IPP-Report

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Interaction between an electron and an electromagnetic wave

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1. Introduction

This report is intended as an addition to chapter 9.6 in the book of M. Kaufmann, "Plasmaphysik und Fusionsforschung", 2nd edition 2013, Springer Spektrum Verlag.

While the propagation of waves in plasmas can often be described in the fluid picture, this is not the case for the absorption of waves, which may lead to heating or current drive. For this a kinetic treatment is necessary describing not only the interaction between the particles and the wave fields, but also between the particles themselves, resulting in a change of their distribution function, [1,2].

Here in this report we consider only a single electron interacting with an electromagnetic wave in the presence of a dc-magnetic field. It is also valid for single ions with corresponding corrections for the sign of the charge, the mass and the direction of cyclotron motion. We use arbitrary parameters for the electromagnetic field, i.e. they do not satisfy any dispersion relation. However, they are not totally unrealistic: in a plasma waves with an elliptical polarisation in the direction of propagation can exist, or electrostatic waves with an electric field directed along the k-vector. And in a gyrotron cavity relativistic electrons interact with a rotating electric field through the cyclotron resonance. The intention of this report is only to indicate what kind of response can be expected, with different basic assumptions, when an electron is perpendicularly accelerated by an rf-field.

In section 2 we discuss the equation of motion as appropriate to our problem. In section 3 we assume that the electric field is small and that changes in the electron velocity and the electron trajectory are negligible small. From this we conclude that perpendicular absorption can occur at the cyclotron frequency and its harmonics. In section 4 we treat the case of a field with infinite wavelength, $k = 0$, for which there is an analytical solution of the equation of motion. With $k \neq 0$ and long interaction time, such that changes in the electron velocity and orbit are no longer negligible, we rely on numerical calculations, which are presented in section 5. Finally, in section 6 we include in the calculations a relativistic dependence of the electron mass on the energy.

Definitions and initial conditions
We introduce the following quantities
- the electron charge and mass $-e$ and m_e
- the stationary magnetic field in z-direction $\vec{B}_0 = B_0 \cdot \{0;0;1\} = B_0 \cdot \hat{b}$
- the electron cyclotron frequency $\omega_c = eB_0 / m_e$
- the radius of gyration $\rho(t) = v_\perp(t)/\omega_c$ (1.1)
- the phase of the gyration $\varphi(t) = \omega_c t + \varphi_0$ (1.2)
- the wave vector $\vec{k} = \{k_x;0;k_z\} = \{\vec{k}_\perp;\vec{k}_\parallel\}$
- the electric field of the wave

$$\vec{E}(\vec{r},t) = \left\{ E_{0x}\cos(\omega t - \vec{k}\vec{r} - \psi_x); E_{0y}\cos(\omega t - \vec{k}\vec{r} - \psi_y); E_{0z}\cos(\omega t - \vec{k}\vec{r} - \psi_z) \right\} = \left\{ \vec{E}_\perp ; \vec{E}_\parallel \right\}$$
     (1.3)

- the electron velocity $\vec{v}(t) = \{v_x(t);v_y(t);v_z(t)\} = \{v_\perp(t)\cos\varphi(t);v_\perp(t)\sin\varphi(t);v_\parallel(t)\} = \{\vec{v}_\perp;\vec{v}_\parallel\}$

     with initial values $\vec{v}(t=0) = \{v_{0\perp}\cos\varphi_0; v_{0\perp}\sin\varphi_0; v_{0\parallel}\}$ (1.4)
- the electron trajectory $\vec{r}(t) = \{\rho(t)\sin\varphi(t); -\rho(t)\cos\varphi(t); v_\parallel(t)t\}$

     with initial values $\vec{r}(t=0) = \{\rho_0\sin\varphi_0; -\rho_0\cos\varphi_0; 0\}$.   (1.5)

2. The equation of motion and the effect of the wave magnetic field

The equation of motion for the electron is

$$\dot{\vec{v}} = -\frac{e}{m_e}\left[\vec{E} + \vec{v}\times\vec{B} + \vec{v}\times\vec{B}_0\right] \qquad (2.1)$$

where $\vec{E}$ and $\vec{B}$ are the wave electric and magnetic fields.

From Maxwell's equations we can express the wave magnetic field as $\vec{B} = \vec{k}\times\vec{E}/\omega$. The acceleration then reads

$$\dot{\vec{v}} = -\frac{e}{m_e}\left[\vec{E} + \vec{v}\times(\vec{k}\times\vec{E})/\omega + \vec{v}\times\vec{B}_0\right] . \qquad (2.2)$$

Multiplying this equation with $\vec{v}_\parallel$ we obtain the energy gain in parallel direction

$$P_\parallel = m_e\dot{\vec{v}}_\parallel\vec{v}_\parallel = -e\vec{v}_\parallel\vec{E}_\parallel - e\left(\vec{k}_\parallel\vec{v}_\parallel\cdot\vec{v}_\perp\vec{E}_\perp - \vec{k}_\perp\vec{v}_\perp\cdot\vec{v}_\parallel\vec{E}_\parallel\right)/\omega . \qquad (2.3)$$

Multiplying the acceleration with $\vec{v}_\perp$ we obtain the perpendicular energy gain

$$P_\perp = m_e\dot{\vec{v}}_\perp\vec{v}_\perp = -e\vec{v}_\perp\vec{E}_\perp + e\left(\vec{k}_\parallel\vec{v}_\parallel\cdot\vec{v}_\perp\vec{E}_\perp - \vec{k}_\perp\vec{v}_\perp\cdot\vec{v}_\parallel\vec{E}_\parallel\right)/\omega . \qquad (2.4)$$

The first terms in both equations describe the energy gain through the wave electric field, while the terms in brackets describe the energy gain through the wave magnetic field. The latter are only nonzero if $v_\perp \neq 0$; $v_\parallel \neq 0$; $k \neq 0$; They are equal in magnitude, but of opposite sign in perpendicular and parallel direction. Hence the total energy gain is unaffected by the wave magnetic field. Its effect is to redistribute part of the parallel energy gain into perpendicular energy gain and vice versa. In cases where the phase velocity of the wave $\omega/k \approx c_0$ is comparable to the velocity of light and when $v \ll c_0$ then the effect of the wave magnetic field is small compared to that of the wave electric field.

In the following we therefore neglect the wave magnetic field and consider only the remaining equation of motion

$$\dot{\vec{v}} = -\frac{e}{m_e}(\vec{E} + \vec{v}\times\vec{B}_0) = -\frac{e}{m_e}\vec{E} - \omega_c(\vec{v}\times\hat{b}) . \qquad (2.5)$$

The electric field along the electron trajectory is

$$E_{x,y,z}(t) = E_{0x,y,z}\cos[\omega t - k_\perp\rho(t)\sin\varphi(t) - k_\parallel v_\parallel(t)t - \psi_{x,y,z}] . \qquad (2.6)$$

Using the identity

$$\cos(\alpha \pm \xi\sin\varphi) = \sum_{m=-\infty}^{+\infty}J_m(\xi)\cos(\alpha \pm m\varphi)$$

we get

$$E_{x,y,z}(t) = E_{0x,y,z}\sum_{m=-\infty}^{+\infty}J_m(k_\perp\rho(t))\cos[\omega t - m\varphi(t) - k_\parallel v_\parallel(t)t - \psi_{x,y,z}]. \qquad (2.7)$$

Inserted into the equation of motion we obtain

$$\dot{v}_x(t) = -\frac{e}{m_e}E_{0x}\sum_{m=-\infty}^{+\infty}J_m(k_\perp\rho(t))\cos\left[\omega t - m\varphi(t) - k_\parallel v_\parallel(t)t - \psi_x\right] - \omega_c v_\perp\sin\varphi(t)$$

$$\dot{v}_y(t) = -\frac{e}{m_e}E_{0y}\sum_{m=-\infty}^{+\infty}J_m(k_\perp\rho(t))\cos\left[\omega t - m\varphi(t) - k_\parallel v_\parallel(t)t - \psi_y\right] + \omega_c v_\perp\cos\varphi(t). \qquad (2.8)$$

$$\dot{v}_z(t) = -\frac{e}{m_e}E_{0z}\sum_{m=-\infty}^{+\infty}J_m(k_\perp\rho(t))\cos\left[\omega t - m\varphi(t) - k_\parallel v_\parallel(t)t - \psi_z\right]$$

This equation is still generally valid as long as $\rho(t)$ and $v(t)$ are functions of time..

3. Small electric field and unperturbed orbit

In the following we assume that the electric field is small enough so that the time dependence of ρ and v can be neglected for some gyro cycles and that we can set $\rho(t) = \rho_0$ and $v_{x,y,z}(t) = v_{0x,y,z}$, [3, 4]. Because the z-component does not contain a Lorentz term we have to discuss perpendicular and parallel acceleration separately.

3.1 acceleration perpendicular to $\vec{B}_0$

Persistent perpendicular acceleration requires that all terms have the same time dependence $\omega_c t$ as given by the Lorentz force term, hence $(\omega - m\omega_c - k_\parallel v_{\parallel 0})t = \omega_c t$, leading to the familiar wave particle resonance condition

$$\omega - (m+1)\omega_c - k_\parallel v_\parallel = 0 \ . \tag{3.1}$$

In this case the electron moving along the orbit experiences an electric field oscillating at a frequency ω_c rather than ω. Such a resonant interaction can take place at all cyclotron harmonics with a Doppler shift according to the parallel velocity.
The perpendicular power absorption is obtained from

$$P_\perp = m_e \vec{\dot{v}}_\perp \vec{v}_\perp = m_e (\dot{v}_x v_x + \dot{v}_y v_y) \tag{3.2}$$

Inserting the above results and averaging over an integer number of gyro cycles we obtain the time averaged perpendicular power absorption

$$\langle P_\perp \rangle = -\frac{e}{m_e} v_{0\perp} J_m(k_\perp \rho_0) \cdot \{ E_{0x} \cos[(m+1)\varphi_0 + \psi_x] + E_{0y} \cos[(m+1)\varphi_0 + \psi_y] \} \ . \tag{3.3}$$

From this we see that only the electric field component parallel to the velocity vector contributes to the energy gain.
For an elliptic polarisation we take $\psi_x = 0$ and $\psi_y = \pm\pi/2$ and we obtain

$$\langle P_\perp \rangle_{ell.pol.} = -\frac{e}{m_e} v_{0\perp} J_m(k_\perp \rho_0) \cdot (E_{0x} \pm E_{0y}) \cos(m+1)\varphi_0 \ . \tag{3.4}$$

The perpendicular absorption is maximum for a co-rotating circularly polarised electric field with $E_{0x} = E_{0y}$ and $\psi_y = +\pi/2$, and is zero for a counter-rotating field with $E_{0x} = E_{0y}$ and $\psi_y = -\pi/2$.
The power absorption scales with $J_m(k_\perp \rho_0)$. For very small argument $k_\perp \rho_0 \approx 0$ only the Bessel function $J_{m=0}$ is nonzero, corresponding to interaction at $\omega \approx \omega_c$. Interaction at harmonics $n\omega_c$ requires $k_\perp \rho_0 \neq 0$.

While the term m in the sum has a time dependence $\omega_c t$ and leads to resonant acceleration, the term $(m+1)$ has no time dependence. In this case the electron moving along the orbit experiences a dc-electric field which leads to a particle drift at a velocity

$$\vec{v}_{drift} = \left(\vec{E}_{0\perp} J_{m+1}(k_\perp \rho_0) e^{i(m+1)\varphi_0} \times \vec{B}_0 \right) / B_0^2 \tag{3.5}$$

in $\vec{E}_{\perp 0} \times \vec{B}_0$ direction. There is no power absorption associated with the drift motion. Heating and drift occur simultaneously. For example: with $m = 0$ we have cyclotron absorption near the fundamental frequency $\omega \approx \omega_c$ with a scaling factor $J_0(k_\perp \rho_0)$ while the term with $m = 1$ produces a drift with a scaling factor of $J_1(k_\perp \rho_0)$. So the drift requires $k_\perp \rho_0 \neq 0$. The drift is usually small and can be neglected.

3.2 acceleration parallel to $\vec{B}_0$

Since there is no Lorentz force term in the equation for $\dot{v}_z$ we get persistent acceleration if there are no oscillating terms in the equation. Only the term of the sum contributes for which

$$(\omega - m\omega_c - k_{\parallel}v_{0\parallel}) = 0 \ . \tag{3.6}$$

With this condition satisfied the parallel power absorption is

$$P_{\parallel} = m_e \dot{v}_z v_z = -ev_{0\parallel}E_{0z}J_m(k_{\perp}\rho_0)\cos(m\varphi_0 + \psi_z) \ . \tag{3.7}$$

In particular, for $k_{\perp}\rho_0 = 0$ only the term with $m = 0$ leads to acceleration since $J_0(0) = 1$ while all other terms $J_{m\neq 0}(0) = 0$. This concerns particles satisfying $\omega - k_{\parallel}v_{0\parallel} = 0$ and corresponds standard Landau damping of plasma waves.

This discussion with the assumption of a small amplitude is meaningful only in the initial phase of the electron wave interaction as long as the electron orbit does not change much. For this time period we can conclude
- energy exchange is only possible when the wave particle resonance condition is satisfied
- energy exchange requires an electric field component in the direction of the velocity vector
- energy exchange at harmonics of the cyclotron frequency is possible only when $k_{\perp}\rho_0 \neq 0$.
- the interaction leads to a small drift motion.

4. Interaction with a perpendicular homogenous circularly polarised electric field with $k = 0$

For the case $k = 0$ there is an analytical solution to the equation of motion and we do not need the assumption of an unperturbed orbit, [5]. Here we present the solution for an elliptically polarised field with components E_{0x} and E_{0y}, and we take $\psi_x = 0$ and $\psi_y = \pi/2$ for a co-rotating field. This solution describes also a linearly polarised field if either E_{x0} or E_{y0} are set to zero, and also a counter-rotating field if E_{y0} is negative.

$$v_x = v_{0\perp}\cos(\omega_c t + \varphi_0) - \frac{e}{m_e}\frac{(\omega E_{0x} + \omega_c E_{0y})\sin\omega t - (\omega_c E_{0x} + \omega E_{0y})\sin\omega_c t}{\omega^2 - \omega_c^2} \quad \text{and}$$

$$v_y = v_{0\perp}\sin(\omega_c t + \varphi_0) - \frac{e}{m_e}\frac{(\omega_c E_{0x} + \omega E_{0y})\cos\omega t - (\omega_c E_{0x} + \omega E_{0y})\cos\omega_c t}{\omega^2 - \omega_c^2} \ . \tag{4.1}$$

4.1 general case $\omega \neq \omega_c$, circularly polarised field co-rotating
In the following we consider a circularly co-rotating polarised field with $E_{0x} = E_{0y} = E_0$. Then equ. 4.1 can be reformulated as

$$v_x = v_{0\perp}\cos(\omega_c t + \varphi_0) - \frac{e}{m_e}E_0\frac{2}{\omega - \omega_c}\cos\frac{(\omega + \omega_c)t}{2}\sin\frac{(\omega - \omega_c)t}{2} \quad \text{and}$$

$$v_y = v_{0\perp}\sin(\omega_c t + \varphi_0) - \frac{e}{m_e}E_0\frac{2}{\omega - \omega_c}\sin\frac{(\omega + \omega_c)t}{2}\sin\frac{(\omega - \omega_c)t}{2} \ . \tag{4.2}$$

The first terms describe the unperturbed gyro motion of the electron with frequency ω_c. The second terms describe the change due to the electric field with a frequency $\omega \neq \omega_c$. They represent an oscillation at a high frequency $(\omega + \omega_c)/2$ modulated with a low beat frequency $(\omega - \omega_c)/2$. An example is shown in fig. 4.1. Calculated orbits are shown in fig. 4.2.
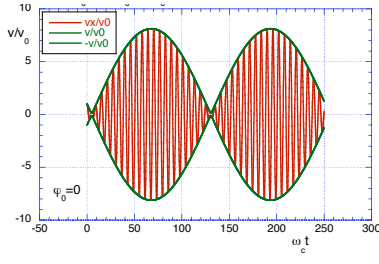
Fig. 4.1 v_x/v_0 and $|v/v_0|$ as function of time

Parameters: $\omega = 1.05\,\omega_c$; $\vec{E} = [0.1;0.1;0]$; $\vec{k} = [0;0;0]$; $\vec{v}_0 = [1;0;0]$; $\varphi_0 = \pi$;

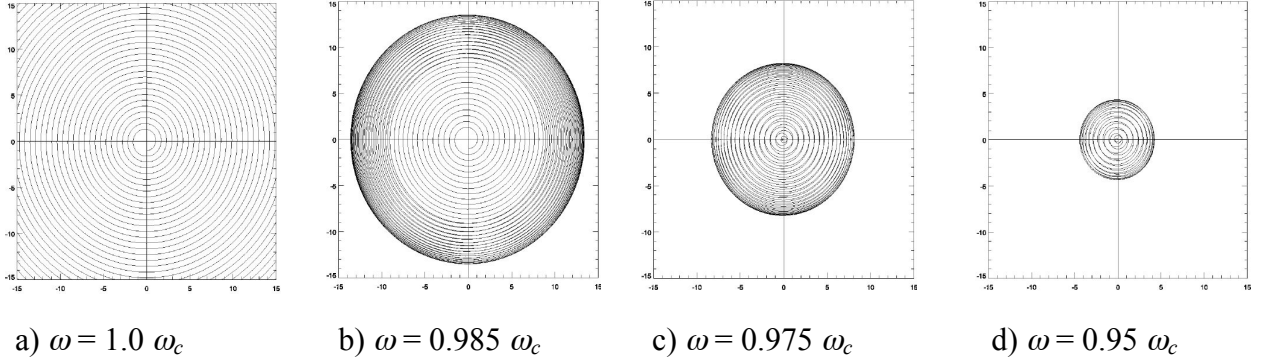| a) $\omega = 1.0\,\omega_c$ | b) $\omega = 0.985\,\omega_c$ | c) $\omega = 0.975\,\omega_c$ | d) $\omega = 0.95\,\omega_c$ |

Fig. 4.2 Calculated orbits $\vec{r}(t)$, 50 cycles, variation of frequency

Parameters: $\omega = \text{var.}$; $\vec{E} = [0.1;0.1;0]$; $\vec{k} = [0;0;0]$; $\vec{v}_0 = [1;0;0]$; $\varphi_0 = \pi$;

A frequency matching exactly the gyro frequency leads to continuous heating. However, when the frequency deviates from the gyro frequency the phase between the gyromotion and the electric field changes continuously and therefore the electron experiences a periodic heating and cooling with the beat frequency $\omega - \omega_c$.

The energy of the electron is

$$\frac{v^2}{v_{0\perp}^2} = \frac{v_x^2 + v_y^2}{v_{0\perp}^2} = 1 + A^2 \sin^2 \frac{(\omega - \omega_c)t}{2} - 2A\sin\frac{(\omega - \omega_c)t}{2}\cos[\frac{(\omega - \omega_c)t}{2} - \varphi_0] \qquad (4.3)$$

with $A = \dfrac{e}{m}\dfrac{E_0}{v_{0\perp}}\dfrac{2}{\omega - \omega_c}$.

Fig. 4.3 shows v^2/v_0^2 as function of time at different phases φ_0. Although the electron energy increases and decreases periodically, there remains a limited time averaged net energy gain which depends on the initial phase. This energy gain is higher the closer the frequency is to the exact cyclotron frequency as shown in Fig. 4.4. Finally, at $\omega/\omega_c = 1$ the periodic character is lost and the energy grows about quadratic with time, due to the $\sin(\omega-\omega_c)t\,/(\omega-\omega_c)$ term.
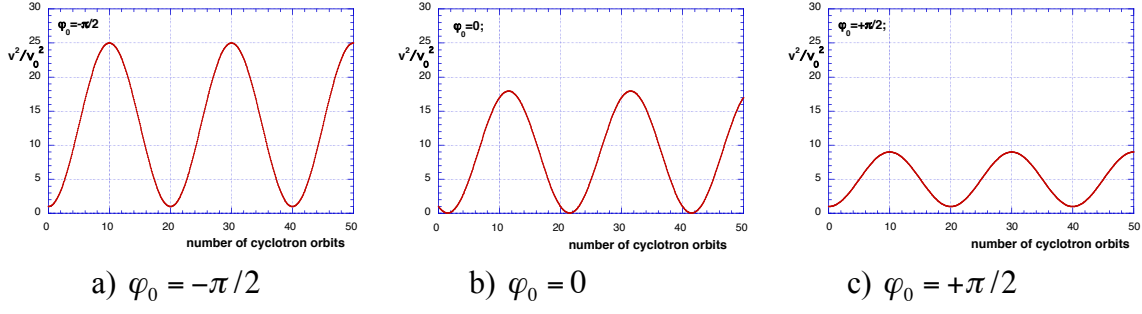
a) $\varphi_0 = -\pi/2$ b) $\varphi_0 = 0$ c) $\varphi_0 = +\pi/2$

Fig. 4.3 $v_\perp^2/v_0^2$ as function of time at different phases.
   Parameters: $\omega = 1.05\omega_c$; $\vec{E} = [0.1;0.1;0]$; $\vec{k} = [0;0;0]$; $\vec{v}_0 = [1;0;0]$; $\varphi_0 = $ var;
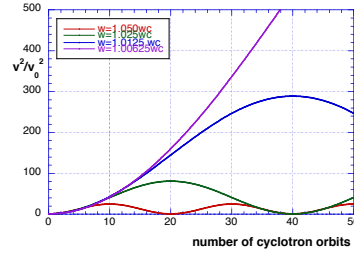

Fig. 4.4 Variation of $v_\perp^2/v_0^2$ with frequency, $\omega/\omega c=1.05$; 1.025, 1.0125; 1.00625;
   Parameters: $\omega = $ var; $\vec{E} = [0.1;0.1;0]$; $\vec{k} = [0;0;0]$; $\vec{v}_0 = [1;0;0]$; $\varphi_0 = -\pi/2$;

Time averaged energy gain is in this model thus not only possible when the wave particle resonance condition $\omega = \omega_c$ is fulfilled exactly, but also close to this frequency.

We can perform similar calculations for a frequency slightly different from the second harmonic, $\omega \approx 2\omega_c$. As function of frequency there is only a much weaker dependence of v^2/v_0^2 since near the second harmonic the term $1/(\omega-\omega_c)$ does not go to infinity. The results of a phase variation are shown in fig. 4.5. In case a) the amplitude is periodically varying around v_0. In cases b) and c) the amplitude is also periodically varying, but remains lower resp. higher than v_0, i.e. the electron looses resp. gains time averaged energy. Still there remains a phase and time averaged energy gain.
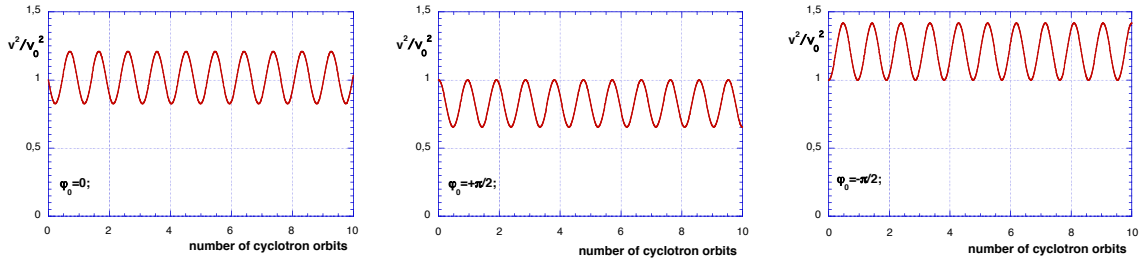

Fig.4.5 v^2 as function of time; a) $\varphi_0 = 0$, b) $\varphi_0 = +\pi/2$, c) $\varphi_0 = -\pi/2$.
   Constant: $\omega = 2.05 \omega_c$; $k = 0$; $E = [0.1; 0.1; 0]$; $v_0 = [1; 0; 0]$; Circ. pol. field co-rotating.

4.2 special case $\omega = \omega_c$, co-rotating circularly polarised electric field
Now we analyse the special case of $\omega = \omega_c$. For the velocity we get

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$$v_x = v_{0\perp}\cos(\omega_c t + \varphi_0) - \frac{e}{m_e} E_0 \cdot t \cdot \cos\omega_c t \quad \text{and}$$

$$v_y = v_{0\perp}\sin(\omega_c t + \varphi_0) - \frac{e}{m_e} E_0 \cdot t \cdot \sin\omega_c t \; . \tag{4.4}$$

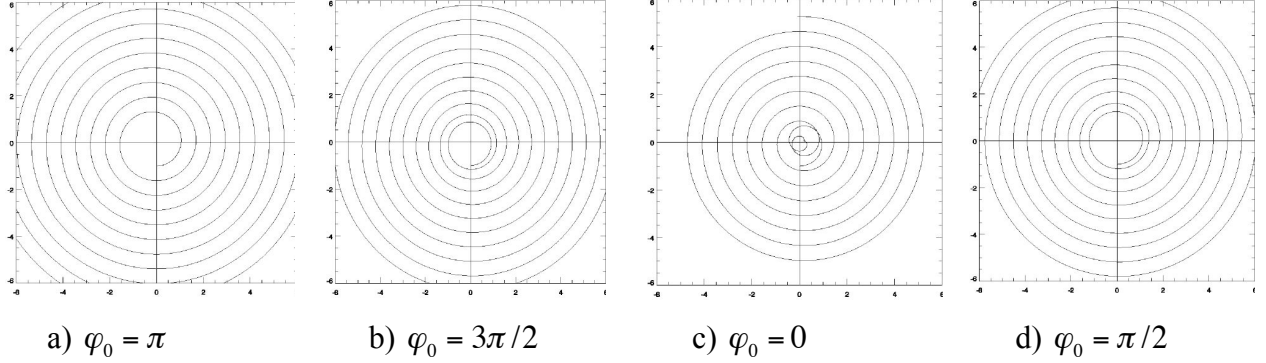Calculated orbits are shown in fig.4.6 for different initial phases.

a) $\varphi_0 = \pi$ b) $\varphi_0 = 3\pi/2$ c) $\varphi_0 = 0$ d) $\varphi_0 = \pi/2$

Fig. 4.6 Calculated orbits for different initial phases
Parameters: $\omega = \omega_c; \vec{E}_0 = [0.1;+0.1;0]; \psi_y = +\pi/2; \vec{k} = [0;0;0]; \vec{v}_0 = [1;0;0]; \varphi_0 = \text{var}$.

In fig. 4.6a the electric field vector is initially parallel to the velocity vector, but of opposite sign and therefore the electron gains immediately energy. In fig. 4.6c the electric field is initially in the same direction as the velocity, and therefore the electron first looses energy until it comes to rest, and then gains energy like in case a.
In figs. 4.6b and 4.6d the electric field vector is initially perpendicular to the electron velocity, and therefore there is no immediate energy exchange. Nevertheless, in both cases the electron finally gains energy at about the same rate as in case a.

The total velocity is

$$\left| v(t) \right| = v_{0\perp}\sqrt{1 + (t/\tau_{ph})^2 - 2(t/\tau_{ph})\cos\varphi_0} \tag{4.5}$$

where we have introduced a characteristic time

$$\tau_{ph} = \frac{m_e v_{0\perp}}{eE_0} \; . \tag{4.6}$$
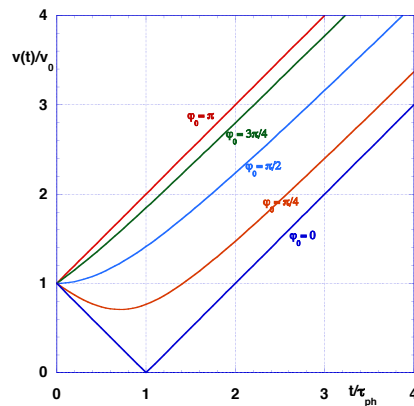
This is plotted in fig. 4.7.

Fig.4.7 Total velocity as a function of time for different initial phases, [6]

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Depending on the phase between electric field and gyro motion the electron can initially be accelerated ($\varphi_0 = \pi$) or decelerated ($\varphi_0 = 0$). For $\varphi_0 = \pi/2$ there is no initial acceleration. However, after about one characteristic time τ_{ph} all electrons are accelerated, and after two characteristic times all electrons have a velocity greater than the initial one increasing further at the same rate independent of the initial phase. Obviously the phases must have been shifted and synchronised such that $\varphi = \pi$ is reached. In the cases $\varphi_0 = \pi/2$ and $\varphi_0 = 3\pi/2$ the electric field, being initially perpendicular to the velocity vector, drives the electron to a smaller, respectively larger, radius, while the absolute value of the velocity does not change. Hence the angular frequency of the electron gyration is no longer equal to the cyclotron frequency with which the electric field is rotating, leading to a phase shift until finally the electric field is opposite to the velocity vector and thus accelerating.

Such phase synchronisation can occur only as long as the electron motion is correlated with the perpendicular electric field. There are a number of ways how this correlation can be destroyed:

- de-correlation due to finite transit time through an electric field of finite spatial extent
    In practice the electric field is limited in space, like in a gyrotron cavity of finite length, or in plasma heating experiments, where the wave electric field is given by a microwave beam. Here we assume a homogenous microwave beam with a diameter d_b and power P propagating perpendicular to the stationary magnetic field. We then get for the electric field

$$E_0 = 1/d_b \cdot \sqrt{8 P Z_F / \pi} \qquad (4.7)$$

with the wave impedance $Z_F = \sqrt{\mu_0 / \varepsilon_0}$. Assuming further equal parallel and perpendicular initial velocities $v_{0\perp} = v_{0\parallel} = v_0$ we get for the transit time through the beam

$$\tau_{tr} = d_b / v_0 = d_b / \sqrt{2eU / m_e} \qquad (4.8)$$

where the electron velocity is expressed in terms of its energy $eU = 1/2 \cdot m_e v_0^2$. Inserting this into τ_{ph} we get

$$\frac{\tau_{ph}}{\tau_{tr}} = \frac{U}{\sqrt{2 P Z_F / \pi}} \qquad (4.9)$$

which depends only on the initial electron energy and the power in the microwave beam. From fig. 4.7 we see that it takes two characteristic times τ_{ph} until also the electrons with the most unfavourable initial phase reach a velocity above their initial velocity.
Taking as example $B_0 = 2.5$ T, $f_{ce} = 70$ GHz and a microwave beam focused $d_b = 5$ cm we get the result shown in fig. 4.8. Phase synchronisation at $\omega = \omega_c$ may be important as long as $\tau_{ph}/\tau_{tr} < 1$, i.e. at electron energies below a few keV, depending on power, as is shown in fig. 4.8.
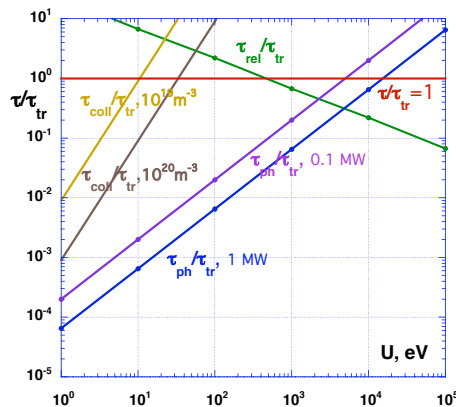

Fig. 4.8 τ_{ph}/τ_{tr} , τ_{rel}/τ_{tr} , and τ_{coll}/τ_{tr} as a function of the electron energy, assuming $v_{\parallel} = v_0$.

- de-correlation by collisions

In a plasma there is another limitation due to collisions. Electrons must not collide during a transit through the beam. With the collision time $\tau_{coll} = 10.4 m_e^{1/2} U^{3/2} \varepsilon_0^2 /(e^{5/2} \ln \Lambda n_e)$, n_e is the plasma density, we get the ratio

$$\frac{\tau_{coll}}{\tau_{tr}} = 10.4 \frac{\varepsilon_0^2 U^2}{e^2 \ln \Lambda n_e d_b} \ . \tag{4.10}$$

This is also plotted in fig. 4.8 assuming a 5 cm diameter beam, a density of $n_e = 10^{20}$ m^{-3}, and lnΛ = 10 for a low temperature plasma. For a possible phase de-correlation τ_{coll} must be smaller than τ_{tr} which in this example is the case for electron energies lower than $\approx$ 50 eV.

- de-correlation by the relativistic velocity dependence of mass

The phase relation between the constant frequency electric field oscillating at the non-relativistic cyclotron frequency $\omega_{c0} = eB_0/m_{e0}$ and the gyro motion of the electron varies due to the relativistic change of the cyclotron frequency with electron energy (see section 6).
The relativistic cyclotron frequency is

$$\omega_c = \omega_{c0} \sqrt{1 - v^2/c^2} \approx \omega_{c0}(1 - v^2/2c^2) \tag{4.11}$$

Assuming a constant velocity v the variation of this phase difference is

$$\Delta(d\phi/dt) = \omega_{c0} - \omega_c = 1/2 \cdot \omega_{c0} \cdot v^2/c^2 \tag{4.12}$$

To arrive at a phase difference of π/2, which means that the angle between the electric field vector and the electron velocity vector changes by π/2, this needs a time

$$\tau_{rel} = 1/(2 \cdot v^2/c^2 \cdot f_{c0}) \tag{4.13}$$

This relates to the transit time through a microwave beam as

$$\tau_{rel}/\tau_{tr} = v_{\parallel}/(2 \cdot v^2/c^2 \cdot f_{c0} \cdot d_b) . \tag{4.14}$$

Note that the time τ_{rel} is only an estimate based on a constant velocity. This condition is also indicated in fig. 4.8.
Relativistic de-phasing during one transit can occur if $\tau_{rel}/\tau_{tr} < 1$. In our example for a 5 cm diameter beam this occurs for electron energies higher than $\approx$ 400 eV.
Thus in our example of fig. 4.8, classical phase synchronisation during one transit may occur for electron energies up to $\approx$ 5 keV. Taking relativistic effects into account phase synchronisation is limited to energies up to $\approx$ 400 eV. Collisions require an energy larger than $\approx$ 50 eV. So there is only a small gap in electron energy where phase synchronisation may be possible. The data shown in the figure are, however, for equal parallel and perpendicular velocities. Higher or lower parallel velocity leads to a shorter or longer transit time. Furthermore τ_{ph} was estimated assuming k = 0.

- there are further ways how the phase relation between electron gyration and electric field can be changed: an inhomogeneous magnetic field (like in toroidal plasma confinement), a dc or low frequency electric field, or scattering of the incident microwave beam due to density fluctuations in a plasma.

4.3 special case $\omega = \omega_c$, circularly polarised field counter-rotating
We obtain a counter-rotating field by setting $\psi_y = -\pi/2$ and $E_{0y} = -E_{0x} = -E_0$. The solution is

$$v_x = v_{0\perp} \cos(\omega_c t + \varphi_0) \qquad \text{and}$$
$$v_y = v_{0\perp} \sin(\omega_c t + \varphi_0) - \frac{e}{m_e} E_0 \frac{1}{\omega_c} \cos\omega_c t \ . \tag{4.15}$$

The velocity is purely periodic and does not grow in time. Hence there is no time averaged absorption. This is true for all initial phases φ_0 . The orbits are closed orbits consisting of the unperturbed orbit with a superimposed perturbation due to the electric field, as shown in fig. 4.9.

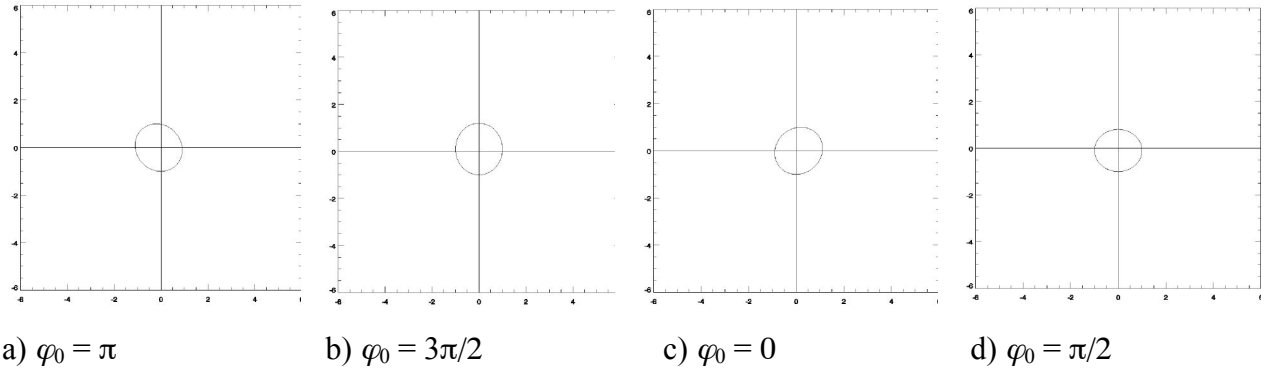| a) $\varphi_0 = \pi$ | b) $\varphi_0 = 3\pi/2$ | c) $\varphi_0 = 0$ | d) $\varphi_0 = \pi/2$ |

Fig. 4.9 Calculated orbits in the x-y-plane for circularly polarised field rotating opposite to the electron gyration with different initial phases

Parameters: $\omega = \omega_c; \vec{E}_0 = [0.1; -0.1; 0]; \psi_y = -\pi/2; \vec{k} = [0;0;0]; \vec{v}_0 = [1;0;0]; \varphi_0 = var$.

4.4 special case $\omega = n\omega_c$, circularly polarised field co-rotating

For harmonic frequencies we have

$$v_x = v_{0\perp} \cos(\omega_c t + \varphi_0) - \frac{e}{m_e} E_0 \frac{2}{(n-1)\omega_c} \cos\frac{(n+1)\omega_c t}{2} \sin\frac{(n-1)\omega_c t}{2} \quad \text{and}$$

$$v_y = v_{0\perp} \sin(\omega_c t + \varphi_0) - \frac{e}{m_e} E_0 \frac{2}{(n-1)\omega_c} \sin\frac{(n+1)\omega_c t}{2} \sin\frac{(n-1)\omega_c t}{2} . \quad (4.16)$$

We see that for $n \neq 1$ the perturbation to the initial gyration is purely periodic without any time averaged energy exchange between electron and wave. The orbits remain closed orbits as shown in fig. 4.10 for the case $\omega = 2\omega_c$.
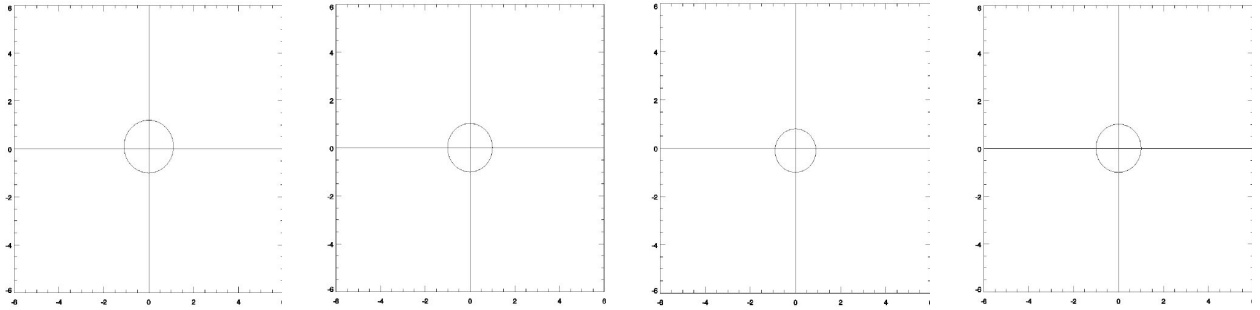

Fig. 4.10 Calculated orbits in the x-y-plane for different initial phases

Parameters: $\omega = 2\omega_c; \vec{E}_0 = [0.1; +0.1; 0]; \psi_y = +\pi/2; \vec{k} = [0;0;0]; \vec{v}_0 = [1;0;0]; \varphi_0 = var$.

4.5 summary for perpendicular electric field with $k = 0$

- time averaged power absorption occurs not only at the wave particle resonance frequency, but in a limited way also at frequencies close to it.
- at $\omega = \omega_c$ the power absorption increases continuously with time.
- there is no continuously increasing absorption at harmonics $\omega = n\omega_c$, in agreement with the findings of section 2 for $k = 0$.
- there is no drift motion, in agreement with the findings of section 2 for $k = 0$.
- due to phase synchronisation at $\omega = \omega_c$ all electrons, independent of their initial phases, finally may absorb energy from the co-rotating electric field. However, this is strongly restricted by loss of correlation due to finite transit time, collisions, or relativistic change of cyclotron frequency.
- counter-rotating electric fields cannot transfer energy from the field to the particle. Their orbits remain closed, although perturbed.

5. Numerical non-relativistic calculations of particle orbits with circularly polarised field and k ≠ 0,

To calculate time varying orbits due to electric fields with nonzero wave number we have to solve the equation of motion numerically. This was done with a Runge – Kutta procedure.

In the following figures we give some examples of calculated orbits where we have followed mostly M=10 cycles of the oscillating electric field. In some cases we calculated more cycles as mentioned in the figure caption. As in the previous chapter we will use dimensionless numbers for all quantities.

5.1 $\omega = \omega_c$, co-rotating field

With $k \neq 0$ the electron experiences already during its gyro motion a phase shift relative to the electric field.

Fig. 5.1 $\omega = \omega_c$, circularly polarised field rotating in the direction of electron gyration;
  Parameters: $\omega = \omega_c; \vec{E}_0 = [0.1;+0.1;0]; \psi_y = +\pi/2; \vec{k} = [0.5;0;0]; \vec{v}_0 = [1;0;0]; \varphi_0 = \text{var}$.

   a) $\varphi_0 = \pi$ b) $\varphi_0 = 3\pi/2$ c) $\varphi_0 = 0$ d) $\varphi_0 = \pi/2$
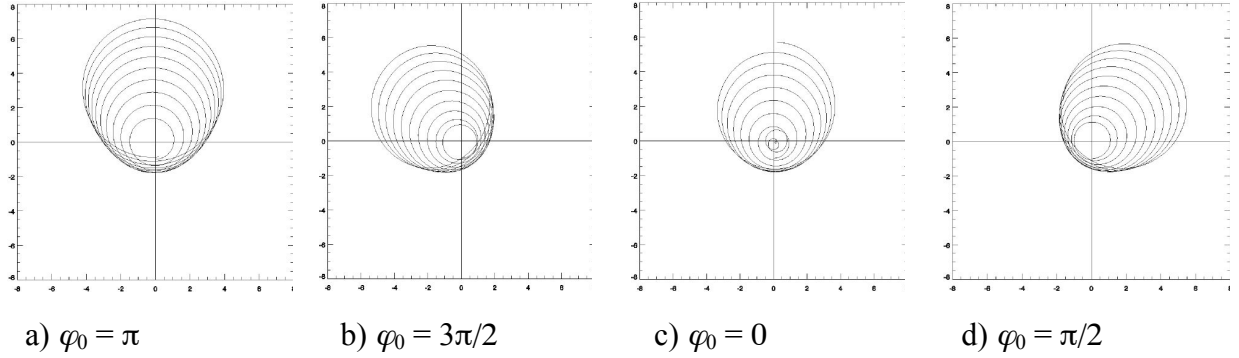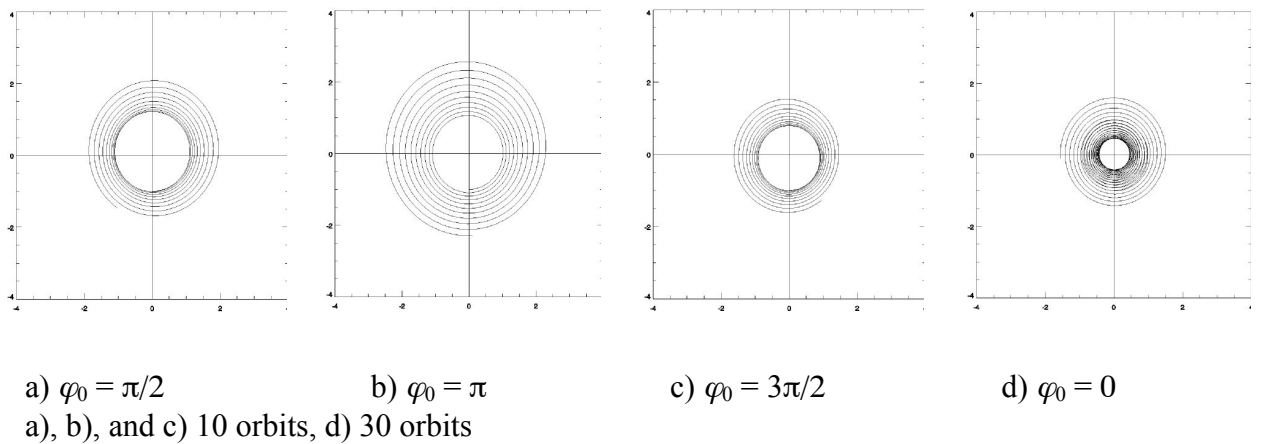
In all cases the electrons finally gain energy. The orbits differ only initially depending on phase similar to the case $k = 0$. Since here $k \neq 0$ there is also a drift superimposed.

5.2 $\omega = 2\omega_c$, co-rotating field

Fig. 5.2 $\omega = 2\omega_c$; circularly polarised field rotating in the direction of electron gyration.
  Parameters: $\omega = \omega_c; \vec{E}_0 = [0.1;+0.1;0]; \psi_y = +\pi/2; \vec{k} = [0.5;0;0]; \vec{v}_0 = [1;0;0]; \varphi_0 = \text{var}$.

   a) $\varphi_0 = \pi/2$ b) $\varphi_0 = \pi$ c) $\varphi_0 = 3\pi/2$ d) $\varphi_0 = 0$
   a), b), and c) 10 orbits, d) 30 orbits

In case b) continuous heating occurs from the very beginning. In the other cases continuous

heating occurs only after phase synchronisation, even if at first a deceleration happens. In case d) with initial deceleration the velocity does not decrease to zero before rising again.

A drift is superimposed since $k \neq 0$. The drift is weak since according to section 2 for $\omega = 2\omega_c$ it is associated with the Bessel function $J_2(k_\perp \rho)$.

# 6. Numerical relativistic calculations of particle orbits

In the preceding sections we have seen that a changing phase relation between the gyro motion and the constant frequency electric field affects the power absorption. The relativistic dependence of the electron mass on velocity makes such phase changes unavoidable.

6.1 numerical relativistic calculations of orbits and velocity

For these calculations we use in the equation of motion an electron mass depending on the electron energy $m_e = m_{e0}/\sqrt{1 - v^2/c^2}$, affecting the cyclotron frequency and the electric field force. A changing relativistic cyclotron frequency leads to a changing phase relative to the constant frequency electric field. Here we can no longer use normalised dimensionless quantities.

In the following calculations we use $B_0 = 2.5$ T, $k = \{0;0;0\}$, $U_0 = 10$ keV, $v_{\parallel 0} = 0$ and a circularly polarized electric field with $E_0 = 10 \; 10^5$ V/m rotating in the direction of the electron gyration with the non-relativistic cyclotron frequency ω_{c0}. We use high values for U_0 and E_0 so that the interaction shows up within a small number of orbits. We follow the orbit for 60 cyclotron periods.

In fig. 6.1 we vary the initial phase. (Our estimate for τ_{rel}, where we assume a constant velocity, gives for this case $\tau_{rel} = 12.5$ cyclotron orbits).
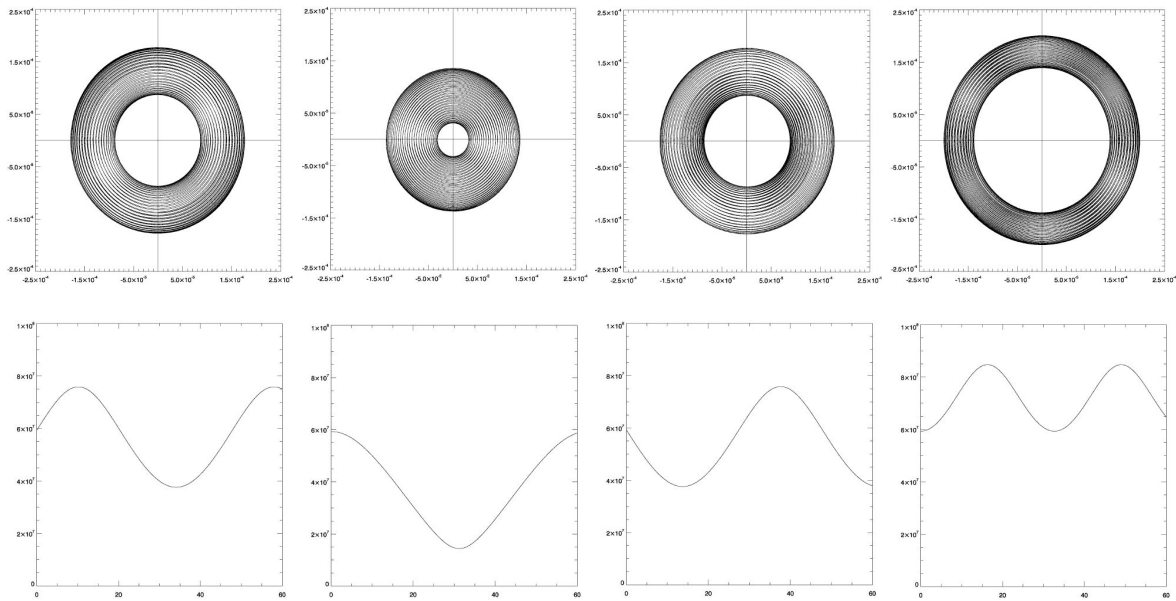
a) $\varphi = \{\pi; \pi/2; 0\}$ b) $\varphi = \{3\pi/2; \pi; 0\}$ c) $\varphi = \{2\pi; 3\pi/2; 0\}$ d) $\varphi = \{\pi/2; 0; 0\}$

Fig. 6.1 top: orbits in x-y-plane; bottom: velocity as function of time (number of orbits)

In all cases there is never a situation where the electron energy increases continuously as in the non-relativistic case with $k = 0$ and $\omega = \omega_{c0}$. As expected we find instead a periodic behaviour of the velocity. In case a) the electron is initially accelerated, in case c) it is initially decelerated, while in cases b) and d) the electric field vector is initially perpendicular to the velocity vector. In cases a) and c) the velocity oscillates around v_0. In case b) the velocity is always smaller than

v_0, and in case d) the velocity is always greater v_0. The periodicity changes with the phase.
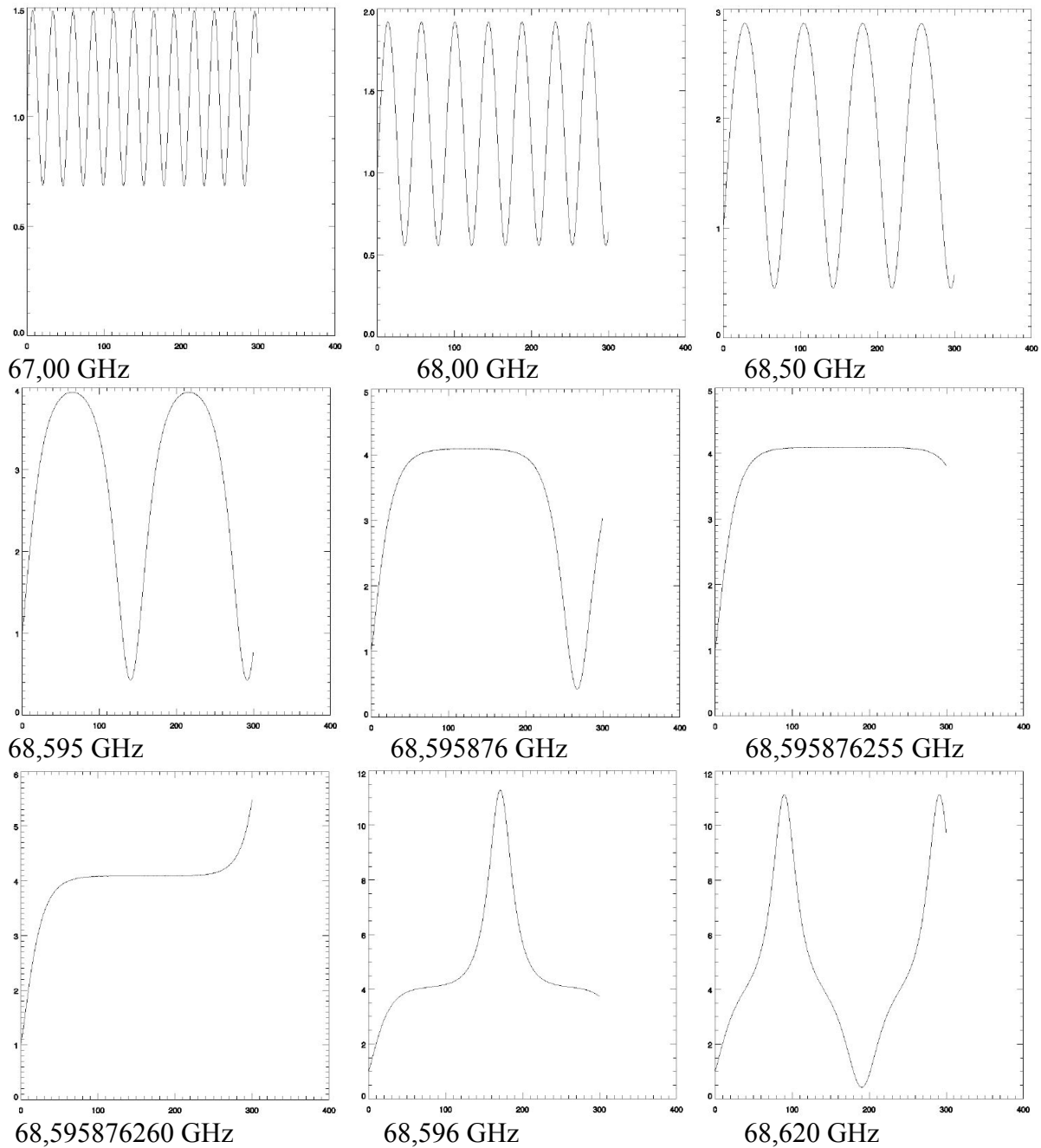
6.2 numerical relativistic calculations of the electron energy gain for $\omega \approx \omega_c$.

Here we present calculations of the time evolution of the energy $v_\perp^2/v_0^2$ of an electron including the relativistic mass dependence. In a toroidal plasma, like in fusion research, such a situation would only be possible for launching the microwave beam from the high field side. However in a gyrotron cavity the electrons may experience a comparable situation.

For all examples the following parameters are taken as constant
$B_0 = 2.5$ T; $k_x = 0.5$ k_0; $k_y = k_z = 0$; $v_{\|0} = 0$; $f_{c0} = 69,9804$ GHz; $e/m_e = 1,7588$ 10^{11} Cb/kg;
Here we show only one example where we vary the frequency, and with $E_0 = 5$ 10^5 V/m and $U_0 = 2$ keV, which corresponds to a 5 cm diameter microwave beam of 1 MW power. In further calculations we varied φ_0, U_0, E_0.
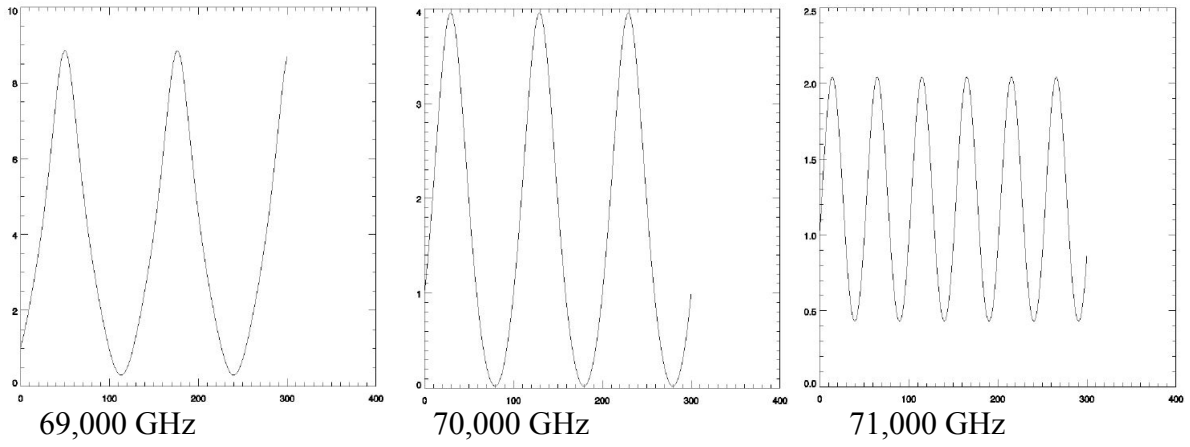

67,00 GHz	68,00 GHz	68,50 GHz
68,595 GHz	68,595876 GHz	68,595876255 GHz
68,595876260 GHz	68,596 GHz	68,620 GHz

| 69,000 GHz | 70,000 GHz | 71,000 GHz |

Fig. 6.2 time traces of of $v^2/v_0{}^2$. Constant: $U_0 = 2$ keV, $E_0 = 5 \; 10^5$ V/m; $\varphi_0 = (\pi; \pi/2; 0)$;

There is never a continuous energy gain as results from the non-relativistic calculation for $\omega = \omega_c$ with $k = 0$. The energy gain or loss is always limited and periodic in time. This is due to the varying frequency difference between the gyromotion and the constant frequency electric field.
The associated phase difference is continuously changing with time, therefore the electron will experience periodically an acceleration and a deceleration.

There is a frequency f_{res} at which a sudden transition in the $v^2/v_0{}^2$ – plots occurs, like in fig. 6.2 near 68.595876 GHz: the structure of the time traces is different for frequencies below and above this transition frequency. In all these cases $v_\perp{}^2$ assumes a long lasting flat portion preceeding the transition. This means that during this time the E-vector is nearly perpendicular to the velocity vector and rotates at very closely the same speed. Under these conditions the gyration frequency of the electron is determined by the combined action of the Lorentz force and the radial electric field force as

$$\omega_{gyr} = \omega_c (1 - \frac{eE_0}{m_e \omega_c v_0}) \; . \tag{6.1}$$

As an example we take plots f) and g) of fig. 6.2. The electron energy in the flat region before the transition is ≈ 8.2 keV, resulting in a relativistic cyclotron frequency of $f_c = 68,85$ GHz. Inserting this in the above equation with $E_0 = 5 \; 10^5$ V/m we get $f_{gyr} = 68,59$ GHz which is close to the transition frequency found in the numerically calculated patterns. Reducing the electric field amplitude E_0 shifts this resonant transition to a higher frequency.

There also occur minima in the electron energy where the electron has lost nearly all of its initial energy.

Summaries of such calculations where different parameters were varied are shown in figs. 6.3 to 6.5 displaying the maximum and minimum energy of plots like above.
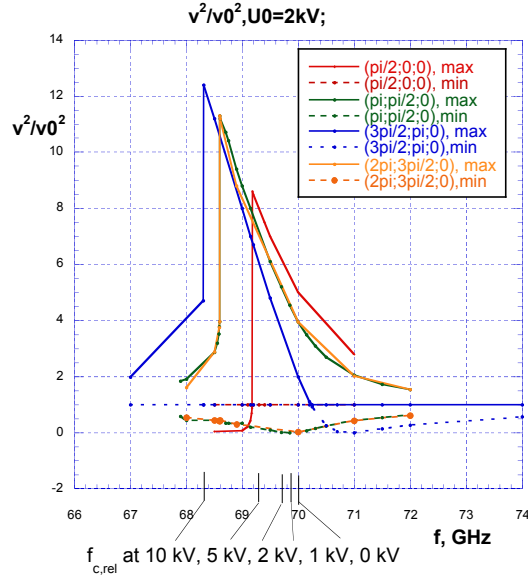
Fig. 6.3 Maxima and minima of $v_\perp^2/v_0^2$ as a function of frequency for different initial phases φ_0.
Constant: $B_0 = 2.5$ T; $U_0 = 2$ kV; $E_0 = 5\ 10^5$ V/m; co-rotating field.

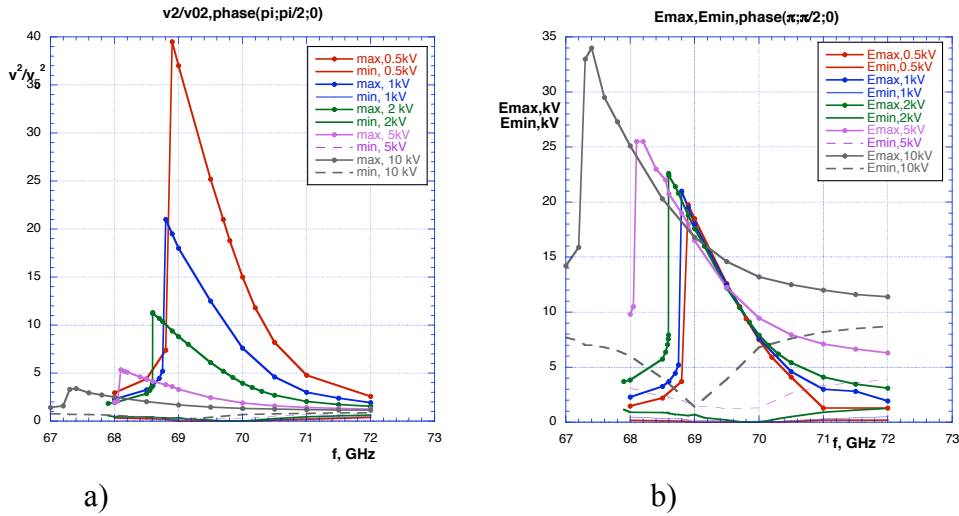a)   b)

Fig. 6.4 a) Maxima and minima of $v_\perp^2/v_0^2$ and b) Maximum and minimum energy in keV as
function of frequency for different initial energies U_0.
Constant: $B_0 = 2.5$ T; $E_0 = 5\ 10^5$ V/m, $\varphi_0 = (\pi; \pi/2; 0)$; co-rotating field.

An increasing initial energy shifts the relativistic resonance to a lower frequency. Although the relative energy gain, $v_\perp^2/v_0^2$, is highest for low initial energy, the maximum possible final energy increases with the initial energy. The difference between the possible maximum and the initial energy is comparable in all cases.
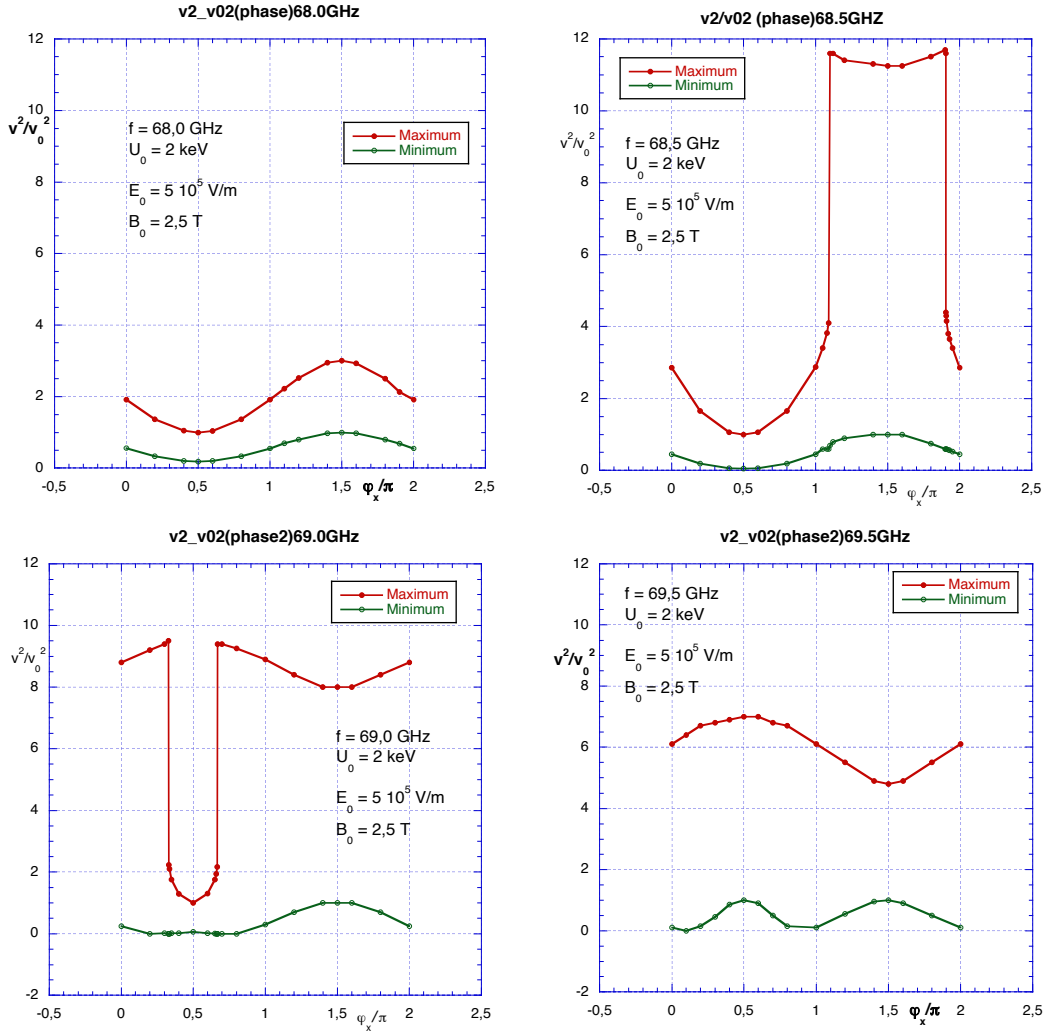
Fig. 6.5 Maxima and minima of $v_\perp^2/v_0^2$ as function of initial phase at different frequencies.
a) f = 68,0 GHz; b) f = 68.5 GHz; c) f = 69.0 GHz; d) f = 69.5 GHz;
Constant: B_0 = 2.5 T; E_0 = 5 10^5 V/m, U_0 = 2 kV; $\varphi_0 = (\varphi_x; \varphi_x - \pi/2; 0)$;

Heating can occur for all initial phases, but not always at the same frequency. In a situation with an inhomogenous magnetic field, like in a tokamak plasma, this corresponds to absorption at a different radius. A high energy gain is possible only when a transition to the relativistic cyclotron resonance can occur. In this case there are two regions in phase space where either no or little heating occurs, and where very high heating can occur. The frequency where a maximum energy gain occurs depends on initial phase, initial velocity and wave amplitude. The frequency range with high energy gain is on the low frequency side of the non-relativistic cyclotron frequency, and also lower than the relativistic cyclotron frequency related to the initial electron energy. Energy loss may also occur but is less propable.

The non-relativistic continuous energy gain at the classical cyclotron resonance does not exist relativistically.

The actual energy gain or loss of the electron depends not only on the electric field strength, on the initial energy and on the initial phase, but also on the time for one transit through the microwave beam, i.e. the parallel velocity. And the possible maxima or minima in the electron energy as shown in figs. 6.3 to 6.5 may not be reached if the electron stays not long enough in the beam.

In the above examples the parallel velocity was zero. In practice the electron will however have a non-zero parallel velocity. This will not only change the wave particle resonance due to the Doppler shift, but determines also the transit time through a finite size beam. An electron for which the interaction starts with a high initial acceleration may eventually leave the finite size beam at a lower than initial velocity, and an electron, which is initially decelerated, may finally end up with an appreciable energy gain. This is shown in fig. 6.6. The transit time through the beam, normalized to the classical cyclotron period, is

$$\frac{\tau_{tr}}{T_c} = \frac{d_b B_0 \sqrt{e}}{\pi \sqrt{8 U_{\parallel} m_e}}$$

For a 5cm diameter beam and a magnetic field of $B_0 = 2.5$ T, $\tau_{tr}/T_c = 50$, 100, 200 and 300 correspond to a parallel electron energy of 14 keV, 3.5 keV, 0.9 keV and 0.4 keV. Note, however, that in calculating the relativistic mass we did not include the parallel energy $U_{\parallel}$.
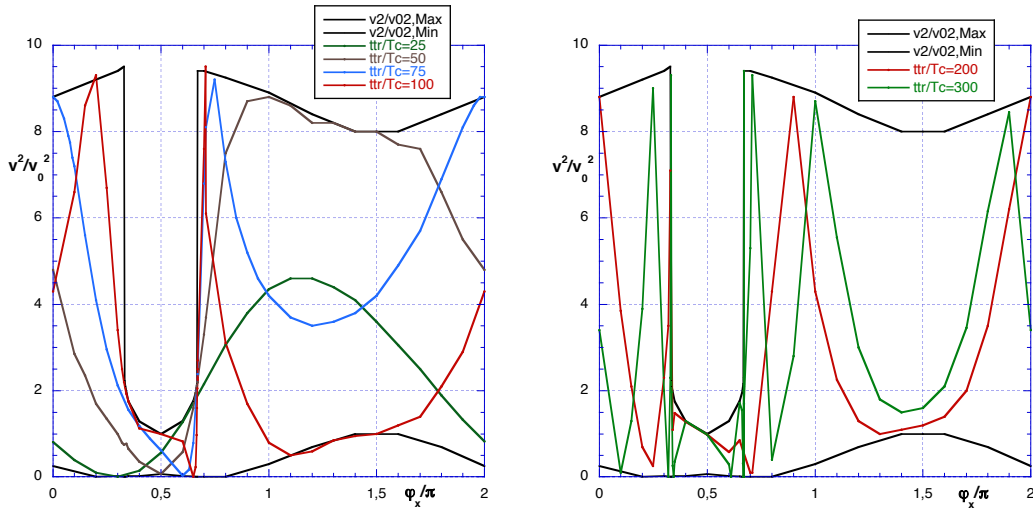

Fig. 6.6 Maxima and minima of $v_{\perp}^2/v_0^2$, and $v_{\perp}^2/v_0^2$ at different transit times $t_{tr}/T_c = 25$, 50, 75, 100, 200 and 300, normalized to the classical time for one gyro orbit,
Constant: $f = 69.0$ GHz, $B_0 = 2.5$ T; $E_0 = 5 \cdot 10^5$ V/m, $U_0 = 2$ kV; $\varphi_0 = (\varphi_x; \varphi_x - \pi/2; 0)$; $k_x = 0.5 \, k_0$;

Another example, with parameters more close to the situation in a gyrotron cavity, is shown in fig. 6.7.
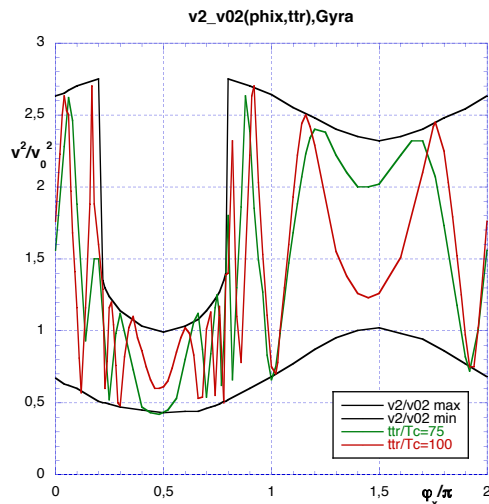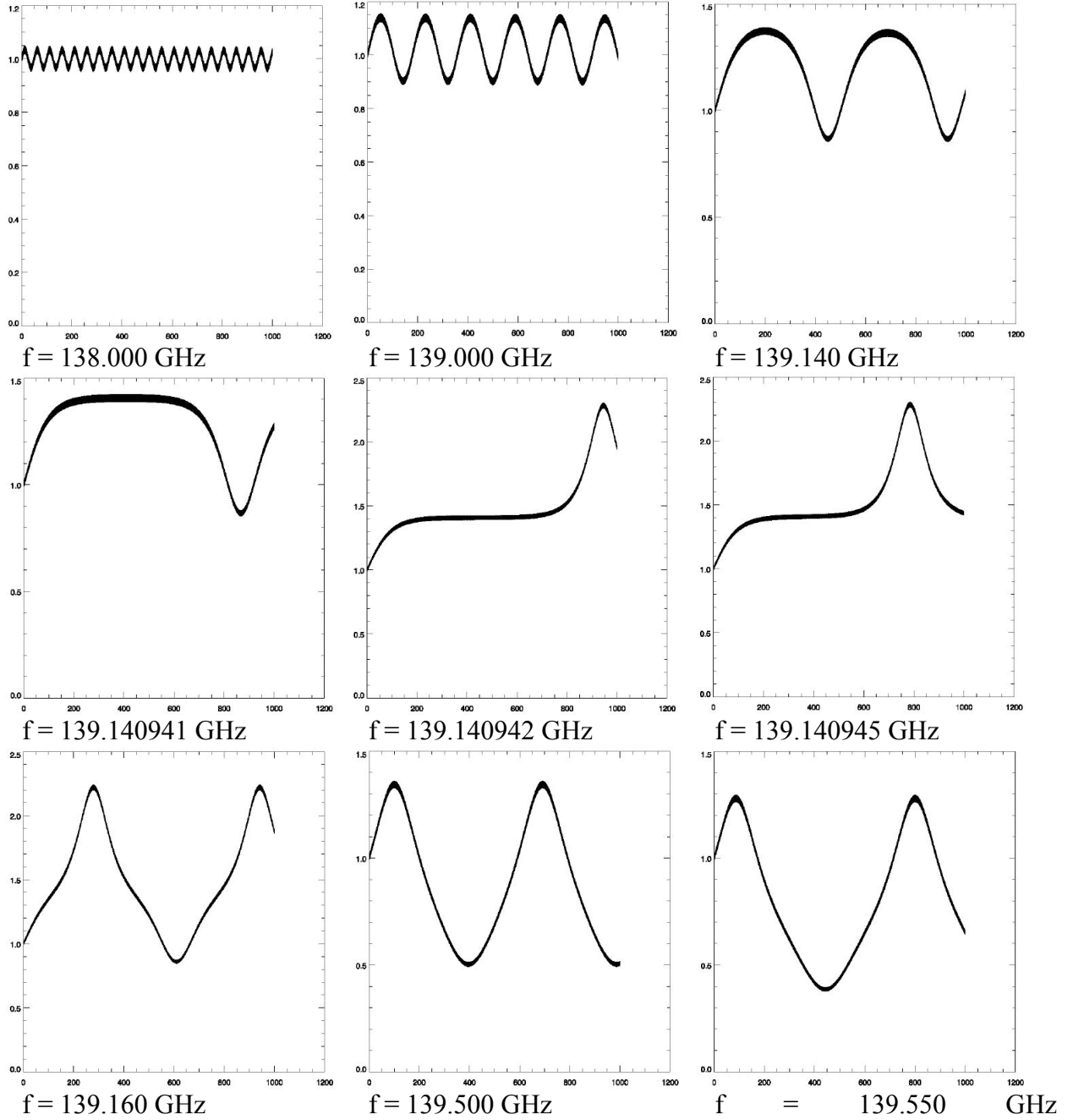

Fig. 6.7 Maxima and minima of v^2/v_0^2 and v^2/v_0^2 at different transit times $t_{tr}/T_c = 75$, 100,

Constant: $f = 140.0$ GHz, $B_0 = 5.777$ T; $E_0 = 1 \cdot 10^7$ V/m, $U_{0\perp} = 40$ kV; $U_{0tot} = 80$ kV; $v_0^2/c^2 = 0.25$; $\varphi_0 = (\varphi_x; \varphi_x - \pi/2; 0)$; $k_x = 0.9 \, k_0$;

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6.3 numerical relativistic calculations of the electron energy gain for $\omega \approx 2\omega_c$.

Here we show the energy gain of electrons subject to a co-rotating electric field oscillating at a frequency close to the second harmonic of the cyclotron frequency. This is a situation comparable to second harmonic heating in fusion plasmas. Like in the previous case we take the following parameters as constant: $k_x = 0.5\ k_0$; $B_0 = 2.5$ T; $f_{c0} = 69{,}9804$ GHz; $e/m_e = 1{,}7588\ 10^{11}$ Cb/kg; $v_{\|0} = 0$;

In fig.6.8 we show one example where we vary the frequency, and with $E_0 = 5\ 10^5$ V/m, which corresponds to a 5 cm diameter microwave beam of 1 MW power.
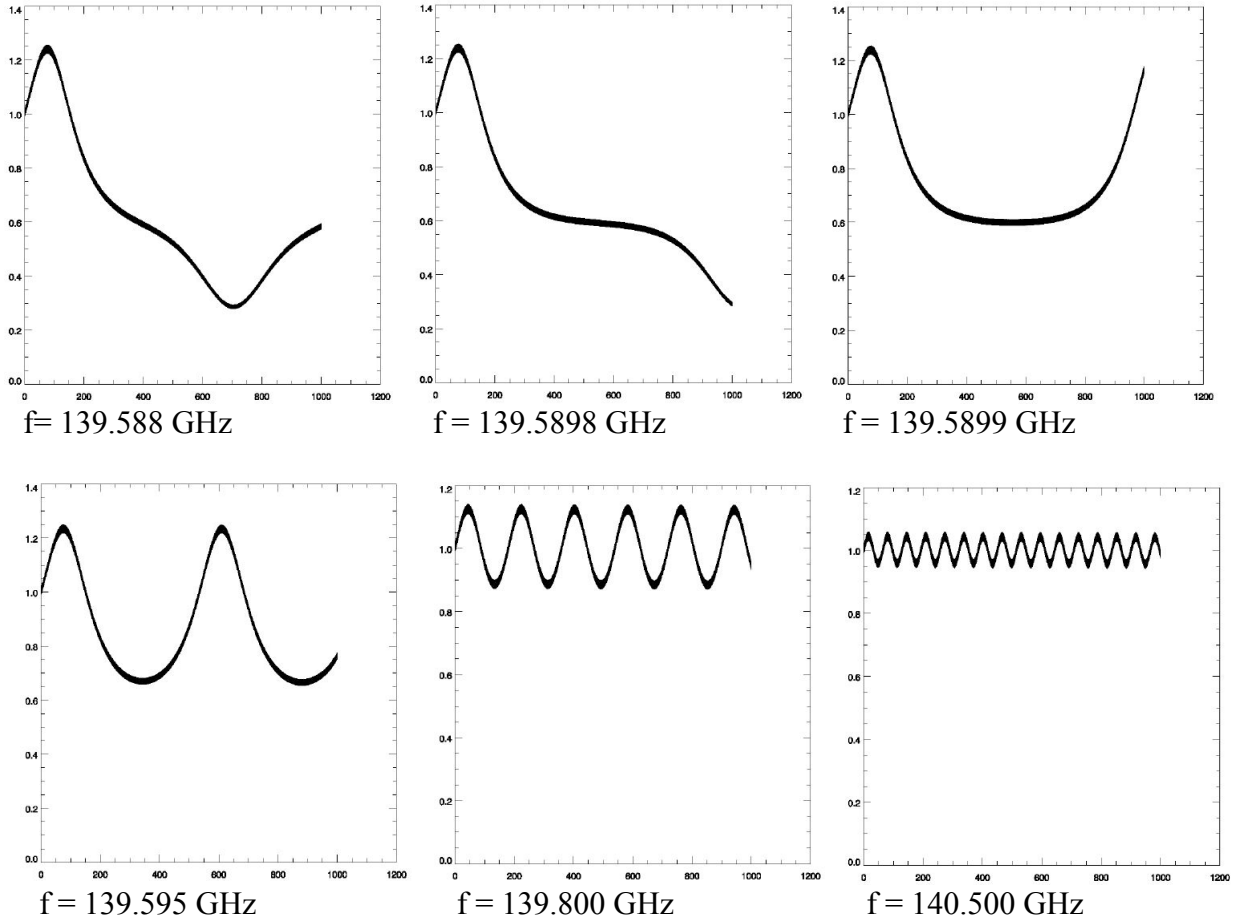

f = 138.000 GHz   f = 139.000 GHz   f = 139.140 GHz

f = 139.140941 GHz   f = 139.140942 GHz   f = 139.140945 GHz

f = 139.160 GHz   f = 139.500 GHz   f   =   139.550   GHz

Fig. 6.8 time traces of of $v_\perp^2/v_0^2$ at frequencies near the second harmonic. Circular polarized field co-rotating. Constant: $U_0 = 2$ kV, $E_0 = 5 \ 10^5$ V/m; $\varphi_0 = (\pi; \pi/2; 0)$; $k_x = 0.5 \ k_0$;

An expanded view of the case 139.0 GHz is shown in fig. 6.9 where we see that the energy gain shows a $2\omega_c$ oscillation superimposed on a slower variation as seen in fig. 6.8.
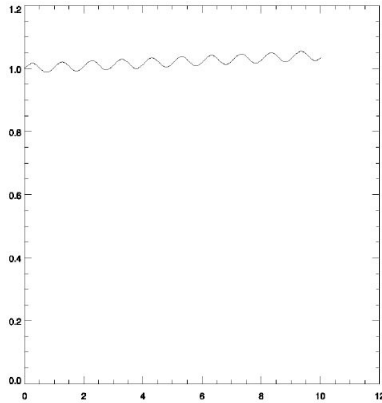

Fig. 6.9 Expanded time scale of case 139.0 GHz of fig. 6.8 shows $2\omega_c$ oscillation.

The energy gain is considerably less than in the case of fundamental cyclotron interaction like in fig. 6.2. This is because the force which the electron experiences at the second harmonic scales with $J_1(k_\perp\rho)$ rather than with $J_0(k_\perp\rho)$.

Similar to a resonant maximum possible energy gain at 139.140942 GHz there is also a resonant maximum energy loss at 139.5898 GHz. The result of calculations at different initial phases is shown in fig. 6.10.
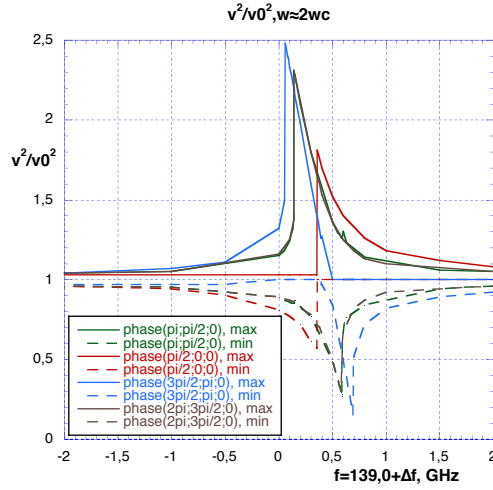
Fig. 6.10 maxima and minima of the energy gain as function of frequency near the second harmonic at different initial phases. Circular polarized field co-rotating.
Constant: $U_0 = 2$ kV, $E_0 = 5 \ 10^5$ V/m; $k_x = 0.5 \ k_0$;

At the second harmonic the resonant maximum or resonant minimum energy can be reached only after a long enough transit time through a microwave beam (longer than at the fundamental frequency) and is hence accessible only for lower parallel velocity. And a microwave beam at a given total power designed with a larger diameter will, however, have a corresponding lower electric field amplitude.

In fig. 6.11 we show $v_\perp^2/v_0^2$ at different transit times $t_{tr}/T_c = 50, 100, 200$ and 300, normalized to the classical time for one gyro orbit, corresponding to parallel electron energies of 14 keV, 3.5 keV, 0.9 keV and 0.4 keV. There are initial phase regions where energy loss or energy gain occurs. Averaged over phase there is an energy gain. Here too, there is a transit time for which the phase averaged energy gain is a maximum.
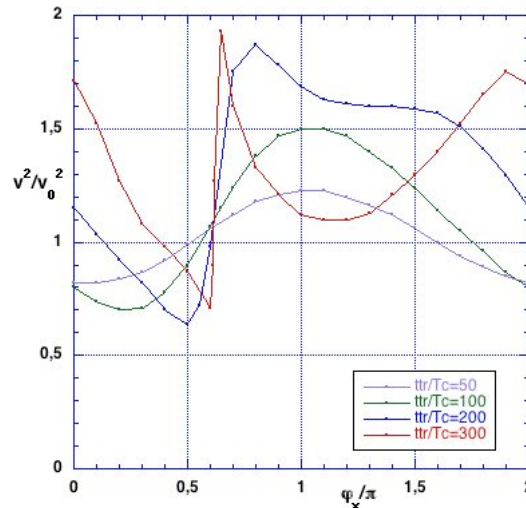

Fig. 6.11 v^2/v_0^2 at different transit times $t_{tr}/T_c = 25, 50, 75, 100, 200$ and 300, normalized to the classical time for one gyro orbit, as function of the initial phase. Circular polarized field co-rotating.
Const.: $f = 139.300$ GHz, $E_0 = 5 \ 10^5$ V/m, $U_0 = 2$ kV; $\varphi_0 = (\varphi_x; \ \varphi_x - \pi/2; \ 0)$; $k_x = 0.5 \ k_0$;

An interesting example for interaction at $\omega = 2\omega_c$ is the case of breakdown experiments, [7]. If $k_\perp \rho = 0$ there is no energy gain as explained in section 3. However, even for very small $k_\perp \rho$ there can be a considerable energy gain if the electron remains long enough within the

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microwave beam. In fig. 6.12 we show the result of a relativistic calculation of the electron energy assuming an initial energy of only 0.03 eV, corresponding to room temperature. It can easily reach an energy of more than 100 eV during one transit through a microwave beam, sufficient for ionization. For a 5 cm diameter beam and $B_0 = 2.5$ T the normalized transit time $t_{tr}/T_c = 6 \cdot 10^3$ and $3.5 \cdot 10^4$ for a parallel energy of 1 eV resp. 0.03 eV.
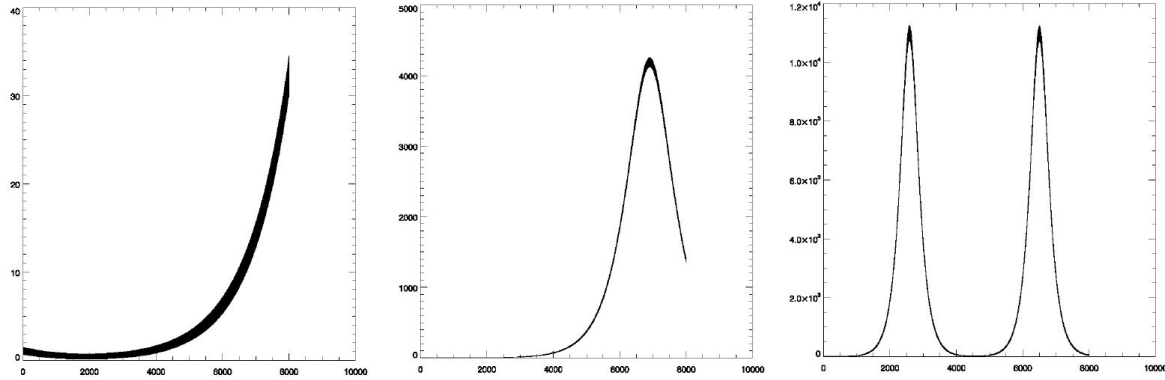

Fig. 6.12 $v_\perp^2/v_0^2$ as function of t_{tr}/T_c with an initial energy of 0.03 eV. Linear polarized field.
a) $E_{x0} = 1 \cdot 10^5$ V/m; b) $E_{x0} = 2 \cdot 10^5$ V/m; $E_{x0} = 5 \cdot 10^5$ V/m;
Const.: $f = 139.962$ GHz $= 2 f_c$, $B_0 = 2.5$ T; $U_0 = 0.03$ eV; $\varphi_0 = (0; 0; 0)$; $k = (k_0; 0; 0)$

6.4 summary of numerical relativistic calculations of energy gain

- the relativistic cyclotron frequency change with velocity leads to a changing phase shift between the rotating constant frequency electric field and the gyro motion of the electron
- there is never a continuous energy gain as results from the non-relativistic calculation for $\omega = \omega_c$ with $k = 0$.
- the energy gain or loss is always limited and periodic in time in an infinitely extended field.
- a positive energy gain can occur for all initial phases, but not always at the same frequency.
- the frequency where a maximum energy gain occurs depends on initial phase, initial velocity and wave amplitude.
- energy loss may also occur but is less propable.
- in case of an electric field with limited spacial extent, like in a mirowave beam or in a microwave cavity, the energy gain or loss depends also essentially on the transit time through the beam
- due to the finite Larmor radius effect even an electron at room temperature subject to a second harmonic field can gain sufficient energy for ionisation.
- treating a single electron only we cannot describe phase bunching effects as in [8].
- in a plasma the heating of the electron ensemble is the result of averaging over all initial phases and velocities, both in parallel and perpendicular direction with respect to the magnetic field, taking the finite size of the injected microwave beam into account.

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