that they agree after the rewriting in $K(E_{10})$ variables is to look at a fixed particular real root, say $P_{\alpha_{1|12345678}} \equiv P_{1|12345678}^{(3)}$. In (4.6) this term has contributions from both expressions via $\frac{2}{8!}(8! - 4 \times 7!) = 1$, and in (4.7) one similarly has $\frac{1}{8!} \times 8! = 1$. This example illustrates well how the $K(E_{10})$ properties can be obscured by insisting on $SO(10)$ invariant expressions.

which agrees exactly with what was found before in [2]. Note, however, that starting from the supersymmetry constraint (4.1), the first term on the r.h.s. only appears for the null root at level $\ell = 3$, whereas this term is missing for the higher level null roots, because the supersymmetry constraint only goes up to $\ell = 3$. By contrast, the null roots appearing in the combinations $\alpha + \beta$ can go up to $\ell = 6$. This is a clear signal of the incompleteness of the supersymmetry constraint (4.1) as derived from supergravity. We note also that there is only *one* constraint per null root δ, whereas there are eight root generators E_δ^r.

We note that the fermion ϕ^a appears as a *matter fermion* in the one-dimensional model even though it transforms in a vector-spinor representation and descends from the $D = 11$ gravitino. This can for instance be seen by considering the transformation of ϕ^a under $\mathcal{S}$ of (3.35) which does not contain any derivatives of the transformation parameter (these would come from $D_a\phi^a$ terms that were truncated away in the derivation from supergravity). The one-dimensional gravitino that is the supersymmetry partner of the one-dimensional lapse function was set to zero.

B. (In)compatibility of supersymmetry and $K(E_{10})$

We can now also investigate the transformation properties of the constraint $\mathcal{S}$ under $K(E_{10})$. Because $\mathcal{S}$ is "built" out of objects that do transform properly under $K(E_{10})$, namely the coset quantities P_α on the one hand, and the unfaithful vector spinor representation ϕ^a on the other, one would naively expect this constraint to transform in the Dirac representation, that is, $\delta_\alpha\mathcal{S} = \frac{1}{2}\Gamma(\alpha)\mathcal{S}$. However, there appears a basic clash: as we will now show very explicitly, $\mathcal{S}$ *fails to transform properly under* $K(E_{10})$. There are two reasons for this, namely first the presence of imaginary roots in E$_{10}$ and $K(E_{10})$ (and thus the fact that both algebras are infinite dimensional), and second the unfaithfulness of the vector-spinor representation. For the variation of $\mathcal{S}$ under a $K(E_{10})$ transformation generated by k_α we use the formulas

$$\delta_\alpha\pi^a = -2\alpha^a P_\alpha,$$
$$\delta_\alpha P_\beta = \delta_{\alpha,\beta}\alpha_a\pi^a + c_{\beta-\alpha,\alpha}P_{\beta-\alpha} - c_{\alpha+\beta,-\alpha}P_{\alpha+\beta},$$
$$\delta_\alpha\phi^a = \frac{1}{2}\tilde{\Gamma}(\alpha)\phi^a - \alpha^a\tilde{\Gamma}(\alpha)\phi(\alpha), \qquad (4.9)$$

restricting to positive real α, β for simplicity. For the first two lines we have evaluated $[P, k_\alpha]$ and projected onto the H_a and $E_\beta + E_{-\beta}$ components.[10] We emphasize that it is not

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[10]For other roots the first two lines would generalize to

$$\delta_\alpha^r\pi^a = -2\alpha^a P_\alpha^r,$$
$$\delta_\alpha^r P_\beta^s = \delta^{rs}\delta_{\alpha,\beta}\alpha_a\pi^a + \sum_t c_{\beta-\alpha,\alpha}^{rst}P_{\beta-\alpha}^t - \sum_t c_{\alpha+\beta,-\alpha}^{rst}P_{\alpha+\beta}^t,$$

but no general formula is available for $\delta_\alpha^r\phi^a$. A conjectural formula is given in Appendix A.

known whether the P_α, when supplemented by the higher root partners P_α^r, transform in an irreducible representation of $K(E_{10})$, or whether this representation is reducible under $K(E_{10})$. Substituting these formulas into the variation of $\mathcal{S}$ some further calculation leads to

$$\delta_\alpha \mathcal{S} = \frac{1}{2}\tilde{\Gamma}(\alpha)\mathcal{S} + \frac{1}{2}\sum_{\beta>0}P_\beta[\tilde{\Gamma}(\beta),\tilde{\Gamma}(\alpha)]\phi(\beta)$$
$$+ \sum_{\beta>0,\beta\neq\alpha}[-(\alpha\cdot\beta)P_\beta\tilde{\Gamma}(\beta)\tilde{\Gamma}(\alpha)\phi(\alpha)$$
$$+ c_{\beta-\alpha,\alpha}P_{\beta-\alpha}\tilde{\Gamma}(\beta)\phi(\beta) - c_{\alpha+\beta,-\alpha}P_{\alpha+\beta}\tilde{\Gamma}(\beta)\phi(\beta)].$$
$$(4.10)$$

The result would thus have the desired structure if we could show that all terms on the r.h.s. cancel except for the first. However, as we will now demonstrate by explicit computation this is the case only for finite dimensional K, but no longer for $K(E_{10})$. To do so we first rewrite the last term in brackets as

$$-\sum_{\beta>\alpha}c_{\beta,-\alpha}P_\beta\tilde{\Gamma}(\beta-\alpha)\phi(\beta-\alpha) \qquad (4.11)$$

and shift the sums in the second term such a way that only P_β with $\beta > 0$ appear. So in the second term in brackets above we consider the partial sum[11]

$$\sum_{0<\beta<\alpha}c_{\beta-\alpha,\alpha}P_{\beta-\alpha}\tilde{\Gamma}(\beta)\phi(\beta)$$
$$= \sum_{0<\beta<\alpha}c_{-\beta,\alpha}P_{-\beta}\tilde{\Gamma}(-\beta+\alpha)\phi(-\beta+\alpha). \qquad (4.12)$$

Next, using $P_{-\beta} = P_\beta$ and $\phi(-\beta+\alpha) = -\phi(\beta-\alpha)$ as well as (2.3) and not forgetting the extra minus sign from the definition of $\tilde{\Gamma}$ in (3.17) for negative roots this term becomes equal to

$$-\sum_{0<\beta<\alpha}c_{\beta,-\alpha}\tilde{\Gamma}(\beta-\alpha)\phi(\beta-\alpha) \qquad (4.13)$$

and therefore combines with the above term to give a full sum over $\beta > 0$ [because $\phi(0) = 0$, there is no contribution for $\beta = \alpha$]. Finally we obtain

$$\delta_\alpha\mathcal{S} = \frac{1}{2}\tilde{\Gamma}(\alpha)\mathcal{S} + \frac{1}{2}\sum_{\beta>0}P_\beta[\tilde{\Gamma}(\beta),\tilde{\Gamma}(\alpha)]\phi(\beta)$$
$$+ \sum_{\beta>0,\beta\neq\alpha}[-(\alpha\cdot\beta)P_\beta\tilde{\Gamma}(\beta)\tilde{\Gamma}(\alpha)\phi(\alpha)$$
$$+ c_{\beta,\alpha}P_\beta\tilde{\Gamma}(\alpha+\beta)\phi(\alpha+\beta)$$
$$- c_{\beta,-\alpha}P_\beta\tilde{\Gamma}(\beta-\alpha)\phi(\beta-\alpha)], \qquad (4.14)$$

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[11]The inequalities in the sums are taken to imply that the corresponding elements are roots of the algebra (as an ordering cannot be generally defined for arbitrary pairs of roots α and β).

where the sign in (3.17) is again essential. Let us now inspect the different terms here: using $\phi(\alpha+\beta) = \phi(\alpha) + \phi(\beta)$ the terms containing $\phi(\alpha)$ become

$$-\sum_{\beta>0,\beta\neq\alpha}P_\beta\{(\alpha\cdot\beta)\tilde{\Gamma}(\beta)\tilde{\Gamma}(\alpha) - c_{\beta,\alpha}\tilde{\Gamma}(\alpha+\beta)$$
$$- c_{\beta,-\alpha}\tilde{\Gamma}(\beta-\alpha)\}\phi(\alpha). \qquad (4.15)$$

The expression inside brackets does indeed cancel if α and β are real roots such that $(\alpha\pm\beta)$ are also real roots (in which case $\alpha\cdot\beta = \mp1$); for $\alpha\cdot\beta = 0$ all terms vanish. This covers all possible cases for $GL(n)$, but for indefinite G there are infinitely many more possibilities because $\alpha\cdot\beta$ can assume any value, and then the extra terms no longer obviously cancel. We note that there is some room for modifications of the argument coming from the values of $c_{\alpha,\beta}$ when $\alpha+\beta$ is imaginary and also from terms associated with imaginary roots in the ansatz (4.2). The calculation in [2] shows that the (truncated) expression (3.35), involving some terms from null roots, does not transform covariantly.

For the terms containing $\phi(\beta)$ the argument is similar; in this case we end up with

$$\frac{1}{2}\sum_{\beta>0,\beta\neq\alpha}P_\beta\{[\tilde{\Gamma}(\beta),\tilde{\Gamma}(\alpha)] + 2c_{\beta,\alpha}\tilde{\Gamma}(\alpha+\beta)$$
$$- 2c_{\beta,-\alpha}\tilde{\Gamma}(\beta-\alpha)\}\phi(\beta) \qquad (4.16)$$

and a case by case analysis analogous to the one above shows again that these terms cancel under the same conditions as before. To sum up, the extra terms do cancel for finite-dimensional G, when we need only consider the cases $\alpha\cdot\beta = \pm1$ or $= 0$; in this case the supersymmetry constraint indeed transforms properly under K. This need no longer be true for infinite-dimensional G, where we have only insufficient knowledge of the structure constants $c_{\alpha,\beta}$. Let us also emphasize that this problem arises already at linear order in the fermions, so the addition of cubic or even higher order fermion terms cannot remedy this problem.

V. OUTLOOK

In this section, we discuss various possible extensions of our results. One pressing challenge is the correct treatment of the full E_{10} algebra beyond level $\ell = 3$ when trying to construct a $K(E_{10})$ covariant supersymmetry constraint. This will be discussed in Sec. VA. Irrespective of the construction of a supersymmetric model one can consider the spinning particle of [2] and how the fermionic degrees of freedom influence the canonical structures discussed in Sec. II. We offer some comments on this in Sec. VB below.

A. Tentative generalization beyond $\ell = 3$

The expression (4.1) is very suggestive of a generalization beyond level $\ell = 3$, so we are tempted to propose

$$\mathcal{S}_A = \pi_a \phi_A^a + \sum_{\substack{\alpha^2 = 2 \\ \ell \leq 3}} P_\alpha (\Gamma(\alpha)\phi(\alpha))_A$$

$$+ \sum_{\substack{\delta^2 = 0 \\ \ell = 3}} P_\delta^r (\Gamma(\delta)\phi(\epsilon^r))_A + \cdots, \qquad (5.1)$$

where the dots could stand for (at least) three kinds of additional terms, namely

(1) additional terms linear in fermions associated with higher ($\ell > 3$) level roots, coming either from real or imaginary roots;

(2) additional terms cubic in fermions;

(3) terms involving new "higher spin" or other unfaithful realizations of $K(\mathrm{E}_{10})$.

In the following, we will concentrate only on the first extension. This already represents an extension beyond the truncated supergravity constraints. We note that the arguments of Sec. IV B show that such a generalization will not be $K(\mathrm{E}_{10})$ covariant. Nevertheless we can find some constraints on the possible form by demanding at least Weyl invariance of the known terms.

Since the Weyl group $W(\mathrm{E}_{10})$ of E$_{10}$ can be embedded in $K(\mathrm{E}_{10})$ it would seem like a minimal requirement to extend the expression (4.1) by complete Weyl orbits of roots. As the real roots of E$_{10}$ form a single Weyl orbit (E$_{10}$ is simply laced), this would lead to the following expression for $\mathcal{S}_A$:

$$\mathcal{S}_A = \pi \cdot \phi_A + \frac{1}{2} \sum_{\alpha^2 = 2} P_\alpha (\tilde{\Gamma}(\alpha)\phi(\alpha))_A + \cdots,$$

$$\equiv \pi \cdot \phi_A + \frac{1}{2} \sum_{w \in W(\mathrm{E}_{10})} P_{w(\alpha_0)} (\tilde{\Gamma}(w(\alpha_0))\phi(w(\alpha_0)))_A$$

$$+ \cdots, \qquad (5.2)$$

where the dots now indicate terms associated with imaginary roots. In the second line α_0 represents an arbitrary real root. We see again that the minus sign in (3.15) is essential; otherwise the contributions from positive and negative roots would cancel in the sum. As we showed in Sec. IV B, this expression containing only the real roots is incompatible with $K(\mathrm{E}_{10})$. The expression is, however, compatible with the E$_{10}$ Weyl group. But the anticommutator would now give rise to an infinity of new terms that do not seem to make sense.

We know from supergravity that we require also contributions from null imaginary roots ($\alpha^2 = 0$), and these would need to be covariantized under the Weyl group as well. We will not investigate the effect of this covariantization here since already the real roots are problematic. A uniform treatment of all E$_{10}$ roots requires also the inclusion of timelike imaginary roots ($\alpha^2 < 0$). These come with higher multiplicity, and their addition to $\mathcal{S}_A$ might necessitate higher spin realizations of the type constructed in [24], so as to be able to contract the relevant polarization tensors with the fermions.

B. Adding fermions

We now consider some aspects of the inclusion of fermionic degrees of freedom (at lowest order). Let Ψ be a spinorial representation Ψ of the compact subgroup and consider the Lagrangian

$$L = L_B + L_F = \frac{1}{2}\langle P|P\rangle - \frac{i}{2}\langle \Psi|D\Psi\rangle, \qquad (5.3)$$

where $D\Psi$ is the K-covariant derivative with the composite connection Q constructed out of V. In triangular gauge one has $Q_\alpha = P_\alpha$ for all positive root components.

We can write out the covariant derivative in the vector-spinor representation for $\alpha > 0$ (real or imaginary) as follows:

$$L_F = -\frac{i}{2}\langle \Psi|D\Psi\rangle = -\frac{i}{2}G_{ab}\phi^a\partial\phi^b + \frac{i}{2}\sum_{\alpha > 0}\sum_{r=1}^{\mathrm{mult}(\alpha)} P_\alpha^r j_\alpha^r, \qquad (5.4)$$

where j_α^r denotes the fermion bilinear constructed out of the action of the k_α^r in the vector-spinor representation and then contracted in the invariant bilinear form: $j_\alpha^r = G_{ab}\phi^a\delta_\alpha^r\phi^b = G_{ab}\phi^a(k_\alpha^r \cdot \phi^b) = -2J_\alpha^r$. We will suppress the multiplicity index r in our schematic discussion below in order to avoid cluttering the expressions.

The canonical fermionic momentum from (5.3) is

$$\varpi_a = \frac{\partial^L L}{\partial\partial\phi^a} = \frac{i}{2}G_{ab}\phi^b, \qquad (5.5)$$

where we are using left Grassmann derivatives. The momentum satisfies the Poisson bracket

$$\{\phi^a, \varpi_b\} = -1. \qquad (5.6)$$

The corresponding (classical) Dirac bracket is therefore

$$\{\phi^a, \phi^b\} = iG^{ab}. \qquad (5.7)$$

Above we were using this bracket without the factor of i by thinking of the ϕ^a as quantum operators. The additional i here implies that at the classical level

$$\{j_\alpha, j_\beta\} = 2i(c_{\alpha,\beta}j_{\alpha+\beta} - c_{\alpha,-\beta}j_{\alpha-\beta}). \qquad (5.8)$$

Let us denote the bosonic conjugate momenta in the theory with fermions by $\hat{\Pi}$. Then we get

$$\hat{\pi}_a = \frac{\partial L}{\partial\partial q^a} = G_{ab}\partial q^b = \pi_a, \qquad (5.9)$$

$$\hat{\Pi}_\alpha = \frac{\partial L}{\partial\partial A_\alpha} = \sum_{\beta > 0}\left(2P_\beta + \frac{i}{2}j_\beta\right)\frac{\partial P_\beta}{\partial\partial A_\alpha}. \qquad (5.10)$$

We note that the momenta conjugate to the Cartan sub-algebra variables q^a do not change (since these do not couple to the fermions) and that the matrix $\frac{\partial P_\beta}{\partial \partial A_\alpha}$ relating the conjugate momenta to the P_β is identical to the purely bosonic theory. This means that the inversion proceeds in exactly the same way as in (2.19), leading to

$$P_\alpha + \frac{i}{4}j_\alpha = e^{-q^a \alpha_a} \left(\hat{\Pi}_\alpha - \frac{1}{2} \sum_\beta c_{\beta,\alpha} A_\beta \hat{\Pi}_{\alpha+\beta} + \cdots \right). \quad (5.11)$$

We now introduce the notation

$$\hat{P}_\alpha \equiv P_\alpha + \frac{i}{4}j_\alpha. \quad (5.12)$$

Since $\hat{\Pi}_\alpha$ and A_α are conjugate variables as before, we deduce that we have the following canonical commutation relations:

$$\{\hat{P}_\alpha, \hat{P}_\beta\} = c_{\alpha,\beta} \hat{P}_{\alpha+\beta},$$
$$\{\hat{\pi}_a, \hat{P}_\alpha\} = \alpha_a \hat{P}_\alpha. \quad (5.13)$$

The new "supercovariant" velocity components $\hat{P}_\alpha$ therefore satisfy the same Borel algebra as the P_α in the purely bosonic theory.[12] In terms of the original velocities, and in view of (5.8), one therefore has

$$\{P_\alpha, P_\beta\} = c_{\alpha,\beta} \hat{P}_{\alpha+\beta} - \frac{i}{8} c_{\alpha,\beta} j_{\alpha+\beta} + \frac{i}{8} c_{\alpha,-\beta} j_{\alpha-\beta},$$
$$\{\pi_a, P_\alpha\} = \alpha_a \hat{P}_\alpha. \quad (5.14)$$

The appearance of $\hat{P}_\alpha$ on the r.h.s. in these equations is important. For deriving this, we used that $\hat{P}_\alpha$ and j_α commute whence P_α and j_α satisfy

$$\{P_\alpha, j_\beta\} = \left\{ -\frac{i}{4} j_\alpha, j_\beta \right\} = \frac{1}{2} c_{\alpha,\beta} j_{\alpha+\beta} - \frac{1}{2} c_{\alpha,-\beta} j_{\alpha-\beta}. \quad (5.15)$$

Let us verify the consistency of the relation (5.13) and (5.15) in the equations of motion. In the model (5.3) one has, on the one hand, the Euler–Lagrange equations

$$\partial P_\alpha = -\pi^a \alpha_a P_\alpha + 2 \sum_{\beta>0} c_{\alpha,\beta} P_\beta P_{\alpha+\beta} - \frac{i}{4} (\alpha \cdot \pi) j_\alpha$$

$$+ \frac{i}{4} \sum_{\beta>0} P_\beta (c_{\alpha,\beta} j_{\alpha+\beta} + c_{\alpha,-\beta} j_{\alpha-\beta})$$

$$= -(\alpha \cdot \pi) \hat{P}_\alpha + 2 \sum_{\beta>0} c_{\alpha,\beta} P_\beta \hat{P}_{\alpha+\beta}$$

$$- \frac{i}{4} \sum_{\beta>0} P_\beta (c_{\alpha,\beta} j_{\alpha+\beta} - c_{\alpha,-\beta} j_{\alpha-\beta}) \quad (5.16)$$

The Hamiltonian, on the other hand, is (as before)

$$H = \frac{1}{2} \langle P | P \rangle = \frac{1}{2} \hat{\pi}_a G^{ab} \hat{\pi}_b + \sum_{\alpha>0} P_\alpha P_\alpha, \quad (5.17)$$

in terms of the "old" purely bosonic P.[13] The Hamiltonian equations of motion for P_α are then

$$\partial P_\alpha = \{P_\alpha, H\} = -\pi_a \{\pi^a, P_\alpha\} + 2 \sum_{\beta>0} P_\beta \{P_\alpha, P_\beta\}$$

$$= -(\alpha \cdot \pi) \hat{P}_\alpha + 2 \sum_{\beta>0} c_{\alpha,\beta} P_\beta \hat{P}_{\alpha+\beta}$$

$$- \frac{i}{4} \sum_{\beta>0} P_\beta (c_{\alpha,\beta} j_{\alpha+\beta} - c_{\alpha,-\beta} j_{\alpha-\beta}) \quad (5.18)$$

in complete agreement with the Lagrangian equations.

C. Final comments

The underlying problem of the non-$K(E_{10})$ covariance of the supersymmetry constraint $\mathcal{S}$ appears to be the unfaithfulness of the spinor representation that was used to construct $\mathcal{S}$. A full understanding of this issue requires a more detailed understanding of the representation theory of $K(E_{10})$. This involves not only the construction of faithful fermionic representations but also a study of the properties of the "coset representation" P and the decomposition of its tensor products with fermionic representations.

Finding a supersymmetric E_{10} model might exhibit a feature similar to one of the hallmarks of superstring theory. In superstring theory, supersymmetry is implemented only on the two-dimensional world sheet but the consistency conditions of the theory imply that there is also supersymmetry in the target space-time, leading to supergravity at low energies. It is not inconceivable that a supersymmetric E_{10} model on a worldline would similarly induce supersymmetry in the algebraically generated space-time. The close connection between the fermionic E_{10} model on the worldline and the space-time supergravity equations found in [2–4] could be viewed as evidence for this idea.

The problem of finding a $K(E_{10})$ covariant supersymmetry constraint (1.1) can be phrased representation theoretically as follows. Both the coset velocity P and the vector spinor Ψ are honest $K(E_{10})$ representations. Their tensor product $P \otimes \Psi$ is also a $K(E_{10})$ representation, and the question is what the invariant subspaces of this tensor product are, in particular, if there is a Dirac-spinor representation $\mathcal{S}$ contained in it. To the best of our knowledge very little is known about these kinds of questions since $K(E_{10})$ is not a Kac-Moody algebra.

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[12]The terminology "supercovariant" is not fully adequate here since the fermionic fields ϕ^a are more properly thought of as matter fermions rather than gravitino fields. Nevertheless, we will use the term for brevity.

[13]That this is true can be seen in the following simple example involving a derivative coupling. Let $L = \frac{1}{2}\dot{q}^2 + \dot{q}j$. The conjugate momentum is $\hat{p} = \dot{q} + j \equiv p + j$, and the Hamiltonian is $H = \hat{p}\dot{q} - L = \frac{1}{2}\dot{q}^2 = \frac{1}{2}p^2$.

Already the decomposability (or not) of the coset velocity P itself is an open question. If P *was* decomposable, this could have important consequences for the construction of invariant Lagrangians.

Finally, we note that similar issues already arise for the affine case [28,29], where $K(E_{10})$ is replaced by the simpler (but still infinite-dimensional) involutory subgroup $K(E_9) \subset E_9$. In that case one is dealing with a *field theory* in two dimensions, rather than a worldline model, and the faithfulness of the $K(E_9)$ representations is ensured *on shell* by the additional dependence on the space coordinate and the differential relations obeyed by the transformation coefficients. For the *off shell* theory, however, the existence and construction of faithful representations remains an open problem there as well.

APPENDIX A: THE SECOND QUANTIZED VECTOR SPINOR FOR IMAGINARY ROOTS

The Dirac-spinor representation is insensitive to the polarization (i.e., multiplicity) of the imaginary roots but the more faithful vector-spinor representation can be used to derive partial information on the structure constants of $K(E_{10})$. The more faithful higher-spin realizations of [24] in principle capture even more information on the imaginary roots.

In this appendix, we study in more detail the representation of the vector spinor that was completely determined by its values on the real roots by (3.25). For any real root α of E_{10} we recall that the canonical $K(E_{10})$ generators are given by

$$k_\alpha = X_{ab}(\alpha)\phi^a\tilde{\Gamma}(\alpha)\phi^b, \quad \text{with} \quad X_{ab}(\alpha) = -\frac{1}{2}\alpha_a\alpha_b + \frac{1}{4}G_{ab}. \tag{A1}$$

We seek to obtain similar general expression for null roots δ, satisfying $\delta^2 = 0$ and timelike root Λ with $\Lambda^2 = -2$.

1. Null roots δ

All null roots δ of E_{10} have multiplicity $\text{mult}(\delta) = 8$, and we therefore require the representation of eight generators k_α^r. To arrive at the expression, we decompose $\delta = \alpha + (\delta - \alpha)$ for a *real* root α. Then $(\delta - \alpha)$ is also real and $\delta \cdot \alpha = 0$. Employing then the commutator

$$[k_\alpha, k_{\delta-\alpha}] = \tilde{\varepsilon}_{\alpha,\delta-\alpha}k_\delta^{(\alpha)}, \tag{A2}$$

where we have indicated that there are different possibilities for $k_\delta^{(\alpha)}$. One knows *a priori* that there are at most eight independent generators.

Substituting in the explicit expression for the real root generators (A1) one finds that in the (second quantized) vector-spinor representation

$$k_\delta^{(\alpha)} = -2\alpha_{[a}\delta_{b]}\phi^a\tilde{\Gamma}(\delta)\phi^b, \tag{A3}$$

where $\alpha \cdot \delta = 0$. To bring this into a form that brings out the multiplicity $\text{mult}(\delta) = 8$, we note that shifting $\alpha \to \alpha + \delta$ does not change the expression, so that we can also summarize it by

$$k_\delta^r = \epsilon_{[a}^r\delta_{b]}\phi^a\Gamma(\delta)\phi^b, \tag{A4}$$

where the "polarization vector" ϵ^r is orthogonal (transverse) to δ in the DeWitt metric and there is also a gauge invariance $\epsilon^r \to \epsilon^r + \delta$. This leaves eight independent choices which agrees with the multiplicity of the null root of E_{10}. Note that the transversality of the polarization vector is *not* manifest in (A4); it is rather a consequence of the way the generator is constructed from commutators of real roots.

2. Imaginary roots $\Lambda^2 = -2$

It is also possible to derive the general form of the $\Lambda^2 = -2$ generators from commuting two real root generators in a way similar to above. Let $\Lambda = \alpha + (\Lambda - \alpha)$ with $\alpha^2 = (\Lambda - \alpha)^2 = 2$, and then $\alpha \cdot \Lambda = -1$. We know that

$$[k_\alpha, k_\beta] = \tilde{\varepsilon}_{\alpha,\beta}k_\Lambda^{(\alpha)}. \tag{A5}$$

By substituting in the explicit form for the real root generators one finds

$$k_\Lambda^{(\alpha)} = 2Y_{ab}(\alpha)\phi^a\Gamma(\Lambda)\phi^b \tag{A6}$$

with

$$Y_{ab}(\alpha) = Y_{ba}(\alpha) = -\alpha_{(a}\Lambda_{b)} + \alpha_a\alpha_b - \frac{1}{4}\Lambda_a\Lambda_b + \frac{1}{8}G_{ab}$$
$$= v_{(a}\Lambda_{b)} + a_{ab} \tag{A7}$$

for

$$v_a = -\alpha_a - \frac{5}{8}\Lambda_a, \tag{A8}$$

$$a_{ab} = \alpha_a\alpha_b + \frac{3}{8}\Lambda_a\Lambda_b + \frac{1}{8}G_{ab}. \tag{A9}$$

The separation of the $\Lambda_{(a}\Lambda_{b)}$ here was chosen in such a way that

$$\Lambda^b a_{ba} = v_a \tag{A10}$$

and is motivated by vertex operator algebra (VOA) constructions. The gauge symmetries of the parametrization (A7) are

$$a_{ab} \to a_{ab} + 2\epsilon_{(a}\Lambda_{b)}, \tag{A11}$$

$$v_a \to v_a - 2\epsilon_a, \tag{A12}$$

and also leave the above condition $\Lambda^b a_{ba} = v_a$ invariant. The parameter ϵ here is chosen orthogonal to Λ. We also note that $\Lambda^a\Lambda^b Y_{ab} = 19/4$ and $G^{ab}Y_{ab} = -9/4$ are gauge invariant and constrain the tensor Y_{ab}. The count is

55	Components of a_{ab}
−9	Components of ϵ such that $\epsilon \cdot \Lambda = 0$
−2	Norm conditions on Y_{ab}
44	Multiplicity of root space

This count does not completely parallel the VOA construction, and it would be desirable to have an interpretation in terms of Young symmetries similar to the null case above.

3. Conjectural form for any generator

Similar to the formula for roots satisfying $\Lambda^2 = -2$ as above, we can give a tentative form of the action of any generator k_Λ^r on the vector spinor ϕ^a for an arbitrary imaginary root Λ. This form rests on the assumption that any generator in the root space of Λ can be written as the commutator of two *real* root generators, that is,

$$c_{\alpha,\beta}k_\Lambda^{(\alpha)} = [k_\alpha, k_\beta] \tag{A13}$$

for $\alpha, \beta > 0$ and real with $\alpha + \beta = \Lambda$. [We note that there the second term in the commutation relation (3.9) vanishes automatically for the configuration chosen here since $\alpha - \beta$ is not a root.] The conjecture is that as α and β traverse all possible decompositions of Λ, their commutators contain a basis of the root space of Λ. The number of decompositions of Λ is larger than mult(Λ), and many of the commutators will be linearly dependent. What we require is that the space generated by all possible commutators is equal to the full root space of Λ:

$$\langle k_\Lambda^{(\alpha)} | \alpha > 0, \alpha^2 = 2, (\Lambda - \alpha)^2 = 2 \rangle$$
$$= \langle k_\Lambda^r | r = 1, \ldots, \text{mult}(\Lambda) \rangle. \tag{A14}$$

This is a stronger version of a conjecture already contained [30] which only addressed the decomposition of the imaginary root vector $\Lambda = \alpha + \beta$ into two real roots. In all cases that we checked the assumption we are making is true but we are not aware of a general proof.

Under this assumption, we can find the following formula for $k_\Lambda^{(\alpha)}$ *in the vector-spinor representation*. We have to distinguish the cases $\Lambda^2 = -4k$ and $\Lambda^2 = 2 - 4k$ for integer $k \geq 0$ because of the (anti-)symmetry of $\tilde{\Gamma}(\Lambda)$. The condition that α and $\Lambda - \alpha$ be real implies that $\alpha \cdot (\Lambda - \alpha) = \frac{1}{2}(\Lambda^2 - 4)$. Then the calculations are completely analogous to the two cases described above, leading to

$$\Lambda^2 = -4k \quad : \quad k_\Lambda^{(\alpha)} = 2(k-1)\alpha_{[a}\Lambda_{b]}\phi^a\tilde{\Gamma}(\Lambda)\phi^b, \tag{A15a}$$

$$\Lambda^2 = 2 - 4k \quad :$$
$$k_\Lambda^{(\alpha)} = \left(-2k\alpha_{(a}\Lambda_{b)} + 2k\alpha_a\alpha_b - \frac{1}{2}\Lambda_a\Lambda_b + \frac{1}{4}G_{ab}\right)\phi^a\tilde{\Gamma}(\Lambda)\phi^b. \tag{A15b}$$

These expressions were derived under the assumption that $c_{\alpha,\Lambda-\alpha} = -\tilde{\varepsilon}_{\alpha,\Lambda-\alpha}$.

APPENDIX B: SOME MORE EXPLICIT RESULTS FOR $GL(3,\mathbb{R})/SO(3)$

For $GL(3,\mathbb{R})$ the coset element is

$$V = AN = \begin{pmatrix} e^{q^1} & & \\ & e^{q^2} & \\ & & e^{q^3} \end{pmatrix} \begin{pmatrix} 1 & N^1{}_{\tilde{2}} & N^1{}_{\tilde{3}} \\ & 1 & N^2{}_{\tilde{3}} \\ & & 1 \end{pmatrix}. \tag{B1}$$

The notation here is such that an index value with a tilde refers to a curved (world) index and an index value without a tilde to a flat (tangent space) direction. The inverse of N is given by

$$N^{-1} = \begin{pmatrix} 1 & N^{\tilde{1}}{}_2 & N^{\tilde{1}}{}_3 \\ & 1 & N^2{}_3 \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & -N^1{}_{\tilde{2}} & -N^1{}_{\tilde{3}} + N^1{}_{\tilde{2}}N^2{}_{\tilde{3}} \\ & 1 & -N^2{}_{\tilde{3}} \\ & & 1 \end{pmatrix}. \tag{B2}$$

With this parametrization it is straightforward to compute the coset velocity from $\partial V V^{-1}$,

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} = \begin{pmatrix} \partial q^1 & \frac{1}{2}e^{q^1-q^2}\partial N^1{}_{\tilde{2}} & \frac{1}{2}e^{q^1-q^3}(\partial N^1{}_{\tilde{3}} + \partial N^1{}_{\tilde{2}}N^2{}_{\tilde{3}}) \\ \frac{1}{2}e^{q^1-q^2}\partial N^1{}_{\tilde{2}} & \partial q^2 & \frac{1}{2}e^{q^2-q^3}\partial N^2{}_{\tilde{3}} \\ \frac{1}{2}e^{q^1-q^3}(\partial N^1{}_{\tilde{3}} + \partial N^1{}_{\tilde{2}}N^2{}_{\tilde{3}}) & \frac{1}{2}e^{q^2-q^3}\partial N^2{}_{\tilde{3}} & \partial q^3 \end{pmatrix}, \tag{B3}$$

where, of course, $P_{ab} = P_{ba}$.

The bosonic Lagrangian is

$$L = \frac{1}{2}[\text{Tr}(P^2) - (\text{Tr}P)^2]$$

$$= \frac{1}{2}G_{ab}\partial q^a \partial q^b + P_{12}^2 + P_{13}^2 + P_{23}^2$$

$$= \frac{1}{2}G_{ab}\partial q^a \partial q^b + \frac{1}{4}e^{2q^1-2q^2}(\partial N^1{}_{\tilde{2}})^2 + \frac{1}{4}e^{2q^2-2q^3}(\partial N^2{}_{\tilde{3}})^2 + \frac{1}{4}e^{2q^1-2q^3}(\partial N^1{}_{\tilde{3}} + \partial N^1{}_{\tilde{2}}N^{\tilde{2}}{}_3)^2, \quad (B4)$$

where $\frac{1}{2}G_{ab}\partial q^a \partial q^b = -\partial q^1 \partial q^2 - \partial q^1 \partial q^3 - \partial q^2 \partial q^3$. The conjugate momenta are

$$\pi_a = G_{ab}\partial q^b,$$

$$\Pi^{\tilde{2}}{}_1 = \frac{\partial L}{\partial \partial N^1{}_{\tilde{2}}} = \frac{1}{2}e^{2q^1-2q^2}\partial N^1{}_{\tilde{2}} - \frac{1}{2}e^{2q^1-2q^3}N^2{}_{\tilde{3}}(\partial N^1{}_{\tilde{3}} + \partial N^1{}_{\tilde{2}}N^{\tilde{2}}{}_3)$$

$$= e^{q^1-q^2}P_{21} + e^{q^1-q^3}N^{\tilde{2}}{}_3 P_{31}, \quad (B5)$$

$$\Pi^{\tilde{3}}{}_1 = \frac{\partial L}{\partial \partial N^1{}_{\tilde{3}}} = \frac{1}{2}e^{2q^1-2q^3}(\partial N^1{}_{\tilde{3}} + \partial N^1{}_{\tilde{2}}N^{\tilde{2}}{}_3) = e^{q^1-q^3}P_{31} \quad (B6)$$

$$\Pi^{\tilde{3}}{}_2 = \frac{\partial L}{\partial \partial N^2{}_{\tilde{3}}} = \frac{1}{2}e^{2q^2-2q^3}\partial N^2{}_{\tilde{3}} = e^{q^2-q^3}P_{32}. \quad (B7)$$

These relations can be inverted to give $\partial q^a = G^{ab}\pi_b$ and

$$\partial N^1{}_{\tilde{2}} = 2e^{-2q^1+2q^2}(\Pi^{\tilde{2}}{}_1 + N^2{}_{\tilde{3}}\Pi^{\tilde{3}}{}_1),$$

$$\partial N^1{}_{\tilde{3}} = 2e^{-2q^1+2q^3}\Pi^{\tilde{3}}{}_1 + 2e^{-2q^1+2q^2}N^2{}_{\tilde{3}}(\Pi^{\tilde{2}}{}_1 + N^2{}_{\tilde{3}}\Pi^{\tilde{3}}{}_1),$$

$$\partial N^2{}_{\tilde{3}} = 2e^{-2q^2+2q^3}\Pi^{\tilde{3}}{}_2. \quad (B8)$$

Equivalently,

$$P_{12} = e^{-q^1+q^2}(\Pi^{\tilde{2}}{}_1 + N^2{}_{\tilde{3}}\Pi^{\tilde{3}}{}_1),$$

$$P_{23} = e^{-q^2+q^3}\Pi^{\tilde{3}}{}_2,$$

$$P_{13} = e^{-q^1+q^3}\Pi^{\tilde{3}}{}_1, \quad (B9)$$

in agreement with the general formula (2.38), up to an overall factor. Let us denote

$$P(\alpha_{12}) \equiv P_{12}, \qquad P(\alpha_{23}) \equiv P_{23}, \qquad P(\alpha_{13}) \equiv P_{13}, \quad (B10)$$

and

$$\alpha_{12} = (1, -1, 0), \qquad \alpha_{13} = (1, 0, -1), \qquad \alpha_{23} = (0, 1, -1). \quad (B11)$$

Then the Hamiltonian is

$$H = \frac{1}{2}\pi_a G^{ab}\pi_b + e^{-2q^1+2q^2}(\Pi^{\tilde{2}}{}_1 + N^2{}_{\tilde{3}}\Pi^{\tilde{3}}{}_1)^2 + e^{-2q^1+2q^3}(\Pi^{\tilde{3}}{}_1)^2 + e^{-2q^2+2q^3}(\Pi^{\tilde{3}}{}_2)^2$$

$$= \frac{1}{2}\pi_a G^{ab}\pi_b + \sum_{a<b}P(\alpha_{ab})^2. \quad (B12)$$

Using the canonical brackets $\{q, p\} = 1$ between the conjugate variables we recover the relations already previously derived (for $a < b$ and any $\alpha > 0$)

$$\{\pi_a, \pi_b\} = 0,$$

$$\{\pi_a, P(\alpha)\} = \alpha_a P(\alpha),$$

$$\{P(\alpha_{ab}), P(\alpha_{cd})\} = \begin{cases} \epsilon_{\alpha_{ab},\alpha_{cd}} P(\alpha_{ab} + \alpha_{cd}) & \text{if } \alpha_{ab} + \alpha_{cd} \text{ is a root,} \\ 0 & \text{otherwise.} \end{cases} \tag{B13}$$

The conserved current is

$$J = V^{-1} P V = \begin{pmatrix} J^{\tilde{1}}_{\tilde{1}} & J^{\tilde{1}}_{\tilde{2}} & J^{\tilde{1}}_{\tilde{3}} \\ J^{\tilde{2}}_{\tilde{1}} & J^{\tilde{2}}_{\tilde{2}} & J^{\tilde{2}}_{\tilde{3}} \\ J^{\tilde{3}}_{\tilde{1}} & J^{\tilde{3}}_{\tilde{2}} & J^{\tilde{3}}_{\tilde{3}} \end{pmatrix} \tag{B14}$$

with

$$J^{\tilde{2}}_{\tilde{1}} = \Pi^{\tilde{2}}_{1},$$

$$J^{\tilde{3}}_{\tilde{1}} = \Pi^{\tilde{3}}_{1},$$

$$J^{\tilde{3}}_{\tilde{2}} = \Pi^{\tilde{3}}_{2} + N^{1}_{\tilde{2}}\Pi^{\tilde{3}}_{1}, \tag{B15}$$

below the diagonal and the following diagonal and upper triangular components

$$J^{\tilde{1}}_{\tilde{1}} = G^{1a}\pi_a - N^{1}_{\tilde{2}}\Pi^{\tilde{2}}_{1} - N^{1}_{\tilde{3}}\Pi^{\tilde{3}}_{1},$$

$$J^{\tilde{2}}_{\tilde{2}} = G^{2a}\pi_a + N^{1}_{\tilde{2}}\Pi^{\tilde{2}}_{1} - N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2},$$

$$J^{\tilde{3}}_{\tilde{3}} = G^{3a}\pi_a + N^{1}_{\tilde{3}}\Pi^{\tilde{3}}_{1} + N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2},$$

$$J^{\tilde{1}}_{\tilde{2}} = e^{-2q^1 + 2q^2}(\Pi^{\tilde{2}}_{1} + N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{1}) + N^{1}_{\tilde{2}}(\pi_{\tilde{1}} - \pi_{\tilde{2}} - N^{1}_{\tilde{3}}\Pi^{\tilde{3}}_{1} + N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2}) - N^{1}_{\tilde{3}}\Pi^{\tilde{3}}_{2} - N^{1}_{\tilde{2}}N^{1}_{\tilde{2}}\Pi^{\tilde{2}}_{1},$$

$$J^{\tilde{1}}_{\tilde{3}} = e^{-2q^1 + 2q^3}\Pi^{\tilde{3}}_{1} + e^{-2q^1 + 2q^2}N^{2}_{\tilde{3}}(\Pi^{\tilde{2}}_{1} + N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{1}) - e^{-2q^2 + 2q^3}N^{1}_{\tilde{2}}\Pi^{\tilde{3}}_{2}$$

$$+ N^{1}_{\tilde{3}}(\pi_{\tilde{1}} - \pi_{\tilde{3}} - N^{1}_{\tilde{2}}\Pi^{\tilde{2}}_{2} - N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2}) - N^{1}_{\tilde{2}}N^{2}_{\tilde{3}}(\pi_{\tilde{2}} - \pi_{\tilde{3}} - N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2}) - N^{1}_{\tilde{3}}N^{1}_{\tilde{3}}\Pi^{\tilde{3}}_{1},$$

$$J^{\tilde{2}}_{\tilde{3}} = e^{-2q^2 + 2q^3}\Pi^{\tilde{3}}_{2} + N^{2}_{\tilde{3}}(\pi_{\tilde{2}} - \pi_{\tilde{3}}) + N^{1}_{\tilde{3}}\Pi^{\tilde{2}}_{1} - N^{2}_{\tilde{3}}N^{2}_{\tilde{3}}\Pi^{\tilde{3}}_{2}. \tag{B16}$$

The relation of the components of the conserved charge to the canonical momenta was already discussed in [17]. The "lowest" components of J are just identical to the canonical momenta, and the structure gets increasingly complicated for higher and higher components. For infinite-dimensional algebras (without a lowest component) this description breaks down without a suitable truncation.

One can now check that

$$H = \frac{1}{2}[\text{Tr}(J^2) - (\text{Tr}J)^2] \tag{B17}$$

and that the components of the current satisfy the $GL(3)$ algebra,

$$\{J^{i}_{j}, J^{k}_{l}\} = \delta^{k}_{j}J^{i}_{l} - \delta^{i}_{l}J^{k}_{j}. \tag{B18}$$

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