Weakly differentiable functions on varifolds

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Abstract

The present paper is intended to provide the basis for the study of weakly differentiable functions on rectifiable varifolds with locally bounded first variation. The concept proposed here is defined by means of integration by parts identities for certain compositions with smooth functions. In this class the idea of zero boundary values is realised using the relative perimeter of superlevel sets. Results include a variety of Sobolev Poincaré type embeddings, embeddings into spaces of continuous and sometimes Hölder continuous functions, pointwise differentiability results both of approximate and integral type as well as coarea formulae.

As prerequisite for this study decomposition properties of such varifolds and a relative isoperimetric inequality are established. Both involve a concept of distributional boundary of a set introduced for this purpose.

As applications the finiteness of the geodesic distance associated to varifolds with suitable summability of the mean curvature and a characterisation of curvature varifolds are obtained.

Contents

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Introduction

Overview

The main purpose of this paper is to present a concept of weakly differentiable functions on nonsmooth "surfaces" in Euclidean space with arbitrary dimension and codimension arising in variational problems involving the area functional. The model used for such surfaces are rectifiable varifolds whose first variation with respect to area is representable by integration (that is, in the terminology of Simon [Sim83, 39.2], rectifiable varifolds with locally bounded first variation). This includes area minimising rectifiable currents[1], in particular perimeter minimising "Caccioppoli sets", or almost every time slice of Brakke's mean curvature flow[2] just as well as surfaces occurring in diffuse interface models[3] or image restoration models[4]. The envisioned concept should be defined without reference to an approximation by smooth functions and it should be as broad as possible so as to still allow for substantial positive results.

In order to integrate well the first variation of the varifold into the concept of weakly differentiable function, it appeared necessary to provide an entirely new notion rather than to adapt one of the many concepts of weakly differentiable functions which have been invented for different purposes. For instance, to study the support of the varifold as metric space with its geodesic distance, stronger conditions on the first variation are needed, see Section 14.

Description of results

Setup and basic results

To describe the results obtained, consider the following set of hypotheses; the notation is explained in Section 1.

General hypothesis. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, V is an m dimensional rectifiable varifold in U whose first variation δV is representable by integration.*

[1] See Allard [All72, 4.8 (4)].
[2] See Brakke [Bra78, §4].
[3] See for instance Hutchinson and Tonegawa [HT00] or Röger and Tonegawa [RT08].
[4] See for example Ambrosio and Masnou [AM03].

The study of weakly differentiable functions is closely related to the study of connectedness properties of the underlying space or varifold. Therefore it is instructive to begin with the latter.

Definition (see 6.2)**.** If V is as in the general hypothesis, it is called *indecomposable* if and only if there is no $\|V\| + \|\delta V\|$ measurable set E such that $\|V\|(E) > 0$, $\|V\|(U \sim E) > 0$ and $\delta(V \llcorner E \times \mathbf{G}(n,m)) = (\delta V) \llcorner E$.

The basic theorem involving this notion is the following.

Decomposition theorem, see 6.10 and 6.12. *If m, n, U, and V are as in the general hypothesis, then there exists a countable disjointed collection G of Borel sets whose union is U such that $V \llcorner E \times \mathbf{G}(n,m)$ is nonzero and indecomposable and $\delta(V \llcorner E \times \mathbf{G}(n,m)) = (\delta V) \llcorner E$ whenever $E \in G$.*

Employing the following definition, the Borel partition G of U is required to consist of members whose distributional V boundary vanishes and which cannot be split nontrivially into smaller Borel sets with that property.

Definition (see 5.1)**.** If m, n, U, and V are as in the general hypothesis and E is $\|V\| + \|\delta V\|$ measurable, then the *distributional V boundary of E* is defined by

$$V\,\partial E = (\delta V) \llcorner E - \delta(V \llcorner E \times \mathbf{G}(n,m)) \in \mathscr{D}'(U, \mathbf{R}^n).$$

If the varifold is sufficiently regular a version of the Gauss Green theorem can be proven which both justifies the terminology and links the concept of boundary to the one for currents, see 5.10 and 5.11.

In the terminology of 6.6 and 6.9 the varifolds $V \llcorner E \times \mathbf{G}(n,m)$ occurring in the decomposition theorem are components of V and the family $\{V \llcorner E \times \mathbf{G}(n,m) : E \in G\}$ is a decomposition of V. However, unlike the decomposition into connected components for topological spaces, the preceding decomposition is nonunique in an essential way. In fact, the varifold corresponding to the union of three lines in $\mathbf{R}^2$ meeting at the origin at equal angles may also be decomposed into two "Y-shaped" varifolds each consisting of three rays meeting at equal angles, see 6.13.

The seemingly most natural definition of weakly differentiable function would be to require an integration by parts identity involving the first variation of the varifold. However, the resulting class of real valued functions is neither stable under truncation of its members nor does a coarea formula hold, see 8.26 and 8.31. Therefore, instead one requires an integration by parts identity for the composition of the function in question with smooth functions whose derivative has compact support, see 8.3. Whenever Y is a finite dimensional normed vectorspace the resulting class of functions of Y valued functions is denoted by

$$\mathbf{T}(V, Y).$$

Whenever f belongs to that class there exists a $\|V\|$ measurable $\operatorname{Hom}(\mathbf{R}^n, Y)$ valued function $V\,\mathbf{D}f$, called the (generalised) weak derivative of f, which is $\|V\|$ almost uniquely determined by the integration by parts identity. In defining $\mathbf{T}(V, Y)$, it seems natural not to require local summability of $V\,\mathbf{D}f$ but only

$$\textstyle\int_{K \cap \{x : |f(x)| \leq s\}} |V\,\mathbf{D}f| \,\mathrm{d}\|V\| < \infty$$

whenever K is a compact subset of U and $0 \leq s < \infty$; this is in analogy with definition of the space "$\mathscr{T}^{1,1}_{\mathrm{loc}}(U)$" introduced by Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vázquez in [BBG+95, p. 244] for the case of Lebesgue measure, see 8.19. In both cases the letter "T" in the name of the space stands for truncation.

Stability properties under composition (for example truncation) then follow readily, see 8.12 and 8.15. Also, it is evident that members of $\mathbf{T}(V, Y)$ may be defined on components separately, see 8.24. The space $\mathbf{T}(V, Y)$ is stable under addition of a locally Lipschitzian function but it is not closed with respect to addition in general, see 8.20 (3) and 8.25. A similar statement holds for multiplication of functions, see 8.20 (4) and 8.25. Adopting an axiomatic viewpoint, consider the class of functions which satisfies the following three conditions for a given varifold:

(1) Each function which is constant on the components of some decomposition of the varifold belongs to the class.

(2) The class is closed under addition.

(3) The class is closed under truncation.

Then there exists a stationary one dimensional integral varifold in $\mathbf{R}^2$ such that the associated class necessarily contains characteristic functions with nonvanishing distributional derivative representable by integration, see 8.27. These characteristic functions should not belong to a class of "weakly differentiable" functions but rather to a class of functions of "bounded variation". The reason for this phenomenon is the afore-mentioned nonuniqueness of decompositions of varifolds.

Much of the development of the theory rests on the following basic theorem.

Coarea formula, functional analytic form, see 8.5 and 8.29. *Suppose m, n, U, and V are as in the general hypothesis, $f \in \mathbf{T}(V)$, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.*

Then there holds

$$\int \langle \phi(x, f(x)), V\,\mathbf{D}f(x)\rangle \,\mathrm{d}\|V\|x = \int V\,\partial E(y)(\phi(\cdot, y))\,\mathrm{d}\mathscr{L}^1 y,$$
$$\int g(x, f(x))|V\,\mathbf{D}f(x)|\,\mathrm{d}\|V\|x = \int\int g(x, y)\,\mathrm{d}\|V\,\partial E(y)\|x\,\mathrm{d}\mathscr{L}^1 y.$$

whenever $\phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n)$ and $g : U \times \mathbf{R} \to \mathbf{R}$ is a continuous function with compact support.

An example of such a development is provided by passing from the notion of zero boundary values on a relatively open part G of the boundary of U for sets to a similar notion for weakly differentiable functions. For $\|V\| + \|\delta V\|$ measurable sets E such that $V\,\partial E$ is representable by integration, the condition is defined in 7.1. In the special case that $\|V\|$ is associated to $M \cap U$ for some properly embedded m dimensional submanifold M of $\mathbf{R}^n \sim ((\operatorname{Bdry} U) \sim G)$ of class 2 without boundary it is equivalent to E being of locally finite perimeter in M and its distributional boundary being measure theoretically contained in U, see 7.2. In order to define such a notion for nonnegative members f of $\mathbf{T}(V, \mathbf{R})$, one requires for $\mathscr{L}^1$ almost all $0 < y < \infty$ that the set $E(y) = \{x : f(x) > y\}$ satisfies the corresponding zero boundary value condition.

This gives rise to the subspaces $\mathbf{T}_G(V)$ of $\mathbf{T}(V,\mathbf{R}) \cap \{f : f \geq 0\}$ with $|f| \in \mathbf{T}_\varnothing(V)$ whenever $f \in \mathbf{T}(V,Y)$, see 9.1 and 9.2. The space $\mathbf{T}_G(V)$ satisfies useful truncation and closure properties, see 9.9 and 9.13. Moreover, under a natural summability hypothesis, the multiplication of a member of $\mathbf{T}_G(V)$ by a nonnegative Lipschitzian function belongs to $\mathbf{T}_G(V)$, see 9.16. Whereas the usage of level sets in the definition of $\mathbf{T}_G(V)$ is tailored to work nicely in the proofs of embedding results in Section 10, the stability property under multiplication requires a more delicate proof in turn.

Embedding results and structural results

To proceed to deeper results on functions in $\mathbf{T}(V,Y)$, the usage of the isoperimetric inequality for varifolds seems indispensable. The latter works best under the following additional hypothesis.

Density hypothesis. *Suppose m, n, U, and V are as in the general hypothesis and satisfy*

$$\boldsymbol{\Theta}^m(\|V\|, x) \geq 1 \quad \textit{for } \|V\| \textit{ almost all } x.$$

The key to prove effective versions of Sobolev Poincaré type embedding results is the following theorem.

Relative isoperimetric inquality, see 7.9. *Suppose m, n, U, and V satisfy the general hypothesis and the density hypothesis, E is $\|V\| + \|\delta V\|$ measurable, $1 \leq Q \leq M < \infty$, $n \leq M$, $\Lambda = \Gamma_{7.8}(M)$, $0 < r < \infty$,*

$$\|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \quad \|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Lambda^{-1} r^m,$$

and $A = \{x : \mathbf{U}(x,r) \subset U\}$.

Then there holds

$$\|V\|(E \cap A)^{1-1/m} \leq \Lambda\big(\|V\,\partial E\|(U) + \|\delta V\|(E)\big),$$

where $0^0 = 0$.

In the special case that $V\,\partial E = 0$, $\|\delta V\|(E) = 0$ and

$$\boldsymbol{\Theta}^m(\|V\|, x) \geq Q \quad \text{for } \|V\| \text{ almost all } x$$

the varifold $V \llcorner E \times \mathbf{G}(n,m)$ is stationary and the support of its weight measure, $\operatorname{spt}(\|V\| \llcorner E)$, cannot intersect A by the monotonicity identity and the upper bound on $\|V\|(E)$. The value of the theorem lies in quantifying this behaviour. Much of the usefulness of the result for the purposes of the present paper stems from the fact that values of Q larger than 1 are permitted. This allows to effectively apply the result in neighbourhoods of points where the density function $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ has a value larger than 1 and is approximately continuous. The case $Q = 1$ is partly contained in a result of Hutchinson [Hut90, Theorem 1] which treats Lipschitzian functions.

If the set E satisfies a zero boundary value condition, see 7.1, on a relatively open subset G of $\operatorname{Bdry} U$, the conclusion may be sharpened by replacing $A = \{x : \mathbf{U}(x,r) \subset U\}$ by

$$A' = U \cap \{x : \mathbf{U}(x,r) \subset \mathbf{R}^n \sim B\}, \quad \text{where } B = (\operatorname{Bdry} U) \sim G.$$

In order to state a version of the resulting Sobolev Poincaré inequalities, recall a less known notation from Federer's treatise on geometric measure theory, see [Fed69, 2.4.12].

Definition. Suppose μ measures X and f is a μ measurable function with values in some Banach space Y.

Then one defines $\mu_{(p)}(f)$ for $1 \leq p \leq \infty$ by the formulae

$$\begin{aligned} \mu_{(p)}(f) &= \left(\textstyle\int |f|^p \,\mathrm{d}\mu\right)^{1/p} \quad \text{in case } 1 \leq p < \infty, \\ \mu_{(\infty)}(f) &= \inf\big\{s : s \geq 0,\ \mu(\{x : |f(x)| > s\}) = 0\big\}. \end{aligned}$$

In comparison to the more common notation $\|f\|_{\mathbf{L}_p(\mu,Y)}$ it puts the measure in focus and avoids iterated subscripts.

Sobolev Poincaré inequality, zero median version, see 8.16, 9.2, and 10.1 (1a)**.** *Suppose* $1 \leq M < \infty$.

Then there exists a positive, finite number Γ *with the following property.*

If m, n, U, *and* V *satisfy the general hypothesis and the density hypothesis,* $n \leq M$, Y *is a finite dimensional normed vectorspace,* $f \in \mathbf{T}(V,Y)$, $1 \leq Q \leq M$, $0 < r < \infty$, $E = U \cap \{x : f(x) \neq 0\}$,

$$\begin{gathered} \|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \\ \|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m, \\ \beta = \infty \text{ if } m = 1, \quad \beta = m/(m-1) \text{ if } m > 1, \end{gathered}$$

$A = \{x : \mathbf{U}(x,r) \subset U\}$, *then*

$$(\|V\| \llcorner A)_{(\beta)}(f) \leq \Gamma\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|_{(1)}(f)\big).$$

Again, apart from extending the result to the class $\mathbf{T}(V,Y)$, the main improvement upon known results such as Hutchinson [Hut90, Theorem 1] is the applicability with values $Q > 1$. If f belongs to $\mathbf{T}_G(V)$, then A may be replaced by A' as in the relative isoperimetric inequality. Not surprisingly, there also exists a version of the Sobolev inequality for members of $\mathbf{T}_{\mathrm{Bdry}\,U}(V)$.

Sobolev inequality, see 10.1 (2a)**.** *Suppose* $1 \leq M < \infty$.

Then there exists a positive, finite number Γ *with the following property.*

If m, n, U, *and* V *satisfy the general hypothesis and the density hypothesis,* $n \leq M$, $f \in \mathbf{T}_{\mathrm{Bdry}\,U}(V)$,

$$\begin{gathered} E = U \cap \{x : f(x) \neq 0\}, \quad \|V\|(E) < \infty, \\ \beta = \infty \text{ if } m = 1, \quad \beta = m/(m-1) \text{ if } m > 1, \end{gathered}$$

then there holds

$$\|V\|_{(\beta)}(f) \leq \Gamma\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|_{(1)}(f)\big).$$

For Lipschitzian functions Sobolev inequalities were obtained by Allard [All72, 7.1] and Michael and Simon [MS73, Theorem 2.1] for varifolds and by Federer in [Fed75, §2] for rectifiable currents which are absolutely minimising with respect to a positive, parametric integrand.

Coming back the Sobolev Poincaré inequalities, one may also establish a version with several "medians". The number of medians needed is controlled by the total weight measure of the varifold in a natural way.

Sobolev Poincaré inequality, several medians, see 10.7 (1). *Suppose* $1 \leq M < \infty$.

Then there exists a positive, finite number Γ *with the following property.*

If m, n, U, *and* V *satisfy the general hypothesis and the density hypothesis,* Y *is a finite dimensional normed vectorspace,* $\sup\{\dim Y, n\} \leq M$, $f \in \mathbf{T}(V, Y)$, $1 \leq Q \leq M$, N *is a positive integer,* $0 < r < \infty$,

$$\begin{gathered} \|V\|(U) \leq (Q - M^{-1})(N+1)\boldsymbol{\alpha}(m) r^m, \\ \|V\|(\{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m, \\ \beta = \infty \text{ if } m = 1, \quad \beta = m/(m-1) \text{ if } m > 1, \end{gathered}$$

and $A = \{x : \mathbf{U}(x, r) \subset U\}$, *then there exists a subset* Υ *of* Y *such that* $1 \leq \operatorname{card} \Upsilon \leq N$ *and*

$$(\|V\| \llcorner A)_{(\beta)}(f_\Upsilon) \leq \Gamma N^{1/\beta} \big(\|V\|_{(1)}(V\, \mathbf{D} f) + \|\delta V\|_{(1)}(f_\Upsilon) \big),$$

where $f_\Upsilon(x) = \operatorname{dist}(f(x), \Upsilon)$ *for* $x \in \operatorname{dmn} f$.

If $Y = \mathbf{R}$, the approach of Hutchinson, see [Hut90, Theorem 3], carries over unchanged and, in fact, yields a somewhat sharper estimate, see 10.9 (1). If $\dim Y \geq 2$ and $N \geq 2$ the selection procedure for Υ is more delicate.

In order to precisely state the next result, the concept of approximate tangent vectors and approximate differentiability (see [Fed69, 3.2.16]) will be recalled.

Definition. Suppose μ measures an open subset U of a normed vectorspace X, $a \in U$, and m is a positive integer.

Then $\operatorname{Tan}^m(\mu, a)$ denotes the closed cone of (μ, m) *approximate tangent vectors* at a consisting of all $v \in X$ such that

$$\begin{gathered} \boldsymbol{\Theta}^{*m}(\mu \llcorner \mathbf{E}(a, v, \varepsilon), a) > 0 \quad \text{for every } \varepsilon > 0, \\ \text{where } \mathbf{E}(a, v, \varepsilon) = X \cap \{x : |r(x-a) - v| < \varepsilon \text{ for some } r > 0\}. \end{gathered}$$

Moreover, if X is an inner product space, then the cone of (μ, m) *approximate normal vectors* at a is defined to be

$$\operatorname{Nor}^m(\mu, a) = X \cap \{u : u \bullet v \leq 0 \text{ for } v \in \operatorname{Tan}^m(\mu, a)\}.$$

If V is an m dimensional rectifiable varifold in an open subset U of $\mathbf{R}^n$, then at $\|V\|$ almost all a, $T = \operatorname{Tan}^m(\|V\|, a)$ is an m dimensional plane such that

$$r^{-m} \textstyle\int f(r^{-1}(x-a)) \,\mathrm{d}\|V\| x \to \boldsymbol{\Theta}^m(\|V\|, a) \int_T f \,\mathrm{d}\mathscr{H}^m \quad \text{as } r \to 0+$$

whenever $f : U \to \mathbf{R}$ is a continuous function with compact support. However, at individual points the requirement that $\operatorname{Tan}^m(\|V\|, a)$ forms an m dimensional plane differs from the condition that $\|V\|$ admits an m dimensional approximate tangent plane in the sense of Simon [Sim83, 11.8].

Definition. Suppose μ, U, a and m are as in the preceding definition and f maps a subset of X into another normed vectorspace Y.

Then f is called (μ, m) *approximately differentiable* at a if and only if there exist $b \in Y$ and a continuous linear map $L : X \to Y$ such that

$$\boldsymbol{\Theta}^m(\mu \llcorner X \sim \{x : |f(x) - b - L(x-a)| \leq \varepsilon |x-a|\}, a) = 0 \quad \text{for every } \varepsilon > 0.$$

In this case $L|\operatorname{Tan}^m(\mu,a)$ is unique and it is called the (μ,m) *approximate differential* of f at a, denoted

$$(\mu,m)\operatorname{ap} Df(a).$$

Also, the following notation will be convenient, see Almgren [Alm00, T.1 (9)].

Definition. Whenever P is an m dimensional plane in $\mathbf{R}^n$, the orthogonal projection of $\mathbf{R}^n$ onto P will be denoted by $P_\natural$.

The Sobolev Poincaré inequality with zero median and a suitably chosen number Q is the key to prove the following structural result for weakly differentiable functions.

Approximate differentiability, see 11.2. *Suppose m, n, U, and V satisfy the general hypothesis and the density hypothesis, Y is a finite dimensional normed vectorspace, and $f \in \mathbf{T}(V,Y)$.*

Then f is $(\|V\|,m)$ approximately differentiable with

$$V\,\mathbf{D}f(a) = (\|V\|,m)\operatorname{ap} Df(a) \circ \operatorname{Tan}^m(\|V\|,a)_\natural$$

at $\|V\|$ almost all a.

The preceding assertion consists of two parts. Firstly, it yields that

$$V\,\mathbf{D}f(a)|\operatorname{Nor}^m(\|V\|,a) = 0 \quad \text{for } \|V\| \text{ almost all } a;$$

a property that is not required by the definition. Secondly, it asserts that

$$V\,\mathbf{D}f(a)|\operatorname{Tan}^m(\|V\|,a) = (\|V\|,m)\operatorname{ap} Df(a) \quad \text{for } \|V\| \text{ almost all } a.$$

This is somewhat similar to the situation for the generalised mean curvature of an integral m varifold where it was first obtained by Brakke in [Bra78, 5.8] that its tangential component vanishes and secondly by the author in [Men13, 4.8] that its normal component is induced by approximate quantities.

Differentiability in Lebesgue spaces, see 11.4 (1). *Suppose m, n, U, and V satisfy the general hypothesis and the density hypothesis, Y is a finite dimensional normed vectorspace, $f \in \mathbf{T}(V,Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|\delta V\|,Y)$, and $V\,\mathbf{D}f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|,\operatorname{Hom}(\mathbf{R}^n,Y))$.*

If $m>1$ and $\beta = m/(m-1)$, then

$$\lim_{r\to 0+} r^{-m} \textstyle\int_{\mathbf{B}(a,r)} (|f(x)-f(a)-V\,\mathbf{D}f(a)(x-a)|/|x-a|)^\beta \,\mathrm{d}\|V\|x = 0$$

for $\|V\|$ almost all a.

The result is derived from the approximate differentiability result mainly by means of the zero median version of the Sobolev Poincaré inequality. Another consequence of the approximate differentiability result is the rectifiability of the distributional boundary of almost all superlevel sets of weakly differentiable functions which supplements the functional analytic form of the coarea formula.

Coarea formula, measure theoretic form, see 12.2. *Suppose m, n, U, and V satisfy the general hypothesis and the density hypothesis, $f \in \mathbf{T}(V,\mathbf{R})$, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.*

Then there exists an $\mathscr{L}^1$ measurable function W with values in the weakly topologised space of $m-1$ dimensional rectifiable varifolds in U such that for $\mathscr{L}^1$ almost all y there holds

$$\operatorname{Tan}^{m-1}(\|W(y)\|, x) = \operatorname{Tan}^m(\|V\|, x) \cap \ker V\,\mathbf{D}f(x) \in \mathbf{G}(n, m-1),$$
$$\boldsymbol{\Theta}^{m-1}(\|W(y)\|, x) = \boldsymbol{\Theta}^m(\|V\|, x)$$

for $\|W(y)\|$ almost all x and

$$V\,\partial E(y)(\theta) = \textstyle\int |V\,\mathbf{D}f(x)|^{-1} V\,\mathbf{D}f(x)(\theta(x))\,\mathrm{d}\|W(y)\|x \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n),$$

in particular $\|V\,\partial E(y)\| = \|W(y)\|$ for such y.

The proof of an appropriate form of the coarea formula was the original motivation for the author to establish the new relative isoperimetric inequality and its corresponding Sobolev Poincaré inequalities as well as the approximate differentiability result. In this respect notice that the proof of the measure theoretic form of the coarea formula could not be based on the more elementary functional analytic form of the coarea formula in conjunction with an extension of the Gauss Green theorem (see [Fed69, 4.5.6]) to sets whose distributional V boundary is representable by integration. In fact, for general sets E whose distributional V boundary is representable by integration it may happen that there is no $m-1$ dimensional rectifiable varifold whose weight measure equals $\|V\,\partial E\|$, see 12.3. This is in contrast to the behaviour of sets of locally finite perimeter in Euclidean space.

Critical mean curvature

Several of the preceding estimates and structural results may be sharpened in case the generalised mean curvature satisfies an appropriate summability condition.

Mean curvature hypothesis. *Suppose m, n, U and V are as in the general hypothesis and satisfies the following condition.*

If $m > 1$ then for some $h \in \mathbf{L}_m^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ there holds

$$(\delta V)(\theta) = -\textstyle\int h \bullet \theta\,\mathrm{d}\|V\| \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

In this case ψ will denote the Radon measure over U characterised by the condition $\psi(X) = \int_X |h|^m\,\mathrm{d}\|V\|$ whenever X is a Borel subset of U.

Clearly, if this condition is satisfied, then the function h is $\|V\|$ almost equal to the generalised mean curvature vector $\mathbf{h}(V, \cdot)$ of V. The exponent m is "critical" with respect to homothetic rescaling of the varifold. Replacing it by a slightly larger number would entail upper semicontinuity of $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ and the applicability of Allard's regularity theory, see Allard [All72, §8]. In contrast, replacing the exponent by a slightly smaller number would allow for examples in which the varifold is locally highly disconnected, see the author [Men09, 1.2].

The importance of this hypothesis for the present considerations lies in the fact that in the relative isoperimetric inequality, the summand "$\|\delta V\|(U)$" may be dropped provided V satisfies additionally the mean curvature hypothesis with $\psi(E)^{1/m} \leq \Lambda^{-1}$, see 7.11. As a first consequence, one obtains the following version of the Sobolev Poincaré inequality.

Sobolev Poincaré inequality, zero median, critical mean curvature, see 8.16, 9.2, 10.1 (1c) (1d)**.** *Suppose* $1 \leq M < \infty$.

Then there exists a positive, finite number Γ *with the following property.*

If m, n, U, V, *and* ψ *satisfy the general hypothesis, the density hypothesis and the mean curvature hypothesis,* $n \leq M$, Y *is a finite dimensional normed vectorspace,* $f \in \mathbf{T}(V, Y)$, $1 \leq Q \leq M$, $0 < r < \infty$, $E = U \cap \{x : f(x) \neq 0\}$,

$$\|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \quad \psi(E) \leq \Gamma^{-1},$$
$$\|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m,$$

and $A = \{x : \mathbf{U}(x, r) \subset U\}$, *then the following two statements hold:*

(1) *If* $1 \leq q < m$, *then*

$$(\|V\| \llcorner A)_{(mq/(m-q))}(f) \leq \Gamma (m-q)^{-1} \|V\|_{(q)}(V\,\mathbf{D}f).$$

(2) *If* $1 < m < q \leq \infty$, *then*

$$(\|V\| \llcorner A)_{(\infty)}(f) \leq \Gamma^{1/(1/m-1/q)} \|V\|(E)^{1/m-1/q} \|V\|_{(q)}(V\,\mathbf{D}f).$$

Even if $Q = q = 1$ and f and V correspond to smooth objects, this estimate appears to be new; at least, it seems not to be straightforward to derive from Hutchinson [Hut90, Theorem 1].

A similar result holds for $m = 1$, see 10.1 (1b). Again, if f belongs to $\mathbf{T}_G(V)$, then A may be replaced by A'. Also, appropriate versions of the Sobolev inequality may be furnished, see 10.1 (2b)–(2d). The same is true with respect to the Sobolev Poincaré inequalities with several medians, see 10.7 (2)–(4) and 10.9 (2)–(4).

Before stating the stronger differentiability properties that result from the mean curvature hypothesis, recall the definition of relative differential from [Fed69, 3.1.21, 3.1.22].

Definition. Suppose X and Y are normed vectorspaces, $A \subset X$, and $a \in \operatorname{Clos} A$, and $f : A \to Y$.

Then the *tangent cone* of A at a, denoted $\operatorname{Tan}(A, a)$, is the set of all $v \in X$ such that for every $\varepsilon > 0$ there exist $x \in A$ and $0 < r \in \mathbf{R}$ with $|x - a| < \varepsilon$ and $|r(x-a) - v| < \varepsilon$. Moreover, f is called *differentiable relative to* A *at* a if and only if there exist $b \in Y$ and a continuous linear map $L : X \to Y$ such that

$$|f(x) - b - L(x-a)|/|x-a| \to 0 \quad \text{as } A \ni x \to a.$$

In this case $L|\operatorname{Tan}(A, a)$ is unique and denoted $Df(a)$.

Differentiability in Lebesgue spaces, critical mean curvature, see 11.4 (2)–(4)**.** *Suppose* m, n, U, *and* V *satisfy the general hypothesis, the density hypothesis and the mean curvature hypothesis,* Y *a finite dimensional normed vectorspace,* $f \in \mathbf{T}(V, Y)$, $1 \leq q \leq \infty$, *and* $V\,\mathbf{D}f \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$.

Then the following two statements hold.

(1) *If* $q < m$ *and* $\iota = mq/(m-q)$, *then*

$$\lim_{r \to 0+} r^{-m} {\textstyle\int_{\mathbf{B}(a,r)}} (|f(x) - f(a) - V\,\mathbf{D}f(a)(x-a)|/|x-a|)^\iota \,\mathrm{d}\|V\| x = 0$$

for $\|V\|$ *almost all* a.

(2) *If $m = 1$ or $m < q$, then there exists a subset A of U with $\|V\|(U \sim A) = 0$ such that $f|A$ is differentiable relative to A at a with*

$$D(f|A)(a) = V\,\mathbf{D}f(a)|\operatorname{Tan}^m(\|V\|, a) \quad \textit{for } \|V\| \textit{ almost all } a,$$

in particular $\operatorname{Tan}(A, a) = \operatorname{Tan}^m(\|V\|, a)$ *for such a.*

Considering V associated to two crossing lines, it is evident that one cannot expect a function in $\mathbf{T}(V, \mathbf{R})$ to be $\|V\|$ almost equal to a continuous function even if $\delta V = 0$ and $V\,\mathbf{D}f = 0$. Yet, in case f is continuous, its modulus of continuity may be locally estimated by its weak derivative. This estimate depends on V but not on f as formulated in the next theorem.

Oscillation estimate, see 13.1 (2)**.** *Suppose m, n, U, and V satisfy the general hypothesis, the density hypothesis and the mean curvature hypothesis, K is a compact subset of U, $0 < \varepsilon \leq \operatorname{dist}(K, \mathbf{R}^n \sim U)$, $\varepsilon < \infty$, and $1 < m < q$.*

Then there exists a positive, finite number Γ with the following property.

If Y is a finite dimensional normed vectorspace, $f : \operatorname{spt}\|V\| \to Y$ is a continuous function, $f \in \mathbf{T}(V, Y)$, and $\kappa = \sup\{(\|V\| \llcorner \mathbf{U}(a, \varepsilon))_{(q)}(V\,\mathbf{D}f) : a \in K\}$, then

$$|f(x) - f(\chi)| \leq \varepsilon\kappa \quad \textit{whenever } x, \chi \in K \cap \operatorname{spt}\|V\| \textit{ and } |x - \chi| \leq \Gamma^{-1}.$$

A similar result holds for $m = 1$, see 13.1 (1). This theorem rests on the fact that connected components of $\operatorname{spt}\|V\|$ are relatively open in $\operatorname{spt}\|V\|$, see 6.14 (3). If a varifold V satisfies the general hypothesis, the density hypothesis and the mean curvature hypothesis then the decomposition of $\operatorname{spt}\|V\|$ into connected components yields a locally finite decomposition into relatively open and closed subsets whose distributional V boundary vanishes, see 6.14 (2)–(4). Moreover, any decomposition of the varifold V will refine the decomposition of the topological space $\operatorname{spt}\|V\|$ into connected components, see 6.14 (1).

When the amount of total weight measure (“mass”) available excludes the possibility of two separate sheets, the oscillation estimate may be sharpened to yield Hölder continuity even without assuming a priori the continuity of the function, see 13.3.

Two applications

Despite the necessarily rather weak oscillation estimate in the general case, this estimate is still sufficient to prove that the geodesic distance between any two points in the same connected component of the support is finite.

Geodesic distance, see 14.2. *Suppose m, n, U, and V satisfy the general hypothesis, the density hypothesis and the mean curvature hypothesis, C is a connected component of* $\operatorname{spt}\|V\|$*, and $a, x \in C$.*

Then there exist $-\infty < b \leq y < \infty$ and a Lipschitzian function $g : \{\upsilon : b \leq \upsilon \leq y\} \to \operatorname{spt}\|V\|$ such that $g(b) = a$ and $g(y) = x$.

The proof follows a pattern common in theory of metric spaces, see 14.3.

Finally, the presently introduced notion of weak differentiability may also be used to reformulate the defining condition for curvature varifolds, see 15.4 and 15.5, introduced by Hutchinson in [Hut86, 5.2.1].

Characterisation of curvature varifolds, see 15.6. *Suppose m and n are positive integers, $m \leq n$, U is an open subset of $\mathbf{R}^n$, V is an m dimensional integral varifold in U,*

$$X = U \cap \{x : \operatorname{Tan}^m(\|V\|, x) \in \mathbf{G}(n,m)\}, \quad Y = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{\sigma : \sigma = \sigma^*\},$$

and $\tau : X \to Y$ satisfies

$$\tau(x) = \operatorname{Tan}^m(\|V\|, x)_\natural \quad \textit{whenever } x \in X.$$

Then V is a curvature varifold if and only if $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$ and $\tau \in \mathbf{T}(V, Y)$.

The condition is readily verified to be a necessary one for curvature varifolds. The key to prove its sufficiency is to relate the mean curvature vector of V to the weak differential of the tangent plane map τ. This may be accomplished, for instance, by means of the approximate differentiability result, see 11.2, applied to τ, in conjunction with the previously obtained second order rectifiability result for such varifolds, see [Men13, 4.8].

Possible lines of further study

Sobolev spaces. The results obtained by the author on the area formula for the Gauss map, see [Men12b, Theorem 3], rest on several estimates for Lipschitzian solutions of certain linear elliptic equations on varifolds satisfying the mean curvature hypothesis. The formulation of these estimates necessitated several ad hoc formulations of concepts such as zero boundary values which would not seem natural from the point of view of partial differential equations. In order to avoid repetition, the author decided to directly build an adequate framework for these results rather than first publish the Lipschitzian case along with a proof of the results announced in [Men12b].[5] The first part of such framework is provided in the present paper. Continuing this programme, a notion of Sobolev spaces, complete vectorspaces contained in $\mathbf{T}(V, Y)$ in which locally Lipschitzian functions are suitably dense, has already been developed but is not included here for length considerations.

Functions of locally bounded variation. It seems worthwhile to investigate to which extent results of the present paper for weakly differentiable functions extend to a class of "functions of locally bounded variation." A possible definition is suggested in 8.10.

Intermediate conditions on the mean curvature. The mean curvature hypothesis may be weakened by replacing $\mathbf{L}_m^{\mathrm{loc}}$ by $\mathbf{L}_p^{\mathrm{loc}}$ for some $1 < p < m$. In view of the Sobolev Poincaré type inequalities obtained for height functions by the author in [Men10, Theorem 4.4], one might seek for adequate formulations of the Sobolev Poincaré inequalities and the resulting differentiability results for functions belonging to $\mathbf{T}(V, Y)$ in these intermediate cases. This could potentially have applications to structural results for curvature varifolds, see 15.11 and 15.12. Additionally, the case $p = m - 1$ seems to be related to the study of the geodesic distance for indecomposable varifolds, see 14.4.

[5]Would the author have fully anticipated the effort needed for such a construction he might have reconciled himself with some repetition.

Multiple valued weakly differentiable functions. For convergence considerations it appears useful to extend the concept of weakly differentiable functions to a more general class of "multiple-valued" functions in the spirit of Moser [Mos01, §4], see 8.11.

Organisation of the paper

In Section 1 the notation is introduced. In Section 2 some basic terminology and results on topological vector spaces are collected. Sections 3 and 4 contain preliminary results concerning respectively certain distributions representable by integration and consequences of the monotonicity identity. Section 5 introduces the concept of distributional boundary of a set with respect to a varifold. In Section 6 the decomposition properties for varifolds are established and Section 7 contains the relative isoperimetric inequality. In Sections 8–13 the theory of weakly differentiable functions is presented. Finally, in Sections 14 and 15 the applications to the study of the geodesic distance associated to certain varifolds and to curvature varifolds are discussed briefly.

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1 Notation

The notation of Federer [Fed69] and Allard [All72] will be used throughout.

Less common symbols. The set of positive integers is denoted by $\mathscr{P}$, see [Fed69, 2.2.6]. For the open and closed ball with centre a and radius r the symbols $\mathbf{U}(a,r)$ and $\mathbf{B}(a,r)$ are employed, see [Fed69, 2.8.1]. Whenever f is a linear map and v belongs to its domain the alternate notation $\langle v, f\rangle$ for $f(v)$ are used, see [Fed69, 1.10.1]. Inner products, in contrast, are denoted by "$\bullet$", see [Fed69, 1.7.1]. For integration the alternate notations $\int f \,\mathrm{d}\mu$, $\int f(x)\,\mathrm{d}\mu x$ and $\mu(f)$ are employed, see [Fed69, 2.4.2]. Moreover, for evaluating a distribution T at ϕ are alternately denoted by $T(\phi)$ and $T_x(\phi(x))$, see [Fed69, 4.1.1].

Modifications. If f is a relation, then $f[A] = \{y : (x,y) \in f \text{ for some } x \in A\}$ whenever A is a set, see Kelley [Kel75, p. 8]. Following Almgren [Alm00, T.1 (9)], the symbol $P_\natural$ will denote the orthogonal projection of $\mathbf{R}^n$ onto P whenever P is a plane in $\mathbf{R}^n$. Extending Federer [Fed69, 3.2.16], whenever μ measures an open subset of a normed vectorspace X, $\iota : U \to X$ is the inclusion map, $a \in U$ and m is a positive integer notions of tangent vectors, normal vectors and differentials of (μ, m) approximate type will refer to the corresponding $(\iota_\# \mu, m)$ approximate notion, for instance $\operatorname{Tan}^m(\mu, a)$ will denote $\operatorname{Tan}^m(\iota_\#\mu, a)$. Following Schwartz, see [Sch66, Chapitre III, §1], the vectorspace $\mathscr{D}(U,Y)$ is given the (usual) locally convex topology, see 2.10, which differs from the topology employed by Federer in [Fed69, 4.1.1], see 2.14. Moreover, the usage of $\|S\|$ for distributions S is chosen to be in accordance with Allard [All72, 4.2] which agrees with Federer's usage in [Fed69, 4.1.5] in most but not in all cases, see 2.15 and 2.16.

Extending Allard [All72, 2.5 (2)], whenever M is a submanifold of $\mathbf{R}^n$ of class 2 and $a \in M$ the *mean curvature vector of* M *at* a is the unique $\mathbf{h}(M,a) \in \operatorname{Nor}(M,a)$ such that

$$\operatorname{Tan}(M,a)_\natural \bullet (Dg(a) \circ \operatorname{Tan}(M,a)_\natural) = -g(a) \bullet \mathbf{h}(M,a)$$

whenever $g : M \to \mathbf{R}^n$ is of class 1 and $g(x) \in \operatorname{Nor}(M,x)$ for $x \in M$. If V is an m dimensional varifold in U and $\|\delta V\|$ is a Radon measure, then the *generalised mean curvature vector of* V *at* x is the unique $\mathbf{h}(V,x) \in \mathbf{R}^n$ such that

$$\mathbf{h}(V,x) \bullet u = -\lim_{r \to 0+} \frac{(\delta V)(b_{x,r} \cdot u)}{\|V\| \, \mathbf{B}(x,r)} \quad \text{for } u \in \mathbf{R}^n,$$

where $b_{x,r}$ is the characteristic function of $\mathbf{B}(x,r)$; hence $x \in \operatorname{dmn} \mathbf{h}(V,\cdot)$ if and only if the above limit exists for every $u \in \mathbf{R}^n$, see [Men12a, p. 9].

Additional notation. If $1 \le p \le \infty$, μ is a Radon measure over a locally compact Hausdorff space X, and Y is a Banach space, then $\mathbf{L}_p^{\mathrm{loc}}(\mu, Y)$ consists of all f such that $f \in \mathbf{L}_p(\mu \llcorner K, Y)$ whenever K is a compact subset of X. Concerning the Besicovitch Federer covering theorem, whenever $n \in \mathscr{P}$ the number $\boldsymbol{\beta}(n)$ denotes the least positive integer with the following property, see Almgren [Alm86, p. 464]: If G is a family of closed balls in $\mathbf{R}^n$ with $\sup\{\operatorname{diam} B : B \in G\} < \infty$, then there exist disjointed subfamilies $G_1, \ldots, G_{\boldsymbol{\beta}(n)}$ such that

$$\{x : \mathbf{B}(x,r) \in G \text{ for some } 0 < r < \infty\} \subset \bigcup \bigcup \{G_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}.$$

Concerning the isoperimetric inequality, whenever $m \in \mathscr{P}$ the smallest number with the following property is denoted by $\boldsymbol{\gamma}(m)$, see [Men09, 2.2–2.4]: If $n \in \mathscr{P}$, $m \le n$, $V \in \mathbf{RV}_m(\mathbf{R}^n)$, $\|V\|(\mathbf{R}^n) < \infty$, and $\|\delta V\|(\mathbf{R}^n) < \infty$, then

$$\|V\|(\mathbf{R}^n \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) \ge 1\}) \le \boldsymbol{\gamma}(m) \|V\|(\mathbf{R}^n)^{1/m} \|\delta V\|(\mathbf{R}^n).$$

Definitions in the text. The notions of *Lusin space*, *locally convex space* and *locally convex topology*, *strict inductive limit* as well as *final topology* and *locally convex final topology* are defined in 2.1, 2.3, 2.4 and 2.6 respectively. The topologies on $\mathscr{K}(X)$ and $\mathscr{D}(U,Y)$ are defined in 2.7 and 2.10. The *restriction* of a distribution representable by integration to a set will be defined in 2.17. In 5.1, the notion of *distributional boundary* of a set with respect to certain varifolds is defined. The notions of *indecomposability*, *component*, and *decomposition* for certain varifolds are defined in 6.2, 6.6 and 6.9 respectively. The notion of *generalised weakly differentiable functions* with respect to certain varifolds and the corresponding *generalised weak derivatives*, $V \, \mathbf{D} f$, and the associated space $\mathbf{T}(V,Y)$ are introduced in 8.3. The space $\mathbf{T}_G(V)$ is defined in 9.1. Finally, the notion of *curvature varifold* is explained in 15.4.

A convention. Each statement asserting the existence of a positive, finite number Γ will give rise to a function depending on the listed parameters whose name is $\Gamma_{\mathrm{x.y}}$, where x.y denotes the number of the statement. This is a refinement of a concept employed by Almgren in [Alm00].

2 Topological vector spaces

Some basic results on topological vector spaces and Lusin spaces are gathered here mainly from Schwartz [Sch73] and Bourbaki [Bou87].

2.1 Definition (see [Sch73, Chapter II, Definition 2, p. 94])**.** Suppose X is a topological space.

Then X is called a *Lusin space* if and only if X is a Hausdorff topological space and there exists a complete, separable metric space W and a continuous univalent map $f : W \to X$ whose image is X.

2.2 Remark. Any subset of a Lusin space is sequentially separable.

2.3 Definition (see [Bou87, II, p. 23, def. 1])**.** A topological vector space is called a *locally convex space* if and only if there exists a fundamental system of neighbourhoods of 0 that are convex sets; its topology is called *locally convex topology.*

2.4 Definition. The locally convex spaces form a category; its morphisms are the continuous linear maps. An inductive limit in this category is called *strict* if and only if the morphisms of the corresponding inductive system are homeomorphic embeddings.

2.5 Remark. The notion of inductive limit is employed in accordance with [ML98, p. 67–68].

2.6 Definition (see [Bou98, I, §2.4, prop. 6], [Bou87, II, p. 27, prop. 5])**.** Suppose E is a set [vector space], E_i are topological spaces [topological vector spaces] and $f_i : E_i \to E$ are functions [linear maps] for $i \in I$.

Then there exists a unique topology [unique locally convex topology] on E such that a function [linear map] $g : E \to F$ into a topological space [locally convex space] F is continuous if and only if $g \circ f_i$ are continuous for $i \in I$. This topology is called *final topology* [*locally convex final topology*] on E with respect to the family f_i.

2.7 Definition (see [Bou87, II, p. 29, Example II][6])**.** Suppose X is a locally compact Hausdorff space. Consider the inductive system consisting of the locally convex spaces $\mathscr{K}(X) \cap \{f : \operatorname{spt} f \subset K\}$ with the topology of uniform convergence corresponding to all compact subsets K of U and its inclusion maps.

Then $\mathscr{K}(X)$ endowed with the locally convex final topology with respect to the inclusions of the topological vector spaces $\mathscr{K}(X) \cap \{f : \operatorname{spt} f \subset K\}$ is the inductive limit of the above system in the category of locally convex spaces.

2.8 Remark (see [Bou87, II, p. 29, Example II]). The locally convex topology on $\mathscr{K}(X)$ is Hausdorff and induces the given topology on each closed subspace $\mathscr{K}(X) \cap \{f : \operatorname{spt} f \subset K\}$. Moreover, the space $\mathscr{K}(X)^*$ of Daniell integrals on $\mathscr{K}(X)$ agrees with the space of continuous linear functionals on $\mathscr{K}(X)$.

2.9 Remark. If $K(i)$ is a sequence of compact subsets of X with $K(i) \subset \operatorname{Int} K(i+1)$ for $i \in \mathscr{P}$ and $X = \bigcup_{i=1}^{\infty} K(i)$, then $\mathscr{K}(X)$ is the strict inductive limit of the sequence of locally convex spaces $\mathscr{K}(X) \cap \{f : \operatorname{spt} f \subset K(i)\}$.

[6]In the terminology of [Bou87, II, p. 29, Example II] a locally compact Hausdorff space is called a locally compact space.

2.10 Definition. Suppose U is an open subset of $\mathbf{R}^n$ and Z is a Banach space. Consider the inductive system consisting of the locally convex spaces $\mathscr{D}_K(U,Z)$ with the topology induced from $\mathscr{E}(U,Z)$ corresponding to all compact subsets K of U and its inclusion maps.

Then $\mathscr{D}(U,Z)$ endowed with the locally convex final topology with respect to the inclusions of the topological vector spaces $\mathscr{D}_K(U,Z)$ is the inductive limit of the above inductive system in the category of locally convex spaces.

2.11 Remark. The locally convex topology on $\mathscr{D}(U,Z)$ is Hausdorff and induces the given topology on each closed subspace $\mathscr{D}_K(U,Z)$.

2.12 Remark. If $K(i)$ is a sequence of compact subsets of U with $K(i) \subset \operatorname{Int} K(i+1)$ for $i \in \mathscr{P}$ and $U = \bigcup_{i=1}^\infty K(i)$, then $\mathscr{D}(U,Z)$ is the strict inductive limit of the sequence of locally convex spaces $\mathscr{D}_{K(i)}(U,Z)$. In particular, the convergent sequences in $\mathscr{D}(U,Z)$ are precisely the convergent sequences in some $\mathscr{D}_K(U,Z)$, see [Bou87, III, p. 3, prop. 2; II, p. 32, prop. 9 (i) (ii); III, p. 5, prop. 6].

2.13 Remark. The locally convex topology on $\mathscr{D}(U,Z)$ is also the inductive limit topology in the category of topological vector spaces, see [Bou87, I, p. 9, Lemma to prop. 7; II, p. 75, exerc. 14].

2.14 Remark. Consider the inductive system consisting of the topological spaces $\mathscr{D}_K(U,Z)$ corresponding to all compact subsets K of U and its inclusion maps. Then $\mathscr{D}(U,Z)$ endowed with the final topology with respect to the inclusions of the topological vector spaces $\mathscr{D}_K(U,Z)$ is the inductive limit of this inductive system in the category of topological spaces.

Denoting this topology by S and the topology described in 2.10 by T, the following three statements hold.

(1) Amongst all locally convex topologies on $\mathscr{D}(U,Z)$, the topology T is characterised by the following property: A linear map from $\mathscr{D}(U,Z)$ into some locally convex space is T continuous if and only if it is S continuous.

(2) The S closed sets are precisely the T sequentially closed sets, see 2.12.

(3) If U is nonempty and $\dim Z > 0$, then S is strictly finer than T, compare Shirai [Shi59, Théorème 5].[7] In particular, in this case S is not compatible with the vector space structure of $\mathscr{D}(U,Z)$ by 2.13.

2.15 Definition. Suppose U is an open subset of $\mathbf{R}^n$, Z is a separable Banach space, and $S : \mathscr{D}(U,Z) \to \mathbf{R}$ is linear.

Then $\|S\|$ is defined to be the largest Borel regular measure over U such that

$$\|S\|(G) = \sup\{S(\theta) : \theta \in \mathscr{D}(U,Z) \text{ with } \operatorname{spt}\theta \subset G \text{ and } |\theta(x)| \leq 1 \text{ for } x \in U\}$$

whenever G is an open subset of U.[8]

2.16 Remark. This concept agrees with [Fed69, 4.1.5] in case $\|S\|$ is a Radon measure (which is equivalent to S being a distribution in U of type Z representable by integration) in which case $S(\theta)$ continuous to denote the value of

[7] The topological preliminaries of [Shi59] are quite general. For the purpose of verifying only the cited result, one may replace the terms «topologie» [respectively «véritable-topologie»] with operators satisfying conditions (a), (b) and (d) [respectively all] of the Kuratowski closure axioms, see [Kel75, p. 43]. (Extending [Shi59, Théorème 5] which treats the case $U = \mathbf{R}^n$ and $Z = \mathbf{R}$ to the case stated poses no difficulty.)

[8] Existence may be shown by use of the techniques occurring in [Fed69, 2.5.14].

the unique $\|S\|_{(1)}$ continuous extension of S to $\mathbf{L}_1(\|S\|, Z)$ at $\theta \in \mathbf{L}_1(\|S\|, Z)$. In general the concepts differ since it may happen that $|S(\theta)| > \|S\|(|\theta|)$ for some $\theta \in \mathscr{D}(U, Z)$; an example is provided by taking $U = Z = \mathbf{R}$ and $S = \partial(\boldsymbol{\delta}_0 \wedge \mathbf{e}_1)$, see [Fed69, 4.1.16].

2.17 Definition. Suppose U is an open subset of $\mathbf{R}^n$, Z is a separable Banach space, $S \in \mathscr{D}'(U, Z)$ is representable by integration, and A is $\|S\|$ measurable.

Then the *restriction of S to A*, $S \llcorner A \in \mathscr{D}'(U, Z)$, is defined by

$$(S \llcorner A)(\theta) = S(\theta_A) \quad \text{whenever } \theta \in \mathscr{D}(U, Z),$$

where $\theta_A(x) = \theta(x)$ for $x \in A$ and $\theta_A(x) = 0$ for $x \in U \sim A$.

2.18 Remark. This extends the notion for currents in [Fed69, 4.1.7].

2.19 Theorem. *Suppose E is the strict inductive limit of an increasing sequence of locally convex spaces E_i and the spaces E_i are separable, complete (with respect to its topological vector space structure) and metrisable for $i \in \mathscr{P}$.*[9]

Then E and the dual E' of E endowed with the compact open topology are Lusin spaces and whenever D is a dense subset of E the Borel family of E' is generated by the family

$$\big\{E' \cap \{u : u(x) < t\} : x \in D \text{ and } t \in \mathbf{R}\big\}.$$

Proof. First, note that E_i is a Lusin space for $i \in \mathscr{P}$, since it may be metrised by a translation invariant metric by [Bou87, II, p. 34, prop. 11]. Second, note that E is Hausdorff and induces the given topology on E_i by [Bou87, II, p. 32, prop. 9 (i)]. Therefore E is a Lusin space by [Sch73, Chapter II, Corollary 2 to Theorem 5, p. 102]. Since every compact subset K of E is a compact subset of E_i for some i by [Bou87, III, p. 3, prop. 2; II, p. 32, prop. 9 (ii); III, p. 5, prop. 6], it follows that E' is a Lusin space from [Sch73, Chapter 2, Theorem 7, p. 112]. Defining the continuous, univalent map $\iota : E' \to \mathbf{R}^D$ by $\iota(u) = u|D$ for $u \in E'$, the initial topology on E' induced by ι is a Hausdorff topology coarser than the compact open topology, hence their Borel families agree by [Sch73, Chapter II, Corollary 2 to Theorem 4, p. 101]. ☐

2.20 Example. Suppose X is a locally compact Hausdorff space which admits a countable base of its topology.

Then $\mathscr{K}(X)$ with its locally convex topology (see 2.7) and $\mathscr{K}(X)^*$ with its weak topology are Lusin spaces and the Borel family of $\mathscr{K}(X)^*$ is generated by the sets $\mathscr{K}(X)^* \cap \{\mu : \mu(f) < t\}$ corresponding to $f \in \mathscr{K}(X)$ and $t \in \mathbf{R}$. Moreover, the topological space

$$\mathscr{K}(X)^* \cap \{\mu : \mu \text{ is monotone}\}$$

is homeomorphic to a complete, separable metric space;[10] in fact, choosing a countable dense subset D of $\mathscr{K}(X)^+$, the image of its homeomorphic embedding into $\mathbf{R}^D$ is closed.

[9] That is, E_i are separable Fréchet spaces in the terminology of [Bou87, II, p. 24].
[10] Such spaces are termed polish spaces in [Sch73, Chapter II, Definition 1].

2.21 Example. Suppose U is an open subset of $\mathbf{R}^n$ and Z is a separable Banach space.

Then $\mathscr{D}(U,Z)$ with its locally convex topology (see 2.10) and $\mathscr{D}'(U,Z)$ with its weak topology are Lusin spaces and the Borel family of $\mathscr{D}'(U,Z)$ is generated by the sets $\mathscr{D}'(U,Z) \cap \{S : S(\theta) < t\}$ corresponding to $\theta \in \mathscr{D}(U,Z)$ and $t \in \mathbf{R}$. Moreover, recalling 2.2 and [Fed69, 4.1.5], it follows that

$$\big(\mathscr{D}'(U,Z) \times \mathscr{K}(U)^*\big) \cap \{(S, \|S\|) : S \text{ is representable by integration}\}$$

is a Borel function whose domain is a Borel set (with respect to the weak topology on both spaces).

3 Distributions on products

The purpose of the present section is to separate functional analytic considerations from those employing properties of the varifold or the weakly differentiable function in deriving the various coarea type formulae in 8.1, 8.29, 12.1, and 12.2.

3.1. Suppose U and V are open subsets of Euclidean spaces and Z is a Banach space. Then the image of $\mathscr{D}(U,\mathbf{R}) \otimes \mathscr{D}(V,\mathbf{R}) \otimes Z$ in $\mathscr{D}(U \times V, Z)$ is sequentially dense, compare [Fed69, 1.1.3, 4.1.2, 4.1.3].

3.2. If J is an open subset of $\mathbf{R}$, $R \in \mathscr{D}'(J,\mathbf{R})$ is representable by integration, and $\|R\|$ is absolutely continuous with respect to $\mathscr{L}^1 | \mathbf{2}^J$, then there exists $k \in \mathbf{L}_1^{\mathrm{loc}}(\mathscr{L}^1 | \mathbf{2}^J)$ such that

$$R(\omega) = \textstyle\int_J \omega k \, \mathrm{d}\mathscr{L}^1 \quad \text{whenever } \omega \in \mathbf{L}_1(\|R\|),$$
$$k(y) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} R(i_{y,\varepsilon}) \quad \text{for } \mathscr{L}^1 \text{ almost all } y \in J,$$

where $i_{y,\varepsilon}$ is the characteristic function of the interval $\{\upsilon : y < \upsilon \leq y + \varepsilon\}$; in fact, localising the problem, one employs [Fed69, 2.5.8] to construct k satisfying the first condition which implies the second one by [Fed69, 2.8.17, 2.9.8].

3.3 Lemma. *Suppose $n \in \mathscr{P}$, μ is a Radon measure over an open subset U of $\mathbf{R}^n$, J is an open subset of $\mathbf{R}$, f is a μ measurable real valued function, $A = f^{-1}[J]$, F is a μ measurable $\mathrm{Hom}(\mathbf{R}^n, \mathbf{R})$ valued function with*

$$\textstyle\int_{K \cap \{x : f(x) \in I\}} |F| \, \mathrm{d}\mu < \infty$$

whenever K is a compact subset of U and I is a compact subset of J, and $T \in \mathscr{D}'(U \times J, \mathbf{R}^n)$ satisfies

$$T(\phi) = \textstyle\int_A \langle \phi(x, f(x)), F(x) \rangle \, \mathrm{d}\mu x \quad \text{for } \phi \in \mathscr{D}(U \times J, \mathbf{R}^n)$$

Then T is representable by integration and

$$T(\phi) = \textstyle\int_A \langle \phi(x, f(x)), F(x) \rangle \, \mathrm{d}\mu x, \quad \int g \, \mathrm{d}\|T\| = \int_A g(x, f(x)) |F(x)| \, \mathrm{d}\mu x$$

whenever $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function.

Proof. Define $p : U \times J \to U$ by

$$p(x, y) = x \quad \text{for } (x, y) \in U \times J.$$

Define the measure ν over U by $\nu(B) = \int^*_{A \cap B} |F| \,\mathrm{d}\mu$ for $B \subset U$. Let $G : A \to U \times J$ be defined by $G(x) = (x, f(x))$ for $x \in A$. Employing [Fed69, 2.2.2, 2.2.3, 2.4.10] yields that $\nu \llcorner \{x : f(x) \in I\}$ is a Radon measure whenever I is a compact subset of J, hence so is the measure $G_\#(\nu|\mathbf{2}^A)$ over $U \times J$ by [Fed69, 2.2.2, 2.2.3, 2.2.17, 2.3.5]. One deduces that

$$T(\phi) = \int \langle \phi, |F \circ p|^{-1} F \circ p \rangle \,\mathrm{d}G_\#(\nu|\mathbf{2}^A) \quad \text{for } \phi \in \mathscr{D}(U \times J, \mathbf{R}^n);$$

in fact, $F|A = F \circ p \circ G$, hence both sides equal $\int \langle \phi \circ G, |F|^{-1} F \rangle \,\mathrm{d}\nu$ by [Fed69, 2.4.10, 2.4.18 (1)]. Consequently, $\|T\| = G_\#(\nu|\mathbf{2}^A)$ and the conclusion follows by means of [Fed69, 2.2.2, 2.2.3, 2.4.10, 2.4.18 (1)]. □

3.4 Theorem. *Suppose U is an open subset of $\mathbf{R}^n$, J is an open subset of $\mathbf{R}$, Z is a separable Banach space, $T \in \mathscr{D}'(U \times J, Z)$ is representable by integration, $R_\theta \in \mathscr{D}'(J, \mathbf{R})$ satisfy*

$$R_\theta(\omega) = T_{(x,y)}(\omega(y)\theta(x)) \quad \textit{whenever } \omega \in \mathscr{D}(J, \mathbf{R}) \textit{ and } \theta \in \mathscr{D}(U, Z).$$

and $S(y) : \mathscr{D}(U, Z) \to \mathbf{R}$ satisfy, see 3.2,

$$S(y)(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} R_\theta(i_{y,\varepsilon}) \in \mathbf{R} \quad \textit{for } \theta \in \mathscr{D}(U, Z)$$

whenever $y \in J$, that is $y \in \operatorname{dmn} S$ if and only if the limit exists and belongs to $\mathbf{R}$ for $\theta \in \mathscr{D}(U, Z)$.

Then S is an $\mathscr{L}^1 \llcorner J$ measurable function with respect to the weak topology on $\mathscr{D}'(U, Z)$ and the following two statements hold.

(1) If g is an $\{u : 0 \leq u \leq \infty\}$ valued $\|T\|$ measurable function, then

$$\textstyle\int_J \int g(x, y) \,\mathrm{d}\|S(y)\| x \,\mathrm{d}\mathscr{L}^1 y \leq \int g \,\mathrm{d}\|T\|.$$

(2) If $\|R_\theta\|$ is absolutely continuous with respect to $\mathscr{L}^1|\mathbf{2}^J$ for $\theta \in \mathscr{D}(U, Z)$, then

$$T(\phi) = \textstyle\int_J S(y)(\phi(\cdot, y)) \,\mathrm{d}\mathscr{L}^1 y, \quad \int_J g \,\mathrm{d}\|T\| = \int_J \int g(x, y) \,\mathrm{d}\|S(y)\| x \,\mathrm{d}\mathscr{L}^1 y.$$

whenever $\phi \in \mathbf{L}_1(\|T\|, Z)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function.

Proof. Define $p : U \times J \to \mathbf{R}$ and $q : U \times J \to U$ by

$$p(x, y) = x \quad \text{and} \quad q(x, y) = y \quad \text{for } (x, y) \in U \times J.$$

First, one derives that S is an $\mathscr{L}^1 \llcorner J$ measurable function with values in

$$\mathscr{D}'(U, Z) \cap \{\Sigma : \Sigma \text{ is representable by integration}\},$$

where the weak topology on $\mathscr{D}'(U, Z)$ is employed; in fact, choosing countable, sequentially dense subsets C and D of $\mathscr{K}(U)^+$ and $\mathscr{D}(U, Z)$ respectively (see 2.2, 2.20, and 2.21) and noting that $S(y)$ belongs to set in question whenever

$$\lim_{\varepsilon \to 0+} \varepsilon^{-1} \|T\|((f \circ p)(i_{y,\varepsilon} \circ q)) \in \mathbf{R}, \quad \lim_{\varepsilon \to 0+} \varepsilon^{-1} R_\theta(i_{y,\varepsilon}) \in \mathbf{R}$$

for $f \in C$ and $\theta \in D$ by means of 2.12, the assertion follows from [Fed69, 2.9.19] and 2.21.

In order to prove (1), one may assume $g \in \mathscr{K}(U \times J)^+$. Suppose $f \in \mathscr{K}(U)^+$, $h \in \mathscr{K}(J)^+$, and $g = (f \circ p)(h \circ q)$ and β denotes the Radon measure over $U \times J$ defined by $\beta(B) = \int_B^* f \circ p \, \mathrm{d}\|T\|$ for $B \subset U \times J$. Noting

$$\begin{gathered}\|S(y)\|(f) \leq \mathbf{D}(q_\# \beta, \mathscr{L}^1 | \mathbf{2}^J, V, y) \quad \text{for } \mathscr{L}^1 \text{ almost all } y \in J, \\ \text{where } V = \{(b, I) : b \in I \subset J \text{ and } I \text{ is a compact interval}\}\end{gathered}$$

and employing [Fed69, 2.4.10, 2.8.17, 2.9.7] and 2.21, one infers that

$$\begin{aligned}\textstyle\int_J \|S(y)\|(f) h(y) \, \mathrm{d}\mathscr{L}^1 y &\leq \textstyle\int_J \mathbf{D}(q_\# \beta, \mathscr{L}^1 | \mathbf{2}^J, V, y) h(y) \, \mathrm{d}\mathscr{L}^1 y \\ &\leq (q_\# \beta)(h) = \|T\|(g).\end{aligned}$$

An arbitrary $g \in \mathscr{K}(U \times J)^+$ may be approximated by a sequence of functions which are nonnegative linear combinations of functions of the previously considered type, compare [Fed69, 4.1.2, 4.1.3].

To prove the first equation in (2), it is sufficient to exhibit a sequentially dense subset E of $\mathscr{D}(U \times J, Z)$ such that

$$T(\phi) = \textstyle\int_J S(y)(\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y \quad \text{whenever } \phi \in E$$

by (1) and 2.12. If $\theta \in \mathscr{D}(U, Z)$, then

$$T_{(x,y)}(\omega(y)\theta(x)) = R_\theta(\omega) = \textstyle\int_J S(y)_x(\omega(y)\theta(x)) \, \mathrm{d}\mathscr{L}^1 y$$

for $\omega \in \mathscr{D}(J, \mathbf{R})$ by 3.2. One may now take E to be the image of $\mathscr{D}(J, \mathbf{R}) \otimes \mathscr{D}(U, Z)$ in $\mathscr{D}(U \times J, Z)$, see 3.1.

The second equation in (2) follows from (1) and the first equation in (2). □

4 Monotonicity identity

The purpose of this section is to derive the modifications and consequences of the monotonicity identity which will be employed in 6.12, 8.33, 13.1, and 14.2.

4.1 Lemma. *Suppose $1 \leq m < \infty$, $0 \leq s < r < \infty$, $0 \leq \kappa < \infty$, $I = \{t : s < t \leq r\}$, and $f : I \to \{y : 0 < y < \infty\}$ is a function satisfying*

$$\limsup_{u \to t-} f(u) \leq f(t) \leq f(r) + \kappa f(r)^{1/m} \textstyle\int_t^r u^{-1} f(u)^{1-1/m} \, \mathrm{d}\mathscr{L}^1 u \quad \text{for } t \in I.$$

Then there holds

$$f(t) \leq \big(1 + m^{-1}\kappa \log(r/t)\big)^m f(r) \quad \text{for } t \in I.$$

Proof. Suppose $f(r) < y < \infty$ and $v = m^{-1}\kappa f(r)^{1/m} y^{-1/m}$ and consider the set J of all $t \in I$ such that

$$f(u) \leq (1 + v \log(r/u))^m y \quad \text{whenever } t \leq u \leq r.$$

Clearly, J is an interval and r belongs to the interior of J relative to I. The same holds for t with $s < t \in \operatorname{Clos} J$ since

$$\begin{aligned}f(t) &\leq f(r) + \kappa f(r)^{1/m} y^{1-1/m} \textstyle\int_t^r u^{-1}(1 + v\log(r/u))^{m-1} \, \mathrm{d}\mathscr{L}^1 u \\ &= f(r) + \big((1 + v\log(r/t))^m - 1\big) y < (1 + v \log(r/t))^m y.\end{aligned}$$

Therefore I equals J. □

4.2 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $V \in \mathbf{V}_m(\mathbf{U}(a,r))$, and $\varrho \in \mathscr{D}(\{s : 0 < s < r\}, \mathbf{R})$.*

Then there holds

$$
\begin{aligned}
&- \textstyle\int_0^r \varrho'(s) s^{-m} \|V\| \, \mathbf{B}(a,s) \, \mathrm{d}\mathscr{L}^1 s \\
&\qquad = \textstyle\int_{(\mathbf{U}(a,r) \sim \{a\}) \times \mathbf{G}(n,m)} \varrho(|x-a|)|x-a|^{-m-2}|P_\natural^\perp(x-a)|^2 \, \mathrm{d}V(x,P) \\
&\qquad\quad - (\delta V)_x \Big(\big(\textstyle\int_{|x-a|}^r s^{-m-1} \varrho(s) \, \mathrm{d}\mathscr{L}^1 s \big)(x-a) \Big).
\end{aligned}
$$

Proof. Assume $a = 0$ and let $I = \{s : -\infty < s < r\}$ and $J = \{s : 0 < s < r\}$.

If $\omega \in \mathscr{E}(I, \mathbf{R})$, $\sup \operatorname{spt} \omega < r$, $0 \notin \operatorname{spt} \omega'$ and $\theta : \mathbf{U}(a,r) \to \mathbf{R}^n$ is associated to ω by $\theta(x) = \omega(|x|)x$ for $x \in \mathbf{U}(a,r)$, then $D\theta(0) \bullet P_\natural = m\omega(0)$ and

$$
\begin{aligned}
D\theta(x) \bullet P_\natural &= |P_\natural(x)|^2 |x|^{-1} \omega'(|x|) + m\omega(|x|) \\
&= -|P_\natural^\perp(x)|^2 |x|^{-1} \omega'(|x|) + |x|\omega'(|x|) + m\omega(|x|)
\end{aligned}
$$

whenever $x \in \mathbf{R}^n$, $0 < |x| < r$, and $P \in \mathbf{G}(n,m)$. Define $\omega, \gamma \in \mathscr{E}(I, \mathbf{R})$ by

$$
\omega(s) = - \textstyle\int_{\sup\{s,0\}}^r u^{-m-1} \varrho(u) \, \mathrm{d}\mathscr{L}^1 u \quad \text{and} \quad \gamma(s) = s\omega'(s) + m\omega(s)
$$

for $s \in I$, hence $\sup \operatorname{spt} \gamma \leq \sup \operatorname{spt} \omega < r$, $0 \notin \operatorname{spt} \omega'$, and

$$
\omega'(s) = s^{-m-1} \varrho(s), \quad \omega''(s) = -(m+1) s^{-m-2} \varrho(s) + s^{-m-1} \varrho'(s)
$$

for $s \in J$. Using Fubini's theorem, one computes with θ as before that

$$
\begin{aligned}
&\delta V(\theta) + \textstyle\int_{(\mathbf{R}^n \times \mathbf{G}(n,m)) \cap \{(x,P) : 0 < |x| < r\}} \varrho(|x|)|x|^{-m-2} |P_\natural^\perp(x)|^2 \, \mathrm{d}V(x,P) \\
&\quad = \textstyle\int \gamma(|x|) \, \mathrm{d}\|V\| x = - \int \int_{|x|}^r \gamma'(s) \, \mathrm{d}\mathscr{L}^1 s \, \mathrm{d}\|V\| x = - \int_0^r \gamma'(s) \|V\| \, \mathbf{B}(a,s) \, \mathrm{d}\mathscr{L}^1 s.
\end{aligned}
$$

Finally, notice that $\gamma'(s) = s\omega''(s) + (m+1)\omega'(s) = s^{-m}\varrho'(s)$ for $s \in J$. □

4.3 Remark. This is a slight generalisation of Simon's version of the monotonicity identity, see [Sim83, 17.3], included here for the convenience of the reader.

4.4. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and $\eta(V, \cdot)$ is a $\|\delta V\|$ measurable $\mathbf{S}^{n-1}$ valued function satisfying

$$
(\delta V)(\theta) = \textstyle\int \eta(V,x) \bullet \theta(x) \, \mathrm{d}\|\delta V\| x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n),
$$

see Allard [All72, 4.3].

4.5 Corollary. *Suppose m, n, U, V, and η are as in 4.4.*

Then there holds

$$
\begin{aligned}
&s^{-m} \|V\| \, \mathbf{B}(a,s) + \textstyle\int_{(\mathbf{B}(a,r) \sim \mathbf{B}(a,s)) \times \mathbf{G}(n,m)} |x-a|^{-m-2} |P_\natural^\perp(x-a)|^2 \, \mathrm{d}V(x,P) \\
&\qquad = r^{-m} \|V\| \, \mathbf{B}(a,r) \\
&\qquad\quad + m^{-1} \textstyle\int_{\mathbf{B}(a,r)} (\sup\{|x-a|, s\}^{-m} - r^{-m})(x-a) \bullet \eta(V,x) \, \mathrm{d}\|\delta V\| x
\end{aligned}
$$

whenever $a \in \mathbf{R}^n$, $0 < s \leq r < \infty$, and $\mathbf{B}(a,r) \subset U$.

Proof. Letting ζ approximate the characteristic function of $\{t : s < t \leq r\}$, the assertion is a consequence of 4.2. ☐

4.6 Remark. Using Fubini's theorem, the last summand can be expressed as

$$\textstyle\int_s^r t^{-m-1} \int_{\mathbf{B}(a,t)} (x-a) \bullet \eta(V,x) \, \mathrm{d}\|\delta V\| x \, \mathrm{d}\mathscr{L}^1 t.$$

4.7 Remark. 4.5 and 4.6 are a minor variations of Simon [Sim83, 17.3, 17.4].

4.8 Corollary. *Suppose m, n, U, and V are as in 4.4, $m = 1$, and $X = U \cap \{a : \|\delta V\|(\{a\}) > 0\}$.*

Then the following three statements hold.

(1) *If $a \in \mathbf{R}^n$, $0 < s \leq r < \infty$, and $\mathbf{B}(a,r) \subset U$, then*

$$\begin{aligned} s^{-1} \|V\| \, \mathbf{B}(a,s) + \textstyle\int_{(\mathbf{B}(a,r) \sim \mathbf{B}(a,s)) \times \mathbf{G}(n,m)} |x-a|^{-3} |P_\natural^\perp(x-a)|^2 \, \mathrm{d}V(x,P) \\ \leq r^{-1} \|V\| \, \mathbf{B}(a,r) + \|\delta V\|(\mathbf{B}(a,r) \sim \{a\}) \end{aligned}$$

(2) *$\boldsymbol{\Theta}^1(\|V\|, \cdot)$ is a real valued function whose domain is U.*

(3) *$\boldsymbol{\Theta}^1(\|V\|, \cdot)$ is upper semicontinuous at a whenever $a \in U \sim X$.*

If additionally $\boldsymbol{\Theta}^1(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, then the following two statements hold.

(4) *If $a \in \operatorname{spt} \|V\|$, then*

$$\boldsymbol{\Theta}^1(\|V\|, a) \geq 1 \quad \text{if } a \notin X \qquad \text{and} \qquad \boldsymbol{\Theta}^1(\|V\|, a) \geq 1/2 \quad \text{if } a \in X.$$

(5) *If $a \in \operatorname{spt} \|V\|$, $0 < s \leq r < \infty$, $\mathbf{U}(a,r) \subset U$, and $\|\delta V\|(\mathbf{U}(a,r) \sim \{a\}) \leq \varepsilon$, then*

$$\|V\| \, \mathbf{U}(x,s) \geq 2^{-1}(1-\varepsilon)s \quad \text{whenever } x \in \operatorname{spt} \|V\| \text{ and } |x-a| + s \leq r.$$

Proof. If $a \in U$, $0 < s < r < \infty$ and $\mathbf{B}(a,r) \subset U$, then

$$\big| \sup\{|x-a|, s\}^{-1} - r^{-1} \big| |x-a| \leq 1 \quad \text{whenever } x \in \mathbf{B}(a,r).$$

Therefore (1) follows from 4.5. (1) readily implies (2) and (3) and the first half of (4). To prove the second half, choose η as in 4.4 and consider $a \in X$. One may assume $a = 0$ and in view of Allard [All72, 4.10 (2)] also $U = \mathbf{R}^n$. Abbreviating $v = \eta(V,0) \in \mathbf{S}^{n-1}$ and defining the reflection $f : \mathbf{R}^n \to \mathbf{R}^n$ by $f(x) = x - 2(x \bullet v)v$ for $x \in \mathbf{R}^n$, one infers the second half of (4) by applying the first half of (4) to the varifold $V + f_\# V$.

If a, s, r, ε and x satisfy the conditions of (5), then

$$\begin{gathered} \|V\| \, \mathbf{U}(x,s) \geq (1-\varepsilon)s \quad \text{if either } s \leq |x-a| \text{ or } x = a, \\ \|V\| \, \mathbf{U}(x,s) \geq \|V\| \, \mathbf{U}(a,s/2) \geq 2^{-1}(1-\varepsilon)s \quad \text{if } 2|x-a| \leq s \end{gathered}$$

by (1) and (4), the case $|x-a| < s < 2|x-a|$ then follows. ☐

4.9 Corollary. *Suppose m, n, U, and V are as in 4.4, $a \in \mathbf{R}^n$, $0 < s < r < \infty$, $\mathbf{B}(a,r) \subset U$, $0 \leq \kappa < \infty$, and*

$$\|\delta V\| \, \mathbf{B}(a,t) \leq \kappa r^{-1} \|V\|(\mathbf{B}(a,r))^{1/m} \|V\|(\mathbf{B}(a,t))^{1-1/m} \quad \text{for } s < t < r,$$

where $0^0 = 1$.

Then there holds

$$s^{-m} \|V\| \, \mathbf{B}(a,s) \leq \big(1 + m^{-1} \kappa \log(r/s)\big)^m r^{-m} \|V\| \, \mathbf{B}(a,r).$$

Proof. Assume $\|V\| \, \mathbf{B}(a,r) > 0$ and define $t = \inf\{u : \|V\| \, \mathbf{B}(a,u) > 0\}$ and $f(u) = u^{-m} \|V\| \, \mathbf{B}(a,u)$ for $\sup\{s,t\} < u \leq r$. Then, in view of 4.5 and 4.6, one may apply 4.1 with s replaced by $\sup\{s,t\}$ to infer the conclusion. □

5 Distributional boundary

In this section the notion of distributional boundary of a set with respect to certain varifolds is introduced, see 5.1. Moreover, a basic structural theorem is proven, see 5.10, which allows to compare this notion to a similar one employed by Bombieri and Giusti in the context of area minimising currents in [BG72, Theorem 2], see 7.14.

5.1 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and E is $\|V\| + \|\delta V\|$ measurable.

Then the *distributional V boundary of E* is given by (see 2.17)

$$V \, \partial E = (\delta V) \llcorner E - \delta(V \llcorner E \times \mathbf{G}(n,m)) \in \mathscr{D}'(U, \mathbf{R}^n).$$

5.2 Remark. If $W \in \mathbf{V}_m(U)$, $\|\delta W\|$ is a Radon measure and E is additionally $\|W\| + \|\delta W\|$ measurable, then

$$(V + W) \, \partial E = V \, \partial E + W \, \partial E.$$

5.3 Remark. If E and F are $\|V\| + \|\delta V\|$ measurable sets and $E \subset F$, then

$$V \, \partial(F \sim E) = (V \, \partial F) - (V \, \partial E).$$

5.4 Remark. If $V \, \partial E$ is representable by integration, $W = V \llcorner E \times \mathbf{G}(n,m)$, and F is a Borel set, then

$$V \, \partial(E \cap F) = W \, \partial F + (V \, \partial E) \llcorner F.$$

5.5 Remark. If G is a countable, disjointed collection of $\|V\| + \|\delta V\|$ measurable sets with $V \, \partial E = 0$ for $E \in G$ and $\|V\|(U \sim \bigcup G) = 0$, then

$$\|\delta V\|(U \sim \textstyle\bigcup G) = 0;$$

in fact $\delta V = \sum_{E \in G} \delta(V \llcorner E \times \mathbf{G}(n,m)) = \sum_{E \in G} (\delta V) \llcorner E = (\delta V) \llcorner \bigcup G$.

5.6 Example. Suppose E and P are distinct members of $\mathbf{G}(2,1)$ and $V \in \mathbf{IV}_1(\mathbf{R}^2)$ is characterised by $\|V\| = \mathscr{H}^1 \llcorner (E \cup P)$.

Then $\delta V = 0$ and $V \, \partial E = 0$ but there exists no sequence of locally Lipschitzian functions $f_i : \mathbf{R}^2 \to \mathbf{R}$ satisfying

$$\textstyle\int |f_i - f| + |(\|V\|, 1) \operatorname{ap} D f_i| \, \mathrm{d} \|V\| \to 0 \quad \text{as } i \to \infty,$$

where f is the characteristic function of E.

5.7 Remark. Since it will follow from 8.24 (3) that f is a generalised weakly differentiable function with vanishing generalised weak derivative, the preceding example shows that the theory of generalised weakly differentiable functions cannot be developed using approximation by locally Lipschitzian functions. Instead, the relevant properties of sets are studied first in Sections 5–7 before proceeding to the theory of generalised weakly differentiable functions in Sections 8–13.

5.8 Lemma. *Suppose $n \in \mathscr{P}$, $1 \leq m \leq n$, U is an open subset of $\mathbf{R}^n$, μ is a Radon measure over U, K is a compact subset of U, A is compact subset of $\operatorname{Int} K$, $\mathscr{H}^{m-1}(A) = 0$, and*

$$\lim_{r \to 0+} \sup\{s^{-m} \mu\, \mathbf{B}(x,s) : x \in A \text{ and } 0 < s \leq r\} < \infty.$$

Then there exists a sequence $f_i \in \mathscr{E}(U, \mathbf{R})$ satisfying

$$0 \leq f_i \leq 1, \quad A \subset \operatorname{Int}\{x : f_i(x) = 0\}, \quad \{x : f_i(x) < 1\} \subset K \qquad \text{for } i \in \mathscr{P},$$
$$\lim_{i \to \infty} f_i(x) = 1 \quad \text{for } \mathscr{H}^{m-1} \text{ almost all } x \in U, \qquad \lim_{i \to \infty} \textstyle\int |Df_i| \,\mathrm{d}\mu = 0.$$

Proof. Let ϕ_∞ denote the size ∞ approximating measure for $\mathscr{H}^{m-1}$ over $\mathbf{R}^n$. Observe that it is sufficient to prove that for $\varepsilon > 0$ there exists $f \in \mathscr{E}(U, \mathbf{R})$ with

$$0 \leq f \leq 1, \quad A \subset \operatorname{Int}\{x : f(x) = 0\}, \quad \{x : f(x) < 1\} \subset K,$$
$$\phi_\infty(\{x : f(x) < 1\}) < \varepsilon, \quad \textstyle\int |Df| \,\mathrm{d}\mu < \varepsilon.$$

For this purpose assume $A \neq \varnothing$, denote the limit in the hypotheses of the lemma by Q, and choose $j \in \mathscr{P}$ and $x_i \in A$, $0 < r_i < \infty$ for $i \in \{1, \ldots, j\}$ with

$$\mu\, \mathbf{B}(x_i, r_i) \leq (Q+1) r_i^m \quad \text{and} \quad \mathbf{B}(x_i, r_i) \subset K \quad \text{for } i \in \{1, \ldots, j\},$$
$$A \subset \textstyle\bigcup_{i=1}^j \mathbf{U}(x_i, r_i/2), \quad \sum_{i=1}^j r_i^{m-1} < \varepsilon/\Delta,$$

where $\Delta = \sup\{\boldsymbol{\alpha}(m-1), 4(Q+1)\}$. Selecting $f_i \in \mathscr{E}(U, \mathbf{R})$ with $0 \leq f_i \leq 1$, $|Df_i| \leq 4 r_i^{-1}$, and

$$\mathbf{U}(x_i, r_i/2) \subset \{x : f_i(x) = 0\}, \quad \{x : f_i(x) < 1\} \subset \mathbf{B}(x_i, r_i)$$

whenever $i \in \{1, \ldots, j\}$, one may take $f = \prod_{i=1}^j f_i$. ☐

5.9 (see [Fed69, 1.7.5])**.** Suppose $m, n \in \mathscr{P}$ and $m \leq n$. Then one defines the linear map $\gamma_m : \Lambda_m \mathbf{R}^n \to \Lambda^m \mathbf{R}^n$ by the equation

$$\langle \xi, \gamma_m(\eta) \rangle = \xi \bullet \eta \quad \text{whenever } \xi, \eta \in \Lambda_m \mathbf{R}^n.$$

5.10 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, $0 \leq \kappa < \infty$, $\|\delta V\| \leq \kappa \|V\|$, M is a relatively open subset of $\operatorname{spt} \|V\|$, $\mathscr{H}^{m-1}((\operatorname{spt}\|V\|) \sim M) = 0$, M is an m dimensional submanifold of class 2, $\tau : M \to \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)$ satisfies $\tau(x) = \operatorname{Tan}(M, x)_\natural$ for $x \in M$, E is a $\|V\|$ measurable set, and*

$$B = M \sim \{x : \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M \cap E, x) = 0 \text{ or } \boldsymbol{\Theta}^m(\mathscr{H}^m \llcorner M \sim E, x) = 0\}.$$

Then the following three statements hold.

(1) If $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ then

$$V\,\partial E(\theta) = -{\textstyle\int_{E\cap M}} \tau(x) \bullet D\,\langle \theta, \tau \rangle\,(x) \boldsymbol{\Theta}^m(\|V\|, x)\,\mathrm{d}\mathscr{H}^m x.$$

(2) If ξ is an m vectorfield orienting M and γ_m is as in 5.9, then

$$V\,\partial E(\theta) = (-1)^m {\textstyle\int_{E\cap M}} \langle \xi(x), d(\theta \mathbin{\lrcorner} (\gamma_m \circ \xi))(x) \rangle\, \boldsymbol{\Theta}^m(\|V\|, x)\,\mathrm{d}\mathscr{H}^m x$$

for $\theta \in \mathscr{D}(U, \mathbf{R}^n)$.

(3) The distribution $V\,\partial E$ is representable by integration if and only if

$$\mathscr{H}^{m-1}(K \cap B) < \infty \quad \textit{whenever } K \textit{ is a compact subset of } U;$$

in this case B is $\mathscr{H}^{m-1}$ almost equal to $\{x : \mathbf{n}(M; E, x) \in \mathbf{S}^{n-1}\}$ and meets every compact subset of U is a $(\mathscr{H}^{m-1}, m-1)$ rectifiable set and

$$V\,\partial E(\theta) = -{\textstyle\int} \mathbf{n}(M; E, x) \bullet \theta(x) \boldsymbol{\Theta}^m(\|V\|, x)\,\mathrm{d}\mathscr{H}^{m-1} x$$

for $\theta \in \mathscr{D}(U, \mathbf{R}^n)$.

Proof. Notice that the results of Allard [All72, 2.5, 4.5–4.7] stated for submanifolds of class ∞ have analogous formulations for submanifolds of class 2. The meaning of $\operatorname{Tan}(M, \theta)$, $\operatorname{Nor}(M, \theta)$, $\mathbf{G}_m(M)$, and $\mathbf{IV}_m(M)$ defined in [All72, 2.5, 3.1] will be extended accordingly.

The following assertion will be proven. *If ξ is as in* (2)*, then $\phi = \gamma_m \circ \xi$ satisfies*

$$\langle \xi, d(\theta \mathbin{\lrcorner} \phi)(x) \rangle = (-1)^{m-1} \tau(x) \bullet D\,\langle \theta, \tau \rangle\,(x) \quad \textit{for } x \in M$$

whenever $\theta : U \to \mathbf{R}^n$ is a vectorfield of class 1*;* in fact, assuming $\theta | M = \operatorname{Tan}(M, \theta)$ and noting $|\xi(x)| = 1$ for $x \in M$, one infers

$$\langle \xi(x), \langle u, D\phi(x) \rangle \rangle = \xi(x) \bullet \langle u, D\xi(x) \rangle = 0 \quad \text{for } x \in M,\ u \in \operatorname{Tan}(M, x),$$

hence for $x \in M$ one expresses $\xi(x) = u_1 \wedge \cdots \wedge u_m$ for some orthonormal basis $u_1, \ldots, u_m$ of $\operatorname{Tan}(M, x)$ and computes

$$\begin{aligned}
&\langle \xi(x), d(\theta \mathbin{\lrcorner} \phi)(x) \rangle \\
&\quad = {\textstyle\sum_{i=1}^m} (-1)^{i-1} \langle u_1 \wedge \cdots \wedge u_{i-1} \wedge u_{i+1} \wedge \cdots \wedge u_m, \langle u_i, D\theta(x) \rangle \mathbin{\lrcorner} \phi(x) \rangle \\
&\quad = (-1)^{m-1} {\textstyle\sum_{i=1}^m} \langle u_i, D\theta(x) \rangle \bullet u_i = (-1)^{m-1} \tau(x) \bullet D\theta(x).
\end{aligned}$$

Next, the case $M = \operatorname{spt} \|V\|$ will be considered. In this case one may assume M to be connected. Since

$$\delta V(\theta) = 0 \quad \text{whenever } \theta \in \mathscr{D}(U, \mathbf{R}^n) \text{ and } \operatorname{Nor}(M, \theta) = 0$$

for instance by Brakke [Bra78, 5.8] (or [Men13, 4.8]), Allard [All72, 4.6 (3)] then implies for some $0 < \lambda < \infty$ that

$$V(k) = \lambda {\textstyle\int_M} k(x, \operatorname{Tan}(M, x))\,\mathrm{d}\mathscr{H}^m x \quad \text{for } k \in \mathscr{K}(U \times \mathbf{G}(n, m)).$$

Define $W \in \mathbf{IV}_m(M)$ by $W(k) = \int_{E \times \mathbf{G}(n,m)} k \,\mathrm{d}V$ for $k \in \mathscr{K}(\mathbf{G}_m(M))$ and notice that

$$\begin{aligned} V\,\partial E(\theta) &= -\textstyle\int_E \mathbf{h}(V,x) \bullet \theta(x) \,\mathrm{d}\|V\|x - \int_{E \times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x) \,\mathrm{d}V(x,P) \\ &= -\textstyle\int_{E \times \mathbf{G}(n,m)} P_\natural \bullet (D\,\mathrm{Tan}(M,\theta)(x) \circ P_\natural) \,\mathrm{d}V(x,P) = -\delta W(\mathrm{Tan}(M,\theta)) \end{aligned}$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ by Allard [All72, 2.5 (2)]. (1) is now evident and implies (2) by the assertion of the preceding paragraph. Concerning (3), the subcase $M = U$ follows from [Fed69, 4.5.6, 4.5.11] (recalling also [Fed69, 4.1.28 (5), 4.2.1]) to which the case $M = \operatorname{spt} \|V\|$ may be reduced by applying Allard [All72, 4.5] to W, see also Allard [All72, 4.7].

To treat the general case, first notice that, in view of Allard [All72, 5.1 (3)], 5.8 is applicable with $\mu = \|V\|$ and $A = (\operatorname{spt}\theta) \cap (\operatorname{spt}\|V\|) \sim M$ whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, K is a compact subset of U, and $\operatorname{spt}\theta \subset \operatorname{Int} K$. The resulting functions $f_i \in \mathscr{E}(U, \mathbf{R})$ satisfy

$$\textstyle\int |\theta - f_i\theta| + |D\theta - D(f_i\theta)| \,\mathrm{d}\|V\| + \int_X |\theta - f_i\theta| \,\mathrm{d}\mathscr{H}^{m-1} \to 0$$

as $i \to \infty$ whenever X is a $\mathscr{H}^{m-1}$ measurable subset of U with $\mathscr{H}^{m-1}(X) < \infty$. It follows that $V\,\partial E(\theta) = \lim_{i\to\infty} V\,\partial E(f_i\theta)$ and, if $\mathscr{H}^{m-1}(K \cap B) < \infty$, then

$$\lim_{i\to\infty} \textstyle\int \mathbf{n}(M; E, x) \bullet (\theta(x) - (f_i\theta)(x)) \boldsymbol{\Theta}^m(\|V\|, x) \,\mathrm{d}\mathscr{H}^{m-1}x = 0.$$

Therefore one now readily verifies the assertion. □

5.11 Remark. If $S \in \mathscr{R}_m^{\mathrm{loc}}(\mathbf{R}^n)$ is absolutely area minimising with respect to $\mathbf{R}^n$, $\partial S = 0$, and $n - m = 1$, then $V \in \mathbf{IV}_m(\mathbf{R}^n)$ characterised by $\|S\| = \|V\|$ satisfies the hypotheses of 5.10 (2) with $U = \mathbf{R}^n$, $\kappa = 0$ for some M and ξ by Allard [All72, 4.8 (4)], [Fed69, 5.4.15], and Federer [Fed70, Theorem 1], and in this case $\|\partial(S \llcorner E)\| = \|V\,\partial E\|$ as may be verified using 5.10 (2) (3) in conjunction with [Fed69, 3.1.19, 4.1.14, 4.1.20, 4.1.30, 4.5.6].[11]

5.12 Remark. Considering the situation $m, n \in \mathscr{P}$, $1 < m < n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$ and $\delta V = 0$, few properties of V are known to hold near $\mathscr{H}^{m-1}$ almost all $x \in \operatorname{spt}\|V\|$. Consequently, it appears difficult to obtain a structural description similar to 5.10 (3) for $\|V\|$ measurable sets whose distributional V boundary is representable by integration in this more general situation. However, for $\mathscr{L}^1$ almost all superlevel sets of a real valued generalised weakly differential function such a description will be proven under even milder hypotheses on V in 12.2.

6 Decompositions of varifolds

In this section the existence of a decomposition of rectifiable varifolds whose first variation is representable by integration is established in 6.12. If the first variation is sufficiently well behaved, this decomposition may be linked to the decomposition of the support of the weight measure into connected components, see 6.14.

[11] Referring additionally to [Fed69, 5.3.20] and Almgren [Alm00, 5.22], the hypothesis $n - m = 1$ could have been omitted. However, the author has not checked Almgren's result and its consequences will not be used in the present paper.

6.1. A useful set of hypotheses is gathered here for later reference.

Suppose $m, n \in \mathscr{P}$, $m \le n$, $1 \le p \le m$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x. If $p > 1$, then suppose additionally that $\mathbf{h}(V, \cdot) \in \mathbf{L}_p^{\mathrm{loc}}(\|V\|, \mathbf{R}^n)$ and

$$\delta V(\theta) = -\textstyle\int \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\| x \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Therefore $V \in \mathbf{RV}_m(U)$ by Allard [All72, 5.5 (1)]. If $p = 1$ let $\psi = \|\delta V\|$. If $p > 1$ define a Radon measure ψ over U by $\psi(A) = \int_A^* |\mathbf{h}(V, x)|^p \,\mathrm{d}\|V\| x$ for $A \subset U$.

6.2 Definition. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$ and $\|\delta V\|$ is a Radon measure.

Then V is called *indecomposable* if there exists no $\|V\| + \|\delta V\|$ measurable set E such that

$$\|V\|(E) > 0, \quad \|V\|(U \sim E) > 0, \quad V\,\partial E = 0.$$

6.3 Remark. The same definition results if E is required to be a Borel set.

6.4 Remark. If V is indecomposable then so is λV for $0 < \lambda < \infty$. This is in contrast to a similar notion employed by Mondino in [Mon14, 2.15].

6.5 Lemma. *Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, E_0 and E_1 are nonempty, disjoint, relatively closed subset of* $\operatorname{spt}\|V\|$, *and $E_0 \cup E_1 = \operatorname{spt}\|V\|$.*

Then there holds

$$\|V\|(E_i) > 0 \quad \textit{and} \quad V\,\partial E_i = 0$$

for $i \in \{0, 1\}$. In particular, if V is indecomposable then $\operatorname{spt}\|V\|$ *is connected.*

Proof. Notice that $U \sim E_0$ and $U \sim E_1$ are open, hence

$$\|V\|(E_0) = \|V\|(U \sim E_1) > 0, \quad \|V\|(E_1) = \|V\|(U \sim E_0) > 0.$$

Next, one constructs $w \in \mathscr{E}(U, \mathbf{R})$ such that

$$E_i \subset \operatorname{Int}\{x : w(x) = i\} \quad \text{for } i = \{0, 1\};$$

in fact, applying [Fed69, 3.1.13] with $\Phi = \{U \sim E_0, U \sim E_1\}$, one obtains h, S and v_s, notices that either $\mathbf{B}(s, 10h(s)) \subset \mathbf{R}^n \sim E_0$ or $\mathbf{B}(s, 10h(s)) \subset \mathbf{R}^n \sim E_1$ whenever $s \in S$, lets $T = S \cap \big\{s : \mathbf{B}(s, 10h(s)) \subset \mathbf{R}^n \sim E_0\big\}$ and takes

$$w(x) = \sum_{t \in T} v_t(x) \quad \text{for } x \in U.$$

This yields $V\,\partial E_i = 0$ for $i = \{0, 1\}$ by Allard [All72, 4.10 (1)]. ☐

6.6 Definition. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, and $\|\delta V\|$ is a Radon measure.

Then W is called a *component of* V if and only if $0 \ne W \in \mathbf{V}_m(U)$ is indecomposable and there exists a $\|V\| + \|\delta V\|$ measurable set E such that

$$W = V \llcorner E \times \mathbf{G}(n, m), \quad V\,\partial E = 0.$$

6.7 Remark. Suppose F is a $\|V\| + \|\delta V\|$ measurable set. Then E is $\|V\| + \|\delta V\|$ almost equal to F if and only if

$$W = V \llcorner F \times \mathbf{G}(n,m), \quad V\,\partial F = 0.$$

6.8 Remark. If C is a connected component of $\operatorname{spt}\|V\|$ and W is a component of V with $C \cap \operatorname{spt}\|W\| \neq \varnothing$, then $\operatorname{spt}\|W\| \subset C$ by 6.5.

6.9 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, and $\Xi \subset \mathbf{V}_m(U)$.

Then Ξ is called a *decomposition of V* if and only if the following three conditions are satisfied:

(1) Each member of Ξ is a component of V.

(2) Whenever W and X are distinct members of Ξ there exist disjoint $\|V\| + \|\delta V\|$ measurable sets E and F with $V\,\partial E = 0 = V\,\partial F$ and

$$W = V \llcorner E \times \mathbf{G}(n,m), \quad X = V \llcorner F \times \mathbf{G}(n,m).$$

(3) $V(k) = \sum_{W \in \Xi} W(k)$ whenever $k \in \mathscr{K}(U \times \mathbf{G}(n,m))$.

6.10 Remark. Clearly, Ξ is countable.

Moreover, using 6.7 one constructs a function ξ mapping Ξ into the class of all Borel subsets of U such that distinct members of Ξ are mapped onto disjoint sets and

$$W = V \llcorner \xi(W) \times \mathbf{G}(n,m), \quad V\,\partial\xi(W) = 0$$

whenever $W \in \Xi$. Consequently, in view of 5.5, one infers

$$(\|V\| + \|\delta V\|)\big(U \sim \textstyle\bigcup \operatorname{im}\xi\big) = 0.$$

Also notice that $V\,\partial(\bigcup \xi[N]) = 0$ whenever $N \subset \Xi$.

6.11 Remark. Suppose m, n, p, U and V are as in 6.1, $p = m$, and Ξ is a decomposition of W. Observe that 4.8 (5) and [Men09, 2.5] imply

$$\operatorname{card}(\Xi \cap \{W : K \cap \operatorname{spt}\|W\| \neq \varnothing\}) < \infty$$

whenever K is a compact subset of U, hence

$$\operatorname{spt}\|V\| = \textstyle\bigcup\{\operatorname{spt}\|W\| : W \in \Xi\}.$$

Notice that [Men09, 1.2] readily shows that both assertions need not to hold in case $p < m$.

6.12 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, and $\|\delta V\|$ is a Radon measure.*

Then there exists a decomposition of V.

Proof. Assume $V \neq 0$.

Denote by R the family of Borel subsets E of U such that $V\,\partial E = 0$. Notice that

$$\textstyle\bigcap_{i=1}^{\infty} E_i \in R \qquad \text{whenever } E_i \text{ is a sequence in } R \text{ with } E_{i+1} \subset E_i \text{ for } i \in \mathscr{P},$$
$$E \in R \quad \text{if and only if} \quad E \sim F \in R \qquad \text{whenever } E \supset F \in R$$

by 5.3. Let $P = R \cap \{E : \|V\|(E) > 0\}$. Next, define

$$\delta_i = \boldsymbol{\alpha}(m) 2^{-m-1} i^{-1-2m}, \quad \varepsilon_i = 2^{-1} i^{-2}$$

for $i \in \mathscr{P}$ and let A_i denote the Borel set of $a \in \mathbf{R}^n$ satisfying

$$|a| \le i, \quad \mathbf{U}(a, 2\varepsilon_i) \subset U, \quad \boldsymbol{\Theta}^m(\|V\|, a) \ge 1/i,$$
$$\|\delta V\| \, \mathbf{B}(a, r) \le \boldsymbol{\alpha}(m) i r^m \quad \text{for } 0 < r < \varepsilon_i$$

whenever $i \in \mathscr{P}$. Clearly, $A_i \subset A_{i+1}$ for $i \in \mathscr{P}$ and $\|V\|(U \sim \bigcup_{i=1}^\infty A_i) = 0$ by Allard [All72, 3.5 (1a)] and [Fed69, 2.8.18, 2.9.5]. Moreover, define

$$P_i = R \cap \{E : \|V\|(E \cap A_i) > 0\}$$

and notice that $P_i \subset P_{i+1}$ for $i \in \mathscr{P}$ and $P = \bigcup_{i=1}^\infty P_i$. One observes the lower bound given by

$$\|V\|(E \cap \mathbf{B}(a, \varepsilon_i)) \ge \delta_i$$

whenever $E \in R$, $i \in \mathscr{P}$, $a \in A_i$ and $\boldsymbol{\Theta}^{*m}(\|V\| \llcorner E, a) \ge 1/i$; in fact, noting

$$\textstyle\int_0^{\varepsilon_i} r^{-m} \|\delta(V \llcorner E \times \mathbf{G}(n,m))\| \, \mathbf{B}(a,r) \, \mathrm{d}\mathscr{L}^1 r \le \boldsymbol{\alpha}(m) i \varepsilon_i,$$

the inequality follows from 4.5 and 4.6. Let Q_i denote the set of $E \in P$ such that there is no F satisfying

$$F \subset E, \quad F \in P_i, \quad E \sim F \in P_i.$$

Denote by Ω the class of Borel partitions H of U with $H \subset P$ and let $G_0 = \{U\} \in \Omega$. The previously observed lower bound implies

$$\delta_i \operatorname{card}(H \cap P_i) \le \|V\|(U \cap \{x : \operatorname{dist}(x, A_i) \le \varepsilon_i\}) < \infty$$

whenever H is a disjointed subfamily of P, since for each $E \in H \cap P_i$ there exists $a \in A_i$ with $\boldsymbol{\Theta}^m(\|V\| \llcorner E, a) = \boldsymbol{\Theta}^m(\|V\|, a) \ge 1/i$ by [Fed69, 2.8.18, 2.9.11], hence

$$\|V\|(E \cap \{x : \operatorname{dist}(x, A_i) \le \varepsilon_i\}) \ge \|V\|(E \cap \mathbf{B}(a, \varepsilon_i)) \ge \delta_i.$$

In particular, such H is countable.

Next, one inductively (for $i \in \mathscr{P}$) defines Ω_i to be the class of all $H \in \Omega$ such that every $E \in G_{i-1}$ is the union of some subfamily of H and chooses $G_i \in \Omega_i$ such that

$$\operatorname{card}(G_i \cap P_i) \ge \operatorname{card}(H \cap P_i) \quad \text{whenever } H \in \Omega_i.$$

The maximality of G_i implies $G_i \subset Q_i$; in fact, if there would exist $E \in G_i \sim Q_i$ there would exist F satisfying

$$F \subset E, \quad F \in P_i, \quad E \sim F \in P_i$$

and $H = (G_i \sim \{E\}) \cup \{F, E \sim F\}$ would belong to Ω_i with

$$\operatorname{card}(H \cap P_i) > \operatorname{card}(G_i \cap P_i).$$

Moreover, it is evident that to each $x \in U$ there corresponds a sequence E_i uniquely characterised by the requirements $x \in \bigcap_{i=1}^\infty E_i$ and $E_{i+1} \subset E_i \in G_i$ for $i \in \mathscr{P}$.

Define $G = \bigcup_{i=1}^\infty G_i$ and notice that G is countable. Define C to be the collection of sets $\bigcap_{i=1}^\infty E_i$ with positive $\|V\|$ measure corresponding to all sequences E_i with $E_{i+1} \subset E_i \in G_i$ for $i \in \mathscr{P}$. Clearly, C is a disjointed subfamily of P, hence C is countable. Next, it will be shown that

$$\|V\|(U \sim \textstyle\bigcup C) = 0.$$

In view of [Fed69, 2.8.18, 2.9.11] it is sufficient to prove

$$A_i \sim \textstyle\bigcup C \subset \bigcup\big\{E \cap \{x : \mathbf{\Theta}^{*m}(\|V\| \llcorner E, x) < \mathbf{\Theta}^{*m}(\|V\|, x)\} : E \in G\big\}$$

for $i \in \mathscr{P}$. For this purpose consider $a \in A_i \sim \bigcup C$ with corresponding sequence E_j. It follows that $\|V\|(\bigcap_{j=1}^\infty E_j) = 0$, hence there exists j with $\|V\|(E_j \cap \mathbf{B}(a, \varepsilon_i)) < \delta_i$ and the lower bound implies

$$\mathbf{\Theta}^{*m}(\|V\| \llcorner E_j, a) < 1/i \leq \mathbf{\Theta}^m(\|V\|, a).$$

It remains to prove that each varifold $V \llcorner E \times \mathbf{G}(n, m)$ corresponding to $E \in C$ is indecomposable. If this were not the case, then there would exist $E = \bigcap_{i=1} E_i \in C$ with $E_{i+1} \subset E_i \in G_i$ for $i \in \mathscr{P}$ and a Borel set F such that

$$\|V\|(E \cap F) > 0, \quad \|V\|(E \sim F) > 0, \quad V\,\partial(E \cap F) = 0$$

by 5.4 and 6.3. This would imply $E \cap F \in P$ and $E \sim F \in P$, hence for some i also $E \cap F \in P_i$ and $E \sim F \in P_i$ which would yield

$$E \cap F \subset E_i, \quad E_i \sim (E \cap F) \in P_i,$$

since $E \sim F \subset E_i \sim (E \cap F) \in R$; a contradiction to $E_i \in Q_i$. □

6.13 Remark. The decomposition of V may be nonunique. In fact, considering the six rays

$$R_j = \{t \exp(\pi \mathbf{i} j/3) : 0 < t < \infty\} \subset \mathbf{C} = \mathbf{R}^2, \quad \text{where } \pi = \mathbf{\Gamma}(1/2)^2,$$

corresponding to $j \in \{0, 1, 2, 3, 4, 5\}$ and their associated varifolds $V_j \in \mathbf{IV}_1(\mathbf{R}^2)$ with $\|V_j\| = \mathscr{H}^1 \llcorner R_j$, one notices that $V = \sum_{j=0}^5 V_j \in \mathbf{IV}_1(\mathbf{R}^2)$ is a stationary varifold such that

$$\{V_0 + V_2 + V_4, V_1 + V_3 + V_5\} \quad \text{and} \quad \{V_0 + V_3, V_1 + V_4, V_2 + V_5\}$$

are distinct decompositions of V.

6.14 Corollary. *Suppose m, n, p, U and V are as in 6.1, $p = m$, and Φ is the family of all connected components of* $\operatorname{spt} \|V\|$.

Then the following four statements hold.

(1) If $C \in \Phi$, then

$$C = \textstyle\bigcup\{\operatorname{spt} \|W\| : W \in \Xi, C \cap \operatorname{spt} \|W\| \neq \varnothing\}$$

whenever Ξ is decomposition of V.

(2) $\operatorname{card}(\Phi \cap \{C : C \cap K \neq \varnothing\}) < \infty$ *whenever K is a compact subset of U.*

(3) If $C \in \Phi$, then C is open relative to $\operatorname{spt} \|V\|$.

(4) If $C \in \Phi$, then $\operatorname{spt}(\|V\| \llcorner C) = C$ *and* $V\,\partial C = 0$.

Proof. (1) is a consequence of 6.8 and 6.11. In view of (1) and 6.12, (2) is a consequence of 6.11. Next, (2) implies (3). Finally, (3) and 6.5 yield (4). ☐

6.15 Remark. If V is stationary, then (4) implies that $V \llcorner C \times \mathbf{G}(n,m)$ is stationary. This fact might prove useful in considerations involving a strong maximum principle such as Wickramasekera [Wic14, Theorem 1.1].

7 Relative isoperimetric inequality

In this section a general isoperimetric inequality for varifolds satisfying a lower density bound is established, see 7.8. As corollaries one obtains two relative isoperimetric inequalities under the relevant conditions on the first variation of the varifold, see 7.9 and 7.11.

7.1. Suppose $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|V\|, x) \ge 1$ for $\|V\|$ almost all x, E is $\|V\| + \|\delta V\|$ measurable, B is a closed subset of $\operatorname{Bdry} U$, and, see 2.15–2.17,

$$(\|V\| + \|\delta V\|)(E \cap K) + \|V\,\partial E\|(U \cap K) < \infty,$$
$$\textstyle\int_{E \times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x)\,\mathrm{d}V(x,P) = ((\delta V) \llcorner E)(\theta|U) - (V\,\partial E)(\theta|U)$$

whenever K is compact subset of $\mathbf{R}^n \sim B$ and $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$. Defining $W \in \mathbf{V}_m(\mathbf{R}^n \sim B)$ by

$$W(A) = V(A \cap (E \times \mathbf{G}(n,m))) \quad \text{for } A \subset (\mathbf{R}^n \sim B) \times \mathbf{G}(n,m),$$

this implies

$$\|\delta W\|(A) \le \|\delta V\|(E \cap A) + \|V\,\partial E\|(U \cap A) \quad \text{for } A \subset \mathbf{R}^n \sim B.$$

7.2 Example. Using 5.10 (1) (3) one verifies the following statement. *If $m, n \in \mathscr{P}$, $m \le n$, U is an open subset of $\mathbf{R}^n$, B is a closed subset of* $\operatorname{Bdry} U$, *M is an m dimensional submanifold of $\mathbf{R}^n$ of class* 2, *$M \subset \mathbf{R}^n \sim B$,* $(\operatorname{Clos} M) \sim M \subset B$, *$V \in \mathbf{V}_m(U)$ and $V' \in \mathbf{V}_m(\mathbf{R}^n \sim B)$ satisfy*

$$V(k) = \textstyle\int_{M \cap U} k(x, \operatorname{Tan}(M,x))\,\mathrm{d}\mathscr{H}^m x \quad \textit{for } k \in \mathscr{K}(U \times \mathbf{G}(n,m)),$$
$$V'(k) = \textstyle\int_{M} k(x, \operatorname{Tan}(M,x))\,\mathrm{d}\mathscr{H}^m x \quad \textit{for } k \in \mathscr{K}((\mathbf{R}^n \sim B) \times \mathbf{G}(n,m)),$$

and E is an $\mathscr{H}^m$ measurable subset of $M \cap U$, then E satisfies the conditions of 7.1 if and only if $V'\,\partial E$ is representable by integration and $\mathbf{n}(M; E, x) = 0$ *for $\mathscr{H}^{m-1}$ almost all $x \in (\mathbf{R}^n \sim B) \sim U$; in this case*

$$V\,\partial E(\theta|U) = -\textstyle\int_U \mathbf{n}(M; E, x) \bullet \theta(x)\,\mathrm{d}\mathscr{H}^{m-1} x \quad \textit{for } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n).$$

Since in the situation of 7.1 there no varifold V' available which extends V in a canonical way, the condition on E is formulated in terms of the behaviour W.

7.3 Lemma. *Suppose $1 \leq M < \infty$.*

Then there exists a positive, finite number Γ with the following property.

If $m, n \in \mathscr{P}$, $m \leq n \leq M$, $1 \leq Q \leq M$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $W \in \mathbf{V}_m(\mathbf{U}(a,r))$, $\|\delta W\|$ is a Radon measure, $\boldsymbol{\Theta}^m(\|W\|, x) \geq 1$ for $\|W\|$ almost all x, $a \in \operatorname{spt} \|W\|$, and

$$\|\delta W\| \, \mathbf{B}(a,s) \leq \Gamma^{-1} \|W\| (\mathbf{B}(a,s))^{1-1/m} \quad \text{for } 0 < s < r,$$
$$\|W\|(\{x : \boldsymbol{\Theta}^m(\|W\|, x) < Q\}) \leq \Gamma^{-1} \|W\| \, \mathbf{U}(a,r),$$

then there holds

$$\|W\| \, \mathbf{U}(a,r) \geq (Q - M^{-1}) \boldsymbol{\alpha}(m) r^m.$$

Proof. If the lemma were false for some M, there would exist a sequence Γ_i with $\Gamma_i \to \infty$ as $i \to \infty$, and sequences m_i, n_i, Q_i, a_i, r_i, and W_i showing that $\Gamma = \Gamma_i$ does not have the asserted property.

One could assume for some $m, n \in \mathscr{P}$, $1 \leq Q \leq M$ that $m \leq n \leq M$,

$$m = m_i, \quad n = n_i, \quad a_i = 0, \quad r_i = 1$$

for $i \in \mathscr{P}$ and $Q_i \to Q$ as $i \to \infty$ and, by [Men09, 2.5],

$$\|W_i\| \, \mathbf{B}(0,s) \geq (2m\boldsymbol{\gamma}(m))^{-m} s^m \quad \text{whenever } 0 < s < 1 \text{ and } i \in \mathscr{P}.$$

Defining $W \in \mathbf{V}_m(\mathbf{R}^n \cap \mathbf{U}(0,1))$ to be the limit of some subsequence of W_i, one would obtain

$$\|W\| \, \mathbf{U}(0,1) \leq (Q - M^{-1}) \boldsymbol{\alpha}(m), \quad 0 \in \operatorname{spt} \|W\|, \quad \delta W = 0.$$

Finally, using Allard [All72, 5.1 (2), 5.4, 8.6], one would then conclude that

$$\boldsymbol{\Theta}^m(\|W\|, x) \geq Q \quad \text{for } \|W\| \text{ almost all } x,$$
$$\boldsymbol{\Theta}^m(\|W\|, 0) \geq Q, \quad \|W\| \, \mathbf{U}(0,1) \geq Q \boldsymbol{\alpha}(m),$$

a contradiction. ☐

7.4 Remark. Considering stationary varifolds whose support is contained in two affine planes with $\boldsymbol{\Theta}^m(\|W\|, a)$ a small positive number, shows that the hypotheses "$\boldsymbol{\Theta}^m(\|W\|, x) \geq 1$ for $\|W\|$ almost all x" cannot be omitted.

7.5 Remark. Even for smooth functions, 7.3 is the key observation which – through the relative isoperimetric inequalities 7.9 and 7.11 – leads to Sobolev Poincaré type estimates which are applicable near $\|V\|$ almost all points of $\{x : \boldsymbol{\Theta}^m(\|V\|, x) \geq 2\}$, see 10.1. For generalised weakly differentiable functions, these estimates in turn provide an important ingredient for the differentiability results obtained in 11.2 and 11.4 and the coarea formula in 12.2.

7.6 Remark. Taking $Q = 1$ in 7.3 (or applying [Men09, 2.6]) yields the following proposition: *If m, n, p, U, V, and ψ are as in 6.1, $p = m$, $a \in \operatorname{spt} \|V\|$, and $\psi(\{a\}) = 0$, then $\boldsymbol{\Theta}^m_*(\|V\|, a) \geq 1$.* If $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$ then the condition $\psi(\{a\}) = 0$ is redundant.

If $m = n$ and $f : \mathbf{R}^n \to \{y : 1 \leq y < \infty\}$ is a weakly differentiable function with $\mathbf{D}f \in \mathbf{L}_n^{\mathrm{loc}}(\mathscr{L}^n, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}))$, then the varifold $V \in \mathbf{RV}_n(\mathbf{R}^n)$ defined by the requirement $\|V\|(B) = \int_B f \,\mathrm{d}\mathscr{L}^n$ whenever B is a Borel subset of $\mathbf{R}^n$ satisfies the conditions of 6.1 with $p = n$ since

$$\delta V(g) = -\textstyle\int \langle g(x), \mathbf{D}(\log \circ f)(x)\rangle \,\mathrm{d}\|V\|x \quad \text{for } g \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n).$$

If $n > 1$ and C is a countable subset of $\mathbf{R}^n$, one may use well known properties of Sobolev functions, in particular example [AF03, 4.43], to construct f such that $C \subset \{x : \boldsymbol{\Theta}^n(\|V\|, x) = \infty\}$. It is therefore evident that the conditions of 6.1 with $p = m > 1$ are insufficient to guarantee finiteness or upper semicontinuity of $\boldsymbol{\Theta}^m(\|V\|, \cdot)$ at each point of U. However, the following proposition was obtained by Kuwert and Schätzle in [KS04, Appendix A]: *If m, n, p, U, and V are as in 6.1, $p = m = 2$, and $V \in \mathbf{IV}_2(U)$, then $\boldsymbol{\Theta}^2(\|V\|, \cdot)$ is a real valued, upper semicontinuous function whose domain is U.*

7.7 Remark. The preceding remark is a corrected and extended version of the author's remark in [Men09, 2.7] where the last two sentences should have referred to integral varifolds.

7.8 Theorem. *Suppose $1 \leq M < \infty$.*

Then there exists a positive, finite number Γ with the following property.

If $m, n \in \mathscr{P}$, $m \leq n \leq M$, $1 \leq Q \leq M$, U is an open subset of $\mathbf{R}^n$, $W \in \mathbf{V}_m(U)$, $S, \Sigma \in \mathscr{D}'(U, \mathbf{R}^n)$ are representable by integration, $\delta W = S + \Sigma$, $\boldsymbol{\Theta}^m(\|W\|, x) \geq 1$ for $\|W\|$ almost all x, $0 < r < \infty$, and

$$\begin{gathered} \|S\|(U) \leq \Gamma^{-1} \qquad \textit{if } m = 1, \\ S(\theta) \leq \Gamma^{-1}\|W\|_{(m/(m-1))}(\theta) \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n) \qquad \textit{if } m > 1, \\ \|W\|(U) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \quad \|W\|(\{x : \boldsymbol{\Theta}^m(\|W\|, x) < Q\}) \leq \Gamma^{-1} r^m, \end{gathered}$$

then there holds

$$\|W\|(\{x : \mathbf{U}(x, r) \subset U\})^{1-1/m} \leq \Gamma\,\|\Sigma\|(U),$$

where $0^0 = 0$.

Proof. Define

$$\begin{gathered} \Delta_1 = \Gamma_{7.3}(2M)^{-1}, \quad \Delta_2 = \inf\{(2\boldsymbol{\gamma}(m))^{-1} : M \geq m \in \mathscr{P}\}, \\ \Delta_3 = \Delta_1 \inf\{(2m\boldsymbol{\gamma}(m))^{-m} : M \geq m \in \mathscr{P}\}, \quad \Delta_4 = \sup\{\boldsymbol{\beta}(n) : M \geq n \in \mathscr{P}\}, \\ \Delta_5 = (1/2)\inf\{\Delta_2, \Delta_3\}, \quad \Gamma = \Delta_4 \Delta_5^{-1}. \end{gathered}$$

Notice that $\boldsymbol{\gamma}(1) \geq 1/2$, hence $2\Delta_5 \leq \Gamma_{7.3}(2M)^{-1}$.

Suppose m, n, Q, U, W, S, Σ, and r satisfy the hypotheses in the body of the theorem with Γ.

Abbreviate $A = \{x : \mathbf{U}(x, r) \subset U\}$. Clearly, if $m = 1$ then $\|S\|(U) \leq \Delta_5$. Observe, if $m > 1$ then

$$\|S\|(X) \leq \Delta_5 \|W\|(X)^{1-1/m} \quad \text{whenever } X \subset U.$$

Next, the following assertion will be shown: *If $a \in A \cap \operatorname{spt}\|W\|$ then there exists $0 < s < r$ such that*

$$\Delta_5 \|W\|(\mathbf{B}(a, s))^{1-1/m} < \|\Sigma\|\,\mathbf{B}(a, s).$$

Since $\|\delta W\| \mathbf{B}(a,s) \leq \Delta_5 \|W\|(\mathbf{B}(a,s))^{1-1/m} + \|\Sigma\| \mathbf{B}(a,s)$, it is sufficient to exhibit $0 < s < r$ with

$$2\Delta_5 \|W\|(\mathbf{B}(a,s))^{1-1/m} < \|\delta W\| \mathbf{B}(a,s).$$

As $\|W\| \mathbf{U}(a,r) \leq \|W\|(U) < \big(Q - (2M)^{-1}\big)\boldsymbol{\alpha}(m) r^m$ the nonexistence of such s would imply by use of [Men09, 2.5] and 7.3

$$\begin{aligned}\Delta_1 (2m\boldsymbol{\gamma}(m))^{-m} r^m &\leq \Delta_1 \|W\| \mathbf{U}(a,r) \\ &< \|W\|(\mathbf{U}(a,r) \cap \{x : \boldsymbol{\Theta}^m(\|W\|, x) < Q\}) \leq \Delta_5 r^m,\end{aligned}$$

a contradiction.

By the assertion of the preceding paragraph there exist countable disjointed families of closed balls $G_1, \ldots, G_{\boldsymbol{\beta}(n)}$ such that

$$A \cap \operatorname{spt} \|W\| \subset \bigcup \bigcup \{G_i : i = 1, \ldots, \boldsymbol{\beta}(n)\} \subset U,$$
$$\|W\|(B)^{1-1/m} \leq \Delta_5^{-1} \|\Sigma\|(B) \quad \text{whenever } B \in G_i \text{ and } i \in \{1, \ldots, \boldsymbol{\beta}(n)\}.$$

If $m > 1$ then, defining $\beta = m/(m-1)$, one estimates

$$\begin{aligned}\|W\|(A) &\leq \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{B \in G_i} \|W\|(B) \leq \Delta_5^{-\beta} \sum_{i=1}^{\boldsymbol{\beta}(n)} \sum_{B \in G_i} \|\Sigma\|(B)^\beta \\ &\leq \Delta_5^{-\beta} \textstyle\sum_{i=1}^{\boldsymbol{\beta}(n)} \big(\sum_{B \in G_i} \|\Sigma\|(B)\big)^\beta \leq \Delta_5^{-\beta} \boldsymbol{\beta}(n) \|\Sigma\|(U)^\beta.\end{aligned}$$

If $m = 1$ and $\|W\|(A) > 0$ then $\Delta_5 \leq \|\Sigma\|(B) \leq \|\Sigma\|(U)$ for some $B \in \bigcup \{G_i : i = 1, \ldots, \boldsymbol{\beta}(n)\}$. ☐

7.9 Corollary. *Suppose m, n, U, V, E, and B are as in 7.1, $1 \leq Q \leq M < \infty$, $n \leq M$, $\Lambda = \Gamma_{7.8}(M)$, $0 < r < \infty$, and*

$$\|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \quad \|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Lambda^{-1} r^m.$$

Then there holds

$$\|V\|(E \cap \{x : \mathbf{U}(x,r) \subset \mathbf{R}^n \sim B\})^{1-1/m} \leq \Lambda\big(\|V \, \partial E\|(U) + \|\delta V\|(E)\big),$$

where $0^0 = 0$.

Proof. Define W as in 7.1 and note $\boldsymbol{\Theta}^m(\|W\|, x) = \boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|W\|$ almost all x by [Fed69, 2.8.18, 2.9.11]. Therefore applying 7.8 with U, S, and Σ replaced by $\mathbf{R}^n \sim B$, 0, and δW yields the conclusion. ☐

7.10 Remark. The case $B = \operatorname{Bdry} U$ and $Q = 1$ corresponds to Hutchinson [Hut90, Theorem 1] which is formulated in the context of functions rather than sets.

7.11 Corollary. *Suppose m, n, p, U, V, and ψ are as in 6.1, $p = m$, $n \leq M$, E and B are related to m, n, U, and V, as in 7.1, $1 \leq Q \leq M < \infty$, $\Lambda = \Gamma_{7.8}(M)$, $0 < r < \infty$, and*

$$\|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m, \quad \psi(E)^{1/m} \leq \Lambda^{-1},$$
$$\|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Lambda^{-1} r^m.$$

Then there holds

$$\|V\|(E \cap \{x : \mathbf{U}(x,r) \subset \mathbf{R}^n \sim B\})^{1-1/m} \leq \Lambda \, \|V \, \partial E\|(U),$$

where $0^0 = 0$.

Proof. Define W as in 7.1 and note $\boldsymbol{\Theta}^m(\|W\|, x) = \boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|W\|$ almost all x by [Fed69, 2.8.18, 2.9.11]. If $m > 1$ then approximation shows

$$((\delta V) \llcorner E)(\theta|U) = -\textstyle\int_E \mathbf{h}(V, x) \bullet \theta(x) \,\mathrm{d}\|V\| x \quad \text{for } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n);$$

in fact since $\|(\delta V) \llcorner E\| = \|\delta V\| \llcorner E$ and $\theta|U \in \mathbf{L}_1(\|\delta V\| \llcorner E, \mathbf{R}^n)$ the problem reduces to the case $\operatorname{spt} \theta \subset U$ which is readily treated. Therefore applying 7.8 with U, $S(\theta)$ and $\Sigma(\theta)$ replaced by $\mathbf{R}^n \sim B$, $((\delta V) \llcorner E)(\theta|U)$ and $-(V \,\partial E)(\theta|U)$ yields the conclusion. □

7.12 Remark. Evidently, considering $E = U$, $B = \operatorname{Bdry} U$, $Q = 1$, and stationary varifolds V such that $\|V\|(U)$ is a small positive number shows that the intersection with $\{x : \mathbf{U}(x, r) \subset \mathbf{R}^n \sim B\}$ cannot be omitted in the conclusion.

7.13 Remark. Since estimating $\|\delta V\|(E)$ by means of Hölder's inequality seems insufficient to derive 7.11 from 7.9 by means of "absorption", this procedure was implemented at an earlier stage and led to the formulation of 7.8.

7.14 Remark. It is instructive to consider the following situation. Suppose $m, n \in \mathscr{P}$, $1 \leq m \leq n$, $1 \leq M < \infty$, $a \in \mathbf{R}^n$, $0 < r < \infty$, $0 \leq \kappa < \infty$, $1 < \lambda < \infty$, $U = \mathbf{U}(a, \lambda r)$, $B = \operatorname{Bdry} U$, $A = \{x : \mathbf{U}(x, r) \subset \mathbf{R}^n \sim B\}$, hence $A = \mathbf{B}(a, (\lambda - 1)r)$, $V \in \mathbf{IV}_m(U)$ is a stationary varifold, and E is a $\|V\|$ measurable set satisfying the relative isoperimetric estimate

$$\inf\{\|V\|(A \cap E), \|V\|(A \sim E)\}^{1-1/m} \leq \kappa \|V \,\partial E\|(U),$$

where $0^0 = 0$. Then $\|V\|(A \cap E) \leq (1 - M^{-1})\|V\|(A)$ implies

$$\|V\|(A \cap E)^{1-1/m} \leq M\kappa \|V \,\partial E\|(U).$$

Exhibiting a suitable class of V and E such that the relative isoperimetric estimate holds with a uniform number κ is complicated by the absence of such an estimate on the catenoid.[12] If V corresponds to an absolutely area minimising locally rectifiable current in codimension one, then such uniform control was obtained for some λ in Bombieri and Giusti [BG72, Theorem 2] (and attributed by the authors to De Giorgi); see 5.11 for the link of the concept of distributional boundary employed by Bombieri and Giusti and the one of the present paper.

Finally, the term "relative isoperimetric inequality" (or estimate) is chosen in accordance with the usage of that term in the case $\|V\| = \mathscr{L}^n$ by Ambrosio, Fusco and Pallara, see [AFP00, (3.43), p. 152].

8 Basic properties of weakly differentiable functions

In this section generalised weakly differentiable functions are defined in 8.3. Properties studied include behaviour under composition, see 8.12, 8.13 and 8.15, addition and multiplication, see 8.20 (3) (4), and decomposition of the varifold, see 8.24 and 8.33. Moreover, coarea formulae in terms of the distributional boundary of superlevel sets are established, see 8.1 and 8.29. A measure theoretic description of the boundary of the superlevel sets will appear in 12.2. The theory is illustrated by examples in 8.25 and 8.27.

[12] See for instance [Oss86, p. 18] for a description of the catenoid.

8.1 Lemma. *Suppose $m, n \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{V}_m(U)$, $\|\delta V\|$ is a Radon measure, f is a real valued $\|V\| + \|\delta V\|$ measurable function with* $\operatorname{dmn} f \subset U$*, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.*

Then there exists a unique $T \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ such that, see 2.16,

$$(\delta V)((\omega \circ f)\theta) = \textstyle\int \omega(f(x)) P_\natural \bullet D\theta(x) \, \mathrm{d}V(x, P) + T_{(x,y)}(\omega'(y)\theta(x))$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\omega \in \mathscr{E}(\mathbf{R}, \mathbf{R})$, $\operatorname{spt} \omega'$ is compact and $\inf \operatorname{spt} \omega > -\infty$. Moreover, there holds

$$T(\phi) = \textstyle\int V \, \partial E(y)(\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y \quad \textit{for } \phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n).$$

Proof. Define $C = \{(x, y) : x \in E(y)\}$ and $g : (U \times \mathbf{G}(n, m)) \times \mathbf{R} \to U \times \mathbf{R}$ by $g((x, P), y) = (x, y)$ for $x \in U$, $P \in \mathbf{G}(n, m)$ and $y \in \mathbf{R}$. Using [Fed69, 2.2.2, 2.2.3, 2.2.17, 2.6.2], one obtains that C is $\|V\| \times \mathscr{L}^1$ and $\|\delta V\| \times \mathscr{L}^1$ measurable, hence that $g^{-1}[C]$ is $V \times \mathscr{L}^1$ measurable since $\|V\| \times \mathscr{L}^1 = g_\#(V \times \mathscr{L}^1)$.

Define $p : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ by $p(x, y) = x$ for $(x, y) \in \mathbf{R}^n \times \mathbf{R}$ and let $T \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ be defined by

$$\begin{aligned} T(\phi) = {} & \textstyle\int_C \eta(V, x) \bullet \phi(x, y) \, \mathrm{d}(\|\delta V\| \times \mathscr{L}^1)(x, y) \\ & - \textstyle\int_{g^{-1}[C]} P_\natural \bullet (D\phi(x, y) \circ p^*) \, \mathrm{d}(V \times \mathscr{L}^1)((x, P), y) \end{aligned}$$

whenever $\phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n)$, see 4.4. Fubini's theorem then yields the two equations. The uniqueness of T follows from 3.1. □

8.2 Remark. Notice that characterising equation for T also holds if the requirement $\inf \operatorname{spt} \omega > -\infty$ is dropped.

8.3 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and Y is a finite dimensional normed vectorspace.

Then a Y valued $\|V\| + \|\delta V\|$ measurable function f with $\operatorname{dmn} f \subset U$ is called *generalised V weakly differentiable* if and only if for some $\|V\|$ measurable $\operatorname{Hom}(\mathbf{R}^n, Y)$ valued function F the following two conditions hold:

(1) If K is a compact subset of U and $0 \leq s < \infty$, then

$$\textstyle\int_{K \cap \{x : |f(x)| \leq s\}} \|F\| \, \mathrm{d}\|V\| < \infty.$$

(2) If $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} D\gamma$ is compact then

$$\begin{aligned} & (\delta V)((\gamma \circ f)\theta) \\ & \quad = \textstyle\int \gamma(f(x)) P_\natural \bullet D\theta(x) \, \mathrm{d}V(x, P) + \int \langle \theta(x), D\gamma(f(x)) \circ F(x) \rangle \, \mathrm{d}\|V\| x. \end{aligned}$$

The function F is $\|V\|$ almost unique. Therefore, one may define the *generalised V weak derivative of f* to be the function $V\,\mathbf{D}f$ characterised by $a \in \operatorname{dmn} V\,\mathbf{D}f$ if and only if

$$(\|V\|, C) \operatorname*{ap\,lim}_{x \to a} F(x) = \sigma \quad \text{for some } \sigma \in \operatorname{Hom}(\mathbf{R}^n, Y)$$

and in this case $V\,\mathbf{D}f(a) = \sigma$, where $C = \{(a, \mathbf{B}(a, r)) : \mathbf{B}(a, r) \subset U\}$. Moreover, the set of all Y valued generalised V weakly differentiable functions will be denoted by $\mathbf{T}(V, Y)$ and $\mathbf{T}(V) = \mathbf{T}(V, \mathbf{R})$.

8.4 Remark. Condition (2) is equivalent to the following condition:

(2)′ If $\zeta \in \mathscr{D}(U, \mathbf{R})$, $u \in \mathbf{R}^n$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} D\gamma$ is compact, then

$$\begin{aligned}&(\delta V)((\gamma \circ f)\zeta \cdot u)\\ &\qquad = \textstyle\int \gamma(f(x)) \langle P_\natural(u), D\zeta(x) \rangle + \zeta(x) \langle u, D\gamma(f(x)) \circ F(x) \rangle \, \mathrm{d}V(x, P).\end{aligned}$$

If $\dim Y \geq 2$ or f is locally bounded then one may require $\operatorname{spt} \gamma$ to be compact in (2) or (2)′ as $\dim Y \geq 2$ implies that $Y \sim \mathbf{B}(0, s)$ is connected for $0 < s < \infty$.

8.5 Remark. If $f \in \mathbf{T}(V)$ then the distribution T associated to f in 8.1 is representable by integration and, see 2.12, 3.1, and 3.3 with $J = \mathbf{R}$,

$$\begin{aligned} T(\phi) &= \textstyle\int \langle \phi(x, f(x)), V\,\mathbf{D}f(x) \rangle \, \mathrm{d}\|V\|x, \\ \textstyle\int g \, \mathrm{d}\|T\| &= \textstyle\int g(x, f(x)) |V\,\mathbf{D}f(x)| \, \mathrm{d}\|V\|x \end{aligned}$$

whenever $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function.

8.6 Remark. If $f \in \mathbf{T}(V, Y)$, $\theta : U \to \mathbf{R}^n$ is Lipschitzian with compact support, $\gamma : Y \to \mathbf{R}$ is of class 1, and either $\operatorname{spt} D\gamma$ is compact of f is locally bounded, then

$$\begin{aligned}(\delta V)((\gamma \circ f)\theta) = \textstyle\int \gamma(f(x)) P_\natural \bullet ((\|V\|, m) \operatorname{ap} D\theta(x) \circ P_\natural) \, \mathrm{d}V(x, P) \\ + \textstyle\int \langle \theta(x), D\gamma(f(x)) \circ V\,\mathbf{D}f(x) \rangle \, \mathrm{d}\|V\|x\end{aligned}$$

as may be verified by means of approximation and [Men12a, 4.5 (3)]. Consequently, if $f \in \mathbf{T}(V, Y)$ is locally bounded, Z is a finite dimensional normed vectorspace, and $g : Y \to Z$ is of class 1, then $g \circ f \in \mathbf{T}(V, Z)$ with

$$V\,\mathbf{D}(g \circ f)(x) = Dg(f(x)) \circ V\,\mathbf{D}f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

8.7 Example. If $f : U \to Y$ is a locally Lipschitzian function then f is generalised V weakly differentiable with

$$V\,\mathbf{D}f(x) = (\|V\|, m) \operatorname{ap} Df(x) \circ \operatorname{Tan}^m(\|V\|, x)_\natural \quad \text{for } \|V\| \text{ almost all } x,$$

as may be verified by means of [Men12a, 4.5 (4)]. Moreover, if $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, then the equality holds for any $f \in \mathbf{T}(V, Y)$ as will be shown in 11.2.

8.8 Remark. The prefix “generalised” has been chosen in analogy with the notion of “generalised function of bounded variation” treated in [AFP00, §4.5] originating from De Giorgi and Ambrosio [DGA88].

8.9 Remark. The usefulness of partial integration identities involving the first variation in defining a concept of weakly differentiable functions on varifolds has already been “expected” by Anzellotti, Delladio and Scianna who developed two notions of functions of bounded variation on integral currents, see [ADS96, p. 261],

8.10 Remark. In order to define a concept of “generalised (real valued) function of bounded variation” with respect to a varifold, it could be of interest to study the class of those functions f satisfying the hypotheses of 8.1 such that the associated function T is representable by integration.

8.11 Remark. A concept related to the present one has been proposed by Moser in [Mos01, Definition 4.1] in the context of curvature varifolds (see 15.4 and 15.5); in fact, it allows for certain "multiple-valued" functions. In studying convergence of pairs of varifolds and weakly differentiable functions, it would seem natural to investigate the extension of the present concept to such functions. (Notice that the usage of the term "multiple-valued" here is different but related to the one of Almgren in [Alm00, §1]).

8.12 Lemma. *Suppose m, n, U, V, and Y are as in 8.3, $f \in \mathbf{T}(V, Y)$, Z is a finite dimensional normed vectorspace, $0 \leq \kappa < \infty$, Υ is a closed subset of Y, $g : Y \to Z$, $H : Y \to \operatorname{Hom}(Y, Z)$, $g_i : Y \to Z$ is a sequence of functions of class 1, $\operatorname{spt} Dg_i \subset \Upsilon$, $\operatorname{Lip} g_i \leq \kappa$, $g|\Upsilon$ is proper, and*

$$g(y) = \lim_{i\to\infty} g_i(y) \quad \textit{uniformly in } y \in Y,$$
$$H(y) = \lim_{i\to\infty} Dg_i(y) \quad \textit{for } y \in Y.$$

Then $g \circ f \in \mathbf{T}(V, Z)$ and

$$V\,\mathbf{D}(g \circ f)(x) = H(f(x)) \circ V\,\mathbf{D}f(x) \quad \textit{for } \|V\| \textit{ almost all } x.$$

Proof. Note $\operatorname{Lip} g \leq \kappa$, $\|H(y)\| \leq \kappa$ for $y \in Y$, and $H|Y \sim \Upsilon = 0$, hence

$$\{y : |g(y)| \leq s\} \subset \{y : H(y) = 0\} \cup (g|\Upsilon)^{-1}[\mathbf{B}(0, s)],$$
$$\textstyle\int_{K \cap \{x : |g(f(x))| \leq s\}} \|H(f(x)) \circ V\,\mathbf{D}f(x)\| \,\mathrm{d}\|V\|x < \infty$$

whenever K is a compact subset of U and $0 \leq s < \infty$.

Suppose $\gamma \in \mathscr{E}(Z, \mathbf{R})$ and $0 \leq s < \infty$ with $\operatorname{spt} D\gamma \subset \mathbf{U}(0, s)$ and $C = (g|\Upsilon)^{-1}[\mathbf{B}(0, s)]$. Then $\operatorname{im} \gamma$ is bounded, C is compact and

$$Y \cap \{y : D\gamma(g_i(y)) \circ Dg_i(y) \neq 0\} \subset (g_i|\Upsilon)^{-1}[\operatorname{spt} D\gamma] \subset C \quad \text{for large } i,$$

in particular $\operatorname{spt} D(\gamma \circ g_i) \subset C$ for such i. Using 8.6, it follows

$$\begin{aligned}(\delta V)((\gamma \circ g_i \circ f)\theta) = {} & \textstyle\int \gamma(g_i(f(x))) P_\natural \bullet D\theta(x) \,\mathrm{d}V(x, P) \\ & + \textstyle\int \langle \theta(x), D\gamma(g_i(f(x))) \circ Dg_i(f(x)) \circ V\,\mathbf{D}f(x) \rangle \,\mathrm{d}\|V\|x\end{aligned}$$

for $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and considering the limit $i \to \infty$ yields the conclusion. □

8.13 Example. Amongst the functions g and H admitting an approximation as in 8.12 are the following:

(1) If $L : Y \to Y$ is a linear automorphism of Y, then $g = L$ and $H = DL$ is admissible.

(2) If $b \in Y$ then $g = \boldsymbol{\tau}_b$ with $H = D\boldsymbol{\tau}_b$ is admissible.

(3) If Y is an inner product space and $b \in Y$ then one may take g and H such that $g(y) = |y - b|$ for $y \in Y$,

$$H(y)(v) = |y - b|^{-1}(y - b) \bullet v \text{ if } y \neq b, \quad H(y) = 0 \text{ if } y = b$$

whenever $v, y \in Y$.

(4) If $Y = \mathbf{R}$ and $b \in \mathbf{R}$ then one may take g and H such that

$$g(y) = \sup\{y, b\}, \quad H(y)(v) = v \text{ if } y > b, \quad H(y) = 0 \text{ if } y \leq b$$

whenever $v, y \in \mathbf{R}$.

(1) and (2) are trivial.

To prove (3), assume $Y = \mathbf{R}^l$ for some $l \in \mathscr{P}$ by (1), choose $\varrho \in \mathscr{D}(Y, \mathbf{R})^+$ with $\int \varrho \, \mathrm{d}\mathscr{L}^l = 1$ and $\varrho(y) = \varrho(-y)$ for $y \in Y$ and take $\kappa = 1$, $\Upsilon = Y$, and $g_i = \varrho_{1/i} * g$ in 8.12 noting $(\varrho_{1/i} * g)(b - y) = (\varrho_{1/i} * g)(b + y)$ for $y \in Y$, hence $D(\varrho_{1/i} * g)(b) = 0$.

To prove (4), choose $\varrho \in \mathscr{D}(\mathbf{R}, \mathbf{R})^+$ with $\int \varrho \, \mathrm{d}\mathscr{L}^1 = 1$, $\operatorname{spt} \varrho \subset \mathbf{B}(0, 1)$ and $\varepsilon = \inf \operatorname{spt} \varrho > 0$, and take $\kappa = 1$, $\Upsilon = \mathbf{R} \cap \{y : y \geq b\}$, and $g_i = \varrho_{1/i} * g$ in 8.12 noting $g_i(y) = b$ if $-\infty < y \leq b + \varepsilon/i$, hence $Dg_i(y) = 0$ for $-\infty < y \leq b$.

8.14 Lemma. *Suppose U is an open subset of $\mathbf{R}^n$, μ is a Radon measure over U, Y is a finite dimensional normed vectorspace, $g \in \mathbf{L}_1(\mu)$, K denotes the set of all $f \in \mathbf{L}_1(\mu, Y)$ such that*

$$|f(x)| \leq g(x) \quad \text{for } \mu \text{ almost all } x,$$

$L_1(\mu, Y) = \mathbf{L}_1(\mu, Y)/\{f : \mu_{(1)}(f) = 0\}$ is the (usual) quotient Banach space, and $\pi : \mathbf{L}_1(\mu, Y) \to L_1(\mu, Y)$ denotes the canonical projection.

Then $\pi[K]$ with the topology induced by the weak topology on $L_1(\mu, Y)$ is compact and metrisable.

Proof. First, notice that $\pi[K]$ is convex and closed, hence weakly closed by [DS88, V.3.13]. Therefore one may assume that $Y = \mathbf{R}$ as a basis of Y induces a linear homeomorphism $L_1(\mu, \mathbf{R})^{\dim Y} \simeq L_1(\mu, Y)$ with respect to the weak topologies on $L_1(\mu, \mathbf{R})$ and $L_1(\mu, Y)$. Since $L_1(\mu, \mathbf{R})$ is separable, the conclusion now follows combining [DS88, IV.8.9, V.6.1, V.6.3]. □

8.15 Lemma. *Suppose m, n, U, V, and Y are as in 8.3, $f \in \mathbf{T}(V, Y)$, Z is a finite dimensional normed vectorspace, Υ is a closed subset of Y, c is the characteristic function of $f^{-1}[\Upsilon]$, and $g : Y \to Z$ is a Lipschitzian function such that $g|\Upsilon$ is proper and $g|Y \sim \Upsilon$ is locally constant.*

Then $g \circ f \in \mathbf{T}(V, Z)$ and

$$\|V \, \mathbf{D}(g \circ f)(x)\| \leq \operatorname{Lip}(g) c(x) \|V \, \mathbf{D} f(x)\| \quad \text{for } \|V\| \text{ almost all } x.$$

Proof. Suppose $0 < \varepsilon \leq 1$ and abbreviate $\kappa = \operatorname{Lip} g$.

Define $B = Y \cap \{y : \operatorname{dist}(y, \Upsilon) \leq \varepsilon\}$ and let b denote the characteristic function of $f^{-1}[B]$. Since $g|B$ is proper, one may employ convolution to construct $g_i \in \mathscr{E}(Y, Z)$ satisfying $\operatorname{Lip} g_i \leq \kappa$, $\operatorname{spt} Dg_i \subset B$ and

$$\delta_i = \sup\{|(g - g_i)(y)| : y \in Y\} \to 0 \quad \text{as } i \to \infty.$$

Therefore, if $\delta_i < \infty$ then $g_i|B$ is proper and $g_i \circ f \in \mathbf{T}(V, Z)$ with

$$\|V \, \mathbf{D}(g_i \circ f)(x)\| \leq \kappa \, b(x) \|V \, \mathbf{D} f(x)\| \quad \text{for } \|V\| \text{ almost all } x$$

by 8.12 with Υ, g, and H replaced by B, g_i, and Dg_i. Choose a sequence of compact sets K_j such that $K_j \subset \operatorname{Int} K_{j+1}$ for $j \in \mathscr{P}$ and $U = \bigcup_{j=1}^{\infty} K_j$

and define $E(j) = K_j \cap \{x : |f(x)| < j\}$ for $j \in \mathscr{P}$. In view of 8.14, possibly passing to a subsequence by means of a diagonal process, there exist functions $F_j \in \mathbf{L}_1(\|V\| \llcorner E(j), \operatorname{Hom}(\mathbf{R}^n, Z))$ such that

$$\|F_j(x)\| \leq \kappa\, b(x) \|V\, \mathbf{D} f(x)\| \quad \text{for } \|V\| \text{ almost all } x \in E_j,$$
$$\textstyle\int_{E(j)} \langle V\, \mathbf{D}(g_i \circ f), G\rangle \,\mathrm{d}\|V\| \to \int_{E(j)} \langle F_j, G\rangle \,\mathrm{d}\|V\| \quad \text{as } i \to \infty$$

whenever $G \in \mathbf{L}_\infty\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Z)^*\big)$ and $j \in \mathscr{P}$. Noting $E_j \subset E_{j+1}$ and $F_j(x) = F_{j+1}(x)$ for $\|V\|$ almost all $E(j)$ for $j \in \mathscr{P}$, one may define a $\|V\|$ measurable function F by $F(x) = \lim_{j \to \infty} F_j(x)$ whenever $x \in U$.

In order to verify $g \circ f \in \mathbf{T}(V, Z)$ with $V\, \mathbf{D}(g \circ f)(x) = F(x)$ for $\|V\|$ almost all x, suppose $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\gamma \in \mathscr{E}(Z, \mathbf{R})$ with $\operatorname{spt} D\gamma$ compact. Then there exists $j \in \mathscr{P}$ with $\operatorname{spt} \theta \subset K_j$ and $(g|B)^{-1}[\operatorname{spt} D\gamma] \subset \mathbf{U}(0, j)$. Define $G_i, G \in \mathbf{L}_\infty\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Z)^*\big)$ by the requirements

$$\langle \sigma, G_i(x)\rangle = \langle \theta(x), D\gamma(g_i(f(x))) \circ \sigma\rangle, \quad \langle \sigma, G(x)\rangle = \langle \theta(x), D\gamma(g(f(x))) \circ \sigma\rangle$$

whenever $x \in \operatorname{dmn} f$ and $\sigma \in \operatorname{Hom}(\mathbf{R}^n, Z)$, hence

$$\|V\|_{(\infty)}(G_i - G) \to 0 \quad \text{as } i \to \infty.$$

Observing $(g_i|B)^{-1}[\operatorname{spt} D\gamma] \subset \mathbf{U}(0, j)$ for large i, one infers

$$\begin{aligned}
&\textstyle\int \langle \theta(x), D\gamma(g(f(x))) \circ F(x)\rangle \,\mathrm{d}\|V\| x = \int_{E(j)} \langle F, G\rangle \,\mathrm{d}\|V\| \\
&\quad = \lim_{i \to \infty} \textstyle\int_{E(j)} \langle V\, \mathbf{D}(g_i \circ f)(x), G_i(x)\rangle \,\mathrm{d}\|V\| x \\
&\quad = \lim_{i \to \infty} \textstyle\int \langle \theta(x), D\gamma(g_i(f(x)) \circ V\, \mathbf{D}(g_i \circ f)(x)\rangle \,\mathrm{d}\|V\| x
\end{aligned}$$

as $\langle F, G\rangle(x) = 0 = \langle V\, \mathbf{D}(g_i \circ f), G_i\rangle(x)$ for $\|V\|$ almost all $x \in U \sim E(j)$. ☐

8.16 Remark. Taking $\Upsilon = Y$ and $g(y) = |y|$ for $y \in Y$ yields $|f| \in \mathbf{T}(V)$ and

$$\|V\, \mathbf{D}|f|(x)\| \leq \|V\, \mathbf{D} f(x)\| \quad \text{for } \|V\| \text{ almost all } x.$$

8.17. *Whenever Y is a finite dimensional normed vectorspace, there exists a family of functions $g_s \in \mathscr{D}(Y, Y)$ with $0 < s < \infty$ satisfying*

$$g_s(y) = y \quad \textit{whenever } y \in Y \cap \mathbf{B}(0, s),\ 0 < s < \infty,$$
$$\sup\{\operatorname{Lip} g_s : 0 < s < \infty\} < \infty;$$

in fact, one may assume $Y = \mathbf{R}^l$ for some $l \in \mathscr{P}$, select $\omega \in \mathscr{D}(\mathbf{R}, \mathbf{R})$ such that

$$0 \leq \omega(t) \leq t \quad \text{for } 0 \leq t < \infty, \qquad \omega(t) = t \quad \text{for } -1 \leq t \leq 1,$$

define $\omega_s = s\omega \circ \boldsymbol{\mu}_{1/s}$ and $g_s \in \mathscr{D}(Y, Y)$ by

$$g_s(y) = 0 \quad \text{if } y = 0, \qquad g_s(y) = \omega_s(|y|)|y|^{-1} y \quad \text{if } y \neq 0,$$

whenever $y \in Y$ and $0 < s < \infty$, and conclude

$$\omega_s(t) = t \quad \text{for } -s \leq t \leq s, \qquad \operatorname{Lip} \omega_s = \operatorname{Lip} \omega,$$
$$D g_s(y)(v) = \omega_s'(|y|)(|y|^{-1} y) \bullet v(|y|^{-1} y) + \omega_s(|y|)|y|^{-1}\big(v - (|y|^{-1} y) \bullet v(|y|^{-1} y)\big)$$

whenever $y \in Y \sim \{0\}$, $v \in Y$, and $0 < s < \infty$, hence $\operatorname{Lip} g_s \leq 2 \operatorname{Lip} \omega < \infty$.

8.18 Lemma. *Suppose m, n, U, V, and Y are as in 8.3, $f \in \mathbf{T}(V, Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$, $V\,\mathbf{D} f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$, and $\alpha \in \mathrm{Hom}(Y, \mathbf{R})$.*

Then $\alpha \circ f \in \mathbf{T}(V)$ and

$$V\,\mathbf{D}(\alpha \circ f)(x) = \alpha \circ V\,\mathbf{D} f(x) \quad \text{for } \|V\| \text{ almost all } x,$$
$$(\delta V)((\alpha \circ f)\theta) = \alpha\big(\textstyle\int (P_\natural \bullet D\theta(x)) f(x) + \langle \theta(x), V\,\mathbf{D} f(x)\rangle \,\mathrm{d}V(x, P)\big)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$.

Proof. If f is bounded the conclusion follows from 8.6. The general case may be treated by approximation based on 8.12 and 8.17. □

8.19 Remark. If $l = 1$, $m = n$, and $\|V\|(A) = \mathscr{L}^m(A)$ for $A \subset U$, then $f \in \mathbf{T}(V)$ if and only if f belongs to the class $\mathscr{T}^{1,1}_{\mathrm{loc}}(U)$ introduced by Bénilan, Boccardo, Gallouët, Gariepy, Pierre and Vázquez in [BBG+95, p. 244] and in this case $V\,\mathbf{D} f$ corresponds to “the derivative Df of $f \in \mathscr{T}^{1,1}_{\mathrm{loc}}(U)$” of [BBG+95, p. 246] as may be verified by use of 8.12, 8.13 (1) (4), 8.18, and [BBG+95, 2.1, 2.3].

8.20 Theorem. *Suppose m, n, U, V, and Y are as in 8.3, and $f \in \mathbf{T}(V, Y)$.*

Then the following four statements hold:

(1) If $A = \{x : f(x) = 0\}$, then

$$V\,\mathbf{D} f(x) = 0 \quad \text{for } \|V\| \text{ almost all } x \in A.$$

(2) If Z is a finite dimensional normed vectorspace, $g : U \to Z$ is locally Lipschitzian, and $h(x) = (f(x), g(x))$ for $x \in \operatorname{dmn} f$, then $h \in \mathbf{T}(V, Y \times Z)$ and

$$V\,\mathbf{D} h(x)(u) = (V\,\mathbf{D} f(x)(u), V\,\mathbf{D} g(x)(u)) \quad \text{whenever } u \in \mathbf{R}^n$$

for $\|V\|$ almost all x.

(3) If $g : U \to Y$ is locally Lipschitzian, then $f + g \in \mathbf{T}(V, Y)$ and

$$V\,\mathbf{D}(f + g)(x) = V\,\mathbf{D} f(x) + V\,\mathbf{D} g(x) \quad \text{for } \|V\| \text{ almost all } x.$$

(4) If $f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, Y)$, $V\,\mathbf{D} f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$, and $g : U \to \mathbf{R}$ is locally Lipschitzian, then $gf \in \mathbf{T}(V, Y)$ and

$$V\,\mathbf{D}(gf)(x) = V\,\mathbf{D} g(x)\, f(x) + g(x) V\,\mathbf{D} f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

Proof of (1). By 8.17 in conjunction with 8.12 one may assume f to be bounded and $V\,\mathbf{D} f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$, hence by 8.18 also $Y = \mathbf{R}$. In this case it follows from 8.12 and 8.13 (1) (4) that f^+ and f^- satisfy the same hypotheses as f, hence

$$(\delta V)(g\theta) = \textstyle\int (P_\natural \bullet D\theta(x)) g(x) + \langle \theta(x), V\,\mathbf{D} g(x)\rangle \,\mathrm{d}V(x, P)$$

for $\theta \in \mathscr{D}(U, \mathbf{R})$ and $g \in \{f, f^+, f^-\}$ by 8.18. Since $f = f^+ - f^-$, this implies

$$V\,\mathbf{D} f(x) = V\,\mathbf{D} f^+(x) - V\,\mathbf{D} f^-(x) \quad \text{for } \|V\| \text{ almost all } x$$

and the formulae derived in 8.13 (4) yield the conclusion. □

Proof of (2). Assume $\dim Y > 0$. Define a $\|V\|$ measurable function H with values in $\operatorname{Hom}(\mathbf{R}^n, Y \times Z)$ by

$$H(x)(u) = (V\,\mathbf{D}f(x)(u), V\,\mathbf{D}g(x)(u)) \quad \text{for } u \in \mathbf{R}^n$$

whenever $x \in \operatorname{dmn} V\,\mathbf{D}f \cap \operatorname{dmn} V\,\mathbf{D}g$. It will proven that

$$(\delta V)((\gamma \circ h)\theta) = \textstyle\int \gamma(h(x))P_\natural \bullet D\theta(x) + \langle \theta(x), D\gamma(h(x)) \circ H(x)\rangle \,\mathrm{d}V(x, P)$$

whenever $\gamma \in \mathscr{D}(Y \times Z, \mathbf{R})$ and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$; in fact, in view of 2.12 and 3.1 the problem reduces to the case that, for some $\mu \in \mathscr{D}(Y, \mathbf{R})$ and some $\nu \in \mathscr{D}(Z, \mathbf{R})$,

$$\gamma(y, z) = \mu(y)\nu(z) \quad \text{for } (y, z) \in Y \times Z$$

in which case one computes, using 8.6 and 8.7,

$$\begin{aligned}
&(\delta V)((\gamma \circ h)\theta) = (\delta V)((\mu \circ f)(\nu \circ g)\theta) \\
&\quad = \textstyle\int \mu(f(x))\big(\langle \theta(x), D\nu(g(x)) \circ V\,\mathbf{D}g(x)\rangle + \nu(g(x))P_\natural \bullet D\theta(x)\big) \,\mathrm{d}V(x, P) \\
&\qquad + \textstyle\int \nu(g(x))\,\langle \theta(x), D\mu(f(x)) \circ V\,\mathbf{D}f(x)\rangle \,\mathrm{d}\|V\|x \\
&\quad = \textstyle\int \gamma(h(x))P_\natural \bullet D\theta(x) + \langle \theta(x), D\gamma(h(x)) \circ H(x)\rangle \,\mathrm{d}V(x, P).
\end{aligned}$$

If $\dim Y \geq 2$ or f is bounded the conclusion now follows from 8.4. Finally, to approximate f in case $\dim Y = 1$, one assumes $Y = \mathbf{R}$ and employs the functions f_i defined by $f_i(x) = \sup\{\inf\{f(x), i\}, -i\}$ for $x \in \operatorname{dmn} f$ and $i \in \mathscr{P}$ and notices that $f_i \in \mathbf{T}(V)$ and

$$\begin{gathered}
|V\,\mathbf{D}f_i(x)| \leq |V\,\mathbf{D}f(x)|, \qquad V\,\mathbf{D}f_i(x) \to V\,\mathbf{D}f(x) \quad \text{as } i \to \infty, \\
|f_i(x)| < |f(x)| \text{ implies } V\,\mathbf{D}f_i(x) = 0
\end{gathered}$$

for $\|V\|$ almost all x by 8.12, 8.13 (1) (4). $\square$

Proof of (3). Assume $\dim Y > 0$ and that $\operatorname{im} g \subset \mathbf{B}(0, t)$ for some $0 < t < \infty$. Define h as in (2) and let $L : Y \times Y \to Y$ denote addition.

The following assertion will be shown. *If $\gamma \in \mathscr{D}(Y, \mathbf{R})$ and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, then*

$$\begin{aligned}
&(\delta V)((\gamma \circ L \circ h)\theta) \\
&\quad = \textstyle\int \gamma(L(h(x)))P_\natural \bullet D\theta(x) + \langle \theta(x), D(\gamma \circ L)(h(x)) \circ V\,\mathbf{D}h(x)\rangle \,\mathrm{d}V(x, P).
\end{aligned}$$

For this purpose define $D = Y \times (Y \cap \mathbf{B}(0, 2t))$ and choose $\varrho \in \mathscr{D}(Y, \mathbf{R})$ with

$$\mathbf{B}(0, t) \subset \operatorname{Int}\{z : \varrho(z) = 1\}, \quad \operatorname{spt} \varrho \subset \mathbf{B}(0, 2t).$$

Let $\phi : Y \times Y \to Y$ be defined by

$$\phi(y, z) = \varrho(z)(\gamma \circ L)(y, z) \quad \text{for } (y, z) \in Y \times Y.$$

Noting that

$$\begin{gathered}
L|D \text{ is proper}, \quad \operatorname{spt} \phi \subset D \cap L^{-1}[\operatorname{spt} \gamma], \quad \operatorname{spt} D\phi \text{ is compact}, \\
\phi(y, z) = (\gamma \circ L)(y, z) \quad \text{and} \quad D\phi(y, z) = D(\gamma \circ L)(y, z)
\end{gathered}$$

for $y \in Y$ and $z \in Y \cap \mathbf{B}(0, t)$, one uses (2) and 8.3 (2) with f and γ replaced by h and ϕ to infer the assertion.

If $\dim Y \geq 2$ or f is bounded the conclusion now follows from the assertion of the preceding paragraph in conjunction with 8.4. Finally, the case $\dim Y = 1$ may be treated by means of approximation as in (2). $\square$

Proof of (4). Assume g to be bounded, define h as in (2) and let $\mu : Y \times \mathbf{R} \to Y$ be defined by $\mu(y,t) = ty$ for $y \in Y$ and $t \in \mathbf{R}$. If f is bounded the conclusion follows from (2) and 8.6 with f and g replaced by h and μ. The general case then follows by approximation using 8.12 and 8.17. $\square$

8.21 Remark. The approximation procedure in the proof of (2) uses ideas from [Fed69, 4.1.2, 4.1.3].

8.22 Remark. The need for some strong restriction on g in (2) and (3) will be illustrated in 8.25.

8.23. If ϕ is a measure, A is ϕ measurable and B is a $\phi \llcorner A$ measurable subset of A, then B is ϕ measurable.

8.24 Theorem. *Suppose m, n, U, V, and Y are as in 8.3, Ξ is a decomposition of V, ξ is associated to Ξ as in 6.10, $f_W \in \mathbf{T}(W,Y)$ for $W \in \Xi$, and*

$$f = \bigcup \{ f_W | \xi(W) : W \in \Xi \}, \quad F = \bigcup \{ W \, \mathbf{D} f_W | \xi(W) : W \in \Xi \}.$$

Then the following three statements hold:

(1) f is $\|V\| + \|\delta V\|$ measurable.

(2) F is $\|V\|$ measurable.

(3) If $\int_{K \cap \{x : |f(x)| \leq s\}} \|F\| \, \mathrm{d}\|V\| < \infty$ whenever K is a compact subset of U and $0 \leq s < \infty$, then $f \in \mathbf{T}(V,Y)$ and

$$V \, \mathbf{D} f(x) = F(x) \quad \text{for } \|V\| \text{ almost all } x.$$

Proof. Clearly, $f | \xi(W) = f_W | \xi(W)$ and $F | \xi(W) = W \, \mathbf{D} f_W | \xi(W)$ for $W \in \Xi$. (1) and (2) readily follow by means of 8.23, since $\|W\| = \|V\| \llcorner \xi(W)$ and $\|\delta W\| = \|\delta V\| \llcorner \xi(W)$ for $W \in \Xi$. The hypothesis of (3) implies

$$\begin{aligned}
&(\delta V)((\gamma \circ f)\theta) = \textstyle\sum_{W \in \Xi} (\delta W)((\gamma \circ f_W)\theta) \\
&\quad = \textstyle\sum_{W \in \Xi} \int \gamma(f_W(x)) P_\natural \bullet D\theta(x) + \langle \theta(x), D\gamma(f_W(x)) \circ W \, \mathbf{D} f_W(x) \rangle \, \mathrm{d}W(x,P) \\
&\quad = \textstyle\sum_{W \in \Xi} \int_{\xi(W) \times \mathbf{G}(n,m)} \gamma(f(x)) P_\natural \bullet D\theta(x) + \langle \theta(x), D\gamma(f(x)) \circ F(x) \rangle \, \mathrm{d}V(x,P) \\
&\quad = \textstyle\int \gamma(f(x)) P_\natural \bullet D\theta(x) + \langle \theta(x), D\gamma(f(x)) \circ F(x) \rangle \, \mathrm{d}V(x,P)
\end{aligned}$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $\operatorname{spt} D\gamma$ is compact. $\square$

8.25 Example. Suppose R_j, V_j, and V are as in 6.13. Define $f : \mathbf{C} \to \mathbf{R}$, $g : \mathbf{C} \to \mathbf{R}$, and $h : \mathbf{C} \to \mathbf{R}$ by

$$\begin{gathered}
f(x) = 1 \quad \text{if } x \in R_1 \cup R_3 \cup R_5, \quad f(x) = 0 \quad \text{else}, \\
g(x) = 1 \quad \text{if } x \in R_1 \cup R_4, \qquad g(x) = 0 \quad \text{else}, \\
h(x) = (f(x), g(x))
\end{gathered}$$

whenever $x \in \mathbf{C}$.

Then $f \in \mathbf{T}(V_1 + V_3 + V_5)$, $g \in \mathbf{T}(V_1 + V_4)$, hence $f, g \in \mathbf{T}(V)$ with

$$V \, \mathbf{D} f(x) = 0 = V \, \mathbf{D} g(x) \quad \text{for } \|V\| \text{ almost all } x$$

by 8.24. However, neither $h \notin \mathbf{T}(V, \mathbf{R} \times \mathbf{R})$ nor $f + g \in \mathbf{T}(V)$ nor $gf \in \mathbf{T}(V)$; in fact the characteristic function of R_1 which equals gf does not belong to $\mathbf{T}(V)$, hence $f + g \notin \mathbf{T}(V)$ by 8.12 and $h \in \mathbf{T}(V, \mathbf{R} \times \mathbf{R})$ would imply $f + g \in \mathbf{T}(V)$ by 8.6 with f and g replaced by h and the addition on $\mathbf{R}$.

8.26 Remark. Here some properties of the class $\mathbf{W}(V,Y)$ consisting of all $f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$ such that for some $F \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$

$$(\delta V)((\alpha \circ f)\theta) = \alpha\big(\textstyle\int (P_\natural \bullet D\theta(x)) f(x) + \langle \theta(x), F(x) \rangle \, \mathrm{d}V(x,P)\big)$$

whenever $\theta \in \mathscr{D}(U, \mathbf{R}^n)$ and $\alpha \in \operatorname{Hom}(Y, \mathbf{R})$, associated with V and Y whenever m, n, U, V, and Y are as in 8.3 will be discussed briefly.

Clearly, F is $\|V\|$ almost unique and $\mathbf{W}(V,Y)$ is a vectorspace. Note that

$$\mathbf{T}(V,Y) \cap \mathbf{L}_1^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y) \cap \{ f : V\,\mathbf{D}f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)\}$$

is contained in $\mathbf{W}(V,Y)$ by 8.18. However, it may happen that $f \in \mathbf{W}(V,Y)$ but $\varrho \circ f \notin \mathbf{W}(V,Y)$ for some $\varrho \in \mathscr{D}(Y, \mathbf{R})$; in fact the function $f+g$ constructed in 8.25 provides an example.

8.27 Example. The considerations of 8.25 may be axiomatised as follows.

Suppose Φ denotes the family of stationary one dimensional integral varifolds in $\mathbf{R}^2$ and, whenever $V \in \Phi$, $C(V)$ is a class of real valued functions on $\mathbf{R}^2$ satisfying the following conditions.

(1) If Ξ is a decomposition of V, ξ is associated to Ξ as in 6.10, $\upsilon : \Xi \to \mathbf{R}$ and $f : \mathbf{R}^2 \to \mathbf{R}$ satisfies

$$f(x) = \upsilon(W) \quad \text{if } x \in \xi(W),\ W \in \Xi, \qquad f(x) = 0 \quad \text{if } x \in \mathbf{R}^2 \sim \textstyle\bigcup \operatorname{im} \xi,$$

then $f \in C(V)$.

(2) If $f, g \in C(V)$, then $f + g \in C(V)$.

(3) If $f \in C(V)$, $\omega \in \mathscr{E}(\mathbf{R}, \mathbf{R})$ and $\operatorname{spt} \omega'$ is compact, then $\omega \circ f \in C(V)$.

Then, using the terminology of 6.13, the characteristic function of any ray R_j belongs to $C(V)$. Moreover, the same holds, if the conditions (2) and (3) are replaced by the following condition.

(4) If $f, g \in C(V)$, then $fg \in C(V)$.

8.28 Lemma. *Suppose m, n, U, and V are as in 8.3, $f \in \mathbf{T}(V)$, $y \in \mathbf{R}$, and $E = \{x : f(x) > y\}$.*

Then there holds

$$V\,\partial E(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} \textstyle\int_{\{x \,:\, y < f(x) \le y + \varepsilon\}} \langle \theta, V\,\mathbf{D}f \rangle \, \mathrm{d}\|V\| \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Proof. Suppose $\theta \in \mathscr{D}(U, \mathbf{R}^n)$. Define $g_\varepsilon : \mathbf{R} \to \mathbf{R}$ by $g_\varepsilon(\upsilon) = \varepsilon^{-1} \inf\{\varepsilon, \sup\{\upsilon - y, 0\}\}$ whenever $\upsilon \in \mathbf{R}$ and $0 < \varepsilon \le 1$. Notice that

$$g_\varepsilon(\upsilon) = 0 \quad \text{if } \upsilon \le y, \qquad g_\varepsilon(\upsilon) = 1 \quad \text{if } \upsilon \ge y + \varepsilon,$$
$$g_\varepsilon(\upsilon) \uparrow 1 \quad \text{as } \varepsilon \to 0+ \text{ if } \upsilon > y$$

whenever $\upsilon \in \mathbf{R}$. Consequently, one infers $g_\varepsilon \circ f \in \mathbf{T}(V)$ with

$$V\,\mathbf{D}(g_\varepsilon \circ f)(x) = \varepsilon^{-1} V\,\mathbf{D}f(x) \text{ if } y < f(x) \le y + \varepsilon, \quad V\,\mathbf{D}(g_\varepsilon \circ f)(x) = 0 \text{ else}$$

for $\|V\|$ almost all x from 8.12, 8.13 (1) (2) (4) and 8.20 (1). It follows

$$\begin{aligned} V\,\partial E(\theta) &= \lim_{\varepsilon \to 0+} (\delta V)((g_\varepsilon \circ f)\theta) - \lim_{\varepsilon \to 0+} \textstyle\int (g_\varepsilon \circ f)(x) P_\natural \bullet D\theta(x) \, \mathrm{d}V(x,P) \\ &= \lim_{\varepsilon \to 0+} \varepsilon^{-1} \textstyle\int_{\{x \,:\, y < f(x) \le y + \varepsilon\}} \langle \theta(x), V\,\mathbf{D}f(x) \rangle \, \mathrm{d}\|V\| x, \end{aligned}$$

where 8.18 with α and f replaced by $\mathbf{1}_{\mathbf{R}}$ and $g_\varepsilon \circ f$ was employed. ☐

8.29 Theorem. *Suppose m, n, U, and V are as in 8.3, $f \in \mathbf{T}(V)$, $T \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ satisfies*

$$T(\phi) = \textstyle\int \langle \phi(x, f(x)), V\,\mathbf{D}f(x) \rangle \,\mathrm{d}\|V\| x \quad \text{for } \phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n),$$

and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.

Then T is representable by integration and

$$T(\phi) = \textstyle\int V\,\partial E(y)(\phi(\cdot, y)) \,\mathrm{d}\mathscr{L}^1 y, \quad \textstyle\int g \,\mathrm{d}\|T\| = \int\int g(x, y) \,\mathrm{d}\|V\,\partial E(y)\| x \,\mathrm{d}\mathscr{L}^1 y$$

whenever $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function. In particular, for $\mathscr{L}^1$ almost all y, the distribution $V\,\partial E(y)$ is representable by integration and $\|V\,\partial E(y)\|(U \cap \{x : f(x) \neq y\}) = 0$.

Proof. Taking 8.5 into account, 8.1 yields

$$\textstyle\int \omega \circ f \,\langle \theta, V\,\mathbf{D}f \rangle \,\mathrm{d}\|V\| = T_{(x,y)}(\omega(y)\theta(x)) = \int \omega(y) V\,\partial E(y)(\theta) \,\mathrm{d}\mathscr{L}^1 y$$

for $\omega \in \mathscr{D}(\mathbf{R}, \mathbf{R})$ and $\theta \in \mathscr{D}(U, \mathbf{R}^n)$. In view of 8.28, the principal conclusion is implied by 3.4 (1) (2) with $J = \mathbf{R}$ and $Z = \mathbf{R}^n$. The postscript follows employing 8.5 and noting that $(U \times \mathbf{R}) \cap \{(x, y) : f(x) \neq y\}$ is $\|T\|$ measurable since f is $\|V\|$ almost equal to a Borel function by [Fed69, 2.3.6]. □

8.30 Remark. The formulation of 8.29 is modelled on [Fed69, 4.5.9 (13)].

8.31 Remark. The equalities in 8.29 are not valid for functions in $\mathbf{W}(V)$ with $V\,\mathbf{D}f$ in the definition of T replaced by the function F occurring in the definition of $\mathbf{W}(V)$, see 8.26; in fact, the function $f + g$ constructed in 8.25 provides an example.

8.32 Corollary. *Suppose m, n, U, V, and Y are as in 8.3 and $f \in \mathbf{T}(V, Y)$.*

Then there holds

$$(\|\delta V\| \llcorner X)_{(\infty)}(f) \leq (\|V\| \llcorner X)_{(\infty)}(f)$$

whenever X is an open subset of U.

Proof. Assume $X = U$ and abbreviate $s = \|V\|_{(\infty)}(f)$. Recalling 8.16, one applies 8.29 with f replaced by $|f|$ to infer

$$V\,\partial E(t) = 0 \quad \text{for } \mathscr{L}^1 \text{ almost all } s < t < \infty,$$

where $E(t) = \{x : |f(x)| > t\}$, hence $\|\delta V\|(E(t)) = 0$ for those t. □

8.33 Theorem. *Suppose m, n, U, V, and Y are as in 8.3, $f \in \mathbf{T}(V, Y)$, and*

$$V\,\mathbf{D}f(x) = 0 \quad \text{for } \|V\| \text{ almost all } x.$$

Then there exists a decomposition Ξ of V and $\upsilon : \Xi \to Y$ such that

$$f(x) = \upsilon(W) \quad \text{for } \|W\| + \|\delta W\| \text{ almost all } x$$

whenever $W \in \Xi$.

Proof. Define $B(y,r) = \{x : |f(x) - y| \leq r\}$ for $y \in Y$ and $0 \leq r < \infty$. First, one observes that 8.29 in conjunction with 8.12, 8.13 (2) and 8.16 implies

$$V\,\partial B(y,r) = 0 \quad \text{for } y \in Y \text{ and } 0 \leq r < \infty.$$

Next, a countable subset Υ of Y such that

$$f(x) \in \Upsilon \quad \text{for } \|V\| \text{ almost all } x$$

will be constructed. For this purpose define $\delta_i = \boldsymbol{\alpha}(m) 2^{-m-1} i^{-1-2m}$, $\varepsilon_i = 2^{-1} i^{-2}$ and Borel sets A_i consisting of all $a \in \mathbf{R}^n$ satisfying

$$|a| \leq i, \quad \mathbf{U}(a, 2\varepsilon_i) \subset U, \quad \boldsymbol{\Theta}^m(\|V\|, a) \geq 1/i,$$
$$\|\delta V\|\,\mathbf{B}(a,s) \leq \boldsymbol{\alpha}(m) i s^m \quad \text{for } 0 < s < \varepsilon_i$$

whenever $i \in \mathscr{P}$. Clearly, $A_i \subset A_{i+1}$ for $i \in \mathscr{P}$ and $\|V\|(U \sim \bigcup_{i=1}^{\infty} A_i) = 0$ by Allard [All72, 3.5 (1a)] and [Fed69, 2.8.18, 2.9.5]. Abbreviating $\mu_i = f_{\#}(\|V\| \llcorner A_i)$, one obtains

$$\|V\|(B(y,r) \cap \{x : \operatorname{dist}(x, A_i) \leq \varepsilon_i\}) \geq \delta_i$$

whenever $y \in \operatorname{spt} \mu_i$, $0 \leq r < \infty$ and $i \in \mathscr{P}$; in fact, assuming $r > 0$ there exists $a \in A_i$ with $\boldsymbol{\Theta}^m(\|V\| \llcorner B(y,r), a) = \boldsymbol{\Theta}^m(\|V\|, a) \geq 1/i$ by [Fed69, 2.8.18, 2.9.11], hence

$$\textstyle\int_0^{\varepsilon_i} s^{-m} \|\delta(V \llcorner B(y,r) \times \mathbf{G}(n,m))\|\,\mathbf{B}(a,s)\,\mathrm{d}\mathscr{L}^1 s \leq \boldsymbol{\alpha}(m) i \varepsilon_i$$

implying $\|V\|(B(y,r) \cap \mathbf{B}(a, \varepsilon_i)) \geq \delta_i$ by 4.5 and 4.6. As $\{B(y,0) : y \in \operatorname{spt} \mu_i\}$ is disjointed, one deduces

$$\delta_i \operatorname{card} \operatorname{spt} \mu_i \leq \|V\|(U \cap \{x : \operatorname{dist}(x, A_i) \leq \varepsilon_i\}) < \infty$$

and one may take $\Upsilon = \bigcup_{i=1}^{\infty} \operatorname{spt} \mu_i$, since $\mu_i(Y \sim \operatorname{spt} \mu_i) = 0$.

Applying 6.12 to $V \llcorner B(y,0) \times \mathbf{G}(n,m)$ for $y \in \Upsilon$ and recalling 5.4 and 6.10, one constructs a decomposition Ξ of V and a function ξ mapping Ξ into the class of all $\|V\| + \|\delta V\|$ measurable sets such that distinct members of Ξ are mapped onto disjoint sets,

$$W = V \llcorner \xi(W) \times \mathbf{G}(n,m), \quad V\,\partial \xi(W) = 0$$

whenever $W \in \Xi$, and each $\xi(W)$ is contained in some $B(y,0)$. Defining $\upsilon : \Xi \to Y$ by the requirement $\{\upsilon(W)\} = f[\xi(W)]$ for $W \in \Xi$ and noting $(\|W\| + \|\delta W\|)(U \sim \xi(W)) = 0$ for $W \in \Xi$, the conclusion is now evident. □

8.34 Remark. Clearly, the second paragraph of the proof has conceptual overlap with the second and third paragraph of the proof of 6.12.

9 Zero boundary values

In this section a notion of zero boundary values for nonnegative weakly differentiable functions based on the behaviour of superlevel sets is introduced. Stability of this class under composition, see 9.9, convergence, see 9.13 and 9.14, and multiplication by a nonnegative Lipschitzian function, see 9.16, are investigated. The deeper parts of this study rest on a characterisation of such functions in terms of an associated distribution built from certain distributional boundaries of superlevel sets, see 9.12.

9.1 Definition. Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, $\|\delta V\|$ is a Radon measure, and G is a relatively open subset of $\operatorname{Bdry} U$.

Then $\mathbf{T}_G(V)$ is defined to be the set of all nonnegative functions $f \in \mathbf{T}(V)$ such that with

$$B = (\operatorname{Bdry} U) \sim G \qquad \text{and} \qquad E(y) = \{x : f(x) > y\} \quad \text{for } 0 < y < \infty$$

the following conditions hold for $\mathscr{L}^1$ almost all $0 < y < \infty$, see 2.15–2.17,

$$(\|V\| + \|\delta V\|)(E(y) \cap K) + \|V\, \partial E(y)\|(U \cap K) < \infty,$$
$$\textstyle\int_{E(y) \times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x) \,\mathrm{d}V(x,P) = ((\delta V) \llcorner E(y))(\theta|U) - V\, \partial E(y)(\theta|U)$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$ and $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$.

9.2 Remark. Notice that $|f| \in \mathbf{T}_\varnothing(V)$ whenever Y is a finite dimensional normed vectorspace and $f \in \mathbf{T}(V, Y)$ by 8.16 and 8.29.

9.3 Remark. The condition on $E(y)$ has been studied in Section 7 under the supplementary hypothesis on V that $\boldsymbol{\Theta}^m(\|V\|, x) \geq 1$ for $\|V\|$ almost all x, see 7.1. In the present section this hypothesis will not occur leaving those properties of $\mathbf{T}_G(V)$ employing the additionally hypothesis on V to Section 10.

9.4 Lemma. *Suppose m, n, U, V, G, B, and $E(y)$ are as in 9.1, $f \in \mathbf{T}_G(V)$, and*

$$\operatorname{Clos} \operatorname{spt} \big((\|V\| + \|\delta V\|) \llcorner E(y) \big) \subset \mathbf{R}^n \sim B \quad \textit{for } \mathscr{L}^1 \textit{ almost all } 0 < y < \infty.$$

Then $f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$.

Proof. Define $A(y) = \operatorname{Clos} \operatorname{spt} \big((\|V\| + \|\delta V\|) \llcorner E(y) \big)$ for $0 < y < \infty$. Suppose y satisfies the conditions of 9.1 and $A(y) \subset \mathbf{R}^n \sim B$. One concludes

$$(\|V\| + \|\delta V\|)(E(y) \cap K) + \|V\, \partial E(y)\|(U \cap K) < \infty$$

whenever K is a compact subset of $\mathbf{R}^n$. Hence one may define $S \in \mathscr{D}'(\mathbf{R}^n, \mathbf{R}^n)$ by

$$S(\theta) = \textstyle\int_{E(y) \times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x) \,\mathrm{d}V(x,P) - ((\delta V) \llcorner E(y))(\theta|U) + V\, \partial E(y)(\theta|U)$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$. Notice that $\operatorname{spt} S \subset A(y) \subset \mathbf{R}^n \sim B$. On the other hand the conditions of 9.1 imply $\operatorname{spt} S \subset B$. It follows $\operatorname{spt} S = \varnothing$ and $S = 0$. $\square$

9.5 Lemma. *Suppose m, n, U, V, G, and B are as in 9.1, $f \in \mathbf{T}_G(V)$, X is an open subset of $\mathbf{R}^n \sim B$, $H = X \cap \operatorname{Bdry} U$, and $W = V | \mathbf{2}^{(U \cap X) \times \mathbf{G}(n,m)}$.*

Then $f|X \in \mathbf{T}_H(W)$.

Proof. Notice that $H = X \cap \operatorname{Bdry}(U \cap X)$. In particular, H is a relatively open subset of $\operatorname{Bdry}(U \cap X)$. Let $C = (\operatorname{Bdry}(U \cap X)) \sim H$ and observe the inclusions

$$(\mathbf{R}^n \sim C) \cap \operatorname{Clos}(U \cap X) \subset X \subset \mathbf{R}^n \sim (B \cup C).$$

Suppose $0 < y < \infty$ satisfies the conditions of 9.1. Define $E = \{x : f(x) > y\}$ and notice that

$$(\|W\| + \|\delta W\|)(E \cap X \cap K) + \|W\, \partial(E \cap X)\|(U \cap X \cap K) < \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim C$ by the first inclusion of the first paragraph. Therefore one may define $S \in \mathscr{D}'(\mathbf{R}^n \sim C, \mathbf{R}^n)$ by

$$\begin{aligned} S(\theta) = {} & \textstyle\int_{(E\cap X)\times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x)\,\mathrm{d}W(x,P) \\ & - ((\delta W) \llcorner E \cap X)(\theta | U \cap X) + (W\,\partial(E\cap X))(\theta|U\cap X) \end{aligned}$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^n \sim C, \mathbf{R}^n)$. Notice that $\operatorname{spt} S \subset (\mathbf{R}^n \sim C) \cap \operatorname{Clos}(U \cap X) \subset X$. On the other hand the conditions of 9.1 in conjunction with the second inclusion of the first paragraph imply

$$\begin{aligned} & S(\theta|\mathbf{R}^n \sim C) \\ & \quad = \textstyle\int_{E\times\mathbf{G}(n,m)} P_\natural \bullet D\theta(x)\,\mathrm{d}V(x,P) - ((\delta V)\llcorner E)(\theta|U) + V\,\partial E(\theta|U) = 0 \end{aligned}$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^n, \mathbf{R}^n)$ and $\operatorname{spt}\theta \subset X$, hence $X \cap \operatorname{spt} S = \varnothing$. It follows $\operatorname{spt} S = \varnothing$ and $S = 0$. □

9.6 Remark. Recalling the first paragraph of the proof of 9.5, one notices that

$$U \cap \{x : \mathbf{U}(x,r) \subset X\} \subset U \cap X \cap \{x : \mathbf{U}(x,r) \subset \mathbf{R}^n \sim C\} \quad \text{for } 0 < r < \infty;$$

this fact will be useful for localisation in 10.1 (1d).

9.7 Example. Suppose $m = n = 1$, $U = \mathbf{R} \sim \{0\}$, $V \in \mathbf{RV}_m(U)$ with $\|V\|(A) = \mathscr{L}^1(A)$ for $A \subset U$, $f : U \to \mathbf{R}$ with $f(x) = 1$ for $x \in U$, $X = U \cap \{x : x < 0\}$, and $W = V|\mathbf{2}^{X\times\mathbf{G}(n,m)}$.

Then $\delta V = 0$ and $f \in \mathbf{T}_{\{0\}}(V)$ but $f|X \notin \mathbf{T}_{\{0\}}(W)$.

9.8 Remark. The preceding example shows that 9.5 may not be sharpened by allowing H to be an arbitrary relatively open subset of $\operatorname{Bdry}(U \cap X)$ contained in G.

9.9 Lemma. *Suppose m, n, U, V, and G are as in 9.1, $f \in \mathbf{T}_G(V)$, Υ is a closed subset of $\{y : 0 \leq y < \infty\}$, $g : \{y : 0 \leq y < \infty\} \to \{z : 0 \leq z < \infty\}$ is a Lipschitzian function such that $g(0) = 0$, $g|\Upsilon$ is proper and $g|\mathbf{R} \sim \Upsilon$ is locally constant.*

Then $g \circ f \in \mathbf{T}_G(V)$.

Proof. Let $h = g \cup \{(y,0) : -\infty < y < 0\}$. Since $\operatorname{Lip} h = \operatorname{Lip} g$ and $h|\mathbf{R} \sim \Upsilon$ is locally constant, 8.15 yields $g \circ f = h \circ f \in \mathbf{T}(V)$.

Define F to be the class of all Borel subsets Y of $\{y : 0 < y < \infty\}$ such that $E = f^{-1}[Y]$ satisfies

$$\begin{gathered} (\|V\| + \|\delta V\|)(E \cap K) + \|V\,\partial E\|(U \cap K) < \infty, \\ \textstyle\int_{E\times\mathbf{G}(n,m)} P_\natural \bullet D\theta(x)\,\mathrm{d}V(x,P) = ((\delta V)\llcorner E)(\theta|U) - (V\,\partial E)(\theta|U) \end{gathered}$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$ and $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$. If $Y \in F$, G is a finite disjointed subfamily of F and $\bigcup G \subset Y$, then $Y \sim \bigcup G \in F$; as one readily verifies using 5.3. Let O denote the set of $0 < b < \infty$ such that either $\{y : b < y < \infty\}$ or $\{y : b \leq y < \infty\}$ does not belong to F. Since $(\|V\| + \|\delta V\|)(f^{-1}[\{b\}]) = 0$ for all but countably many b, one obtains $\mathscr{L}^1(O) = 0$.

Next, it will be shown that

$$D = g^{-1}[\{z : c < z < \infty\}] \in F$$

whenever c satisfies $0 < c \in \mathbf{R} \sim g[O]$ and $N(g,c) < \infty$. For this purpose assume $D \neq \varnothing$. Since $\inf D > 0$ and $\operatorname{Bdry} D \subset \{y : g(y) = c\}$, one infers that $\operatorname{Bdry} D$ is a finite subset of $\{b : 0 < b < \infty\} \sim O$, in particular $Y = \{y : \inf D < y < \infty\} \in F$. Let G denote the family of connected components of $Y \sim D$. Observe that G is a finite disjointed family of possibly degenerated closed intervals. Since

$$\operatorname{Bdry} I \subset \operatorname{Bdry} D, \quad I = \{y : \inf I \leq y < \infty\} \sim \{y : \sup I < y < \infty\}$$

whenever $I \in G$, it follows $G \subset F$ and $D = Y \sim \bigcup G \in F$.

Finally, notice that $\mathscr{L}^1$ almost all $0 < c < \infty$ satisfy $N(g|\{y : y \leq i\}, c) < \infty$ for $i \in \mathscr{P}$ by [Fed69, 3.2.3 (1)], hence $\infty > N(g|\Upsilon, c) = N(g,c)$ for such c. ☐

9.10 Example. Suppose $m = n = 1$, $U = \mathbf{R} \sim \{0\}$, and $V \in \mathbf{RV}_m(U)$ with $\|V\|(A) = \mathscr{L}^1(A)$ for $A \subset U$.

Then $\delta V = 0$ and the following two statements hold.

(1) If $f = \operatorname{sign}|U$ then $f \in \mathbf{T}(V)$ and $|f| \in \mathbf{T}_{\{0\}}(V)$ but $f^+ \notin \mathbf{T}_{\{0\}}(V)$.

(2) If $y, b \in \mathbf{R}^2$, $|y| = |b|$, ν is a norm on $\mathbf{R}^2$, $\nu(y) \neq \nu(b)$, $f(x) = y$ for $0 > x \in \mathbf{R}$ and $f(x) = b$ for $0 < x \in \mathbf{R}$, then $f \in \mathbf{T}(V, \mathbf{R}^2)$ and $|f| \in \mathbf{T}_{\{0\}}(V)$ but $\nu \circ f \notin \mathbf{T}_{\{0\}}(V)$.

9.11 Remark. The preceding example shows that the class $\mathbf{T}(V,Y) \cap \{f : |f| \in \mathbf{T}_G(V)\}$, where Y is a finite dimensional normed vectorspace, does not satisfy stability properties similar to those proven for $\mathbf{T}_G(V)$ in 9.9.

9.12 Theorem. *Suppose m, n, U, V, G, and B are as in 9.1, $f \in \mathbf{T}(V)$, f is nonnegative, $J = \{y : 0 < y < \infty\}$, $A = f^{-1}[J]$, $E(y) = U \cap \{x : f(x) > y\}$ for $y \in J$,*

$$(\|V\| + \|\delta V\|)(K \cap E(y)) < \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$ and $y \in J$, and the distributions $S(y) \in \mathscr{D}'(\mathbf{R}^n \sim B, \mathbf{R}^n)$ and $T \in \mathscr{D}'((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ satisfy

$$S(y)(\theta) = ((\delta V) \llcorner E(y))(\theta|U) - \textstyle\int_{E(y) \times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x)\, \mathrm{d}V(x,P) \quad \text{for } y \in J,$$
$$T(\phi) = \textstyle\int_J S(y)(\phi(\cdot, y))\, \mathrm{d}\mathscr{L}^1 y$$

whenever $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$ and $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$.

Then the following three conditions are equivalent:

(1) $\int_{K \cap \{x : f(x) \in I\}} |V\, \mathbf{D} f(x)|\, \mathrm{d}\|V\| x < \infty$ whenever K is a compact subset of $\mathbf{R}^n \sim B$ and I is a compact subset of J, and $f \in \mathbf{T}_G(V)$.

(2) $\int_I \|S(y)\|(K)\, \mathrm{d}\mathscr{L}^1 y < \infty$ whenever K is a compact subset of $\mathbf{R}^n \sim B$ and I is a compact subset of J, and $\|S(y)\|((\mathbf{R}^n \sim B) \sim U) = 0$ for $\mathscr{L}^1$ almost all $y \in J$.

(3) T is representable by integration and $\|T\|(((\mathbf{R}^n \sim B) \sim U) \times J) = 0$.

If these conditions are satisfied then the following three statements hold:

(4) For $\mathscr{L}^1$ almost all $y \in J$, there holds

$$S(y)(\theta) = V\, \partial E(y)(\theta|U) \quad \text{whenever } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n).$$

(5) *If* $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$, *then*

$$T(\phi) = \textstyle\int_J V\,\partial E(y)(\phi(\cdot, y)|U)\,\mathrm{d}\mathscr{L}^1 y = \int_A \langle \phi(x, f(x)), V\,\mathbf{D} f(x)\rangle\,\mathrm{d}\|V\| x.$$

(6) *If* g *is an* $\overline{\mathbf{R}}$ *valued* $\|T\|$ *integrable function, then*

$$\begin{aligned}\textstyle\int g\,\mathrm{d}\|T\| &= \textstyle\int_J \int g(x, y)\,\mathrm{d}\|V\,\partial E(y)\| x\,\mathrm{d}\mathscr{L}^1 y \\ &= \textstyle\int_A g(x, f(x))|V\,\mathbf{D} f(x)|\,\mathrm{d}\|V\| x.\end{aligned}$$

Proof. Notice that $S(y)(\theta) = V\,\partial E(y)(\theta|U)$ whenever $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$ and $\operatorname{spt}\theta \subset U$. From 8.5 and 8.29 one infers

$$\textstyle\int_{K \cap \{x : f(x) \in I\}} |V\,\mathbf{D} f(x)|\,\mathrm{d}\|V\| x = \int_I \|V\,\partial E(y)\|(U \cap K)\,\mathrm{d}\mathscr{L}^1 y$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$ and I is a compact subset of J. Consequently, (1) and (2) are equivalent.

Moreover, one remarks that S is $\mathscr{L}^1 \llcorner J$ measurable by 2.21 and, employing a countable sequentially dense subset of $\mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$, one obtains that, for $\mathscr{L}^1$ almost all $y \in J$,

$$S(y)(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} \textstyle\int_y^{y+\varepsilon} S(v)(\theta)\,\mathrm{d}\mathscr{L}^1 v \quad \text{whenever } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$$

by 2.2, 2.12, and [Fed69, 2.8.17, 2.9.8]. Defining $R_\theta \in \mathscr{D}'(J, \mathbf{R})$ by

$$R_\theta(\omega) = T_{(x,y)}(\omega(y)\theta(x)) \quad \text{whenever } \omega \in \mathscr{D}(J, \mathbf{R}) \text{ and } \theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n),$$

one notes that $\|R_\theta\|$ is absolutely continuous with respect to $\mathscr{L}^1 | \mathbf{2}^J$ for $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$. If (2) holds, then T is representable by integration and 3.4 (2) with U and Z replaced by $\mathbf{R}^n \sim B$ and $\mathbf{R}^n$ yields (3). Conversely, if (3) holds, then (2) follows similarly from 3.4 (1).

Suppose now (1)–(3) hold. Then (4) is evident from 9.1 and implies

$$T(\phi) = \textstyle\int_A \langle \phi(x, f(x)), V\,\mathbf{D} f(x)\rangle\,\mathrm{d}\|V\| x \quad \text{for } \phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$$

by 8.5 and 8.29. Finally, (5) and (6) follow from 3.3 and 3.4 (2) with U replaced by $\mathbf{R}^n \sim B$. ☐

9.13 Lemma. *Suppose* m, n, U, V, G, *and* B *are as in 9.1,* $f \in \mathbf{T}(V)$, f_i *is a sequence in* $\mathbf{T}_G(V)$, $J = \{y : 0 < y < \infty\}$, *and*

$$\begin{gathered}(\|V\| + \|\delta V\|)(K \cap \{x : f(x) > b\}) < \infty, \\ f_i \to f \quad \text{as } i \to \infty \text{ in } (\|V\| + \|\delta V\|) \llcorner U \cap K \text{ measure}, \\ \varrho(K, I, b, \delta) < \infty \quad \text{for } 0 < \delta < \infty, \qquad \varrho(K, I, b, \delta) \to 0 \quad \text{as } \delta \to 0+\end{gathered}$$

whenever K *is a compact subset of* $\mathbf{R}^n \sim B$, I *is a compact subset of* J, *and* $\inf I > b \in J$, *where* $\varrho(K, I, b, \delta)$ *denotes the supremum of all numbers*

$$\limsup_{i \to \infty} \textstyle\int_{K \cap A \cap \{x : f_i(x) \in I\}} |V\,\mathbf{D} f_i|\,\mathrm{d}\|V\|$$

corresponding to $\|V\|$ *measurable sets* A *with* $\|V\|(A \cap K \cap \{x : f(x) > b\}) \le \delta$.

Then $f \in \mathbf{T}_G(V)$.

Proof. Define $C = (U \times J) \cap \{(x,y) : f(x) > y\}$, $C_i = (U \times J) \cap \{(x,y) : f_i(x) > y\}$ for $i \in \mathscr{P}$, $E(y) = \{x : (x,y) \in C\}$ and $E_i(y) = \{x : (x,y) \in C_i\}$ for $y \in J$ and $i \in \mathscr{P}$, and $g : (U \times \mathbf{G}(n,m)) \times J \to U \times J$ by $g((x,P),y) = (x,y)$ for $x \in U$, $P \in \mathbf{G}(n,m)$, and $y \in J$. As in 8.1 one derives that C and C_i are $\|\delta V\| \times \mathscr{L}^1$ measurable and that $g^{-1}[C]$ and $g^{-1}[C_i]$ are $V \times \mathscr{L}^1$ measurable. Defining $T \in \mathscr{D}'(\mathbf{R}^n \sim B, \mathbf{R}^n)$ as in 9.12, Fubini's theorem yields that (see 4.4)

$$\begin{aligned} T(\phi) = {} & \textstyle\int_C \eta(V,x) \bullet \phi(x,y) \,\mathrm{d}(\|\delta V\| \times \mathscr{L}^1)(x,y) \\ & - \textstyle\int_{g^{-1}[C]} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}(V \times \mathscr{L}^1)((x,P),y) \end{aligned}$$

for $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$. Observe that

$$\begin{gathered} \big((\|V\| + \|\delta V\|) \times \mathscr{L}^1\big)\big(((U \cap K) \times I) \cap C\big) < \infty, \\ \lim_{i \to \infty} \big((\|V\| + \|\delta V\|) \times \mathscr{L}^1\big)\big(((U \cap K) \times I) \cap ((C \sim C_i) \cup (C_i \sim C))\big) = 0 \end{gathered}$$

whenever K is a compact subset of U and I a compact subset of J; in fact, if $b, \varepsilon \in J$, $b + \varepsilon < \inf I$, and $A = \{x : |f(x) - f_i(x)| > \varepsilon\}$, then

$$\begin{aligned} (U \times I) \cap (C \sim C_i) &\subset (A \times I) \cup \{(x,y) : b < f(x) - \varepsilon \le y < f(x)\}, \\ (U \times I) \cap (C_i \sim C) &\subset (A \times I) \cup \{(x,y) : b < f(x) \le y < f(x) + \varepsilon\}. \end{aligned}$$

Consequently, one employs Fubini's theorem and 9.1 to compute

$$\begin{aligned} T(\phi) &= \lim_{i \to \infty} \Big(\textstyle\int_{C_i} \eta(V,x) \bullet \phi(x,y) \,\mathrm{d}(\|\delta V\| \times \mathscr{L}^1)(x,y) \\ &\qquad - \textstyle\int_{g^{-1}[C_i]} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}(V \times \mathscr{L}^1)((x,P),y) \Big) \\ &= \lim_{i \to \infty} \textstyle\int_J \big(((\delta V) \llcorner E_i(y))(\phi(\cdot,y)|U) \\ &\qquad - \textstyle\int_{E_i(y) \times \mathbf{G}(n,m)} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}V(x,P) \big) \,\mathrm{d}\mathscr{L}^1 y \\ &= \lim_{i \to \infty} \textstyle\int_J V\, \partial E_i(y)(\phi(\cdot,y)|U) \,\mathrm{d}\mathscr{L}^1 y. \end{aligned}$$

In view of 8.29, one infers

$$\|T\|\big(((\operatorname{Int} K) \sim A) \times \operatorname{Int} I\big) \le \varrho(K, I, b, \delta)$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$, I is a compact subset of J, $\inf I > b \in J$, $0 < \delta < \infty$, A is a compact subset of U, and $\|V\|((K \sim A) \cap E(b)) \le \delta$. In particular, taking $A = \varnothing$ and δ sufficiently large, one concludes that T is representable by integration and taking A such that $\|V\|((K \sim A) \cap E(b))$ is small yields

$$\|T\|(((\mathbf{R}^n \sim B) \sim U) \times J) = 0.$$

The conclusion now follows from 9.12 (1) (3). ☐

9.14 Remark. The conditions on ϱ are satisfied for instance if for any compact subset K of $\mathbf{R}^n \sim B$ there holds

$$\begin{aligned} \text{either} \quad & \textstyle\int_{U \cap K} |V\, \mathbf{D} f| \,\mathrm{d}\|V\| < \infty, \quad \lim_{i \to \infty} (\|V\| \llcorner U \cap K)_{(1)}(V\, \mathbf{D} f - V\, \mathbf{D} f_i) = 0, \\ \text{or} \quad & \limsup_{i \to \infty} (\|V\| \llcorner U \cap K)_{(q)}(V\, \mathbf{D} f_i) < \infty \quad \text{for some } 1 < q \le \infty; \end{aligned}$$

in fact if I is a compact subset of J and $\inf I > b \in J$ then, in the first case,

$$\limsup_{i\to\infty} \textstyle\int_{K\cap A\cap\{x\,:\,f_i(x)\in I\}} |V\,\mathbf{D}f_i| \,\mathrm{d}\|V\| \leq \int_{K\cap A\cap\{x\,:\,f(x)>b\}} |V\,\mathbf{D}f| \,\mathrm{d}\|V\|$$

whenever A is $\|V\|$ measurable and, in the second case,

$$\varrho(K,I,b,\delta) \leq \delta^{1-1/q} \limsup_{i\to\infty} (\|V\| \llcorner U\cap K)_{(q)}(V\,\mathbf{D}f_i) \quad \text{for } 0<\delta<\infty.$$

9.15. If f is a nonnegative $\mathscr{L}^1$ measurable $\overline{\mathbf{R}}$ valued function, $O \subset \mathbf{R}$, $\mathscr{L}^1(O) = 0$, $\varepsilon > 0$, and $j \in \mathscr{P}$, then there exist $b_1,\ldots,b_j$ such that

$$\begin{gathered} \varepsilon(i-1) < b_i < \varepsilon i \quad \text{and} \quad b_i \notin O \qquad \text{for } i=1,\ldots,j, \\ \textstyle\sum_{i=1}^j (b_i - b_{i-1}) f(b_i) \leq 2\int f \,\mathrm{d}\mathscr{L}^1, \end{gathered}$$

where $b_0 = 0$; in fact, it is sufficient to choose b_i such that

$$\varepsilon(i-1) < b_i < \varepsilon i, \quad b_i \notin O, \quad \varepsilon f(b_i) \leq \textstyle\int_{\varepsilon(i-1)}^{\varepsilon i} f \,\mathrm{d}\mathscr{L}^1$$

for $i = 1,\ldots,j$, and note $b_i - b_{i-1} \leq 2\varepsilon$.

9.16 Theorem. *Suppose m, n, U, V, G, and B are as in 9.1, $f \in \mathbf{T}_G(V)$, $g : U \to \{y : 0 \leq y < \infty\}$, and*

$$\textstyle\int_{U\cap K} |f| + |V\,\mathbf{D}f| \,\mathrm{d}\|V\| < \infty, \quad \operatorname{Lip}(g|K) < \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$.
Then $gf \in \mathbf{T}_G(V)$.

Proof. Define $h = gf$ and note $h \in \mathbf{T}(V)$ by 8.20 (4). Define a function c by $c = ((\mathbf{R}^n \sim B) \times \mathbf{R}) \cap \operatorname{Clos} g$ and note

$$\operatorname{dmn} c = U \cup G, \quad c|U = g, \quad \operatorname{Lip}(c|K) < \infty$$

whenever K is a compact subset of $\mathbf{R}^n \sim B$. Moreover, let

$$J = \{y : 0 < y < \infty\}, \quad A = (U \times J) \cap \{(x,y) : h(x) > y\}.$$

and define $p : (\mathbf{R}^n \sim B) \times J \to \mathbf{R}^n \sim B$ by

$$p(x,y) = x \quad \text{for } x \in \mathbf{R}^n \sim B \text{ and } y \in J.$$

Noting $(\|V\| + \|\delta V\|)(K \cap \{x : h(x) > y\}) < \infty$ whenever K is a compact subset of $\mathbf{R}^n \sim B$ and $y \in J$, the proof may be carried out by showing that the distribution $T \in \mathscr{D}'((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ defined by

$$\begin{aligned} T(\phi) = \textstyle\int_J \big(& ((\delta V) \llcorner \{x : h(x) > y\})(\phi(\cdot,y)|U) \\ & - \textstyle\int_{\{x\,:\,h(x)>y\}\times \mathbf{G}(n,m)} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}V(x,P) \big) \,\mathrm{d}\mathscr{L}^1 y \end{aligned}$$

for $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ satisfies the conditions of 9.12 (3) with f replaced by h.

For this purpose, define subsets $D(y)$, $E(\upsilon)$, and $F(y,\upsilon)$ of $U \cup G$, varifolds $W_y \in \mathbf{RV}_m(\mathbf{R}^n \sim B)$, and distributions $S(y) \in \mathscr{D}'(\mathbf{R}^n \sim B, \mathbf{R}^n)$ and $\Sigma(y,\upsilon) \in \mathscr{D}'(\mathbf{R}^n \sim B, \mathbf{R}^n)$ by

$$\begin{gathered} D(y) = \{x : f(x) > y\}, \quad E(\upsilon) = \{x : c(x) > \upsilon\}, \\ F(y,\upsilon) = D(y) \cap E(\upsilon), \quad W_y(k) = \textstyle\int_{D(y)\times \mathbf{G}(n,m)} k \,\mathrm{d}V, \\ S(y)(\theta) = ((\delta V) \llcorner D(y))(\theta|U) - \textstyle\int_{D(y)\times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x) \,\mathrm{d}V(x,P), \\ \Sigma(y,\upsilon)(\theta) = ((\delta V) \llcorner F(y,\upsilon))(\theta|U) - \textstyle\int_{F(y,\upsilon)\times \mathbf{G}(n,m)} P_\natural \bullet D\theta(x) \,\mathrm{d}V(x,P) \end{gathered}$$

whenever $y,\upsilon \in J$, $k \in \mathscr{K}((\mathbf{R}^n \sim B) \times \mathbf{G}(n,m))$, and $\theta \in \mathscr{D}(\mathbf{R}^n \sim B, \mathbf{R}^n)$. Let O consist of all $b \in J$ violating the following condition:

$$S(b) \text{ is representable by integration and } \|S(b)\|((\mathbf{R}^n \sim B) \sim U) = 0.$$

Note $\mathscr{L}^1(J \sim O) = 0$ by 9.1 and

$$\|\delta W_b\| \text{ is a Radon measure}, \quad \Sigma(b,y) = S(b) \llcorner E(y) + W_b \,\partial E(y)$$

whenever $b \in J \sim O$ and $y \in J$. One readily verifies by means of 8.7 in conjunction with [Fed69, 2.10.19 (4), 2.10.43] that $c \in \mathbf{T}(W)$ and

$$W\,\mathbf{D}c(x) = V\,\mathbf{D}g(x) \quad \text{for } \|V\| \text{ almost all } x \in D$$

whenever D is $\|V\|$ measurable, $W \in \mathbf{RV}_m(\mathbf{R}^n \sim B)$, $W(k) = \int_{D\times \mathbf{G}(n,m)} k \,\mathrm{d}V$ for $k \in \mathscr{K}((\mathbf{R}^n \sim B) \times \mathbf{G}(n,m))$ and $\|\delta W\|$ is a Radon measure.

Whenever N is a nonempty finite subset of $J \sim O$ define functions $f_N : \operatorname{dmn} f \to \{y : 0 \le y < \infty\}$ and $h_N : \operatorname{dmn} f \to \{y : 0 \le y < \infty\}$ by

$$f_N(x) = \sup(\{0\} \cup (N \cap \{y : x \in D(y)\})), \quad h_N(x) = f_N(x) g(x)$$

whenever $x \in \operatorname{dmn} f$ and distributions $\Theta_N \in \mathscr{D}'((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ and $T_N \in \mathscr{D}'((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ by

$$\begin{aligned} \Theta_N(\phi) = \textstyle\int_J \big(& ((\delta V) \llcorner \{x : f_N(x) > y\})(\phi(\cdot,y)|U) \\ & - \textstyle\int_{\{x : f_N(x) > y\}\times \mathbf{G}(n,m)} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}V(x,P) \big) \,\mathrm{d}\mathscr{L}^1 y, \\ T_N(\phi) = \textstyle\int_J \big(& ((\delta V) \llcorner \{x : h_N(x) > y\})(\phi(\cdot,y)|U) \\ & - \textstyle\int_{\{x : h_N(x) > y\}\times \mathbf{G}(n,m)} P_\natural \bullet D\phi(\cdot,y)(x) \,\mathrm{d}V(x,P) \big) \,\mathrm{d}\mathscr{L}^1 y \end{aligned}$$

whenever $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$.

Next, it will be shown *if N is a nonempty finite subset of $J \sim O$ then*

$$\|T_N\|(X \times J) \le ((p_\# \|\Theta_N\|) \llcorner X)(c) + \textstyle\int_{U \cap X} f |V\,\mathbf{D}g| \,\mathrm{d}\|V\|$$

whenever X is an open subset of $\mathbf{R}^n \sim B$. For this purpose suppose $j \in \mathscr{P}$ and $0 = b_0 < b_1 < \ldots < b_j < \infty$ satisfy $N = \{b_i : i = 1, \ldots, j\}$ and notice that

$$\begin{gathered} f_N(x) = b_i \quad \text{if } x \in D(b_i) \sim D(b_{i+1}) \text{ for some } i = 1, \ldots, j-1, \\ f_N(x) = 0 \quad \text{if } x \in (\operatorname{dmn} f) \sim D(b_1), \qquad f_N(x) = b_j \quad \text{if } x \in D(b_j) \end{gathered}$$

and define distributions $\Psi_1, \Psi_2 \in \mathscr{D}'((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$ by

$$\begin{aligned}
\Psi_1(\phi) = {\textstyle\int_J} \big(\big(& S(b_1) \llcorner E(y/b_1) \\
& + {\textstyle\sum_{i=2}^j} (S(b_i) \llcorner E(y/b_i) \sim E(y/b_{i-1})) \big) (\phi(\cdot, y)) \big) \, \mathrm{d}\mathscr{L}^1 y, \\
\Psi_2(\phi) = {\textstyle\int_J} \big(& {\textstyle\sum_{i=1}^{j-1}} (W_{b_i} - W_{b_{i+1}}) \, \partial E(y/b_i) + W_{b_j} \, \partial E(y/b_j) \big) (\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y
\end{aligned}$$

for $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$. One computes

$$\Theta_N(\phi) = {\textstyle\sum_{i=1}^j \int_{b_{i-1}}^{b_i}} S(b_i)(\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y \quad \text{for } \phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$$

and deduces from 3.4 (2) with U replaced by $\mathbf{R}^n \sim B$ that

$$\|\Theta_N\|(d) = {\textstyle\sum_{i=1}^j \int_{b_{i-1}}^{b_i}} \|S(b_i)\|(d(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y$$

whenever d is an $\overline{\mathbf{R}}$ valued $\|\Theta_N\|$ integrable function. Noting

$$\begin{gathered}
\{x : h_N(x) > y\} \cap (D(b_i) \sim D(b_{i+1})) = F(b_i, y/b_i) \sim F(b_{i+1}, y/b_i), \\
\{x : h_N(x) > y\} \cap (U \sim D(b_1)) = \varnothing, \quad \{x : h_N(x) > y\} \cap D(b_j) = F(b_j, y/b_j)
\end{gathered}$$

for $i = 1, \ldots, j-1$ and $y \in J$, one obtains

$$T_N(\phi) = {\textstyle\int_J} \big({\textstyle\sum_{i=1}^{j-1}} (\Sigma(b_i, y/b_i) - \Sigma(b_{i+1}, y/b_i)) + \Sigma(b_j, y/b_j) \big) (\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y$$

whenever $\phi \in \mathscr{D}((\mathbf{R}^n \sim B) \times J, \mathbf{R}^n)$. Computing with the help of 5.2 that

$$\begin{aligned}
& {\textstyle\sum_{i=1}^{j-1}} (\Sigma(b_i, y/b_i) - \Sigma(b_{i+1}, y/b_i)) + \Sigma(b_j, y/b_j) \\
& \quad = {\textstyle\sum_{i=1}^j} S(b_i) \llcorner E(y/b_i) + {\textstyle\sum_{i=1}^j} W_{b_i} \, \partial E(y/b_i) \\
& \qquad - {\textstyle\sum_{i=1}^{j-1}} S(b_{i+1}) \llcorner E(y/b_i) - {\textstyle\sum_{i=1}^{j-1}} W_{b_{i+1}} \, \partial E(y/b_i) \\
& \quad = S(b_1) \llcorner E(y/b_1) + {\textstyle\sum_{i=2}^j} S(b_i) \llcorner (E(y/b_i) \sim E(y/b_{i-1})) \\
& \qquad + {\textstyle\sum_{i=1}^{j-1}} (W_{b_i} - W_{b_{i+1}}) \, \partial E(y/b_i) + W_{b_j} \, \partial E(y/b_j)
\end{aligned}$$

whenever $y \in J$ yields

$$T_N = \Psi_1 + \Psi_2.$$

Moreover, the quantity $\|\Psi_1\|(X \times J)$ does not exceed

$$\begin{aligned}
& {\textstyle\int_J} \big(\|S(b_1)\| \llcorner E(y/b_1) + {\textstyle\sum_{i=2}^j} \|S(b_i)\| \llcorner (E(y/b_i) \sim E(y/b_{i-1})) \big) (X) \, \mathrm{d}\mathscr{L}^1 y \\
& \quad = {\textstyle\sum_{i=1}^j \int_J} (\|S(b_i)\| \llcorner X)(E(y/b_i)) \, \mathrm{d}\mathscr{L}^1 y \\
& \qquad - {\textstyle\sum_{i=2}^j \int_J} (\|S(b_i)\| \llcorner X)(E(y/b_{i-1})) \, \mathrm{d}\mathscr{L}^1 y \\
& \quad = {\textstyle\sum_{i=1}^j} (b_i - b_{i-1})(\|S(b_i)\| \llcorner X)(c) = ((p_\# \|\Theta_N\|) \llcorner X)(c)
\end{aligned}$$

and, using 8.5 and 8.29, the quantity $\|\Psi_2\|(X \times J)$ may be bounded by

$$\begin{aligned}
& {\textstyle\int_J} \big({\textstyle\sum_{i=1}^{j-1}} \|(W_{b_i} - W_{b_{i+1}}) \, \partial E(y/b_i)\| + \|W_{b_j} \, \partial E(y/b_j)\| \big)(X) \, \mathrm{d}\mathscr{L}^1 y \\
& \quad = {\textstyle\sum_{i=1}^{j-1}} b_i {\textstyle\int_X} |(W_{b_i} - W_{b_{i+1}}) \, \mathbf{D} c| \, \mathrm{d}\|W_{b_i} - W_{b_{i+1}}\| + b_j {\textstyle\int_X} |W_{b_j} \, \mathbf{D} c| \, \mathrm{d}\|W_{b_j}\| \\
& \quad = {\textstyle\sum_{i=1}^{j-1}} b_i {\textstyle\int_{X \cap (D(b_i) \sim D(b_{i+1}))}} |V \, \mathbf{D} g| \, \mathrm{d}\|V\| + b_j {\textstyle\int_{X \cap D(b_j)}} |V \, \mathbf{D} g| \, \mathrm{d}\|V\| \\
& \quad = {\textstyle\int_{U \cap X}} f |V \, \mathbf{D} g| \, \mathrm{d}\|V\|.
\end{aligned}$$

Next, *it will be proven*

$$\|T\|(X \times J) \leq \textstyle\int_{U \cap X} 2g|V\,\mathbf{D}f| + f|V\,\mathbf{D}g| \,\mathrm{d}\|V\|$$

whenever X is an open subset of $\mathbf{R}^n \sim B$. Recalling the formula for $\|\Theta_N\|$, one may use 9.15 with $f(y)$ replaced by $(\|S(y)\| \llcorner X)(c)$ for $y \in J$ to construct a sequence $N(i)$ of nonempty finite subsets of $J \sim O$ such that

$$((p_\# \|\Theta_{N(i)}\|) \llcorner X)(c) \leq 2 \textstyle\int_J (\|S(y)\| \llcorner X)(c) \,\mathrm{d}\mathscr{L}^1 y,$$
$$\operatorname{dist}(y, N(i)) \to 0 \quad \text{as } i \to \infty \text{ for } y \in J.$$

Define $A(i) = (U \times J) \cap \{(x, y) : h_{N(i)}(x) > y\}$ for $i \in \mathscr{P}$. Noting

$$h_{N(i)}(x) \to h(x) \quad \text{as } i \to \infty \text{ for } x \in \operatorname{dmn} f$$

and recalling $(\|V\| + \|\delta V\|)(K \cap \{x : h(x) > y\}) < \infty$ for $y \in J$ whenever K is a compact subset of $\mathbf{R}^n \sim B$, one infers

$$\big((\|V\| + \|\delta V\|) \times \mathscr{L}^1\big)(C \cap A \sim A(i)) \to 0$$

as $i \to \infty$ whenever C is a compact subset of $(\mathbf{R}^n \sim B) \times J$. Since $A(i) \subset A$ for $i \in \mathscr{P}$, it follows by means of Fubini's theorem that

$$T_{N(i)} \to T \quad \text{as } i \to \infty$$

and, in conjunction with the assertion of the preceding paragraph,

$$\|T\|(X \times J) \leq 2 \textstyle\int_J (\|S(y)\| \llcorner X)(c) \,\mathrm{d}\mathscr{L}^1 y + \int_{U \cap X} f|V\,\mathbf{D}g| \,\mathrm{d}\|V\|.$$

Therefore the assertion of the present paragraph is implied by 9.12 (1) (4) (6).

Finally, the assertion of the preceding paragraph extends to all Borel subsets X of $\mathbf{R}^n \sim B$ by approximation and the conclusion follows. ☐

10 Embeddings into Lebesgue spaces

In this section a variety of Sobolev Poincaré type inequalities for weakly differentiable functions are established by means of the relative isoperimetric inequalities 7.9 and 7.11. The key are local estimates under a smallness condition on set of points where the nonnegative function is positive, see 10.1 (1). These estimates are formulated in such a way as to improve in case the function satisfies a zero boundary value condition on an open part of the boundary. Consequently, Sobolev inequalities are essentially a special case, see 10.1 (2). Local summability results also follow, see 10.3. Finally, versions without the previously hypothesised smallness condition are derived in 10.7 and 10.9.

The differentiability results which will be derived in 11.2 and 11.4 are based on 10.1 (1) whereas the oscillation estimates which will be proven in 13.1 and 13.3 employ 10.9.

10.1 Theorem. *Suppose* $1 \leq M < \infty$.

Then there exists a positive, finite number Γ *with the following property.*

If m, n, p, U, V, and ψ are as in 6.1, $n \leq M$, G is a relatively open subset of $\operatorname{Bdry} U$, $B = (\operatorname{Bdry} U) \sim G$, *and* $f \in \mathbf{T}_G(V)$, *then the following two statements hold:*

(1) Suppose $1 \leq Q \leq M$, $0 < r < \infty$, $E = \{x : f(x) > 0\}$,

$$\|V\|(E) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) r^m,$$
$$\|V\|(E \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m,$$

and $A = U \cap \{x : \mathbf{U}(x, r) \subset \mathbf{R}^n \sim B\}$. *Then the following four implications hold:*

(a) If $p = 1$, $\beta = \infty$ *if* $m = 1$ *and* $\beta = m/(m-1)$ *if* $m > 1$, *then*

$$(\|V\| \llcorner A)_{(\beta)}(f) \leq \Gamma\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(f)\big).$$

(b) If $p = m = 1$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$(\|V\| \llcorner A)_{(\infty)}(f) \leq \Gamma\, \|V\|_{(1)}(V\,\mathbf{D}f).$$

(c) If $1 \leq q < m = p$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$(\|V\| \llcorner A)_{(mq/(m-q))}(f) \leq \Gamma(m-q)^{-1} \|V\|_{(q)}(V\,\mathbf{D}f).$$

(d) If $1 < p = m < q \leq \infty$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$(\|V\| \llcorner A)_{(\infty)}(f) \leq \Gamma^{1/(1/m-1/q)} \|V\|(E)^{1/m-1/q} \|V\|_{(q)}(V\,\mathbf{D}f).$$

(2) Suppose $G = \operatorname{Bdry} U$, $E = \{x : f(x) > 0\}$, *and* $\|V\|(E) < \infty$. *Then the following four implications hold:*

(a) If $p = 1$, $\beta = \infty$ *if* $m = 1$ *and* $\beta = m/(m-1)$ *if* $m > 1$, *then*

$$\|V\|_{(\beta)}(f) \leq \Gamma\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(f)\big).$$

(b) If $p = m = 1$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$\|V\|_{(\infty)}(f) \leq \Gamma\, \|V\|_{(1)}(V\,\mathbf{D}f).$$

(c) If $1 \leq q < m = p$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$\|V\|_{(mq/(m-q))}(f) \leq \Gamma(m-q)^{-1} \|V\|_{(q)}(V\,\mathbf{D}f).$$

(d) If $1 < p = m < q \leq \infty$ *and* $\psi(E) \leq \Gamma^{-1}$, *then*

$$\|V\|_{(\infty)}(f) \leq \Gamma^{1/(1/m-1/q)} \|V\|(E)^{1/m-1/q} \|V\|_{(q)}(V\,\mathbf{D}f).$$

Proof. Denote by (1c)$'$ [respectively (2c)$'$] the implication resulting from (1c) [respectively (2c)] through omission of the factor $(m-q)^{-1}$ and addition of the requirement $q = 1$. It is sufficient to construct functions $\Gamma_{(1a)}, \Gamma_{(1b)}, \Gamma_{(1c)}, \Gamma_{(1c)'}$, $\Gamma_{(1d)}, \Gamma_{(2a)}, \Gamma_{(2b)}, \Gamma_{(2c)}, \Gamma_{(2c)'}$, and $\Gamma_{(2d)}$ corresponding to the implications (1a), (1b), (1c), (1c)$'$, (1d), (2a), (2b), (2c), (2c)$'$, and (2d) whose value at M for

$1 \le M < \infty$ is a positive, finite number such that the respective implication is true for M with Γ replaced by this value. Define

$$\Gamma_{(1a)}(M) = \Gamma_{(1b)}(M) = \Gamma_{(1c)'}(M) = \Gamma_{7.8}(M), \quad \Gamma_{(2a)}(M) = \Gamma_{(1a)}(\sup\{2, M\}),$$
$$\Gamma_{(2b)}(M) = \Gamma_{(1b)}(\sup\{2, M\}), \quad \Gamma_{(2c)'}(M) = \Gamma_{(1c)'}(\sup\{2, M\}),$$
$$\Gamma_{(2c)}(M) = M^2 \Gamma_{(2c)'}(M),$$
$$\Delta_1(M) = (\sup\{1/2, 1 - 1/(2M^2 - 1)\})^{1/M}, \quad \Delta_2(M) = 2M/(1 - \Delta_1(M)),$$
$$\Delta_3(M) = M \sup\{\boldsymbol{\alpha}(m) : M \ge m \in \mathscr{P}\},$$
$$\Delta_4(M) = \inf\{\boldsymbol{\alpha}(m) : M \ge m \in \mathscr{P}\}/2,$$
$$\Delta_5(M) = \sup\{1, \Gamma_{(2c)}(M)(\Delta_2(M)\Delta_3(M) + 1)\},$$
$$\Delta_6(M) = 4^{M+1} \Delta_3(M)^M \Delta_5(M)^{M+1},$$
$$\Gamma_{(1c)}(M) = \sup\{2\Gamma_{(1c)'}(2M), \Delta_6(M)(1 + \Delta_3(M)\Gamma_{(1c)'}(2M))\},$$
$$\Delta_7(M) = \sup\{1, \Gamma_{(1c)'}(2M)\},$$
$$\Delta_8(M) = \inf\big\{ \inf\{1, \Delta_3(M)^{-1}\Delta_4(M)\Delta_1(M)^M\}(1 - \Delta_1(M))^M, 1/2\big\},$$
$$\Delta_9(M) = 2\Delta_7(M)\Delta_8(M)^{-1},$$
$$\Gamma_{(1d)}(M) = \sup\big\{\Delta_9(M), \Delta_1(M)^{-M}\Gamma_{(1c)'}(2M)\big\},$$
$$\Gamma_{(2d)}(M) = \Gamma_{(1d)}(\sup\{2, M\})$$

whenever $1 \le M < \infty$.

In order to verify the asserted properties suppose $1 \le M < \infty$ and abbreviate $\delta_i = \Delta_i(M)$ whenever $i = 1, \ldots, 9$. In the verification of assertion (X) [respectively $(X)'$], where X is one of 1a, 1b, 1c, 1d, 2a, 2b, 2c, and 2d, [respectively 1c and 2c] it will be assumed that the quantities occurring in its hypotheses are defined and satisfy these hypotheses with Γ replaced by $\Gamma_{(X)}(M)$ [respectively $\Gamma_{(X)'}(M)$].

Step 1. Verification of the property of $\Gamma_{(1a)}$.

Define $E(b) = \{x : f(x) > b\}$ for $0 \le b < \infty$. If $m = 1$ then $\Gamma_{7.8}(M)^{-1} \le \|V\,\partial E(b)\|(U) + \|\delta V\|(E(b))$ for $\mathscr{L}^1$ almost all b with $0 < b < (\|V\| \llcorner A)_{(\beta)}(f)$ by 7.9 and the conclusion follows from 8.29. If $m > 1$, define

$$f_b = \inf\{f, b\}, \quad g(b) = (\|V\| \llcorner A)_{(\beta)}(f_b) \le b\, \|V\|(E)^{1/\beta} < \infty$$

for $0 \le b < \infty$ and use Minkowski's inequality to conclude

$$0 \le g(b + y) - g(b) \le (\|V\| \llcorner A)_{(\beta)}(f_{b+y} - f_b) \le y\, \|V\|(A \cap E(b))^{1/\beta}$$

for $0 \le b < \infty$ and $0 < y < \infty$. Therefore g is Lipschitzian and one infers from [Fed69, 2.9.19] and 7.9 that

$$0 \le g'(b) \le \|V\|(A \cap E(b))^{1-1/m} \le \Gamma_{7.8}(M)\big(\|V\,\partial E(b)\|(U) + \|\delta V\|(E(b))\big)$$

for $\mathscr{L}^1$ almost all $0 < b < \infty$, hence $(\|V\| \llcorner A)_{(\beta)}(f) = \lim_{b \to \infty} g(b) = \int_0^\infty g'\, \mathrm{d}\mathscr{L}^1$ by [Fed69, 2.9.20] and 8.29 implies the conclusion.

Step 2. Verification of the properties of $\Gamma_{(1b)}$ and $\Gamma_{(1c)'}$.

Omitting the terms involving δV from the proof of Step 1 and using 7.11 instead of 7.9, a proof of Step 2 results.

Step 3. Verification of the properties of $\Gamma_{(2a)}$, $\Gamma_{(2b)}$, and $\Gamma_{(2c)'}$.

Taking $Q = 1$, one may apply (1a), (1b), and (1c)$'$ with M replaced by $\sup\{2, M\}$ and a sufficiently large number r.

Step 4. Verification of the property of $\Gamma_{(2c)}$.

One may assume f to be bounded by 8.12, 8.13 (4). Noting $f^{q(m-1)/(m-q)} \in \mathbf{T}_{\operatorname{Bdry} U}(V)$ by 9.9, one now applies (2c)$'$ with f replaced by $f^{q(m-1)/(m-q)}$ to deduce the assertion by the method of [Fed69, 4.5.15].

Step 5. Verification of the property of $\Gamma_{(1c)}$.

One may assume $\|V\|_{(q)}(V\,\mathbf{D}f) < \infty$ and, possibly using homotheties, also $r = 1$. Moreover, one may assume f to be bounded by 8.12, 8.13 (4), hence

$$\textstyle\int |f| + |V\,\mathbf{D}f| \,\mathrm{d}\|V\| < \infty$$

by 8.20 (1) and Hölder's inequality. Define

$$r_i = \delta_1 + (i-1)(1-\delta_1)/M, \quad A_i = U \cap \{x : \mathbf{U}(x, r_i) \subset \mathbf{R}^n \sim B\}$$

whenever $i = 1, \ldots, m$. Observe that

$$Q - M^{-1} \leq r_1^m \big(Q - (2M)^{-1}\big), \quad r_1^m \geq 1/2, \quad r_m \leq 1,$$
$$r_{i+1} - r_i = 2/\delta_2 \quad \text{for } i = 1, \ldots, m-1.$$

Choose $g_i \in \mathscr{E}(U, \mathbf{R})$ with

$$\operatorname{spt} g_i \subset A_i, \qquad g_i(x) = 1 \quad \text{for } x \in A_{i+1},$$
$$0 \leq g_i(x) \leq 1 \quad \text{for } x \in U, \qquad \operatorname{Lip} g_i \leq \delta_2$$

whenever $i = 1, \ldots, m-1$. Notice that $g_i f \in \mathbf{T}(V)$ by 8.20 (4) with

$$V\,\mathbf{D}(g_i f)(x) = V\,\mathbf{D}g_i(x) f(x) + g_i(x) V\,\mathbf{D}f(x) \quad \text{for } \|V\| \text{ almost all } x.$$

Moreover, one infers $g_i f \in \mathbf{T}_{\operatorname{Bdry} U}(V)$ by 9.16 and 9.4.

If $i \in \{1, \ldots, m-1\}$ and $1 - (i-1)/m \geq 1/q \geq 1 - i/m$ then by (2c) and Hölder's inequality

$$\begin{aligned}
(\|V\| \llcorner A_{i+1})_{(mq/(m-q))}(f) &\leq \|V\|_{(mq/(m-q))}(g_i f) \\
&\leq \Gamma_{(2c)}(M)(m-q)^{-1}\big(\delta_2 (\|V\| \llcorner A_i)_{(q)}(f) + \|V\|_{(q)}(V\,\mathbf{D}f)\big) \\
&\leq \delta_5 (m-q)^{-1}\big((\|V\| \llcorner A_i)_{(m/(m-i))}(f) + \|V\|_{(q)}(V\,\mathbf{D}f)\big).
\end{aligned}$$

In particular, replacing q by $m/(m-i)$ and using Hölder's inequality in conjunction with 8.20 (1), one infers

$$(\|V\| \llcorner A_{i+1})_{(m/(m-i-1))}(f) \leq 2\delta_3\delta_5\big((\|V\| \llcorner A_i)_{(m/(m-i))}(f) + \|V\|_{(q)}(V\,\mathbf{D}f)\big)$$

whenever $i \in \{1, \ldots, m-2\}$ and $1 - i/m \geq 1/q$, choosing $j \in \{1, \ldots, m-1\}$ with $1 - (j-1)/m \geq 1/q > 1 - j/m$ and iterating this $j-1$ times, also

$$\begin{aligned}
&(\|V\| \llcorner A_j)_{(m/(m-j))}(f) \\
&\qquad \leq (4\delta_3\delta_5)^{j-1}\big((\|V\| \llcorner A_1)_{(m/(m-1))}(f) + \|V\|_{(q)}(V\,\mathbf{D}f)\big).
\end{aligned}$$

Together this yields

$$\begin{aligned}
&(\|V\| \llcorner A_{j+1})_{(mq/(m-q))}(f) \\
&\qquad \leq \delta_6 (m-q)^{-1} \big((\|V\| \llcorner A_1)_{(m/(m-1))}(f) + \|V\|_{(q)}(V \mathbf{D} f) \big).
\end{aligned}$$

and the conclusion then follows from (1c)′ with M and r replaced by $2M$ and r_1.

Step 6. Verification of the property of $\Gamma_{(1d)}$.

Assume

$$r = 1, \quad \|V\|(E) > 0, \quad \|V\|_{(q)}(V \mathbf{D} f) < \infty,$$

abbreviate $\beta = m/(m-1)$ and $\alpha = 1/(1/m - 1/q)$, and suppose

$$\lambda = \|V\|(E), \quad \|V\|_{(q)}(V \mathbf{D} f) < \kappa < \infty.$$

Notice that $0 < \delta_1 < 1$, $0 < \delta_8 < 1$, $\alpha \geq 1$ and $\lambda > 0$, and define

$$\begin{gathered}
r_i = 1 - (1-\delta_1)^i, \quad s_i = \delta_9^\alpha \lambda^{1/m - 1/q} \kappa (1 - 2^{-i}), \\
X_i = \mathbf{R}^n \cap \{ x : \operatorname{dist}(x, B) > r_i \}, \quad U_i = U \cap X_i, \\
H_i = X_i \cap \operatorname{Bdry} U, \quad C_i = (\operatorname{Bdry} U_i) \sim H_i, \quad W_i = V | \mathbf{2}^{U_i \times \mathbf{G}(n,m)}, \\
t_i = \|V\|(U_i \cap \{ x : f(x) > s_i \}), \\
f_i(x) = \sup \{ f(x) - s_i, 0 \} \quad \text{whenever } x \in \operatorname{dmn} f
\end{gathered}$$

whenever i is a nonnegative integer. Notice that

$$r_{i+1} - r_i = \delta_1 (1-\delta_1)^i, \quad U_{i+1} \subset U_i, \quad s_{i+1} - s_i = \delta_9^\alpha \lambda^{1/m - 1/q} \kappa 2^{-i-1} > 0$$

whenever i is a nonnegative integer. The conclusion is readily deduced from the assertion,

$$t_i \leq \lambda \delta_8^{\alpha i} \quad \textit{whenever } i \textit{ is a nonnegative integer},$$

which will be proven by induction.

The case $i = 0$ is trivial. To prove the case $i = 1$, one applies (1c)′ with M and r replaced by $2M$ and δ_1 and Hölder's inequality in conjunction with 8.20 (1) to obtain

$$(\|V\| \llcorner U_1)_{(\beta)}(f) \leq \Gamma_{(1c)'}(2M) \|V\|_{(1)}(V \mathbf{D} f) \leq \delta_7 \lambda^{1-1/q} \kappa,$$

hence

$$t_1^{1-1/m} \leq (s_1 - s_0)^{-1} (\|V\| \llcorner U_1)_{(\beta)}(f) \leq \delta_9^{-\alpha} \lambda^{1-1/m} 2 \delta_7 \leq \lambda^{1-1/m} \delta_8^{\alpha(1-1/m)}.$$

Assuming the assertion to be true for some $i \in \mathscr{P}$, notice that

$$\begin{gathered}
\lambda \leq \delta_3, \quad \alpha i \geq 1, \quad t_i \leq \lambda \delta_8^{\alpha i} \leq \delta_4 \delta_1^m (1-\delta_1)^{im} \leq (1/2) \boldsymbol{\alpha}(m) (r_{i+1} - r_i)^m, \\
f_i \in \mathbf{T}_G(V), \quad f_i | X_i \in \mathbf{T}_{H_i}(W_i), \\
H_i = X_i \cap \operatorname{Bdry} U_i, \quad U_{i+1} \subset U_i \cap \{ x : \mathbf{U}(x, r_{i+1} - r_i) \subset \mathbf{R}^n \sim C_i \},
\end{gathered}$$

by 8.12, 8.13 (4) and 9.5, 9.6 with f, X, H, W, C, and r replaced by f_i, X_i, H_i, W_i, C_i, and $r_{i+1} - r_i$. Therefore (1c)′ applied with U, V, Q, M, G, B, f, and r replaced by U_i, X_i, 1, $2M$, H_i, C_i, $f_i|X_i$, and $r_{i+1} - r_i$ yields

$$(\|V\| \llcorner U_{i+1})_{(\beta)}(f_i) \leq \Gamma_{(1c)'}(2M)(\|V\| \llcorner U_i)_{(1)}(V\,\mathbf{D}f_i),$$

hence, using Hölder's inequality in conjunction with 8.12, 8.13 (4) and 8.20 (1),

$$(\|V\| \llcorner U_{i+1})_{(\beta)}(f_i) \leq \delta_7 t_i^{1-1/q}\kappa \leq \delta_7 \lambda^{1-1/q}\delta_8^{\alpha i(1-1/q)}\kappa.$$

Noting $\delta_9^{-\alpha} 2\delta_7 \leq \delta_8^{\alpha(1-1/m)}$ and $2\delta_8^{\alpha(1-1/q)} \leq \delta_8^{\alpha(1-1/m)}$, it follows

$$\begin{aligned} t_{i+1}^{1-1/m} &\leq (s_{i+1} - s_i)^{-1}(\|V\| \llcorner U_{i+1})_{(\beta)}(f_i) \\ &\leq \lambda^{1-1/m}\delta_9^{-\alpha}2\delta_7\big(2\delta_8^{\alpha(1-1/q)}\big)^i \leq \lambda^{1-1/m}\delta_8^{\alpha(1-1/m)(i+1)} \end{aligned}$$

and the assertion is proven.

Step 7. Verification of the property of $\Gamma_{(2d)}$.

Taking $Q = 1$, one may apply (1d) with M replaced by $\sup\{2, M\}$ and a sufficiently large number r. □

10.2 Remark. The role of 7.9 and 7.11 respectively in the Steps 1 and 2 of the preceding proof is identical to the one of Allard [All72, 7.1] in Allard [All72, 7.3].

The method of deduction of (2c) from (2c)′ is classical, see [Fed69, 4.5.15].

In the context of Lipschitz functions the method of deduction of (1c) from (1c)′ and (2c) is outlined by Hutchinson in [Hut90, pp. 59–60].

The iteration procedure employed in the proof (1d) bears formal resemblance with Stampacchia [Sta66, Lemma 4.1 (i)]. However, here the estimate of t_{i+1} in terms of t_i requires a smallness hypothesis on t_i.

10.3 Corollary. *Suppose m, n, U, V, and p are as in 6.1, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{T}(V, Y)$, and $V\,\mathbf{D}f \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \mathrm{Hom}(\mathbf{R}^n, Y))$.*

Then the following four statements hold:

(1) *If $m > 1$ and $f \in \mathbf{L}_1^{\mathrm{loc}}(\|\delta V\|, Y)$, then $f \in \mathbf{L}_{m/(m-1)}^{\mathrm{loc}}(\|V\|, Y)$.*

(2) *If $m = 1$, then $f \in \mathbf{L}_\infty^{\mathrm{loc}}(\|V\| + \|\delta V\|, Y)$.*

(3) *If $1 \leq q < m = p$, then $f \in \mathbf{L}_{mq/(m-q)}^{\mathrm{loc}}(\|V\|, Y)$.*

(4) *If $1 < m = p < q \leq \infty$, then $f \in \mathbf{L}_\infty^{\mathrm{loc}}(\|V\|, Y)$.*

Proof. In view of 8.16 and 9.2 it is sufficient to consider the case $Y = \mathbf{R}$ and $f \in \mathbf{T}_\varnothing(V)$. Assume $p = 1$ in case of (1) or (2) and define ψ as in 6.1. Moreover, assume $(\|V\| + \psi)(U) + \|V\|_{(q)}(V\,\mathbf{D}f) < \infty$ and, in case of (1), also $\|\delta V\|_{(1)}(f) < \infty$. Suppose K is a compact subset of U. Choose $0 < r < \infty$ with $\mathbf{U}(x, r) \subset U$ for $x \in K$ and $0 < s < \infty$ with

$$\|V\|(E) \leq (1/2)\boldsymbol{\alpha}(m)r^m, \quad \psi(E) \leq \Gamma_{10.1}(\sup\{2, n\})^{-1},$$

where $E = \{x : f(x) > s\}$. Select $g \in \mathscr{E}(\mathbf{R}, \mathbf{R})$ with $g(t) = 0$ if $t \leq s$ and $g(t) = t$ if $t \geq s + 1$ and notice that $g \circ f \in \mathbf{T}_{\varnothing}(V)$ and

$$\|V\,\mathbf{D}(g \circ f)(x)\| \leq (\operatorname{Lip} g)\|V\,\mathbf{D}f\|(x) \quad \text{for } \|V\| \text{ almost all } x$$

by 8.12 with $\Upsilon = \{t : t \leq s\}$ and 9.2. Applying 10.1 (1) with M, G, f, and Q replaced by $\sup\{2, n\}$, $\varnothing$, $g \circ f$, and 1 and, in case of (2), noting 8.32, the conclusion follows. □

10.4. *If $1 \leq q \leq p \leq \infty$, $q < \infty$, μ measures X, G is a countable collection of μ measurable $\{t : 0 \leq t \leq \infty\}$ valued functions, $1 \leq \kappa < \infty$,*

$$\operatorname{card}(G \cap \{g : g(x) > 0\}) \leq \kappa \quad \textit{for } \mu \textit{ almost all } x,$$

and $f(x) = \sum_{g \in G} g(x)$ for μ almost all x, then

$$\mu_{(p)}(f) \leq \kappa \big(\textstyle\sum_{g \in G} \mu_{(p)}(g)^q\big)^{1/q}, \quad \mu_{(\infty)}(f) \leq \kappa \sup(\{0\} \cup \{\mu_{(\infty)}(g) : g \in G\});$$

in fact, abbreviating $h(x) = \sum_{g \in G} g(x)^q$ for μ almost all x, one estimates

$$\mu_{(p)}(f)^q \leq \kappa^q \big(\textstyle\sum_{g \in G} \int g^p \,\mathrm{d}\mu\big)^{q/p} \leq \kappa^q \textstyle\sum_{g \in G} \mu_{(p)}(g)^q \quad \text{if } p < \infty,$$
$$\mu_{(\infty)}(f)^q \leq \kappa^q \mu_{(\infty)}(h) \leq \kappa^q \textstyle\sum_{g \in G} \mu_{(\infty)}(g)^q.$$

10.5 Lemma. *Suppose Y is a finite dimensional normed vectorspace, Φ is a family of open subsets Y, and $h : \bigcup \Phi \to \{t : 0 < t < \infty\}$ satisfies*

$$h(y) = \tfrac{1}{20} \sup\{\inf\{1, \operatorname{dist}(y, Y \sim \Upsilon)\} : \Upsilon \in \Phi\} \quad \textit{whenever } y \in \textstyle\bigcup \Phi.$$

Then there exists a countable subset B of $\bigcup \Phi$ and for each $b \in B$ an associated function $g_b : Y \to \mathbf{R}$ with the following properties.

(1) If $y \in B$, then $B_y = B \cap \{b : \mathbf{B}(b, 10h(b)) \cap \mathbf{B}(y, 10h(y)) \neq \varnothing\}$ satisfies

$$\operatorname{card} B_y \leq (129)^{\dim Y}.$$

(2) If $b \in B$, then $0 \leq g_b(y) \leq 1$ for $y \in Y$, $\operatorname{spt} g_b \subset \mathbf{B}(b, 10h(b))$, and

$$\operatorname{Lip}(g_b | \mathbf{B}(y, 10h(y))) \leq (130)^{\dim Y} h(y)^{-1} \quad \textit{whenever } y \in \textstyle\bigcup \Phi.$$

(3) If $y \in Y$, then $\sum_{b \in B} g_b(y) = 1$.

Proof. Assume $\dim Y > 0$ and observe that [Fed69, 3.1.13] remains valid with $\mathbf{R}^m$ and m replaced by Y and $\dim Y$; in fact, one modifies the proof by choosing a nonzero translation invariant Radon measure ν over Y and replacing $\boldsymbol{\alpha}(m)$ by $\nu\,\mathbf{B}(0, 1)$. Then one modifies the proof of [Fed69, 3.1.14] by taking B to be the set named "S" there, $\gamma(t) = \sup\{0, \inf\{1, 2 - t\}\}$ for $t \in \mathbf{R}$, hence $\operatorname{Lip} \mu = 1$, and adding the estimates

$$\operatorname{Lip} u_b \leq h(y)^{-1}, \quad \operatorname{Lip}(\sigma | \mathbf{B}(y, 10h(y))) \leq (129)^{\dim Y} h(y)^{-1},$$
$$\operatorname{Lip}(v_b | \mathbf{B}(y, 10h(y))) \leq \operatorname{Lip} u_b + \operatorname{Lip}(\sigma | \mathbf{B}(y, 10h(y))) \leq (130)^{\dim Y} h(y)^{-1}$$

whenever $y \in \bigcup \Phi$ and $b \in B_y$; hence one may take $g_b = v_b$ for $b \in B$. □

10.6 Lemma. *Suppose Y is a finite dimensional normed vectorspace, D is a closed subset of Y, and $0 < s < \infty$.*

Then there exists a countable family G with the following properties.

(1) *If $g \in G$, then $g : Y \to \mathbf{R}$ and $\operatorname{spt} g \subset \mathbf{U}(d, s)$ for some $d \in D$.*

(2) *If $y \in Y$ then $\operatorname{card}(G \cap \{g : \mathbf{B}(y, s/4) \cap \operatorname{spt} g \neq \varnothing\}) \leq (129)^{\dim Y}$.*

(3) *There holds $D \subset \operatorname{Int}\big\{\sum_{g\in G} g(y) = 1\big\}$ and $\sum_{g \in G} g(y) \leq 1$ for $y \in Y$.*

(4) *If $g \in G$, then $0 \leq g(y) \leq 1$ for $y \in Y$ and $\operatorname{Lip} g \leq 40 \cdot (130)^{\dim Y} s^{-1}$.*

Proof. Assume $s = 1$. Defining $\Phi = \{Y \sim D\} \cup \{\mathbf{U}(d, 1) : d \in D\}$, observe that the function h resulting from 10.5 satisfies $h(y) \geq \frac{1}{40}$ for $y \in Y$, hence one verifies that one can take $G = \{g_b : D \cap \operatorname{spt} g_b \neq \varnothing\}$. ☐

10.7 Theorem. *Suppose $1 \leq M < \infty$.*

Then there exists a positive, finite number Γ with the following property.

If m, n, p, U, V, and ψ are as in 6.1, Y is a finite dimensional normed vectorspace, $\sup\{\dim Y, n\} \leq M$, $f \in \mathbf{T}(V, Y)$, $1 \leq Q \leq M$, $N \in \mathscr{P}$, $0 < r < \infty$,

$$\|V\|(U) \leq (Q - M^{-1})(N + 1)\boldsymbol{\alpha}(m) r^m,$$
$$\|V\|(\{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m,$$

$A = \{x : \mathbf{U}(x, r) \subset U\}$, and $f_\Upsilon : \operatorname{dmn} f \to \mathbf{R}$ is defined by

$$f_\Upsilon(x) = \operatorname{dist}(f(x), \Upsilon) \quad \text{whenever } x \in \operatorname{dmn} f,\ \varnothing \neq \Upsilon \subset Y,$$

then the following four statements hold.

(1) *If $p = 1$, $\beta = \infty$ if $m = 1$ and $\beta = m/(m-1)$ if $m > 1$, then there exists a subset Υ of Y such that $1 \leq \operatorname{card} \Upsilon \leq N$ and*

$$(\|V\| \llcorner A)_{(\beta)}(f_\Upsilon) \leq \Gamma N^{1/\beta}\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(f_\Upsilon)\big).$$

(2) *If $p = m = 1$ and $\psi(U) \leq \Gamma^{-1}$, then there exists a subset Υ of Y such that $1 \leq \operatorname{card} \Upsilon \leq N$ and*

$$(\|V\| \llcorner A)_{(\infty)}(f_\Upsilon) \leq \Gamma\, \|V\|_{(1)}(V\,\mathbf{D}f).$$

(3) *If $1 \leq q < m = p$ and $\psi(U) \leq \Gamma^{-1}$, then there exists a subset Υ of Y such that $1 \leq \operatorname{card} \Upsilon \leq N$ and*

$$(\|V\| \llcorner A)_{(mq/(m-q))}(f_\Upsilon) \leq \Gamma N^{1/q - 1/m}(m - q)^{-1}\|V\|_{(q)}(V\,\mathbf{D}f).$$

(4) *If $1 < p = m < q \leq \infty$ and $\psi(U) \leq \Gamma^{-1}$, then there exists a subset Υ of Y such that $1 \leq \operatorname{card} \Upsilon \leq N$ and*

$$(\|V\| \llcorner A)_{(\infty)}(f_\Upsilon) \leq \Gamma^{1/(1/m - 1/q)} r^{1 - m/q}\|V\|_{(q)}(V\,\mathbf{D}f).$$

Proof. Define

$$\Delta_1 = \sup\{1, \Gamma_{7.8}(M)\}^M, \quad \Delta_2 = (\sup\{1/2, 1 - 1/(2M^2 - 1)\})^{1/M},$$
$$\Delta_3 = 40 \cdot (130)^M, \quad \Delta_4 = \Delta_2^{-M} \Gamma_{10.1}(2M),$$
$$\Delta_5 = (1/2) \inf\{\boldsymbol{\alpha}(m) : M \geq m \in \mathscr{P}\},$$
$$\Delta_6 = \sup\{1, \Gamma_{10.1}(2M)\}(\Delta_3^3 + 1)\Delta_5^{-1}(1 - \Delta_2)^{-M},$$
$$\Delta_7 = \sup\{\boldsymbol{\alpha}(m) : M \geq m \in \mathscr{P}\}, \quad \Delta_8 = 4M\Delta_6\Delta_7 + \Gamma_{10.1}(2M),$$
$$\Delta_9 = M\Delta_7\Gamma_{10.1}(2M), \quad \Gamma = \sup\{\Delta_1, \Delta_4, \Delta_8, 2\Delta_6(1 + \Delta_9)\}$$

and notice that $\Delta_5 \leq 1 \leq \inf\{\Delta_6, \Delta_7\}$.

In order to verify that Γ has the asserted property, suppose m, n, p, U, V, ψ, Y, f, Q, N, r, A, and f_Υ are related to Γ as in the body of the theorem. Abbreviate $\iota = mq/(m - q)$ in case of (3) and $\alpha = 1/m - 1/q$ in case of (4). Moreover, define

$$\begin{aligned} &\kappa = \|V\|_{(1)}(V\,\mathbf{D}f) && \text{in case of (1) or (2),} \\ &\kappa = (m - q)^{-1}\|V\|_{(q)}(V\,\mathbf{D}f) && \text{in case of (3),} \\ &\kappa = \Delta_9^{1/\alpha}\|V\|_{(q)}(V\,\mathbf{D}f) && \text{in case of (4)} \end{aligned}$$

and assume $r = 1$ and $\operatorname{dmn} f = U$.

First, the *case* $\kappa = 0$ will be considered.

For this purpose choose Ξ and υ as in 8.33 and let

$$\Pi = \Xi \cap \{W : \|W\|(U) \leq (Q - M^{-1})\boldsymbol{\alpha}(m)\}.$$

Applying 7.9 and 7.11 with V, E, and B replaced by W, U, and $\operatorname{Bdry} U$, one infers

$$\begin{aligned} &\|W\|(A)^{1/\beta} \leq \Delta_1\,\|\delta W\|(U) && \text{in case of (1), where } 0^0 = 0, \\ &\|W\|(A) = 0 && \text{in case of (2) or (3) or (4)} \end{aligned}$$

whenever $W \in \Pi$. Since $\operatorname{card}(\Xi \sim \Pi) \leq N$, there exists Υ such that

$$\upsilon[\Xi \sim \Pi] \subset \Upsilon \subset Y \quad \text{and} \quad 1 \leq \operatorname{card} \Upsilon \leq N.$$

In case of (1), it follows that, using 6.10,

$$\begin{aligned} (\|V\| \llcorner A)_{(\beta)}(f_\Upsilon) &\leq \textstyle\sum_{W \in \Pi} \operatorname{dist}(\upsilon(W), \Upsilon)\|W\|(A)^{1/\beta} \\ &\leq \Delta_1 \textstyle\sum_{W \in \Pi} \operatorname{dist}(\upsilon(W), \Upsilon)\|\delta W\|(U) = \Delta_1\|\delta V\|(f_\Upsilon). \end{aligned}$$

In case of (2) or (3) or (4), the corresponding estimate is trivial.

Second, the *case* $\kappa > 0$ will be considered.

For this purpose assume $\kappa < \infty$ and define

$$X = \{x : \mathbf{B}(x, \Delta_2) \subset U\}, \quad s = \Delta_6\kappa,$$
$$B = Y \cap \big\{y : \|V\|(f^{-1}[\mathbf{U}(y, s)]) > (Q - M^{-1})\boldsymbol{\alpha}(m)\big\}.$$

Choose $\Upsilon \subset Y$ satisfying

$$1 \leq \operatorname{card} \Upsilon \leq N \quad \text{and} \quad B \subset \{y : \operatorname{dist}(y, \Upsilon) < 2s\};$$

in fact, if $B \neq \varnothing$ then one may take Υ to be a maximal subset of B with respect to inclusion such that $\{\mathbf{U}(y,s) : y \in \Upsilon\}$ is disjointed. Abbreviate $\gamma = \operatorname{dist}(\cdot, \Upsilon)$ and, in case of (1), define

$$\lambda = \kappa + \|\delta V\|(\gamma \circ f)$$

and assume $\lambda < \infty$. In case of (2) or (3) or (4), define $\lambda = \kappa$. Let

$$D = Y \cap \{y : \gamma(y) \geq 2\Delta_6 \lambda\}, \quad E = f^{-1}[D].$$

Next, *it will be shown that*

$$\begin{aligned} &\|V\|(X \cap E) \leq (1/2)\boldsymbol{\alpha}(m)(1-\Delta_2)^m && \textit{in case of (1) or (3)}, \\ &\|V\|(X \cap E) = 0 && \textit{in case of (2) or (4)}. \end{aligned}$$

To prove this assertion, choose G as in 10.6 and let $E_g = \{x : g(f(x)) > 0\}$ for $g \in G$. Since $D \cap B = \varnothing$ as $\Delta_6 \lambda \geq s$, it follows that

$$\begin{gathered} \|V\|(E_g) \leq (Q - M^{-1})\boldsymbol{\alpha}(m) \leq \big(Q - (2M)^{-1}\big)\boldsymbol{\alpha}(m)\Delta_2^m, \\ \|V\|(E_g \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} \leq \Delta_4^{-1} \leq \Gamma_{10.1}(2M)^{-1}\Delta_2^m, \\ \psi(E_g) \leq \Gamma^{-1} \leq \Gamma_{10.1}(2M)^{-1} \quad \text{in case of (2) or (3) or (4)} \end{gathered}$$

whenever $g \in G$. Denoting by c_g the characteristic function of E_g, applying 8.15 with Υ replaced by $\operatorname{spt} g$ and noting 8.20 (1) and 9.2 yields $g \circ f \in \mathbf{T}_\varnothing(V)$ with

$$|V\,\mathbf{D}(g \circ f)(x)| \leq \Delta_3 s^{-1} c_g(x) \|V\,\mathbf{D}f(x)\| \quad \text{for } \|V\| \text{ almost all } x$$

whenever $g \in G$. Moreover, denoting the function whose domain is U and whose value at x equals $\sum_{g\in G}(g \circ f)(x)$ by $\sum_{g \in G} g \circ f$, notice that

$$\begin{gathered} \operatorname{card}(G \cap \{g : x \in E_g\}) \leq \Delta_3 \quad \text{for } x \in U, \\ \big(\textstyle\sum_{g\in G} g \circ f\big)|E = 1, \qquad \operatorname{spt} g \subset \{y : \gamma(y) \geq \Delta_6\lambda\} \quad \text{for } g \in G. \end{gathered}$$

In case of (1), one estimates, using 10.1 (1a) with M, G, f, and r replaced by $2M$, $\varnothing$, $g \circ f$, and Δ_2 and [Fed69, 2.4.18 (1)],

$$\begin{aligned} \|V\|(X \cap E)^{1/\beta} &\leq (\|V\| \llcorner X)_{(\beta)}(\textstyle\sum_{g\in G} g \circ f) \leq \sum_{g \in G}(\|V\| \llcorner X)_{(\beta)}(g \circ f) \\ &\leq \Gamma_{10.1}(2M)\textstyle\sum_{g\in G}\big(\Delta_3 s^{-1}(\|V\| \llcorner E_g)_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(g \circ f)\big), \\ &\leq \Gamma_{10.1}(2M)\big(\Delta_6^{-1}\Delta_3^2 + (f_\# \|\delta V\|)(\{y : \gamma(y) \geq \Delta_6\lambda\})\big) \\ &\leq \Gamma_{10.1}(2M)(\Delta_3^2 + 1)\Delta_6^{-1} \leq \Delta_5(1-\Delta_2)^M \leq \big((1/2)\boldsymbol{\alpha}(m)(1-\Delta_2)^m\big)^{1/\beta}, \end{aligned}$$

where $0^0 = 0$. In case of (2), using 10.1 (1b) with M, G, f, and r replaced by $2M$, $\varnothing$, $g \circ f$, and Δ_2, one estimates

$$\begin{aligned} (\|V\| \llcorner X)_{(\infty)}(\textstyle\sum_{g\in G} g \circ f) &\leq \textstyle\sum_{g\in G}(\|V\| \llcorner X)_{(\infty)}(g \circ f) \\ &\leq \Gamma_{10.1}(2M)\textstyle\sum_{g\in G}\Delta_3 s^{-1}(\|V\| \llcorner E_g)_{(1)}(V\,\mathbf{D}f) \\ &\leq \Gamma_{10.1}(2M)\Delta_3^2\Delta_6^{-1} \leq \Delta_5(1-\Delta_2)^M < 1, \end{aligned}$$

hence $\|V\|(X \cap E) = 0$. In case of (3), using 10.4 and 10.1 (1c) with M, G, f, and r replaced by $2M$, $\varnothing$, $g \circ f$, and Δ_2, one estimates

$$\begin{aligned} \|V\|(X \cap E)^{1/\iota} &\leq (\|V\| \llcorner X)_{(\iota)}(\textstyle\sum_{g \in G} g \circ f) \\ &\leq \Delta_3 \big(\textstyle\sum_{g \in G} (\|V\| \llcorner X)_{(\iota)}(g \circ f)^q\big)^{1/q} \\ &\leq \Gamma_{10.1}(2M)(m-q)^{-1}\Delta_3^2 s^{-1}\big(\textstyle\sum_{g \in G}(\|V\| \llcorner E_g)_{(q)}(V\,\mathbf{D}f)^q\big)^{1/q} \\ &\leq \Gamma_{10.1}(2M)\Delta_3^3\Delta_6^{-1} \leq \Delta_5(1-\Delta_2)^M \leq \big((1/2)\boldsymbol{\alpha}(m)(1-\Delta_2)^m\big)^{1/\iota}. \end{aligned}$$

In case of (4), using 10.1 (1d) with M, G, f, and r replaced by $2M$, $\varnothing$, $g \circ f$, and Δ_2 and noting $\Gamma_{10.1}(2M)^{1/\alpha}(M\Delta_7)^\alpha \leq \Delta_9^{1/\alpha}$, one obtains

$$(\|V\| \llcorner X)_{(\infty)}(g \circ f) \leq \Delta_9^{1/\alpha}\Delta_3 s^{-1}(\|V\| \llcorner E_g)_{(q)}(V\,\mathbf{D}f) \quad \text{for } g \in G$$

and therefore $\|V\|(X \cap E) = 0$, since if $q < \infty$ then, using 10.4,

$$\begin{aligned} (\|V\| \llcorner X)_{(\infty)}(\textstyle\sum_{g \in G} g \circ f) &\leq \Delta_3\big(\textstyle\sum_{g \in G}(\|V\| \llcorner X)_{(\infty)}(g \circ f)^q\big)^{1/q} \\ &\leq \Delta_9^{1/\alpha}\Delta_3^2 s^{-1}\big(\textstyle\sum_{g \in G}(\|V\| \llcorner E_g)_{(q)}(V\,\mathbf{D}f)^q\big)^{1/q} \\ &\leq \Delta_3^3\Delta_6^{-1} \leq \Delta_5(1-\Delta_2)^M < 1, \end{aligned}$$

and if $q = \infty$ then $(\|V\| \llcorner X)_{(\infty)}(\sum_{g \in G} g \circ f) < 1$ follows similarly.

In case of (2) or (4), noting $2\Delta_6\Delta_9^{1/\alpha} \leq \Gamma^{1/\alpha}$ in case of (4), the conclusion is evident from the assertion of the preceding paragraph. Now consider the case of (1) or (3) and define $h : Y \to \mathbf{R}$ by

$$h(y) = \sup\{0, \gamma(y) - 2\Delta_6\lambda\} \quad \text{for } y \in Y.$$

Notice that $h \circ f \in \mathbf{T}_\varnothing\big(V|\mathbf{2}^{X \times \mathbf{G}(n,m)}\big)$ with

$$|V\,\mathbf{D}(h \circ f)(x)| \leq \|V\,\mathbf{D}f(x)\| \quad \text{for } \|V\| \text{ almost all } x$$

by 8.15 with Υ replaced by Y and 9.2. In case of (1), applying 10.1 (1a) with M, U, V, G, f, Q, and r replaced by $2M$, X, $V|\mathbf{2}^{X \times \mathbf{G}(n,m)}$, $\varnothing$, $h \circ f|X$, 1, and $1 - \Delta_2$ implies

$$\begin{aligned} (\|V\| \llcorner A)_{(\beta)}(h \circ f) &\leq \Gamma_{10.1}(2M)\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(h \circ f)\big), \\ (\|V\| \llcorner A)_{(\beta)}(\gamma \circ f) &\leq 4M\Delta_6\Delta_7 N^{1/\beta}\lambda + \Gamma_{10.1}(2M)\lambda \leq \Gamma N^{1/\beta}\lambda \end{aligned}$$

since $\|V\|(A)^{1/\beta} \leq 2M\Delta_7 N^{1/\beta}$. In case of (3), applying 10.1 (1c) with M, U, V, G, f, Q, and r replaced by $2M$, X, $V|\mathbf{2}^{X \times \mathbf{G}(n,m)}$, $\varnothing$, $h \circ f|X$, 1, and $1 - \Delta_2$ implies

$$\begin{aligned} (\|V\| \llcorner A)_{(\iota)}(h \circ f) &\leq \Gamma_{10.1}(2M)\kappa, \\ (\|V\| \llcorner A)_{(\iota)}(\gamma \circ f) &\leq 4M\Delta_6\Delta_7 N^{1/\iota}\kappa + \Gamma_{10.1}(2M)\kappa \leq \Gamma N^{1/\iota}\kappa \end{aligned}$$

since $\|V\|(A)^{1/\iota} \leq 2M\Delta_7 N^{1/\iota}$. □

10.8 Remark. Comparing the preceding theorem to previous Sobolev Poincaré inequalities of the author, see [Men10, 4.4–4.6], which treated the particular case of f being the orthogonal projection onto an $n-m$ dimensional plane in the context of integral varifolds, the present approach is significantly more general. Yet, for the case of orthonormal projections the previous results give somewhat more precise information as they include, for instance, the intermediate cases $1 < p < m$ as well as Lorentz spaces. Evidently, the question arises whether the present theorem can be correspondingly extended.

10.9 Theorem. *Suppose m, n, p, U, V, and ψ are as in 6.1, $n \leq M < \infty$, $N \in \mathscr{P}$, $1 \leq Q \leq M$, $f \in \mathbf{T}(V)$, $0 < r < \infty$, X is a finite subset of U,*

$$\|V\|(U) \leq (Q - M^{-1})(N+1)\boldsymbol{\alpha}(m)r^m,$$
$$\|V\|(\{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Gamma^{-1} r^m,$$

and $A = \{x : \mathbf{U}(x,r) \subset U\}$, then there exists a subset Υ of $\mathbf{R}$ with $1 \leq \operatorname{card} \Upsilon \leq N + \operatorname{card} X$ such that the following four statements hold with $g = \operatorname{dist}(\cdot, \Upsilon) \circ f$:

(1) *If $p = 1$, $\beta = \infty$ if $m = 1$ and $\beta = m/(m-1)$ if $m > 1$, then*

$$(\|V\| \llcorner A)_{(\beta)}(g) \leq \Gamma_{10.1}(M)\big(\|V\|_{(1)}(V\,\mathbf{D}f) + \|\delta V\|(g)\big).$$

(2) *If $p = m = 1$ and $\psi(U \sim X) \leq \Gamma_{10.1}(M)^{-1}$, then*

$$(\|V\| \llcorner A)_{(\infty)}(g) \leq \Gamma_{10.1}(M)\,\|V\|_{(1)}(V\,\mathbf{D}f).$$

(3) *If $1 \leq q < m = p$ and $\psi(U) \leq \Gamma_{10.1}(M)^{-1}$, then*

$$(\|V\| \llcorner A)_{(mq/(m-q))}(g) \leq \Gamma_{10.1}(M)(m-q)^{-1}\|V\|_{(q)}(V\,\mathbf{D}f).$$

(4) *If $1 < p = m < q \leq \infty$ and $\psi(U) \leq \Gamma_{10.1}(M)^{-1}$, then*

$$(\|V\| \llcorner A)_{(\infty)}(g) \leq \Gamma^{1/(1/m-1/q)} Q^{1/m-1/q} r^{1-m/q} \|V\|_{(q)}(V\,\mathbf{D}f),$$

where $\Gamma = \Gamma_{10.1}(M)\sup\{\boldsymbol{\alpha}(i) : M \geq i \in \mathscr{P}\}$.

Proof. Choose a nonempty subset Υ of $\mathbf{R}$ satisfying

$$\operatorname{card} \Upsilon \leq N + \operatorname{card} X, \quad f[X] \subset \Upsilon,$$
$$\|V\|(U \cap \{x : f(x) \in I\}) \leq (Q - M^{-1})\boldsymbol{\alpha}(m)r^m \quad \text{for } I \in \Phi,$$

where Φ is the family of connected components of $\mathbf{R} \sim \Upsilon$. Notice that

$$\operatorname{dist}(b, \Upsilon) = \operatorname{dist}(b, \mathbf{R} \sim I) \quad \text{and} \quad \operatorname{dist}(b, \mathbf{R} \sim J) = 0$$

whenever $b \in I \in \Phi$ and $I \neq J \in \Phi$, and

$$\operatorname{dist}(b, \Upsilon) = \operatorname{dist}(b, \mathbf{R} \sim I) = 0 \quad \text{whenever } b \in \Upsilon \text{ and } I \in \Phi.$$

Defining $f_I = \operatorname{dist}(\mathbf{R} \sim I) \circ f$, one infers $f_I \in \mathbf{T}_\varnothing(V)$ and

$$|V\,\mathbf{D}f(x)| = |V\,\mathbf{D}f_I(x)| \quad \text{for } \|V\| \text{ almost all } x \in f^{-1}[I],$$
$$V\,\mathbf{D}f(x) = 0 \quad \text{for } \|V\| \text{ almost all } x \in f^{-1}[\Upsilon],$$
$$V\,\mathbf{D}f_I(x) = 0 \quad \text{for } \|V\| \text{ almost all } x \in f^{-1}[\mathbf{R} \sim I]$$

whenever $I \in \Phi$ by 8.12, 8.13, 8.20 (1), and 9.2.

The conclusion now will be obtained employing 10.4 and applying 10.1 (1) with G and f replaced by $\varnothing$ and f_I for $I \in \Phi$. For instance, in case of (3) one estimates

$$\begin{aligned}(\|V\| \llcorner A)_{(mq/(m-q))}(g) &\leq \big(\textstyle\sum_{I\in\Phi}(\|V\| \llcorner A)_{(mq/(m-q))}(f_I)^q\big)^{1/q} \\ &\leq \lambda\big(\textstyle\sum_{I\in\Phi}\|V\|_{(q)}(V\,\mathbf{D}f_I)^q\big)^{1/q} = \lambda\,\|V\|_{(q)}(V\,\mathbf{D}f),\end{aligned}$$

where $\lambda = \Gamma_{10.1}(M)(m-q)^{-1}$. The cases (1) and (2) follow similarly. Defining $\Delta = \sup\{\boldsymbol{\alpha}(i) : M \geq i \in \mathscr{P}\}$ and noting $\Delta \geq 1$, hence $\boldsymbol{\alpha}(m)^{(1/m-1/q)^2} \leq \Delta^{(1/m-1/q)^2} \leq \Delta$, the same holds for (4). $\square$

10.10 Remark. The method of deduction of 10.9 from 10.1 is that of Hutchinson [Hut90, Theorem 3] which is derived from Hutchinson [Hut90, Theorem 1].

10.11 Remark. A nonempty choice of X will occur in 13.1.

11 Differentiability properties

In this section approximate differentiability, see 11.2, and differentiability in Lebesgue spaces, see 11.4, are established for weakly differentiable functions. The primary ingredient is the Sobolev Poincaré inequality 10.1 (1).

11.1 Lemma. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of U, $V \in \mathbf{RV}_m(U)$, Y is a finite dimensional normed vectorspace, f is a $\|V\|$ measurable Y valued function, and A is the set of points at which f is $(\|V\|, m)$ approximately differentiable.*

Then the following four statements hold.

1. *The set A is $\|V\|$ measurable and $(\|V\|, m)\operatorname{ap} Df(x) \circ \operatorname{Tan}^m(\|V\|, x)_\natural$ depends $\|V\| \llcorner A$ measurably on x.*
2. *There exists a sequence of functions $f_i : U \to Y$ of class 1 such that*
$$\|V\|(A \sim \{x : f(x) = f_i(x) \text{ for some } i\}) = 0.$$
3. *If $g : U \to Y$ is locally Lipschitzian, then*
$$\|V\|(U \cap \{x : f(x) = g(x)\} \sim A) = 0.$$
4. *If g is a $\|V\|$ measurable Y valued function and $B = U \cap \{x : f(x) = g(x)\}$, then $B \cap A$ is $\|V\|$ almost equal to $B \cap \operatorname{dmn}(\|V\|, m)\operatorname{ap} Dg$ and*
$$(\|V\|, m)\operatorname{ap} Df(x) = (\|V\|, m)\operatorname{ap} Dg(x) \quad \text{for } \|V\| \text{ almost all } x \in B \cap A.$$

Proof. Assume $Y = \mathbf{R}$. Then (1) is [Men12a, 4.5 (1)].

In order to prove (2), one may reduce the problem. Firstly, to the case that $\|V\|(U \sim M) = 0$ for some m dimensional submanifold M of $\mathbf{R}^n$ of class 1 by [Fed69, 2.10.19 (4), 3.2.29]. Secondly, to the case that for some $1 < \lambda < \infty$ the varifold satisfies additionally that $\lambda^{-1} \leq \boldsymbol{\Theta}^m(\|V\|, x) \leq \lambda$ for $\|V\|$ almost all x by [Fed69, 2.10.19 (4)] and Allard [All72, 3.5 (1a)], hence thirdly to the case that $\boldsymbol{\Theta}^m(\|V\|, x) = 1$ for $\|V\|$ almost all x by [Fed69, 2.10.19 (1) (3)]. Finally, to the case $\|V\| = \mathscr{L}^m$ by [Fed69, 3.1.19 (4), 3.2.3, 2.8.18, 2.9.11] which may be treated by means of [Fed69, 3.1.16].

(4) follows from [Fed69, 2.10.19 (4)] and implies (3) by [Men12a, 4.5 (2)]. $\square$

11.2 Theorem. *Suppose m, n, U, and V are as in 6.1, Y is a finite dimensional normed vectorspace, and $f \in \mathbf{T}(V, Y)$.*

Then f is $(\|V\|, m)$ approximately differentiable with

$$V\,\mathbf{D}f(a) = (\|V\|, m)\operatorname{ap} Df(a) \circ \operatorname{Tan}^m(\|V\|, a)_\natural$$

at $\|V\|$ almost all a.

Proof. In view of 11.1 (4), one employs 8.17 and 8.12 to reduce the problem to the case that f is bounded and $V\,\mathbf{D}f \in \mathbf{L}_1^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$. In particular, one may assume $Y = \mathbf{R}$ by 8.18. Define $\beta = \infty$ if $m = 1$ and $\beta = m/(m-1)$ if $m > 1$. Let $g_a : \operatorname{dmn} f \to \mathbf{R}$ be defined by $g_a(x) = f(x) - f(a)$ for $a, x \in \operatorname{dmn} f$.

First, *it will be shown that*

$$\limsup_{s \to 0+} s^{-1} \|V\|(\mathbf{B}(a,s))^{-1} {\textstyle\int_{\mathbf{B}(a,s)}} g_a \,\mathrm{d}\|V\| < \infty \quad \textit{for } \|V\| \textit{ almost all } a.$$

For this purpose define $C = \{(a, \mathbf{B}(a,r)) : \mathbf{B}(a,r) \subset U\}$ and consider a point a satisfying for some M the conditions

$$\sup\{4, n\} \leq M < \infty, \quad 1 \leq \boldsymbol{\Theta}^m(\|V\|, a) \leq M, \quad (\|V\| + \|\delta V\|)_{(\infty)}(f) \leq M,$$
$$\limsup_{s \to 0+} s^{-m} \big({\textstyle\int_{\mathbf{B}(a,s)}} |V\,\mathbf{D}f| \,\mathrm{d}\|V\| + \|\delta V\|\,\mathbf{B}(a,s) \big) < M,$$
$$\boldsymbol{\Theta}^m(\|V\|, \cdot) \text{ and } f \text{ are } (\|V\|, C) \text{ approximately continuous at } a$$

which are met by $\|V\|$ almost all a by [Fed69, 2.10.19 (1) (3), 2.8.18, 2.9.13]. Define $\Delta = \Gamma_{10.1}(M)$. Choose $1 \leq Q \leq M$ and $1 < \lambda \leq 2$ subject to the requirements $\boldsymbol{\Theta}^m(\|V\|, a) < 2\lambda^{-m}(Q - 1/4)$ and

$$\text{either } Q = \boldsymbol{\Theta}^m(\|V\|, a) = 1 \quad \text{or } Q < \boldsymbol{\Theta}^m(\|V\|, a).$$

Then pick $0 < r < \infty$ such that $\mathbf{B}(a, \lambda r) \subset U$ and

$$\|V\|\,\mathbf{B}(a,s) \geq (1/2)\boldsymbol{\alpha}(m) s^m, \quad \|V\|\,\mathbf{U}(a, \lambda s) \leq 2(Q - M^{-1})\boldsymbol{\alpha}(m) s^m,$$
$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : \boldsymbol{\Theta}^m(\|V\|, x) < Q\}) \leq \Delta^{-1} s^m,$$
$${\textstyle\int_{\mathbf{B}(a,\lambda s)}} |V\,\mathbf{D}f| \,\mathrm{d}\|V\| + \|\delta V\|\,\mathbf{B}(a, \lambda s) \leq M \lambda^m s^m$$

for $0 < s \leq r$. Choose $y(s) \in \mathbf{R}$ such that

$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : f(x) < y(s)\}) \leq (1/2)\|V\|\,\mathbf{U}(a, \lambda s),$$
$$\|V\|(\mathbf{U}(a, \lambda s) \cap \{x : f(x) > y(s)\}) \leq (1/2)\|V\|\,\mathbf{U}(a, \lambda s)$$

for $0 < s \leq r$, in particular

$$|y(s)| \leq M \quad \text{and} \quad f(a) = \lim_{s \to 0+} y(s).$$

Define $f_s(x) = f(x) - y(s)$ whenever $0 < s \leq r$ and $x \in \operatorname{dmn} f$. Recalling 8.12, 8.13 (4), and 9.2, one applies 10.1 (1a) with U, G, f, and r replaced by $\mathbf{U}(a, \lambda s)$, $\varnothing$, f_s^+ respectively f_s^-, and s to infer

$$\begin{aligned}(\|V\| \llcorner \mathbf{B}(a,(\lambda-1)s))_{(\beta)}(f_s) &\leq \Delta \textstyle\int_{\mathbf{B}(a,\lambda s)} |V \mathbf{D} f_s| \,\mathrm{d}\|V\| + \int_{\mathbf{B}(a,\lambda s)} |f_s| \,\mathrm{d}\|\delta V\| \\ &\leq \kappa s^m\end{aligned}$$

for $0 < s \leq r$, where $\kappa = \Delta M \lambda^m$. Noting that

$$\begin{gathered}|y(s) - y(s/2)| \cdot \|V\|(\mathbf{B}(a,(\lambda-1)s/2))^{1/\beta} \\ \leq (\|V\| \llcorner \mathbf{B}(a,(\lambda-1)s/2))_{(\beta)}(f_{s/2}) + (\|V\| \llcorner \mathbf{B}(a,(\lambda-1)s))_{(\beta)}(f_s) \leq 2\kappa s^m, \\ |y(s) - y(s/2)| \leq 2^{m+1} \kappa \boldsymbol{\alpha}(m)^{-1/\beta} (\lambda-1)^{1-m} s,\end{gathered}$$

one obtains for $0 < s \leq r$ that

$$\begin{gathered}|y(s) - f(a)| \leq 2^{m+2} \kappa \boldsymbol{\alpha}(m)^{-1/\beta} (\lambda-1)^{1-m} s, \\ (\|V\| \llcorner \mathbf{B}(a,(\lambda-1)s))_{(\beta)}(g_a) \leq \kappa \big(1 + 2^{m+3} M (\lambda-1)^{1-m}\big) s^m.\end{gathered}$$

Combining the assertion of the preceding paragraph with [Men09, 3.7 (i)] applied with α, q, and r replaced by 1, 1, and ∞, one obtains a sequence of locally Lipschitzian functions $f_i : U \to \mathbf{R}$ such that

$$\|V\|\big(U \sim \textstyle\bigcup \{B_i : i \in \mathscr{P}\}\big) = 0, \quad \text{where } B_i = U \cap \{x : f(x) = f_i(x)\}.$$

Since $f - f_i \in \mathbf{T}(V)$ with

$$V \mathbf{D} f(a) - V \mathbf{D} f_i(a) = V \mathbf{D}(f - f_i)(a) = 0 \quad \text{for } \|V\| \text{ almost all } a \in B_i$$

by 8.20 (1) (3), the conclusion follows from of 8.7 and 11.1 (4). □

11.3. *If m, n, U, V, ψ, and p are as in 6.1, $p = m$, and $a \in U$, then*

$$\operatorname{Tan}^m(\|V\|, a) = \operatorname{Tan}(\operatorname{spt} \|V\|, a);$$

in fact, if $0 < r < \infty$, $\mathbf{U}(a, 2r) \subset U$, $\psi(\mathbf{U}(a,2r) \sim \{a\})^{1/m} \leq (2\boldsymbol{\gamma}(m))^{-1}$, and $x \in \mathbf{U}(a,r) \cap \operatorname{spt} \|V\|$, then $\|V\| \mathbf{B}(x,s) \geq (2m\boldsymbol{\gamma}(m))^{-m} s^m$ for $0 < s < |x - a|$ by [Men09, 2.5].

11.4 Theorem. *Suppose m, n, p, U, V are as in 6.1, $1 \leq q \leq \infty$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{T}(V, Y)$, $V \mathbf{D} f \in \mathbf{L}_q^{\mathrm{loc}}(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y))$,*

$$C = \{(a, \mathbf{B}(a,r)) : \mathbf{B}(a,r) \subset U\},$$

and X is the set of points in $\operatorname{spt} \|V\|$ at which f is $(\|V\|, C)$ approximately continuous.

Then $\|V\|(U \sim X) = 0$ and the following four statements hold.

(1) If $m > 1$, $\beta = m/(m-1)$, and $f \in \mathbf{L}_1^{\mathrm{loc}}(\|\delta V\|, Y)$, then

$$\lim_{r \to 0+} r^{-m} \textstyle\int_{\mathbf{B}(a,r)} (|f(x) - f(a) - \langle x - a, V \mathbf{D} f(a) \rangle| / |x - a|)^\beta \,\mathrm{d}\|V\| x = 0$$

for $\|V\|$ almost all a.

(2) If $m = 1$, then $f|X$ is differentiable relative to X at a with

$$D(f|X)(a) = V\,\mathbf{D}f(a)|\operatorname{Tan}^m(\|V\|, a) \quad \textit{for } \|V\| \textit{ almost all } a.$$

(3) If $q < m = p$ and $\iota = mq/(m-q)$, then

$$\lim_{r\to 0+} r^{-m} {\textstyle\int_{\mathbf{B}(a,r)}} (|f(x) - f(a) - \langle x-a, V\,\mathbf{D}f(a)\rangle|/|x-a|)^\iota \,\mathrm{d}\|V\|x = 0$$

for $\|V\|$ almost all a.

(4) If $p = m < q$, then $f|X$ is differentiable relative to X at a with

$$D(f|X)(a) = V\,\mathbf{D}f(a)|\operatorname{Tan}^m(\|V\|, a) \quad \textit{for } \|V\| \textit{ almost all } a.$$

Proof. Clearly, $\|V\|(U \sim X) = 0$ by [Fed69, 2.8.18, 2.9.13].

Assume $p = q = 1$ in case of (1) or (2). In case of (4) also assume $p > 1$ and $q < \infty$. Define $\alpha = \beta$ in case of (1), $\alpha = \iota$ in case of (3), and $\alpha = \infty$ in case of (2) or (4). Moreover, let $h_a : \operatorname{dmn} f \to Y$ be defined by

$$h_a(x) = f(x) - f(a) - \langle x-a, V\,\mathbf{D}f(a)\rangle$$

whenever $a \in \operatorname{dmn} f \cap \operatorname{dmn} V\,\mathbf{D}f$ and $x \in \operatorname{dmn} f$.

The following assertion will be shown. *There holds*

$$\lim_{s\to 0+} s^{-1}(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(h_a) = 0 \quad \textit{for } \|V\| \textit{ almost all } a.$$

In the special case that f is of class 1, in view of 8.7, it is sufficient to prove that

$$\lim_{s\to 0+} s^{-1}(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(\operatorname{Nor}^m(\|V\|, a)_\natural(\cdot - a)) = 0$$

for $\|V\|$ almost all a; and if $\varepsilon > 0$ and $\operatorname{Tan}^m(\|V\|, a) \in \mathbf{G}(n, m)$, then

$$\boldsymbol{\Theta}^m(\|V\| \llcorner U \cap \{x : |\operatorname{Nor}^m(\|V\|, a)_\natural(x-a)| > \varepsilon|x-a|\}, a) = 0$$

in case of (1) by [Fed69, 3.2.16] and

$$\mathbf{B}(a,s) \cap \operatorname{spt}\|V\| \subset \{x : |\operatorname{Nor}^m(\|V\|, a)_\natural(x-a)| \le \varepsilon|x-a|\} \quad \text{for some } s > 0$$

in case of (2) or (3) or (4) by 11.3 and [Fed69, 3.1.21]. To treat the general case, one obtains a sequence of functions $f_i : U \to Y$ of class 1 such that

$$\|V\|\big(U \sim {\textstyle\bigcup}\{B_i : i \in \mathscr{P}\}\big) = 0, \quad \text{where } B_i = U \cap \{x : f(x) = f_i(x)\}$$

from 11.2 and 11.1 (2). Define $g_i = f - f_i$ and notice that $g_i \in \mathbf{T}(V, Y)$ and

$$V\,\mathbf{D}g_i(a) = V\,\mathbf{D}f(a) - V\,\mathbf{D}f_i(a) \quad \text{for } \|V\| \text{ almost all } a$$

for $i \in \mathscr{P}$ by 8.20 (3). Define Radon measures μ_i over U by

$$\mu_i(A) = {\textstyle\int_A^*} |V\,\mathbf{D}g_i| \,\mathrm{d}\|V\| + {\textstyle\int_A^*} |g_i| \,\mathrm{d}\|\delta V\| \quad \text{in case of (1)},$$
$$\mu_i(A) = {\textstyle\int_A^*} |V\,\mathbf{D}g_i|^q \,\mathrm{d}\|V\| \quad \text{in case of (2) or (3) or (4)}$$

whenever $A \subset U$ and $i \in \mathscr{P}$ and notice that $\mu_i(B_i) = 0$ by 8.20 (1) (or alternately by 11.2 and 11.1 (4)). The assertion will be shown to hold at a point a satisfying for some $i \in \mathscr{P}$ that

$$f(a) = f_i(a), \quad V\,\mathbf{D}f(a) = V\,\mathbf{D}f_i(a),$$
$$\lim_{s \to 0+} s^{-1}(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(f_i(\cdot) - f_i(a) - \langle \cdot - a, V\,\mathbf{D}f_i(a)\rangle) = 0,$$
$$\boldsymbol{\Theta}^m(\|V\| \llcorner U \sim B_i, a) = 0, \quad \psi(\{a\}) = 0, \quad \boldsymbol{\Theta}^m(\mu_i, a) = 0.$$

These conditions are met by $\|V\|$ almost all a in view of the special case, [Fed69, 2.10.19 (4)]. Choosing $0 < r < \infty$ with $\mathbf{B}(a, 2r) \subset U$ and

$$\|V\|(\mathbf{U}(a,2s) \sim B_i) \le (1/2)\boldsymbol{\alpha}(m)s^m,$$
$$\psi\,\mathbf{U}(a,2r) \le \Gamma_{10.1}(2n)^{-1} \quad \text{in case of (2) or (3) or (4)}$$

for $0 < s \le r$, one infers from 9.2 and 10.1 (1) with U, M, G, f, Q, and r replaced by $\mathbf{U}(a, 2s)$, $2n$, $\varnothing$, g_i, 1, and s that

$$(\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(g_i) \le \Delta\mu_i(\mathbf{B}(a,2s))^{1/q} \quad \text{in case of (1) or (2) or (3),}$$
$$(\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(g_i) \le \Delta s^{1-m/q}\mu_i(\mathbf{B}(a,2s))^{1/q} \quad \text{in case of (4)}$$

for $0 < s \le r$, where $\Delta = \Gamma_{10.1}(2n)$ in case of (1) or (2), $\Delta = (m - q)^{-1}\Gamma_{10.1}(2n)$ in case of (3), and $\Delta = \Gamma_{10.1}(2n)^{1/(1/m-1/q)}\boldsymbol{\alpha}(m)^{1/m-1/q}$ in case of (4). Consequently,

$$\lim_{s \to 0+} s^{-1}(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(g_i) = 0$$

and the assertion follows.

From the assertion of the preceding paragraph one obtains

$$\lim_{s \to 0+} (s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(|\cdot - a|^{-1}h_a(\cdot)) = 0 \quad \text{for } \|V\| \text{ almost all } a;$$

in fact, if $\mathbf{B}(a,r) \subset U$ and $\kappa = \sup\{s^{-1}(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(h_a) : 0 < s \le r\}$, then one estimates

$$\begin{aligned}
&(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(|\cdot - a|^{-1}h_a) \\
&\quad \le s^{-m/\alpha}\textstyle\sum_{i=1}^\infty 2^i s^{-1}(\|V\| \llcorner (\mathbf{B}(a,2^{1-i}s) \sim \mathbf{B}(a,2^{-i}s)))_{(\alpha)}(h_a) \\
&\quad \le 2\kappa\textstyle\sum_{i=1}^\infty 2^{(1-i)m/\alpha} = 2\kappa/(1-2^{-m/\alpha})
\end{aligned}$$

in case of (1) or (3) and $(s^{-m}\|V\| \llcorner \mathbf{B}(a,s))_{(\alpha)}(|\cdot - a|^{-1}h_a) \le \kappa$ in case of (2) or (4) for $0 < s \le r$. This yields the conclusion in case of (1) or (3) and that $f|X$ is differentiable relative to X at a with

$$D(f|X)(a) = V\,\mathbf{D}f(a)|\operatorname{Tan}(X,a) \quad \text{for } \|V\| \text{ almost all } a$$

in case of (2) or (4) by [Fed69, 3.1.22]. To complete the proof, note that X is dense in $\operatorname{spt}\|V\|$ and hence if $p = m$, then

$$\operatorname{Tan}(X,a) = \operatorname{Tan}(\operatorname{spt}\|V\|, a) = \operatorname{Tan}^m(\|V\|, a) \quad \text{for } a \in U$$

by [Fed69, 3.1.21] and 11.3. □

11.5 Remark. The usage of h_a in the last paragraph of the proof is adapted from the proof of [Fed69, 4.5.9 (26) (II)].

12 Coarea formula

In this section rectifiability properties of the distributional boundary of almost all superlevel sets of real valued weakly differentiable functions are established, see 12.2. The result rests on the approximate differentiability of such functions, see 11.2. To underline this fact, it is derived as corollary to a general result for approximately differentiable functions, see 12.1.

12.1 Theorem. *Suppose $m, n \in \mathscr{P}$, $m \leq n$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{RV}_m(U)$, f is a $\|V\|$ measurable real valued function which is $(\|V\|, m)$ approximately differentiable at $\|V\|$ almost all points, F is a $\|V\|$ measurable $\operatorname{Hom}(\mathbf{R}^n, \mathbf{R})$ valued function with*

$$F(x) = (\|V\|, m) \operatorname{ap} Df(x) \circ \operatorname{Tan}^m(\|V\|, x)_\natural \quad \text{for } \|V\| \text{ almost all } x,$$

and $\int_{K \cap \{x : |f(x)| \leq s\}} |F| \, \mathrm{d}\|V\| < \infty$ whenever K is a compact subset of U and $0 \leq s < \infty$, and $T \in \mathscr{D}'(U \times \mathbf{R}, \mathbf{R}^n)$ and $S(y) : \mathscr{D}(U, \mathbf{R}^n) \to \mathbf{R}$ satisfy

$$T(\phi) = \int \langle \phi(x, f(x)), F(x) \rangle \, \mathrm{d}\|V\| x \quad \text{for } \phi \in \mathscr{D}(U \times \mathbf{R}, \mathbf{R}^n),$$
$$S(y)(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} \int_{\{x : y < f(x) \leq y + \varepsilon\}} \langle \theta, F \rangle \, \mathrm{d}\|V\| \in \mathbf{R} \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n)$$

whenever $y \in \mathbf{R}$, that is $y \in \operatorname{dmn} S$ if and only if the limit exists and belongs to $\mathbf{R}$ for $\theta \in \mathscr{D}(U, \mathbf{R}^n)$.

Then the following two statements hold.

(1) *If $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function, then*

$$T(\phi) = \int S(y)(\phi(\cdot, y)) \, \mathrm{d}\mathscr{L}^1 y, \quad \int g \, \mathrm{d}\|T\| = \int\int g(x, y) \, \mathrm{d}\|S(y)\| x \, \mathrm{d}\mathscr{L}^1 y.$$

(2) *There exists an $\mathscr{L}^1$ measurable function W with values in $\mathbf{RV}_{m-1}(U)$ endowed with the weak topology such that for $\mathscr{L}^1$ almost all y there holds*

$$\operatorname{Tan}^{m-1}(\|W(y)\|, x) = \operatorname{Tan}^m(\|V\|, x) \cap \ker F(x) \in \mathbf{G}(n, m-1),$$
$$\boldsymbol{\Theta}^{m-1}(\|W(y)\|, x) = \boldsymbol{\Theta}^m(\|V\|, x)$$

for $\|W(y)\|$ almost all x and

$$S(y)(\theta) = \int \left\langle \theta, |F|^{-1} F \right\rangle \mathrm{d}\|W(y)\| \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Proof. First, notice that 3.3 with $J = \mathbf{R}$ implies that T is representable by integration and

$$T(\phi) = \int \langle \phi(x, f(x)), F(x) \rangle \, \mathrm{d}\|V\| x, \quad \|T\|(g) = \int g(x, f(x)) |F(x)| \, \mathrm{d}\|V\| x$$

whenever $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function.

Next, the following assertion will be shown. *There exists an $\mathscr{L}^1$ measurable function W with values in $\mathbf{RV}_{m-1}(U)$ such that for $\mathscr{L}^1$ almost all y there holds*

$$\operatorname{Tan}^{m-1}(\|W(y)\|, x) = \operatorname{Tan}^m(\|V\|, x) \cap \ker F(x) \in \mathbf{G}(n, m-1),$$
$$\boldsymbol{\Theta}^{m-1}(\|W(y)\|, x) = \boldsymbol{\Theta}^m(\|V\|, x)$$

for $\|W(y)\|$ almost all x and

$$T(\phi) = \int\int \langle \phi(x, f(x)), |F(x)|^{-1} F(x) \rangle \,\mathrm{d}\|W(y)\| x \,\mathrm{d}\mathscr{L}^1 y,$$
$$\int g \,\mathrm{d}\|T\| = \int\int g(x,y) \,\mathrm{d}\|W(y)\| x \,\mathrm{d}\mathscr{L}^1 y$$

whenever $\phi \in \mathbf{L}_1(\|T\|, \mathbf{R}^n)$ and g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function. For this purpose choose a disjoint sequence of Borel subsets B_i of U, sequences M_i of m dimensional submanifolds of $\mathbf{R}^n$ of class 1 and functions $f_i : M_i \to \mathbf{R}$ of class 1 satisfying $\|V\|\big(U \sim \bigcup_{i=1}^\infty B_i\big) = 0$ and

$$B_i \subset M_i, \quad f_i(x) = f(x), \quad Df_i(x) = (\|V\|, m) \operatorname{ap} Df(x) = F(x) | \operatorname{Tan}^m(\|V\|, x)$$

whenever $i \in \mathscr{P}$ and $x \in B_i$, see 11.1, [Fed69, 2.8.18, 2.9.11, 3.2.17, 3.2.29] and Allard [All72, 3.5 (2)]. Let $B = \bigcup_{i=1}^\infty B_i$. If $y \in \mathbf{R}$ satisfies

$$\mathscr{H}^{m-1}(B \cap \{x : f(x) = y \text{ and } (\|V\|, m) \operatorname{ap} Df(x) = 0\}) = 0,$$
$$\int_{B \cap K \cap \{x : f(x) = y\}} \boldsymbol{\Theta}^m(\|V\|, x) \,\mathrm{d}\mathscr{H}^{m-1} x < \infty$$

whenever K is a compact subset of U, then define $W(y) \in \mathbf{RV}_{m-1}(U)$ by

$$W(y)(k) = \int_{B \cap \{x : f(x) = y\}} k(x, \ker (\|V\|, m) \operatorname{ap} Df(x)) \boldsymbol{\Theta}^{m-1}(\|V\|, x) \,\mathrm{d}\mathscr{H}^{m-1} x$$

for $k \in \mathscr{K}(U \times \mathbf{G}(n, m-1))$. Applying [Fed69, 3.2.22] with f replaced by f_i and summing over i, one infers that

$$\int_{A \cap \{x : s < f(x) < t\}} |F| \,\mathrm{d}\|V\| = \int_s^t \int_{A \cap B \cap \{x : f(x) = y\}} \boldsymbol{\Theta}^m(\|V\|, x) \,\mathrm{d}\mathscr{H}^{m-1} x \,\mathrm{d}\mathscr{L}^1 y$$

whenever A is $\|V\|$ measurable and $-\infty < s < t < \infty$, hence

$$\int g(x, f(x)) |F(x)| \,\mathrm{d}\|V\| x = \int\int g(x,y) \,\mathrm{d}\|W(y)\| x \,\mathrm{d}\mathscr{L}^1 y$$

whenever g is an $\overline{\mathbf{R}}$ valued $\|T\|$ integrable function. The remaining parts of the assertion now follow by considering appropriate choices of g and recalling 2.20.

Consequently, (1) is implied by 3.4 (2) with $J = \mathbf{R}$ and $Z = \mathbf{R}^n$. Noting

$$S(y)(\theta) = \lim_{\varepsilon \to 0+} \varepsilon^{-1} \int_y^{y+\varepsilon} \int \langle \theta, |F|^{-1} F \rangle \,\mathrm{d}\|W(v)\| \,\mathrm{d}\mathscr{L}^1 v \quad \text{for } \theta \in \mathscr{D}(U, \mathbf{R}^n)$$

whenever $y \in \operatorname{dmn} S$, (2) follows using 2.2, 2.21, and [Fed69, 2.8.17, 2.9.8]. □

12.2 Corollary. *Suppose m, n, U, and V are as in 6.1, $f \in \mathbf{T}(V)$, and $E(y) = \{x : f(x) > y\}$ for $y \in \mathbf{R}$.*

Then there exists an $\mathscr{L}^1$ measurable function W with values in $\mathbf{RV}_{m-1}(U)$ endowed with the weak topology such that for $\mathscr{L}^1$ almost all y there holds

$$\operatorname{Tan}^{m-1}(\|W(y)\|, x) = \operatorname{Tan}^m(\|V\|, x) \cap \ker V\, \mathbf{D} f(x) \in \mathbf{G}(n, m-1),$$
$$\boldsymbol{\Theta}^{m-1}(\|W(y)\|, x) = \boldsymbol{\Theta}^m(\|V\|, x)$$

for $\|W(y)\|$ almost all x and

$$V\, \partial E(y)(\theta) = \int \langle \theta, |V\, \mathbf{D} f|^{-1} V\, \mathbf{D} f \rangle \,\mathrm{d}\|W(y)\| \quad \textit{for } \theta \in \mathscr{D}(U, \mathbf{R}^n).$$

Proof. In view of 8.28 and 11.2, this is a consequence of 12.1. □

12.3 Remark. The formulation of 12.2 is modelled on similar results for sets of locally finite perimeter, see [Fed69, 4.5.6 (1), 4.5.9 (12)]. Observe however that the structural description of $V\,\partial E(y)$ given here for $\mathscr{L}^1$ almost all y does not extend to arbitrary $\|V\| + \|\delta V\|$ measurable sets E such that $V\,\partial E$ is representable by integration; in fact, $\|V\,\partial E\|$ does not even need to be the weight measure of some member of $\mathbf{RV}_{m-1}(U)$. (Using [Fed69, 2.10.28] to construct $V \in \mathbf{IV}_1(\mathbf{R}^2)$ such that $\|\delta V\|$ is a nonzero Radon measure with $\|\delta V\|(\{x\}) = 0$ for $x \in \mathbf{R}^2$ and $\|V\|(\operatorname{spt}\delta V) = 0$, one may take $E = \operatorname{spt}\delta V$.)

13 Oscillation estimates

In this section two situations are studied where the oscillation of a generalised weakly differentiable function may be controlled by its weak derivative assuming a suitable summability of the mean curvature of the underlying varifold. In general, such control is necessarily rather weak, see 13.1 and 13.2, but under special circumstances one may obtain Hölder continuity, see 13.3. The main ingredients are the analysis of the connectedness structure of a varifold, see 6.14, and the Sobolev Poincaré inequalities with several medians, see 10.9.

13.1 Theorem. *Suppose m, n, p, U, and V are as in 6.1, $p = m$, K is a compact subset of U, $0 < \varepsilon \leq \operatorname{dist}(K, \mathbf{R}^n \sim U)$, $\varepsilon < \infty$, and either*

(1) $1 = m = q$ and $\lambda = 2^{m+5}\boldsymbol{\alpha}(m)^{-1}\Gamma_{10.1}(2n)\sup\{\boldsymbol{\Theta}^1(\|V\|, a) : a \in K\}$, or

(2) $1 < m < q$ and $\lambda = \varepsilon$.

Then there exists a positive, finite number Γ with the following property.

If Y is a finite dimensional normed vectorspace, $f : \operatorname{spt}\|V\| \to Y$ is a continuous function, $f \in \mathbf{T}(V, Y)$, and $\kappa = \sup\{(\|V\| \llcorner \mathbf{U}(a, \varepsilon))_{(q)}(V\,\mathbf{D}f) : a \in K\}$, then

$$|f(x) - f(\chi)| \leq \lambda\kappa \quad \textit{whenever } x, \chi \in K \cap \operatorname{spt}\|V\| \textit{ and } |x - \chi| \leq \Gamma^{-1}.$$

Proof. Let ψ be as in 6.1. Abbreviate $A = \operatorname{spt}\|V\|$ and $\delta = 1 - m/q$. Abbreviate

$$\Delta_1 = 2^{m+3}\boldsymbol{\alpha}(m)^{-1}, \quad \Delta_2 = \Gamma_{10.1}(2n).$$

Notice that $\boldsymbol{\Theta}^m_*(\|V\|, a) \geq 1/2$ for $a \in A$ by 4.8 (4) and 7.6. If $m > 1$, then

$$\operatorname{dmn}\boldsymbol{\Theta}^1(\|V\|, \cdot) = U, \quad \Delta_3 = \sup\{\boldsymbol{\Theta}^1(\|V\|, a) : a \in K\} < \infty$$

by 4.8 (1) (2). If $m > 1$, then $\boldsymbol{\Theta}^{m-\delta}(\|V\|, a) = 0$ for $a \in A$ by 4.9. Moreover, if $a \in A$, $0 < r < \infty$, and $\mathbf{U}(a, r) \subset U$ then, by 6.14 (3), there exists $0 < s \leq r$ such that $A \cap \mathbf{U}(a, s)$ is a subset of the connected component of $A \cap \mathbf{U}(a, r/2)$ which contains a. Since $K \cap A$ is compact, one may therefore construct $j \in \mathscr{P}$ and a_i, s_i and r_i for $i = 1, \ldots, j$ such that $K \cap A \subset \bigcup_{i=1}^j \mathbf{U}(a_i, s_i)$ and

$$a_i \in K \cap A, \quad 0 < s_i \leq r_i \leq \varepsilon, \quad A \cap \mathbf{U}(a_i, s_i) \subset C_i, \quad \mathbf{U}(a_i, r_i) \subset U,$$
$$\psi(\mathbf{U}(a_i, r_i) \sim \{a_i\}) \leq \Delta_2^{-1}, \quad \|V\|\,\mathbf{U}(a_i, r_i) \geq (1/2)\boldsymbol{\alpha}(m)(r_i/2)^m,$$
$$r_i^{-1}\|V\|\,\mathbf{B}(a_i, r_i) \leq 4\Delta_3 \quad \text{if } m = 1,$$
$$\Delta_1\Gamma_{10.9\,(4)}(2n)^{m/\delta} r_i^{\delta - m}\|V\|\,\mathbf{U}(a_i, r_i) \leq \varepsilon \quad \text{if } m > 1$$

for $i = 1, \ldots, j$, where C_i is the connected component of $A \cap \mathbf{U}(a_i, r_i/2)$ which contains a_i. Since $K \cap A$ is compact, there exists a positive, finite number Γ with the following property. If $x, \chi \in K \cap A$ and $|x - \chi| \leq \Gamma^{-1}$ then $\{x, \chi\} \subset \mathbf{U}(a_i, s_i)$ for some i.

In order to verify that Γ has the asserted property, suppose that Y, f, κ, x, and χ are related to Γ as in the body of the theorem.

Assume $\kappa < \infty$. Since $|y| = \sup\{\alpha(y) : \alpha \in \operatorname{Hom}(Y, \mathbf{R}), \|\alpha\| \leq 1\}$ for $y \in Y$ by [Fed69, 2.4.12], one may also assume $Y = \mathbf{R}$ by 8.18. Choose i such that $\{x, \chi\} \subset C_i$. Choose $N \in \mathscr{P}$ satisfying

$$(1/2) N \boldsymbol{\alpha}(m) (r_i/2)^m \leq \|V\| \, \mathbf{U}(a_i, r_i) \leq (1/2)(N+1) \boldsymbol{\alpha}(m)(r_i/2)^m.$$

Applying 10.9 (2) (4) with U, M, Q, r, and X replaced by $\mathbf{U}(a_i, r_i)$, $2n$, 1, $r_i/2$, and $\{a_i\}$ yields a subset Υ of $\mathbf{R}$ such that $\operatorname{card} \Upsilon \leq N + 1$ and

$$f[\mathbf{U}(a_i, r_i/2)] \subset \bigcup \{\mathbf{B}(y, \kappa_i) : y \in \Upsilon\},$$

where $\kappa_i = \Delta_2 \kappa$ if $m = 1$ and $\kappa_i = \Gamma_{10.9\,(4)} (2n)^{m/\delta} r_i^\delta \kappa$ if $m > 1$. Defining I to be the connected component of $\bigcup \{\mathbf{B}(y, \kappa_i) : y \in \Upsilon\}$ which contains $f(a)$, one infers

$$|f(x) - f(\chi)| \leq \operatorname{diam} I = \mathscr{L}^1(I) \leq 2(N+1)\kappa_i \leq \Delta_1 \kappa_i r_i^{-m} \|V\| \, \mathbf{U}(a_i, r_i) \leq \lambda \kappa$$

since $\{f(x), f(\chi)\} \subset f[C_i] \subset I$ and I is an interval. $\square$

13.2 Remark. Considering varifolds corresponding to two parallel planes, it is clear that Γ may not be chosen independently of V. Also the continuity hypothesis on f is essential as may be seen considering V associated to two transversely intersecting lines.

13.3 Theorem. *Suppose $1 \leq M < \infty$.*

Then there exists a positive, finite number Γ with the following property.

If m, n, p, U, V, and ψ are as in 6.1, $p = m$, $n \leq M$, $0 < r < \infty$, $A = \{x : \mathbf{U}(x, r) \subset U\}$,

$$\psi \, \mathbf{U}(a, r) \leq \Gamma^{-1}, \qquad \|V\| \, \mathbf{U}(a, s) \leq (2 - M^{-1}) \boldsymbol{\alpha}(m) s^m \quad \text{for } 0 < s \leq r$$

whenever $a \in A \cap \operatorname{spt} \|V\|$, Y is a finite dimensional normed vectorspace, $f \in \mathbf{T}(V, Y)$, $C = \{(x, \mathbf{B}(x, s)) : \mathbf{B}(x, s) \subset U\}$, X is the set of $a \in A \cap \operatorname{spt} \|V\|$ such that f is $(\|V\|, C)$ approximately continuous at a, $m < q \leq \infty$, $\delta = 1 - m/q$, and $\kappa = \sup\{(\|V\| \llcorner \mathbf{U}(a, r))_{(q)}(V \, \mathbf{D} f) : a \in A \cap \operatorname{spt} \|V\|\}$, then

$$|f(x) - f(\chi)| \leq \lambda |x - \chi|^\delta \kappa \quad \text{whenever } x, \chi \in X \text{ and } |x - \chi| \leq r/\Gamma,$$

where $\lambda = \Gamma$ if $m = 1$ and $\lambda = \Gamma^{1/\delta}$ if $m > 1$.

Proof. Define

$$\Delta_1 = (1 - 1/(4M - 1))^{1/M}, \quad \Delta_2 = \sup\{1, \Gamma_{10.9\,(4)}(2M)\}^M,$$
$$\Gamma = 4(1 - \Delta_1)^{-1} \sup\{4 \Gamma_{10.1}(2M), \Delta_2\}$$

and notice that $\Gamma \geq 4(1 - \Delta_1)^{-1}$.

In order to verify that Γ has the asserted property, suppose that m, n, p, U, V, ψ, M, r, A, Y, f, C, X, q, δ, κ, x, χ, and λ are related to Γ as in the body of the theorem.

In view of 8.18 and 10.3 (2) (4) one may assume $Y = \mathbf{R}$. Define $s = 2|x-\chi|$ and $t = (1-\Delta_1)^{-1}s$ and notice that $0 < s < t \le r$ and

$$\begin{gathered}\|V\|\,\mathbf{U}(x,t) \le (2-M^{-1})\boldsymbol{\alpha}(m)(1-\Delta_1)^{-m}s^m \\ \le \big(2-(2M)^{-1}\big)\boldsymbol{\alpha}(m)\Delta_1^m(1-\Delta_1)^{-m}s^m = \big(2-(2M)^{-1}\big)\boldsymbol{\alpha}(m)(t-s)^m, \\ \psi\,\mathbf{U}(x,t) \le \Gamma^{-1} \le \Gamma_{10.1}(2M)^{-1}.\end{gathered}$$

Therefore applying 10.9 (2) (4) with U, M, N, Q, r, and X replaced by $\mathbf{U}(x,t)$, $2M$, 1, 1, $t-s$, and $\varnothing$ yields a subset Υ of $\mathbf{R}$ with $\operatorname{card}\Upsilon = 1$ such that $(\|V\| \llcorner \mathbf{U}(x,s))_{(\infty)}(f_\Upsilon)$ is bounded by, using Hölder's inequality,

$$\begin{gathered}\Gamma_{10.1}(2M)(\|V\| \llcorner \mathbf{U}(x,t))_{(1)}(V\,\mathbf{D}f) \le 4\Gamma_{10.1}(2M)t^\delta\kappa \quad \text{if } m=1, \text{ and by} \\ \Gamma_{10.9\,(4)}(2M)^{m/\delta}t^\delta(\|V\| \llcorner \mathbf{U}(x,t))_{(q)}(V\,\mathbf{D}f) \le \Delta_2^{1/\delta}t^\delta\kappa \quad \text{if } m>1.\end{gathered}$$

Noting $|f(x)-f(\chi)| \le 2(\|V\| \llcorner \mathbf{U}(x,s))_{(\infty)}(f_\Upsilon)$ and $t = 2(1-\Delta_1)^{-1}|x-\chi|$, the conclusion follows. □

14 Geodesic distance

Reconsidering the support of the weight measure of a varifold whose mean curvature satisfies a suitable summability of the mean curvature, the oscillation estimate 13.1 is used to establish in this section that its connected components agree with the components induced by its geodesic distance, see 14.2.

14.1 Example. Whenever $1 \le p < m < n$, U is an open subset of $\mathbf{R}^n$ and X is an open subset of U, there exists V related to m, n, U, and p as in 6.1 such that $\operatorname{spt}\|V\|$ equals the closure of X relative to U as is readily seen taking the into account the behaviour of ψ and V under homotheties; compare [Men09, 1.2].

14.2 Theorem. *Suppose m, n, U, V, and p are as in 6.1, $p = m$, and C is a connected component of $\operatorname{spt}\|V\|$, and $a, x \in C$.*

Then there exist $-\infty < b \le y < \infty$ and a Lipschitzian function $g : \{\upsilon : b \le \upsilon \le y\} \to \operatorname{spt}\|V\|$ such that $g(b) = a$ and $g(y) = x$.

Proof. In view of 6.14 (4) one may assume $C = \operatorname{spt}\|V\|$. Whenever $c \in C$ denote by $X(c)$ the set of $\chi \in C$ such that there exist $-\infty < b \le y < \infty$ and a Lipschitzian function $g : \{\upsilon : b \le \upsilon \le y\} \to C$ such that $g(b) = c$ and $g(y) = \chi$. Observe that it is sufficient to prove that c belongs to the interior of $X(c)$ relative to C whenever $c \in C$.

For this purpose suppose $c \in C$, choose $0 < r < \infty$ with $\mathbf{U}(c, 2r) \subset U$ and let $K = \mathbf{B}(c,r)$. If $m = 1$, define λ as in 13.1 (1) and notice that $\lambda < \infty$ by 4.8 (1). Define $q = 1$ if $m = 1$ and $q = 2m$ if $m > 1$. Choose $0 < \varepsilon \le r$ such that

$$\lambda \sup\{\|V\|(\mathbf{U}(\chi,\varepsilon))^{1/q} : \chi \in K\} \le r,$$

where $\lambda = \varepsilon$ in accordance with 13.1 (2) if $m > 1$, and define

$$s = \inf\{\Gamma_{13.1}(m,n,m,U,V,K,\varepsilon,q)^{-1}, r\}.$$

Next, define functions $f_\delta : C \to \overline{\mathbf{R}}$ by letting $f_\delta(\chi)$ for $\chi \in C$ and $\delta > 0$ equal the infimum of sums

$$\sum_{i=1}^{j} |x_i - x_{i-1}|$$

corresponding to all finite sequences $x_0, x_1, \ldots, x_j$ in C with $x_0 = c$, $x_j = \chi$ and $|x_i - x_{i-1}| \leq \delta$ for $i = 1, \ldots, j$ and $j \in \mathscr{P}$. Notice that

$$f_\delta(\chi) \leq f_\delta(\zeta) + |\zeta - \chi| \quad \text{whenever } \zeta, \chi \in C \text{ and } |\zeta - \chi| \leq \delta.$$

Since C is connected and $f_\delta(c) = 0$, it follows that f_δ is a locally Lipschitzian real valued function satisfying

$$\operatorname{Lip}(f_\delta | A) \leq 1 \quad \text{whenever } A \subset U \text{ and } \operatorname{diam} A \leq \delta,$$

in particular $f_\delta \in \mathbf{T}(V)$ and $\|V\|_{(\infty)}(V\,\mathbf{D} f_\delta) \leq 1$ by 8.7. One infers

$$f_\delta(\chi) \leq r \quad \text{whenever } \chi \in C \cap \mathbf{B}(c,s) \text{ and } \delta > 0$$

from 13.1. It follows that $C \cap \mathbf{B}(c,s) \subset X(c)$; in fact, if $\chi \in C \cap \mathbf{B}(c,s)$ one readily constructs $g_\delta : \{u : 0 \leq \upsilon \leq r\} \to \mathbf{R}^n$ satisfying

$$\begin{gathered} g_\delta(0) = c, \quad g_\delta(r) = \chi, \quad \operatorname{Lip} g_\delta \leq 1 + \delta, \\ \operatorname{dist}(g_\delta(\upsilon), C) \leq \delta \quad \text{whenever } 0 \leq \upsilon \leq r, \end{gathered}$$

hence, noting that $\operatorname{im} g_\delta \subset \mathbf{B}(c, (1+\delta)r)$, the existence of $g : \{\upsilon : 0 \leq \upsilon \leq r\} \to C$ satisfying $g(0) = c$, $g(r) = \chi$, and $\operatorname{Lip} g \leq 1$ now is a consequence of [Fed69, 2.10.21]. □

14.3 Remark. The deduction of 14.2 from 13.1 is adapted from Cheeger [Che99, §17] who attributes the argument to Semmes, see also David and Semmes [DS93].

14.4 Remark. In view of 14.1 it is not hard to construct examples showing that the hypothesis “$p = m$” in 13.1 may not be replaced by “$p \geq q$” for any $1 \leq q < m$ if $m < n$. Yet, for indecomposable V, the study of possible extensions of 14.2 as well as related questions seems to be most natural under the hypothesis “$p = m - 1$”, see Topping [Top08].

15 Curvature varifolds

In this section Hutchinson’s concept of curvature varifold is rephrased in terms of the concept of weakly differentiable function proposed in the present paper, see 15.6. To indicate possible benefits of this perspective, a result on the differentiability of the tangent plane map is included, see 15.9.

15.1. Suppose $n \in \mathscr{P}$ and $Y = \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n) \cap \{\sigma : \sigma = \sigma^*\}$. Then $T : \operatorname{Hom}(\mathbf{R}^n, Y) \to \mathbf{R}^n$ denotes the linear map which is given by the composition (see [Fed69, 1.1.1, 1.1.2, 1.1.4, 1.7.9])

$$\begin{aligned} &\operatorname{Hom}(\mathbf{R}^n, Y) \xrightarrow{\subset} \operatorname{Hom}(\mathbf{R}^n, \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}^n)) \\ &\qquad \simeq \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}) \otimes \operatorname{Hom}(\mathbf{R}^n, \mathbf{R}) \otimes \mathbf{R}^n \xrightarrow{L \otimes \mathbf{1}_{\mathbf{R}^n}} \mathbf{R} \otimes \mathbf{R}^n \simeq \mathbf{R}^n, \end{aligned}$$

where the linear map L is induced by the inner product on $\operatorname{Hom}(\mathbf{R}^n,\mathbf{R})$, hence $T(g) = \sum_{i=1}^n \langle u_i, g(u_i)\rangle$ whenever $g \in \operatorname{Hom}(\mathbf{R}^n, Y)$ and $u_1,\ldots,u_n$ form an orthonormal basis of $\mathbf{R}^n$.

15.2. Suppose n, Y and T are as in 15.1, $n \geq m \in \mathscr{P}$, M is an m dimensional submanifold of $\mathbf{R}^n$ of class 2, and $\tau : M \to Y$ is defined by $\tau(x) = \operatorname{Tan}(M,x)_\natural$ for $x \in M$. Then one computes

$$\mathbf{h}(M,x) = T(D\tau(x)\circ\tau(x)) \quad \text{whenever } x \in M;$$

in fact, differentiating the equation $\tau(x)\circ\tau(x) = \tau(x)$ for $x \in M$, one obtains

$$\begin{gathered}\tau(x)\circ\langle u, D\tau(x)\rangle\circ\tau(x) = 0 \quad \text{for } u \in \operatorname{Tan}(M,x),\\ T(D\tau(x)\circ\tau(x)) \in \operatorname{Nor}(M,x)\end{gathered}$$

for $x \in M$ and, denoting by $u_1,\ldots,u_n$ an orthonormal base of $\mathbf{R}^n$, one computes

$$\begin{aligned}\tau(x)\bullet(Dg(x)\circ\tau(x)) &= \textstyle\sum_{i=1}^n \langle\tau(x)(u_i), Dg(x)\rangle\bullet\tau(x)(u_i)\\ &= -g(x)\bullet\textstyle\sum_{i=1}^n \langle\tau(x)(u_i), D\tau(x)\rangle\,(u_i) = -g(x)\bullet T(D\tau(x)\circ\tau(x))\end{aligned}$$

whenever $g : M \to \mathbf{R}^n$ is of class 1 and $g(x) \in \operatorname{Nor}(M,x)$ for $x \in M$.

15.3 Lemma. *Suppose n and Y are as in 15.1 and $n > m \in \mathscr{P}$.*
Then the vectorspace Y is generated by $\{P_\natural : P \in \mathbf{G}(n,m)\}$.

Proof. If $u_1,\ldots,u_{m+1}$ are orthonormal vectors in $\mathbf{R}^n$, $u_{m+1} \in L \in \mathbf{G}(n,1)$, and

$$P_i = \operatorname{span}\big\{u_j : j \in \{1,\ldots,m+1\} \text{ and } j \neq i\big\} \in \mathbf{G}(n,m)$$

for $i = 1,\ldots,m+1$, then $L_\natural + (P_{m+1})_\natural = \frac{1}{m}\sum_{i=1}^{m+1}(P_i)_\natural$. Hence one may assume $m = 1$ in which case [Fed69, 1.7.3] yields the assertion. ☐

15.4 Definition. Suppose n, Y and T are as in 15.1, $n \geq m \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, $X = U \cap \{x : \operatorname{Tan}^m(\|V\|,x) \in \mathbf{G}(n,m)\}$, and $\tau : X \to Y$ satisfies

$$\tau(x) = \operatorname{Tan}^m(\|V\|,x)_\natural \quad \text{whenever } x \in X.$$

Then the varifold V is called a *curvature varifold* if and only if there exists $F \in \mathbf{L}_1^{\mathrm{loc}}\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)\big)$ satisfying

$$\textstyle\int \langle(\tau(x)(u), F(x)(u)), D\phi(x,\tau(x))\rangle + \phi(x,\tau(x))T(F(x))\bullet u \,\mathrm{d}\|V\|x = 0$$

whenever $u \in \mathbf{R}^n$ and $\phi \in \mathscr{D}(U\times Y, \mathbf{R})$.

15.5 Remark. Using approximation and the fact that τ is a bounded function, one may verify that the same definition results if the equation is required for every $\phi : U \times Y \to \mathbf{R}$ of class 1 such that

$$\operatorname{Clos}(U \cap \{(x,\sigma) : \phi(x,\sigma) \neq 0 \text{ for some } \sigma \in Y\})$$

is compact. Extending T accordingly, one may also replace Y by $\operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$ in the definition; in fact, if $F \in \mathbf{L}_1^{\mathrm{loc}}\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n))\big)$ satisfies the condition of the modified definition then $\operatorname{im} F(x) \subset Y$ for $\|V\|$ almost all x as may be seen by enlarging the class of ϕ as before and considering $\phi(x,\sigma) = \zeta(x)\alpha(\sigma)$ for $x \in U$, $\sigma \in \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n)$, $\zeta \in \mathscr{D}(U,\mathbf{R})$ and $\alpha : \operatorname{Hom}(\mathbf{R}^n,\mathbf{R}^n) \to \mathbf{R}$ is linear with $Y \subset \ker\alpha$, compare Hutchinson [Hut86, 5.2.4 (i)].

Consequently, the definition 15.4 is equivalent to Hutchinson's definition in [Hut86, 5.2.1]. The present formulation is motivated by 15.3.

15.6 Theorem. *Suppose n, Y and T are as in 15.1, $n \geq m \in \mathscr{P}$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$, $X = U \cap \{x : \operatorname{Tan}^m(\|V\|, x) \in \mathbf{G}(n,m)\}$, and $\tau : X \to Y$ satisfies*

$$\tau(x) = \operatorname{Tan}^m(\|V\|, x)_\natural \quad \textit{whenever } x \in X.$$

Then V is a curvature varifold if and only if $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$ and $\tau \in \mathbf{T}(V, Y)$. In this case, there holds

$$\mathbf{h}(V,x) = T(V\,\mathbf{D}\tau(x)) \quad \textit{for } \|V\| \textit{ almost all } x,$$
$$(\delta V)_x(\phi(x,\tau(x))u) = \textstyle\int \langle (\tau(x)(u), V\,\mathbf{D}\tau(x)(u)), D\phi(x,\tau(x))\rangle \,\mathrm{d}\|V\|x$$

whenever $\phi : U \times Y \to \mathbf{R}$ is of class 1 such that

$$\operatorname{Clos}(U \cap \{(x,\sigma) : \phi(x,\sigma) \neq 0 \textit{ for some } \sigma \in Y\})$$

is compact and $u \in \mathbf{R}^n$.

Proof. Suppose V is a curvature varifold and F is as in 15.4. If $\zeta \in \mathscr{D}(U, \mathbf{R})$, $\gamma \in \mathscr{E}(Y, \mathbf{R})$ and $u \in \mathbf{R}^n$, one may take $\phi(x,\sigma) = \zeta(x)\gamma(\sigma)$ in 15.4, 15.5 to obtain

$$\begin{aligned}\textstyle\int \gamma(\tau(x))\,\langle (\tau(x)(u), D\zeta(x)\rangle + \zeta(x)\,\langle F(x)(u), D\gamma(\tau(x))\rangle \,\mathrm{d}\|V\|x \\ = -\textstyle\int \zeta(x)\gamma(\tau(x))T(F(x)) \bullet u \,\mathrm{d}\|V\|x.\end{aligned}$$

In particular, taking $\gamma = 1$, one infers that $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$ with

$$T(F(x)) = \mathbf{h}(V,x) \quad \|V\| \text{ almost all } x.$$

It follows that

$$(\delta V)((\gamma \circ \tau)\zeta \cdot u) = -\textstyle\int \zeta(x)\gamma(\tau(x))T(F(x)) \bullet u \,\mathrm{d}\|V\|x$$

for $\zeta \in \mathscr{D}(U,\mathbf{R})$, $\gamma \in \mathscr{E}(Y,\mathbf{R})$ and $u \in \mathbf{R}^n$. Together with the first equation this implies that $\tau \in \mathbf{T}(V,Y)$ with $V\,\mathbf{D}\tau(x) = F(x)$ for $\|V\|$ almost all x by 8.4.

To prove the converse, suppose $\|\delta V\|$ is a Radon measure absolutely continuous with respect to $\|V\|$ and $\tau \in \mathbf{T}(V,Y)$. Since τ is a bounded function, $V\,\mathbf{D}\tau \in \mathbf{L}_1^{\mathrm{loc}}\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)\big)$ by 8.3 (1). In order to prove the equation for the generalised mean curvature vector of V, in view of 11.2 and [Men13, 4.8], it is sufficient to prove that

$$\mathbf{h}(M,x) = T\big((\|V\|, m)\operatorname{ap} D\tau(x) \circ \tau(x)\big) \quad \text{for } \|V\| \text{ almost all } x \in U \cap M$$

whenever M is an m dimensional submanifold of $\mathbf{R}^n$ of class 2. The latter equation however is evident from 11.1 (4) and 15.2 since

$$\operatorname{Tan}(M,x) = \operatorname{Tan}^m(\|V\|, x) \quad \text{for } \|V\| \text{ almost all } x \in U \cap M$$

by [Fed69, 2.8.18, 2.9.11, 3.2.17] and Allard [All72, 3.5 (2)].

Next, define $f : X \to \mathbf{R}^n \times Y$ by

$$f(x) = (x, \tau(x)) \quad \text{for } x \in \operatorname{dmn} \tau$$

and notice that $f \in \mathbf{T}(V, \mathbf{R}^n \times Y)$ with

$$V\,\mathbf{D}f(x)(u) = (\tau(x)(u), V\,\mathbf{D}\tau(x)(u)) \quad \text{for } u \in \mathbf{R}^n$$

for $\|V\|$ almost all x by 8.20 (2) and 8.7. The proof may be concluded by establishing the last equation of the postscript. Using approximation and the fact that τ is a bounded function yields that is sufficient to consider $\phi \in \mathscr{D}(U \times Y, \mathbf{R})$. Choose $\zeta \in \mathscr{D}(U, \mathbf{R})$ with

$$\operatorname{Clos}(U \cap \{x : \phi(x,\sigma) \neq 0 \text{ for some } \sigma \in Y\}) \subset \operatorname{Int}\{x : \zeta(x) = 1\}$$

and denote by γ the extension of ϕ to $\mathbf{R}^n \times Y$ by 0. Now, applying 8.4 with Y replaced by $\mathbf{R}^n \times Y$ yields the equation in question. □

15.7 Remark. The first paragraph of the proof is essentially contained in Hutchinson in [Hut86, 5.2.2, 5.2.3] and is included here for completeness.

15.8 Remark. If $m = n$ then $\mathbf{\Theta}^m(\|V\|, \cdot)$ is a locally constant, integer valued function whose domain is U; in fact, τ is constant, hence V is stationary by 15.6 and the asserted structure follows from Allard [All72, 4.6 (3)].

15.9 Corollary. *Suppose $m, n \in \mathscr{P}$, $1 < m < n$, $\beta = m/(m-1)$, U is an open subset of $\mathbf{R}^n$, $V \in \mathbf{IV}_m(U)$ is a curvature varifold, $X = U \cap \{x : \operatorname{Tan}^m(\|V\|, x) \in \mathbf{G}(n,m)\}$, and $\tau : X \to Y$ satisfies*

$$\tau(x) = \operatorname{Tan}^m(\|V\|, x)_\natural \quad \textit{whenever } x \in X.$$

Then $\|V\|$ almost all a satisfy

$$\lim_{r \to 0+} r^{-m} \textstyle\int_{\mathbf{B}(a,r)} (|\tau(x) - \tau(a) - \langle x - a, V\,\mathbf{D}\tau(a)\rangle| / |x - a|)^\beta \,\mathrm{d}\|V\|x = 0.$$

Proof. This is an immediate consequence of 15.6 and 11.4 (1). □

15.10 Remark. Using 11.4 (2), one may formulate a corresponding result for the case $m = 1$.

15.11 Remark. Notice that one may deduce decay results for height quantities from this result by use of [Men10, 4.11 (1)].

15.12 Remark. If $1 < p < m$ and $V\,\mathbf{D}\tau \in \mathbf{L}_p^{\mathrm{loc}}\big(\|V\|, \operatorname{Hom}(\mathbf{R}^n, Y)\big)$, see 15.1, one may investigate whether the conclusion still holds with β replaced by $mp/(m-p)$.

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