Filtrations in Dyson–Schwinger equations: Next-to$^{\{j\}}$-leading log expansions systematically

Olaf Krüger [a,b,*], Dirk Kreimer [b]

[a] Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany
[b] Humboldt-Universität zu Berlin, Institut für Physik, Newtonstraße 15, D-12489 Berlin, Germany

ARTICLE INFO

ABSTRACT

Dyson–Schwinger equations determine the Green functions $G^r(\alpha, L)$ in quantum field theory. Their solutions are triangular series in a coupling constant α and an external scale parameter L for a chosen amplitude r, with the order in L bounded by the order in the coupling.

Perturbation theory calculates the first few orders in α. On the other hand, Dyson–Schwinger equations determine next-to$^{\{j\}}$-leading log expansions, $G^r(\alpha, L) = 1 + \sum_{j=0}^{\infty} \sum_{\mathcal{M}} p_j^{\mathcal{M}} \alpha^j \mathcal{M}(u)$. $\sum_{\mathcal{M}}$ sums for any finite j a finite number of functions $\mathcal{M}$ in u. Here, u is the one-loop approximation to G^r, for example, for the (inverse) propagator in massless Yukawa theory, $u = \alpha L/2$.

The leading logs come then from the trivial representation, e.g. $\mathcal{M}(u) = 1 - \sqrt{1 - 2u}$ in massless Yukawa theory. The respective period at $j = 0$ is $p_0^{\mathcal{M}(u)} = 1$. All non-leading logs are organized by corresponding suppressions in powers α^j.

We describe an algebraic method to derive all next-to$^{\{j\}}$-leading log terms from the knowledge of the first $(j + 1)$ terms in perturbation theory and their filtrations. This implies the calculation of the functions $\mathcal{M}(u)$ and periods $p_j^{\mathcal{M}}$.

In the first part of our paper, we investigate the structure of Dyson–Schwinger equations and develop a method to filter their solutions. Applying renormalized Feynman rules maps each filtered term to a certain power of α and L in the log-expansion.

Based on this, the second part derives the next-to$^{\{j\}}$-leading log expansions. Our method is general. Here, we exemplify it using

---

* Corresponding author at: Max-Planck-Institut für Gravitationsphysik, Albert-Einstein-Institut, Am Mühlenberg 1, D-14476 Potsdam, Germany.

E-mail addresses: olaf.krueger@aei.mpg.de (O. Krüger), kreimer@physik.hu-berlin.de (D. Kreimer).

the examples of the propagator in Yukawa theory and the photon self-energy in quantum electrodynamics. In particular, we give explicit formulas for the leading log, next-to-leading log and next-to-next-to-leading log orders in terms of at most three-loop Feynman integrals. The reader may apply our method to any (set of) Dyson–Schwinger equation(s) appearing in renormalizable quantum field theories.

1. Introduction and results

In this section, we give a short introduction as well as a presentation and discussion on the type of results we obtain in this paper.

1.1. Introduction

The usual way to compute a physical probability amplitude is to replace each term of the perturbative series in the coupling α by a sum of Feynman graphs. Applying renormalized Feynman rules Φ_R to all such graphs translates this sum of graphs to the physically observable renormalized Feynman amplitude, say a Green function $G_R(\alpha, L, \theta)$, at least as a formal series.

Φ_R evaluated on a graph is a polynomial in a suitably chosen external scale parameter $L = \log S/S_0$, $\Phi_R = \Phi_R(L)$. L depends for example on the center of mass energy S given by the underlying process, with S_0 fixing a reference scale for renormalization.

Further dependences are scattering angles, collected in $\Phi_R(L, \theta)$ by a set of variables θ. These are dimensionless parameters incorporating dependences on scalar products $p_i \cdot p_j/S$ or masses m_i^2/S. Throughout, we assume we leave those scattering angles unchanged for the renormalization point. A discussion of this point can be found in [1,2].

Therefore, any renormalized Green function G_R can be written as a triangular expansion

$$G_R(\alpha, L, \theta) = 1 + \sum_{j=0}^{\infty} \sum_{i=1}^{\infty} \gamma_{i+j,i}(\theta)\alpha^{i+j}L^i = 1 + \sum_{j=0}^{\infty} \sum_{\mathcal{M}} p_j^{\mathcal{M}}(\theta)\alpha^j \mathcal{M}(u), \qquad (1)$$

called the 'log-expansion'. Here, α denotes the perturbative parameter, a coupling constant say, and $\gamma_{i+j,i}$ are some functions in θ. We have $p_0^{\mathcal{M}} = 1$. For $j \geq 1$, the $p_j^{\mathcal{M}}(\theta)$ may depend on θ and are obtained from the first $(j+1)$ terms of a perturbative expansion in the coupling. $\mathcal{M}(u) = \sum_{i=1}^{\infty} q_{i:j}^{\mathcal{M}} u^i$ is a series in $u = \alpha L/2$[1] with $q_{i:j}^{\mathcal{M}} \in \mathbb{Q}$. These series are determined from their counterparts in the universal enveloping algebra of Feynman graphs that we detail below, see also [1].

Such a form of the Green functions in a renormalizable field theory is a consequence of the Hopf algebra structure of Feynman graphs [3]. One starts from the fact that all superficially divergent one-particle irreducible (1PI) Feynman graphs of any physical quantum field theory generate a Hopf algebra $\mathcal{H}$.

This allows us to introduce Dyson–Schwinger equations (DSEs) as fix-point equations for Feynman graphs upon using Hochschild cohomology [6]. See [7] for an effective application of these mathematical structures to the automatization of perturbative renormalization, graph generation and graph counting.

---

[1] Note that in general, $u \propto \alpha L$. Since we mainly consider the Green function for the Yukawa fermion propagator, the proportionality factor is 1/2. For the QED photon self-energy Green function that is also considered in this paper, one finds $u = 4/3\,\alpha L$.

Given a quantum field theory, one always finds DSEs whose solution is simply related to the log-expansion (Eq. (1)) by applying renormalized Feynman rules Φ_R. Loïc Foissy classified exhaustively the structure of possible DSEs [8].

We apply our approach to two exemplary cases: first, to the fermion propagator

$$S(q) = \frac{1}{\not{q}(1 - \Sigma(\alpha, L))} = \frac{1}{\not{q}\Phi_R(X_{\mathrm{Yuk}})} \tag{2}$$

occurring in Yukawa theory. X_{Yuk} represents an infinite sum of graphs that satisfies the DSE

$$X_{\mathrm{Yuk}} = \mathbb{I} - \sum_{j \geq 1} \alpha^j B_+^{\Gamma_j} \left(X_{\mathrm{Yuk}}^{(1-2j)} \right). \tag{3}$$

Secondly, to the photon self-energy

$$\Pi_{\mu\nu}(q) = \frac{g_{\mu\nu} - \hat{q}_\mu \hat{q}_\nu}{q^2(1 - \Pi(\alpha, L))} = \frac{g_{\mu\nu} - \hat{q}_\mu \hat{q}_\nu}{q^2 \Phi_R(X_{\mathrm{QED}})} \tag{4}$$

occurring in quantum electrodynamics (QED). The infinite sum of graphs X_{QED} satisfies the DSE

$$X_{\mathrm{QED}} = \mathbb{I} - \sum_{j \geq 1} \alpha^j B_+^{\Gamma_j} \left(X_{\mathrm{QED}}^{(1-j)} \right). \tag{5}$$

Note that acting with renormalized Feynman rules Φ_R on X_{Yuk} and X_{QED} yields the log expansions (Eq. (1)).

In Yukawa theory, one should consider systems of DSEs because there are no Ward identities. Here, we restrict to a truncation eliminating vertex divergences for purposes of presentation.

Our paper consists of two parts. First, in Section 2, we give a brief overview on Hopf algebras and DSEs. In particular, we relate the Hopf algebra of Feynman graphs to the Hopf algebra of words by a morphism of Hopf algebras. Once the solution of a DSE is given in the Hopf algebra of words, we describe the filtration method in Section 3.1. There, we rely on properties of renormalized Feynman rules that we derive in Section 3.2. Finally, we present the filtration algorithm in Section 3.3.

In the second part of our paper, we describe a general method to derive the next-to$^{\{j\}}$-leading log expansion in Section 4. In particular, we exemplify up to $j \leq 2$ for the Yukawa fermion propagator. In Appendix A, we collect the respective results for the QED photon self-energy. In the filtrations of the Yukawa fermion propagator X_{Yuk} and the QED photon self-energy X_{QED}, each term comes with a multiplicity. We list some resulting series in these multiplicities in Appendix B.

In the remainder of this section, we summarize and discuss our results.

1.2. Results and discussion

Let us first concentrate on Yukawa theory, Eq. (3). To find the leading log expansion we can simplify to

$$X_{\mathrm{Yuk}} = \mathbb{I} - \alpha B_+^{\Gamma_1} \left(X_{\mathrm{Yuk}}^{-1} \right). \tag{6}$$

Feynman rules Φ_R with massless internal propagators and a momentum scheme for subtraction turn Eq. (3) into a differential equation for the corresponding anomalous dimension, which can be solved implicitly [10]. If we are only interested in the leading log expansion the situation is even simpler. We only have to solve

$$\mathcal{M}(u) = 1 - \int^u \frac{\mathrm{d}x}{1 - \mathcal{M}(x)}, \qquad \mathcal{M}(0) = 1 \qquad \Rightarrow \qquad \mathcal{M}(u) = 1 - \sqrt{1 - 2u}. \tag{7}$$

Indeed, only inserting the one-loop correction for the Yukawa fermion propagator,

$$\Phi_R(\Gamma_1) = \frac{1}{2}L \tag{8}$$

into itself in all possible ways give graphs that contain leading log contributions.

In our paper, we introduce a convenient matrix notation. []-bracketed matrices with a dot in the upper left entry denote the functions $\mathcal{M}$ in one variable, say $z \in \mathbb{R}$. These functions occur in the log-expansions (Eq. (1)) setting

$$z \to \alpha \Phi_R(\Gamma_1) = \frac{\alpha L}{2} = u. \tag{9}$$

In Section 4, we explain the notation and derive ordinary first order differential equations for these objects. The respective differential equations depend on the corresponding DSEs (in our case, Eq. (3)) and are solved for Yukawa theory in Section 4. For the leading log order, we define

$$\left[\bullet \right](u) = 1 - \sqrt{1 - 2u} \tag{10}$$

with corresponding period $p_0^{[\bullet]} = 1$. The leading log expansion ($j = 0$ in Eq. (1)) finally yields

$$G_R(X_{\text{Yuk}})|_{\text{l.l.}} = p_0^{[\bullet]} \left[\bullet \right](u). \tag{11}$$

$p_0^{[\bullet]}$ and $[\bullet](u)$ are also given in the first line of Table 1.

Eq. (7) is obvious as Feynman rules in a momentum scheme map the Hochschild one-co-cycle $B_+^{\Gamma_1}$ to the Hochschild one-co-cycle $\int^u dx$ on polynomials in the variable $u = \alpha L/2$ [11].

To get the next-to-leading log expansion, we consider two contributions: the insertion of the one-loop propagator graph into itself gives a contribution $u^2/2 + p_1^{\left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix} \right]} u$. The first term corresponds to the leading log and the second term to the next-to-leading log expansion. There is also a contribution $p_1^{[\bullet 1]} u$ from the next Hochschild one-co-cycle provided by Γ_2. The latter contribution occurs in Eq. (3) that have a single appearance of Γ_2 (beside several Γ_1). We can therefore simplify to

$$X_{\text{Yuk}} = \mathbb{I} - \alpha B_+^{\Gamma_1} \left(X_{\text{Yuk}}^{-1} \right) - \alpha^2 B_+^{\Gamma_2} \left(X_{\text{Yuk}}^{-3} \right). \tag{12}$$

This gives the next-to-leading log expansion ($j = 1$ in Eq. (1)),

$$G_R(X_{\text{Yuk}})|_{\text{n.l.l.}} = \alpha p_1^{[\bullet 1]} \left[\bullet \quad 1 \right](u) + \alpha p_1^{\left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix} \right]} \left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix} \right](u). \tag{13}$$

The functions $[\bullet \ 1](u)$ and $\left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix} \right](u)$ as well as the corresponding periods are listed in lines (ii) and (iii) of Table 1.

For the next-to-next-to-leading log expansion, there are several contributions, which we do not summarize here. In Section 4.6.3, we derive

$$G_R(X_{\text{Yuk}})|_{\text{n.n.l.l.}} = \alpha^2 p_2^{[\bullet \, 0 \, 1]} \left[\bullet \quad 0 \quad 1 \right](u) + \alpha^2 p_2^{[\bullet \, 2]} \left[\bullet \quad 2 \right](u)$$

$$+ \alpha^2 p_2^{\left[\begin{smallmatrix} \bullet \\ 3 \end{smallmatrix} \right]} \left[\begin{smallmatrix} \bullet \\ 3 \end{smallmatrix} \right](u) + \alpha^2 p_2^{\left[\begin{smallmatrix} \bullet \, 0 \\ 1 \, 1 \end{smallmatrix} \right]} \left[\begin{smallmatrix} \bullet & 0 \\ 1 & 1 \end{smallmatrix} \right](u) + \alpha^2 p_2^{\left[\begin{smallmatrix} 2 \\ \bullet \\ 2 \end{smallmatrix} \right]} \left[\begin{smallmatrix} \bullet \\ 2 \\ 2 \end{smallmatrix} \right](u)$$

$$+ \alpha^2 p_2^{\left[\begin{smallmatrix} \bullet \, 1 \\ 2 \, 0 \end{smallmatrix} \right]} \left[\begin{smallmatrix} \bullet & 1 \\ 2 & 0 \end{smallmatrix} \right](u)$$

$$+ \alpha^2 p_2^{[\bullet]_{[a_1, a_2]}} \left[\bullet \right]_{[a_1, a_2]}(u) + \alpha^2 p_2^{[\bullet]_{[a_1, \Theta(a_1, a_1)]}} \left[\bullet \right]_{[a_1, \Theta(a_1, a_1)]}(u) \tag{14}$$

with generating functions and periods given in lines (iv)–(xi) of Table 1.

Eqs. (11), (13) and (14) and the respective periods (see the first column of Table 1) are in general, valid for any DSE. However, the generating functions (2nd and 3rd column of Table 1) depend on the DSE.

For the QED photon self-energy, the corresponding DSE is given in Eq. (5). The respective generating functions are listed in the third column of Table 1 and are much simpler. This is because there is no insertion point in the Hochschild one-co-cycle $B_+^{\Gamma_1}$ in QED. It follows that the mentioned differential

Table 1

The results obtained in this paper: a renormalized Green function G_R is given by the log-expansion in Eq. (1). We calculate G_R up to next-to-next-to-leading log order ($j \leq 2$). This table gives the periods $p_j^{\mathcal{M}}(\theta)$ in the first column. These are general for any Green function in any quantum field theory. The second column shows the generating functions $\mathcal{M}(u)$ for the Yukawa fermion propagator. Here, $u = \alpha L/2$ and we abbreviate $x = 1/\sqrt{1 - 2u}$ and $y = 1/(1 - u)$. The periods are calculated implicitly in Eqs. (220), (226) and (243). They can also be obtained independently in three ways: first, from the Feynman rules acting on primitive elements $\Gamma_j \in \mathcal{H}$, which define an accompanying period. Secondly, they are obtained from primitives generated by the Dynkin operator $S \star Y$ applied to shuffles of primitives. Finally, they are obtained from concatenation multi-commutators of primitives of either sort. The generating functions in Yukawa theory are obtained from Eqs. (143), (151), (159), (153), (155), (164), (168), (175), (182), (207) and (216). The generating functions for the QED photon self-energy Green function are given in Eqs. (A.4), (A.7)–(A.10), (A.12), (A.14), (A.15) and (A.17).

	Periods $p_j^{\mathcal{M}}$	Yukawa generating functions $\mathcal{M}(u)$	QED generating functions $\mathcal{M}(u)$
(i)	$p_0^{[\bullet]} = 1$	$[\bullet](u) = 1 - \frac{1}{x}$	$[\bullet](u) = u$
(ii)	$p_1^{[\bullet\ 1\ 1]} = \alpha \Phi_R(\Gamma_2)$	$[\bullet\ 1](u) = x \log x$	$[\bullet\ 1](u) = \log y$
(iii)	$p_1^{[\bullet\ 2]} = \alpha \Phi_R(\Gamma_1)^2 - 2\alpha \Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)$	$\begin{bmatrix}\bullet\\2\end{bmatrix}(u) = -\frac{x}{2}\log x$	$\begin{bmatrix}\bullet\\2\end{bmatrix}(u) = 0$
(iv)	$p_2^{[\bullet\ 0\ 1]} = \alpha \Phi_R(\Gamma_3)$	$[\bullet\ 0\ 1](u) = -\frac{x}{2} + \frac{x^3}{2}$	$[\bullet\ 0\ 1](u) = y - 1$
(v)	$p_2^{[\bullet\ 2]} = \alpha^2 \Phi_R(\Gamma_2)^2$	$[\bullet\ 2](u) = \frac{x}{2} - \frac{x^3}{2} + x^3 \log x + \frac{x^3}{2}\log^2 x$	$[\bullet\ 2](u) = 1 - y + y \log y$
(vi)	$p_2^{[\bullet\ 3]} = \alpha\left(3\Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - 3\Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) + \Phi_R(\Gamma_1)^3\right)$	$\begin{bmatrix}\bullet\\3\end{bmatrix}(u) = \frac{x^3}{2} - \frac{x}{2u}\log x$	$\begin{bmatrix}\bullet\\3\end{bmatrix}(u) = 0$
(vii)	$p_2^{[\bullet\ 0\atop 1\ 1]} = \alpha\left(-\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right) + \Phi_R(\Gamma_1)\Phi_R(\Gamma_2)\right)$	$\begin{bmatrix}\bullet&0\\1&1\end{bmatrix}(u) = -\frac{x}{8} - \frac{3x^3}{8} + \frac{x^3}{4}\log x + \frac{x}{2u}\log x + \frac{x^3}{8}\log^2 x$	$\begin{bmatrix}\bullet&0\\1&1\end{bmatrix}(u) = -y + \frac{1}{u} + \log y$
(viii)	$p_2^{[\bullet\ 2\atop 2]} = \alpha^2\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right)^2$	$\begin{bmatrix}\bullet\\2\ 2\end{bmatrix}(u) = -\frac{x}{2} - \frac{3x^3}{8} + \frac{x^3}{4}\log x + \frac{x}{2u}\log x + \frac{x^3}{8}\log^2 x$	$\begin{bmatrix}\bullet\\2\ 2\end{bmatrix}(u) = 0$
(ix)	$p_2^{[\bullet\ 1\atop 2\ 0]} = \alpha^2 \Phi_R(\Gamma_2)\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right)$	$\begin{bmatrix}\bullet&1\\2&0\end{bmatrix}(u) = x^3 - x^3 \log x - \frac{x}{u}\log x - \frac{x^3}{2}\log^2 x$	$\begin{bmatrix}\bullet&1\\2&0\end{bmatrix}(u) = \frac{1}{2} + \frac{u}{2} - \frac{1}{u}\log y$
(x)	$p_2^{[\bullet\,]_{[\alpha_1,\alpha_2]}} = \alpha\left(\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right)\right)$	$[\bullet]_{[\alpha_1,\alpha_2]}(u) = \frac{x^3}{4} - \frac{x^5}{4} + \frac{x^3}{4}\log x - \frac{3x^5}{4}\log x$	$[\bullet]_{[\alpha_1,\alpha_2]}(u) = \frac{y}{2} - \frac{y^2}{2}$
(xi)	$p_2^{[\bullet\,]_{[\alpha_1,\Theta(\alpha_1,\alpha_1)]}} = \alpha\left(2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1)\right) - \Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - \Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right)$	$[\bullet]_{[\alpha_1,\Theta(\alpha_1,\alpha_1)]}(u) = -\frac{x^3}{8} + \frac{x^5}{8} + \frac{x^5}{8} - \frac{x^3}{8}\log x + \frac{3x^5}{8}\log x$	$[\bullet]_{[\alpha_1,\Theta(\alpha_1,\alpha_1)]}(u) = 0$

equations for generating functions turn out to be ordinary equations. Note that in QED, we have $u = 4/3\,\alpha L$ instead of $\alpha L/2$.

The usual perturbative approach to quantum field theory does not suffice in high-energy regimes, where $\alpha L \sim 1$. There, it becomes significant to consider the log-expansion instead of perturbation theory.

Our results simplify the calculation of the log-expansion in Eq. (1) drastically. One usually needs to compute an infinite number of Feynman integrals of any loop-number. Using our results, the complete next-to-$[j]$-leading log expansion only depends on at most $(j+1)$-loop graphs. For example, the leading log order (Eq. (11)) only depends on the Feynman graph Γ_1. The next-to-leading log Green function in Eq. (13) only depends on the graphs Γ_1, Γ_2 and $B_+^{\Gamma_1}(\Gamma_1)$. The next-to-next-to-leading log expansion only depends on the Feynman graphs

$$
\begin{aligned}
&\Gamma_1, \qquad \Gamma_2, \qquad B_+^{\Gamma_1}(\Gamma_1), \qquad \Gamma_3, \qquad B_+^{\Gamma_1}(\Gamma_2), \qquad B_+^{\Gamma_2}(\Gamma_1), \\
&B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right), \qquad B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1),
\end{aligned}
\tag{15}
$$

see Eq. (14) and the periods in the first column of Table 1.

In summary, we filter the images of Feynman graphs (as in Eq. (15)) in a suitable universal enveloping algebra that we construct below. This decomposition then yields ordinary first order differential equations for generating functions $\mathcal{M}$. We solve for the functions $\mathcal{M}$, which are the coefficients for the periods $p_j^{\mathcal{M}}$ obtained in these filtrations.

The methods presented here are valid in general. One can apply the described techniques to any DSE in any quantum field theory. One could also compute the log-expansion for systems of DSEs. We expect that the generating functions then become solutions in systems of ordinary first order differential equations. Here, we are content in exhibiting our approach. A structural analysis of its mathematical underpinnings is left to future work.

We start with the introduction of necessary preliminaries.

2. Preliminaries

2.1. Hopf algebra of Feynman graphs

Let $\mathcal{H}$ be the vector space of 1PI Feynman graphs and their disjoint unions in a given quantum field theory. $(\mathcal{H}, m, \mathbb{I})$ forms an associative and unital algebra, where $\mathbb{I}$ denotes the empty graph and m is the disjoint union of graphs, serving as a product.

This algebra is graded by the loop number (the first Betti number) as an infinite sum of finite dimensional vector spaces

$$
\mathcal{H} = \oplus_{j=0}^{\infty} \mathcal{H}^{(j)},
\tag{16}
$$

with $\mathcal{H}^{(0)} = \mathbb{Q}\mathbb{I}$ and augmentation ideal

$$
\mathcal{A}_H = \oplus_{j=1}^{\infty} \mathcal{H}^{(j)}.
\tag{17}
$$

Note that Cartesian products $\mathcal{A}_H^j := \mathcal{A}_H^{\times j} \subset \mathcal{H}$ deliver a decreasing filtration $\mathcal{A}_H^{j+1} \subsetneq \mathcal{A}_H^j$.

The associated graded spaces $\mathfrak{gr}_j(\mathcal{A}_H) = \mathcal{A}_H^j / \mathcal{A}_H^{j+1}$ contain $\langle \Gamma \rangle := \mathcal{A}_H / \mathcal{A}_H^{\times 2} \equiv \mathfrak{gr}_1(\mathcal{A}_H)$ as the linear span of graphs in first degree. We set $\mathfrak{gr}_\bullet(\mathcal{A}_H) = \oplus_j \mathfrak{gr}_j(\mathcal{A}_H)$.

$\mathcal{H}$ acquires a co-algebraic structure by introducing a co-product $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ that acts on 1PI graphs Γ as

$$
\Delta(\Gamma) = \mathbb{I} \otimes \Gamma + \Gamma \otimes \mathbb{I} + \sum_{\gamma \in \mathcal{P}(\Gamma)} \gamma \otimes \Gamma/\gamma.
\tag{18}
$$

$\mathcal{P}(\Gamma)$ is the set of all proper sub-graphs $\gamma \subsetneq \Gamma$ such that γ is the disjoint union of divergent 1PI sub-graphs in Γ. The action of Δ on $\mathbb{I}$ and products of graphs is given by

$$
\Delta \mathbb{I} = \mathbb{I} \otimes \mathbb{I}, \quad \Delta \circ m = (m \otimes m) \circ \tau_{(2,3)} \circ (\Delta \otimes \Delta),
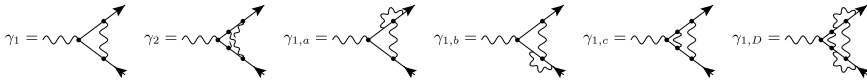\tag{19}
$$

Fig. 1. Some QED Feynman graphs that are used within this paper.

where $\tau_{(2,3)}$ flips the second and third element of the fourfold tensor product. The co-product is co-associative [3]. Together with a co-unit $\hat{\mathbb{I}} : \mathcal{H} \to \mathbb{K}$ that assigns a non-zero value only for $\mathbb{I}$ and $\hat{\mathbb{I}}(\mathbb{I}) = 1$, $\left(\mathcal{H}, \Delta, \hat{\mathbb{I}}\right)$ forms a co-associative and co-unital co-algebra.

The given construction implies that $(\mathcal{H}, m, \Delta, \mathbb{I}, \hat{\mathbb{I}})$ forms a bi-algebra as well.

Finally, $(\mathcal{H}, m, \Delta, \mathbb{I}, \hat{\mathbb{I}}, S)$ forms a Hopf algebra, abbreviated by $\mathcal{H}$. $S : \mathcal{H} \to \mathcal{H}$ is the antipode that fulfills

$$m \circ (S \otimes \text{id}) \circ \Delta = m \circ (\text{id} \otimes S) \circ \Delta = \mathbb{I} \circ \hat{\mathbb{I}}. \tag{20}$$

$\mathcal{H}$ is called the Hopf algebra of Feynman graphs. There have been various introductions to Hopf algebras and in particular, to the Hopf algebra of Feynman graphs [7].

Graphs without sub-divergences form primitive elements. By Eq. (18), their reduced co-product,

$$\tilde{\Delta}(\Gamma) := \Delta(\Gamma) - \mathbb{I} \otimes \Gamma - \Gamma \otimes \mathbb{I} \tag{21}$$

vanishes.

For each such primitive element $\Gamma \in \mathcal{H}$, $\tilde{\Delta}(\Gamma) = 0$, we define a grafting operator $B_+^\Gamma : \mathcal{H} \to \mathcal{H}$ that linearly inserts its argument graph(s) into Γ. For example,

$$B_+^{\gamma_1}(\gamma_1 \cup \gamma_1 \cup \gamma_1) = \gamma_{1,D}, \tag{22}$$

see Fig. 1 for the QED Feynman graphs γ_1 and $\gamma_{1,D}$.

Furthermore, we have $B_+^\Gamma(\mathbb{I}) := \Gamma$. If the insertion is not unique, the result is a sum over all possibilities, for example

$$B_+^{\gamma_1}(\gamma_1) = \frac{1}{3}\left(\gamma_{1,a} + \gamma_{1,b} + \gamma_{1,c}\right). \tag{23}$$

The graphs γ_1, $\gamma_{1,a}$, $\gamma_{1,b}$ and $\gamma_{1,c}$ are defined in Fig. 1.

The B_+^Γ are Hochschild one-co-cycles by definition [13,14] (this quite typically implies that one works in a quotient Hopf algebra with division by an ideal and co-ideal defined by suitable Ward identities, see [14] and references there)

$$\Delta \circ B_+^\Gamma(\cdot) = B_+^\Gamma(\cdot) \otimes \mathbb{I} + \left(\text{id} \otimes B_+^\Gamma\right) \circ \Delta(\cdot). \tag{24}$$

They generate the co-radical filtration and the associated grading by sub-divergences. To define the co-radical filtration, set $G_{-1}^H = \emptyset$ and for $j \geq 0$ set

$$G_j^H = \Delta^{-1}\left(\mathcal{H} \otimes \mathcal{H}^{(0)} + G_{j-1}^H \otimes \mathcal{H}\right). \tag{25}$$

This is an increasing filtration, $G_j^H \subsetneq G_{j+1}^H$, and we set $\mathfrak{gr}^\bullet(\mathcal{H}) = \oplus_{j=0}^\infty G_j^H/G_{j-1}^H$, so the first degree elements are given as

$$\mathfrak{gr}^1(\mathcal{H}) = G_1^H/G_0^H = \{\Gamma \in \mathcal{A}_H | \tilde{\Delta}(\Gamma) = 0\}. \tag{26}$$

In this filtration,

$$2B_+^\Gamma \circ B_+^\Gamma(\mathbb{I}) \sim m\left(B_+^\Gamma(\mathbb{I}) \otimes B_+^\Gamma(\mathbb{I})\right), \tag{27}$$

so they are the same element in $\mathfrak{gr}^2(\mathcal{H}) = G_2^H/G_1^H$, an algebraic fact that by Φ_R is the starting point for the existence of the renormalization group [11,2].

Hochschild closedness Eq. (24) will also be essential when we relate a general DSE and its solution to the Hopf algebra of words in Section 2.3. This relation is reminiscent of the flag-decomposition [11] which appears when analyzing renormalized amplitudes as a limiting mixed Hodge structure.

We now turn to the Hopf algebra of words, for which the decreasing and increasing filtrations $\mathfrak{gr}_\bullet$ and $\mathfrak{gr}^\bullet$ above exist analogously.

2.2. Hopf algebra of words

Let $\mathcal{H}_W$ be the vector space of words and $\mathcal{H}_L \subset \mathcal{H}_W$ be the sub-space of letters. We need to collect some properties of $\mathcal{H}_W$. In the remainder of our paper, we abbreviate letters by $a, b, c, a_1, a_2, \ldots$ and words by $u, v, w, w_0, w_1, w_2, \ldots$.[2] Concatenation of letters creates words, e.g., ab and au form new words.

Let furthermore $\Theta : \mathcal{H}_L \times \mathcal{H}_L \to \mathcal{H}_L$ be a commutative and associative map that assigns a new letter to any two given letters. It is always assumed that $\mathcal{H}_L$ is completed if necessary so that it contains all images of Θ.

Using this map, we can recursively define the shuffle product $m_W : \mathcal{H}_W \otimes \mathcal{H}_W \to \mathcal{H}_W$ (also denoted by $\sqcup\kern-0.6em\sqcup_\Theta$) as

$$m_W(au \otimes bv) := au \sqcup\kern-0.6em\sqcup_\Theta bv = a(u \sqcup\kern-0.6em\sqcup_\Theta bv) + b(au \sqcup\kern-0.6em\sqcup_\Theta v) + \Theta(a,b)(u \sqcup\kern-0.6em\sqcup_\Theta v), \tag{28}$$

$$m_W(u \otimes \mathbb{I}_W) := u =: m_W(\mathbb{I}_W \otimes u), \tag{29}$$

where on the rhs of Eq. (28) any word in brackets is concatenated from right to the respective letter. The so defined shuffle product is commutative and associative[3] [15]. Thus, $(\mathcal{H}_W, m_W = \sqcup\kern-0.6em\sqcup_\Theta, \mathbb{I}_W)$ forms an associative and unital algebra.

$\mathcal{H}_W$ acquires a co-algebraic structure by introducing the co-product $\Delta_W : \mathcal{H}_W \to \mathcal{H}_W \otimes \mathcal{H}_W$ and the co-unit $\hat{\mathbb{I}}_W : \mathcal{H}_W \to \mathbb{K}$ acting on words as

$$\Delta_W(w) = \sum_{vu=w} u \otimes v, \qquad \hat{\mathbb{I}}_W(w) = \begin{cases} 1, & w = \mathbb{I}_W \\ 0, & \text{else.} \end{cases} \tag{30}$$

For example, the co-product of the word abc is

$$\Delta_W(abc) = \mathbb{I}_W \otimes abc + c \otimes ab + bc \otimes a + abc \otimes \mathbb{I}_W. \tag{31}$$

$\left(\mathcal{H}_W, m_W, \mathbb{I}_W, \Delta_W, \hat{\mathbb{I}}_W\right)$ forms a bi-algebra as well.

Finally, $(\mathcal{H}_W, m_W, \Delta_W, \mathbb{I}_W, \hat{\mathbb{I}}_W, S_W)$ forms a Hopf algebra, called the Hopf algebra of words. The antipode $S_W : \mathcal{H}_W \to \mathcal{H}_W$ fulfills

$$m_W \circ (S_W \otimes \mathrm{id}) \circ \Delta_W = m_W \circ (\mathrm{id} \otimes S_W) \circ \Delta_W = \mathbb{I}_W \circ \hat{\mathbb{I}}_W. \tag{32}$$

For more details on the Hopf algebra of words, the reader may consult the textbook of Reutenauer [15].

We finally introduce the grafting operators. The primitive elements of $\mathcal{H}_W$ are all letters, since $\Delta_W(a) = \mathbb{I}_W \otimes a + a \otimes \mathbb{I}$. We then define

$$B_+^a(u) := au, \tag{33}$$

which means that B_+^a concatenates the letter a from left to its argument word (or sum of words since B_+^a acts linearly). The reader may check using Eqs. (30) and (33), that the B_+^a are indeed, Hochschild one-co-cycles,

$$\Delta_W \circ B_+^a(u) = B_+^a(u) \otimes \mathbb{I}_W + \left(\mathrm{id} \otimes B_+^a\right) \circ \Delta_W(u). \tag{34}$$

Note that $\mathfrak{gr}_1(\mathcal{A}_{\mathcal{H}_W})$ is the linear span of words that cannot be written as a shuffle product, in analogy to the definitions for $\mathcal{H}$. Similarly, $\mathfrak{gr}^1(\mathcal{H}_W)$ can be identified with the completed set of letters.

---

[2] Note that the symbol u is used twice in this paper: It denotes the word $u \in \mathcal{H}_W$ and it abbreviates $u = \alpha L/2$. In the following sections, we only use $u \in \mathcal{H}_W$.

[3] In some references, this product is called a quasi-shuffle product.

2.3. Isomorphism between $\mathcal{H}$ and $\mathcal{H}_W$

We can now relate the Hopf algebra of Feynman graphs to the Hopf algebra of words. Indeed, there exists a unique Hopf algebra morphism $\Upsilon : \mathcal{H} \to \mathcal{H}_W$ that fulfills

$$\Upsilon(\mathbb{I}) = \mathbb{I}_W, \tag{35}$$

$$m_W \circ (\Upsilon \otimes \Upsilon) = \Upsilon \circ m, \tag{36}$$

$$\hat{\mathbb{I}}_W \circ \Upsilon = \Upsilon \circ \hat{\mathbb{I}}, \tag{37}$$

$$\Delta_W \circ \Upsilon = (\Upsilon \otimes \Upsilon) \circ \Delta, \tag{38}$$

$$S_W \circ \Upsilon = \Upsilon \circ S, \tag{39}$$

$$B_+^{a_n} \circ \Upsilon = \Upsilon \circ B_+^{\Gamma_n}. \tag{40}$$

It respects the Hopf algebraic structures. The existence of such a morphism is guaranteed by the fact that the grafting operators are Hochschild one-co-cycles [16]. For example, Eqs. (35) and (40) give

$$\Upsilon(\Gamma_n) = \Upsilon \circ B_+^{\Gamma_n}(\mathbb{I}) = B_+^{a_n} \circ \Upsilon(\mathbb{I}) = B_+^{a_n}(\mathbb{I}_W) = a_n \tag{41}$$

(Υ assigns the letter a_n to the primitive Feynman graph Γ_n).

We give another example. Consider the last graph in Fig. 1. With $\Theta(a_1, a_1, a_1) := \Theta(a_1, \Theta(a_1, a_1))$, we find

$$\begin{aligned}
\Upsilon(\gamma_{1,D}) &= \Upsilon \circ B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1 \cup \Gamma_1) \\
&= B_+^{a_1} \circ \Upsilon \circ m(\Gamma_1 \otimes m(\Gamma_1 \otimes \Gamma_1)) \\
&= B_+^{a_1} \circ m_W \circ (\Upsilon(\Gamma_1) \otimes \Upsilon \circ m(\Gamma_1 \otimes \Gamma_1)) \\
&= B_+^{a_1} \circ m_W \circ (a_1 \otimes m_W(a_1 \otimes a_1)) \\
&= B_+^{a_1}(a_1 \sqcup\!\sqcup_\Theta (a_1 \sqcup\!\sqcup_\Theta a_1)) \\
&= B_+^{a_1}(a_1 \sqcup\!\sqcup_\Theta (2a_1 a_1 + \Theta(a_1, a_1))) \\
&= B_+^{a_1}(6a_1 a_1 a_1 + 3a_1 \Theta(a_1, a_1) + 3\Theta(a_1, a_1)a_1 + \Theta(a_1, a_1, a_1)) \\
&= 6a_1 a_1 a_1 a_1 + 3a_1 a_1 \Theta(a_1, a_1) + 3a_1 \Theta(a_1, a_1)a_1 + a_1 \Theta(a_1, a_1, a_1).
\end{aligned} \tag{42}$$

The Hopf algebra morphism Υ allows us to translate any DSE to the Hopf algebra of words. In particular, applying Υ to Eqs. (3) and (5) and using Eqs. (35)–(40) yields

$$W_{\text{Yuk}} := \Upsilon(X_{\text{Yuk}}) = \mathbb{I}_W - \sum_{j \geq 1} \alpha^j B_+^{a_j}\left(W_{\text{Yuk}}^{\sqcup\!\sqcup_\Theta (1-2j)}\right), \tag{43}$$

$$W_{\text{QED}} := \Upsilon(X_{\text{QED}}) = \mathbb{I}_W - \sum_{j \geq 1} \alpha^j B_+^{a_j}\left(W_{\text{QED}}^{\sqcup\!\sqcup_\Theta (1-j)}\right). \tag{44}$$

Here, we drop any further subscript on letters, hence $a_j^{\text{QED}} = a_j$ and $a_j^{\text{Yuk}} = a_j$.

We finally solve Eqs. (43) and (44) via the Ansätze

$$W_{\text{Yuk}} = w_0^{\text{Yuk}} - \sum_{n \geq 1} \alpha^n w_n^{\text{Yuk}}, \qquad W_{\text{QED}} = w_0^{\text{QED}} - \sum_{n \geq 1} \alpha^n w_n^{\text{QED}} \tag{45}$$

and obtain

$$w_0^{\text{Yuk}} = \mathbb{I}_W, \qquad w_n^{\text{Yuk}} = a_n + \sum_{j=1}^{n-1} \sum_{k=1}^{n-j} \binom{2j-2+k}{k} B_+^{a_j}\left(\sum_{\substack{t_i \geq 1 \\ t_1 + \cdots + t_k = n-j}} w_{t_1}^{\text{Yuk}} \sqcup\!\sqcup_\Theta \cdots \sqcup\!\sqcup_\Theta w_{t_k}^{\text{Yuk}}\right), \tag{46}$$

$$w_0^{\text{QED}} = \mathbb{I}_W, \qquad w_n^{\text{QED}} = a_n + \sum_{j=2}^{n-1} \sum_{k=1}^{n-j} \binom{j-2+k}{k} B_+^{a_j}\left(\sum_{\substack{t_i \geq 1 \\ t_1 + \cdots + t_k = n-j}} w_{t_1}^{\text{QED}} \sqcup\!\sqcup_\Theta \cdots \sqcup\!\sqcup_\Theta w_{t_k}^{\text{QED}}\right). \tag{47}$$

The first orders are $w_1^{\text{QED}} = a_1$, $w_2^{\text{QED}} = a_2$ and $w_1^{\text{Yuk}} = a_1$, all others are recursively given. *This Ansatz can also be used for any other DSE* [6].

Renormalized Feynman rules linearly act on words as

$$\Psi_R = \Phi_R \circ \Upsilon^{-1} \tag{48}$$

and $\Psi_R (\alpha^n w_n) = \alpha^n \Psi_R(w_n) \propto \alpha^n$ in the log-expansion (Eq. (1)). The remaining question is now: which part of W maps to which power of the external scale parameter L in the log-expansion (Eq. (1))? We will answer this question in the next section, in full accordance with the blow-ups needed for Feynman integrands from the viewpoint of algebraic geometry [11,2].

A remark is in order concerning the map Υ^{-1}. Its existence as a Hopf algebra morphism is not obvious from the outset. Indeed, the map Υ from graphs to words can have a kernel precisely when the map Θ is trivial and the quasi-shuffle $\sqcup\!\sqcup_\Theta$ becomes the ordinary shuffle $\sqcup\!\sqcup$ (see Lemma 2.1 and Remark 2.2 in [11]).

Here, we assume that Θ is non-degenerate and Υ acts on complete sums of graphs given by iterations of maps $B_+^{\Gamma_i}$. Then Υ is invertible as a Hopf morphism.

3. Filtrations in Dyson–Schwinger equations

In the following, we filter the coefficients w_n in the solution of any DSE (as occurring in Eq. (45)). Each filtered term then maps to a certain power of L in the log-expansion (Eq. (1)).

We derive filtration rules for words (in general, w_n is a sum of words) by considering their dual elements in the universal enveloping algebra $\mathcal{U}_{\mathscr{L}}$. We introduce $\mathcal{U}_{\mathscr{L}}$ as the dual Hopf algebra to $\mathcal{H}_W$ in the next section. Section 3.2 states and proves the two most important properties of renormalized Feynman rules. This also explains how the filtration of words works. In Section 3.3, we finally present a canonical filtration algorithm for arbitrary words and prove that it is free of redundancies.

3.1. The dual Hopf algebra to $\mathcal{H}_W$

Let $\mathscr{L}$ be a vector space over a field $\mathbb{K}$. Let furthermore $[\cdot, \cdot] : \mathscr{L} \otimes \mathscr{L} \to \mathscr{L}$ be a bi-linear map that fulfills $[x, x] = 0$ as well as the Jacobi-identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \tag{49}$$

$\forall x, y, z \in \mathscr{L}$. $(\mathscr{L}, [\cdot, \cdot])$ (or shortly $\mathscr{L}$) is called a Lie algebra and the bi-linear map $[\cdot, \cdot]$ (called Lie bracket) is antisymmetric.

$\mathscr{L}$ acquires a descending series of sub-algebras $\mathscr{L} = \mathscr{L}_1 \trianglerighteq \mathscr{L}_2 \trianglerighteq \mathscr{L}_3 \trianglerighteq \ldots$, where $\mathscr{L}_{n+1}$ is generated by all $[x, y]$ with $x \in \mathscr{L}$ and $y \in \mathscr{L}_n$. In the following, we denote all basis elements of some Lie algebra $\mathscr{L}$ that are not in $\mathscr{L}_2$ by $x_1, x_2, x_3, \ldots$ and ask for a complete basis in terms of these x_i. For example, is $[x_1, x_2]$ or $[x_2, x_1] = -[x_1, x_2]$ a basis element of $\mathscr{L}$? Both are linearly dependent.

We therefore consider the Hall basis [17], which exists for any Lie algebra. It requires a lexicographical ordering of all elements in $\mathscr{L}$, for example, let $x_1 < x_2 < \cdots < [x_1, x_2] < \cdots$ (it does not matter which ordering we take as long as we choose one). We then define $[x, x']$ to be a (Hall) basis element of $\mathscr{L}$ iff both,

1. $x, x' \in \mathscr{L}$ are (Hall) basis elements with $x < x'$,
2. if $x' = [x'', x''']$, then $x \geq x''$

are fulfilled. For example, $[x_1, x_2]$, $[x_2, [x_1, x_3]]$ and $[x_3, [x_1, x_2]]$ are (Hall) basis elements while $[x_2, x_1]$, $[x_1, [x_2, x_3]]$ and $[[x_1, x_2], x_3]$ etc. are not.

The Lie algebra $(\mathscr{L}, [\cdot, \cdot])$ is not necessarily an algebra. In general, it only comes with a Lie bracket, not with an associative product. However, one can construct enveloping algebras, i.e. an *algebra* $(\mathcal{A}_{\mathscr{L}}, m_{\mathcal{A}}, \mathbb{I}_{\mathcal{A}})$ such that there exists a homomorphism $\rho_{\mathcal{A}} : \mathscr{L} \to \mathcal{A}_{\mathscr{L}}$ fulfilling

$$m_{\mathcal{A}}(\rho_{\mathcal{A}}(x) \otimes \rho_{\mathcal{A}}(y)) - m_{\mathcal{A}}(\rho_{\mathcal{A}}(y) \otimes \rho_{\mathcal{A}}(x)) = \rho_{\mathcal{A}}([x, y]) \tag{50}$$

$\forall x, y \in \mathscr{L}$.

There may be several enveloping algebras but we can always find a unique *universal enveloping algebra* $(\mathcal{U}(\mathcal{L}), m_{\mathcal{U}}, \mathbb{I}_{\mathcal{U}})$ up to isomorphism: for each enveloping algebra $(\mathcal{A}_{\mathcal{L}}, m_{\mathcal{A}}, \mathbb{I}_{\mathcal{A}})$ there exists a unique algebra homomorphism $\rho_{\mathcal{U} \to \mathcal{A}}$ such that the following diagram

$$\begin{array}{ccc} \mathcal{L} & \xrightarrow{\ \rho_{\mathcal{U}}\ } & \mathcal{U}(\mathcal{L}) \\ & \rho_{\mathcal{A}}\searrow & \ \downarrow \rho_{\mathcal{U} \to \mathcal{A}} \\ & & \mathcal{A}_{\mathcal{L}} \end{array} \tag{51}$$

commutes. $\mathcal{U}(\mathcal{L})$ is unique (assume that there are two universal enveloping algebras $\mathcal{U}_1(\mathcal{L})$ and $\mathcal{U}_2(\mathcal{L})$, then the homomorphism $\rho_{\mathcal{U}_1 \to \mathcal{U}_2}$ turns out to be an isomorphism).

Let us now construct the universal enveloping algebra for any Lie algebra $\mathcal{L}$, which will prove its existence as well. Consider the tensor algebra $(T(\mathcal{L}), \otimes, 1)$, where

$$T(\mathcal{L}) = \bigoplus_{n \geq 0} \mathcal{L}^{\otimes n} = \mathbb{K} \oplus \mathcal{L} \oplus (\mathcal{L} \otimes \mathcal{L}) \oplus \ldots \tag{52}$$

We define the sub-space $I \subset T(\mathcal{L})$ as

$$I := \{ s \otimes (x \otimes y - y \otimes x - [x, y]) \otimes t \,|\, x, y \in \mathcal{L};\ s, t \in T(\mathcal{L}) \} \tag{53}$$

(I is a 2-sided ideal) and introduce equivalent classes of $T(\mathcal{L})$,

$$[t] = \{ s \in T(\mathcal{L}) \,|\, s - t \in I \}. \tag{54}$$

For example, $[[x_1, x_2]] = \{[x_1, x_2], x_1 \otimes x_2 - x_2 \otimes x_1, \ldots\}$ and $[x_1 \otimes x_2 \otimes x_3] = \{x_1 \otimes x_2 \otimes x_3, [x_1, x_2] \otimes x_3 + x_2 \otimes x_1 \otimes x_3, \ldots\}$ etc.

All such equivalent classes together form a vector space, denoted by $T(\mathcal{L})/I$. The sum of two elements $[s], [t] \in T(\mathcal{L})/I$ is well defined as $[s + t]$. We carefully abbreviate $T(\mathcal{L})/I$ by $\mathcal{U}(\mathcal{L})$ and define an associative product

$$m_{\mathcal{U}} : \mathcal{U}(\mathcal{L}) \otimes \mathcal{U}(\mathcal{L}) \to \mathcal{U}(\mathcal{L}) \tag{55}$$

acting on equivalent classes as

$$m_{\mathcal{U}}([s] \otimes [t]) := [s \otimes t]. \tag{56}$$

Thus, $(\mathcal{U}(\mathcal{L}), m_{\mathcal{U}}, [1])$ forms an algebra and together with the homomorphism $\rho_{\mathcal{U}} : \mathcal{L} \to \mathcal{U}(\mathcal{L})$ defined as $\rho_{\mathcal{U}}(x) = [x] \forall x \in \mathcal{L}$, it is an enveloping algebra (Eq. (50) holds). Finally, $\mathcal{U}(\mathcal{L})$ is even the universal enveloping algebra of $\mathcal{L}$, since $T(\mathcal{L})$ fulfills the universality property in Eq. (51) as well [18].

It turns out that the universal enveloping algebra of a Lie algebra acquires a Hopf algebra structure. Upon setting

$$\Delta_{\mathcal{U}}([x]) = [x] \otimes \mathbb{I} + \mathbb{I} \otimes [x] \qquad \forall x \in \mathcal{L}, \tag{57}$$

it is determined from compatibility with the product $m_{\mathcal{U}}$.

For the Hopf algebra of words $\mathcal{H}_W$, there exists a Lie algebra $\mathcal{L}$ such that $\mathcal{U}(\mathcal{L})$ is dual to $\mathcal{H}_W$ (Milnor–Moore theorem [18]).

The indicated Lie algebra is easily constructed. For each letter $a_1, a_2, \ldots, \Theta(a_1.a_2), \ldots \in \mathcal{H}_L \subset \mathcal{H}_W$, we name one respective element $x_1, x_2, \ldots, \Theta(x_1, x_2), \ldots \in \mathcal{L}/\mathcal{L}_2$. Note that $\mathcal{L}$ contains more elements than $\mathcal{H}_L$ (for example, there is no element $[l_1, l_2]$ in $\mathcal{H}_L$, but $[x_1, x_2] \in \mathcal{L}$). This will be crucial in the following.

Duality between $\mathcal{H}_W$ and $\mathcal{U}(\mathcal{L})$ allows us to uniquely define a linear and invertible map $\eta : \mathcal{H}_W \to \mathcal{U}(\mathcal{L})$ (see Section 3.3.2), such that

$$\eta(a_i) = [x_i], \qquad \eta(\Theta(a_i, a_j)) = [\Theta(x_i, x_j)], \qquad \eta(a_i a_j) = [x_i \otimes x_j], \ldots \tag{58}$$

$\forall i, j \in \mathbb{N}$. In general, concatenation of words is the dual operation of multiplication in $\mathcal{U}(\mathcal{L})$.

3.2. Renormalized Feynman rules: how the filtration of words works

We now give two crucial properties of renormalized Feynman rules, which we need for our filtration method. The proofs are collected below.

1. Let $u \in \mathcal{H}_W$ and $[x] \in \mathcal{U}(\mathcal{L})$ be its dual element ($\eta(u) = [x]$). If $x \in T(\mathcal{L})$ is also an element of $\mathcal{L} \subset T(\mathcal{L})$, then renormalized Feynman rules map u to the L-linear part of the log-expansion in Eq. (1),

$$\Psi_R(u) \propto L. \tag{59}$$

2. Renormalized Feynman rules are character-like,

$$\Psi_R(u \sqcup\!\sqcup_\Theta v) = \Psi_R(u) \cdot \Psi_R(v) \tag{60}$$

$\forall u, v \in \mathcal{H}_W$, where the dot on the rhs of Eq. (60) represents usual multiplication. Furthermore, $\Psi_R(a)/L$ is a period $\forall a \in \mathcal{H}_L$.

Let us consider some examples before we turn to the proofs (in the following, $i, j \in \mathbb{N}$). Each letter $a \in \mathcal{H}_L$ has a dual element $\eta(a) = [x] \in \mathcal{U}(\mathcal{L})$ such that $x \in \mathcal{L}$ (by construction of the dual elements, see Eq. (58)). Thus, $\Psi_R(a_i) \propto L$ and $\Psi_R(\Theta(\cdot, \cdot)) \propto L$. More interesting is the dual element of $a_i a_j - a_j a_i \in \mathcal{H}_W$. It is

$$\eta(a_i a_j - a_j a_i) = \eta(a_i a_j) - \eta(a_j a_i) = [x_i \otimes x_j] - [x_j \otimes x_i] = [[x_i, x_j]] \tag{61}$$

and $[x_i, x_j] \in \mathcal{L}$. Thus, $(\Psi_R(a_i a_j - a_j a_i)) \propto L^1$ although $\Psi_R(a_i a_j)$ also contains terms $\propto L^2$ [1].

Finally, we *filter* the word $a_i a_j$ to

$$a_i a_j = \frac{1}{2} a_i \sqcup\!\sqcup_\Theta a_j + \frac{1}{2}[a_i, a_j] - \frac{1}{2}\Theta(a_i, a_j). \tag{62}$$

Here, we abbreviated $a_i a_j - a_j a_i$ by the concatenation commutator $[a_i, a_j]$ discussed in Section 3.3.2. We treat concatenation (multi-)commutators of letters as a letter itself, since the respective dual Hopf algebra element is primitive.

Note the information content of our filtration. The first term of Eq. (62) maps to L^2 under renormalized Feynman rules and Eq. (60) tells us how to determine this L^2-term simply by calculating $\Psi_R(a_i)$ and $\Psi_R(a_j)$. The other two terms of Eq. (62) map to L.

The possibility to calculate the L^2-term in Eq. (62) out of the renormalized Feynman amplitudes for a_i and a_j (Eq. (60)) finally leads to the desired relations between next-to$^{\{j\}}$-leading log orders and terms up to $\mathcal{O}(\alpha^{j+1})$ in the log-expansion (Eq. (1)). We explore this in great detail in Section 4.

However, we first give the proofs for the necessary properties of Feynman rules stated above in light in particular of the duality of $\mathcal{H}_W$ and $\mathcal{U}(\mathcal{L})$. Terms linear in L can be interpreted in $\rho_\mathcal{U}(\mathcal{L}) \subset \mathcal{U}(\mathcal{L})$, and higher powers in L reflect terms in the quotient algebra $\mathcal{U}(\mathcal{L})$.

3.2.1. Proof of Claim 1

In the following, we assume that $B_+^\Gamma(X) = 0$ implies $X = 0, \forall X$. If not, we have a degeneracy which can appear in gauge theories and can be adopted by enlarging the Hopf algebra suitably.

As we stated before, renormalized Feynman rules map an element u that is dual to a Lie algebra element as above to the L-linear part of the log-expansion in Eq. (1),

$$\Psi_R(u) \propto L, \tag{63}$$

see Eq. (59).

This is a direct consequence of the renormalization group action on a single graph. In fact, let Γ be a Hopf algebra element of fixed co-radical degree r_Γ, $\Gamma \in \mathfrak{gr}_{r_\Gamma}(G)$. Then, it allows for an expansion

$$\Phi_R(\Gamma) = \sum_{j=1}^{r_\Gamma} c_j^\Gamma(\theta) L^j. \tag{64}$$

One way of phrasing the renormalization group is as saying that the leading log comes from all primitive renormalization parts of a diagram (see the scattering type formula in [4] and the $S \star Y$ iteration of [5]):

$$c_j^\Gamma = \mathbf{c}_1^{\otimes j} \tilde{\Delta}^{j-1}(\Gamma), \tag{65}$$

where $\mathbf{c}_1$ is the function $\mathbf{c}_1 : \Gamma \to c_1^\Gamma$. Here, we identified the tensor-product of values with their product ($\mathbb{C} \otimes_\mathbb{C} \mathbb{C} \simeq \mathbb{C}$):

$$\mathbf{c}_1^{\otimes j} : \underbrace{\mathcal{H} \otimes \cdots \otimes \mathcal{H}}_{j \text{ times}} \to \mathbb{C}. \tag{66}$$

Note that $\mathbf{c}_1^{\otimes j}$ is a symmetric function by construction.

This leads to a strict inequality on the co-radical degrees

$$r_{[\Gamma_1, \Gamma_2]} < r_{\Gamma_1} + r_{\Gamma_2}, \tag{67}$$

which implies the result, Eq. (59).

3.2.2. Proof of Claim 2

We also have that renormalized Feynman rules are character-like, (Eq. (60)),

$$\Psi_R(u \sqcup_\Theta v) = \Psi_R(u) \cdot \Psi_R(v) \tag{68}$$

$\forall u, v \in \mathcal{H}_W$. This is a direct consequence of Chen's Lemma [2,12] in this context.

To show that $\Psi_R(a)/L$ is a period $\forall a \in \mathcal{H}_L$ is non-trivial only for letters in the image of Θ. Therefore, it suffices to consider Feynman graphs (with fixed labels on their edges) that are nested insertions of primitive graphs into each other: a Hopf algebra element Γ is a flag if there exists a sequence of primitive graphs γ_i, $1 \leq i \leq r_\Gamma$, with

$$\tilde{\Delta}^{r_\Gamma - 1}(\Gamma) = \gamma_1 \otimes \cdots \otimes \gamma_{r_\Gamma}. \tag{69}$$

Similarly, we say that a sum G of r_Γ flags G_i,

$$G = \sum_i q_i G_i, \quad q_i \in \mathbb{Q}, \tag{70}$$

is a symmetrized flag if there exists a sequence of primitive graphs γ_i, $1 \leq i \leq r_\Gamma$, with

$$\tilde{\Delta}^{r_\Gamma - 1}(G) = \sum_\sigma \gamma_{\sigma(1)} \otimes \cdots \otimes \gamma_{\sigma(r_\Gamma)}. \tag{71}$$

The $\sum_\sigma$-sum is over $r_\Gamma!$ unsigned permutations. Instead of having the full permutation group acting, one could also make do with permutations so as to make the rhs co-commutative, if so desired.

For a given flag Γ, and hence given sequence of primitive graphs γ_i, $1 \leq i \leq r_\Gamma$, let n_Γ be the cardinality of the set

$$X_\Gamma := \{\Gamma \text{ a flag} | \tilde{\Delta}^{r_\Gamma - 1}(\Gamma) = \gamma_1 \otimes \cdots \otimes \gamma_{r_\Gamma}\}, \tag{72}$$

so $n_\Gamma = |X_\Gamma|$.

A symmetrized flag is complete if $q_i = 1/n_{G_i}$ in Eq. (70).

Finally, renormalized Feynman rules are a forest sum [2] in graph polynomials ψ, ϕ (see [2] for notation):

$$\Phi_R(\Gamma) = \int \sum_{f \in \mathcal{F}_\Gamma} (-1)^{|f|} \frac{\log \frac{\phi_{\Gamma/f} \psi_f + \phi_f^0 \psi_{\Gamma/f}}{\phi_{\Gamma/f}^0 \psi_f + \phi_f^0 \psi_{\Gamma/f}}}{\psi_{\Gamma/f}^2 \psi_f^2}. \tag{73}$$

Here, $\phi_\emptyset = 0$, $\psi_\emptyset = 1$.

The coefficient Φ_R^1 of the L-linear term is

$$\Phi_R^1(\Gamma) = \int \sum_{f \in \mathscr{F}_\Gamma} (-1)^{|f|} \frac{1}{\psi_{\Gamma/f}^2 \psi_f^2} \frac{\phi_{\Gamma/f} \psi_f}{\phi_{\Gamma/f} \psi_f + \phi_f \psi_{\Gamma/f}}, \tag{74}$$

if the renormalization point preserves scattering angles.

We then have the following result on the angle-independence of symmetrized flags: for any symmetrized flag G,

$$\Phi_R^1(G) := \sum_i q_i \Phi_R^1(\Gamma_i) = \sum_i q_i \int \sum_{f \in \mathscr{F}_{\Gamma_i}} (-1)^{|f|} \frac{1}{\psi_{\Gamma_i/f}^2 \psi_f^2}. \tag{75}$$

This justifies that $\Upsilon^{-1}(\Theta(\cdot, \cdot))$ is primitive in the Hopf algebra of Feynman graphs: to any set S of letters, we can assign a unique complete symmetrized flag G_S of Feynman graphs corresponding to the letters in S. We set $\Theta(S)$ such that

$$\Psi_R(\Theta(S)) = -\Phi_R^1(G_S). \tag{76}$$

From Eq. (75) follows that $\Theta(S)$ can indeed be regarded as a new letter in $\mathcal{H}_W$ because it is independent of scattering angles by construction.

We now prove Eq. (75): it follows immediately from writing Eq. (74) as elementary symmetric polynomials in the variables ϕ_x, ψ_x, with x ranging over all necessary forests and co-forests, which is possible precisely for symmetrized flags. Indeed, the denominator in Eq. (74)

$$\phi_{\Gamma/f} \psi_f + \phi_f \psi_{\Gamma/f} \tag{77}$$

is symmetric under exchange of $\phi \leftrightarrow \psi$, while in symmetrized flags, we also have co-commutativity which ensures symmetry under $\Gamma/f \leftrightarrow f$. Hence, in the sum for symmetrized flags, the factor

$$\frac{\phi_{\Gamma/f} \psi_f}{\phi_{\Gamma/f} \psi_f + \phi_f \psi_{\Gamma/f}} \tag{78}$$

in Eq. (74) turns to unity. This proves Eq. (75).

Let us consider an example: Have a look at the first two graphs in Fig. 1. The graph γ_1 has three vertices, the graph γ_2 has five.

Accordingly, there are three graphs $\gamma_{1,i}$, $1 \le i \le 3$, obtained by replacing a vertex of γ_1 by γ_2, and five graphs $\gamma_{2,i}$, $1 \le i \le 5$, obtained by replacing a vertex of γ_2 by γ_1.

We have the reduced co-products

$$\tilde{\Delta}\gamma_{1,i} = \gamma_2 \otimes \gamma_1, \quad \forall 1 \le i \le 3, \qquad \tilde{\Delta}\gamma_{2,i} = \gamma_1 \otimes \gamma_2, \quad \forall 1 \le i \le 5. \tag{79}$$

Set

$$X = \frac{1}{3}\left(\sum_{i=1}^3 \gamma_{1,i}\right) + \frac{1}{5}\left(\sum_{i=1}^5 \gamma_{2,i}\right). \tag{80}$$

We have $\tilde{\Delta}(X) = \gamma_1 \otimes \gamma_2 + \gamma_2 \otimes \gamma_1$, so X is a symmetrized flag, and it is complete. Then,

$$\Phi_R(X) = \frac{1}{3}\left(\sum_{i=1}^3 \Phi_R(\gamma_{1,i})\right) + \frac{1}{5}\left(\sum_{i=1}^5 \Phi_R(\gamma_{2,i})\right). \tag{81}$$

Using Eq. (74) and the reduced co-products above, we indeed find that the second Symanzik polynomial appearing in Eq. (74) drops out in this co-commutative sum of symmetrized graph insertions

$$\Phi_R^1(X) = \int \left(\frac{1}{3}\left(\sum_{i=1}^3 \frac{1}{\psi^2(\gamma_{1,i})}\right) + \frac{1}{5}\left(\sum_{i=1}^5 \frac{1}{\psi^2(\gamma_{2,i})}\right) - \frac{1}{\psi^2(\gamma_1)\psi^2(\gamma_2)}\right), \tag{82}$$

which is an example of the above result (see also [1]).

3.3. Filtration algorithm

3.3.1. Presentation of the filtration algorithm

We present a filtration algorithm that filters any Hopf algebra element in $\mathcal{H}_W$ to shuffle products of L-linear terms. A filtered word will then only consists of full shuffle products of letters.[4] It is not difficult to filter w_1 and w_2 in the solution of any DSE (as occurring in Eq. (45)). However, to filter higher order coefficients (w_n for $n > 2$) requires a canonical algorithm that we give here.

Consider for example w_n^{Yuk} for $n = 1, 2, 3$ (see Eq. (46)),

$$w_1^{\text{Yuk}} = a_1, \tag{83}$$

$$w_2^{\text{Yuk}} = a_2 + a_1 a_1 = \frac{1}{2} a_1 \sqcup_\Theta a_1 - \frac{1}{2} \Theta(a_1, a_1) + a_2, \tag{84}$$

$$w_3^{\text{Yuk}} = a_3 + 3 a_2 a_1 + a_1 a_2 + 3 a_1 a_1 a_1 + a_1 \Theta(a_1, a_1). \tag{85}$$

We already filtered w_1^{Yuk} and w_2^{Yuk} without much effort but it is not obvious to see the filtration for w_3^{Yuk}. The required filtration algorithm for a given expression in $\mathcal{H}_W$ is the following loop over the length k of occurring words. It starts with the maximal occurring length down to $k = 2$. Hence, in the case of w_n, we perform the loop below for $k = n, \ldots, 2$.

1. Bring all words with length k into lexicographical order[5] using the concatenation commutator. For example, the word $a_2 a_1 a_1$ is replaced by

$$a_2 a_1 a_1 = [a_2, a_1] a_1 + a_1 [a_2, a_1] + a_1 a_1 a_2. \tag{86}$$

Note that all words containing newly introduced commutator letters are of length $(k-1)$. We bring them into lexicographical order within the next cycle.
2. Write all newly introduced commutators into the Hall basis. This is always possible using the Jacobi identity and the antisymmetry of the Lie bracket. In the example above, we must simply replace $[a_2, a_1] = -[a_1, a_2]$.
3. All words with k letters are now lexicographically ordered. For each of those, compute the full shuffle product of the k corresponding letters and repeat steps 1 and 2 with the resulting expressions. In the above example, we must compute

$$a_1 \sqcup_\Theta a_1 \sqcup_\Theta a_2 = 2 a_1 a_1 a_2 + 2 a_1 a_2 a_1 + 2 a_2 a_1 a_1 + 2 a_1 \Theta(a_1, a_2) + 2 \Theta(a_1, a_2) a_1$$
$$+ a_2 \Theta(a_1, a_1) + \Theta(a_1, a_1) a_2 + \Theta(a_1, a_1, a_2). \tag{87}$$

After step 1 and 2, we obtain

$$a_1 \sqcup_\Theta a_1 \sqcup_\Theta a_2 = 6 a_1 a_1 a_2 - 4 a_1 [a_1, a_2] - 2 [a_1, a_2] a_1 + 2 a_1 \Theta(a_1, a_2) + 2 \Theta(a_1, a_2) a_1$$
$$+ a_2 \Theta(a_1, a_1) + \Theta(a_1, a_1) a_2 + \Theta(a_1, a_1, a_2). \tag{88}$$

4. We can now introduce these full shuffle products into the original expression such that all words with length k drop out. In our example, we insert Eq. (88) into Eq. (86) such that the word $a_1 a_1 a_2$ drops out. Hence, the entire cycle results in

$$a_2 a_1 a_1 = \frac{1}{6} a_1 \sqcup_\Theta a_1 \sqcup_\Theta a_2 - \frac{1}{3} a_1 [a_1, a_2] - \frac{2}{3} [a_1, a_2] a_1 - \frac{1}{3} a_1 \Theta(a_1, a_2) - \frac{1}{3} \Theta(a_1, a_2) a_1$$
$$- \frac{1}{6} a_2 \Theta(a_1, a_1) - \frac{1}{6} \Theta(a_1, a_1) a_2 - \frac{1}{6} \Theta(a_1, a_1, a_2). \tag{89}$$

The introduced full shuffle products remain untouched in the following cycles. In our example, the word $a_2 a_1 a_1$ is not entirely filtered but the next cycle ($k = 2$) will close the filtration.

---

[4] Letters are $a_1, a_2, a_3 \ldots$, all $\Theta(\cdot, \cdot)$ and concatenation commutator letters $[\cdot, \cdot]$.

[5] In most literature, the phrase 'lexicographical order' is used for sorting words. We rather use it for the ordering of letters within a word. For example, the word $a_1 a_2 a_3$ is lexicographically ordered but $a_1 a_3 a_2$ is not.

Let us illustrate this algorithm for w_3^{Yuk}. The only word with length $k = 3$ in Eq. (85) is $a_1 a_1 a_1$ and it is already given in lexicographical order. We thus, skip steps 1 and 2 of the above algorithm and continue with step 3. We calculate the corresponding full shuffle product

$$a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1 = 6a_1 a_1 a_1 + 3a_1 \Theta(a_1, a_1) + 3\Theta(a_1, a_1)a_1 + \Theta(a_1, a_1, a_1) \tag{90}$$

and insert it into w_3^{Yuk} such that the word $a_1 a_1 a_1$ drops out. Hence,

$$w_3^{\text{Yuk}} = a_3 - \frac{3}{2}\Theta(a_1, a_1)a_1 - \frac{1}{2}a_1\Theta(a_1, a_1) + 3a_2 a_1 + a_1 a_2$$
$$- \frac{1}{2}\Theta(a_1, a_1, a_1) + \frac{1}{2}a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1 \tag{91}$$

and we proceed with the next cycle ($k = 2$). The term $a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1$ remains untouched during the rest of the filtration.

The words $a_2 a_1$ and $\Theta(a_1, a_1)a_1$ are not in lexicographical order and we write

$$w_3^{\text{Yuk}} = a_3 + \frac{3}{2}[a_1, \Theta(a_1, a_1)] - 2a_1\Theta(a_1, a_1) - 3[a_1, a_2] + 4a_1 a_2$$
$$- \frac{1}{2}\Theta(a_1, a_1, a_1) + \frac{1}{2}a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1. \tag{92}$$

Here, we already performed step 2 of the above algorithm and only introduced Hall basis elements ($[a_1, a_2]$ instead of $[a_2, a_1]$ etc.). Let us carry on with step 3. The respective shuffle products are

$$a_1 \sqcup\!\sqcup_\Theta a_2 = 2a_1 a_2 - [a_1, a_2] + \Theta(a_1, a_2), \tag{93}$$
$$a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) = 2a_1\Theta(a_1, a_1) - [a_1, \Theta(a_1, a_1)] + \Theta(a_1, a_1, a_1) \tag{94}$$

(they are already brought into lexicographical order using the concatenation commutator, steps 1 and 2). Inserting Eqs. (93) and (94) into Eq. (92) such that the words $a_1 a_2$ and $a_1\Theta(a_1, a_1)$ drop out finally results in

$$w_3^{\text{Yuk}} = \frac{1}{2}a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1 + 2a_1 \sqcup\!\sqcup_\Theta a_2 - a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) + a_3$$
$$+ \frac{1}{2}[a_1, \Theta(a_1, a_1)] - [a_1, a_2] + \frac{1}{2}\Theta(a_1, a_1, a_1) - 2\Theta(a_1, a_2). \tag{95}$$

The explicit filtration algorithm is the basis for Section 4. There, we derive the relations for next-to$^{\{j\}}$-leading log terms in the log expansion (Eq. (1)). However, we first give some basics to Hall words and concatenation (multi-) commutator letters. This explains, why the filtration algorithm described above is redundancy-free.

3.3.2. Hall words and concatenation (multi-)commutators

In the Hopf algebra of Feynman graphs $\mathcal{H}$, any $\Gamma \in \mathcal{H}$ evaluates to $\Phi_R(\Gamma) = \sum_{j=1}^{r_\Gamma} c_j^\Gamma L^j$. Here, $\Gamma \in \mathfrak{gr}^{r_\Gamma}(\mathcal{H})$ and the c_j^Γ are given through Eq. (65). In particular, this amounts for $j = r_\Gamma$ to an evaluation of products of primitive elements.

Through Υ, we inherit the same properties for words [11]: any $u \in \mathcal{H}_W$ evaluates to $\Psi_R(u) = \sum_{j=1}^{r_u} d_j^u L^j$, where $u \in \mathfrak{gr}^{r_u}(\mathcal{H}_W)$ and

$$d_j^u = c_j^{\Upsilon^{-1}(u)}. \tag{96}$$

In particular, this amounts for $j = r_u$ to an evaluation of products of letters. Note that Υ preserves the co-radical degree.

The above filtration algorithm answers the question of how to obtain non-leading logs, $j < r_u$, from the letters that constitute u. For this, we first have to consider the lower central series filtration $\mathfrak{gr}_\bullet(\mathcal{L})$ of the Lie algebra $\mathcal{L}$, $\mathfrak{gr}_k(\mathcal{L}) = \mathcal{L}_k/\mathcal{L}_{k+1}$ and its associated grading. Secondly, we have to consider the filtrations and gradings of the universal enveloping algebra: $\mathfrak{gr}_\bullet(\mathcal{U}_\mathcal{L})$ by its augmentation and $\mathfrak{gr}^\bullet(\mathcal{U}_\mathcal{L})$ by its co-radical.

We will use that $\mathfrak{gr}_k(\mathcal{U}_{\mathcal{L}})$ is isomorphic to the k-fold symmetric tensor-power of $\mathcal{L}$ by the Poincaré–Birkhoff–Witt theorem:

$$\mathfrak{gr}_k(\mathcal{U}_{\mathcal{L}}) \sim \mathfrak{Sym}\left(\rho_{\mathcal{U}}(\mathcal{L})^{\otimes k}\right). \tag{97}$$

Let $\eta : \mathscr{H}_W \to \mathcal{U}_{\mathcal{L}}$ as before. We have the commutative diagrams

$$\tag{98}$$

and two more, by replacing $\eta \to \eta^{-1}$ and downward pointing arrows by upward pointing ones. These determine the action of η once we have defined it on letters $a \in \mathscr{H}_L$ ($\eta(a_i) = [x_i]$, $\eta(\Theta(a_i, a_j)) = [\Theta(x_i, x_j)] \ldots$). For example, the degree two image $\eta(aa)$ of the word aa with respect to $\mathfrak{gr}_{\bullet}(\mathcal{U}_{\mathcal{L}})$ is $\frac{1}{2}[x \otimes x]$, with $\eta(a) = [x]$.

We define Feynman rules for elements $[s] \in \mathcal{U}_{\mathcal{L}}$ by

$$\Psi_R^{\eta} : [s] \to \Psi_R(\eta^{-1}([s])). \tag{99}$$

In particular for homogeneous elements $[s] \in \mathfrak{gr}_j(\mathcal{U}_{\mathcal{L}})$ we have $\Psi_R^{\eta}([s]) \sim L^j$, by construction.

The structure of renormalized Feynman rules then allows us to regain the above filtration algorithm on words w as

$$w \to \Theta_{\mathcal{U}}\left(\sum_{j=1}^{|w|} P_j\left(\eta^{\otimes |w|} \tilde{\Delta}^{|w|-1}(w)\right)\right). \tag{100}$$

Here, P_j is the projection into the grade j piece, $P_j(\eta(w)) \in \mathfrak{gr}_j(\mathcal{U}_{\mathcal{L}})$, $\forall \eta(w) \in \mathcal{U}_{\mathcal{L}}$. $\Theta_{\mathcal{U}}$ is the map $[x_1 \otimes \cdots \otimes x_j] \to \eta \circ \Theta(\eta^{-1}([x_1]), \ldots, \eta^{-1}([x_j]))$, where the $\eta^{-1}([x_i]) \in \mathscr{H}_L$ are letters by construction ($x_i \in \mathcal{L}$).

The lower central series filtration $\mathfrak{gr}_k(\mathcal{L}) = \mathcal{L}_k / \mathcal{L}_{k+1}$ filters in particular $\mathfrak{gr}_1(\mathcal{U}_{\mathcal{L}}) \sim \mathcal{L}$. Thus, using the Hall basis of $\mathcal{U}_{\mathcal{L}}$ and the invertibility of η finally allows us to write the filtration algorithm in the word algebra $\mathscr{H}_W$ with concatenation multi-commutators.

Indeed, the rhs of Eq. (100) is of degree one by construction as it is in the image of $\Theta_{\mathcal{U}}$. This suffices, as the degree-j piece is a product of the corresponding j degree-one pieces obtained in $\tilde{\Delta}^{j-1}(w)$.

Let us consider an example. Words on three letters a_1, a_2, a_3 have a Hall basis, which *for their degree one part* can be written in

$$\{y_1 := \Theta(a_1, a_2, a_3),\ y_2 := \Theta(a_1, [a_2, a_3]),\ y_3 := \Theta(a_2, [a_1, a_3]),$$
$$y_4 := \Theta(a_3, [a_1, a_2]),\ y_5 := [a_2, [a_1, a_3]],\ y_6 := [a_3, [a_1, a_2]]\}. \tag{101}$$

In degree one this is the inverse image η^{-1} of the elements

$$\{x_1 x_2 x_3,\ x_1[x_2, x_3],\ x_2[x_1, x_3],\ x_3[x_1, x_2],\ [x_2, [x_1, x_3]],\ [x_3, [x_1, x_2]]\} \tag{102}$$

in $\mathcal{U}_{\mathcal{L}}$ written in Hall basis notation (ordered and omitting the symmetric tensor product). These form a standard Hall basis on three 'letters' x_1, x_2, x_3 dual to a_1, a_2, a_3 in $\mathcal{U}_{\mathcal{L}}$.

There are six words on three distinct letters a_1, a_2, a_3. For their degree one part, these can be written in the basis above:

$$a_1 a_2 a_3 = (y_1 + 3y_2 + 3y_3 + 3y_4 + 2y_5 + 4y_6)/6, \tag{103}$$
$$a_1 a_3 a_2 = (y_1 - 3y_2 + 3y_3 + 3y_4 + 2y_5 + 4y_6)/6, \tag{104}$$
$$a_2 a_1 a_3 = (y_1 + 3y_2 + 3y_3 - 3y_4 - 4y_5 - 2y_6)/6, \tag{105}$$
$$a_2 a_3 a_1 = (y_1 + 3y_2 - 3y_3 - 3y_4 - 4y_5 - 2y_6)/6, \tag{106}$$
$$a_3 a_1 a_2 = (y_1 - 3y_2 - 3y_3 + 3y_4 + 2y_5 - 2y_6)/6, \tag{107}$$
$$a_3 a_2 a_1 = (y_1 - 3y_2 - 3y_3 - 3y_4 + 2y_5 - 2y_6)/6. \tag{108}$$

Inverting these equations expresses the degree one elements y_i through the six words on the left. These correspond to linear combinations of Feynman graphs $B_+^{\Gamma_i} \circ B_+^{\Gamma_j} \circ B_+^{\Gamma_k}(\mathbb{I})$, $i, j, k \in \{1, 2, 3\}$ which map under Υ to the corresponding words. For example,

$$y_2 \equiv \Theta(a_1, [a_2, a_3]) = \eta^{-1} P_1 \eta (a_1 a_2 a_3 - a_1 a_3 a_2 + a_2 a_3 a_1 - a_3 a_2 a_1), \tag{109}$$

with

$$\Psi_R(y_2) = \Psi_R^1 (a_1 a_2 a_3 - a_1 a_3 a_2 + a_2 a_3 a_1 - a_3 a_2 a_1). \tag{110}$$

Furthermore $P_3(a_1 a_2 a_3 - a_1 a_3 a_2 + a_2 a_3 a_1 - a_3 a_2 a_1) = 0$ so there is no term $\sim L^3$, whilst the term in L^2 is

$$\Psi_R^1(a_1) \Psi_R^1([a_2, a_3]) - \frac{1}{2} \left(\Psi_R^1(a_2) \Psi_R^1([a_1, a_3]) + \Psi_R^1(a_3) \Psi_R^1([a_2, a_1]) \right). \tag{111}$$

This gives us a definition in terms of Feynman diagrams for

$$
\begin{aligned}
\Upsilon^{-1}(y_2) = {} & \left(B_+^{\Gamma_1} B_+^{\Gamma_2} B_+^{\Gamma_3}(\mathbb{I}) - B_+^{\Gamma_1} B_+^{\Gamma_3} B_+^{\Gamma_2}(\mathbb{I}) + B_+^{\Gamma_2} B_+^{\Gamma_3} B_+^{\Gamma_1}(\mathbb{I}) - B_+^{\Gamma_3} B_+^{\Gamma_2} B_+^{\Gamma_1}(\mathbb{I}) \right) \\
& - B_+^{\Gamma_1}(\mathbb{I}) \left(B_+^{\Gamma_2} B_+^{\Gamma_3}(\mathbb{I}) - B_+^{\Gamma_3} B_+^{\Gamma_2}(\mathbb{I}) \right) \\
& + \frac{1}{2} \left(B_+^{\Gamma_2}(\mathbb{I}) \left(B_+^{\Gamma_1} B_+^{\Gamma_3}(\mathbb{I}) - B_+^{\Gamma_3} B_+^{\Gamma_1}(\mathbb{I}) \right) + B_+^{\Gamma_3}(\mathbb{I}) \left(B_+^{\Gamma_2} B_+^{\Gamma_1}(\mathbb{I}) - B_+^{\Gamma_1} B_+^{\Gamma_2}(\mathbb{I}) \right) \right),
\end{aligned} \tag{112}
$$

which defines a L-linear term.

Let us now describe the standard Hall basis for a set of words on n distinct letters in general. The case of repeated letters follows easily. Our result is in agreement with the study of heap-ordered forests in [9]. There are $n!$ words we can form. First, we count with the help of the Möbius function μ the number of available concatenation multi-commutators C_n.

$$C_n = \sum_{j=2}^{n} \binom{n}{j} (-1)^{n-j} C_n^j = (n-1)!, \tag{113}$$

with $C_n^j = \frac{1}{n} \sum_{d|n} \mu(d) j^{n/d}$ the well-known number of multi-commutators of degree n on an alphabet of size j.

Let $\mathfrak{P}(n)$ be the set of partitions p of the integer n with the following properties:

$$n = p_1 + \cdots + p_k, \quad p_i \le p_{i+1}, \ p_i \ge 2, i \ge 2. \tag{114}$$

We allow for at most one such p_i to be marked, which we indicate as $\dot{p}_i$. Furthermore, if $p_1 = 1, p_1$ must be marked. If p contains such a distinguished marked $\dot{p}_i$, we say p is marked at i, else it is unmarked.

The marking reflects the fact that $\mathfrak{gr}_1(\mathcal{L}) = \mathcal{L}/[\mathcal{L}, \mathcal{L}]$ is distinguished amongst all $\mathfrak{gr}_j(\mathcal{L})$.

We say that a partition q of a set A_n of n letters is compatible with the partition p of the integer n, if it is a disjoint union of sets A_{p_i} of p_i letters according to p. Letters in $A_{\dot{p}_i}$ are completely symmetrized, while all other sit in multi-commutators of degree p_j.

Assume $p \in \mathfrak{P}(n)$ is unmarked. Then we assign a set of letters

$$X_q = \{\Theta_{\mathcal{U}}(l_1, \ldots, l_k)\} \tag{115}$$

on k multi-commutators $l_i \in C_{p_i}$ on letters $p_i \in A_{p_i}$ to it.

If p is marked at i, we assign a set of letters

$$X_q = \{\Theta_{\mathcal{U}}(l_1, \ldots, l_k, a_1, \ldots, a_{p_i}), a_j \in A_{\dot{p}_i}\}. \tag{116}$$

Then, summing over all partitions p and all partitions of letters q compatible with it, we get $n!$ different words which form a base for the degree one elements of $\mathcal{U}_{\mathcal{L}}$.

If a partition p contains an unmarked integer p_i say r_i times, the symmetry factor $S(p)$ of p is $S(p) := \prod_i r_i!$. Then, we indeed count

$$n! = \sum_p \binom{n}{p_1 \cdots p_k} \frac{1}{S(p)} \prod_{i=1}^{k} N_i, \tag{117}$$

where $N_i = C_{p_i}$ if p is not marked at i, and $N_i = 1$ if it is marked at i.

We complete this section by giving a final example. For four distinct letters, we can have the partition $p = 4 + 0$ with p unmarked. So it will provide six elements in C_4. For the partition $p = \dot{1} + 3$ we have $4 = \frac{4!}{3!}$ possibilities to choose three letters for C_3 which itself has two elements, whilst the fourth letter belongs to $\dot{1}$. For the partition $p = \dot{2} + 2$ we have $6 = \frac{4!}{2!2!}$ possibilities to choose two letters for a one-element C_2 while the other two letters constitute $\dot{2}$. For $p = 2 + 2$ we get a non-trivial symmetry factor and have $3 = \frac{4!}{2!2!} \frac{1}{2!}$ possibilities to form the product $C_2 \times C_2$. Finally, we have the partition $\dot{4}$. This gives a single element − the symmetric sum over all permutations of four letters.

Counting, we get

$$24 = 1 \times 6 + 4 \times 2 + 6 \times 1 + 3 \times 1 \times 1 + 1. \tag{118}$$

All such words are independent by construction and using Eq. (113) repeatedly there are $n!$ of them. They hence form a base.

4. Relations for the log expansion

We now present the main result of our work: how to write the next-to-$^{[j]}$-leading log order as a function of terms up to $\mathcal{O}(\alpha^{j+1})$ in the log-expansion (Eq. (1))? We first introduce a convenient notation for multiplicities of full shuffle products in a filtered word. Secondly, we derive generating functions for these multiplicities using the example of the Yukawa propagator. We derive the generating functions for the QED photon self-energy in Appendix A.

4.1. Notation

We represent the multiplicity of shuffle products in a filtered word by []-bracketed matrices m, for example

$$w_2^{\text{Yuk}} = m_1 a_1 \sqcup\!\sqcup_\Theta a_1 + m_2 \Theta(a_1, a_1) + m_3 a_2. \tag{119}$$

Each matrix denotes a number, in our case $m_1 = -m_2 = 1/2$, $m_3 = 1$ (see Eq. (84)). In the following we say that a matrix m *belongs to* a shuffle product S, when it gives the multiplicity of S in a filtered word. We also say that S is the *respective shuffle product* to m.

Each matrix m with corresponding shuffle product S is built as follows: the first row contains the numbers of letters $a_1, a_2, \ldots$ in S. The other rows represent one letter $\Theta(\ldots)$ in S each, s.t. $\ldots$ contains m_{ij} letters a_j for the ith row. For example, the filtered word w_{14} contains the term,

$$w_{14} = \begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} a_1^{\sqcup\!\sqcup_\Theta 4} \sqcup\!\sqcup_\Theta a_2 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1, a_3) \sqcup\!\sqcup_\Theta \Theta(a_1, a_2) + \cdots. \tag{120}$$

We index matrices m if the respective shuffle S contains (multi-)commutator letters $[\cdot, \cdot]$ by the commutator letters themselves. For example, the filtered word w_6 contains the term

$$w_6 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{[a_1, a_2]} a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) \sqcup\!\sqcup_\Theta [a_1, a_2] + \cdots. \tag{121}$$

We treat indexed matrices separately in Section 4.5. First though, we only treat index-free matrices and hence, full shuffle products *without (multi-)commutator letters*.

Now, consider an *unfiltered word* w_n that is recursively defined via a DSE. In our case of the Yukawa propagator, this is Eq. (46). A matrix with {} brackets represents the number of words in w_n that consists of a given set of letters. The matrix entries encode the particular set of letters in full analogy to the case of [] brackets above. For example,

$$\begin{Bmatrix} 4 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{Bmatrix} \tag{122}$$

represents the number of words with four letters a_1, one a_2, one $\Theta(a_1, a_1, a_3)$ and one $\Theta(a_1, a_2)$ in the unfiltered word w_n. As another example, w_3^{Yuk} contains the terms $3a_2a_1$ and a_1a_2 (see Eq. (85)), hence $\{1\ 1\} = 3 + 1 = 4$.

We always represent a []-bracketed matrix by a lower case letter and the same matrix with {} brackets by the corresponding capital letter ($m \to M$).

The defined matrices can have any sizes. Filling with zeros does not change the multiplicity of the corresponding shuffle product.

We call a row 'Θ-row' when it is not the first row of a matrix. We consider two matrices with two Θ-rows interchanged to be the same object, since they represent the same number ($\sqcup\!\sqcup_\Theta$ is symmetric). For example

$$\begin{bmatrix} 1 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}. \tag{123}$$

Let two matrices m_1, m_2 with corresponding shuffle products S_1 and S_2 be given. We define the matrix $m_1 \oplus m_2$ such that it belongs to the shuffle product $S_1 \sqcup\!\sqcup_\Theta S_2$. This defines a special summation of matrices, denoted by $\oplus$. This is realized as follows: $\oplus$ adds up each first row as in ordinary matrix summation and writes the Θ-rows one below the other. We can even add matrices with different sizes by filling with zeros. For example,

$$\begin{bmatrix} 4 & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 2 \end{bmatrix}. \tag{124}$$

We define matrices

$$p_j = \begin{bmatrix} \delta_{1j} & \delta_{2j} & \delta_{3j} & \dots \end{bmatrix}, \tag{125}$$

as well as the useful vectors

$$\mathbf{u} = \begin{pmatrix} 1 & 1 & 1 & \dots \end{pmatrix}, \qquad \mathbf{v} = \begin{pmatrix} 1 & 2 & 3 & \dots \end{pmatrix}, \qquad \mathbf{w} = \begin{pmatrix} 1 & 0 & 0 & \dots \end{pmatrix} \tag{126}$$

with appropriate sizes. We use these vectors to notate some properties of matrices. We calculate products of matrices and vectors by ordinary matrix multiplication and not by replacing the matrix by the corresponding number. For example, a matrix m that occurs in the filtered word w_n of a DSE fulfills $\mathbf{u}m\mathbf{v}^T = n$. Let $n_\Theta(m)$ be the number of nonzero Θ-rows. Then, we define

$$|m| := \mathbf{w}m\mathbf{u}^T + n_\Theta(m), \tag{127}$$

which is the number of letters in the respective shuffle product.

Note that once, a matrix representing a number is contracted with a vector, the result is regarded as a real vector,

$$\mathbf{u}m_1 + \mathbf{u}m_2 = \mathbf{u}(m_1 \oplus m_2). \tag{128}$$

We introduce a function $\mathcal{S}$ acting on two matrices, say m_1 and m_2 with respective shuffle products S_1 and S_2 as follows. Work out the shuffle S_1 and let n_1 be the number of words that consist of the letters in S_1. Obviously, $n_1 = |m_1|!$. Now, work out the shuffle S_2 and let n_2 be the number of words that consist of the letters in S_1. Then we define $\mathcal{S}(m_1, m_2) := n_2/n_1 = n_2/|m_1|!$.

For example, let us calculate $\mathcal{S}\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, [3\ 1]\right)$. We have

$$\begin{aligned} S_1 &= a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_2) \\ &= 2a_1a_1\Theta(a_1, a_2) + 2a_1\Theta(a_1, a_2)a_1 + 2\Theta(a_1, a_2)a_1a_1 + \cdots, \end{aligned} \tag{129}$$

where the dots represent all other words. So $n_1 = 6 = \left|\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}\right|! = 3!$ On the other hand,

$$\begin{aligned} S_2 &= a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_2 \\ &= 6a_1a_1\Theta(a_1, a_2) + 6a_1\Theta(a_1, a_2)a_1 + 6\Theta(a_1, a_2)a_1a_1 + \cdots, \end{aligned} \tag{130}$$

where all other words are collected in the dots. Thus, $n_2 = 18$ and we find $\mathcal{S}\left(\begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}, [3\ 1]\right) = 18/6 = 3$.

We relate {}-bracketed matrices to []-bracketed matrices. Consider for example $m = [\,1\ 1\ 1\,]$ and the respective shuffle product $S = a_1 \shuffle_\Theta a_2 \shuffle_\Theta a_3$. Computing S gives 3! words that consists of the letters a_1, a_2 and a_3. Ordering these words using the filtration algorithm,

$$a_2 a_1 \rightarrow a_1 a_2 - [a_1, a_2] \tag{131}$$

does not change the overall number of words with letters a_1, a_2 and a_3. It only introduces new words with commutator letters. We have thus,

$$\left\{ 1 \quad 1 \quad 1 \right\} = 3! \left[1 \quad 1 \quad 1 \right]. \tag{132}$$

In general,

$$M = \sum_{m'} |m|!\, \mathcal{S}(m, m')\, m', \tag{133}$$

where the sum is over all possible matrices m'. If m has only one row, $\mathcal{S}(m, m') = 1$ if $m' = m$ and $\mathcal{S}(m, m') = 0$ otherwise. Eq. (133) then reads $M = |m|!\, m$, as in Eq. (132). If m consists of more than one row, the reader may verify that indeed, Eq. (133) is the correct generalization. Note that Eq. (133) only holds for index-free matrices. Indeed, the number of words that consist of a certain (commutator-free) set of letters does not change when introducing concatenation commutators during the filtration algorithm.

We introduce generating functions for the multiplicities (matrices) m. Let $\mathcal{M}$ be a matrix with integer entries except for the upper left entry, which is just a dot. A matrix m is said to be equivalent to $\mathcal{M}$, $m \sim \mathcal{M}$, if replacing the upper left entry to a dot yields $\mathcal{M}$. We define

$$\mathcal{M} = \mathcal{M}(z) = \sum_{m \sim \mathcal{M}}^{m \neq [\,0\,]} m\, z^{|m|}. \tag{134}$$

$\mathcal{M}(z)$ is a function in z, represented by a matrix. It generates all $m \sim \mathcal{M}$,

$$m = \frac{1}{|m|!} \left(\frac{\mathrm{d}}{\mathrm{d}z} \right)^{|m|} \mathcal{M}(z) \Bigg|_{z=0}. \tag{135}$$

Examples of generating functions are

$$[\bullet] = \sum_{N=1}^{\infty} [N]\, z^N, \qquad \begin{bmatrix} \bullet & 1 \\ 1 & 1 \end{bmatrix} = \sum_{N=0}^{\infty} \begin{bmatrix} N & 1 \\ 1 & 1 \end{bmatrix} z^{N+2},$$

$$\begin{bmatrix} \bullet & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \sum_{N=0}^{\infty} \begin{bmatrix} N & 2 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} z^{N+4}. \tag{136}$$

We translate two properties of matrices m to generating functions $\mathcal{M}$. First, we sum generating functions $\mathcal{M}$ in the same special way as matrices m. The dot remains untouched, for example

$$\begin{bmatrix} \bullet & 1 \\ 1 & 1 \end{bmatrix} \oplus \begin{bmatrix} \bullet & 0 & 0 \\ 2 & 0 & 1 \end{bmatrix} \oplus [\bullet] = \begin{bmatrix} \bullet & 1 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{137}$$

Secondly, let $\tilde{m} \sim \mathcal{M}$ with $\tilde{m}_{11} = 0$. We define $|\mathcal{M}| := |\tilde{m}|$. Summation and absolute value of generating functions will be useful in Section 4.3.

4.2. Derivation of the master differential equation

We find another relation between a {}-bracketed matrix M and []-bracketed matrices by use of the recursive equation Eq. (46). Let the lhs of Eq. (46) be an unfiltered word w_n and let the words w_{t_i} on the rhs be filtered words. Let M be the number of words consisting of a certain set of letters on the lhs, a {}-bracketed matrix hence. On the rhs, only terms that constitute these words are taken into account. Note that we have $\mathbf{u}M\mathbf{v}^T = \mathbf{u}m\mathbf{v}^T = n$.

The recursive relation for M is

$$
M = \delta_{|m|1}\delta_{n_\Theta(m)0} + \sum_{j=1}^{\mathbf{u}m\mathbf{v}^T-1} \left(1 - \delta_{m_{1j}0}\right) \sum_{k=1}^{\mathbf{u}m\mathbf{v}^T-j} \binom{2j-2+k}{k}
$$
$$
\times \sum^{(*)}(|m|-1)! \, \mathscr{S}\left(m \ominus p_j, \bigoplus_i m_i\right) m_1 m_2 \ldots m_k, \tag{138}
$$

where m is still, the same matrix as M but with [] brackets. $(*)$ sums integers t_i as in Eq. (46) and matrices m_i such that

$$
(*): \quad t_i \geq 1, \quad i = 1\ldots k, \quad \sum_{i=1}^k t_i = \mathbf{u}m\mathbf{v}^T - j, \quad \mathbf{u}m_i\mathbf{v}^T = t_i, \quad \mathbf{u}p_j + \sum_i \mathbf{u}m_i = \mathbf{u}m. \tag{139}
$$

We explain the different terms in Eq. (138) individually. First, $\delta_{|m|1}\delta_{n_\Theta(m)0}$ corresponds to the a_n term in Eq. (46). The integers j and k range over the same numbers as in Eq. (46). The term $(1 - \delta_{m_{1j}0})$ gives 1(0) if the respective words to M do (not) contain the letter a_j. Only if they do contain a_j, they may arise from the term $B_+^{a_j}(\ldots)$ in Eq. (46). Since we introduce filtered words w_{t_i} into the rhs in Eq. (46), we obtain expressions $B_+^{a_j}(S)$, where S is a full shuffle product. S is built out of k shuffles, namely terms in $w_{t_1}, w_{t_2}, \ldots, w_{t_k}$. We therefore claim that S is the respective shuffle product to $\bigoplus_{i=1\ldots k} m_i$. Each matrix m_i belongs to a full shuffle product in w_{t_i}. Condition $(*)$ in Eq. (139) consists of two parts. The first three relations correspond to the third sum in Eq. (46). The fourth equation together with the factor $\mathscr{S}(\cdot, \cdot)$ in Eq. (138) ensure that the letters in S together with the letter a_j (matrix p_j) constitute the set of letters in M. S consists of $(|m| - 1)$ letters, which gives rise to the factor of $(|m| - 1)!$ in Eq. (138).

We now derive an inhomogeneous linear differential equation for the corresponding generating function to m, i.e. $\mathcal{M}(z)$ (see Eq. (134)). Therefore, inserting Eq. (133) into Eq. (138) yields

$$
|m|\, m = -\sum_{m' \neq m} |m|\, \mathscr{S}(m, m')m' + \delta_{|m|1}\delta_{n_\Theta(m)0}
$$
$$
+ \sum_{j=1}^{\mathbf{u}m\mathbf{v}^T-1} \left(1 - \delta_{m_{1j}0}\right) \sum_{k=1}^{\mathbf{u}m\mathbf{v}^T-j} \binom{2j-2+k}{k} \sum^{(*)} \mathscr{S}\left(m \ominus p_j, \bigoplus_i m_i\right) m_1 m_2 \ldots m_k. \tag{140}
$$

The final step is to multiply with $z^{|m|-1}$ and to sum over all matrices that are equivalent to m. This gives the master differential equation. Indeed, on the lhs we obtain $\mathcal{M}(z)'$,

$$
\mathcal{M}(z)' = \sum_{m \sim \mathcal{M}} \Bigg(-\sum_{m' \neq m} \frac{\mathrm{d}}{\mathrm{d}z} z^{|m|-|m'|} \mathscr{S}(m, m')m' z^{|m'|} + \delta_{|m|1}\delta_{n_\Theta(m)0} + \sum_{j=1}^{\mathbf{u}m\mathbf{v}^T-1} \sum_{k=1}^{\mathbf{u}m\mathbf{v}^T-j} \binom{2j-2+k}{k}
$$
$$
\times \sum^{(*)} z^{|m|-1-\sum_i |m_i|} \mathscr{S}\left(m \ominus p_j, \bigoplus_i m_i\right) m_1 z^{|m_1|} m_2 z^{|m_2|} \ldots m_k z^{|m_k|}
$$
$$
\times \begin{cases} 1, & j=1 \\ (1 - \delta_{m_{1j}0}), & \text{else} \end{cases} \Bigg). \tag{141}
$$

See Eq. (139) for the summation ($*$) of integers t_i and matrices m_i. From Eq. (134), we read off the initial condition

$$\mathcal{M}(0) = 0. \tag{142}$$

Let us consider a first example: for $\mathcal{M}(z) = [\,\bullet\,]$, the first term in Eq. (141) vanishes. The last term only gives a non-zero value for $j = 1$. $\forall i \leq k$, $m_i = [\,t_i\,]$. The $\mathcal{S}$ function gives 1 for $\bigoplus_i m_i \oplus p_j = m$ and 0 otherwise. Together with the initial condition in Eq. (142), we find

$$[\bullet]' = 1 + \sum_{k \geq 1} [\bullet]^k = \frac{1}{1 - [\bullet]} \qquad \Rightarrow \qquad [\bullet] = 1 - \sqrt{1 - 2z}. \tag{143}$$

We can now derive the homogeneous part of the differential master equation Eq. (141). On the rhs, the only terms including the function $\mathcal{M}(z)$ itself occur in the sum for $j = 1$, when $(k - 1)$ of the matrices m_i are equivalent to $[\,\bullet\,]$ and the kth matrix is equivalent to m. Using Eq. (143), we obtain

$$\mathcal{M}(z)'|_{\text{hom.}} = \sum_{k \geq 1} k [\bullet]^{k-1} \mathcal{M}(z) = \frac{1}{1 - 2z} \mathcal{M}(z). \tag{144}$$

Hence, the differential equation Eq. (141) reduces to an integration using the Ansatz

$$\mathcal{M}(z) = \frac{\mathcal{C}(z)}{\sqrt{1 - 2z}}, \qquad \mathcal{C}(0) = 0. \tag{145}$$

We read off the initial condition for $\mathcal{C}$ from Eq. (142). In particular, the integration is

$$\mathcal{M}(z) = \frac{1}{\sqrt{1 - 2z}} \left(\int \mathcal{M}(z)'|_{\text{inhom.}} \sqrt{1 - 2z}\, dz + c \right), \tag{146}$$

where we obtain $\mathcal{M}(z)'|_{\text{inhom.}}$ from Eq. (141). c is an appropriate integration constant such that $\mathcal{M}(0) = 0$.

Further general simplifications of the differential master equation are not obvious. The problem is that the functions $\mathcal{S}$ in Eq. (141) give individual numbers that do not generalize and so have to be worked out case-by-case. They result in an overall differential operator acting on whatever follows. We demonstrate this in several examples, which will give us next-to and next-to-next-to-leading log generating functions.[6]

4.3. Generating functions for index-free matrices with $n_\Theta(m) = 0$

One exception is the case that $\mathcal{M}$ only contains one row, $n_\Theta(m) = 0$. These generate the matrices m that belong to full shuffle products S without $\Theta(\cdot, \cdot)$ letters. 'Index-free' means that S also does not contain $[\cdot, \cdot]$ letters.

Here, $\mathcal{S}(m, m')$ reduces to 1 if $m = m'$ and to 0 otherwise. Thus, the first term in Eq. (141) is zero. The other $\mathcal{S}$ term constrains

$$m_1 \oplus m_2 \oplus \cdots m_k \oplus p_j = m \tag{147}$$

hence, $\sum_i |m_i| + 1 = |m|$. We denote the generating functions of m_i and p_j by $\mathcal{M}_i$ and $\mathcal{P}_j$ to be consistent. We obtain

$$\mathcal{M}(z)' = \delta_{|\mathcal{M}|0} + \delta_{|\mathcal{M}|1} + \sum_{j,k=1}^{\infty} \binom{2j - 2 + k}{k} \sum^{(**)} \mathcal{M}_1(z)\mathcal{M}_2(z)\ldots\mathcal{M}_k(z)$$

$$\times \begin{cases} 1, & j = 1 \\ (1 - \delta_{\mathcal{M}_{1j}0}), & \text{else,} \end{cases} \tag{148}$$

---

[6] We call a function $\mathcal{M}(z)$ next-to$^{\{j\}}$-leading log generating function when it occurs in the log-expansion (Eq. (1)) for a certain value of j.

where $(\ast\ast)$ sums the generating functions $\mathcal{M}_1, \ldots, \mathcal{M}_k$ such that

$$(\ast\ast): \quad \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k \oplus \mathcal{P}_j = \mathcal{M}. \tag{149}$$

In the following, we give some examples that constitute the next-to$^{[j]}$-leading log expansions.

4.3.1. The generating function [● 1]

The first example is $\mathcal{M}(z) = [\bullet\ 1]$. The respective shuffle products to the matrices $m \sim \mathcal{M}$ are $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta a_2$ for $N \in \mathbb{N}$. The sum in Eq. (148) only gives non-zero values if $j = 1$ or $j = 2$. For $j = 1$, $\mathcal{P}_j = \mathcal{P}_1 = [\bullet]$ and Eq. (149) is only fulfilled if one $\mathcal{M}_i$ matches $\mathcal{M}$. This part belongs to the homogeneous differential equation.

For $j = 2$, $\mathcal{P}_j = \mathcal{P}_2 = \mathcal{M}$ and we find $\mathcal{M}_i(z) = [\bullet]\ \forall i \leq k$ from Eq. (149).

Thus, the inhomogeneous part of Eq. (148) reads

$$\left[\bullet \quad 1\right]'\bigg|_{\text{inhom.}} = 1 + \sum_{k=1}^{\infty} \binom{2+k}{k} [\bullet]^k = \frac{1}{\left(1 - [\bullet]\right)^3} = \frac{1}{\sqrt{1-2z}^3}. \tag{150}$$

We used Eq. (143) in the last line. We insert this result into Eq. (146) and obtain

$$\left[\bullet \quad 1\right] = \frac{1}{\sqrt{1-2z}} \log\left(\frac{1}{\sqrt{1-2z}}\right). \tag{151}$$

4.3.2. The generating function [● 0 1]

Matrices $m \sim \mathcal{M}(z) = [\bullet\ 0\ 1]$ belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta a_3$ for $N \in \mathbb{N}$. The sum in Eq. (148) gives non-zero values only if $j = 1$ or $j = 3$. As in the previous example, the $j = 1$ part belongs to the homogeneous differential equation.

For $j = 3$, $\mathcal{P}_j = \mathcal{P}_3 = \mathcal{M}$ and we find $\mathcal{M}_i(z) = [\bullet]\ \forall i \leq k$ from Eq. (149).

Thus, the inhomogeneous part of Eq. (148) reads

$$\left[\bullet \quad 0 \quad 1\right]'\bigg|_{\text{inhom.}} = 1 + \sum_{k=1}^{\infty} \binom{4+k}{k} [\bullet]^k = \frac{1}{\left(1 - [\bullet]\right)^5} = \frac{1}{\sqrt{1-2z}^5}. \tag{152}$$

We insert this result into Eq. (146) and obtain

$$\left[\bullet \quad 0 \quad 1\right] = -\frac{1}{2\sqrt{1-2z}} + \frac{1}{2\sqrt{1-2z}^3}. \tag{153}$$

4.3.3. The generating function [● 2]

$\mathcal{M}(z) = [\bullet\ 2]$ is the final next-to-next-to-leading log generating function for matrices $m \sim \mathcal{M}$ with $n_\Theta(m) = 0$. The matrices m belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta a_2 \sqcup\!\sqcup_\Theta a_2$ for $N \in \mathbb{N}$. Thus, the sum in Eq. (148) gives non-zero values only if $j = 1$ or $j = 2$.

If $j = 1$, $\mathcal{P}_j = \mathcal{P}_1 = [\bullet]$. There are two possibilities to choose the matrices $\mathcal{M}_i$ such that Eq. (149) is fulfilled. However, only one of these belongs to the inhomogeneous part of the differential equation: for two integers $i \leq k$, $\mathcal{M}_i = [\bullet\ 1]$ and for all other i, $\mathcal{M}_i = [\bullet]$.

If $j = 2$, Eq. (149) is only satisfied if one of the matrices $\mathcal{M}_i$ is $[\bullet\ 1]$ and the others are equal to $[\bullet]$. We finally find the inhomogeneous part of Eq. (148),

$$\left[\bullet \quad 2\right]'\bigg|_{\text{inhom.}} = \sum_{k=1}^{\infty} \binom{k}{2} [\bullet]^{k-2} [\bullet \quad 1]^2 + \sum_{k=1}^{\infty} \binom{2+k}{k} k [\bullet]^{k-1} [\bullet \quad 1]$$

$$= \frac{1}{\left(1 - [\bullet]\right)^3} [\bullet \quad 1]^2 + \frac{3}{\left(1 - [\bullet]\right)^4} [\bullet \quad 1]$$

$$= \frac{1}{\sqrt{1-2z}^5} \left(\log^2\left(\frac{1}{\sqrt{1-2z}}\right) + 3\log\left(\frac{1}{\sqrt{1-2z}}\right)\right). \tag{154}$$

We used the previous results in Eqs. (143) and (151) in the last line. Inserting this into Eq. (146) and performing the integration finally results in

$$
[\bullet \quad 2] = \frac{1}{2\sqrt{1-2z}} - \frac{1}{2\sqrt{1-2z}^3} + \frac{1}{\sqrt{1-2z}^3} \log\left(\frac{1}{\sqrt{1-2z}}\right)
$$

$$
+ \frac{1}{2\sqrt{1-2z}^3} \log^2\left(\frac{1}{\sqrt{1-2z}}\right). \tag{155}
$$

4.4. Generating functions for index-free matrices with $n_\Theta(m) \neq 0$

The respective full shuffle products of index-free matrices m with $n_\Theta(m) \neq 0$ contain at least one letter $\Theta(\cdot, \cdot)$ but no $[\cdot, \cdot]$ letters. Here, we have to proceed from the master differential equation Eq. (141) to obtain $\mathcal{M}(z)'|_{\text{inhom.}}$ in Eq. (146). We treat the different next-to$^{\{j\}}$-leading log generating functions in separate subsections.

4.4.1. The generating function $\left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix}\right]$

The first example is the next-to-leading log generating function $\mathcal{M}(z) = \left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix}\right]$. Matrices $m \sim \mathcal{M}$ belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta \Theta(a_1, a_1)$ for $N \in \mathbb{N}$.

In Eq. (141), we replace the sum over $m \sim \mathcal{M}$ by a sum over $N \in \mathbb{N}$ such that $m = \left[\begin{smallmatrix} N \\ 2 \end{smallmatrix}\right]$. In the first term on the rhs, only $m' = [N+2]$ yields a non-vanishing function $\mathcal{S}$. In particular,

$$
\mathcal{S}\left(\left[\begin{smallmatrix} N \\ 2 \end{smallmatrix}\right], [N+2]\right) = \binom{N+2}{2} = \frac{(N+2)(N+1)}{2}. \tag{156}
$$

The other sum only survives if $j = 1$ and hence $p_j = p_1 = [1]$. The integers t_i in the $(*)$-sum range such that $\sum_i t_i = N + 2 - j = N + 1$ (see Eq. (139)). Furthermore, in the first argument of the function $\mathcal{S}$, $m \ominus p_j = \left[\begin{smallmatrix} N-1 \\ 2 \end{smallmatrix}\right]$. For the inhomogeneous part of the differential equation, this implies that $m_i = [t_i] \, \forall i \le k$ such that $\bigoplus_i m_i = [N+1]$. Then,

$$
\mathcal{S}\left(\left[\begin{smallmatrix} N-1 \\ 2 \end{smallmatrix}\right], [N+1]\right) = \binom{N+1}{2} = \frac{(N+1)N}{2}. \tag{157}
$$

All together, we obtain

$$
\left[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix}\right]'\bigg|_{\text{inhom.}} = \sum_{N=0}^{\infty}\left(-\frac{d}{dz}z^{(N+1)-(N+2)}\frac{(N+2)(N+1)}{2}[N+2]z^{N+2}\right.
$$

$$
\left. + \sum_{k\ge 1}\sum_{\sum_i t_i = N+1} z^{\left((N+1)-1-\sum_i t_i\right)}\frac{(N+1)N}{2}[t_1]z^{t_1}\ldots[t_k]z^{t_k}\right)
$$

$$
= \sum_{N=0}^{\infty}\left(-\frac{d}{dz}\frac{1}{z}\frac{1}{2}z^2\frac{d^2}{dz^2}[N+2]z^{N+2} + \sum_{k\ge 1}\sum_{\sum_i t_i = N+1}\frac{1}{z}\frac{1}{2}z^2\frac{d^2}{dz^2}[t_1]z^{t_1}\ldots[t_k]z^{t_k}\right)
$$

$$
= -\frac{d}{dz}\frac{z}{2}\frac{d^2}{dz^2}[\bullet] + \sum_{k\ge 1}\frac{z}{2}\frac{d^2}{dz^2}[\bullet]^k
$$

$$
= -\frac{1}{2}\frac{d^2}{dz^2}\left(1 - \sqrt{1-2z}\right)
$$

$$
= -\frac{1}{2\sqrt{1-2z}^3}. \tag{158}
$$

In the third line, we used the explicit expression for the generating function $[\bullet]$ (Eq. (143)).

Inserting this result into Eq. (146), we finally obtain

$$\begin{bmatrix} \bullet \\ 2 \end{bmatrix} = -\frac{1}{2\sqrt{1-2z}} \log\left(\frac{1}{\sqrt{1-2z}}\right).$$
(159)

$[\, \bullet \;\; 1\,]$ (Eq. (151)) and $\begin{bmatrix} \bullet \\ 2 \end{bmatrix}$ are the only necessary generating functions to derive relations for the next-to-leading log order. It is surprising that they are related by a factor of $-1/2$,

$$\begin{bmatrix} \bullet \\ 2 \end{bmatrix} = -\frac{1}{2}[\, \bullet \;\; 1\,].$$
(160)

4.4.2. The generating function $\begin{bmatrix} \bullet \\ 3 \end{bmatrix}$

Now, consider the next-to-next-to-leading log generating function $\mathcal{M}(z) = \begin{bmatrix} \bullet \\ 3 \end{bmatrix}$. Matrices $m \sim \mathcal{M}$ belong to the shuffle products $a_1^{\amalg_\Theta N} \amalg_\Theta \Theta(a_1, a_1, a_1)$ for $N \in \mathbb{N}$.

Again in Eq. (141), we replace the sum over $m \sim \mathcal{M}$ by a sum over $N \in \mathbb{N}$ such that $m = \begin{bmatrix} N \\ 3 \end{bmatrix}$. The first term consists of two parts, $m' = [\, N + 3\,]$ and $m' = \begin{bmatrix} N+1 \\ 2 \end{bmatrix}$. For other m', $\mathcal{S}(m, m')$ vanishes. In particular,

$$\mathcal{S}\left(\begin{bmatrix} N \\ 3 \end{bmatrix}, [\, N + 3\,]\right) = \binom{N+3}{3} = \frac{(N+3)(N+2)(N+1)}{6},$$

$$\mathcal{S}\left(\begin{bmatrix} N \\ 3 \end{bmatrix}, \begin{bmatrix} N+1 \\ 2 \end{bmatrix}\right) = \binom{N+1}{1} = N + 1.$$
(161)

The sum over j only gives a non-zero value for $j = 1$. Hence $p_j = p_1 = [\, 1\,]$ and the integers t_i in the $(*)$-sum require $\sum_i t_i = N + 3 - j = N + 2$ (see Eq. (139)). The first argument of the function $\mathcal{S}$ becomes $m \ominus p_j = \begin{bmatrix} N-1 \\ 3 \end{bmatrix}$. For the inhomogeneous part of the differential equation, this implies that either $\bigoplus_i m_i = [\, N + 2\,]$ or $\bigoplus_i m_i = \begin{bmatrix} N \\ 2 \end{bmatrix}$. The first case is realized if $m_i = [\, t_i\,] \; \forall i \leq k$. The second case implies that one of the matrices m_i is equal to $\begin{bmatrix} t_i - 2 \\ 2 \end{bmatrix}$ and the rest of the matrices $m_i = [\, t_i\,]$. We compute

$$\mathcal{S}\left(\begin{bmatrix} N-1 \\ 3 \end{bmatrix}, [\, N + 2\,]\right) = \binom{N+2}{3} = \frac{(N+2)(N+1)N}{6},$$

$$\mathcal{S}\left(\begin{bmatrix} N-1 \\ 3 \end{bmatrix}, \begin{bmatrix} N \\ 2 \end{bmatrix}\right) = \binom{N}{1} = N.$$
(162)

We use all these observations to obtain the inhomogeneous part of the differential equation Eq. (141),

$$\begin{bmatrix} \bullet \\ 3 \end{bmatrix}'\Bigg|_{\text{inhom.}} = \sum_{N=0}^{\infty}\Bigg(-\frac{d}{dz}z^{(N+1)-(N+3)}\frac{(N+3)(N+2)(N+1)}{6}[\, N+3\,]z^{N+3}$$

$$ -\frac{d}{dz}z^{(N+1)-(N+2)}(N+1)\begin{bmatrix} N+1 \\ 2 \end{bmatrix}z^{N+2}$$

$$ +\sum_{k\geq 1}\sum_{\sum_i t_i = N+2}z^{(N+1)-1-\sum_i t_i}\frac{(N+2)(N+1)N}{6}[\, t_1\,]z^{t_1}\ldots[\, t_k\,]z^{t_k}$$

$$ +\sum_{k\geq 1}\sum_{\sum_i t_i = N+2}z^{(N+1)-1-(\sum_i t_i - 1)}Nk\,[\, t_1\,]z^{t_1}\ldots[\, t_{k-1}\,]z^{t_{k-1}}\begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix}z^{t_k - 1}\Bigg)$$

$$ =\sum_{N=0}^{\infty}\Bigg(-\frac{d}{dz}\frac{1}{z^2}\frac{1}{6}z^3\frac{d^3}{dz^3}[\, N+3\,]z^{N+3} -\frac{d}{dz}\frac{1}{z}z^2\frac{d}{dz}\frac{1}{z}\begin{bmatrix} N+1 \\ 2 \end{bmatrix}z^{N+2}$$

$$+\sum_{k\geq1}\sum_{\sum_i t_i=N+2}\frac{1}{z^2}\frac{1}{6}z^3\frac{d^3}{dz^3}\left[t_1\right]z^{t_1}\ldots\left[t_k\right]z^{t_k}$$

$$+\sum_{k\geq1}\sum_{\sum_i t_i=N+2}\frac{1}{z}z^2\frac{d}{dz}\frac{1}{z}k\left[t_1\right]z^{t_1}\ldots\left[t_{k-1}\right]z^{t_{k-1}}\left[\begin{matrix}t_k-2\\2\end{matrix}\right]z^{t_k-1}\Bigg)$$

$$=-\frac{d}{dz}\frac{z}{6}\frac{d^3}{dz^3}\left[\bullet\right]-\frac{d}{dz}z\frac{d}{dz}\frac{1}{z}\left[\begin{matrix}\bullet\\2\end{matrix}\right]+\sum_{k\geq1}\frac{z}{6}\frac{d^3}{dz^3}\left[\bullet\right]^k+\sum_{k\geq1}z\frac{d}{dz}\frac{1}{z}k\left[\bullet\right]^{k-1}\left[\begin{matrix}\bullet\\2\end{matrix}\right]$$

$$=-\frac{d}{dz}\frac{z}{6}\frac{d^3}{dz^3}\left[\bullet\right]-\frac{d}{dz}z\frac{d}{dz}\frac{1}{z}\left[\begin{matrix}\bullet\\2\end{matrix}\right]+\frac{z}{6}\frac{d^3}{dz^3}\frac{1}{1-\left[\bullet\right]}+z\frac{d}{dz}\frac{1}{z}\frac{1}{(1-\left[\bullet\right])^2}\left[\begin{matrix}\bullet\\2\end{matrix}\right]$$

$$=\frac{1}{\sqrt{1-2z}^5}-\frac{1}{2\sqrt{1-2z}}\frac{d}{dz}\left(\frac{1}{z}\log\left(\frac{1}{\sqrt{1-2z}}\right)\right).\tag{163}$$

In the last line, we used the explicit formulas for the generating functions $[\,\bullet\,]$ and $\left[\begin{matrix}\bullet\\2\end{matrix}\right]$ (Eqs. (143) and (159)).

Inserting Eq. (163) into the integration in Eq. (146) finally results in

$$\left[\begin{matrix}\bullet\\3\end{matrix}\right]=\frac{1}{2\sqrt{1-2z}^3}-\frac{1}{2\sqrt{1-2z}}\frac{1}{z}\log\left(\frac{1}{\sqrt{1-2z}}\right).\tag{164}$$

4.4.3. The generating function $\left[\begin{smallmatrix}\bullet&0\\1&1\end{smallmatrix}\right]$

Now, consider $\mathcal{M}(z)=\left[\begin{smallmatrix}\bullet&0\\1&1\end{smallmatrix}\right]$. The respective shuffle products to the matrices $m\sim\mathcal{M}$ are $a_1^{\sqcup\!\sqcup_\Theta N}\sqcup\!\sqcup_\Theta\Theta(a_1,a_2)$ for $N\in\mathbb{N}$.

In Eq. (141), we replace the sum over $m\sim\mathcal{M}$ by a sum over $N\in\mathbb{N}$ such that $m=\left[\begin{smallmatrix}N&0\\1&1\end{smallmatrix}\right]$. Here, only $m'=[N+1\;\;1]$ contributes to the first sum and we calculate

$$\mathscr{S}\left(\left[\begin{matrix}N&0\\1&1\end{matrix}\right],[N+1\quad1]\right)=\binom{N+1}{1}=N+1.\tag{165}$$

In the second sum, $j=1$ and $p_j=p_1=[1]$. The integers t_i in the $(*)$-sum are constraint by $\sum_i t_i=N+3-j=N+2$ (see Eq. (139)). The first argument of the function $\mathscr{S}$ becomes $m\ominus p_j=\left[\begin{smallmatrix}N-1&0\\1&1\end{smallmatrix}\right]$. The second argument must be $\bigoplus_i m_i=[N\;1]$ since we only consider the inhomogeneous part of the differential equation. Thus, one of the matrices m_i is equal to $[t_i-2\;1]$ and the rest of the matrices $m_i=[t_i]$. In particular,

$$\mathscr{S}\left(\left[\begin{matrix}N-1&0\\1&1\end{matrix}\right],[N\quad1]\right)=\binom{N}{1}=N.\tag{166}$$

We deduce the inhomogeneous part of the differential equation Eq. (141),

$$\left[\begin{matrix}\bullet&0\\1&1\end{matrix}\right]'\bigg|_{\text{inhom.}}=\sum_{N=0}^{\infty}\Bigg(-\frac{d}{dz}z^{(N+1)-(N+2)}(N+1)[N+1\quad1]z^{N+2}$$

$$+\sum_{k\geq1}\sum_{\sum_i t_i=N+2}z^{(N+1)-1-(\sum_i t_i-1)}Nk\left[t_1\right]z^{t_1}\ldots\left[t_{k-1}\right]z^{t_{k-1}}\left[t_k-2\quad1\right]z^{t_k-1}\Bigg)$$

$$=\sum_{N=0}^{\infty}\Bigg(-\frac{d}{dz}\frac{1}{z}z^2\frac{d}{dz}\frac{1}{z}[N+1\quad1]z^{N+2}$$

$$
+ \sum_{k \geq 1} \sum_{\sum_i t_i = N+2} \frac{1}{z} z^2 \frac{d}{dz} \frac{1}{z} k \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_{k-1} \end{bmatrix} z^{t_{k-1}} \begin{bmatrix} t_k - 2 & 1 \end{bmatrix} z^{t_k - 1}. \Bigg)
$$

$$
= - \frac{d}{dz} z \frac{d}{dz} \frac{1}{z} \begin{bmatrix} \bullet & 1 \end{bmatrix} + \sum_{k \geq 1} z \frac{d}{dz} \frac{1}{z} k \begin{bmatrix} \bullet \end{bmatrix}^{k-1} \begin{bmatrix} \bullet & 1 \end{bmatrix}
$$

$$
= - \frac{d}{dz} z \frac{d}{dz} \frac{1}{z} \begin{bmatrix} \bullet & 1 \end{bmatrix} + z \frac{d}{dz} \frac{1}{z} \frac{1}{(1 - [\bullet])^2} \begin{bmatrix} \bullet & 1 \end{bmatrix}
$$

$$
= - \frac{3}{\sqrt{1 - 2z}^5} + \frac{1}{\sqrt{1 - 2z}} \frac{d}{dz} \frac{1}{z} \log \left(\frac{1}{\sqrt{1 - 2z}} \right). \tag{167}
$$

In the last line, we used the explicit formulas for the generating functions $[\bullet]$ and $[\bullet\ 1]$ (Eqs. (143) and (151)).

We plug Eq. (167) into Eq. (146) and obtain

$$
\begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix} = \frac{1}{2\sqrt{1 - 2z}} - \frac{3}{2\sqrt{1 - 2z}^3} + \frac{1}{\sqrt{1 - 2z}} \frac{1}{z} \log \left(\frac{1}{\sqrt{1 - 2z}} \right). \tag{168}
$$

4.4.4. The generating function $[\begin{smallmatrix} \bullet \\ 2 \end{smallmatrix}]$

The function $\mathcal{M}(z) = \begin{bmatrix} \bullet \\ 2 \\ 2 \end{bmatrix}$ generates the rationals $m \sim \mathcal{M}$ to the respective shuffle products $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) \sqcup\!\sqcup_\Theta \Theta(a_1, a_1)$ for $N \in \mathbb{N}$.

Consider Eq. (141). We replace the sum over $m \sim \mathcal{M}$ by a sum over $N \in \mathbb{N}$ such that $m = \begin{bmatrix} N \\ 2 \\ 2 \end{bmatrix}$. Here, only the matrices $m' = [N+4]$ and $m' = \begin{bmatrix} N+2 \\ 2 \end{bmatrix}$ contribute to the first sum and we find

$$
\mathscr{S} \left(\begin{bmatrix} N \\ 2 \\ 2 \end{bmatrix}, [N+4] \right) = \frac{1}{2} \binom{N+4}{2} \binom{N+2}{2} = \frac{(N+4)(N+3)(N+2)(N+1)}{8}, \tag{169}
$$

$$
\mathscr{S} \left(\begin{bmatrix} N \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} N+2 \\ 2 \end{bmatrix} \right) = \binom{N+2}{2} = \frac{(N+2)(N+1)}{2}. \tag{170}
$$

In the second sum, $j = 1$ and $p_j = p_1 = [1]$ as before. The integers t_i in the $(*)$-sum range over $\sum_i t_i = N + 4 - j = N + 3$ (see Eq. (139)). The first argument of the function $\mathscr{S}$ is $m \ominus p_j = \begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}$. Here, there are three possibilities for $\bigoplus_i m_i$ in the second sum of Eq. (141). First, $\bigoplus_i m_i = [N+3]$ and $m_i = [t_i] \ \forall i \leq k$. Secondly, $\bigoplus_i m_i = \begin{bmatrix} N+1 \\ 2 \end{bmatrix}$, which implies that one of the matrices m_i is equal to $\begin{bmatrix} t_i - 2 \\ 2 \end{bmatrix}$ and the rest of the matrices $m_i = [t_i]$. In the third case, $\bigoplus_i m_i = \begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}$. Note that $\bigoplus_i m_i \sim \mathcal{M}$ did not occur in the previous examples because it was part of the homogeneous differential equation Eq. (141). Here, we realize $\bigoplus_i m_i = \begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}$ within the inhomogeneous part of the differential equation: two of the matrices m_i are equal to $\begin{bmatrix} t_i - 2 \\ 2 \end{bmatrix}$ and the rest of the matrices $m_i = [t_i]$. The corresponding functions $\mathscr{S}(\cdot, \cdot)$ give

$$
\mathscr{S} \left(\begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}, [N+3] \right) = \frac{1}{2} \binom{N+3}{2} \binom{N+1}{2} = \frac{(N+3)(N+2)(N+1)N}{8}, \tag{171}
$$

$$
\mathscr{S} \left(\begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} N+1 \\ 2 \end{bmatrix} \right) = \binom{N+1}{2} = \frac{(N+1)N}{2}, \tag{172}
$$

$$\mathscr{S}\left(\begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} N-1 \\ 2 \\ 2 \end{bmatrix}\right) = 1. \tag{173}$$

We simplify the inhomogeneous part of the differential equation Eq. (141) as follows:

$$
\begin{aligned}
\begin{bmatrix} \bullet \\ 2 \\ 2 \end{bmatrix}' \bigg|_{\text{inhom.}} &= \sum_{N=0}^{\infty} \Bigg(-\frac{d}{dz} z^{(N+2)-(N+4)} \frac{(N+4)(N+3)(N+2)(N+1)}{8} \begin{bmatrix} N+4 \end{bmatrix} z^{N+4} \\
&\quad - \frac{d}{dz} z^{(N+2)-(N+3)} \frac{(N+2)(N+1)}{2} \begin{bmatrix} N+2 \\ 2 \end{bmatrix} z^{N+3} \\
&\quad + \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} z^{(N+2)-1-\sum_i t_i} \frac{(N+3)(N+2)(N+1)N}{8} \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_k \end{bmatrix} z^{t_k} \\
&\quad + \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} z^{(N+2)-1-(\sum_i t_i - 1)} \frac{(N+1)N}{2} k \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_{k-1} \end{bmatrix} z^{t_{k-1}} \\
&\quad \times \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_k-1} + \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} z^{(N+2)-1-(\sum_i t_i - 2)} \binom{k}{2} \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_{k-2} \end{bmatrix} z^{t_{k-2}} \\
&\quad \times \begin{bmatrix} t_{k-1} - 2 \\ 2 \end{bmatrix} z^{t_{k-1}-1} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_k-1} \Bigg) \\
&= \sum_{N=0}^{\infty} \Bigg(-\frac{d}{dz} \frac{1}{z^2} \frac{1}{8} z^4 \frac{d^4}{dz^4} \begin{bmatrix} N+4 \end{bmatrix} z^{N+4} - \frac{d}{dz} \frac{1}{z} \frac{1}{2} z^3 \frac{d^2}{dz^2} \frac{1}{z} \begin{bmatrix} N+2 \\ 2 \end{bmatrix} z^{N+3} \\
&\quad + \sum_{k\geq 1} \sum_{\sum_{t_i}=N+3} \frac{1}{z^2} \frac{1}{8} z^4 \frac{d^4}{dz^4} \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_k \end{bmatrix} z^{t_k} \\
&\quad + \sum_{k\geq 1} \sum_{\sum_{t_i}=N+3} \frac{1}{z} \frac{1}{2} z^3 \frac{d^2}{dz^2} \frac{1}{z} k \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_{k-1} \end{bmatrix} z^{t_{k-1}} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_k-1} \\
&\quad + \sum_{k\geq 1} \sum_{\sum_{t_i}=N+3} \frac{k(k-1)}{2} \begin{bmatrix} t_1 \end{bmatrix} z^{t_1} \dots \begin{bmatrix} t_{k-2} \end{bmatrix} z^{t_{k-2}} \begin{bmatrix} t_{k-1} - 2 \\ 2 \end{bmatrix} z^{t_{k-1}-1} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_k-1} \Bigg) \\
&= -\frac{d}{dz} \frac{z^2}{8} \frac{d^4}{dz^4} \begin{bmatrix} \bullet \end{bmatrix} - \frac{d}{dz} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} \begin{bmatrix} \bullet \\ 2 \end{bmatrix} + \sum_{k\geq 1} \frac{z^2}{8} \frac{d^4}{dz^4} \begin{bmatrix} \bullet \end{bmatrix}^k \\
&\quad + \sum_{k\geq 1} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} k \begin{bmatrix} \bullet \end{bmatrix}^{k-1} \begin{bmatrix} \bullet \\ 2 \end{bmatrix} + \sum_{k\geq 1} \frac{k(k-1)}{2} \begin{bmatrix} \bullet \end{bmatrix}^{k-2} \begin{bmatrix} \bullet \\ 2 \end{bmatrix}^2 \\
&= -\frac{d}{dz} \frac{z^2}{8} \frac{d^4}{dz^4} \begin{bmatrix} \bullet \end{bmatrix} - \frac{d}{dz} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} \begin{bmatrix} \bullet \\ 2 \end{bmatrix} + \frac{z^2}{8} \frac{d^4}{dz^4} \frac{1}{1-\begin{bmatrix} \bullet \end{bmatrix}} \\
&\quad + \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} \frac{1}{(1-\begin{bmatrix} \bullet \end{bmatrix})^2} \begin{bmatrix} \bullet \\ 2 \end{bmatrix} + \frac{1}{(1-\begin{bmatrix} \bullet \end{bmatrix})^3} \begin{bmatrix} \bullet \\ 2 \end{bmatrix}^2 . \tag{174}
\end{aligned}
$$

We use the previous results in Eqs. (143) and (159) and insert the resulting expression into Eq. (146).

A little calculation yields

$$
\begin{bmatrix} \bullet \\ 2 \\ 2 \end{bmatrix} = -\frac{1}{8\sqrt{1-2z}} - \frac{3}{8\sqrt{1-2z}^3} + \frac{1}{4\sqrt{1-2z}^3} \log\left(\frac{1}{\sqrt{1-2z}}\right)
$$

$$
+ \frac{1}{2\sqrt{1-2z}} \frac{1}{z} \log\left(\frac{1}{\sqrt{1-2z}}\right) + \frac{1}{8\sqrt{1-2z}^3} \log^2\left(\frac{1}{\sqrt{1-2z}}\right). \tag{175}
$$

4.4.5. The generating function $\begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}$

The only next-to-next-to-leading log generating function left is $\mathcal{M}(z) = \begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}$. It generates the rationals $m \sim \mathcal{M}$ with the respective shuffle products $a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta a_2 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1)$ for $N \in \mathbb{N}$.

In Eq. (141), we replace the sum over $m \sim \mathcal{M}$ by a sum over $N \in \mathbb{N}$ such that $m = \begin{bmatrix} N & 1 \\ 2 & 0 \end{bmatrix}$. The function $\mathscr{s}$ in the first sum vanishes except for $m' = [N + 2 \ 1]$:

$$
\mathscr{s}\left(\begin{bmatrix} N & 1 \\ 2 & 0 \end{bmatrix}, [N+2 \quad 1]\right) = \binom{N+2}{2} = \frac{(N+2)(N+1)}{2}. \tag{176}
$$

In the second sum, either $j = 1$ or $j = 2$. This is the main difference to the previous examples. The sum does not vanish for $j = 2$ because $\mathcal{M}_{12} = 1$. For $j = 1$, $p_j = p_1 = [\,1\,]$ as before. Then, the integers t_i in the $(*)$-sum range over $\sum_i t_i = N+4-j = N+3$ (see Eq. (139)). Furthermore, $m \ominus p_j = \begin{bmatrix} N & 1 & 1 \\ 2 & & 0 \end{bmatrix}$ in the first argument of the function $\mathscr{s}$, which implies that either $\bigoplus_i m_i = [N+1 \ 1]$ or $\bigoplus_i m_i = \begin{bmatrix} N & 1 & 1 \\ 2 & & 0 \end{bmatrix}$. In the first case, there is one $i \leq k$ such that $m_i = [\, t_i - 2 \ 1\,]$ and for all other $i \leq k$, $m_i = [\, t_i\,]$. In the second case, one of the matrices m_i must be equal to $[\, t_i - 2 \ 1\,]$ and another one must be equal to $\begin{bmatrix} t_i - 2 \\ 2 \end{bmatrix}$. The rest of the matrices $m_i = [\, t_i\,]$. The corresponding $\mathscr{s}$-function factors are

$$
\mathscr{s}\left(\begin{bmatrix} N-1 & 1 \\ 2 & 0 \end{bmatrix}, [N+1 \quad 1]\right) = \binom{N+1}{2} = \frac{(N+1)N}{2}, \tag{177}
$$

$$
\mathscr{s}\left(\begin{bmatrix} N-1 & 1 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} N-1 & 1 \\ 2 & 0 \end{bmatrix}\right) = 1. \tag{178}
$$

For $j = 2$, $p_j = p_2 = [\,0 \ 1\,]$. The integers t_i in the $(*)$-sum then require $\sum_i t_i = N+4-j = N+2$. Furthermore, $m \ominus p_j = \begin{bmatrix} N \\ 2 \end{bmatrix}$ in the first argument of the function $\mathscr{s}$. Thus, either $\bigoplus_i m_i = [N+2]$ or $\bigoplus_i m_i = \begin{bmatrix} N \\ 2 \end{bmatrix}$. In the former case, $m_i = [\, t_i\,] \ \forall i \leq k$. In the latter case, one of the matrices m_i must be equal to $\begin{bmatrix} t_i - 2 \\ 2 \end{bmatrix}$ and the other matrices $m_i = [\, t_i\,]$. The corresponding functions $\mathscr{s}(\cdot, \cdot)$ give

$$
\mathscr{s}\left(\begin{bmatrix} N \\ 2 \end{bmatrix}, [N+2]\right) = \binom{N+2}{2} = \frac{(N+2)(N+1)}{2}, \tag{179}
$$

$$
\mathscr{s}\left(\begin{bmatrix} N \\ 2 \end{bmatrix}, \begin{bmatrix} N \\ 2 \end{bmatrix}\right) = 1. \tag{180}
$$

Using these observations yields the inhomogeneous part of the differential equation Eq. (141),

$$
\begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}' \bigg|_{\text{inhom.}} = \sum_{N=0}^{\infty} \left(-\frac{d}{dz} z^{(N+2)-(N+3)} \frac{(N+2)(N+1)}{2} [N+2 \quad 1] z^{N+3} \right.
$$

$$
+ \sum_{k \geq 1} \sum_{\sum_i t_i = N+3} z^{(N+2)-1-(\sum_i t_i - 1)} \frac{(N+1)N}{2}
$$

$$
\times k \, [t_1] z^{t_1} \ldots [t_{k-1}] z^{t_{k-1}} [t_k - 2 \quad 1] z^{t_k - 1}
$$

$$+ \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} z^{(N+2)-1-(\sum_i t_i - 2)} k(k-1) [t_1] z^{t_1} \dots [t_{k-2}]$$

$$\times z^{t_{k-2}} [t_{k-1} - 2 \quad 1] z^{t_{k-1}-1} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_{k-1}}$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N+2} z^{(N+2)-1-\sum_i t_i} \frac{(N+2)(N+1)}{2} [t_1] z^{t_1} \dots [t_k] z^{t_k}$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N+2} z^{(N+2)-1-(\sum_i t_i - 1)}$$

$$\times k [t_1] z^{t_1} \dots [t_{k-1}] z^{t_{k-1}} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_{k-1}} \Bigg)$$

$$= \sum_{N=0}^{\infty} \Bigg(-\frac{d}{dz} \frac{1}{z} \frac{1}{2} z^3 \frac{d^2}{dz^2} \frac{1}{z} [N+2 \quad 1] z^{N+3}$$

$$+ \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} \frac{1}{z} \frac{1}{2} z^3 \frac{d^2}{dz^2} \frac{1}{z} k [t_1] z^{t_1} \dots [t_{k-1}] z^{t_{k-1}} [t_k - 2 \quad 1] z^{t_{k-1}}$$

$$+ \sum_{k\geq 1} \sum_{\sum_i t_i = N+3} k(k-1) [t_1] z^{t_1} \dots [t_{k-2}] z^{t_{k-2}}$$

$$\times [t_{k-1} - 2 \quad 1] z^{t_{k-1}-1} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_{k-1}}$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N+2} \frac{1}{z} \frac{1}{2} z^2 \frac{d^2}{dz^2} [t_1] z^{t_1} \dots [t_k] z^{t_k}$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N+2} k [t_1] z^{t_1} \dots [t_{k-1}] z^{t_{k-1}} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_{k-1}} \Bigg)$$

$$= -\frac{d}{dz} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} [\bullet \quad 1] + \sum_{k\geq 1} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} k [\bullet]^{k-1} [\bullet \quad 1]$$

$$+ \sum_{k\geq 1} k(k-1) [\bullet]^{k-2} [\bullet \quad 1] \begin{bmatrix} \bullet \\ 2 \end{bmatrix}$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \frac{z}{2} \frac{d^2}{dz^2} [\bullet]^k + \sum_{k\geq 1} \binom{2+k}{k} k [\bullet]^{k-1} \begin{bmatrix} \bullet \\ 2 \end{bmatrix}$$

$$= -\frac{d}{dz} \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} [\bullet \quad 1] + \frac{z^2}{2} \frac{d^2}{dz^2} \frac{1}{z} \frac{1}{(1-[\bullet])^2} [\bullet \quad 1]$$

$$+ \frac{2}{(1-[\bullet])^3} [\bullet \quad 1] \begin{bmatrix} \bullet \\ 2 \end{bmatrix} + \frac{z}{2} \frac{d^2}{dz^2} \frac{1}{(1-[\bullet])^3} + \frac{3}{(1-[\bullet])^4} \begin{bmatrix} \bullet \\ 2 \end{bmatrix}. \quad (181)$$

We insert this into the integration in Eq. (146). We also need Eqs. (143), (151) and (159) and after some calculation, we obtain

$$\begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix} = \frac{1}{\sqrt{1-2z}^3} - \frac{1}{\sqrt{1-2z}^3} \log\left(\frac{1}{\sqrt{1-2z}}\right) - \frac{1}{\sqrt{1-2z}} \frac{1}{z} \log\left(\frac{1}{\sqrt{1-2z}}\right)$$
$$- \frac{1}{2\sqrt{1-2z}^3} \log^2\left(\frac{1}{\sqrt{1-2z}}\right). \tag{182}$$

4.5. Generating functions for indexed matrices

In this paper, we do not give a general method to obtain the generating functions for indexed matrices. In this section however, we derive the two 'indexed generating functions'

$$[\bullet]_{[a_1,a_2]} = \sum_{N\geq 0} [N]_{[a_1,a_2]} z^{N+1}, \qquad [\bullet]_{[a_1,\Theta(a_1,a_1)]} = \sum_{N\geq 0} [N]_{[a_1,\Theta(a_1,a_1)]} z^{N+1}. \tag{183}$$

These generate the indexed matrices $[N]_{[a_1,a_2]}$ and $[N]_{[a_1,\Theta(a_1,a_1)]}$ that belong to the full shuffle products

$$a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta [a_1, a_2], \qquad a_1^{\sqcup\!\sqcup_\Theta N} \sqcup\!\sqcup_\Theta [a_1, \Theta(a_1, a_1)] \tag{184}$$

respectively. These shuffles make part of the filtered words w_{N+3} (Eq. (46)) and map to the next-to-next-to-leading log order in the log-expansion. In particular, $[\bullet]_{[a_1,a_2]}$ and $[\bullet]_{[a_1,\Theta(a_1,a_1)]}$ complete the set of generating functions that are necessary to obtain the next-to-next-to-leading log expansion. We will work on a general method to derive indexed generating functions in future work.

4.5.1. The generating function $[\bullet]_{[a_1,a_2]}$

Only words with $(N-2)$ letters a_1 and one letter a_2 in the *unfiltered word* w_N can contribute to the term

$$[N-3]_{[a_1,a_2]} a_1^{\sqcup\!\sqcup_\Theta N-3} \sqcup\!\sqcup_\Theta [a_1, a_2] \tag{185}$$

in the *filtered word* w_N. Consider Eq. (46) and let the words w_{t_i} on the rhs be filtered. Then,

$$w_N = B_+^{a_1} \sum_{k\geq 1} \sum_{\sum_i t_i = N-1} k [t_1] a_1^{\sqcup\!\sqcup_\Theta t_1} \sqcup\!\sqcup_\Theta \ldots \sqcup\!\sqcup_\Theta [t_{k-1}] a_1^{\sqcup\!\sqcup_\Theta t_{k-1}} \sqcup\!\sqcup_\Theta$$
$$\times \left([t_k - 3]_{[a_1,a_2]} a_1^{\sqcup\!\sqcup_\Theta t_k-3} \sqcup\!\sqcup_\Theta [a_1, a_2]\right)$$
$$+ B_+^{a_1} \sum_{k\geq 1} \sum_{\sum_i t_i = N-1} k [t_1] a_1^{\sqcup\!\sqcup_\Theta t_1} \sqcup\!\sqcup_\Theta \ldots \sqcup\!\sqcup_\Theta [t_{k-1}] a_1^{\sqcup\!\sqcup_\Theta t_{k-1}} \sqcup\!\sqcup_\Theta$$
$$\times [t_k - 2 \quad 1] a_1^{\sqcup\!\sqcup_\Theta t_k-2} \sqcup\!\sqcup_\Theta a_2$$
$$+ B_+^{a_2} \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N-2} [t_1] a_1^{\sqcup\!\sqcup_\Theta t_1} \sqcup\!\sqcup_\Theta \ldots \sqcup\!\sqcup_\Theta [t_k] a_1^{\sqcup\!\sqcup_\Theta t_k} + \cdots, \tag{186}$$

where the dots represent all missing terms in Eq. (46), for example

$$B_+^{a_1} \sum_{k\geq 1} \sum_{\sum_i t_i = N-1} k [t_1] a_1^{\sqcup\!\sqcup_\Theta t_1} \sqcup\!\sqcup_\Theta \ldots \sqcup\!\sqcup_\Theta [t_{k-1}] a_1^{\sqcup\!\sqcup_\Theta t_{k-1}} \sqcup\!\sqcup_\Theta$$
$$\times \left([t_k - 4]_{[a_1,[a_1,a_2]]} a_1^{\sqcup\!\sqcup_\Theta t_k-4} \sqcup\!\sqcup_\Theta [a_1, [a_1, a_2]]\right). \tag{187}$$

All these other terms do not contribute to $[N-3]_{[a_1,a_2]} a_1^{\sqcup\!\sqcup_\Theta N-3} \sqcup\!\sqcup_\Theta [a_1, a_2]$ in the filtered word. Indeed, the only missing terms in Eq. (186) that give words with $(N-2)$ letters a_1 and one a_2 include

one of the multi-commutator letters $[a_1, \ldots [a_1, a_2] \ldots]$ in the shuffle product. Computing all these shuffles and filtrating the resulting words will not give words with $(N-3)$ letters a_1 and one $[a_1, a_2]$ but words with multi-commutator letters.

Given a filtered word w_N, one regains the original unfiltered word by *first computing all shuffle products in w_N and then, computing all (multi-)commutators*. This can be seen by a look at the filtration algorithm in Section 3.3. For the first term in Eq. (186), this implies that we must first compute the shuffle products in the bracket. Secondly, we have to replace the commutator letter $[a_1, a_2]$ by $a_1 a_2 - a_2 a_1$ and finally, we must compute all remaining shuffle products to obtain the respective terms in the unfiltered word w_N (Eq. (186)).

Let us calculate the unfiltered w_N. We are only interested in words that contribute to $[\bullet]_{[a_1, a_2]}$. Hence, when computing the shuffle products in Eq. (186), we shift all words with $\Theta(\cdot, \cdot)$ letters to the $\ldots$ terms. In the following, it is convenient to define the words

$$A(p, q) := \underbrace{a_1 \ldots a_1}_{p\times} a_2 \underbrace{a_1 \ldots a_1}_{q\times}, \tag{188}$$

$$B(p, q) := \underbrace{a_1 \ldots a_1}_{p\times} [a_1, a_2] \underbrace{a_1 \ldots a_1}_{q\times} \tag{189}$$

for $p, q \in \mathbb{N}$. We note that

$$a_1^{\sqcup\!\sqcup_\Theta p} \sqcup\!\sqcup_\Theta [a_1, a_2] = p!(A(p+1, 0) - A(0, p+1)) + \cdots, \tag{190}$$

$$a_1^{\sqcup\!\sqcup_\Theta p} \sqcup\!\sqcup_\Theta A(q, 0) = p! \sum_{r=0}^{p} \binom{q+r}{r} A(q+r, p-r) + \cdots, \tag{191}$$

$$a_1^{\sqcup\!\sqcup_\Theta p} \sqcup\!\sqcup_\Theta A(0, q) = p! \sum_{r=0}^{p} \binom{q+r}{r} A(p-r, q+r) + \cdots, \tag{192}$$

$$a_1^{\sqcup\!\sqcup_\Theta p} \sqcup\!\sqcup_\Theta a_2 = p! \sum_{q=0}^{p} A(q, p-q) + \cdots. \tag{193}$$

The dots collect all words that consist of other letters than one a_2 and some a_1. Using these relations in Eq. (186), we obtain

$$w_N = \sum_{k\geq 1} \sum_{\sum_i t_i = N-1} k \left[t_1\right] \ldots \left[t_{k-1}\right] \left[t_k - 3\right]_{[a_1, a_2]} (t_k - 3)!(N-1-t_k)! \sum_{p=0}^{N-1-t_k} \binom{t_k - 2 + p}{p}$$

$$\times (A(t_k - 1 + p, N-1-t_k-p) - A(N-t_k-p, t_k-2+p))$$

$$+ \sum_{k\geq 1} \sum_{\sum_i t_i = N-1} k \left[t_1\right] \ldots \left[t_{k-1}\right] \left[t_k - 2 \quad 1\right] (N-3)! \sum_{p=0}^{N-3} A(p+1, N-3-p)$$

$$+ \sum_{k\geq 1} \binom{2+k}{k} \sum_{\sum_i t_i = N-2} \left[t_1\right] \ldots \left[t_k\right] (N-2)! A(0, N-2) + \cdots \tag{194}$$

w_N is still unfiltered. Now, the filtration algorithm brings the words $A(p, q)$ into lexicographical order using the concatenation commutator. In our case, a_2 is sorted to the right,

$$A(p, q) = -B(p, q-1) + A(p+1, q-1),$$

$$\Rightarrow A(p, q) = A(p+q, 0) - \sum_{r=0}^{q-1} B(p+r, q-1-r). \tag{195}$$

In this section, a filtration algorithm that sorts a_2 to the left would be more convenient because the last term in Eq. (194) would already be given in lexicographical order. Since the resulting filtered words

must not depend on the lexicographical order of letters, we now assume throughout this section that we work with a filtration algorithm that sorts a_2 to the left,

$$A(p, q) = B(p - 1, q) + A(p - 1, q + 1),$$

$$\Rightarrow A(p, q) = A(0, p + q) + \sum_{r=0}^{p-1} B(p - 1 - r, q + r). \tag{196}$$

This change does not effect the final generating functions $[\,\bullet\,]_{[a_1,a_2]}$ and the respective generated matrices. Inserting Eq. (196) into Eq. (194) yields

$$
\begin{aligned}
w_N = \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k\,[t_1] \dots [t_{k-1}]\,[t_k - 3]_{[a_1,a_2]}\,(t_k - 3)!(N - 1 - t_k)! \sum_{p=0}^{N-1-t_k} \binom{t_k - 2 + p}{p} \\
\times \left(\sum_{q=0}^{t_k-2+p} B(t_k - 2 + p - q, N - 1 - t_k - p + q) \right. \\
\left. - \sum_{q=0}^{N-t_k-p-1} B(N - t_k - p - 1 - q, t_k - 2 + p + q) \right) \\
+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k\,[t_1] \dots [t_{k-1}]\,[t_k - 2 \quad 1]\,(N - 3)! \sum_{p=0}^{N-3} \sum_{q=0}^{p} B(p - q, N - 3 - p + q) \\
+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k\,[t_1] \dots [t_{k-1}]\,[t_k - 2 \quad 1]\,(N - 2)! A(0, N - 2) \\
+ \sum_{k \geq 1} \binom{2 + k}{k} \sum_{\sum_i t_i = N-2} [t_1] \dots [t_k]\,(N - 2)! A(0, N - 2) + \cdots .
\end{aligned}
\tag{197}
$$

The last two terms together are equal to $\{N - 2 \quad 1\}A(0, N - 2)$, see Section 4.3.1.

Step 3 of the filtration loop in Section 3.3.1 computes

$$a_1^{\sqcup\!\sqcup_{\Theta}N-2} \sqcup\!\sqcup_{\Theta} a_2 = (N - 2)! \sum_{p=0}^{N-2} A(p, N - 2 - p) + \cdots \tag{198}$$

and brings all the words $A(p, N - 2 - p)$ on the rhs into lexicographical order using Eq. (196). Hence, in Eq. (197), $A(0, N - 2)$ is replaced by

$$A(0, N - 2) = \frac{1}{(N - 1)!} a_1^{\sqcup\!\sqcup_{\Theta}N-2} \sqcup\!\sqcup_{\Theta} a_2 - \frac{1}{N - 1} \sum_{p=1}^{N-2} \sum_{q=0}^{p-1} B(p - 1 - q, N - 2 - p + q). \tag{199}$$

Thus, after step 3 and 4 of the respective filtration cycle,

$$
\begin{aligned}
w_N = \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k\,[t_1] \dots [t_{k-1}]\,[t_k - 3]_{[a_1,a_2]}\,(t_k - 3)!(N - 1 - t_k)! \sum_{p=0}^{N-1-t_k} \binom{t_k - 2 + p}{p} \\
\times \left(\sum_{q=0}^{t_k-2+p} B(t_k - 2 + p - q, N - 1 - t_k - p + q) \right. \\
\left. - \sum_{q=0}^{N-t_k-p-1} B(N - t_k - p - 1 - q, t_k - 2 + p + q) \right)
\end{aligned}
$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k \begin{bmatrix} t_1 \end{bmatrix} \ldots \begin{bmatrix} t_{k-1} \end{bmatrix} \begin{bmatrix} t_k - 2 & 1 \end{bmatrix} (N-3)! \sum_{p=0}^{N-3} \sum_{q=0}^{p} B(p-q, N-3-p+q)$$

$$+ \begin{bmatrix} N-2 & 1 \end{bmatrix} a_1^{\sqcup\!\sqcup_\Theta N-2} \sqcup\!\sqcup_\Theta a_2 - \frac{1}{N-1} \begin{Bmatrix} N-2 & 1 \end{Bmatrix}$$

$$\times \sum_{p=1}^{N-2} \sum_{q=0}^{p-1} B(p-1-q, N-2-p+q) + \ldots . \tag{200}$$

Note that we do not take the filtered term $[\, N-2\ 1\,] a_1^{\sqcup\!\sqcup_\Theta N-2} \sqcup\!\sqcup_\Theta a_2$ into account because it does not contribute to the term in Eq. (185).

We now proceed as in the derivation of the master differential equation in Section 4.2. We denote by $\{N-3\}_{[a_1,a_2]}$ the number of words on the rhs of Eq. (200) with $(N-3)$ letters a_1 and one letter $[a_1, a_2]$. As in Section 4.1, $\{N-3\}_{[a_1,a_2]}$ is related to the indexed matrix $[\, N-3\,]_{[a_1,a_2]}$ by

$$[\, N-3\,]_{[a_1,a_2]} = \frac{1}{(N-2)!} \{N-3\}_{[a_1,a_2]} . \tag{201}$$

We obtain $\{N-3\}_{[a_1,a_2]}$ by setting all words $B(\cdot, \cdot)$ on the rhs of Eq. (200) to 1, hence

$$\{N-3\}_{[a_1,a_2]} = \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k[\, t_1\,] \ldots [\, t_{k-1}\,] [\, t_k - 3\,]_{[a_1,a_2]} (t_k-3)!(N-1-t_k)!$$

$$\times \sum_{p=0}^{N-1-t_k} \binom{t_k - 2 + p}{p} (2t_k + 2p - N - 1)$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{(N-1)!}{2} k[\, t_1\,] \ldots [\, t_{k-1}\,] [\, t_k - 2\ 1\,] - \frac{N-2}{2} \{N-2\ 1\} . \tag{202}$$

Standard combinatorial calculation yields

$$(t_k-3)!(N-1-t_k)! \sum_{p=0}^{N-1-t_k} \binom{t_k-2+p}{p} (2t_k + 2p - N - 1) = \frac{(N-1)!}{t_k(t_k-1)} . \tag{203}$$

Furthermore, we find $\{N-2\quad 1\} = (N-1)![\, N-2\ 1\,]$ from Eq. (133). We divide Eq. (202) by $(N-1)!$ and insert Eq. (201) on the lhs. This results in

$$\frac{1}{N-1} [\, N-3\,]_{[a_1,a_2]} = \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k[\, t_1\,] \ldots [\, t_{k-1}\,] \frac{1}{t_k(t_k-1)} [\, t_k - 3\,]_{[a_1,a_2]}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{1}{2} k[\, t_1\,] \ldots [\, t_{k-1}\,][\, t_k - 2\ 1\,] - \frac{N-2}{2} [\, N-2\ 1\,]. \tag{204}$$

Finally, we multiply with z^{N-1} and sum over all $N \in \mathbb{N}$. This will give us $\int [\,\bullet\,]_{[a_1,a_2]}\, dz$ on the lhs with zero integration constant. On the rhs, we obtain

$$\int [\,\bullet\,]_{[a_1,a_2]}\, dz = \sum_{N=0}^{\infty} \Bigg(\sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k[\, t_1\,] z^{t_1} \ldots [\, t_{k-1}\,] z^{t_{k-1}} \frac{1}{t_k(t_k-1)} [\, t_k - 3\,]_{[a_1,a_2]} z^{t_k}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{z}{2} k[\, t_1\,] z^{t_1} \ldots [\, t_{k-1}\,] z^{t_{k-1}} [\, t_k - 2\ 1\,] z^{t_{k-1}}$$

$$
-\frac{z^2}{2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z}[\,N-2\ 1\,]z^{N-1}\Bigg)
$$

$$
=\sum_{k\geq1}k[\,\bullet\,]^{k-1}\int\left(\int[\,\bullet\,]_{[a_1,a_2]}\,\mathrm{d}z\right)\mathrm{d}z
$$

$$
+\sum_{k\geq1}\frac{z}{2}k[\,\bullet\,]^{k-1}[\,\bullet\,1\,]-\frac{z^2}{2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z}[\,\bullet\,1\,]
$$

$$
=\frac{1}{1-2z}\int\left(\int[\,\bullet\,]_{[a_1,a_2]}\,\mathrm{d}z\right)\mathrm{d}z+\frac{1}{4\sqrt{1-2z}}
$$

$$
-\frac{1}{4\sqrt{1-2z}^3}+\frac{1}{2\sqrt{1-2z}}\log\left(\frac{1}{\sqrt{1-2z}}\right),
\tag{205}
$$

where we used the explicit form of the generating functions $[\,\bullet\,]$ and $[\,\bullet\,1\,]$ in the last line (Eqs. (143) and (151)). Again, the integration constants are zero.

Eq. (205) is an ordinary first order differential equation for $\int\left(\int[\,\bullet\,]_{[a_1,a_2]}\,\mathrm{d}z\right)\mathrm{d}z$ with the same homogeneous part as in the case of index-free generating functions, see Section 4.2. We can thus, use Eq. (146) to obtain

$$
\int\left(\int[\,\bullet\,]_{[a_1,a_2]}\,\mathrm{d}z\right)\mathrm{d}z=-\frac{\sqrt{1-2z}}{4}+\frac{1}{4\sqrt{1-2z}}
$$

$$
-\frac{\sqrt{1-2z}}{4}\log\left(\frac{1}{\sqrt{1-2z}}\right)-\frac{1}{4\sqrt{1-2z}}\log\left(\frac{1}{\sqrt{1-2z}}\right).
\tag{206}
$$

The second derivative finally results in the generating function

$$
[\,\bullet\,]_{[a_1,a_2]}=\frac{1}{4\sqrt{1-2z}^3}-\frac{1}{4\sqrt{1-2z}^5}+\frac{1}{4\sqrt{1-2z}^3}\log\left(\frac{1}{\sqrt{1-2z}}\right)
$$

$$
-\frac{3}{4\sqrt{1-2z}^5}\log\left(\frac{1}{\sqrt{1-2z}}\right).
\tag{207}
$$

4.5.2. The generating function $[\bullet]_{[a_1,\Theta(a_1,a_1)]}$

In this section, we derive the generating function $[\,\bullet\,]_{[a_1,\Theta(a_1,a_1)]}$ for the matrices $[\,N-3\,]_{[a_1,\Theta(a_1,a_1)]}$. These belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta\,N-3}\sqcup\!\sqcup_\Theta[a_1,\Theta(a_1,a_1)]$ in the filtered word w_N.

In full analogy to the previous section, we derive an equivalent to Eq. (202). In the partly filtered w_N, we denote the number of words with $(N-3)$ letters a_1 and one letter $[a_1,\Theta(a_1,a_1)]$ by

$$
\{N-3\}_{[a_1,\Theta(a_1,a_1)]}=(N-2)!\,[\,N-3\,]_{[a_1,\Theta(a_1,a_1)]}.
\tag{208}
$$

We then find

$$
\{N-3\}_{[a_1,\Theta(a_1,a_1)]}=\sum_{k\geq1}\sum_{\sum_i t_i=N-1}k[\,t_1\,]\dots[\,t_{k-1}\,][\,t_k-3\,]_{[a_1,\Theta(a_1,a_1)]}
$$

$$
\times\,(t_k-3)!(N-1-t_k)!\sum_{p=0}^{N-1-t_k}\binom{t_k-2+p}{p}
$$

$$
\times\,(2t_k+2p-N-1)+\sum_{k\geq1}\sum_{\sum_i t_i=N-1}\frac{(N-1)!}{2}k[\,t_1\,]\dots[\,t_{k-1}\,]\left[\,t_k-2\atop 2\,\right]
$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \mathscr{S}\left(\begin{bmatrix} N - 3 \\ 2 \end{bmatrix}, [\, N - 1\,] \right) \frac{(N-1)!}{2} [\, t_1\,] \ldots [\, t_k\,]$$

$$- \frac{N-2}{2} \begin{Bmatrix} N - 2 \\ 2 \end{Bmatrix}. \tag{209}$$

Compared to Eq. (202), we made the obvious replacements

$$[\, t_k - 3\,]_{[a_1, a_2]} \rightarrow [\, t_k - 3\,]_{[a_1, \Theta(a_1, a_1)]}, \qquad [\, t_k - 2\ 1\,] \rightarrow \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix},$$

$$\{\, N - 2 \quad 1\,\} \rightarrow \begin{Bmatrix} N - 2 \\ 2 \end{Bmatrix}. \tag{210}$$

The only new term is the third one. It arises because on the rhs of Eq. (186), one must also consider the term

$$B_+^{a_1} \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} [\, t_1\,] a_1^{\sqcup \Theta t_1} \ldots [\, t_k\,] a_1^{\sqcup \Theta t_k} \tag{211}$$

and calculate

$$a_1^{\sqcup \Theta N-1} = (N-1)! \underbrace{a_1 \ldots a_1}_{(N-1) \times} + \mathscr{S}\left(\begin{bmatrix} N - 3 \\ 2 \end{bmatrix}, [\, N - 1\,] \right) a_1^{\sqcup \Theta N-3} \sqcup \Theta \Theta(a_1, a_1). \tag{212}$$

We divide Eq. (209) by $(N-1)!$ and use Eqs. (133), (203) and (208) to obtain

$$\frac{1}{N-1} [\, N - 3\,]_{[a_1, \Theta(a_1, a_1)]} = \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k[\, t_1\,] \ldots [\, t_{k-1}\,] \frac{1}{t_k(t_k - 1)} [\, t_k - 3\,]_{[a_1, \Theta(a_1, a_1)]}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{1}{2} k[\, t_1\,] \ldots [\, t_{k-1}\,] \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \mathscr{S}\left(\begin{bmatrix} N - 3 \\ 2 \end{bmatrix}, [\, N - 1\,] \right) \frac{1}{2} [\, t_1\,] \ldots [\, t_k\,]$$

$$- \frac{N-2}{2} \begin{bmatrix} N - 2 \\ 2 \end{bmatrix} - \frac{N-2}{2} \mathscr{S}\left(\begin{bmatrix} N - 2 \\ 2 \end{bmatrix}, [\, N\,] \right) [\, N\,]. \tag{213}$$

As in the previous section, we multiply with z^{N-1} and sum over all $N \in \mathbb{N}$. With

$$\mathscr{S}\left(\begin{bmatrix} N - 2 \\ 2 \end{bmatrix}, [\, N\,] \right) = \frac{N(N-1)}{2}, \tag{214}$$

we find

$$\int [\, \bullet\,]_{[a_1, \Theta(a_1, a_1)]} \, \mathrm{d}z = \sum_{N=0}^{\infty} \Bigg(\sum_{k \geq 1} \sum_{\sum_i t_i = N-1} k[\, t_1\,] z^{t_1} \ldots [\, t_{k-1}\,] z^{t_{k-1}}$$

$$\times \frac{1}{t_k(t_k - 1)} [\, t_k - 3\,]_{[a_1, \Theta(a_1, a_1)]} z^{t_k}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{z}{2} k[\, t_1\,] z^{t_1} \ldots [\, t_{k-1}\,] z^{t_{k-1}} \begin{bmatrix} t_k - 2 \\ 2 \end{bmatrix} z^{t_{k-1}}$$

$$+ \sum_{k \geq 1} \sum_{\sum_i t_i = N-1} \frac{z^2}{4} \frac{\mathrm{d}^2}{\mathrm{d}z^2} [\, t_1\,] z^{t_1} \ldots [\, t_k\,] z^{t_k}$$

$$-\frac{z^2}{2}\frac{d}{dz}\frac{1}{z}\left[N-2\right]z^{N-1}-\frac{z^2}{4}\frac{d^3}{dz^3}[N]z^N\Bigg)$$

$$=\sum_{k\geq 1}k\,[\bullet]^{k-1}\int\left(\int[\bullet]_{[a_1,\Theta(a_1,a_1)]}\,dz\right)dz$$

$$+\sum_{k\geq 1}\frac{z}{2}k\,[\bullet]^{k-1}\Big[\tfrac{\bullet}{2}\Big]+\sum_{k\geq 1}\frac{z^2}{4}\frac{d^2}{dz^2}[\bullet]^k$$

$$-\frac{z^2}{2}\frac{d}{dz}\frac{1}{z}\Big[\tfrac{\bullet}{2}\Big]-\frac{z^2}{4}\frac{d^3}{dz^3}[\bullet]$$

$$=\frac{1}{1-2z}\int\left(\int[\bullet]_{[a_1,\Theta(a_1,a_1)]}\,dz\right)dz-\frac{1}{8\sqrt{1-2z}}$$

$$+\frac{1}{8\sqrt{1-2z}^3}-\frac{1}{4\sqrt{1-2z}}\log\left(\frac{1}{\sqrt{1-2z}}\right). \tag{215}$$

Note that the third and the fifth term on the rhs of the second equation cancel. This is an interesting incidence. Because of Eq. (160), the inhomogeneous parts of the differential equations Eqs. (205) and (215) only differ by a factor of $-1/2$. We therefore obtain

$$[\bullet]_{[a_1,\Theta(a_1,a_1)]}=-\frac{1}{2}[\bullet]_{[a_1,a_2]}=-\frac{1}{8\sqrt{1-2z}^3}+\frac{1}{8\sqrt{1-2z}^5}$$

$$-\frac{1}{8\sqrt{1-2z}^3}\log\left(\frac{1}{\sqrt{1-2z}}\right)+\frac{3}{8\sqrt{1-2z}^5}\log\left(\frac{1}{\sqrt{1-2z}}\right). \tag{216}$$

4.6. Results

We now demonstrate the power of the generating functions derived in the previous sections: one can write the next-to$^{\{j\}}$-leading log order as a function of terms up to $\mathcal{O}(\alpha^{j+1})$ in the log-expansion (Eq. (1)). We will show this for the Yukawa fermion propagator (Eq. (3)) up to $j=2$ using the explicit generating functions obtained in the previous section. They are also collected in the second column of Table 1. We discuss our results for $j=0,1,2$ separately.

4.6.1. Leading log expansion

Consider the filtered solution W_{Yuk} of the DSE Eq. (43), see Eq. (45). The leading log order is

$$W_{\text{Yuk}}|_{\text{l.l.}}=\sum_{n\geq 1}\left(\alpha^n w_n^{\text{Yuk}}\right)\Bigg|_{\text{l.l.}}. \tag{217}$$

Contributing terms in the filtered words w_n map to L^n under renormalized Feynman rules since the leading log order is $\propto\alpha^n L^n$. These are only the full shuffle products $a_1^{\sqcup\!\!\sqcup_\Theta n}$ (see Section 3.2 that $\Psi_R\left(a_1^{\sqcup\!\!\sqcup_\Theta n}\right)\propto L^n$). Thus,

$$W_{\text{Yuk}}|_{\text{l.l.}}=\sum_{n\geq 0}\alpha^n[n]a_1^{\sqcup\!\!\sqcup_\Theta n}=\alpha a_1+\frac{1}{2}\alpha^2 a_1\sqcup\!\!\sqcup_\Theta a_1+\frac{1}{2}\alpha^3 a_1^{\sqcup\!\!\sqcup_\Theta 3}$$

$$+\frac{5}{8}\alpha^4 a_1^{\sqcup\!\!\sqcup_\Theta 4}+\frac{7}{8}\alpha^5 a_1^{\sqcup\!\!\sqcup_\Theta 5}+\cdots. \tag{218}$$

See the first row of Table 2 for the explicit multiplicities. Acting with renormalized Feynman rules Ψ_R on both sides results in

$$\Psi_R(W_{\text{Yuk}})\Bigg|_{\text{l.l.}}=\sum_{n\geq 0}[n]\alpha^n\Psi_R(a_1)^n=[\bullet]\Bigg|_{z\to\alpha\Psi_R(a_1)}. \tag{219}$$

We write this equation in terms of Feynman graphs. Therefore, set $\Psi_R = \Phi_R \circ \Upsilon^{-1}$. We find that $\Psi_R(a_1) = \Phi_R(\Gamma_1)$ on the rhs using the properties of the Hopf algebra morphism Υ^{-1} in Eqs. (35)–(40). On the lhs, $\Psi_R(W_{\text{Yuk}}) = \Phi_R(X_{\text{Yuk}}) \equiv G_R(X_{\text{Yuk}})$, which is the full Green function of the fermion propagator. Hence, Eq. (219) yields

$$G_R(X_{\text{Yuk}})|_{l.l.} = [\bullet]|_{z \to \alpha \Phi_R(\Gamma_1)}. \tag{220}$$

Using Eq. (143) for the generating function $[\bullet]$, we finally obtain

$$G_R(X_{\text{Yuk}})|_{l.l.} = 1 - \sqrt{1 - 2\alpha \Phi_R(\Gamma_1)}. \tag{221}$$

Without this result, the computation of $G_R(X_{\text{Yuk}})|_{l.l.}$ would be quite more complicated, even impossible. Computing $G_R(X_{\text{Yuk}})|_{l.l.}$ the ordinary way includes to calculate an infinite number of Feynman integrals with any number of loops. For example, the graphs $B_+^{\Gamma_1}\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\dots)\right)\right)$ contribute to $G_R(X_{\text{Yuk}})|_{l.l.}$. Using our formula in Eq. (221), *we only need to compute the one-loop Feynman integral* $\Phi_R(\Gamma_1)$ *to derive the full leading log order Green function* $G_R(X_{\text{Yuk}})|_{l.l.}$.

4.6.2. Next-to-leading log expansion

The next-to-leading log order of Eq. (45) is

$$W_{\text{Yuk}}|_{n.l.l.} = \sum_{n \geq 1} (\alpha^n w_n^{\text{Yuk}})\Big|_{n.l.l.}. \tag{222}$$

Contributing terms of the filtered words w_n map to L^{n-1} under renormalized Feynman rules since the next-to-leading log order is $\propto \alpha^n L^{n-1}$. These are the full shuffle products $a_1^{\sqcup\!\sqcup_\Theta (n-2)} \sqcup\!\sqcup_\Theta a_2$ and $a_1^{\sqcup\!\sqcup_\Theta (n-2)} \sqcup\!\sqcup_\Theta \Theta(a_1, a_1)$. Indeed, renormalized Feynman rules are character-like. Ψ_R acting on a full shuffle product of $n - 1$ letters is $\propto L^{n-1}$ in the log-expansion, see Section 3.2. Thus,

$$W_{\text{Yuk}}|_{n.l.l.} = \sum_{n \geq 2}\Bigg([\, n - 2\ 1\,](\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-2)} \sqcup\!\sqcup_\Theta \left(\alpha^2 a_2\right)$$
$$+ \left[\, n \underset{2}{-}\ 2\,\right](\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-2)} \sqcup\!\sqcup_\Theta \left(\alpha^2 \Theta(a_1, a_1)\right)\Bigg). \tag{223}$$

Acting with renormalized Feynman rules Ψ_R on both sides results in

$$\Psi_R\left(W_{\text{Yuk}}\right)|_{n.l.l.} = \sum_{n \geq 2}\Bigg([\, n - 2\ 1\,]\alpha^{n-2}\Psi_R(a_1)^{n-2}\alpha^2\Psi_R(a_2)$$
$$+ \left[\, n \underset{2}{-}\ 2\,\right]\alpha^{n-2}\Psi_R(a_1)^{n-2}\alpha^2\Psi_R(\Theta(a_1, a_1))\Bigg)$$
$$= [\bullet\ 1]|_{z \to \alpha \Psi_R(a_1)}\alpha^2\Psi_R(a_2) + \left[\substack{\bullet \\ 2}\right]|_{z \to \alpha \Psi_R(a_1)}\alpha^2\Psi_R(\Theta(a_1, a_1)). \tag{224}$$

Again, we write $\Psi_R = \Phi_R \circ \Upsilon^{-1}$ and obtain the full next-to-leading log order renormalized Green function of the Yukawa fermion propagator on the lhs. On the rhs, the only subtle point is that $\Theta(a_1, a_1)$ has no single corresponding Feynman graph. However, we find the period

$$\Psi_R(\Theta(a_1, a_1)) = \Phi_R \circ \Upsilon^{-1}(\Theta(a_1, a_1)) = \Phi_R \circ \Upsilon^{-1}\left((a_1 \sqcup\!\sqcup_\Theta a_1) - 2B_+^{a_1}(a_1)\right)$$
$$= \Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right). \tag{225}$$

Thus,

$$G_R(X_{\text{Yuk}})|_{n.l.l.} = [\bullet\ 1]|_{z \to \alpha \Phi_R(\Gamma_1)}\alpha^2\Phi_R(\Gamma_2)$$
$$+ \left[\substack{\bullet \\ 2}\right]|_{z \to \alpha \Phi_R(\Gamma_1)}\alpha^2\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right). \tag{226}$$

Using the generating functions in Eqs. (151) and (159), we finally derive

$$
\begin{aligned}
G_R(X_{\text{Yuk}})|_{n.l.l.} = & \frac{\alpha^2}{\sqrt{1 - 2\alpha \Phi_R(\Gamma_1)}} \log \left(\frac{1}{\sqrt{1 - 2\alpha \Phi_R(\Gamma_1)}} \right) \\
& \times \left(\Phi_R(\Gamma_2) + \Phi_R \left(B_+^{\Gamma_1}(\Gamma_1) \right) - \frac{1}{2} \Phi_R(\Gamma_1)^2 \right).
\end{aligned}
\tag{227}
$$

This is an enormous simplification: *we only need to compute the one-loop Feynman integral $\Phi_R(\Gamma_1)$ as well as the two-loop integrals $\Phi_R(\Gamma_2)$ and $\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)$ to calculate the full next-to-leading log order Green function $G_R(X_{\text{Yuk}})|_{n.l.l.}$.*

4.6.3. Next-to-next-to-leading log expansion

The next-to-next-to-leading log order of Eq. (45) is

$$
W_{\text{Yuk}}|_{n.n.l.l.} = \sum_{n \geq 1} (\alpha^n w_n^{\text{Yuk}}) \bigg|_{n.n.l.l.}.
\tag{228}
$$

Contributing terms of the filtered words w_n must map to L^{n-2} under renormalized Feynman rules since the next-next-to-leading log order is $\propto \alpha^n L^{n-2}$. These are the full shuffle products

$$
a_1^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta a_3, \qquad a_1^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta a_2^{\sqcup\!\sqcup_\Theta 2},
\tag{229}
$$

$$
a_1^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta \Theta(a_1, a_1, a_1), \qquad a_1^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta \Theta(a_1, a_2),
\tag{230}
$$

$$
a_1^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta \Theta(a_1, a_1)^{\sqcup\!\sqcup_\Theta 2}, \qquad a_1^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta a_2 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1),
\tag{231}
$$

$$
a_1^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta [a_1, a_2], \qquad a_1^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta [a_1, \Theta(a_1, a_1)].
\tag{232}
$$

From Eq. (228),

$$
\begin{aligned}
W_{\text{Yuk}}|_{n.n.l.l.} = \sum_{n \geq 0} \Bigg(& [\, n - 3 \; 0 \; 1 \,](\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta (\alpha^3 a_3) \\
& + [\, n - 4 \; 2 \,](\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta (\alpha^2 a_2)^{\sqcup\!\sqcup_\Theta 2} \\
& + \left[\begin{matrix} n - 3 \\ 3 \end{matrix} \right] (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta (\alpha^3 \Theta(a_1, a_1, a_1)) \\
& + \left[\begin{matrix} n - 3 \; 0 \\ 1 \; 1 \end{matrix} \right] (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta (\alpha^3 \Theta(a_1, a_2)) \\
& + \left[\begin{matrix} n - 4 \\ 2 \\ 2 \end{matrix} \right] (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta (\alpha^2 \Theta(a_1, a_1))^{\sqcup\!\sqcup_\Theta 2} \\
& + \left[\begin{matrix} n - 4 \; 1 \\ 2 \; 0 \end{matrix} \right] (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-4)} \sqcup\!\sqcup_\Theta (\alpha^2 a_2) \sqcup\!\sqcup_\Theta (\alpha^2 \Theta(a_1, a_1)) \\
& + [\, n - 3 \,]_{[a_1, a_2]} (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta (\alpha^3 [a_1, a_2]) \\
& + [\, n - 3 \,]_{[a_1, \Theta(a_1, a_1)]} (\alpha a_1)^{\sqcup\!\sqcup_\Theta (n-3)} \sqcup\!\sqcup_\Theta (\alpha^3 [a_1, \Theta(a_1, a_1)]) \Bigg).
\end{aligned}
\tag{233}
$$

Acting with renormalized Feynman rules Ψ_R on both sides results in

$$
\begin{aligned}
\Psi_R(W_{\text{Yuk}})|_{n.n.l.l.} = & \left([\, \bullet \; 0 \; 1 \,]\alpha^3 \Psi_R(a_3) + [\, \bullet \; 2 \,]\alpha^4 \Psi_R(a_2)^2 \right. \\
& + \left[\begin{matrix} \bullet \\ 3 \end{matrix} \right] \alpha^3 \Psi_R(\Theta(a_1, a_1, a_1)) + \left[\begin{matrix} \bullet \; 0 \\ 1 \; 1 \end{matrix} \right] \alpha^3 \Psi_R(\Theta(a_1, a_2))
\end{aligned}
$$

$$+ \left[\begin{matrix} \bullet \\ 2 \\ 2 \end{matrix} \right] \alpha^4 \Psi_R(\Theta(a_1, a_1))^2 + \left[\begin{matrix} \bullet \ 1 \\ 2 \ 0 \end{matrix} \right] \alpha^4 \Psi_R(a_2) \Psi_R(\Theta(a_1, a_1))$$

$$+ [\, \bullet \,]_{[a_1, a_2]} \, \alpha^3 \Psi_R([a_1, a_2])$$

$$+ [\, \bullet \,]_{[a_1, \Theta(a_1, a_1)]} \, \alpha^3 \Psi_R([a_1, \Theta(a_1, a_1)]) \Bigg) \Bigg|_{z \to \alpha \Psi_R(a_1)} . \tag{234}$$

We write $\Psi_R = \Phi_R \circ \Upsilon^{-1}$ and obtain the full next-to-next-to-leading log order Green function on the lhs. On the rhs, for example $\Psi_R(a_3) = \Phi_R(\Gamma_3)$. The letters $\Theta(a_1, a_1, a_1)$, $\Theta(a_1, a_2)$, $[a_1, a_2]$, $[a_1, \Theta(a_1, a_1)]$ do not correspond to single Feynman graphs but to uniquely determined homogeneous polynomials in Feynman graphs.

We write

$$\Theta(a_1, a_1, a_1) = 3B_+^{a_1}\left(B_+^{a_1}(a_1)\right) + \frac{3}{2} a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) - \frac{1}{2} a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1, \tag{235}$$

$$\Theta(a_1, a_2) = -B_+^{a_1}(a_2) - B_+^{a_2}(a_1) + a_1 \sqcup\!\sqcup_\Theta a_2, \tag{236}$$

$$[a_1, a_2] = B_+^{a_1}(a_2) - B_+^{a_2}(a_1), \tag{237}$$

$$[a_1, \Theta(a_1, a_1)] = 2B_+^{a_1}(a_1 \sqcup\!\sqcup_\Theta a_1) - B_+^{a_1}\left(B_+^{a_1}(a_1)\right) + \frac{1}{2} a_1 \sqcup\!\sqcup_\Theta \Theta(a_1, a_1) - \frac{1}{2} a_1 \sqcup\!\sqcup_\Theta a_1 \sqcup\!\sqcup_\Theta a_1. \tag{238}$$

Now, we act with $\Psi_R = \Phi_R \circ \Upsilon^{-1}$ and use Eq. (225). We use that under Υ^{-1} the shuffle product maps to the disjoint union – the product – on graphs and thus find the periods

$$\Psi_R(\Theta(a_1, a_1, a_1)) = 3\Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - 3\Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) + \Phi_R(\Gamma_1)^3, \tag{239}$$

$$\Psi_R(\Theta(a_1, a_2)) = -\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right) + \Phi_R(\Gamma_1)\Phi_R(\Gamma_2), \tag{240}$$

$$\Psi_R([a_1, a_2]) = \Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right), \tag{241}$$

$$\Psi_R([a_1, \Theta(a_1, a_1)]) = 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1)\right) - \Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - \Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right). \tag{242}$$

Inserting these identities together with Eq. (225) into Eq. (234), we finally obtain the next-to-next-to-leading log order Green function,

$$G_R(X_{\text{Yuk}})|_{n.n.l.l.} = \alpha^3 \Bigg([\, \bullet \ 0 \ 1\,] \Phi_R(\Gamma_3) + \alpha[\, \bullet \ 2\,] \Phi_R(\Gamma_2)^2$$

$$+ \left[\begin{matrix} \bullet \\ 3 \end{matrix} \right] \left(3\Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - 3\Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) + \Phi_R(\Gamma_1)^3 \right)$$

$$+ \left[\begin{matrix} \bullet \ 0 \\ 1 \ 1 \end{matrix} \right] \left(-\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right) + \Phi_R(\Gamma_1)\Phi_R(\Gamma_2) \right)$$

$$+ \alpha \left[\begin{matrix} \bullet \\ 2 \\ 2 \end{matrix} \right] \left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) \right)^2$$

$$+ \alpha \left[\begin{matrix} \bullet \ 1 \\ 2 \ 0 \end{matrix} \right] \Phi_R(\Gamma_2) \left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) \right)$$

$$+ [\, \bullet \,]_{[a_1, a_2]} \left(\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right) \right)$$

$$+ [\, \bullet \,]_{[a_1, \Theta(a_1, a_1)]} \left(2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1)\right) - \Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) \right.$$

$$\left. - \Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) \right) \Bigg) \Bigg) \Bigg|_{z \to \alpha \Phi_R(\Gamma_1)} . \tag{243}$$

We refer to the second column of Table 1 for the explicit expressions of the generating functions. *This shows the explicit dependence of the full next-to-next-to-leading log order Green function $G_R(X_{\text{Yuk}})|_{n.n.l.l.}$ on the Feynman graphs*

$$\Gamma_1, \quad \Gamma_2, \quad B_+^{\Gamma_1}(\Gamma_1), \quad \Gamma_3, \quad B_+^{\Gamma_1}(\Gamma_2),$$
$$B_+^{\Gamma_2}(\Gamma_1), \quad B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right), \quad B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1). \tag{244}$$

These are at most, three-loop graphs.

As a final remark, note that all generating functions have the structure $1/\sqrt{1-2z}$ in common. In particular, this structure is simply related to the leading-log generating function by Eq. (143). One could also argue that all non-leading logs are some functions in the leading-log contribution. This was expected because the non-leading-logs are recursively given in terms of the leading log generating function [•].

In Appendix A we discuss the generating functions for the QED photon self-energy. There we observe again that all generating functions have one same structure in common. However, this structure will be $1/(1-z)$ rather than $1/\sqrt{1-2z}$ because the underlying Dyson–Schwinger equation Eq. (47) differs from the Yukawa case.

Acknowledgments

Appendix A. Relations for the log-expansion of the QED photon self-energy

We relate the next-to-$^{\{j\}}$-leading log order to the first $(j+1)$ terms of perturbation theory in the QED photon self-energy Green function $G_R(X_{\text{QED}})$. There are two differences from the Yukawa propagator Green function (Consider the DSEs equations (46) and (47)). First, there are no insertion points in the one-loop primitive propagator graph. The sum in Eq. (47) starts with $j = 2$ rather than $j = 1$ in Eq. (46). This will simplify the following calculations drastically. Secondly, the term $\binom{2j-2+k}{k}$ in Eq. (46) is replaced by $\binom{j-2+k}{k}$ in Eq. (47). This will change the structure $1/\sqrt{1-2z}$ in the generating functions of Yukawa theory to $1/(1-z)$ in QED.

In the first part, we treat index-free matrices. The corresponding shuffle products do not contain $[\cdot, \cdot]$-letters. We repeat the same steps to derive the master differential equation as in Yukawa theory. The two mentioned differences in the respective DSEs equations (46) and (47) change Eq. (141) to

$$\mathcal{M}(z)' = \sum_{m \sim \mathcal{M}} \left(-\sum_{m' \neq m} \frac{\mathrm{d}}{\mathrm{d}z} z^{|m|-|m'|} \mathscr{S}(m, m') m' z^{|m'|} + \delta_{|m|1} \delta_{n_\Theta(m)0} \right.$$
$$+ \sum_{j \geq 2}^{\mathbf{u}m\mathbf{v}^T - 1} (1 - \delta_{m_1 j 0}) \sum_{k \geq 1}^{\mathbf{u}m\mathbf{v}^T - j} \binom{j-2+k}{k}$$
$$\left. \times \sum^{(*)} z^{|m|-1-\sum_i |m_i|} \mathscr{S}\left(m \ominus p_j, \bigoplus_i m_i\right) m_1 z^{|m_1|} m_2 z^{|m_2|} \dots m_k z^{|m_k|} \right), \tag{A.1}$$

$$(*): \quad t_i \geq 1, \; i = 1 \dots k, \qquad \sum_{i=1}^{k} t_i = \mathbf{u}m\mathbf{v}^T - j, \quad \mathbf{u}m_i\mathbf{v}^T = t_i, \qquad \sum_i \mathbf{u}m_i + \mathbf{u}p_j = \mathbf{u}m. \tag{A.2}$$

This is an ordinary equation for $\mathcal{M}(z)'$ and not a differential equation because the sum starts with $j = 2$. We therefore call Eq. (A.1) *master equation*. We must integrate the master equation to obtain $\mathcal{M}(z)$ such that $\mathcal{M}(0) = 0$ (Eq. (142)). Eqs. (134) and (135) remain valid,

$$\mathcal{M} = \mathcal{M}(z) = \sum_{m \sim \mathcal{M}}^{m \neq (0)} m z^{|m|}, \qquad m = \frac{1}{|m|!} \left(\frac{d}{dz} \right)^{|m|} \mathcal{M}(z)|_{z=0}. \tag{A.3}$$

Consider for example the case $\mathcal{M}(z) = [\, \bullet \,]$. The matrices $m \sim \mathcal{M}$ belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta N}$. Since there is no $B_+^{a_1}$, we already know that $[\, N \,] = 0$ for $N > 1$. In the master equation, all terms vanish except for $\delta_{|m|1}\delta_{n_\Theta(m)0}$. Integrating the remaining equation $[\, \bullet \,]' = 1$ yields

$$[\, \bullet \,] = z, \tag{A.4}$$

which generates the matrices $[\, N \,] = \delta_{N\,1}$ as expected (see Eq. (A.3)).

In the following, we derive the generating functions up to next-to-next-to-leading log order for the QED photon self-energy. As in Section 4, we consider the cases $n_\Theta(m) = 0$ and $n_\Theta(m) \neq 0$ separately. In Appendix A.3, we treat indexed matrices m.

A.1. Generating functions for index-free matrices with $n_\Theta(m) = 0$

Index-free matrices with $n_\Theta(m) = 0$ belong to shuffle products without $\Theta(\cdot, \cdot)$- and $[\cdot, \cdot]$-letters. In full analogy to the Yukawa propagator, only one row in $\mathcal{M}(z)$ reduces the master equation to

$$\mathcal{M}(z)' = \delta_{|\mathcal{M}|0} + \delta_{|\mathcal{M}|1} + \sum_{j \geq 2, k \geq 1}^{\infty} (1 - \delta_{\mathcal{M}_{1j}0}) \binom{j-2+k}{k} \sum^{(**)} \mathcal{M}_1(z)\mathcal{M}_2(z) \ldots \mathcal{M}_k(z), \tag{A.5}$$

where

$$(**): \quad \mathcal{M}_1 \oplus \mathcal{M}_2 \oplus \cdots \oplus \mathcal{M}_k \oplus \mathcal{P}_j = \mathcal{M}. \tag{A.6}$$

A.1.1. The generating function [• 1]

Let $\mathcal{M}(z) = [\, \bullet\, 1 \,]$. The sum in Eq. (A.5) is non-zero only for $j = 2$. Then, $\mathcal{P}_j = \mathcal{P}_2 = \mathcal{M}$ and $\mathcal{M}_i = [\, \bullet \,] = z \,\forall i \leq k$. We obtain

$$[\, \bullet\, 1 \,]' = 1 + \sum_{k \geq 1} [\, \bullet \,]^k = \frac{1}{1-z} \qquad \Rightarrow [\, \bullet\, 1 \,] = \log\left(\frac{1}{1-z} \right). \tag{A.7}$$

A.1.2. The generating function [• 0 1]

Let $\mathcal{M}(z) = [\, \bullet\, 0\, 1 \,]$. The sum in Eq. (A.5) is non-zero only for $j = 3$. Then, $\mathcal{P}_j = \mathcal{P}_3 = \mathcal{M}$ and $\mathcal{M}_i = [\, \bullet \,] = z \,\forall i \leq k$. We obtain

$$[\, \bullet\, 0\, 1 \,]' = 1 + \sum_{k \geq 1} \binom{k+1}{k} [\, \bullet \,]^k = \frac{1}{(1-z)^2} \qquad \Rightarrow [\, \bullet\, 0\, 1 \,] = -1 + \frac{1}{1-z}. \tag{A.8}$$

The term -1 is the integration constant such that $[\, \bullet\, 0\, 1 \,](0) = 0$.

A.1.3. The generating function [• 2]

Let $\mathcal{M}(z) = [\, \bullet\, 2 \,]$. The sum in Eq. (A.5) is non-zero only for $j = 2$. Then, $\mathcal{P}_j = \mathcal{P}_2$. Eq. (A.6) implies that for one $i \leq k$, $\mathcal{M}_i = [\, \bullet\, 1 \,] = \log(1/(1-z))$ and for all other i, $\mathcal{M}_i = [\, \bullet \,] = z$. We obtain

$$[\, \bullet\, 2 \,]' = \sum_{k \geq 1} k [\, \bullet \,]^{k-1}[\, \bullet\, 1 \,] = \frac{1}{(1-z)^2} \log\left(\frac{1}{1-z} \right)$$

$$\Rightarrow [\, \bullet\, 2 \,] = 1 - \frac{1}{1-z} + \frac{1}{1-z} \log\left(\frac{1}{1-z} \right). \tag{A.9}$$

A.2. Generating functions for index-free matrices with $n_\Theta(m) \neq 0$

We now treat full shuffle products that contain $\Theta(\cdot, \cdot)$-letters but no $[\cdot, \cdot]$-letters. Here, we have to proceed from the master equation Eq. (A.1). The following generating functions simplify to zero,

$$\begin{bmatrix} \bullet \\ 2 \end{bmatrix} = \begin{bmatrix} \bullet \\ 3 \end{bmatrix} = \begin{bmatrix} \bullet \\ 2 \\ 2 \end{bmatrix} = 0. \tag{A.10}$$

The reason is that the sum in the master equation starts with $j = 2$.

A.2.1. The generating function $\begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix}$

There is another simplification due to the missing $(j = 1)$-term. If the first row of a matrix m is of the form $[\, N\ 0\ 0\ \ldots\,]$ with arbitrary N, then the second (and more complicated) term on the rhs of the master equation Eq. (A.1) yields zero. For example for the next-to-next-to-leading log generating function $\begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix}$, Eq. (A.1) reduces to

$$\begin{aligned}
\begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix}' &= \sum_{N=0}^{\infty} \left(-\frac{d}{dz} z^{(N+1)-(N+2)} \, \mathcal{S}\left(\begin{bmatrix} N & 0 \\ 1 & 1 \end{bmatrix}, [\, N+1\ 1\,] \right) [\, N+1\ 1\,] z^{N+2} \right) \\
&= \sum_{N=0}^{\infty} \left(-\frac{d}{dz} \frac{1}{z} (N+1) [\, N+1\ 1\,] z^{N+2} \right) \\
&= \sum_{N=0}^{\infty} \left(-\frac{d}{dz} \frac{1}{z} z^2 \frac{d}{dz} \frac{1}{z} [\, N+1\ 1\,] z^{N+2} \right) \\
&= -\frac{d}{dz} z \frac{d}{dz} \frac{1}{z} [\, \bullet\ 1\,].
\end{aligned} \tag{A.11}$$

Integration with suitable initial conditions and using Eq. (A.7) results in

$$\begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix} = -\frac{d}{dz} [\, \bullet\ 1\,] + \frac{1}{z} [\, \bullet\ 1\,] = -\frac{1}{1-z} + \frac{1}{z} \log\left(\frac{1}{1-z} \right). \tag{A.12}$$

A.2.2. The generating function $\begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}$

For $\mathcal{M}(z) = \begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}$, the master equation Eq. (A.1) reduces to

$$\begin{aligned}
\begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix}' &= \sum_{N=0}^{\infty} \Bigg(-\frac{d}{dz} z^{(N+2)-(N+3)} \, \mathcal{S}\left(\begin{bmatrix} N & 1 \\ 2 & 0 \end{bmatrix}, [\, N+2\ 1\,] \right) [\, N+2\ 1\,] z^{N+3} \\
&\quad + \sum_{k \geq 1} \sum_{\sum_i t_i = (N+4)-2} z^{(N+2)-1-\sum_i t_i} \, \mathcal{S}\left(\begin{bmatrix} N \\ 2 \end{bmatrix}, [\, N+2\,] \right) [\, t_1\,] z^{t_1} \ldots [\, t_k\,] z^{t_k} \Bigg) \\
&= \sum_{N=0}^{\infty} \left(-\frac{d}{dz} \frac{1}{z} \binom{N+2}{2} [\, N+2\ 1\,] z^{N+3} \right. \\
&\quad + \sum_{k \geq 1} \sum_{\sum_i t_i = N+2} \frac{1}{z} \binom{N+2}{2} [\, t_1\,] z^{t_1} \ldots [\, t_k\,] z^{t_k} \Bigg) \\
&= \sum_{N=0}^{\infty} \Bigg(-\frac{d}{dz} \frac{1}{z} \frac{z^3}{2} \frac{d^2}{dz^2} \frac{1}{z} [\, N+2\ 1\,] z^{N+3} + \sum_{k \geq 1} \sum_{\sum_i t_i = N+2} \frac{1}{z} \frac{z^2}{2} \frac{d^2}{dz^2} [\, t_1\,] z^{t_1} \ldots [\, t_k\,] z^{t_k} \Bigg)
\end{aligned}$$

$$= -\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{z}[\bullet\, 1] + \sum_{k\geq 1}\frac{z}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}[\bullet\,]^k$$

$$= \frac{\mathrm{d}}{\mathrm{d}z}\left(-\frac{z^2}{2}\frac{\mathrm{d}^2}{\mathrm{d}z^2}\frac{1}{z}\log\left(\frac{1}{1-z}\right) + \frac{z}{2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z}\right). \tag{A.13}$$

In the last line, we used Eqs. (A.4) and (A.7). We integrate with suitable initial conditions and obtain

$$\left[\begin{smallmatrix}\bullet\,1\\2\,0\end{smallmatrix}\right] = \frac{1}{2} + \frac{1}{2(1-z)} - \frac{1}{z}\log\left(\frac{1}{1-z}\right). \tag{A.14}$$

A.3. Generating functions for indexed matrices

As in Yukawa theory, we only calculate the generating functions $[\,\bullet\,]_{[a_1,a_2]}$ and $[\,\bullet\,]_{[a_1,\Theta(a_1,a_1)]}$ that belong to the shuffle products $a_1^{\sqcup\!\sqcup_\Theta N}\sqcup\!\sqcup_\Theta[a_1,a_2]$ and $a_1^{\sqcup\!\sqcup_\Theta N}\sqcup\!\sqcup_\Theta[a_1,\Theta(a_1,a_1)]$.

We find

$$[\,\bullet\,]_{[a_1,\Theta(a_1,a_1)]} = 0 \tag{A.15}$$

because the sum in Eq. (47) starts with $j = 2$.

For the generating function $[\,\bullet\,]_{[a_1,a_2]}$, Eq. (205) reduces to

$$\int [\,\bullet\,]_{[a_1,a_2]}\,\mathrm{d}z = -\frac{z^2}{2}\frac{\mathrm{d}}{\mathrm{d}z}\frac{1}{z}[\bullet\,1] \tag{A.16}$$

because in Eq. (186), the first two terms are missing. Here, the derivation of the generating function is even simpler than in all previous cases. One only needs to differentiate instead of integrating or solving a differential equation.

We differentiate Eq. (A.16) and use the explicit form of $[\,\bullet\,1\,]$ in Eq. (A.7). This results in

$$[\,\bullet\,]_{[a_1,a_2]} = \frac{1}{2(1-z)} - \frac{1}{2(1-z)^2}. \tag{A.17}$$

A.4. Results

We repeat the steps in Section 4.6 to write the next-to-$^{[j]}$-leading log order as a function of terms up to $\mathcal{O}(j+1)$ in the log-expansion (Eq. (1)). We show this up to $j = 2$ and use Eqs. (220), (226) and (243), which are universally valid (in the QED case below several terms vanish):

$$G_R(X_{\text{Yuk}})|_{l.l.} = [\,\bullet\,]|_{z\to\alpha\Phi_R(\Gamma_1)}, \tag{A.18}$$

$$G_R(X_{\text{Yuk}})|_{n.l.l.} = [\,\bullet\,1\,]|_{z\to\alpha\Phi_R(\Gamma_1)}\alpha^2\Phi_R(\Gamma_2) + \left[\begin{smallmatrix}\bullet\\2\end{smallmatrix}\right]\Big|_{z\to\alpha\Phi_R(\Gamma_1)}\alpha^2\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right), \tag{A.19}$$

$$\begin{aligned}
G_R(X_{\text{Yuk}})|_{n.n.l.l.} = \alpha^3\Bigg(&[\,\bullet\,0\,1\,]\Phi_R(\Gamma_3) + \alpha[\,\bullet\,2\,]\Phi_R(\Gamma_2)^2\\
&+ \left[\begin{smallmatrix}\bullet\\3\end{smallmatrix}\right]\left(3\Phi_R\left(B_+^{\Gamma_1}\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right) - 3\Phi_R(\Gamma_1)\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right) + \Phi_R(\Gamma_1)^3\right)\\
&+ \left[\begin{smallmatrix}\bullet\,0\\1\,1\end{smallmatrix}\right]\left(-\Phi_R\left(B_+^{\Gamma_1}(\Gamma_2)\right) - \Phi_R\left(B_+^{\Gamma_2}(\Gamma_1)\right) + \Phi_R(\Gamma_1)\Phi_R(\Gamma_2)\right)\\
&+ \alpha\left[\begin{smallmatrix}\bullet\\2\\2\end{smallmatrix}\right]\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right)^2\\
&+ \alpha\left[\begin{smallmatrix}\bullet\,1\\2\,0\end{smallmatrix}\right]\Phi_R(\Gamma_2)\left(\Phi_R(\Gamma_1)^2 - 2\Phi_R\left(B_+^{\Gamma_1}(\Gamma_1)\right)\right)
\end{aligned}$$

$$+ [\,\bullet\,]_{[a_1,a_2]} \left(\Phi_R \left(B_+^{\Gamma_1}(\Gamma_2) \right) - \Phi_R \left(B_+^{\Gamma_2}(\Gamma_1) \right) \right)$$

$$+ [\,\bullet\,]_{[a_1,\Theta(a_1,a_1)]} \left(2\Phi_R \left(B_+^{\Gamma_1}(\Gamma_1 \cup \Gamma_1) \right) - \Phi_R \left(B_+^{\Gamma_1} \left(B_+^{\Gamma_1}(\Gamma_1) \right) \right) \right.$$

$$\left. \left. - \Phi_R(\Gamma_1)\Phi_R \left(B_+^{\Gamma_1}(\Gamma_1) \right) \right) \right) \Bigg|_{z \to \alpha \Phi_R(\Gamma_1)}. \tag{A.20}$$

For the leading log and next-to-leading log expansions, we use the explicit generating functions in the third column of Table 1. For the next-to-next-to-leading log order, we only discard the zero functions. Thus, we finally obtain

$$G_R(X_{\text{QED}})|_{l.l.} = \alpha \Phi_R(\Gamma_1), \tag{A.21}$$

$$G_R(X_{\text{QED}})|_{n.l.l.} = \alpha^2 \Phi_R(\Gamma_2) \log \left(\frac{1}{1 - \alpha \Phi_R(\Gamma_1)} \right), \tag{A.22}$$

$$G_R(X_{\text{QED}})|_{n.n.l.l.} = \alpha^3 \Bigg([\,\bullet\,0\,1\,]\Phi_R(\Gamma_3) + \alpha[\,\bullet\,2\,]\Phi_R(\Gamma_2)^2$$

$$+ \begin{bmatrix} \bullet & 0 \\ 1 & 1 \end{bmatrix} \left(-\Phi_R \left(B_+^{\Gamma_1}(\Gamma_2) \right) - \Phi_R \left(B_+^{\Gamma_2}(\Gamma_1) \right) + \Phi_R(\Gamma_1)\Phi_R(\Gamma_2) \right)$$

$$+ \alpha \begin{bmatrix} \bullet & 1 \\ 2 & 0 \end{bmatrix} \Phi_R(\Gamma_2) \left(\Phi_R(\Gamma_1)^2 - 2\Phi_R \left(B_+^{\Gamma_1}(\Gamma_1) \right) \right)$$

$$+ [\,\bullet\,]_{[a_1,a_2]} \left(\Phi_R \left(B_+^{\Gamma_1}(\Gamma_2) \right) - \Phi_R \left(B_+^{\Gamma_2}(\Gamma_1) \right) \right) \Bigg) \Bigg|_{z \to \alpha \Phi_R(\Gamma_1)}. \tag{A.23}$$

See the third column of Table 1 for the remaining generating functions.

Appendix B. Some multiplicities of shuffles in filtered words

We list some multiplicities m that are generated by the generating functions obtained so far, see Table 1. We treat the Yukawa fermion propagator and the QED photon self-energy separately. If sequences are known, we say so explicitly and refer to [19].

B.1. Yukawa fermion propagator

The generating functions that are necessary to simplify the log-expansion of the Yukawa fermion propagator up to next-to-next-to-leading log order are collected in the second column of Table 1. Some of the corresponding multiplicities are collected in Table 2.

$[\,\bullet\,]$ generates the exponential sequence of double factorial odd numbers A001147,

$$[\,N\,] = \frac{(2N-3)(2N-5)(2N-7)\dots1}{N!} =: \frac{(2N-3)!!}{N!}. \tag{B.1}$$

Furthermore, $[\,\bullet\,1\,]$ generates the exponential series for the scaled sums of odd reciprocals A004041. The formula is

$$[\,N\,1\,] = \frac{(2N+1)!!}{(N+1)!} \sum_{k=0}^{N} \frac{1}{2k+1}. \tag{B.2}$$

We finally find

$$[\,N\,0\,1\,] = \frac{(2N+2)!}{N!(N+1)!2^{N+1}} \tag{B.3}$$

(exponential sequence A001879). As far as we are concerned, there are no known sequences for the other rationals in Table 2.

Table 2
List of some multiplicities m occurring in the filtration of the Yukawa fermion propagator graphs. These are derived from the generating functions up to next-to-next-to-leading log order in the second column in Table 1.

m	N					
	0	1	2	3	4	5
$[N]$	0	1	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{5}{8}$	$\frac{7}{8}$
$[N\ \ 1]$	1	2	$\frac{23}{6}$	$\frac{22}{3}$	$\frac{563}{40}$	$\frac{1627}{60}$
$\begin{bmatrix}N\\2\end{bmatrix}$	$-\frac{1}{2}$	-1	$-\frac{23}{12}$	$-\frac{11}{3}$	$-\frac{563}{80}$	$-\frac{1627}{120}$
$[N\ \ 0\ \ 1]$	1	3	$\frac{15}{2}$	$\frac{35}{2}$	$\frac{315}{8}$	$\frac{693}{8}$
$[N\ \ 2]$	$\frac{3}{2}$	$\frac{41}{6}$	$\frac{265}{12}$	$\frac{3707}{60}$	$\frac{114961}{720}$	$\frac{219803}{560}$
$\begin{bmatrix}N\\3\end{bmatrix}$	$\frac{1}{2}$	$\frac{11}{6}$	$\frac{61}{12}$	$\frac{253}{20}$	$\frac{7141}{240}$	$\frac{113623}{1680}$
$\begin{bmatrix}N&0\\1&1\end{bmatrix}$	-2	$-\frac{20}{3}$	$-\frac{53}{3}$	$-\frac{214}{5}$	$-\frac{5933}{60}$	$-\frac{46597}{210}$
$\begin{bmatrix}N\\2\\2\end{bmatrix}$	$\frac{1}{24}$	$\frac{3}{8}$	$\frac{389}{240}$	$\frac{1291}{240}$	$\frac{314431}{20160}$	$\frac{93403}{2240}$
$\begin{bmatrix}N&1\\2&0\end{bmatrix}$	$-\frac{5}{6}$	$-\frac{25}{6}$	$-\frac{857}{60}$	$-\frac{833}{20}$	$-\frac{559579}{5040}$	$-\frac{156603}{560}$
$[N]_{[a_1,a_2]}$	-1	-6	$-\frac{71}{3}$	$-\frac{155}{2}$	$-\frac{9129}{40}$	$-\frac{18823}{30}$
$[N]_{[a_1,\Theta(a_1,a_1)]}$	$\frac{1}{2}$	3	$\frac{71}{6}$	$\frac{155}{4}$	$\frac{9129}{80}$	$\frac{18823}{60}$

Table 3
List of some multiplicities m occurring in the filtration of the QED photon self-energy graphs. These are derived from the generating functions up to next-to-next-to-leading log order in the third column in Table 1.

m	N					
	0	1	2	3	4	5
$[N]$	1	0	0	0	0	0
$[N\ \ 1]$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{5}$	$\frac{1}{6}$
$\begin{bmatrix}N\\2\end{bmatrix}$	0	0	0	0	0	0
$[N\ \ 0\ \ 1]$	1	1	1	1	1	1
$[N\ \ 2]$	$\frac{1}{2}$	$\frac{5}{6}$	$\frac{13}{12}$	$\frac{77}{60}$	$\frac{29}{20}$	$\frac{223}{140}$
$\begin{bmatrix}N\\3\end{bmatrix}$	0	0	0	0	0	0
$\begin{bmatrix}N&0\\1&1\end{bmatrix}$	$-\frac{1}{2}$	$-\frac{2}{3}$	$-\frac{3}{4}$	$-\frac{4}{5}$	$-\frac{5}{6}$	$-\frac{6}{7}$
$\begin{bmatrix}N\\2\\2\end{bmatrix}$	0	0	0	0	0	0
$\begin{bmatrix}N&1\\2&0\end{bmatrix}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{3}{10}$	$\frac{1}{3}$	$\frac{5}{14}$	$\frac{3}{8}$

Table 3 (*continued*)

m	N					
	0	1	2	3	4	5
$[N]_{[a_1,a_2]}$	$-\frac{1}{2}$	-1	$-\frac{3}{2}$	-2	$-\frac{5}{2}$	-3
$[N]_{[a_1,\Theta(a_1,a_1)]}$	0	0	0	0	0	0

B.2. QED photon self-energy

The generating functions that are necessary to simplify the log-expansion of the photon self-energy Green function up to next-to-next-to-leading log order are given in the third column of Table 1. Some of the corresponding rationals are listed in Table 3. The reader immediately checks that these numbers look much simpler than in the Yukawa case. Indeed, four rows only contain zero numbers (the first line is almost zero, $[N] = \delta_{N1}$).

We find the trivial sequences

$$[N\ 0\ 1] = 1, \qquad \begin{bmatrix} N\ 0 \\ 1\ 1 \end{bmatrix} = -\frac{N+1}{N+2},$$
$$\begin{bmatrix} N\ 1 \\ 2\ 0 \end{bmatrix} = \frac{N+1}{2(N+3)}, \qquad [N]_{[a_1,a_2]} = -\frac{N+1}{2}. \tag{B.4}$$

One also finds that $[N\ 2]$ is the exponential series of the generalized Stirling numbers A001705,

$$[N\ 2] = \frac{1}{N+2} \sum_{k=0}^{N} \frac{k+1}{N+1-k}. \tag{B.5}$$

References

[1] D. Kreimer, Quantum fields, periods and algebraic geometry, arXiv:1405.4964 [hep-th], in: ICMAT Proceedings Periods and Motives, Madrid, July 2012, Luis Àlvarez-Cónsul et al., eds., AMS Contemporary Mathematics, in press.
[2] F. Brown, D. Kreimer, Lett. Math. Phys. 103 (2013) 933. arXiv:1112.1180 [hep-th].
[3] A. Connes, D. Kreimer, Comm. Math. Phys. 199 (1998) 203. [hep-th/9808042].
[4] A. Connes, D Kreimer, Commun. Math. Phys. 216 (2001) 215–241. e-Print: hep-th/0003188.
[5] D. Kreimer, K Yeats, Nucl. Phys. Proc. Suppl. 160 (2006) 116–121. e-Print: hep-th/0605096.
[6] C. Bergbauer, D. Kreimer, IRMA Lect. Math. Theor. Phys. 10 (2006) 133. [hep-th/0506190].
[7] M. Borinsky, Comput. Phys. Commun. 185 (2014) 3317. arXiv:1402.2613 [hep-th].
[8] L. Foissy, Adv. Math 226 (6) (2011) 4702–4730.
[9] L. Foissy, J. Unterberger, Int. Math. Res. Notices 2013 (2013) 846–885. arXiv:1004.5208.
[10] D. Broadhurst, D. Kreimer, Nuclear Phys. B 600 (2001) 403. [hep-th/0012146].
[11] S. Bloch, D. Kreimer, Commun. Num. Theor. Phys. 2 (2008) 637. arXiv:0804.4399 [hep-th].
[12] T.K. Chen, Bull. Amer. Math. Soc. 83 (1977) 831.
[13] D. Kreimer, Ann. Physics 321 (2006) 2757. [hep-th/0509135].
[14] D. Kreimer, M. Sars, W. van Suijlekom, Ann. Physics 336 (2013) 180. arXiv:1208.6477 [hep-th].
[15] C. Reutenauer, Free Lie Algebras, LMS, 1993.
[16] A. Connes, D. Kreimer, Comm. Math. Phys. 216 (2001) 215. [hep-th/0003188].
[17] P. Hall, The algebra of partitions, in: Proceedings of the 4th Canadian Mathematical Congress, Banff, pp. 147–159.
[18] W. Milnor, J.C. Moore, Ann. of Math. (2) 81 (2) (1965) 211–264.
[19] OEIS Foundation Inc., The On-Line Encyclopedia of Integer Sequences, 2011, http://oeis.org.