

# Kinetic microtearing modes and reconnecting modes in strongly magnetised slab plasmas

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The problem of the linear microtearing mode in a slab magnetised plasma, and its connection to kinetic reconnecting modes, is addressed. Electrons are described using a novel hybrid fluid-kinetic model that captures electron heating, ions are gyrokinetic. Magnetic reconnection can occur as a result of either electron conductivity and inertia, depending on which one predominates. We eschew the use of an energy dependent collision frequency in the collisional operator model, unlike previous works. A model of the electron conductivity that matches the weakly collisional regime to the exact Landau result at zero collisionality and gives the correct electron isothermal response far from the reconnection region is presented. We identify in the breaking of the constant- $A_{\parallel}$  approximation the necessary condition for microtearing instability in the collisional regime. Connections with the theory of collisional non-isothermal (or semicollisional) and collisionless tearing-parity electron temperature gradient driven (ETG) modes are elucidated.

## I. INTRODUCTION

The presence of microtearing modes in fusion plasmas was predicted by Hazeltine, Dobrott and Wang [1] in 1975<sup>1</sup>. These modes are driven unstable by the electron temperature gradient, and rotate in the electron direction with a real frequency of the order of the electron drift frequency. They can drive magnetic reconnection even for magnetic equilibria that are tearing-mode stable.

Microtearing modes have been found unstable in most magnetic confinement systems [3–12]. In some tokamak experiments [13], nonlinear microtearing physics seems to be the key to explaining the favourable scaling of energy confinement with collisionality. There is also growing evidence [8, 14–18] that the simple electrostatic picture of tokamak gyrokinetic turbulence could be inappropriate in many regimes that are operationally relevant, and microtearing certainly plays a role in this. Microtearing activity has also been detected in a reversed-field-pinch configuration [19].

In gyrokinetic simulations, microtearing modes almost inevitably manifest themselves when electromagnetic effects are considered. Their phenomenology is rather complicated, and has been summarised in Ref. [20]. Amongst recent authors involved in microtearing research, some are re-proposing the idea [21] that the mode is responsible for anomalous electron transport in magnetic fusion devices, which is believed to be predominantly electromagnetic [14–16]. To understand better the physics of the mode in conditions relevant to a fusion reactor, microtearing studies have been extended to include toroidal effects [17, 20, 22], finite  $\beta$  (the ratio of plasma kinetic to magnetic pressure) [9, 15, 20], realistic mass ratios [9], and more or less sophisticated model collision operators [6, 14]. In the literature, we can also find claims of quantitative agreement between predicted and measured levels of transport (see Ref. [23] for example). Surprisingly, despite this renaissance in the study of the mode, a slab theory which retains electrostatic perturbations, and small but finite collisionality is still missing. In this work we present such a theory. We avoid imposing constant magnetic perturbations across the reconnection region, a simplification known as “constant- $A_{\parallel}$  approximation” [24] used in most analytical previous works [1, 2, 25–30]. Full ion Larmor orbit effects [31] are retained, even for non-constant magnetic perturbations [32–34]. Finite  $\hat{\beta}_T = 0.5\beta_e L_s^2/L_T^2$  effects are also retained [11, 35]. Here  $\beta_e$  is the ratio of electron to magnetic pressure, whereas  $L_s$  and  $L_T$  are the characteristic magnetic shear and electron temperature gradient scales, respectively. Indeed, as we will prove, a finite  $\hat{\beta}_T$  theory is required to describe unstable microtearing modes. Analytical progress can be made if we combine the approaches introduced in Ref. [34] and in Ref. [36] for ions and electrons, respectively. The first approach is based on the separation between electron and ion scales, and crucially on some results borrowed from the theory of generalized functions.

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[1] For an historical introduction see Ref. [2].

The second is based on a spectral representation of the electron distribution function in Hermite series [36], that leads to a solution of the electron kinetic equation expressed in terms of a continued fraction, reminiscent of earlier theories [27, 28]. We do not make use of an energy dependent collision frequency in the collision operator model. This is crucial to generating a time-dependent thermal force in the parallel momentum equation of the electrons that produces the instability in the highly collisional limit, according to previous studies [28]. However, it is unnecessary in our kinetic theory. In fact, we introduce a new closure for the electron kinetic problem that allows us to study the low and high collisionality limits. We show how our analytical solution of the electron kinetic problem connects to the exact analytical result at zero collisionality obtained using Landau contour integration. The new electron solution reproduces the correct isothermal electron response far from the reconnecting region. We find a dispersion relation that relates the microtearing mode to the collisional, non-isothermal (semicollisional), drift-tearing mode at high collisionality. We identify in the breaking of the constant- $A_{\parallel}$  approximation [24, 37] the necessary condition for instability in the collisional regime. We also prove that the coupling of the drift-tearing mode branch to the kinetic Alfvén wave provides a mechanism to avoid the cancellation of the microtearing mode drive that would happen when the collision frequency is energy independent. This new mechanism that drives unstable the microtearing mode is not excluding other mechanisms already present in the literature such as the energy dependence of the collision frequency.

In the weakly collisional limit, we carry out a numerical analysis with the gyrokinetic code **GS2**. The only electron temperature gradient driven weakly collisional reconnecting mode that we find unstable is the tearing-parity, strongly driven ETG, which happens to be mostly electrostatic. In the semicollisional regime, on the other hand, we identify the microtearing mode and give the explicit analytic expression for its growth rate and real frequency in the cases of large and finite electron temperature gradients.

The paper is organised as follows: In Section II we present the nonlinear and linear model equations. In Section II C, we solve the electron kinetic equation, introduce the new electron conductivity and show how it relates to previous theories in the semicollisional and truly collisionless ( $\nu_{ei} \equiv 0$ ) limits. In Section III we derive a new dispersion relation for drift-kinetic reconnecting modes with gyrokinetic ions. In Section IV and V we study the collisional and weakly collisional limits of drift-kinetic reconnecting modes, and benchmark our results against hybrid fluid-kinetic and gyrokinetic codes. In Section VI we report on our new results on the theory of kinetic microtearing modes. In Section VII a numerical investigation of the existence of collisionless microtearing modes is carried out. Conclusions are presented in Section VIII.

## II. MODEL EQUATIONS

### A. Nonlinear model

For our analysis, a generalisation of the hybrid fluid-kinetic model derived in Ref. [36] is used. Electron temperature gradients are introduced with a maximal ordering that allows us to neglect the electron and ion drift frequencies  $\omega_* \propto L_{n_0}^{-1}$ , where  $L_{n_0}$  is the characteristic background density gradient length scale. We consider

$$\omega \sim \nu_{ei} \sim k_{\parallel} v_{the} \sim k_{\parallel} v_A \sim \omega_T, \quad (1)$$

where  $\omega_T = \eta_e \omega_* \equiv 0.5 k_y v_{the} \rho_e / L_T$ , and  $L_T^{-1} = -T_{0e}^{-1} \partial_x T_{0e}$  defines the characteristic temperature gradient length scale,  $v_{the}$  the electron thermal speed,  $v_A$  the Alfvén speed,  $\omega$  the mode frequency,  $k_{\parallel}$  the wave number parallel to the magnetic field, and  $\nu_{ei}$  the electron-ion collision frequency. In this way we have

$$\omega \frac{h_e}{F_{0e}} \sim k_y \rho_e \frac{v_{the}}{L_{Te}} \frac{e\varphi}{T_{0e}} \sim \frac{v_{the}}{L_{Te}} \varepsilon, \quad (2)$$

where  $\rho_e$  is the electron Larmor radius,  $k_y$  the mode wave number perpendicular to the magnetic field, and  $\varepsilon = k_{\parallel} / k_{\perp} \sim h_e / F_{0e}$  is the small expansion parameter of gyrokinetic theory, with  $k_{\perp}^2 = k_y^2 + k^2$ , where  $k$  is the radial wave vector, and  $h_e$  the non-adiabatic part of the electron distribution function. The ordering in Eq. (2) is derived from the original ordering of Ref. [36], where the electrostatic potential,  $\varphi$ , is ordered as  $e\varphi / T_{0e} \sim \varepsilon / \sqrt{\beta_e}$ , with  $\beta_e = 8\pi n_{0e} T_{0e} / B_0^2$ , and  $\beta_e \sim (m_e / m_i)$ . The ordering in  $\beta_e$  is necessary to include electron inertia and accommodate the kinetic Alfvén wave [36]. From Eq. (1), it follows that  $k_{\parallel} L_T \sim 1$ , therefore we set  $L_{n_{0e}} / L_T \equiv \eta_e \sim \beta_e^{-1/2} \sim (m_i / m_e)^{1/2} \gg 1$ . This is simply a flat density limit, with  $\omega_* \rightarrow 0$  but  $\omega_T = \eta_e \omega_* \sim \mathcal{O}(1)$ . Since electrons are drift-kinetic,  $k_{\perp} \rho_e \sim \sqrt{\beta_e} \sim \sqrt{m_e / m_i} \ll 1$ , the electron diamagnetic drift frequency can only be important, compared to the Alfvén frequency,  $\omega_A = v_A / L_s \equiv \tau_A^{-1}$ , at very low shear; in fact

$$\frac{\omega_T}{\omega_A} = \sqrt{\frac{\beta_e}{2}} \frac{L_s}{L_T} k_y \rho_s,$$

where  $L_s$  is the magnetic shear length,  $\rho_s = \sqrt{Z/(2\tau)}\rho_i$ ,  $\tau = T_{0i}/T_{0e}$ , and  $Z$  is the charge number. The parameter

$$\hat{\beta}_T \equiv (\beta_e/2)L_s^2/L_T^2$$

is the familiar parameter of semicollisional theory [34, 35]. In the following, for ease of comparison of our results with previous results, we will sometimes leave explicit the combination  $\rho_e/L_T$ ; however that should always be regarded as  $\rho_e/L_T \equiv d_e/L_s\sqrt{\beta_e}L_s/L_T$ , with  $\sqrt{\beta_e}L_s/L_T \sim \mathcal{O}(1)$ , where  $d_e = c/\omega_{pe} \propto \sqrt{m_e}$  is the electron inertial length with  $\omega_{pe} = (4\pi n_{0e}e^2/m_e)^{1/2}$  the electron plasma frequency.

Using these orderings, the equations that we obtain can be listed below; their detailed derivation can be found in Appendix A. They are: the electron continuity equation,

$$\frac{d}{dt} \frac{Z}{\tau} (1 - \hat{\Gamma}_0) \frac{e\varphi}{T_{0e}} = \hat{\mathbf{b}} \cdot \nabla \frac{e}{m_e c} d_e^2 \nabla_\perp^2 A_\parallel, \quad (3)$$

with  $\Gamma_0 = I_0(k^2 \rho_i^2/2) \exp[-k^2 \rho_i^2/2]$ , where  $I_0$  is the modified Bessel function (the “hat” is symbolic for the inverse Fourier transform),  $\hat{\mathbf{b}} \cdot \nabla = -\partial_z + B_0^{-1} \{A_\parallel, \cdot\}$ ,  $\{\cdot, \cdot\}$  is the Poisson bracket, and  $A_\parallel$  the parallel component of the magnetic potential, the generalised Ohm’s law,

$$\begin{aligned} \frac{d}{dt} (A_\parallel - d_e^2 \nabla_\perp^2 A_\parallel) &= -c \frac{\partial \varphi}{\partial z} + \frac{T_{0e} c}{e} \hat{\mathbf{b}} \cdot \nabla \left[ \frac{Z}{\tau} (\hat{\Gamma}_0 - 1) \frac{e\varphi}{T_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right] \\ &+ \eta \nabla_\perp^2 A_\parallel - \frac{1}{2} \frac{\rho_e}{L_T} v_{the} \frac{\partial}{\partial y} A_\parallel, \end{aligned} \quad (4)$$

with

$$\frac{\delta T_{\parallel e}}{T_{0e}} \equiv \frac{1}{n_{0e}} \int d^3 \mathbf{v} 2 \frac{v_\parallel^2}{v_{the}^2} g_e, \quad (5)$$

and the kinetic equation

$$\begin{aligned} \frac{dg_e}{dt} + v_\parallel \left[ \hat{\mathbf{b}} \cdot \nabla g_e - F_{0e} \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}} \right] - C[g_e] &= \\ F_{0e} \left( 1 - 2 \frac{v_\parallel^2}{v_{the}^2} \right) \hat{\mathbf{b}} \cdot \nabla \frac{e}{m_e c} d_e^2 \nabla_\perp^2 A_\parallel &+ \\ - \frac{1}{2} \frac{\rho_e}{L_T} v_{the} \frac{\partial}{\partial y} \left[ \left( \frac{m_e v_\parallel^2}{2T_{0e}} - \frac{1}{2} \right) \frac{e\varphi}{T_{0e}} - 2 \frac{v_\parallel}{v_{the}} \left( \frac{m_e v_\parallel^2}{2T_{0e}} - \frac{3}{2} \right) \frac{e}{cm_e} \frac{A_\parallel}{v_{the}} \right] &F_{0e}. \end{aligned} \quad (6)$$

Here  $g_e$  is a kinetic function such that the non-adiabatic part of the electron distribution function can be written as

$$h_e = \left( -\frac{e\varphi}{T_{0e}} + \frac{\delta n_e}{n_{0e}} + \frac{v_\parallel u_{\parallel e}}{T_{0e}} m_e \right) F_{0e} + g_e + \mathcal{O} \left( \frac{m_e}{m_i} \right), \quad (7)$$

with  $\int dv_\parallel (1, v_\parallel) g_e = 0$ . The collision operator is defined as

$$C[g_e] = \left( \frac{\partial h_e}{\partial t} \right)_{coll} - 2 \frac{v_\parallel F_{0e}}{v_{the}^2 n_{0e}} \int d^3 \mathbf{v} v_\parallel \left( \frac{\partial h_e}{\partial t} \right)_{coll}. \quad (8)$$

A collision operator model will be introduced when necessary. Equation (6), in the limit of homogeneous backgrounds, reduces to the result of Ref. [36]. All undefined symbols are standard.

## B. Linear eigenvalue problem

Consider the following magnetic configuration

$$\mathbf{B} = B_0 \hat{\mathbf{z}} + \delta B_y^{(0)}(x) \hat{\mathbf{y}} + \delta \mathbf{B}_\perp^{(1)}, \quad (9)$$

where  $\delta B_{\perp}^{(1)} \ll \delta B_y^{(0)} \ll B_0$ , and  $\delta B_y^{(0)}(x)$  is part of the perturbation of the guide field  $B_0$ . Then  $\delta B_y^{(0)} = -dA_{\parallel}^{(0)}/dx \equiv B_0 f(x)$ . The function  $f$  does not need to be specified at this stage. Let us consider for a moment the solution of Eq. (6) that will be crucial in our following analysis [see Eqs. (26) and (36) later]. We linearize Eqs. (3)-(4)-(6) around (9) to obtain

$$\omega \frac{\delta n_e}{n_{0e}} = k_{\parallel}(x) u_{\parallel e}, \quad (10)$$

$$\begin{aligned} & -i \frac{\omega}{c} \left[ 1 - \frac{\omega_T}{\omega} \right] A_{\parallel} + i k_{\parallel}(x) \varphi = \\ & i k_{\parallel}(x) \frac{T_{0e}}{e} \left( \frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right) + \frac{\eta - i \omega d_e^2}{c} (\partial_x^2 - k_y^2) A_{\parallel}, \end{aligned} \quad (11)$$

and

$$\begin{aligned} \frac{\delta T_{\parallel e}}{T_{0e}} &= \frac{2 k_{\parallel}(x) u_{\parallel e}}{\omega + i \kappa_{\parallel e}(\omega, \nu_{ei}) k_{\parallel}^2(x)} \\ &+ \frac{\kappa_{\parallel e}(\omega) k_{\parallel}(x)}{\omega + i \kappa_{\parallel e}(\omega, \nu_{ei}) k_{\parallel}^2(x)} i \frac{\omega_T}{c} \frac{e A_{\parallel}}{T_{0e}} + \frac{\omega_T}{\omega + i \kappa_{\parallel e}(\omega, \nu_{ei}) k_{\parallel}^2(x)} \frac{e \varphi}{T_{0e}}, \end{aligned} \quad (12)$$

where  $\kappa_{\parallel e} = \kappa_{\parallel e}(\omega, \nu_{ei})$  is a frequency dependent parallel electron thermal conductivity,  $k_{\parallel}(x) = k_y f(x)$ , and we have neglected  $d_e^2 f''(x)/f(x)$  compared to  $d_e^2 \partial_x^2$ , and assumed  $k_z \ll k_y f(x)$ . After using Eqs. (11) and (12) in Eq. (10) we obtain

$$u_{\parallel e} = - \frac{e}{m_e (\nu_{ei} - i \omega)} \left[ i \frac{\omega}{c} A_{\parallel} - i k_{\parallel}(x) \varphi \right] \sigma_e, \quad (13)$$

where  $\sigma_e$  is the electron conductivity. The explicit forms of  $\sigma_e$  and  $\kappa_{\parallel e}$  will be given shortly. If we use  $f(x) \approx x/L_s$  in the neighborhood of the surface at which  $k_{\parallel}(x) = 0$ , (the equilibrium magnetic field is sheared with characteristic length  $L_s$ ), then we obtain the following set of equations for kinetic reconnecting drift modes:

$$-\frac{x}{\delta} \left( A_{\parallel} - \frac{x}{\delta} \hat{\varphi} \right) \sigma_e(x/\delta) = \left( 1 - i \frac{\omega}{\nu_{ei}} \right) \int_{-\infty}^{\infty} dp e^{ipx} F(p \rho_i) \hat{\varphi}(p), \quad (14)$$

$$\frac{1}{\hat{\omega}^2 \hat{\beta}} \left( \frac{d^2}{dx^2} - k_y^2 \right) A_{\parallel} = \frac{1}{\delta^2} \frac{\delta}{x} \int_{-\infty}^{\infty} dp e^{ipx} F(p \rho_i) \hat{\varphi}(p), \quad (15)$$

where  $\hat{\varphi} = (\delta/L_s)(k_y c/\omega) \varphi$ ,  $\hat{\omega} = \omega/\omega_T$ ,  $F = (Z/\tau)(\Gamma_0 - 1)$ , and

$$\delta^2 = \exp[-i\pi/2] 2\omega \nu_e / (k_y^2 v_{the}^2) L_s^2 \quad (16)$$

is the electron semicollisional scale.

Equations (14) and (15) can be solved using a double asymptotic matching technique. By using the separation between ion and electron scales,  $\delta \ll \rho_i$ , the two equations can be simplified in the two regions where  $x \sim \delta \ll L_s$ , and  $x \sim \rho_i \ll L_s$ , and matched in the overlapping regions to give a dispersion relation for the mode. It turns out that, when this separation of scales is allowed for, it is possible to formulate the problem as a single eigenvalue equation for the current density  $J = -\partial_x^2 A_{\parallel}$  [32]. A technical difficulty arises since the electron region equation is naturally formulated (and solved) in real space, whereas for the ion region,  $x \sim \rho_i$ , a formulation in Fourier space is more convenient. Nevertheless, the problem of identifying and matching the correct solutions was solved in Ref. [34], and the same approach will be fruitful here. The case  $k_z \gg k_y f(x)$  will be analysed in the truly collisionless limit,  $\nu_{ei} \equiv 0$ , after integrating exactly Eq. (6).

### C. Electron conductivity

In Ref. [36], a Hermite expansion of the electron distribution function  $g_e$  was introduced. This allowed us to prove the Boltzmann H-theorem, and predict the velocity space spectrum of electron free energy in steady-state and in the presence of growing modes. The use of Hermite polynomials as a basis in velocity space also proved useful for the numerical implementation of the new kinetic description of the electrons (also known as Kinetic Reduced Electron Heating Model) [36, 38]. Here, we find that such a representation is again very powerful. In this section, we give details of the new equations for inhomogeneous backgrounds, and show their implications in the collisional and weakly collisional limit. In particular, we calculate explicitly the electron conductivity  $\sigma_e$  introduced in Eqs. (14) and (15).

### 1. Hermite Series

If we aim at describing the parallel velocity space dynamics of our kinetic system, we are allowed to use a simple model collision operator. As in Ref. [36], it is useful to employ the (modified) Lenard-Bernstein collision operator [36, 39]

$$C[g_e] = \nu_{ei} \left[ \frac{1}{2} \frac{\partial}{\partial \hat{v}_{\parallel}} \left( \frac{\partial}{\partial \hat{v}_{\parallel}} + 2\hat{v}_{\parallel} \right) g_e - (1 - 2\hat{v}_{\parallel}^2) \frac{\delta T_{\parallel e}}{T_{0e}} F_{0e} \right], \quad (17)$$

where  $\nu_{ei} = 4\pi n_{0e} Z^2 e^4 \ln \Lambda / (m_e^2 v_{the}^3)$  is the energy-independent collision frequency. Let us project Eq. (6) onto Hermite polynomials<sup>1</sup>, which are eigenfunctions of the collision operator model in Eq. (17). The resulting equation is

$$\begin{aligned} & \frac{1}{n_{0e}} \frac{d}{dt} n_{0e} \hat{g}_m + \frac{1}{n_{0e}} v_{the} \hat{\mathbf{b}} \cdot \nabla n_{0e} \left( \sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} - \delta_{m,1} \hat{g}_2 \right) \\ &= -\sqrt{2} \delta_{m,2} \left( \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} + \frac{\eta_e}{2} \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} \right) \\ &- \sqrt{3} \delta_{m,3} \frac{\eta_e}{2} \frac{v_{the} \hat{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}} - \nu_{ei} (m \hat{g}_m - 2\delta_{m,2}), \end{aligned} \quad (18)$$

where  $\tilde{\mathbf{b}} \equiv -B_0^{-1} \{A_{\parallel}, \dots\}$ . Here  $\hat{g}_e = 2v_{the}^{-2} \int dv_{\perp} v_{\perp} \exp[-v_{\perp}^2/v_{the}^2] g_e$ , whereas the Hermite inverse transform is defined as

$$\hat{g}_e(v_{\parallel}) = \sum_{m=0}^{\infty} \frac{H_m(\hat{v}_{\parallel})}{\sqrt{2^m m!}} \hat{g}_m F_{0e}(\hat{v}_{\parallel}^2), \quad (19)$$

with coefficients

$$\hat{g}_m = \frac{1}{n_{0e}} \int_{-\infty}^{\infty} d\hat{v}_{\parallel} \frac{H_m(\hat{v}_{\parallel})}{\sqrt{2^m m!}} \hat{g}_e(v_{\parallel}), \quad (20)$$

where  $\hat{v}_{\parallel} = v_{\parallel}/v_{the}$ . Hence, for the first Hermite moments we obtain

$$\begin{aligned} & \frac{d}{dt} \hat{g}_2 + v_{the} \frac{\hat{\mathbf{b}} \cdot \nabla}{n_{0e}} \left( \sqrt{\frac{3}{2}} n_{0e} \hat{g}_3 \right) \\ &= -\sqrt{2} \left( \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} + \frac{\eta_e}{2} \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} \right), \end{aligned} \quad (21)$$

for  $m = 2$ ,

$$\begin{aligned} & \frac{d}{dt} \hat{g}_3 + v_{the} \frac{\hat{\mathbf{b}} \cdot \nabla}{n_{0e}} \left( \sqrt{2} n_{0e} \hat{g}_4 + \sqrt{\frac{3}{2}} n_{0e} \hat{g}_2 \right) = \\ & -\sqrt{3} \frac{\eta_e}{2} \frac{v_{the} \hat{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}} - 3\nu_{ei} \hat{g}_3, \end{aligned} \quad (22)$$

for  $m = 3$  and

$$\frac{d}{dt} \hat{g}_m + \frac{1}{n_{0e}} v_{the} \hat{\mathbf{b}} \cdot \nabla n_{0e} \left( \sqrt{\frac{m+1}{2}} \hat{g}_{m+1} + \sqrt{\frac{m}{2}} \hat{g}_{m-1} \right) = -m\nu_{ei} \hat{g}_m, \quad (23)$$

for  $m \geq 4$ .

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[1] Since the model has no knowledge of perpendicular temperature fluctuations, we consider them fixed to an arbitrary constant. As a consequence, we modify the operator to conserve parallel temperature fluctuations. This is not consistent for  $\nu_{ei} \sim \omega$ , and  $\omega \sim \omega_T$ , however it is correct in the subsidiary limits considered here  $\nu_{ei} \gg \omega$ , and  $N^{-1/2} \ll \nu_{ei}/\omega \ll 1$ , where  $N$  is the highest Hermite moment kept. Other authors have recognised the role of conservation of electron parallel temperature perturbations during collisions in order to obtain a microtearing instability [see App. A of Ref. [27]].

## 2. Collisional limit $\nu_{ei} \gg \omega$

In this limit the Hermite coefficients scale as [36]

$$\frac{\hat{g}_m}{\hat{g}_{m-1}} \sim \frac{k_{\parallel} v_{the}}{\sqrt{m} \nu_{ei}}, \quad (24)$$

and we can truncate the fluid system by neglecting  $\hat{g}_4$  in the  $\hat{g}_3$  equation and invert  $\hat{g}_3$  from Eq. (22), neglecting the time derivative compared to the collision frequency. In this way, we obtain

$$\hat{g}_3 \approx -\frac{1}{3\nu_{ei}} \frac{v_{the}}{n_{0e}} \hat{\mathbf{b}} \cdot \nabla \sqrt{\frac{3}{2}} n_{0e} \hat{g}_2 - \frac{\sqrt{3}}{3\nu_{ei}} \frac{\eta_e}{2} \frac{1}{n_{0e}} v_{the} \tilde{\mathbf{b}} \cdot \nabla n_{0e}. \quad (25)$$

The resulting equation for the temperature perturbation is

$$\begin{aligned} \frac{1}{n_{0e}} \frac{d}{dt} n_{0e} \frac{\delta T_{\parallel e}}{T_{0e}} &= \frac{v_{the}}{n_{0e}} \hat{\mathbf{b}} \cdot \nabla \frac{v_{the}}{2\nu_{ei}} \hat{\mathbf{b}} \cdot \nabla n_{0e} \frac{\delta T_{\parallel e}}{T_{0e}} \\ &+ \frac{\eta_e}{2} \frac{v_{the}}{n_{0e}} \hat{\mathbf{b}} \cdot \nabla \frac{v_{the}}{\nu_{ei}} \frac{\tilde{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}} \\ &- \eta_e \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} - 2\hat{\mathbf{b}} \cdot \nabla u_{\parallel e}. \end{aligned} \quad (26)$$

Equation (26) is coupled to Ohm's law (4) via  $\delta T_{\parallel e}$  in the collisional limit. In this case, electron inertia can be neglected compared to the collisional term, we can use Eq. (26) in Ohm's law and obtain the conductivity for collisional, non-isothermal electrons

$$\sigma_e(x/\delta) = \frac{\sigma_0 + \frac{x^2}{\delta^2}}{1 + 4\frac{x^2}{\delta^2} + \frac{x^4}{\delta^4}}, \quad (27)$$

with

$$\sigma_0 = 1 - \frac{\omega T}{\omega}. \quad (28)$$

The numerical coefficients in Eq. (27) differ from those of fluid theories [40] because of the details of the collisional operator model. This proves mathematically that our model reproduces the results of Ref. [34] in the highly collisional limit.

## 3. General electron conductivity

An alternative way of closing the kinetic hierarchy is by considering

$$\frac{\hat{g}_m}{\hat{g}_{m-1}} \sim \frac{k_{\parallel} v_{the}}{\sqrt{m} \omega} \frac{\omega}{\nu_{ei}} \ll 1, \quad (29)$$

with

$$\frac{k_{\parallel} v_{the}}{\omega \sqrt{m}} \ll 1, \quad \text{but} \quad \frac{k_{\parallel} v_{the}}{\omega} \sim \frac{\omega}{\nu_{ei}} \sim 1, \quad (30)$$

thus the expansion is in large  $m \gg 1$ . This closure scheme works also nonlinearly. Our velocity space representation allows us to avoid taking the  $\omega/\nu_{ei} \ll 1$  limit in order to calculate velocity space integrals, and allows us to study the interesting and realistic limit  $N^{-1/2} \ll \nu_{ei}/\omega \ll 1$ , where  $N$  is the order of the highest Hermite moment kept.

Let us consider an  $N \gg 1$  for which  $\hat{g}_{N+1} \ll \hat{g}_N$  in the sense of Eq. (29). Indeed, there is always one for small and finite  $\nu_{ei}$ . Then, for the  $N$ th component the kinetic equation is

$$(-i\omega + N\nu_{ei})\hat{g}_N = -ik_{\parallel} v_{the} \sqrt{\frac{N}{2}} \hat{g}_{N-1}. \quad (31)$$

We can use this expression for  $\hat{g}_N$  in the equation for the  $N - 1$  component and obtain the  $N - 1$  component as a function of the  $N - 2$  component

$$\left[ -i\omega + (N - 1)\nu_{ei} + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}N/2}{-i\omega + N\nu_{ei}} \right] \hat{g}_{N-1} = -ik_{\parallel}v_{the} \sqrt{\frac{N-1}{2}} \hat{g}_{N-2}. \quad (32)$$

Hence, after  $n$  iterations we have

$$\hat{g}_{N-n} = \frac{-ik_{\parallel}v_{the} \sqrt{\frac{N-n}{2}} \hat{g}_{N-(n+1)} \times 1}{[-i\omega + (N-n)\nu_{ei}] + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}(N-n+1)/2}{[-i\omega + (N-n+1)\nu_{ei}] + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}(N-n+2)/2}{\dots + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}N/2}{-i\omega + N\nu_{ei}}}}. \quad (33)$$

Now, when  $N - n = 4$ , we are able to write  $\hat{g}_3$  in Eq. (21) explicitly as a function of all other  $\hat{g}_m$  up to  $\hat{g}_N$ . By proceeding in the same way as in the collisional case, we obtain a general electron conductivity,

$$\hat{\sigma}_e = \frac{\sigma_0 + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} s^2}{1 + \left[ \frac{3}{1 - i\frac{\omega}{\nu_{ei}}} + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} \right] s^2 + \frac{1}{1 - i\frac{\omega}{\nu_{ei}}} \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} s^4}, \quad (34)$$

where  $s^2 = k_{\parallel}^2 \kappa_{\parallel e}^f / (-i\omega) \equiv x^2 / \delta^2$ ,  $\kappa_{\parallel e}^f = v_{the}^2 / (2\nu_{ei})$  is the fluid parallel electron thermal conductivity, and

$$\Omega(N) = [-i\omega + 4\nu_{ei}] + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}5/2}{[-i\omega + 5\nu_{ei}] + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}6/2}{\dots + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}N/2}{-i\omega + N\nu_{ei}}}}. \quad (35)$$

In the limit  $\omega / \nu_{ei} \ll 1$  Eq. (34) reduces to the semicollisional electron conductivity (27). Equation (34) implies that the electron thermal conductivity is the product of the fluid part times a kinetic contribution that encapsulates the time evolution of all the Hermite moments kept, that is

$$\kappa_{\parallel e}(\omega) = \kappa_{\parallel e}^f \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}4/2}{\dots + ik_{\parallel}v_{the} \frac{-ik_{\parallel}v_{the}N/2}{-i\omega + N\nu_{ei}}}}. \quad (36)$$

If we compare Eq. (27) with Eq. (13) of Ref. [35], we see that the term proportional to the microtearing drive,  $\omega_T / \omega$ , is a different  $\mathcal{O}(1)$  number. This is due to the fact that the model collisional operator, as such, can give correct results up to  $\mathcal{O}(1)$  multiplicative constants. In previous theories of microtearing modes, the frequency dependence of *this* term, originated by a time dependent electron thermal force, has been proposed as crucial to obtain an instability. That is, an electron conductivity of the form

$$\sigma_D = \frac{1 - \frac{\omega_T}{\omega} \left( 1 + \alpha + i\alpha' \frac{\omega}{\nu_{ei}} \right) + d_1 s^2}{1 + d_0 s^2 + d_1 s^4} \quad (37)$$

was used. Here  $\omega \ll \nu_{ei}$ ,  $d_0$ ,  $d_1$ ,  $\alpha$ , and  $\alpha'$  are positive real constants. In this formulation, the term proportional to  $\alpha'$  is responsible for the microtearing instability. If we take the same  $\omega \ll \nu_{ei}$  limit of  $(1 - i\omega / \nu_{ei})^{-1} \hat{\sigma}_e$ , which enters Eqs. (14) and (15), and retain the small collisional correction that couples to  $\omega_T$ , we obtain

$$(1 - i\omega / \nu_{ei})^{-1} \hat{\sigma}_e \approx \frac{1 - \frac{\omega_T}{\omega} \left( 1 + i\frac{\omega}{\nu_{ei}} \right) + s^2}{1 + 4s^2 + s^4}. \quad (38)$$

Then, the electron inertial term  $(1 - i\omega / \nu_{ei})^{-1}$  is coupled to the electron temperature gradient as the time dependent electron thermal force contribution of previous works. Notice that the approximation for the electron conductivity originally used by Hazeltine et al. [1], would give an electron temperature gradient instability in the weakly collisional limit  $\nu_{ei} / \omega \ll 1$  [27], as pointed out by Rosenberg and co-authors [27], while we expect the mode to be marginally



stable in the collisionless limit. In any case, the conductivity, calculated in Ref. [30] to obtain the generalised spatial dependence of the conductivity originally used by Hazeltine et al. [1] [Eq. (27)], does not give the correct isothermal electron response at large distances from the reconnecting layer, that is,  $s^2(1 - i\omega/\nu_{ei})^{-1}\hat{\sigma}_e \neq 1$ , for  $s \gg 1$  [33, 34, 41]. Therefore, it is not appropriate to properly match the electron solution to the ion region solution in a theory with large ion Larmor orbits [33, 34, 41].

As already noticed by Drake et al. [35], in the theory of semicollisional drift tearing modes, the use of a model for the electron conductivity [as in Eq. (37)], instead of the fluid one of Eq. (27), brings about an imaginary correction to the fundamental frequency of the mode which gives an instability in leading order.

While we leave open the question whether the use of such a model can predict results from first principle numerical simulations, we content ourselves with the simple model collisional operator in Eq. (17), and aim at deriving a theory that simultaneously takes account of electron inertia, non-isothermal electrons, the electrostatic potential, and non-constant magnetic perturbations.

#### 4. Finite collisionality and collisionless limit

If we were to neglect collisions completely, we could solve the electron kinetic equation Eq. (6) using Landau integrals [42] and obtain [36, 43, 44]

$$\hat{\sigma}_L \left( \frac{x}{\delta_{c-less}} \right) = -\frac{1}{2} \left\{ \frac{\delta_{c-less}^2}{x^2} Z' \left( \frac{\delta_{c-less}}{|x|} \right) + \frac{1}{2\hat{\omega}} \frac{\delta_{c-less}^3}{|x|^3} Z'' \left( \frac{\delta_{c-less}}{|x|} \right) \right\}, \quad (39)$$

where  $\delta_{c-less} = (\omega/k_y v_{the}) L_s$ ,  $s = x/\delta_{c-less}$ , and  $Z$  is the plasma dispersion function [45]. We can compare the analytical form of the electron conductivity calculated using Landau integrals, and the one calculated using the continued fraction solution (34) for  $N = 100$ ,  $\nu_{ei}/\omega = 0.1/0.5$ , and  $\nu_{ei}/\omega = 0.01/0.5$ . For this value of  $N$ , the continued fraction solution has converged in both cases. The agreement improves with decreasing collisionality, see Figs. (1) and (2). In the following, we will make use of the finite-collisionality formulation, where all scales are

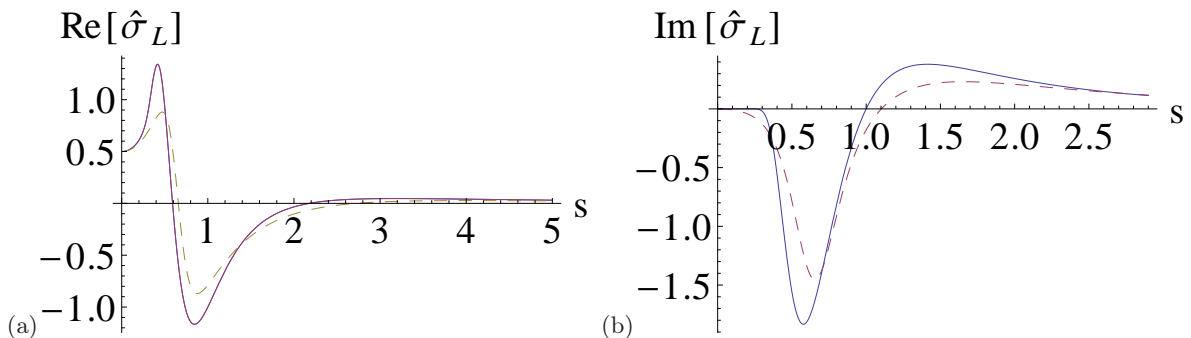


Figure 1: The real (a) and imaginary (b) parts of the electron conductivity  $\hat{\sigma}_L$  in the  $\nu_{ei} = 0$  case (solid line) calculated using Eq. (39), and  $-1/2\hat{\sigma}_e$  for  $N = 100$  (dashed line) calculated using Eq. (34). Here  $\omega/\nu_{ei} = 0.5/0.1$  and  $s \equiv x/\delta_{c-less}$ .

normalised to the semicollisional scale  $\delta^2 = \exp[-i\pi/2] 2\omega\nu_{ei}/(k_y^2 v_{the}^2) L_s^2$ , [Eq. (16)] and the electron conductivity is given by Eq. (34). This turns out to be extremely convenient for numerical and analytical purposes.

### III. LOW- $\hat{\beta}_T$ AND LOW- $k_y\delta$ DISPERSION RELATION

To derive the low- $\hat{\beta}_T$ , low- $k_y\delta$  dispersion relation, we follow Ref. [34]. A general dispersion relation can be written as

$$\frac{\hat{c}_-}{\hat{c}_+} = \frac{\hat{a}_-}{\hat{a}_+} \frac{\delta}{\rho_i}, \quad (40)$$

where the coefficients  $\hat{a}_\pm$  are such that the large asymptotic limit of the solution for the current  $J$  is

$$J(k\rho_i) \sim \hat{a}_+ k\rho_i + \hat{a}_-, \quad \text{for } k\rho_i \gg 1, \quad (41)$$



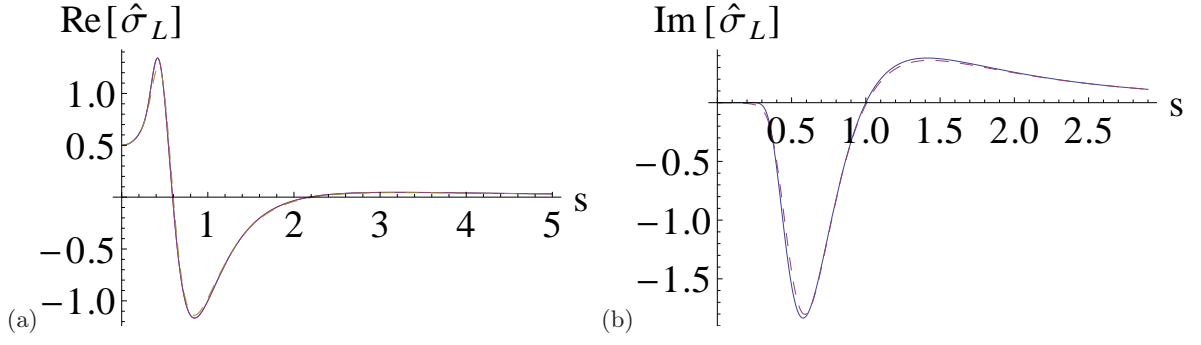


Figure 2: The real (a) and imaginary (b) parts of the electron conductivity  $\hat{\sigma}_L$  in the  $\nu_{ei} = 0$  case (solid line) calculated using Eq. (39), and  $-1/2\hat{\sigma}_e$  for  $N = 100$  (dashed line) calculated using Eq. (34). Here  $\omega/\nu_{ei} = 0.5/0.01$  and  $s \equiv x/\delta_{c-less}$ .

in the ion region  $x \sim \rho_i$ , and

$$J(k\delta) \sim \hat{c}_+ k\delta + \hat{c}_-, \text{ for } k\delta \ll 1, \quad (42)$$

in the electron region  $x \sim \delta$ .

In the electron region when  $x \sim \delta \ll \rho_i$ , we can use the  $k_\perp \rho_i \gg 1$  limit for the RHS of Eqs. (14) and (15), to obtain [34]

$$\frac{d^2}{ds^2} \left[ \frac{\left(1 - i\frac{\omega}{\nu_{ei}}\right) F_\infty - s^2 \hat{\sigma}_e(s)}{F_\infty \hat{\sigma}_e(s)} \right] J(s) = -\hat{\beta}_T \hat{\omega}^2 J(s), \quad (43)$$

where  $\hat{\omega} = \omega/\omega_T$ ,  $s = x/\delta$ , and  $F_\infty = -Z/\tau$ . Equation (43) is valid for any collision frequency provided  $N^{-1/2} \ll \nu_{ei}/\omega \ll 1$ , or  $\nu_{ei} \gg \omega$ , with  $N$  the order of the highest Hermite moment kept in the model. We retain electron inertia in Ohm's law. For this reason we have a new term  $(1 - i\omega/\nu_{ei}) F_\infty - s^2 \hat{\sigma}_e(s)$ , and not  $F_\infty - s^2 \hat{\sigma}_e(s)$ , which was used in the collisional case of Ref. [34].

In order to derive the dispersion relation, we need to study the large argument asymptotic behaviour of the solution of Eq. (43). This is determined by

$$\frac{d^2}{ds^2} s^2 J \sim -\hat{\beta}_T \hat{\omega}^2 \frac{F_\infty}{F_\infty - 1} J. \quad (44)$$

Equation (44) tells us that the solution of Eq. (43) behaves asymptotically as

$$J(s) \sim b_+ s^{-1} + b_- s^{-2}, \text{ for } s \rightarrow \infty, \quad (45)$$

so in  $t$ -space (the Fourier conjugate of  $s$ ) we shall have

$$J(t) \sim \hat{c}_+ t^1 + \hat{c}_- t^0, \text{ for } t \rightarrow 0, \quad (46)$$

with [34, 46]

$$\frac{\hat{c}_-}{\hat{c}_+} = \frac{\Gamma(\mu - \frac{1}{2})}{\Gamma(-\mu - \frac{1}{2})} \tan \left[ \frac{\pi}{2} \left( \frac{1}{2} + \mu \right) \right] \frac{b_+}{b_-}, \quad (47)$$

and

$$\frac{1}{4} - \mu^2 = \hat{\omega}^2 \hat{\beta}_T \frac{F_\infty}{G_\infty}, \quad (48)$$

where  $G_\infty = F_\infty - 1$ . This result is general and *does not* depend on the electron collision model used. Indeed, far from the reconnection region, electrons are isothermal [33, 34, 36, 41]. When we solve Eq. (43) in a low- $\hat{\beta}_T$  expansion,

we notice that the power  $s^{-1}$  in Eq. (45) is not coming from the zeroth order solution of Eq. (43). It is therefore sufficient to solve it to first order. Indeed, we obtain the reconnecting (even) solution

$$J(s) = \frac{F_\infty \hat{\sigma}_e(s)}{\left(1 - i \frac{\omega}{\nu_{ei}}\right) F_\infty - s^2 \hat{\sigma}_e(s)} \left\{ 1 - \hat{\beta}_T \hat{\omega}^2 \int_0^s ds' \int_0^{s'} du \frac{F_\infty \hat{\sigma}_e(u)}{\left(1 - i \frac{\omega}{\nu_{ei}}\right) F_\infty - u^2 \hat{\sigma}_e(u)} \right\}, \quad (49)$$

and the large argument asymptotic behaviour is

$$J(s) \sim \frac{1}{s^2} + \hat{\beta}_T \hat{\omega}^2 I_e \frac{1}{s}, \quad (50)$$

with

$$I_e = - \int_0^\infty ds \frac{F_\infty \hat{\sigma}_e(s)}{\left(1 - i \frac{\omega}{\nu_{ei}}\right) F_\infty - s^2 \hat{\sigma}_e(s)}. \quad (51)$$

Since the matching to the ion solution is performed in Fourier space, in principle we should calculate the Fourier transform of Eq. (49). However, we can apply the analytical formula in Eq. (47) [34] that relates asymptotic leading order coefficients in real space with those in  $k$ -space [46]. Then, in the small  $\hat{\beta}_T$  limit (equivalently  $\mu \rightarrow 1/2^+$ ), we have

$$\frac{\hat{c}_-}{\hat{c}_+} \approx - \frac{2}{\pi} \frac{G_\infty}{F_\infty} I_e. \quad (52)$$

The ion region is treated in the same way as in Ref. [34]. In the limit  $x \gg \delta$ , the product  $s^2 \hat{\sigma}_e$  tends to a constant, and we obtain a differential equation for the current in Fourier space [34]. This can again be solved in a low- $\hat{\beta}_T$  expansion as in Ref. [34]. We report here the result

$$\begin{aligned} J(k) \sim & k^{\hat{\beta}_T \hat{\omega}^2 \frac{F_\infty}{G_\infty}} + \hat{\beta}_T \hat{\omega}^2 \frac{F_\infty}{G_\infty} \frac{\pi}{\Delta' \rho_i} \left\{ k + \frac{1}{G_\infty} \frac{1}{\sqrt{\pi}} \log k \right\} \\ & + \hat{\beta}_T \hat{\omega}^2 \frac{\pi}{\Delta' \rho_i} \bar{I} - \left( \hat{\beta}_T \hat{\omega}^2 \right)^2 \frac{F_\infty}{G_\infty} k \int_0^\infty dk \frac{F}{k^2 G}, \end{aligned} \quad (53)$$

where  $k \equiv k \rho_i$ , and  $\bar{I} = \int_0^\infty dk \left\{ F/G - F_\infty/G_\infty - (F_\infty/G_\infty^2) \pi^{-1/2} / (1+k) \right\}$ , and

$$\Delta' = \frac{1}{A_\parallel^{ext}} \frac{dA_\parallel^{ext}}{dx} \bigg|_{0-}^{0+}$$

is the ideal MHD external solution parameter. So, the coefficients  $\hat{a}_\pm$  in Eq. (40) are

$$\frac{\hat{a}_-}{\hat{a}_+} = \frac{1 + \hat{\beta}_T \hat{\omega}^2 \frac{F_\infty}{G_\infty^2} \frac{\pi}{\Delta' \rho_i} \frac{1}{\sqrt{\pi}} \log \frac{\rho_i}{\delta} + \hat{\beta}_T \hat{\omega}^2 \frac{\pi}{\Delta' \rho_i} \bar{I}}{\hat{\beta}_T \hat{\omega}^2 \frac{F_\infty}{G_\infty} \frac{\pi}{\Delta' \rho_i} - \left( \hat{\beta}_T \hat{\omega}^2 \right)^2 \frac{F_\infty}{G_\infty} \int_0^\infty \frac{F}{k^2 G}} \quad (54)$$

From this, it follows that the dispersion relation is

$$\frac{\delta}{\rho_i} B(\hat{\omega}) - \frac{2}{\pi} I_e \hat{\omega}^2 C(\hat{\omega}) = 0, \quad (55)$$

with

$$B(\hat{\omega}) = \frac{\Delta' \rho_i}{\pi \hat{\beta}_T} - \hat{\omega}^2 \frac{Z/\tau}{[Z/\tau + 1]^2} \frac{1}{\sqrt{\pi}} \log \frac{\rho_i}{\delta} + \hat{\omega}^2 \bar{I}(\tau), \quad (56)$$

and

$$C(\hat{\omega}) = 1 - \frac{\hat{\beta}_T \Delta' \rho_i}{\pi} \hat{\omega}^2 I(\tau). \quad (57)$$

The ion integrals are defined as

$$\bar{I}(\tau) = \int_0^\infty dq \left[ \frac{F}{G} - \frac{Z/\tau}{Z/\tau + 1} + \frac{Z/\tau}{[Z/\tau + 1]^2} \frac{1}{\sqrt{\pi}} \frac{1}{(1+q)} \right], \quad (58)$$

and

$$I(\tau) = \int_0^\infty \frac{dq}{q^2} \frac{F}{G}. \quad (59)$$

We notice that the ion region solution, Eq. (53), has already been matched to a boundary condition at  $k\rho_i \rightarrow 0$ , [32] and the information about the ideal MHD external solution is embedded in the parameter  $\Delta'$ . The authors of Ref. [32] proved analytically that the ideal MHD boundary condition used in (53) is correct even for high wave numbers. This generally corresponds to the substitution  $\Delta' \rightarrow -2k_y$ , when the external solution is of the form  $A_{\parallel}^{ext} \sim e^{-k_y|x|}$ , and is tearing mode stable. Equation (55), for positive  $\Delta'$ , is valid even for  $\Delta'\rho_i \gg 1$ , that is when the so-called constant- $A_{\parallel}$  approximation does not hold. In the present work, we will consider two situations: arbitrary values of  $|\Delta'\rho_i|$  for  $\Delta'\rho_i > 0$ , and  $\Delta'\rho_i < 0$ , with  $|\Delta'\rho_i| \ll 1$ . A thorough investigation of non-constant tearing-stable perturbations is left to future work. For  $\Delta'\rho_i < 0$ , with  $|\Delta'\rho_i| \ll 1$ , we will also find it useful to match directly the electron region solution to an exponentially decaying external solution.

In what follows, we first study Eq. (55) in the highly collisional limit  $\nu_{ei}/\omega \gg 1$ ; we then benchmark our results against numerical hybrid and kinetic simulations and we finally investigate microtearing modes and ETG modes within our theoretical framework.

#### IV. COLLISIONAL LIMIT

##### A. Collisional non-isothermal electrons

By deriving Eq. (27), we proved that, in the collisional limit, the model considered here [Eqs. (3)-(4)-(6)] reproduces the results of Ref. [34]. We then take  $\omega_* \rightarrow 0$ , but  $\omega_*\eta_e \equiv \omega_T \sim \mathcal{O}(1)$ , in Eq. (33) of Ref. [34] (also  $\eta_i \equiv 0$ ) to obtain

$$\begin{aligned} & e^{-i\frac{\pi}{4}} \sqrt{\frac{2\nu_{ei}}{\omega_T}} \frac{\delta_*}{\rho_i} \sqrt{1+1/\tau} \sqrt{1 - \frac{\omega_T}{\omega} + 4/\tau + 2\sqrt{1/\tau(1+1/\tau)}} \times \\ & \left\{ \frac{\Delta'\rho_i}{\pi\hat{\beta}_T} - \frac{\omega^2}{\omega_T^2} \frac{1/\tau}{\sqrt{\pi}(1+1/\tau)^2} \ln \left( e^{i\frac{\pi}{4}} \frac{\rho_i}{\delta_*} \sqrt{\frac{\omega_T}{2\nu_{ei}}} \sqrt{\frac{\omega_T}{\omega}} \right) + \frac{\omega^2}{\omega_T^2} \bar{I}(\tau) \right\} + \\ & \sqrt{\frac{\omega}{\omega_T}} \frac{1}{\tau} \left\{ 1 - \frac{\Delta'\rho_i\hat{\beta}_T}{\pi} \frac{\omega^2}{\omega_T^2} I(\tau) \right\} \left\{ \left( \frac{\omega}{\omega_T} - 1 \right) \sqrt{1+1/\tau} + \frac{\omega}{\omega_T} \sqrt{1/\tau} \right\} = 0, \end{aligned} \quad (60)$$

where  $\delta_* = \omega_T/(k_y v_{the})L_s$ , and the charge number  $Z$  is set to unity for simplicity. This is equivalent to evaluating the electron integral  $I_e$  in Eq. (55) using the electron conductivity defined in Eq. (27).

##### 1. Cold ions limit $\tau \ll 1$ and small $\Delta'$

We know that  $\bar{I} \sim \tau^{1/2}$ , for  $\tau \ll 1$  [34]. We take this limit in Eq. (60). We also consider small  $\Delta'$  for simplicity. Thus we have

$$e^{-i\frac{\pi}{4}} \sqrt{\frac{2\nu_{ei}}{\omega_T}} \frac{\delta_*}{\rho_i} \sqrt{6} \frac{\Delta'\rho_i}{\pi\hat{\beta}_T} = -\sqrt{\frac{\omega}{\omega_T}} \left( 2\frac{\omega}{\omega_T} - 1 \right). \quad (61)$$

For small, negative  $\Delta' = -2k_y$ , Eq. (61) gives a stable solution  $\omega \approx \omega_T/2 + (2/\pi)\sqrt{3}(k_y\delta_*/\hat{\beta}_T)\sqrt{2\nu_{ei}/\omega_T} \exp[-i\pi/4]$ . For small positive  $\Delta'$ , after setting  $\omega = \omega + i\gamma$ , with  $\omega, \gamma \in \mathbb{R}$ , we obtain two equations for the real and imaginary parts of Eq. (61)

$$0 = \frac{\omega}{\omega_T} \left[ \left( 2\frac{\omega}{\omega_T} - 1 \right)^2 - 4\frac{\gamma^2}{\omega_T^2} \right] - 4\frac{\gamma^2}{\omega_T^2} \left( 2\frac{\omega}{\omega_T} - 1 \right) \quad (62)$$

$$-\frac{2\nu_{ei}}{\omega_T} \frac{\delta_*^2}{\rho_i^2} 6 \left( \frac{\Delta' \rho_i}{\pi \hat{\beta}_T} \right)^2 = \frac{\gamma}{\omega_T} \left[ \left( 2 \frac{\omega}{\omega_T} - 1 \right)^2 - 4 \frac{\gamma^2}{\omega_T^2} \right] + \frac{\omega}{\omega_T} 4 \frac{\gamma^2}{\omega_T^2} \left( 2 \frac{\omega}{\omega_T} - 1 \right). \quad (63)$$

We look for a solution  $\omega = \omega_T \left( \frac{1}{2} + \epsilon \right)$ . If  $\frac{\gamma}{\omega_T} > 1 > \epsilon$ , then from Eq. (63) we obtain

$$\left( \frac{\gamma}{\omega_T} \right)^3 \approx \frac{\nu_{ei}}{\omega_T} \frac{\delta_*^2}{\rho_i^2} 3 \left( \frac{\Delta' \rho_i}{\pi \hat{\beta}_T} \right)^2, \quad (64)$$

whereas from Eq. (62) we find  $\epsilon \approx -\frac{1}{6}$ . After converting the growth rate into Alfvénic units, we have

$$\omega \approx \frac{1}{3} \omega_T, \quad (65)$$

$$\frac{\gamma}{\omega_A} \approx 3^{1/3} (k_y L_s)^{2/3} S_\eta^{-1/3} \left( \frac{\Delta' \rho_s}{2\pi} \right)^{2/3}, \quad (66)$$

with  $\rho_s = \sqrt{1/(2\tau)} \rho_i$ , and  $S_\eta = v_A L_s / \eta$  is the Lundquist number. The growth rate is the same as Eq. (98B) of Ref. [36], but now the mode rotates with a frequency  $\omega \approx \frac{1}{3} \omega_T$ . This is the small  $\Delta'$  semicollisional drift-tearing mode [25]. Notice that we are solving for  $\gamma$  real, hence Eq. (66) is not valid for negative  $\Delta'$ .

When  $\Delta' \rho_s$  is large enough, we balance the two terms multiplying  $\Delta'$  in Eq. (60). The ion integral  $I$ , in the cold ion limit, can be calculated analytically after using the Padé approximant  $(Z/\tau)(1 - \hat{\Gamma}_0) = -\rho_s^2 \partial_x^2 / [1 - \rho_i^2 \partial_x^2 / 2]$  for the ion response [33]; then we obtain

$$\left( \frac{\omega}{\omega_T} \right)^{5/2} \left( 2 \frac{\omega}{\omega_T} - 1 \right) \approx \frac{2}{\pi} \sqrt{3} e^{-i\pi/4} \sqrt{2 \frac{\nu_{ei}}{\omega_T} \frac{\delta_*}{\rho_s} \frac{1}{\hat{\beta}_T^2}}. \quad (67)$$

When  $\omega/\omega_T \gg 1$ , we find

$$\left| \frac{\omega}{\omega_T} \right| \approx \left( \frac{\sqrt{3}}{\pi} \sqrt{2 \frac{\nu_{ei}}{\omega_T} \frac{\delta_*}{\rho_s} \frac{1}{\hat{\beta}_T^2}} \right)^{2/7}. \quad (68)$$

This result will be confirmed by the numerical solution of Eq. (60).

## 2. Solution of Equation (60)

We now solve the dispersion relation (60) numerically for arbitrary positive  $\Delta'$ . For this we choose  $\rho_e/L_T = 10^{-3}$ ,  $1/\tau = 100$ ,  $\nu_{ei}/\omega_T = 18$ ,  $d_e/\rho_s = \sqrt{2} \times 0.08$ , and  $k_y L_s = 2$ . In this way  $\delta_0/\rho_i \equiv \sqrt{2\nu_{ei}/\omega_T} \delta_*/\rho_i = 8.5 \times 10^{-4}$ . The electron inertia  $d_e$  is neglected in Ohm's law, but defines  $\hat{\beta}^2 = 2\delta_*/d_e$ , with  $\delta_* = \omega_T/(k_y v_{the}) L_s$  [36]. The result is shown in Fig. (3). The analytical result is reproduced very well. We can also notice that, for  $\Delta' \rho_s \gg 1$ , the growth rate does not depend on  $\Delta'$  [47], and agrees with Eq. (68). For  $\Delta' \rho_s \ll 1$  a diamagnetic stabilisation occurs,  $\gamma < \omega$ , and Eq. (64)-(65) are no longer valid.

## V. WEAKLY COLLISIONAL LIMIT

### A. Fluid Limit

Before studying the microtearing mode, let us verify that Eq. (55), in the limit  $\nu_{ei}/\omega \gg 1$ , agrees with the drift-tearing dispersion relation Eq. (60), which has been derived analytically. The results are shown in Fig. (4) (a) and (b). Here the solid lines represent the solution of Eq. (60) for  $\nu_{ei}/\omega_T = 18$  and  $\nu_{ei}/\omega_T = 180$ , where the electronic integral  $I_e$  defined in Eq. (51) was performed analytically using the semicollisional conductivity in Eq. (27). Other parameters are as in Fig. (3) above. The symbols in Fig. (4) represent the solution calculated using Eq. (55), with  $N = 6$  Hermite moments. For this velocity space resolution, the collisional non-isothermal (fluid) limit is recovered very well and the agreement improves with higher collisionality.

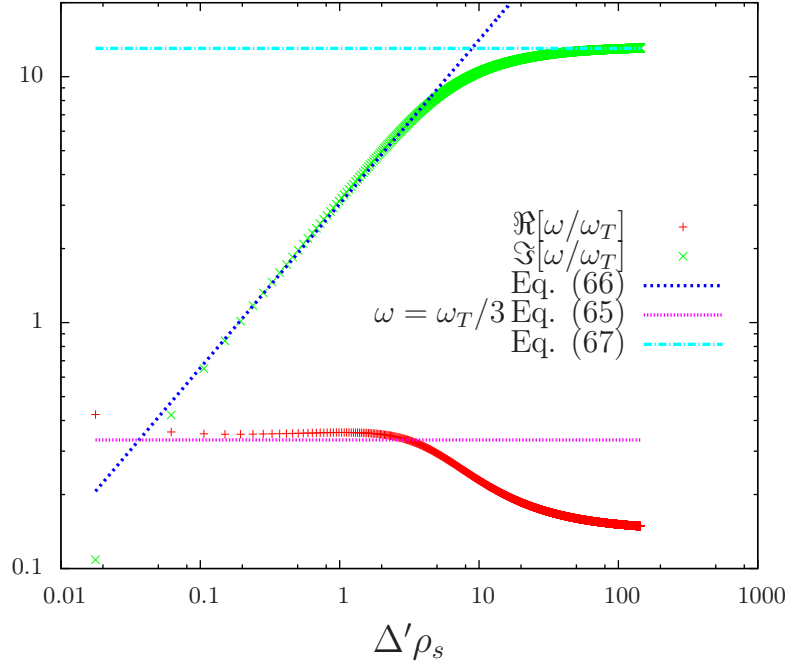


Figure 3: The real and imaginary part of the numerical solution of Eq. (60) as a function of  $\Delta'\rho_s$ . Here,  $\rho_e/L_T = 10^{-3}$ ,  $1/\tau = 100$ ,  $\nu_{ei}/\omega_T = 18$ ,  $d_e/\rho_s = \sqrt{2} \times 0.08$ ,  $k_y L_s = 2$  and  $\delta_0/\rho_i = 8.5 \times 10^{-4}$ . The lines are from the analytical solutions in Eqs. (64), (65) and (67).

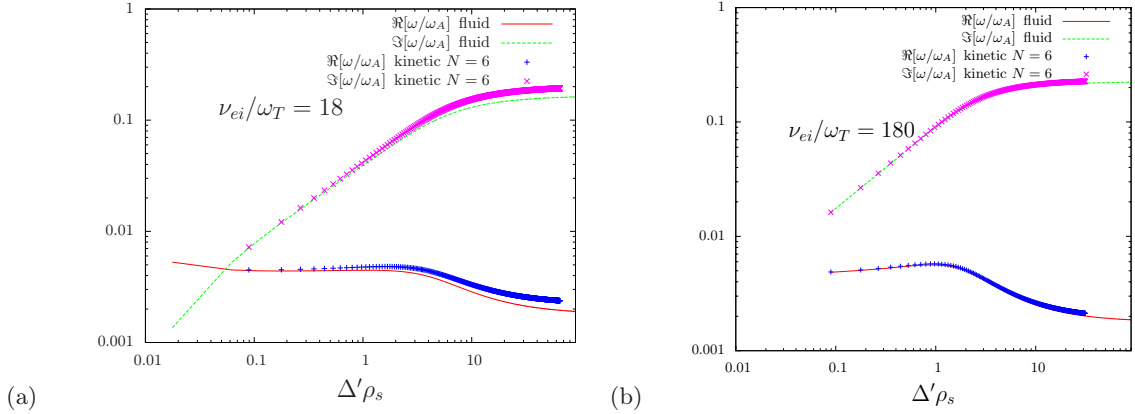


Figure 4: The numerical solution of Eq. (60) (solid lines) and of Eq. (55) (with  $N = 6$ ) (symbols) for  $\rho_e/L_T = 10^{-3}$ ,  $1/\tau = 100$ ,  $d_e/\rho_s = \sqrt{2} \times 0.08$ ,  $k_y L_s = 2$  and  $\delta_0/\rho_i = 8.5 \times 10^{-4}$ . Here  $\nu_{ei}/\omega_T = 18$ , (a) and  $\nu_{ei}/\omega_T = 180$ , (b). The fluid limit (solid lines) is reproduced.

## B. Weakly Collisional Drift-tearing mode

In the weakly collisional limit, we compare the solution of Eq. (55) with the results of the hybrid fluid-kinetic **Viriato** code that solves Eqs. (3), (4) and (6) [38], and the gyrokinetic code **AstroGK** (AGK)[48]. The level of agreement is satisfactory, see Fig. (5).

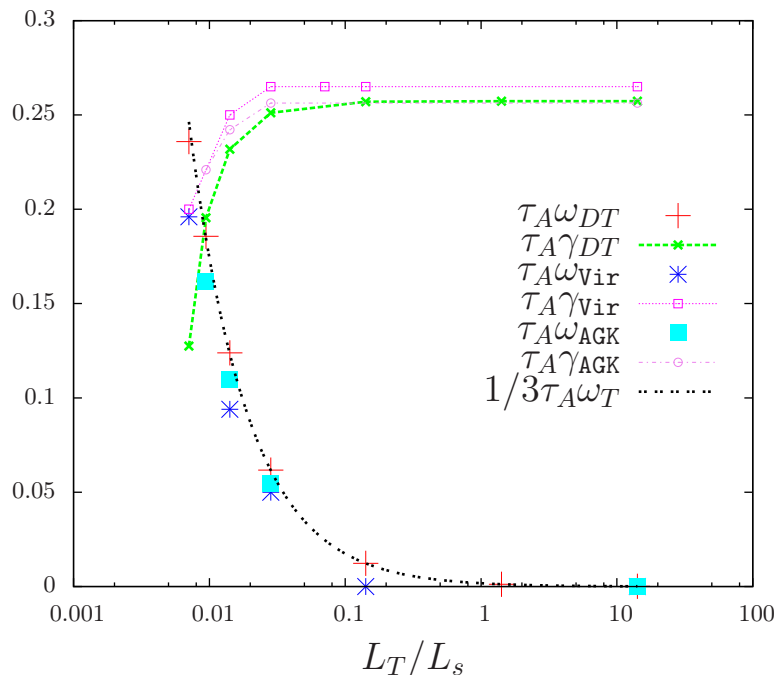


Figure 5: The numerical solution of Eq. (55) ( $N = 20$ ) compared to the solutions obtained by the **Viriato** code and **AstroGK** code. Here  $d_e/L_s = \rho_s/L_s = 0.2$ ,  $\Delta' L_s = 23.2/\sqrt{2}$ ,  $v_{ei}/(v_A/L_s) = .02/\sqrt{2}$ ,  $\tau = 1$ , and  $\tau_A \equiv \omega_A^{-1}$ . Results are presented in Alfvénic units. The subscript *DT* stands for “drift-tearing”.

## VI. MICROTEARING MODE

Now we turn our attention to the microtearing mode, meaning that we analyse the case in which  $\Delta' < 0$ . We start with one example that relates this work to previous theories, thus allowing a comparative approach. Following Gladd et al. [28], we neglect the electrostatic potential in Eqs. (14) and (15) and obtain

$$\left(\frac{d^2}{dx^2} - k_y^2\right) A_{\parallel} \approx -i \frac{\omega}{\nu_{ei}} \frac{1}{d_e^2} \frac{1}{1 - i \frac{\omega}{\nu_{ei}}} \hat{\sigma}_e A_{\parallel}. \quad (69)$$

In the case of constant- $A_{\parallel}$ , this equation is matched to ideal MHD in the usual way, hence [28]

$$\Delta' d_e = -i \frac{\omega}{\nu_{ei}} \frac{2}{d_e} \frac{1}{1 - i \frac{\omega}{\nu_{ei}}} \int_0^{\infty} dx \hat{\sigma}_e(x/\delta). \quad (70)$$

The integral dispersion relation (70) can be studied using different electron conductivity models, and performing several subsidiary expansions; in particular for small  $\omega/\nu_{ei}$  [28]. However, in this limit, the growth rate is always a subdominant correction to the stabilising term which is proportional to  $-|\Delta'| d_e$ . The real frequency of the mode can also be found in some subsidiary expansion. For instance, we could consider the limit  $\Delta' d_e \ll 1$ , in analogy to our treatment of Eq. (60). Yet, the analogy between Eq. (70) and (60), or the more general (55), is not only formal, as we are about to show.

Equation (70) has been derived for unmagnetised ions, and neglecting the electrostatic potential in Ohm’s law. Under these circumstances, one can easily convince oneself that the microtearing theory derived using Eq. (70) is a simplified version of the low- $\hat{\beta}_T$  and low- $\delta k_y$  drift-tearing theory just derived in Section (III). In previous works [31, 43, 49], the fundamental frequency of the drift-tearing mode,  $\omega_{DT}/\omega_T$ , solution of  $I_e(\omega_{DT}/\omega_T) = 0$ , has been derived by neglecting the electrostatic potential, therefore replacing

$$I_e = - \int_0^{\infty} ds \frac{F_{\infty} \hat{\sigma}_e(s)}{\left(1 - i \frac{\omega}{\nu_{ei}}\right) F_{\infty} - s^2 \hat{\sigma}_e(s)} \rightarrow - \frac{1}{\left(1 - i \frac{\omega}{\nu_{ei}}\right)} \int_0^{\infty} ds \hat{\sigma}_e(s),$$

which is exactly the same approximation used to derive Eq. (70). Thus, according to these results, for  $\Delta' = 0$ , to leading order, one expects a marginally stable collisionless drift-tearing mode [31, 43, 49]. Since both drift-tearing and

microtearing modes must be derived from the same equation when the electrostatic potential is not neglected, we infer that, if there were any low- $\hat{\beta}_T$  microtearing mode, this must also be marginally stable for negligible collisionality. Furthermore, if there were any instability arising from Eq. (70), a critical  $\hat{\beta}_T$  for instability could be calculated. However, that would be incorrect, since in Eq. (70) the electrostatic potential was neglected. The parameter  $\hat{\beta}_T$  is in fact a measure of the ratio  $A_{\parallel}/\varphi$ . If the electrostatic potential is neglected with impunity, from Eq. (70) it might seem possible to reach arbitrarily large values of  $\hat{\beta}_T$ . We can understand why this is not appropriate, and a new high- $\hat{\beta}_T$  theory is thus required. Firstly, the microtearing is an electromagnetic mode driven by the electron temperature gradient; hence, increasing  $\hat{\beta}_T$  by decreasing  $L_T$  should enhance the instability. Therefore, we expect the growth rate to be subdominant in a low- $\hat{\beta}_T$  theory. Secondly, from Eq. (55) we notice the following. On the LHS we have the tearing mode driving term (stabilizing for  $\Delta'\rho_i < 0$ ). If one replaces  $\Delta' = -2k_y$ , and approximates  $\omega \approx \omega_T$  in the definition of  $\delta$ , it immediately becomes clear that the whole LHS is proportional to the inverse power of  $\hat{\beta}_T$ . Therefore, stabilising terms can be subdominant for high values of  $\hat{\beta}_T$ . However, the higher the  $\hat{\beta}_T$ , the more important the non-constant- $A_{\parallel}$  contribution on the RHS of Eq. (55) (the  $C$  term). This fact explains why, in microtearing simulations [28], the constant- $A_{\parallel}$  approximation was found to be violated, and invalidates analytical theories that relied on this approximation. We conclude that, at high  $\hat{\beta}_T$  (necessary for instability) the constant- $A_{\parallel}$  approximation cannot be used.

In order to capture these aspects, a new high- $\hat{\beta}_T$  theory is needed. Nevertheless, before embarking on this task, we find it useful to artificially suppress all the stabilising terms on the LHS of Eq. (55) (which was derived in a low- $\hat{\beta}_T$  expansion), and solve for  $I_e = 0$  for kinetic electrons and finite collisionality. Indeed, we are expecting to observe a microtearing mode with a growth rate which is a nonmonotonic function of the collision frequency. Even if Eq. (55) is not enough to describe an unstable microtearing mode, the solution of  $I_e = 0$  can still give an eigenvalue with an imaginary part which is a nonmonotonic function of collisionality when the tearing mode is marginally stable, that is when  $\Delta'$  reaches values that are solutions of the equation  $B = 0$ . We could therefore study the properties of this maximum, which would eventually give an instability for large enough  $\hat{\beta}_T$ . In other words: Eq. (55) surely captures some salient features of a *stable* microtearing mode, but we need a new theory to describe an unstable one in a consistent way.

We then solve for  $I_e = 0$ . Results are shown in Fig. (6). The real frequency is close to the familiar value  $\omega \approx 0.5\omega_T$  [31, 43, 49], and the “growth rate” is indeed a non-monotonic function of collisionality [see Fig. (6)]. This non-monotonic dependence of the imaginary part of the eigenvalue with collisionality remains the invariant feature of the mode in the literature [8, 15, 16, 18]. We verified that the instability is present only if we include electron inertia

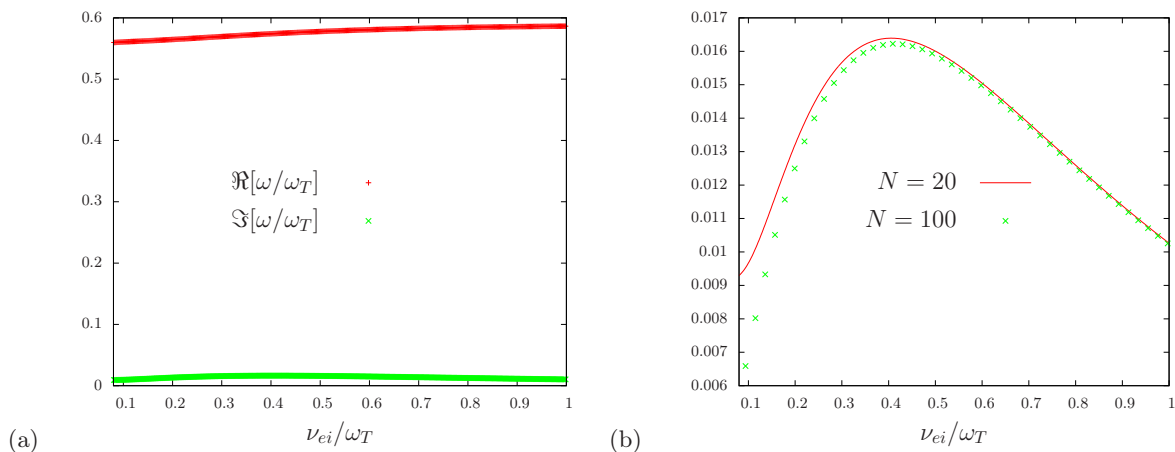


Figure 6: The solution of  $I_e = 0$ . Real and imaginary part (a), and a zoom of the imaginary part (b).

[see Fig (7)], the electrostatic potential and if we consider non-isothermal electrons, that is  $N > 2$ . Electron inertia is neglected when the integral  $I_e$  is replaced with

$$I_e \rightarrow - \int_0^\infty ds \frac{F_\infty \tilde{\sigma}}{F_\infty - s^2 \tilde{\sigma}},$$



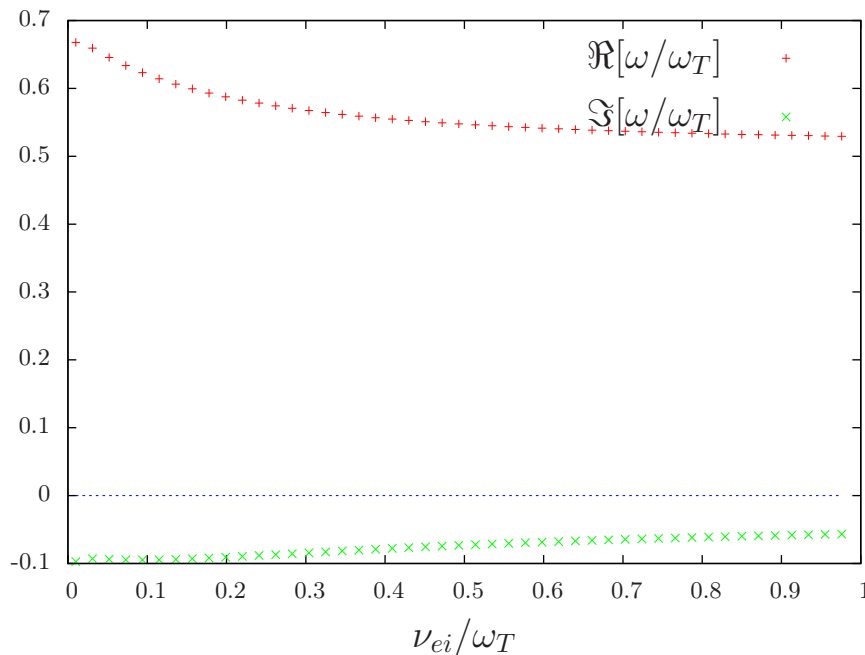


Figure 7: Solution of  $I_e = 0$ . Real and imaginary part of the eigenvalue for  $d_e \equiv 0$  in Ohm's law. Here  $N = 100$ . The mode is always stable.

where

$$\tilde{\sigma} = \frac{\sigma_0 + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} s^2}{1 + \left[ 3 + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} \right] s^2 + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} s^4}. \quad (71)$$

Electron inertia is also required to obtain a frequency that tends to  $\omega \approx 0.5\omega_T$  in the weakly collisional limit [see Fig. (6)]. While the nonmonotonic shape of the growth rate is determined by electron kinetics, the coupling to the kinetic Alfvén wave [ $A_{\parallel}$  non-constant, the second term in Eq. (57)] determines whether the peak will reach positive values. This is the subject of the next Section.

### A. High- $\hat{\beta}_T$ theory

In the previous section, we identified the reason for obtaining a low- $\hat{\beta}_T$  mode with an imaginary part which is a nonmonotonic function of collisionality. To have microtearing instability, high  $\hat{\beta}_T$ 's are needed to overcome the stabilising effect of a negative  $\Delta'$ . As  $\hat{\beta}_T$  is increased, breaking of the constant- $A_{\parallel}$  approximation occurs, and the term  $C$  in Eq. (55) is no longer unity. Therefore, the low- $\hat{\beta}_T$  theory described by Eq. (55) is not sufficient to describe an unstable microtearing mode, even if it captures some salient features. A high- $\hat{\beta}_T$  theory, with non-constant- $A_{\parallel}$ , is a straightforward modification of that presented by Connor et al. [34]. This theory resembles a previous theory formulated by Drake et al. [40]. However, in their work, Connor et al. proved that a high- $\hat{\beta}_T$  theory matches exactly onto a low- $\hat{\beta}_T$  one, as in Eq. (55), when an appropriate "screening factor" is taken into account.

We present such a high- $\hat{\beta}_T$  theory, and give an explicit analytic expression for the growth rate of the microtearing mode that matches the low- $\hat{\beta}_T$  theory. We rewrite Eq. (34) in the following way

$$\hat{\sigma}_e = \frac{\sigma_0 + \sigma_1 s^2}{1 + d_0 s^2 + d_1 s^4}, \quad (72)$$

with

$$\sigma_0 = 1 - \hat{\omega}^{-1}, \quad (73)$$

$$\sigma_1 = \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}}, \quad (74)$$

$$d_0 = \left[ \frac{3}{1 - i\frac{\omega}{\nu_{ei}}} + \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}} \right], \quad (75)$$

and

$$d_1 = \frac{1}{1 - i\frac{\omega}{\nu_{ei}}} \frac{3\nu_{ei}}{-i\omega + 3\nu_{ei} + \frac{4}{2} \frac{k_{\parallel}^2 v_{the}^2}{\Omega(N)}}. \quad (76)$$

We first consider the  $\hat{\beta}_T \gg 1$  limit, with  $\hat{\omega}^2 \sim \hat{\beta}_T^{-1} \ll 1$ , so that  $\hat{\omega}^2 \hat{\beta}_T \sim \mathcal{O}(1)$ . In this limit, we expect diamagnetic effects to screen the resonant layer, thus preventing reconnection. When approaching the ion region, for  $s = x/\delta \gg 1$ , the equation for the current in the electron region [Eq. (43)] becomes

$$\frac{d^2}{ds^2} \bar{J} = \hat{\omega}^2 \hat{\beta}_T \frac{F_{\infty}}{F_{\infty} - 1} \frac{\sigma_0 + \sigma_1^{\infty} s^2}{\sigma_1^{\infty} s^4} \bar{J}, \quad (77)$$

with

$$\bar{J} = \frac{1 - i\frac{\omega}{\nu_{ei}} + \left[ \left(1 - i\frac{\omega}{\nu_{ei}}\right) d_0 - \frac{\sigma_0}{F_{\infty}} \right] s^2 + \frac{F_{\infty}-1}{F_{\infty}} \sigma_1^{\infty} s^4}{\sigma_0 + \sigma_1^{\infty} s^2} J, \quad (78)$$

and

$$\sigma_1^{\infty} = \lim_{s \rightarrow \infty} \sigma_1. \quad (79)$$

We do not need to calculate explicitly the “screening factor” here. We choose the solution of Eq. (77) that is small (completely screened by diamagnetic effects) at  $s = 0$  [34, 35]

$$J = \frac{\sigma_0 + \sigma_1 s^2}{1 - i\frac{\omega}{\nu_{ei}} + \left[ \left(1 - i\frac{\omega}{\nu_{ei}}\right) d_0 - \frac{\sigma_0}{F_{\infty}} \right] s^2 + \frac{F_{\infty}-1}{F_{\infty}} \sigma_1 s^4} \sqrt{\frac{s}{s_t}} K_{\mu} \left( \frac{s_t}{s} \right), \quad (80)$$

where  $K_{\mu}$  is the modified Bessel function,

$$s_t^2 = \left( \mu^2 - \frac{1}{4} \right) \frac{\sigma_0}{\sigma_1^{\infty}} = \hat{\beta}_T \hat{\omega} \frac{Z/\tau}{Z/\tau + 1}, \text{ for } \nu_{ei} \gg \omega, \quad (81)$$

and  $\mu$  is defined as in the low- $\hat{\beta}_T$  case,  $1/4 - \mu^2 = \hat{\omega}^2 \hat{\beta}_T F_{\infty} / (F_{\infty} - 1)$ , but we do not approximate  $\mu \neq 1/2 + \delta\mu$ , with  $\delta\mu \ll 1$ , unlike in the low- $\hat{\beta}_T$  case. Solution (80) has to be matched to the ion region solution. In the high- $\hat{\beta}_T$  regime, a Padé approximant [33, 34] is adequate to describe the ion response [34]. Then, the coefficients  $\hat{a}_{\pm}$  in Eq. (40) are [34]

$$\frac{\hat{a}_{-}}{\hat{a}_{+}} = \left[ \frac{1}{2} \left( 1 + \frac{Z}{\tau} \right) \right]^{-\mu} \frac{\Gamma(-\mu)}{\Gamma(\mu)} \frac{\frac{1}{\Gamma^2(-\frac{\mu}{2} - \frac{1}{4})} - \frac{\pi\sqrt{2}}{8} \frac{\hat{\beta}_T}{\Delta' \rho_{\tau}} \frac{1}{\Gamma^2(-\frac{\mu}{2} - \frac{5}{4})}}{\frac{1}{\Gamma^2(\frac{\mu}{2} - \frac{1}{4})} - \frac{\pi\sqrt{2}}{8} \frac{\hat{\beta}_T}{\Delta' \rho_{\tau}} \frac{1}{\Gamma^2(\frac{\mu}{2} - \frac{5}{4})}}, \quad (82)$$

with  $\rho_{\tau} = \sqrt{\frac{1}{2} \left( 1 + \frac{Z}{\tau} \right)} \rho_i$ . After using Eq. (47), with  $b_{\pm}$  calculated using the large asymptotic expansion of solution (80), we obtain the high- $\hat{\beta}_T$  dispersion relation<sup>1</sup>

$$e^{i\frac{\pi}{2}\mu} \left( \frac{2\sigma_1^{\infty}}{\frac{1}{4} - \mu^2} \frac{\rho_{\tau}^2}{\delta_0^2} \right)^{\mu} = \frac{\mu + \frac{1}{2}}{-\mu + \frac{1}{2}} \frac{\Gamma^2(-\mu)}{\Gamma^2(\mu)} \frac{\mathcal{D} - \cot \left[ \pi \left( \frac{1}{4} + \frac{\mu}{2} \right) \right]}{\mathcal{D} - \cot \left[ \pi \left( \frac{1}{4} - \frac{\mu}{2} \right) \right]}, \quad (83)$$

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[1] After some algebra, it is easy to show that this dispersion relation is the equivalent of Eq. (11) of Pegoraro et al. [33]. We keep it in the form of Ref. [35] to stress the fact that the Fourier space analysis of the ion region [33, 34] gives the same result as the real space analysis.

where

$$\mathcal{D} = \frac{2}{\pi} \Delta' \rho_\tau \frac{\Gamma\left(\frac{5}{4} - \frac{\mu}{2}\right) \Gamma\left(\frac{5}{4} + \frac{\mu}{2}\right)}{\Gamma\left(\frac{3}{4} - \frac{\mu}{2}\right) \Gamma\left(\frac{3}{4} + \frac{\mu}{2}\right)}. \quad (84)$$

This is the large  $\eta_e \gg 1$  ( $\omega_{*e} \ll 1$ ) limit of Eq. (81) of Ref. [34] for kinetic electrons, when the screening factor is set to unity. In the collisional limit,  $\nu_{ei} \gg \omega$ , the low  $\hat{\beta}_T$  limit of Eq. (83) connects to the low  $\hat{\omega}^2 \ll 1$  limit of Eq. (55). Thus, from Eq. (60), using  $\Delta' \rightarrow -2k_y$ , and  $\hat{\omega}^2 \ll 1$ , we have

$$e^{i\frac{\pi}{4}} \sqrt{2 \frac{\tau}{Z} \frac{\nu_{ei}}{\omega_T}} \frac{\delta_*}{\hat{\beta}_T^2 \rho_\tau} \frac{2k_y \rho_\tau \hat{\beta}_T}{\pi} + \frac{\omega}{\omega_T} \left( 1 + k_y \rho_\tau \hat{\beta}_T \frac{1/\tau + 1}{1/\tau} \frac{\omega^2}{\omega_T^2} \right) = 0. \quad (85)$$

When the constant- $A_\parallel$  approximation fails,  $C > 1$  in Eq. (55), we have

$$\hat{\beta}_T > \frac{1}{1 + \tau} \frac{\pi}{2} \frac{1}{k_y \rho_\tau \omega^2 / \omega_T^2}, \quad (86)$$

therefore, for

$$\frac{2}{\pi} (k_y \rho_\tau)^{2/3} \left( \frac{d_e}{\rho_\tau} \right)^{1/3} \left( \frac{2\tau}{\sqrt{\hat{\beta}_T}} \frac{\nu_{ei}}{v_{the}/L_s} \right)^{1/3} > 1, \quad (87)$$

and  $\gamma^2 > \omega_0^2$ , with  $\omega = \omega_0 + i\gamma$ , we find an unstable mode

$$\omega = \omega_T \left( \frac{1}{3} + i \right) \left( 2\tau \frac{\nu_{ei}}{\omega_T} \right)^{1/6} \left( \frac{1}{1 + \tau} \right)^{1/3} \left( \frac{\delta_*}{\hat{\beta}_T^2 \rho_\tau} \right)^{1/3}, \quad (88)$$

the large- $\eta_e$  semicollisional microtearing mode. When electron inertia is the relevant electron scale, we replace  $\nu_{ei}$  by  $-i\omega$  in Eq. (88), to obtain

$$\omega = \omega_T (2\tau)^{1/5} \left( \frac{1}{3} + i \right)^{6/5} e^{i\frac{\pi}{5}} \left( \frac{1}{1 + \tau} \right)^{2/5} \left( \frac{\delta_*}{\hat{\beta}_T^2 \rho_\tau} \right)^{2/5}, \quad (89)$$

the “weakly-collisional” large- $\eta_e$  microtearing mode. Notice that these solutions are based on the expansion in  $\epsilon \sim \omega_0^2/\gamma^2 \sim 1/9$ .

A short digression on the finite  $\eta_e$  theory of Connor et al. [34] is now required. Equation (85) is the equivalent of Eq. (45) of Ref. [34]. In the large  $\hat{\beta}_N = \hat{\beta}_T L_T^2 / L_{n0e}^2$  limit, the real frequency of the drift tearing mode,  $\omega \approx \omega_{*e}(1 + 0.7\eta_e)$ , is driven to  $\omega \approx \omega_{*e}$ . More precisely, for finite  $\eta_e$ , when  $\omega \approx \omega_{*e}$ , equation (85) gives the solution with real frequency

$$\frac{\omega}{\omega_{*e}} = 1 + \frac{2}{1 + \frac{Z}{\tau}} \left( \frac{1}{k_y \rho_i \hat{\beta}_N} \right)^2 + \frac{8\sqrt{2} \sqrt{d_{0,c} + 2\sqrt{d_{1,c}} - 1.71\eta_e/(1 + Z/\tau)}}{1.71\eta_e/(1 + Z/\tau)} \frac{1}{k_y \rho_i \hat{\beta}_N^3} \frac{\delta_N}{\rho_i}, \quad (90)$$

and growth rate

$$\begin{aligned} \frac{\gamma}{\omega_{*e}} = & -\frac{16}{\sqrt{2}\pi} \frac{\sqrt{d_{0,c} + 2\sqrt{d_{1,c}} - 1.71\eta_e/(1 + Z/\tau)}}{1.71\eta_e/(1 + Z/\tau)} \frac{1}{k_y \rho_i \hat{\beta}_N^3} \frac{\delta_N}{\rho_i} \times \\ & \left[ 1 + \frac{1}{\sqrt{2}\pi} \frac{\delta_N}{\rho_i} \frac{k_y \rho_i}{\hat{\beta}_N} \frac{\sqrt{d_{0,c} + 2\sqrt{d_{1,c}} - 1.71\eta_e/(1 + Z/\tau)}}{1.71\eta_e/(1 + Z/\tau)} \right]. \end{aligned} \quad (91)$$

Here  $\delta_N = \delta(\omega = \omega_{*e})$ , while  $d_{0,c}$  and  $d_{1,c}$  are numerical factors that depend on the model collisional operator. They are  $d_{0,c} = 5.08$ , and  $d_{1,c} = 2.13$  in the Braginski collisional model [34, 35, 50], and  $d_{0,c} = 4$ , and  $d_{1,c} = 1$  in our present model. Thus, the growth rate derived in Eq. (88) is equivalent to Eq. (91) in the large  $\eta_e$  limit, and the critical  $k_y \rho_\tau$  in Eq. (87) defines the value above which the constant- $A_\parallel$  approximation breaks down. This is the necessary

condition for instability in the flat density limit. When both  $\omega_{*e}$  and  $\eta_e$  are finite, the root in Eq. (90) is driven unstable for

$$\eta_e > \eta_e^{MT} = \frac{d_{0,c} + 2\sqrt{d_{1,c}}}{1.71/(1 + Z/\tau)}. \quad (92)$$

We call this the finite- $\eta_e$  microtearing mode. The mode is destabilised by electron temperature gradients when the parameter  $\hat{\beta}_N$  or  $\hat{\beta}_T$  are sufficiently large to break the constant- $A_{\parallel}$  approximation. For finite density gradients, the mode is destabilised when  $\eta_e > \eta_e^{MT}$ , but an increasingly large temperature gradient has a *stabilising* effect, since the growth rate scales like  $\gamma \sim \omega_{*e}\eta_e^{-1/2}$ . When density gradients are negligible compared to temperature gradients, the residual mode of Eq. (88) remains.

In this analysis, an energy dependent collision frequency is not required. We show why this is the case. Let us rewrite the electron drift-kinetic Eq. (A1) for  $\delta f_e = h_e + e\varphi/T_{0e}$ , and use a Lorentz collision operator  $\nu\partial_{\xi}(1 - \xi^2)\partial_{\xi}$ , where  $\xi = v_{\parallel}/v$ , and  $\nu = \nu_{ei}\hat{v}^{-3\alpha}$  is the energy-dependent collision frequency. The equation can then be solved by using an expansion in Legendre polynomials [25]. After truncating to first order, calculating the parallel electron current, and using Ampere's law, in the  $\nu_{ei}/\omega \gg 1$  limit one obtains the following electron conductivity

$$\sigma_{DL} \propto \int_0^{\infty} d\hat{v} \hat{v}^{4+3\alpha} \frac{1 + \hat{v}^{3\alpha}\omega/\nu_{ei}}{i\omega\nu_{ei} + \frac{2}{3}s^2\hat{v}^{2+3\alpha}} e^{-\hat{v}^2}. \quad (93)$$

We now evaluate Eq. (70) using Eq. (93). We first calculate the spatial integral, and then the velocity-space integral, to obtain

$$\Delta' \propto a_1 \left(1 - \frac{\omega_{*e}}{\omega}\right) - a_2 \frac{\omega_T}{\omega} + i \frac{\omega}{\nu_{ei}} \left[ a_3 \left(1 - \frac{\omega_{*e}}{\omega}\right) - a_4 \frac{\omega_T}{\omega} \right], \quad (94)$$

where  $a_i$  are real, positive constant functions of  $\alpha$ . The fundamental frequency of the mode is then determined perturbatively in  $\delta/\rho_i$ , as we did for Eq. (60), solving for

$$a_1 \left(1 - \frac{\omega_{*e}}{\omega}\right) - a_2 \frac{\omega_T}{\omega} = 0. \quad (95)$$

When the collision frequency is energy-independent,  $\alpha = 0$ , we obtain the familiar solution

$$\omega = \omega_{*e}(1 + \eta_e/2). \quad (96)$$

In this case,  $a_4 = a_2$ , and  $a_3 = a_1$ , therefore the destabilising collisional term on the RHS of Eq. (94) cancels exactly when evaluated at  $\omega = \omega_{*e}(1 + \eta_e/2)$ . When  $\alpha = 1$ , the destabilising term remains, even when evaluated at the value of  $\omega$  which solves for Eq. (95). We immediately notice that the  $\eta_e$ -term on the RHS of Eq. (94) does not cancel if the fundamental frequency of the mode is *not*  $\omega = \omega_{*e}(1 + \eta_e/2)$ . When  $\omega > \omega_{*e}(1 + \eta_e/2)$ , the  $\eta_e$ -term is actually stabilising. However, when the constant- $A_{\parallel}$  approximation fails,  $\omega \approx \omega_{*e}$  [see Eq. (90)] the destabilising term remains. We want to stress that, from gyrokinetic simulations, we found that the energy-dependent collision frequency can still play an important role in destabilizing an electron temperature gradient driven mode. The simplest way to include this effect in our theory is to modify the coefficient  $\sigma_0$  in Eq. (34) by using a Padé approximant

$$\sigma_0 \rightarrow \sigma_0^P = 1 - \frac{\omega_T}{\omega} \left( 1 + \frac{\nu_{ei}/\omega}{1 - i\nu_{ei}^2/\omega^2} \right) \quad \text{for } \nu_{ei} \sim \omega, \quad (97)$$

so that  $\sigma_0 \rightarrow 1 - \frac{\omega_T}{\omega}$  for  $\nu_{ei} \ll \omega$ , and  $\sigma_0 \rightarrow 1 - \frac{\omega_T}{\omega} (1 + i\omega/\nu_{ei})$  for  $\nu_{ei} \gg \omega$ .

## B. Collisionless limits and ETG

While we have been able to formulate the problem of microtearing modes and relate it to the physics of the drift-tearing mode, the relationship between these reconnecting modes and the collisionless electrostatic ETG (which shares the same drive) still remains unclear. In order to gain some insight into this aspect of the theory, we show firstly that our Eqs. (3)-(4)-(6) support the electrostatic collisionless ETG in 3D shearless geometry. Secondly, we consider the case of finite shear. We return to the eigenvalue equations (14) and (15), and solve them in a sound expansion, keeping  $\hat{\beta}_T$  arbitrary, but considering what we call the “deeply-unstable ETG” ordering, that is

$$\frac{k_{\parallel}^2 v_{the}^2}{\omega^2} = s_{cl}^2 \sim \frac{\omega}{\omega_T} \sim \varepsilon \ll 1.$$

The finite magnetic shear case gives a number of marginally stable modes. We explain how this result relates to previous works [51]. We then perform a series of numerical simulations with the gyrokinetic code **GS2**, and find that the only electron temperature gradient driven collisionless reconnecting mode is a tearing parity ETG, which is mostly electrostatic.

### 1. Collisionless ETG in KREHM: no shear

We Fourier transform the  $\nu_{ei} \rightarrow 0$  limit of Eq. (43), and use Eq. (39), to obtain

$$\frac{k_{\perp}^2 d_e^2}{2} = \left\{ \zeta^2 - \frac{\tau}{Z} \frac{k_{\perp}^2 d_e^2}{2} \right\} \left\{ 1 + \zeta Z(\zeta) + \frac{1}{4} \frac{\omega_T}{\omega} \zeta [Z(\zeta) - 2\zeta(1 + \zeta Z(\zeta))] \right\}, \quad (98)$$

where  $\zeta = \omega/(k_z v_{the})$ . When  $\hat{\omega} = \omega_T/\omega \equiv 0$ , this is Eq. (B.12) of Ref. [36]. In this case, for  $\zeta \ll 1$ , Eq. (98) gives a damped kinetic Alfvén wave [36]. For  $k_{\perp}^2 d_e^2 \gg \zeta^2 = \omega^2/(k_z^2 v_{the}^2) \sim \omega_T/\omega \gg 1$ , we obtain<sup>1</sup>

$$1 \approx -\frac{\tau}{Z} \frac{1}{4} \frac{k_z^2 v_{the}^2 \omega_T}{\omega^3} \left\{ 1 - 2\zeta^5 i \sqrt{\pi} \sigma e^{-\zeta^2} \right\}, \quad (101)$$

where  $\sigma = 0$  for  $\Im[\zeta] > |\Re[\zeta]|^{-1}$ ,  $\sigma = 1$  for  $\Im[\zeta] < |\Re[\zeta]|^{-1}$ , and  $\sigma = 2$  for  $\Im[\zeta] < -|\Re[\zeta]|^{-1}$ . The asymptotic expansion for large  $\zeta$  is in the complex plane, and a direction for the limit has to be specified. If we choose  $\arg \zeta = \pi/3$ , there is always a critical  $\zeta_c$  above which  $\Im[\zeta] > |\Re[\zeta]|^{-1}$  and  $\sigma = 0$ . Therefore, for

$$\frac{\tau}{Z} \frac{1}{4} \frac{\omega_T^2}{k_z^2 v_{the}^2} > \left( \frac{4\sqrt{3}}{3} \right)^{3/2}, \quad (102)$$

we find the electrostatic collisionless ETG mode

$$\omega_0^3 \approx -\frac{\tau}{Z} \frac{1}{4} k_z^2 v_{the}^2 \omega_T. \quad (103)$$

### 2. Electron sound expansion: finite shear

In the case of finite shear and  $k_z \equiv 0$ , the analysis is more complex. We write Eq. (43) for the magnetic potential (from now on  $s_{cl} \equiv s$ )

$$\frac{F_{\infty} - s^2 \hat{\sigma}_L}{F_{\infty} \hat{\sigma}_L} \left( \frac{d^2 A_{\parallel}}{ds^2} - k_y^2 \delta^2 A_{\parallel} \right) = -\hat{\beta}_T \hat{\omega}^2 A_{\parallel}. \quad (104)$$

In the limit  $k_{\parallel}^2 v_{the}^2/\omega^2 = s^2 \sim \frac{\omega}{\omega_T} \sim \varepsilon \ll 1$ , at scales  $s \sim (\omega/\omega_T)^{1/2} \ll 1$ , we obtain

$$\frac{d^2 A_{\parallel}}{d\xi^2} = \delta_{MT}^2 k_y^2 A_{\parallel} - \frac{A_{\parallel}}{1 + \alpha^2 \xi^2}, \quad (105)$$

where  $\alpha^2 = \frac{\tau}{Z} / \left( \hat{\beta}_T \hat{\omega}^2 \right)$ , and

$$\xi^2 = \frac{1}{2} \hat{\omega} \hat{\beta}_T s^2 \rightarrow \xi = \sqrt{\frac{\hat{\beta}_T}{2}} \frac{k_y v_{the}}{\sqrt{\omega \omega_T}} \frac{x}{L_s}. \quad (106)$$

---

[1] The electrostatic limit of our model is found for  $k_{\perp} d_e \gg 1$ . Indeed, by using Ampere's law and the electron continuity equation, we find that

$$k_{\perp}^2 d_e^2 \frac{e}{m_e c} A_{\parallel} = u_{\parallel, e} \sim \frac{\omega}{k_{\parallel}} \frac{e \varphi}{T_{0e}}. \quad (99)$$

When the parallel electron dynamics is included,  $\omega \sim k_{\parallel} v_{the}$ , we obtain

$$\frac{v_{the}}{c} A_{\parallel} \sim \frac{1}{k_{\perp}^2 d_e^2} \varphi, \quad (100)$$

thus, the electromagnetic component of the electron gyrokinetic potential,  $\chi = \varphi - (v_{\parallel}/c) A_{\parallel}$ , is negligible in the  $k_{\perp}^2 d_e^2 \gg 1$  limit.

Equation (106) shows us that there is a new scale,

$$\delta_E = \sqrt{\frac{2}{\hat{\beta}_T} \frac{\sqrt{\omega\omega_T}}{k_y v_{the}}} L_s, \quad (107)$$

that determines the width of the mode. The electron sound expansion is valid for

$$s \sim \sqrt{\omega/\omega_T} \ll 1. \quad (108)$$

When  $\omega \approx \omega_T$ , the new scale  $\delta_E$  coincides with the electron inertial scale  $d_e$ , and Eq. (105) is no longer valid. Equation (105) corresponds to Eq. (3) of Ref. [51]. It has been used to prove the existence of a collisionless microtearing mode when finite  $k_y d_e$  is considered in Ohm's law [51]. We now solve it exactly in the two asymptotic limits  $k_y^2 \delta_E^2 \ll 1$ , and  $k_y^2 \delta_E^2 \sim 1$ .

*Small  $\delta_E^2 k_y^2$  solution* The even parity solution of Eq. (105), for  $\delta_E^2 k_y^2 \ll 1$ , is

$$A_{\parallel}(\xi^2) = {}_2F_1\left(-\frac{1}{4} - \mu, -\frac{1}{4} + \mu; \frac{1}{2}; -\alpha^2 \xi^2\right), \quad (109)$$

where  ${}_2F_1$  is the hypergeometric function, and  $\mu = \sqrt{\alpha^2 - 4}/(4\alpha)$ . We look for an eigenvalue of the form  $\hat{\omega} = \hat{\omega}_0 + i\hat{\gamma}$ , with  $\hat{\gamma} \ll \hat{\omega}_0$ , hence  $\mu < 1/4$ . This forces  $\hat{\beta}_T$  to be small.

For  $\alpha^2 \xi^2 \gg 1$ , when  $1/(\alpha\xi) \sim \delta_E k_y \ll 1$ , the finite wavenumber of the perturbation can be important, and Eq. (105) becomes

$$\frac{d^2 A_{\parallel}}{d\xi^2} = \left(\delta_E^2 k_y^2 - \frac{1}{\alpha^2 \xi^2}\right) A_{\parallel}. \quad (110)$$

The solution of Eq. (110) that decays exponentially at infinity is

$$A_{\parallel} = C \sqrt{\xi} K_{2\mu}(\delta_E k_y \xi),$$

where  $C$  is a constant. After matching to the solution (109), we obtain the following dispersion relation

$$\frac{\Gamma^2(2\mu) \Gamma^2(-\frac{1}{4} - \mu) \frac{1}{4} + \mu}{\Gamma^2(-2\mu) \Gamma^2(-\frac{1}{4} + \mu) \frac{1}{4} - \mu} = \left(\frac{\omega^3}{2\tau v_{the}^2 / L_s^2 \omega_T}\right)^{2\mu}. \quad (111)$$

*Relation with the low- $\hat{\beta}_T$  theory* In the  $\hat{\beta}_T \ll 1$  limit, Eq. (111) reduces to

$$\left(\mu - \frac{1}{4}\right) \pi \sqrt{\frac{1}{2} \frac{\tau}{Z} \frac{1}{\hat{\omega}}} = -\hat{\omega} \delta_* k_y, \quad (112)$$

where we recall  $\delta_* = \omega_T / (k_y v_{the}) L_s$ . It is easy to verify that, for  $\Delta' = -2k_y$ , Eq. (112) is exactly

$$\left(\frac{\Delta' \delta_*}{\pi \hat{\beta}}\right)^2 = \frac{1}{2} \hat{\omega} \frac{Z}{\tau}. \quad (113)$$

We must derive this result from the  $\omega/\omega_T \ll 1$  limit of (55), which was derived in a low- $\hat{\beta}_T$  expansion. When ions are unmagnetised, and the ideal MHD drive is not too large (constant- $A_{\parallel}$  approximation),  $C \approx 1$ , we have

$$\sqrt{2 \frac{\nu_{ei}}{\omega_T}} e^{-i\frac{\pi}{4}} \frac{\delta_* \Delta'}{\pi \hat{\beta}_T} \hat{\omega}^{1/2} = -\frac{2}{\pi} \hat{\omega}^2 \int_0^\infty ds \frac{F_\infty \hat{\sigma}_e}{F_\infty \left(1 - i \frac{\hat{\omega}}{\nu_{ei}}\right) - s^2 \hat{\sigma}_e}. \quad (114)$$

Computing the integral on the RHS is particularly easy using the sound expansion limit [51]  $k_{\parallel} v_{the} / \omega \equiv s \ll 1$ . Let us write,

$$I_e = -\frac{Z}{\tau} \int_0^\infty ds \frac{1 - \hat{\omega}^{-1}}{\left(1 - i \frac{\hat{\omega}}{\nu_{ei}}\right) \frac{Z}{\tau} + s^2 (1 - \hat{\omega}^{-1})}. \quad (115)$$

Within this expansion in  $s \ll 1$ , we consider  $Z/\tau \ll 1$  in order to keep the quadratic term  $s^2(1 - \hat{\omega}^{-1}) \sim Z/\tau \ll 1$  in the denominator of the integrand, which continues to remain convergent in all our subsidiary expansions. Essentially, the electrostatic potential is *never* neglected and the current is *always* proportional to the electric field in the electron region.

Then, we obtain

$$I_e = -\frac{\pi}{2} \sqrt{\frac{Z}{\tau} \left(1 - \frac{1}{\hat{\omega}}\right) / (1 - i\omega/\nu_{ei})},$$

and we obtain the following dispersion relation

$$\sqrt{2 \frac{\nu_{ei}}{\omega_T}} e^{-i\frac{\pi}{4}} \frac{\delta_* \Delta'}{\pi \hat{\beta}_T} \hat{\omega}^{1/2} = -\hat{\omega}^2 \sqrt{\frac{Z}{\tau} \frac{1 - \hat{\omega}^{-1}}{1 - i\frac{\omega}{\nu_{ei}}}}. \quad (116)$$

In the limit  $\frac{1}{\sqrt{N}} \sim \frac{Z}{\tau} \ll \frac{\nu_{ei}}{\omega} \ll 1$ , the collision frequency exactly cancels and we obtain

$$-\left(\frac{\Delta' \delta_*}{\pi \hat{\beta}}\right)^2 = \frac{1}{2} \hat{\omega}^2 \frac{Z}{\tau} \left(1 - \frac{1}{\hat{\omega}}\right). \quad (117)$$

One can get the same result by using the truly collisionless equation (104), and noticing that  $\hat{\sigma}_L(0) = -1/2\hat{\sigma}_e(0)$ .

Taking the  $\omega/\omega_T \ll 1$  limit, we obtain Eq. (113). We have thus proved that the limits  $\hat{\omega} \ll 1$  and  $\hat{\beta}_T \ll 1$  commute, and taking *both* limits results in a stable wave oscillating with the frequency

$$\hat{\omega}_0 = \left(\frac{2}{\pi}\right)^2 2 \frac{\tau}{Z} \left(\frac{k_y \delta_*}{\hat{\beta}_T}\right)^2. \quad (118)$$

We conclude that there is no collisionless microtearing mode in this low  $\hat{\beta}_T$  limit.

*Finite  $k_y^2 \delta_E^2 \sim 1$  limit* Let us prove that a localised perturbation of the type  $\exp[-\sigma x^2]$  fails to give reconnection, even if we retain the electrostatic potential and we consider  $k_y^2 \delta_E^2 \sim 1$ .

For  $\alpha^2 \xi^2 \ll 1$ , Eq. (105) reduces to

$$\frac{d^2 A_{\parallel}}{dY^2} = \frac{1}{\alpha^2} \left\{ \hat{b} - 1 + Y^2 \right\} A_{\parallel}, \quad \text{for } Y \ll 1, \quad (119)$$

which implies  $Y^2 \equiv \alpha^2 \xi^2 \sim 1 - \hat{b} \ll 1$ .

If we use the ansatz

$$A_{\parallel} = e^{-\lambda Y^2}, \quad (120)$$

with  $\Re(\lambda) > 0$ , we obtain

$$\lambda = +\frac{1}{2\alpha} = +\frac{1}{2} \sqrt{\frac{\hat{\beta}_T}{\tau/Z}} \hat{\omega}, \quad (121)$$

for an electron mode, and

$$\hat{b} - 1 = -\sqrt{\frac{\tau/Z}{\hat{\beta}_T}} \frac{1}{\hat{\omega}}. \quad (122)$$

This is equivalent to

$$(\hat{\omega} k_y^2 d_e^2 - 1) \hat{\omega} \sqrt{\frac{\hat{\beta}_T}{\tau/Z}} = -1. \quad (123)$$

There are two marginally stable solutions consistent with the conditions  $\hat{b} \approx 1$ , and  $\hat{\omega} \ll 1$ :

$$\hat{\omega}_1 \approx \frac{1}{k_y^2 d_e^2}, \quad (124)$$



and

$$\hat{\omega}_2 \approx 2\sqrt{\frac{\tau}{\hat{\beta}_T}}. \quad (125)$$

In this case, the eigenfunction decays as  $\exp[-\alpha x^2/(2\delta_E^2)]$  for  $x \gg \delta_E/\alpha$ .

In principle, we could impose the tearing-stable MHD boundary condition at large  $x$ , if we match the low- $k$  solution of Eq. (105) to the Fourier transform of  $A_{\parallel}^{MHD} = \exp[-k_y |x|]$ . Let us Fourier transform Eq. (119), to obtain

$$\frac{d^2 A_{\parallel}(\theta)}{d\theta^2} = \left\{ (\hat{b} - 1) + \alpha^2 \theta^2 \right\} A_{\parallel}(\theta), \quad (126)$$

where  $\theta = \alpha^{-1} \delta_{MT} k$  is the Fourier conjugate of  $Y$ . The solution of this equation which is even for all  $\rho = \frac{1}{4} + \frac{\hat{b}-1}{4\alpha}$  is

$$A_{\parallel}(\theta) = A_0 e^{-\frac{1}{2} \frac{\theta^2}{\alpha}} {}_1F_1 \left( \rho; \frac{1}{2}; \frac{1}{\alpha} \theta^2 \right). \quad (127)$$

On the other hand, the Fourier transform of the tearing-stable boundary condition is

$$A_{\parallel} = \frac{A_0}{k_y^2 + k^2} \approx \frac{A_0}{k_y^2} \left( 1 - \frac{k^2}{k_y^2} \right). \quad (128)$$

Thus, after matching these two expressions in the low- $\theta$  limit, we find

$$\hat{b}(\hat{b} - 1) = -2\alpha^2. \quad (129)$$

Explicitly, we have

$$\sqrt{\frac{2}{\hat{\beta}_T}} \frac{\omega}{v_{the}/L_s} k_y d_e - 1 = -\tau \sqrt{\frac{\hat{\beta}_T}{2}} \frac{k_y d_e}{\omega^3/(v_{the}/L_s)^3}. \quad (130)$$

One of the roots that solves for this equation is connected to the unstable root

$$\frac{\omega^4}{(v_{the}/L_s)^4} \approx -\tau \frac{\hat{\beta}_T}{2}, \quad (131)$$

for  $\hat{b} \gg 1$ . However, we cannot accept this root within our analysis, as the derivation is only valid for  $\hat{b} \approx 1$ .

## VII. GYROKINETIC SIMULATIONS IN THE COLLISIONLESS LIMIT

In Section (VI) we proved that, for a tearing-stable configuration and finite collisions, there exists an unstable microtearing mode. This mode becomes stable in the collisionless limit. In the following, we show numerically that, even if the microtearing mode is stabilised when collisions are small, a tearing-parity, mostly electrostatic ETG, can still cause magnetic reconnection.

We performed a series of gyrokinetic simulations using the code **GS2**. We show the cases in which  $\hat{\beta}_T = 0.1$ , and  $\nu_{ei}/(v_{the}/L_s) = 0.09$ . In slab **GS2**, the quantity  $L_s$  defines the variable  $k_p = L_{ref}/(\hat{s}L_s)$ , which implies that our box has parallel length  $L_z = \hat{s}L_s/(2\pi)$ . Here  $\hat{s}$  is the local magnetic shear, and  $L_{ref}$  a reference length. As the parallel scale of the mode depends on  $k_y$ , we need to change  $L_z$  as we change  $k_y$ . To do this we change  $\hat{s}$ . At the same time we change  $k_p$  to keep  $L_s$  constant. Then, for  $\beta_e = m_e/m_i = 1/2345$ , in terms of input parameters, we have  $\hat{\beta}_T = (m_e/2m_i)(L_{ref}/L_T)^2(1/(k_p\hat{s}))^2$ , with  $L_{ref}/L_T = 50$ , and  $k_p\hat{s} = 1.85$ .

For this collisionality, the mode is basically collisionless. For these parameters, we observe that the response of the ions is nearly adiabatic. We stress that we choose  $\beta_e = m_e/m_i = 1/2345$  in order to enforce the ordering used to derive Eqs. (3)-(4)-(6). A finite amount of collisions is required to ensure regular and convergent eigenfunctions. The eigenfunctions are shown in Figs. (10)-(11). Here  $\phi = (L_{ref}/\rho_{ref})Z_{ref}e\varphi/T_{ref}$ , and  $\hat{A}_{\parallel} = v_{the}(L_{ref}/\rho_{ref})Z_{ref}eA_{\parallel}/T_{ref}$ , with  $T_{ref} = T_e$ ,  $m_{ref} = m_i$ , and  $\theta$  is the conventional angle-like variable of ballooning theory [52]. The velocity space resolution is 8 grid points in energy  $E$ , and 16 in the pitch-angle variable  $\lambda = \mu/E$ , where  $\mu = mv_{\perp}^2/(2B)$  is the magnetic moment. The wave-numbers resolved are in the range  $0.1 \leq k_y d_e \leq 30$ . The spectra are in Figs. (8)-(9).

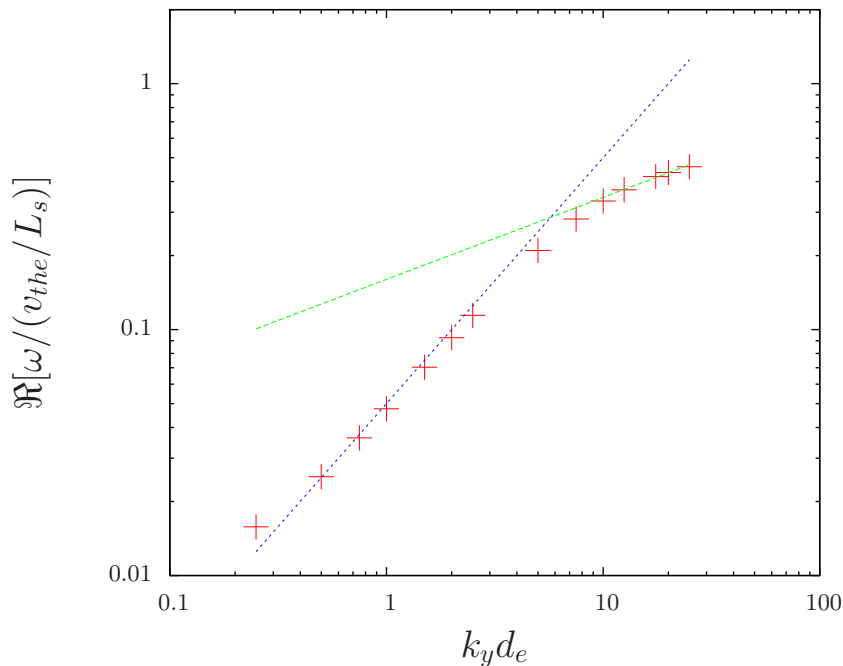


Figure 8: The spectrum of the real part of the eigenvalue obtained from gyrokinetic electromagnetic simulations using the code GS2. Here  $\hat{\beta}_T = 0.1$  and  $\nu_{ei}/(v_{the}/L_s) = 0.09$ . The mode is essentially collisionless. The two lines are the curves  $\omega = 0.05 \omega_T$  and  $\omega/(v_{the}/L_s) = 0.16 [\omega_T/(v_{the}/L_s)]^{1/3}$ .

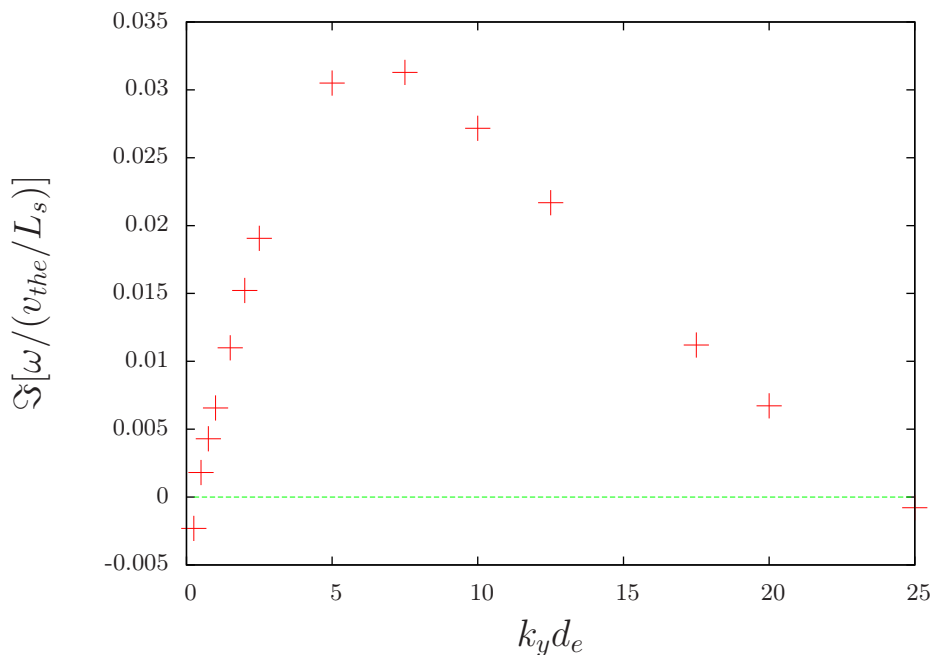


Figure 9: The spectrum of the imaginary part of the eigenvalue obtained from gyrokinetic electromagnetic simulations using the code GS2. Here  $\hat{\beta}_T = 0.1$ , and  $\nu_{ei}/(v_{the}/L_s) = 0.09$ . The mode is essentially collisionless.

After inspection of the real frequency spectrum, we identify two different regimes: a drift-tearing regime  $\omega \sim \omega_T$ , for  $k_y d_e \lesssim 5$ , and an ETG regime  $\omega \sim [(v_{the}/L_s)^2 \omega_T]^{1/3}$ , for  $k_y d_e \gtrsim 10$ , see Fig. (8). In both regimes, it is easy to verify that an electrostatic calculation would give qualitatively the same results as Figs. (8) and (9). We conclude that

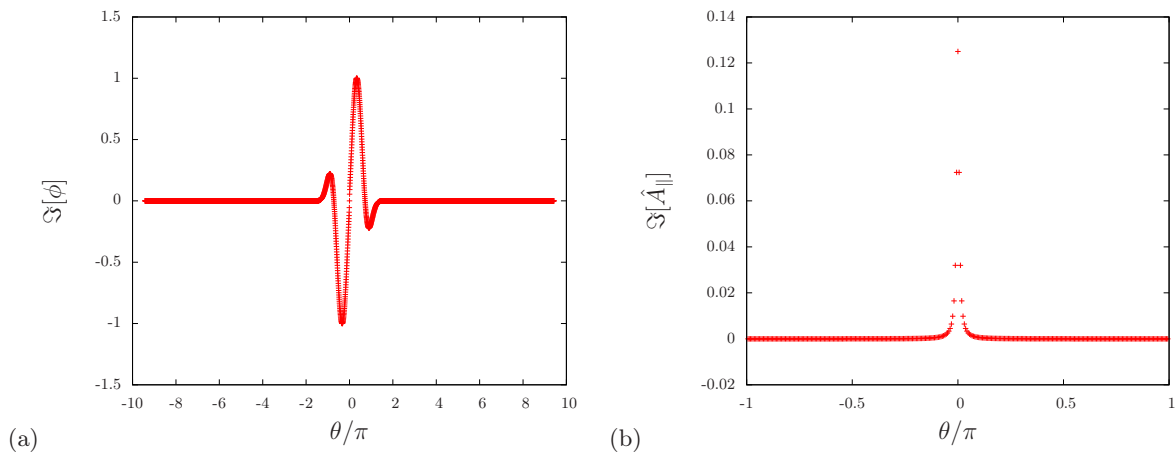


Figure 10: Eigenfunctions in the drift tearing regime:  $k_y d_e = 0.5 \times \sqrt{2}$ . Imaginary part of  $\phi$  (a), and  $\hat{A}_{\parallel}$  (b).

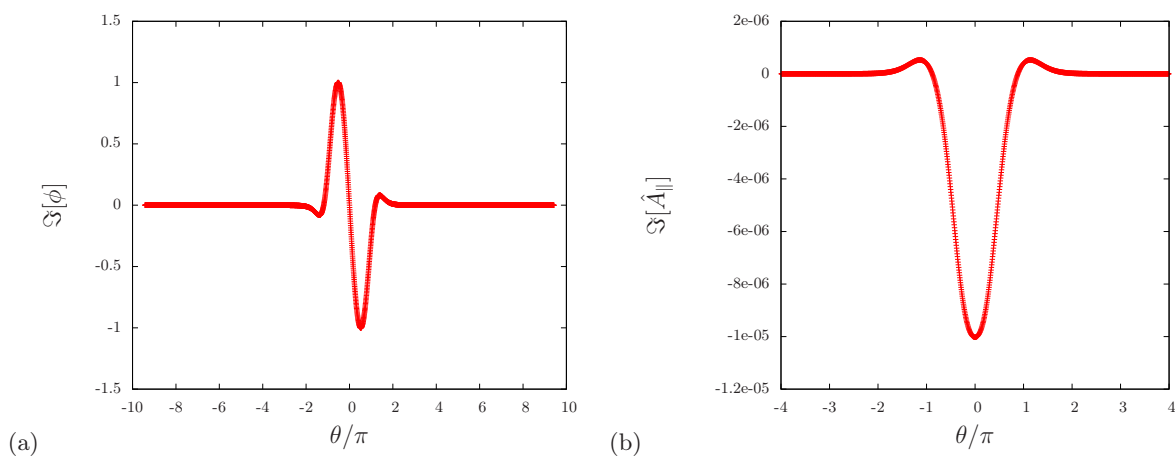


Figure 11: Eigenfunctions in the ETG regime:  $k_y d_e = 25. \times \sqrt{2}$ . Imaginary part of  $\phi$  (a), and  $\hat{A}_{\parallel}$  (b).

the unstable mode presented in Figs. (9) and (8) is a mostly electrostatic tearing parity ETG that generates magnetic reconnection at the electron scale. We notice, however, that the ratio of the amplitudes of the electromagnetic and electrostatic component varies greatly in the different regimes identified here. Indeed,  $\hat{A}_{\parallel}/\phi$  ranges from  $10^{-1}$  in the drift-tearing to  $10^{-5}$  in the ETG regime. The mode is stabilised by collisions (not shown). An analysis similar to that of Connor et al. [53] could predict how much reconnection this mode is actually generating. We leave this question open for the moment, and numerically derive a scaling with  $\hat{\beta}_T$  for the growth rate in the  $\omega \sim \omega_T$  regime. We then choose one simulation from Fig (8) and, for a given  $k_y d_e = 0.1 \times \sqrt{2}$ , we perform a scan in  $\hat{\beta}_T$ , keeping  $\beta_e = m_e/m_i$  constant. The results are shown in Fig. (12). The scaling  $\gamma \sim \omega \sim \hat{\beta}_T^{1/4}$  is observed, even if only for unrealistically large values of  $\hat{\beta}_T$ . To overcome nomenclature issues, we resist the temptation to call the drift-tearing branch a collisionless microtearing mode.

## VIII. CONCLUSIONS AND DISCUSSION

In our efforts to create a new, simple description of fine-scale, kinetic electromagnetic turbulence in plasma, in Sec. (II) we derived a hybrid fluid-kinetic model, for micro and macro reconnecting modes and electron turbulence, which is based on gyrokinetic theory [36]. The model features a gyrokinetic Poisson law for electrostatic perturbations [Eq. (A18) used in Eq. (4)], and the electron parallel momentum equation [Eq. (4)] in the form of a generalised Ohm's law. This is coupled to electron kinetics via electron temperature fluctuations defined by Eqs. (5) and (6).

The derivation of the linearised equation for a sheared magnetic equilibrium [Eqs. (10) and (12)] is carried out

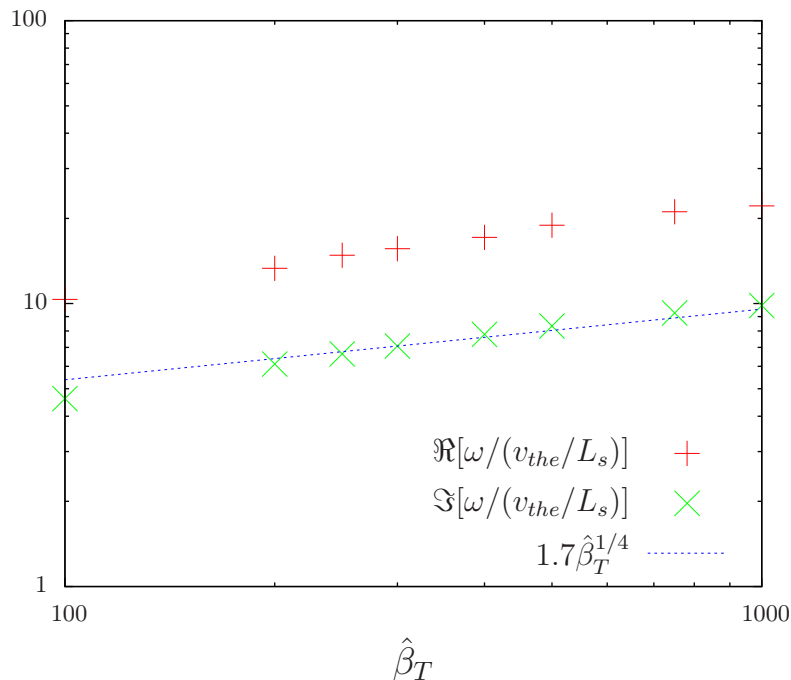


Figure 12: The scaling with  $\hat{\beta}_T$  for the eigenvalue in Figs. (8) and (9) in the regime  $\omega \sim \omega_T$ ,  $k_y d_e = 0.1 \times \sqrt{2}$ .

in Sec. (IIB). In Sec. (IIC) we presented a new solution for the electron kinetic problem [Eq. (33)], based on a Hermite expansion of the electron distribution function [Eq. (18)]. This generated a general electron conductivity [Eq. (34)] which reproduces the well-known highly collisional limit for non-isothermal (semicollisional) electrons [Eq. (27)], and allows us to represent reasonably well the exact collisionless results with a finite number of Hermite moments [Figs. (1)-(2)]. This, in turn, made it possible to include a small but finite amount of collisions in the study of kinetic reconnecting modes, and opened the way to our description of kinetic microtearing modes [Sec. (VI)]. The theory of these is largely based on the theory of tearing modes [Sec. (III)] that features the new electron conductivity, and retains gyrokinetic ions [Eq. (40)]. In the weakly collisional regime, for tearing-unstable configurations, we benchmarked analytical and semianalytical results against a hybrid fluid-kinetic code (**Viriato**) and the gyrokinetic code **AstroGK** [Fig. (5)]. For tearing-stable configurations, our analysis uncovers a series of interesting facts. The inclusion of the electrostatic potential introduces a new fundamental parameter in the theory, which is the ratio of the kinetic electron and ion scales  $d_e/\rho_s$ . Within our formulation, it is particularly easy to see the fundamental role of electron inertia in enabling magnetic reconnection when the mode is unstable [Fig. (7)]. We also see, however, that a finite collisionality is always needed to produce an unstable kinetic microtearing mode, and the mode seems to be marginally stable when the general conductivity is close enough to the exact result at zero collisionality [Eq. (39)]. Our analysis does not require an energy-dependent collision frequency, but does not exclude its importance in driving some electron temperature gradient driven modes. We identified the importance of the breaking of the constant- $A_{\parallel}$  approximation in driving the microtearing mode unstable, derived a high- $\hat{\beta}_T$  theory and gave analytical expressions for the eigenmode in Eqs. (88), (89), and (90)-(91). The critical electron temperature gradient for instability is given in Eq. (92).

When the collisionality becomes smaller, one is justified in asking whether any other micro mode will cause magnetic reconnection, since this kind of microtearing mode does not seem to survive. The answer is yes. In Sec. (VIB1) we show, using the gyrokinetic code **GS2**, that the only electron temperature driven collisionless reconnecting mode we could identify is a tearing parity electromagnetic ETG which happens to reconnect, and is mostly electrostatic. We identified two different branches of this mode: a drift-tearing branch, for which  $\omega \sim \omega_T$ , and an ETG branch, for which  $\omega \sim \omega_T^{1/3} (v_{the}/L_s)^{2/3}$ . In our analysis, we neglected toroidal effects. If we want to consider such effects, our ion response must be modified in order to capture curvature-driven electromagnetic ion modes that can compete with the microtearing mode. Furthermore, if curvature effects were also important for electrons, the Hermite representation of the electron distribution function would be inappropriate, as well as the simple model collision operator used here. Thus, a toroidal extension of our results requires substantial rethinking of some crucial aspects of the theory. While some authors found a toroidal electron temperature gradient driven collisionless reconnecting mode [22], its connection

to the toroidal branch of the electrostatic ETG remains unclear. Other short wavelength temperature gradient driven instabilities are known to exist [54–56], but their relation to our theory is not yet understood. Understanding these aspects, with or without geometry, would require a theory of ETG-driven magnetic reconnection [as was done in Ref. [53] for the ion temperature gradient driven mode]. Only with such a theory, we will be able to quantify how much electromagnetic transport is caused by micro-reconnecting modes, and to investigate the relation between the drift-tearing branch found in our gyrokinetic numerical analysis [Fig. (8)], and the solution of the new dispersion relations Eqs. (55) and (83). The numerical solution of Eqs. (3)–(6) will definitely help us to answer these and other questions.

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### Appendix A: Derivation of the model equations

We derive the equations of the model for a somewhat general case, that is for  $\omega_{*e} = \frac{1}{2}k_y v_{the} \rho_e / L_n \neq 0$ . This requires a considerable amount of algebra to treat the nonlinear ion kinetic equation; however we found no other way to benchmark our model against previous linear results [31, 34, 43, 57].

#### 1. Electrons

We start with the electron kinetic equation [58] for  $h_e = F_e - F_{0e}(1 + e\varphi/T_{0e})$

$$\begin{aligned} \frac{\partial h_e}{\partial t} + \mathbf{v}_E \cdot \nabla h_e + v_{\parallel} \hat{\mathbf{b}} \cdot \nabla h_e = & -\frac{eF_{0e}}{T_{0e}} \frac{\partial}{\partial t} \left( \varphi - \frac{v_{\parallel} A_{\parallel}}{c} \right) \\ & - \frac{c}{B_0} \mathbf{e}_z \cdot \nabla \left( \varphi - \frac{v_{\parallel} A_{\parallel}}{c} \right) \times \nabla F_{0e} + \left( \frac{\partial h_e}{\partial t} \right)_{coll}, \end{aligned} \quad (\text{A1})$$

where  $\mathbf{v}_E = cB_0^{-1}(-\partial_y \varphi \mathbf{e}_x + \partial_x \varphi \mathbf{e}_y)$  is the  $\mathbf{E} \times \mathbf{B}$  drift velocity,  $\hat{\mathbf{b}} \cdot \nabla = \partial_z - B_0^{-1}\{A_{\parallel}, \}$ , and

$$F_{0e} = \frac{n_{0e}(\mathbf{x})}{[\pi v_{the}^2(\mathbf{x})]^{3/2}} e^{-\frac{v_{\parallel}^2 + v_{\perp}^2}{v_{the}^2(\mathbf{x})}} \quad (\text{A2})$$

is the inhomogeneous Maxwellian equilibrium, with temperature  $T_{0e}(\mathbf{x}) = m_e v_{the}^2(\mathbf{x})/2$ . Notice that, simply by balancing  $eF_{0e}T_{0e}^{-1}\partial_t \varphi \sim \mathbf{v}_E \cdot \nabla F_{0e}$ , we obtain

$$\omega \sim \omega_{*e} = \frac{1}{2}k_y v_{the} \frac{\rho_e}{L_n}, \quad (\text{A3})$$

where  $L_n^{-1} \sim F_{0e}^{-1} \nabla F_{0e}$ . Therefore, we now include the electron drift frequency. Since in our fundamental ordering we have  $\omega \sim k_{\parallel} v_{the}$ , together with Eq. (A3), this yields

$$\frac{\rho_e}{L_n} \sim \frac{k_{\parallel}}{k_{\perp}} \equiv \epsilon. \quad (\text{A4})$$

The kinetic equation (A1), when linearized, corresponds to that of Connor et al. in Ref [59].

As in the homogeneous case [36], we introduce a formal mass ratio expansion for the electron gyrocentre distribution, so that to zeroth order

$$h_e = \left( -\frac{e\varphi}{T_{0e}} + \frac{\delta n_e}{n_{0e}} + \frac{v_{\parallel} u_{\parallel e}}{T_{0e}} m_e \right) F_{0e} + g_e + \mathcal{O}\left(\frac{m_e}{m_i}\right), \quad (\text{A5})$$

here  $u_{\parallel e} = n_{0e}^{-1} \int d^3 \mathbf{v} v_{\parallel} h_e$ , and  $\int d^3 \mathbf{v} (1, v_{\parallel}) g_e \equiv 0$ . Using expression (A5) in Eq. (A1), and taking the zeroth moment we obtain the electron continuity equation

$$\frac{d}{dt} \frac{\delta n_e}{n_{0e}} + \hat{\mathbf{b}} \cdot \nabla u_{\parallel e} = -\frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} \quad (\text{A6})$$

with  $d/dt = \partial_t + \mathbf{v}_E \cdot \nabla$ . We recall that every term kept in Eq. (A6) is of order  $\sim \epsilon/\sqrt{\beta_e}$ .

After calculating the first moment of Eq. (A1) we have the generalized Ohm's law

$$\begin{aligned} \frac{d}{dt} (A_{\parallel} - d_e^2 \nabla_{\perp}^2 A_{\parallel}) &= -c \frac{\partial \varphi}{\partial z} + \frac{T_{0e} c}{e} \hat{\mathbf{b}} \cdot \nabla \left[ \frac{\delta n_e}{n_{0e}} + \frac{\delta T_{\parallel e}}{T_{0e}} \right] \\ &- \frac{m_e c}{e} \frac{1}{n_{0e}} \int d^3 \mathbf{v} v_{\parallel} \left( \frac{\partial h_e}{\partial t} \right)_{coll} + \frac{T_{0e} c}{e} (1 + \eta_e) \frac{\tilde{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}} \\ &+ \frac{m_e c}{e} \frac{1}{n_{0e}} \frac{d}{dt} n_{0e} u_{\parallel i}, \end{aligned} \quad (\text{A7})$$

with

$$\eta_e = \frac{n_{oe}}{T_{0e}} \frac{\nabla T_{0e}}{\nabla n_{0e}}, \quad (\text{A8})$$

$$\frac{\delta T_{\parallel e}}{T_{0e}} \equiv \frac{1}{n_{0e}} \int d^3 \mathbf{v} 2 \frac{v_{\parallel}^2}{v_{the}^2} g_e, \quad (\text{A9})$$

and  $\tilde{\mathbf{b}} \cdot \nabla = -B_0^{-1} \{A_{\parallel}, \cdot\}$ . To derive Eq. (A7), we used the fact that

$$u_{\parallel e} = \frac{e}{m_e c} d_e^2 \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i}, \quad (\text{A10})$$

and

$$\nabla F_{0e} = F_{0e} \frac{\nabla n_{0e}}{n_{0e}} \left[ 1 + \eta_e \left( \frac{m_e v^2}{2T_{0e}} - \frac{3}{2} \right) \right]. \quad (\text{A11})$$

We can obtain an equation for  $g_e$  after inserting Eqs. (A6) and (A7) into (A1). The result is

$$\begin{aligned} \frac{dg_e}{dt} + v_{\parallel} \left[ \hat{\mathbf{b}} \cdot \nabla g_e - F_{0e} \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}} \right] - C[g_e] &= \\ F_{0e} \left( 1 - 2 \frac{v_{\parallel}^2}{v_{the}^2} \right) \hat{\mathbf{b}} \cdot \nabla \left[ \left( \frac{e}{m_e c} d_e^2 \nabla_{\perp}^2 A_{\parallel} + u_{\parallel i} \right) \right] &+ \\ - \eta_e F_{0e} \left( \frac{m_e v^2}{2T_{0e}} - \frac{3}{2} \right) \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} &+ \\ - \eta_e F_{0e} v_{\parallel} \left( \frac{m_e v^2}{2T_{0e}} - \frac{5}{2} \right) \frac{v_{the} \tilde{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}}, \end{aligned} \quad (\text{A12})$$

with the notation

$$C[g_e] = \left( \frac{\partial h_e}{\partial t} \right)_{coll} - 2 \frac{v_{\parallel} F_{0e}}{v_{the}^2 n_{oe}} \int d^3 \mathbf{v} v_{\parallel} \left( \frac{\partial h_e}{\partial t} \right)_{coll}. \quad (\text{A13})$$

Equation (A12), in the limit of a homogeneous background, reduces to the result of Ref. [36].

## 2. Ion response and closure

Equations (A6)-(A7)-(A12) [with (A12) replaced by (26) in the collisional limit] are a set of three equations for  $A_{\parallel}$ ,  $\varphi$ ,  $g_e$  (which gives  $\delta T_{\parallel e}/T_{0e}$ ), and  $\delta n_e/n_{0e}$ . To close the system we need an explicit expression for the electron density perturbation  $\delta n_e/n_{0e}$ . This can be readily calculated from the ion gyrokinetic equation. Quasineutrality requires that  $\delta n_e = \delta n_i$ . Under the same orderings that produced Eq. (A12), we have<sup>1</sup>

$$\frac{\partial h_i}{\partial t} + \langle \mathbf{v}_E \rangle \cdot \nabla h_i = \frac{ZeF_{0i}}{T_{0i}} \frac{\partial \langle \varphi \rangle}{\partial t} - \frac{c}{B_0} \mathbf{e}_z \cdot \nabla \langle \varphi \rangle \times \nabla F_{0i}, \quad (\text{A14})$$

where the bracket has the usual meaning of a gyroaverage.

Let us introduce the function  $g_i$  such that

$$h_i = \frac{Ze \langle \varphi \rangle_{\mathbf{R}_i}}{T_{0i}} F_{0i} + g_i. \quad (\text{A15})$$

Thus, we obtain an inhomogeneous equation for  $g_i$

$$\frac{dg_i}{dt} = -\frac{\langle \mathbf{v}_E \rangle \cdot \nabla n_{0i}}{n_{0i}} \left\{ 1 + \eta_i \left( \hat{v}^2 - \frac{3}{2} \right) \right\} \frac{ZeF_{0i}}{T_{0i}}. \quad (\text{A16})$$

This is a 5D equation, however, we can integrate over  $\int dv_{\parallel}$  to obtain

$$\begin{aligned} \frac{\partial \hat{g}_i}{\partial t} + \frac{c}{B_0} \{ \langle \varphi \rangle_{\mathbf{R}_i}, \hat{g}_i \} = \\ + i \sum_{\mathbf{k}'} \omega_{*i}(k'_y) J_0(\mathbf{k}'_{\perp} \rho_i \hat{v}_{\perp}) \frac{Ze\varphi_{\mathbf{k}'}}{T_{0i}} \{ 1 - \eta_i + \eta_i \hat{v}_{\perp}^2 \} \frac{n_{0i} e^{-\hat{v}_{\perp}^2}}{v_{thi}^2 \pi}. \end{aligned} \quad (\text{A17})$$

where  $\hat{g}_i = \int d\hat{v}_{\parallel} g_i$ , and  $\hat{v} = v/v_{thi}$ . When linearized, the ion response is the same as calculated in Ref. [31, 34, 44, 57]; thus, in this case, we have

$$\frac{\delta n_i}{n_{0i}} = \int_{-\infty}^{+\infty} dp e^{ipx} F(p\rho_i) \frac{Ze\varphi}{T_{0i}} \equiv \hat{F} \frac{Ze\varphi_{\mathbf{k}}}{T_{0i}}, \quad (\text{A18})$$

with

$$F(p\rho_i) = -(1 - \Gamma_0) - \frac{\omega_{*i}}{\omega} \left[ \Gamma_0 + \frac{\eta_i}{2} p^2 \rho_i^2 (\Gamma_0 - \Gamma_1) \right], \quad (\text{A19})$$

$\omega_{*i} = -1/2 k_y v_{thi} \rho_i / L_n \equiv -\tau \omega_{*e} < 0$ ,  $\tau = T_{0i}/T_{0e}$ ,  $\Gamma_n = \exp[-p^2 \rho_i^2 / 2] I_n(p^2 \rho_i^2 / 2)$ , where  $I_n$  is the modified Bessel function [60]. The “hat” on  $F(p\rho_i)$  is a short-hand notation for the inverse transform. Notice that we are using  $n_0(x) \approx n_{0i}(1 - x/L_n)$ , as a local approximation for the equilibrium density profile.

A useful way to describe ion kinetics is the following. Let us introduce the representation

$$\hat{g}_i(\mathbf{R}_i, \mathbf{v}_{\perp}, t) = \sum_{\mathbf{k}} \sum_{n=0}^{\infty} g_{\mathbf{k}}^n L_n(\hat{v}_{\perp}^2) F_{0i}(\hat{v}_{\perp}^2) e^{i\mathbf{R}_i \cdot \mathbf{k}}, \quad (\text{A20})$$

where  $L_n$  are the Laguerre polynomials defined through the Rodriguez formula. The velocity space representation is constructed on top of the usual Fourier space representation, here written symbolically as a summation. The coefficients  $g_{\mathbf{k}}^n$  are thus defined as

$$g_{\mathbf{k}}^n = \frac{\pi}{n_{0i}} \int d\mathbf{R}_i \int_0^{\infty} dv_{\perp} v_{\perp} L_n(\hat{v}_{\perp}^2) \hat{g}_i(\mathbf{R}_i, \mathbf{v}_{\perp}, t) e^{-i\mathbf{R}_i \cdot \mathbf{k}}. \quad (\text{A21})$$

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[1] These orderings are discussed at length in Ref. [36]. What is evident is that streaming terms are downgraded compared to those in the electron equation, because  $v_{\parallel} \sim v_{thi} \sim (m_e/m_i)^{1/2} v_{the}$ . Ions do not experience collisions, because they are too heavy to be scattered by electrons [ion self-collisions will be eventually considered in Eq. (A22)].



Using Eq. (A20) to replace for  $\hat{g}_i$  in Eq. (A17), and we operating with  $\pi/n_{0i} \int dv_\perp v_\perp L_m(\hat{v}_\perp^2)$ , all the velocity space integrals can be carried out to obtain

$$\begin{aligned}
& \frac{\partial}{\partial t} g_{\mathbf{k}'}^n - \frac{c}{B_0} \sum_{\mathbf{k}} \sum_{m=0}^{\infty} (-1)^{m+n} \mathbf{z} \cdot \mathbf{k} \times \mathbf{k}' \\
& \times \frac{1}{2} e^{-\frac{k_\perp^2 \rho_i^2}{4}} L_n^{m-n} \left( \frac{1}{4} k_\perp^2 \rho_i^2 \right) \varphi_{\mathbf{k}} \times \\
& L_m^{n-m} \left( \frac{1}{4} k_\perp^2 \rho_i^2 \right) g_{\mathbf{k}'-\mathbf{k}}^n = \\
& \omega_{*i} \frac{1}{2} e^{-\frac{k_\perp'^2 \rho_i^2}{4}} \left[ L_n \left( \frac{1}{4} k_\perp'^2 \rho_i^2 \right) L_0 \left( \frac{1}{4} k_\perp'^2 \rho_i^2 \right) \right. \\
& \left. - \eta_i L_n \left( \frac{1}{4} k_\perp'^2 \rho_i^2 \right) L_1 \left( \frac{1}{4} k_\perp'^2 \rho_i^2 \right) \right] \frac{Z e \varphi_{\mathbf{k}'}}{T_{0i}}.
\end{aligned} \tag{A22}$$

We expect nonlinear phase mixing to play a role now, and to create structures in perpendicular velocity space for the ion distribution function. Hence ion-ion collisions will eventually become important for sufficiently large gradients  $v_{thi}^2 \partial_{v_\perp}^2 \sim \omega/\nu_{ii} \gg 1$  and a simple model collisional operator will need to be considered. This aspect is not crucial for the scope of this paper.

We can now summarize the new set of equations

$$\begin{aligned}
& \frac{d}{dt} (A_\parallel - d_e^2 \nabla_\perp^2 A_\parallel) = -c \frac{\partial \varphi}{\partial z} \\
& \frac{T_{0e} c}{e} \hat{\mathbf{b}} \cdot \nabla \left[ \frac{Z}{\tau} (\hat{\Gamma}_0 - 1) \frac{e \varphi_{\mathbf{k}}}{T_{0e}} + g_i^{(0)} + \frac{\delta T_{\parallel e}}{T_{0e}} \right] \\
& \eta \nabla_\perp^2 A_\parallel + \frac{T_{0e} c}{e} (1 + \eta_e) \frac{\tilde{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}},
\end{aligned} \tag{A23}$$

$$\begin{aligned}
& \frac{d}{dt} \left[ \frac{Z}{\tau} (\hat{\Gamma}_0 - 1) \frac{e \varphi_{\mathbf{k}}}{T_{0e}} + g_i^{(0)} \right] + \hat{\mathbf{b}} \cdot \nabla \frac{e}{m_e c} d_e^2 \nabla_\perp^2 A_\parallel = \\
& - \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}},
\end{aligned} \tag{A24}$$

$$\begin{aligned}
& \frac{dg_e}{dt} + v_\parallel \left[ \hat{\mathbf{b}} \cdot \nabla g_e - F_{0e} \hat{\mathbf{b}} \cdot \nabla \frac{\delta T_{\parallel e}}{T_{0e}} \right] - C[g_e] = \\
& F_{0e} \left( 1 - 2 \frac{v_\parallel^2}{v_{the}^2} \right) \hat{\mathbf{b}} \cdot \nabla \left[ \left( \frac{e}{m_e c} d_e^2 \nabla_\perp^2 A_\parallel + u_{\parallel i} \right) \right] + \\
& - \eta_e F_{0e} \left( \frac{m_e v^2}{2 T_{0e}} - \frac{3}{2} \right) \frac{\mathbf{v}_E \cdot \nabla n_{0e}}{n_{0e}} + \\
& - \eta_e F_{0e} v_\parallel \left( \frac{m_e v^2}{2 T_{0e}} - \frac{5}{2} \right) \frac{v_{the} \tilde{\mathbf{b}} \cdot \nabla n_{0e}}{n_{0e}},
\end{aligned} \tag{A25}$$

with

$$\frac{\delta T_{\parallel e}}{T_{0e}} \equiv \frac{1}{n_{0e}} \int d^3 \mathbf{v} 2 \frac{v_\parallel^2}{v_{the}^2} g_e, \tag{A26}$$

and

$$\begin{aligned}
& g_i^{(0)}(\mathbf{r}, t) = \frac{1}{n_{0i}} \int d^3 \mathbf{v} \langle g_i(\mathbf{R}_i, \mathbf{v}_\perp, t) \rangle_{\mathbf{r}} \\
& = \sum_k \sum_{n=0}^{\infty} \frac{(k_\perp \rho_i)^n}{2^{2n} n!} e^{-\frac{1}{4} k_\perp^2 \rho_i^2} g_{\mathbf{k}}^n e^{i \mathbf{k} \cdot \mathbf{r}}.
\end{aligned} \tag{A27}$$

We calculated explicitly the collision term in Ohm's law (A23), which gives the resistive contribution, with  $\eta = \nu_{ei} d_e^2$  the Spitzer resistivity. We used the fact that

$$\int_0^\infty dx x e^{-x^2} L_n(x^2) J_0(xy) = \frac{2^{-2n}}{2n!} y^{2n} e^{-\frac{1}{4}y^2}. \quad (\text{A28})$$

This result can be proved by using the representation of Bessel functions in terms of Laguerre polynomials

$$J_0(k_\perp \rho_i \hat{v}_\perp) = e^{-\frac{1}{4}k_\perp^2 \rho_i^2} \sum_{n=0}^\infty \frac{(k_\perp^2 \rho_i^2 / 4)^n}{n!} L_n(\hat{v}_\perp^2), \quad (\text{A29})$$

and the orthogonality of these polynomials. We also used the fact that [61]

$$\begin{aligned} \int_0^\infty dx x e^{-x^2} L_n(x^2) L_m(x^2) J_0(xy) = \\ \frac{(-1)^{m+n}}{2} e^{-\frac{1}{4}y^2} L_n^{m-n}\left(\frac{y^2}{4}\right) L_m^{n-m}\left(\frac{y^2}{4}\right). \end{aligned} \quad (\text{A30})$$

The integral (A30) is a simplified version of the integral [62]

$$\begin{aligned} I_{m,n} = \int_0^\infty dx x^{\nu+1} e^{-\alpha x^2} L_m^{\nu-\sigma}(\alpha x^2) L_n^\sigma(\alpha x^2) J_\nu(xy) = \\ \frac{(-1)^{m+n}}{2} e^{-\frac{1}{4}y^2} L_n^{m-n-\sigma}\left(\frac{y^2}{4}\right) L_m^{n-m+\sigma-\nu}\left(\frac{y^2}{4}\right), \end{aligned} \quad (\text{A31})$$

which is the same as given in Ref. [61]. In Ref. [62] we find the following conditions:  $n \neq 0$ ,  $\sigma \neq 0$ ,  $\alpha \neq 1$ . However, if we set  $n = 0$ ,  $\sigma = 0$ ,  $\alpha = 1$ , and  $\nu = 0$ , we calculate analytically term by term for each  $m$ :

$$\begin{aligned} m = 0 \quad I_{m,0} &= \frac{1}{2} e^{-\frac{1}{4}y^2} \\ m = 1 \quad I_{m,0} &= \frac{1}{32} e^{-\frac{1}{4}y^2} (y^2 - 4)^2 \\ m = 1 \quad I_{m,0} &= \frac{1}{2048} e^{-\frac{1}{4}y^2} (y^4 - 16y^2 + 32)^2 \dots, \text{et cetera.} \end{aligned} \quad (\text{A32})$$

Notice that, by construction of the ordering,  $g_i$  is only a function of  $v_\perp$ , thus the ions are not carrying any parallel current.

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