

# Parametric Representation of Rank $d$ Tensorial Group Field Theory: Abelian Models with Kinetic Term $\sum_s |p_s| + \mu$

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## Abstract

We consider the parametric representation of the amplitudes of Abelian models in the so-called framework of rank  $d$  Tensorial Group Field Theory. These models are called Abelian because their fields live on  $U(1)^D$ . We concentrate on the case when these models are endowed with particular kinetic terms involving a linear power in momenta. New dimensional regularization and renormalization schemes are introduced for particular models in this class: a rank 3 tensor model, an infinite tower of matrix models  $\phi^{2n}$  over  $U(1)$ , and a matrix model over  $U(1)^2$ . For all divergent amplitudes, we identify a domain of meromorphicity in a strip determined by the real part of the group dimension  $D$ . From this point, the ordinary subtraction program is applied and leads to convergent and analytic renormalized integrals. Furthermore, we identify and study in depth the Symanzik polynomials provided by the parametric amplitudes of generic rank  $d$  Abelian models. We find that these polynomials do not satisfy the ordinary Tutte's rules (contraction/deletion). By scrutinizing the “face”-structure of these polynomials, we find a generalized polynomial which turns out to be stable only under contraction.

September 2, 2014

Pacs numbers: 11.10.Gh, 04.60.-m, 02.10.Ox

Key words: Renormalization, tensor models, quantum gravity, graph theory

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## 1 Introduction

Tensorial group field theories (TGFTs) provide a background independent framework to quantum gravity which is intimately based on the idea that the fundamental building blocks (quanta) of space-time are discrete [1]–[9]. Within this approach, the fields are rank  $d$  tensors labeled by abstract group representations. From such a discrete structure, one dually associates tensor fields with basic  $d - 1$  dimensional simplexes and their possible interactions with  $d$  dimensional simplicial building blocks. At the level of the partition function, the Feynman diagrams generated by the theory represent discretizations of a manifold in  $d$  dimensions. Thus, in essence, TGFTs which randomly generate topologies and geometries in covariant and algebraic ways, can be rightfully called quantum field theories of spacetime. One of the main efforts in this research program is to seek what types of phases the theory exhibits. More to the point, one may ask if any of these phases give our geometric universe described by General Relativity from the pre-geometric cellular-complex picture that the bare

theory gives [10]. This question is accompanied by a further suggestion that the relevant phase corresponds to a condensate of the microscopic degrees of freedom [1, 5]. Note that this question has found a partial answer in [11, 12].

Because they are field theories, TGFTs can certainly be scrutinized using several different lenses. In particular, one of the main successes of quantum field theory which is a Renormalization Group analysis turns out to have a counterpart in TGFTs. We recall that renormalizability of any quantum field theory is a desirable feature since it ensures that the theory survives after several energy scales. In fact, so far, all known interactions of the standard model are renormalizable. Quantum field theory predictions rely on the fact that, from the Wilsonian renormalization group point of view, the infinities that appear in the theory should locally reflect a change in the form of the theory [13]. In particular, if TGFTs are to describe any physical reality like our spacetime at a low energy scale, one is certainly interested in probing the flow of this theory. The renormalization program suitably provides a mechanism to study the flow of a theory with respect to scales and also might lead to predictions. Within TGFTs, this renormalization program can be addressed in several ways and, indeed, has known important recent developments [14]–[30]. The simplest setting in which one can think within TGFTs is given in purely combinatorial terms as tensor models.

Tensor models, originally introduced in [31]–[35], especially enjoy the knowledge of their lower dimensional cousins: matrix models [36]. These latter models are nowadays well developed and understood through rich statistical tools [37]–[42]. Specifically, the Feynman integral of matrix models generates ribbon graphs organized in a  $1/N$  (or genus) expansion [37]. In short, this statistical sum is analytically well controlled. It is only recent that the notion of large  $N$  expansion was extended to tensor models [43]–[45] (for however the class of colored models [46]–[48]). From this point, important progresses have been unlocked [49]–[63] and a renormalization program for tensor models uncovered (for a review, see [8, 27]).

Back to renormalization and its applicability to TGFTs, one notes that, in anterior works, thriving efforts were developed on the so-called multi-scale renormalization [13]. It is also worth and advantageous to understand how other known tools in renormalization (like the Polchinski equation or Functional Renormalization Group methods [64, 65, 9]) can shed light and even convey more insights in the present class of models and thereby enrich their Physics. Among these well-known renormalization procedures, there is the celebrated dimensional regularization.

In ordinary quantum field theory, dimensional regularization is an important scheme as it delivers finite (regularized) amplitudes and respects, at the same time, the symmetries of gauge theories (preserves field equations and Ward identities) [66, 67]. Of course, in our present class of non-local models, there exists a notion of invariance but it is an open issue to show that their associated Ward identities [29] will be preserved or not after the dimensional regularization and its subtraction program introduced in the present work. Nevertheless, a dimensional regularization is a very interesting tool that one may want to have in TGFTs. It allows one to understand the fine structure of the amplitudes: it makes easy to locate the divergences in any amplitude as it picks out the divergences in the form of poles and exhibit meromorphic structure of these integrals.

- As a first upshot of the present paper, and for a particular class of TGFT models defined over Abelian groups  $U(1)^\delta$ , we show that a dimensional regularization procedure can be defined and yields finite renormalized amplitudes. Under their parametric form and by

complexifying the group dimension  $D \in \mathbb{C}$ , amplitudes are proved to be meromorphic functions in an extended strip  $0 < \Re(D) < \delta + \varepsilon$ , where  $\varepsilon > 0$  is a small parameter. A subtraction operator can be defined at this stage and will provide finite amplitudes. Theorems 1 and 2 contain our main results on this part. During the analysis, it appears possible to introduce another complex parameter associated with the rank  $d$  of the theory. Although, we did not address this issue here, it is a new and interesting fact that another type of regularization (that one can call a “rank regularization” scheme) could be introduced using the parametric amplitudes in tensor models. This will require further investigations elsewhere.

The parametric representation of Feynman amplitudes has several other interesting properties. For instance, it allows one to read off the so-called Symanzik polynomials. In standard quantum field theory [68] and even extended to noncommutative field theory [69], these polynomials satisfy particular contraction/deletion rules like the Tutte polynomial, an important invariant in graph theory.

- As a second set of results, we sort the structure of the “Symanzik polynomials” associated with the parametric amplitudes of any rank  $d$  Abelian models (not only the ones assumed to be renormalizable). We show that these polynomials fail to satisfy a contraction/deletion rule. Under specific assumptions, the first Symanzik polynomial that we found can be mapped onto the invariant by Krajewski and co-workers [69]. As an interesting feature of these polynomials, we will observe that they respect a peculiar “face”-structure of the tensor graph. A way to stabilize the polynomials under some recurrence rules is to fully consider this structure and to enlarge the space under which one must consider the recurrence. Given a graph  $\mathcal{G}$ , we will consider its so-called set of internal faces  $\mathcal{F}_{\text{int}}$  (these are closed loops). The new invariant that we construct is defined over  $\mathcal{G} \times \mathcal{P}(\mathcal{F}_{\text{int}})^{\times 2} \times \{\text{od}, \text{ev}\}^{\times 2}$ , where  $\mathcal{P}(\mathcal{F}_{\text{int}})$  is the power set of  $\mathcal{F}_{\text{int}}$  and  $\{\text{od}, \text{ev}\}$  is a parity set. The new invariant is stable only under contraction operations and this result is new to the best of our knowledge. Theorems 3 and 4 embody our key results on this part.

This paper is organized as follows. Section 2 covers definitions and terminologies associated with important graph concepts used throughout the text, for the rank  $d \geq 3$  colored tensor graphs and the rank  $d = 2$  ribbon graphs. For notations closer to our discussion, we refer to the survey given in Section II in [27]. Section 3 presents the models including rank  $d = 2$  matrix and  $d \geq 3$  tensor models that we shall study. The parametric representation of the amplitudes and the new Symanzik polynomials  $U^{\text{od/ev}}$  and  $\tilde{W}$  are presented. In Section 4, we develop dimensional regularization and renormalization of particular models presented in Section 3. The proof of the amplitude factorization (which is necessary for showing that the pole extraction is equivalent to adding counterterms of the form of the initial theory) and the exploration of meromorphic structure of the amplitudes in the complex dimension parameter  $D$  are undertaken. Then, a paragraph on the subtraction operator and the procedure leading to renormalized amplitudes is sketched. Section 5 explores the properties of the newly found Symanzik polynomials  $U^{\text{od/ev}}$  and  $\tilde{W}$ . Then, we identify a polynomial  $\mathcal{U}^{\varepsilon, \bar{\varepsilon}}$  which is an extended version of  $U^{\text{od/ev}}$  with a stable recurrence relation based only on contraction operations on a graph. Section 6 is devoted to a summary and perspectives of the present work.

## 2 Stranded graphs

Before starting the study of the parametric amplitudes associated with Feynman graphs in tensor models, it is worth fixing the basic definitions of the type of graphs we will be analyzing in these models.

In the following, we shall give a survey of the main ingredients of two types of graphs:

- the so-called rank  $d > 2$  colored tensor graphs which we will describe only from the field theoretical point of view (for a mathematical definition, we will refer to [70]);

- ribbon graphs with half-ribbons also called rank 2 graphs in this paper. These graphs are quite well understood and still intensively investigated. For a complete definition of ribbon graphs, we will refer to one of the following standard references [71, 72, 73, 74, 75] (the last reference offers an up-to-date survey). The case of ribbon graphs with half-edges or half-ribbons and their relation to Physics, the work by Krajewski and co-workers [69] is seminal. However, our notations are closer to [76].

### 2.1 Rank $d > 2$ colored stranded graphs

Colored tensor models [46] expand in perturbation theory as colored Feynman graphs endowed with a rich structure. From these colored tensor graphs, one builds another type of graphs called uncolored [54]–[58]. These graphs will form the useful category of graphs we will be dealing with at the quantum field theoretical level. In this section, we provide a lightning review of the basic definitions of objects in the above references. Most of our illustrations focus on the rank 3 situation which is already a nontrivial case mostly discussed in our following sections; we invite the reader to more illustrations in [27].

**Colored tensor graphs.** In a rank  $d$  colored tensor model, a graph is a collection of edges or lines and vertices with an incidence relation enforced by quantum field theory rules. In such a theory, we call graph a (rank  $d$ ) colored tensor graph. This graph has a stranded structure described by the following properties [70]:

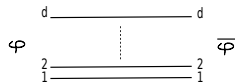


Figure 1: A stranded propagator or line in a rank  $d$  tensor model.

- each edge corresponds to a propagator and is represented by a line with  $d$  strands (see Fig.1). Fields  $\varphi$  are half-lines with the same structure;

- there exists a  $(d + 1)$  edge or line coloring;

- each vertex has coordination or valence  $d + 1$  with each leg connecting all half-lines hooked to the vertex. Due to the stranded structure at the vertex and the existence of an edge coloring, one defines a strand bi-coloring: each strand leaves a leg of color  $a$  and joins a leg of color  $b$ ,  $a \neq b$ , in the vertex;

- there are two types of vertices, black and white, and we require the graph to be bi-partite. Illustrations on rank  $d = 3, 4$  white vertices are depicted in Fig.2. Black vertices, on the other hand, are associated with barred labels and drawn with counterclockwise orientations.

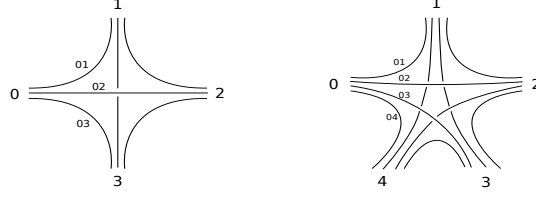


Figure 2: Two vertices in rank  $d = 3$  (left) and  $d = 4$  (right) colored models. Strands have a bi-color label.

We may use a simplified diagram which collapses all the stranded structure into a simple colored graph. The resulting graph still captures all the information of the former. Fig.3 illustrates an example of such a collapsed graph.

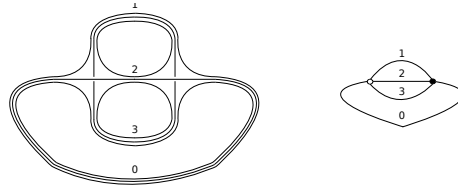


Figure 3: A rank 3 colored tensor graph and its compact colored bi-partite representation (right).

All rank  $d$  tensor graphs (without color) have a nice dual geometrical interpretation. The rank  $d$  vertex determines a  $d$  simplex and the fields represent  $(d - 1)$  simplexes. A generic graph is therefore a  $d$  dimensional simplicial complex obtained from the gluing of  $d$  simplexes along their boundaries. The key role of colors in tensor graphs was put forward in [48]. These colored graphs are dual to simplicial pseudo-manifold in any dimension  $d$ .

**Open and closed graphs.** A graph is said to be closed if it does not contain half-lines (also called half-edges). It is open otherwise. One refers such half-lines to external legs representing external fields in usual field theory. We give an example of a rank 3 open graph in Fig.4.

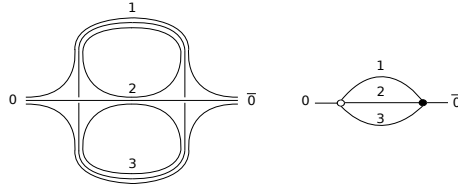


Figure 4: A rank 3 open colored tensor graph and its compact representation with half-edges.

**$p$ -bubbles and faces.** Appearing as one of their most striking features, colored tensor graphs in any rank  $d$  have a homological cellular structure [46]. A  $p$ -bubble is a maximally connected component subgraph of the collapsed colored graph associated with a rank  $d$  colored tensor graph, with  $p$  the number of colors of the edges of that subgraph. For example,

a 0-bubble is a vertex, a 1-bubble is a line. A 2-bubble is called a face. Faces can be viewed in the simplified colored graph as cycles of edges with alternate colors, see Fig.5. They will play a major role in all of our next developments.

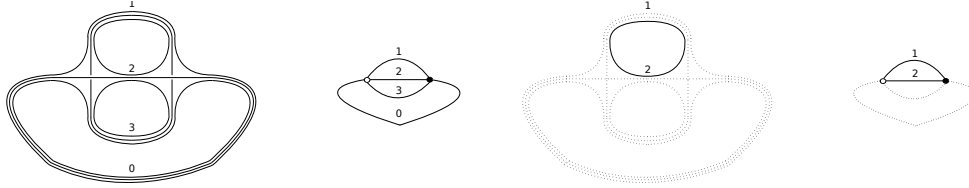


Figure 5: Deleting colors 0 and 3 in the graph on the left, one obtains a 2-bubble, the face  $f_{12}$  (right).

We have few remarks:

- In the full expansion of the colored graph using strands, a face is nothing but a connected component made with one strand. The color of strands alternates when passing through the edges which define the face.

- A  $p$ -bubble is open if it contains an external half-line, otherwise it is closed. For instance, there exist three open 3-bubbles ( $\mathbf{b}_{012}$ ,  $\mathbf{b}_{013}$  and  $\mathbf{b}_{023}$ ) and one closed bubble  $\mathbf{b}_{123}$  in the graph  $\mathcal{G}$  in Fig.4.

**Jackets.** Jackets are ribbon graphs coming from a decomposition of a colored tensor graph. Following [45, 77], a jacket in rank  $d$  colored tensor graph is defined by a permutation of  $\{1, \dots, d\}$  namely  $(0, a_1, \dots, a_d)$ ,  $a_i \in \llbracket 1, d \rrbracket$ , up to orientation. One divides the  $(d+1)$  valent vertex into cycles of colors using the strands with color pairs  $(0a_1)$ ,  $(a_1a_2), \dots, (a_{d-1}a_d)$  and proceed in the same way with rank  $d$  edges. Some jackets are illustrated in Fig.6. Open and closed jackets follow the standard definition of having or not having external legs, respectively.

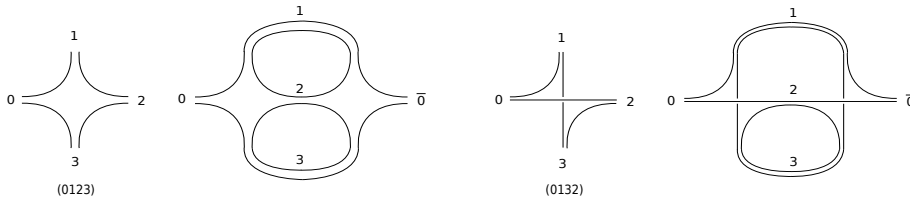


Figure 6: Two open jackets,  $J_{0123}$  (left) and  $J_{0132}$  (right) of the graph in Fig.4. The subscripts stand for a given color cyclic permutation used to decompose the colored tensor vertex in another-ribbon like vertex.

**Boundary graphs.** Tensor graphs with external legs are dual to simplicial complexes with boundaries. This boundary itself inherits a simplicial (even homological) complex structure in the context of colored models [47]. From the field theoretical perspective, we are interested in graphs with external legs,<sup>1</sup> therefore in the present context, in simplicial complexes with boundaries.

<sup>1</sup>External legs allow us to probe events happening at a much higher scale as compared to the scale of their own.

One can map the boundary complex of a rank  $d$  colored graph to a tensor graph with lower rank  $d - 1$  endowed with a vertex-edge coloring [70]. The procedure for achieving this mapping is known as “pinching” (or closing of open tensor graphs): one inserts a  $d$ -valent vertex at each external leg of a rank  $d$  open tensor graph. The boundary  $\partial\mathcal{G}$  of rank  $d$  colored tensor graph  $\mathcal{G}$  is then a graph

- the vertex set of which is one-to-one with the set of external legs of  $\mathcal{G}$  and is the set of  $d$ -valent vertices inserted;
- the edge set of which is one-to-one with the set of open faces of  $\mathcal{G}$ .

As a direct consequence, the boundary graph has a vertex coloring inherited from the edge coloring and has an edge bi-coloring coming from the bi-coloring of the faces of the initial graph. See Fig.7 as an illustration in a rank 3 colored tensor graph. Note, for example, that

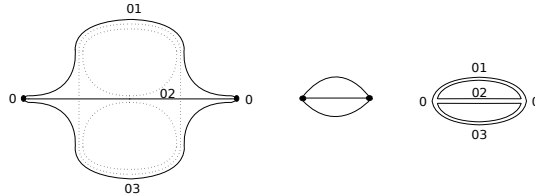


Figure 7: The boundary graph  $\partial\mathcal{G}$  of the graph  $\mathcal{G}$  of Fig.4.  $\partial\mathcal{G}$  (graph in the middle) is obtained by inserting a  $d = 3$  valent vertex at each external leg in  $\mathcal{G}$  and erasing the closed internal faces.  $\partial\mathcal{G}$  has a rank  $d - 1 = 2$  structure (most right).

in rank  $d = 3$ , the boundary of a rank 3 colored tensor graph is a ribbon graph.

**Degree of a colored tensor graph.** Organizing the divergences occurring in the perturbation series of rank  $d$  colored tensor graphs, one introduces the following quantity called degree of the colored tensor graph  $\mathcal{G}$  [43, 44, 45]

$$\omega(\mathcal{G}) = \sum_J g_J, \quad (1)$$

where  $g_J$  is the genus of the jacket  $J$  and the sum is performed over all jackets in the colored tensor graph  $\mathcal{G}$ . For an open graph, one might use instead pinched jackets  $\tilde{J}$  for defining the degree. A graph for which  $\omega(\mathcal{G}) = 0$  is called a “melon” or “melonic” graph [50]. This quantity is at the core of the extension of the notion of genus expansion (t’Hooft large  $N$  expansion in matrix models) now for colored tensor models. It is at the basis of the success of finding a way to analytically resum the perturbation series in colored tensor models at leading order and even beyond [50]–[63].

**Contraction and cut of a stranded edge.** As in ordinary graph theory, an edge can be regular or special (bridge and loop). We will consider the following operations on a tensor graph:

- The **cut operation** on an edge is intuitive: we replace a stranded line by two stranded half-lines on the vertex or vertices where the edge was incident (see Fig.8). Importantly, we respect the bi-coloring of strands during the process. We denote  $\mathcal{G} \vee e$  the resulting graph after cutting  $e$  in  $\mathcal{G}$ . We realize immediately that cutting edges has a strong effect on the boundary graph.



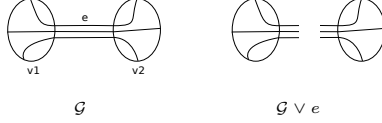


Figure 8: Cut of a stranded line  $e$ .

- The **contraction of a non-loop** rank  $d$  stranded edge is similar to ordinary contraction in graph theory. The important point is, once again, to respect the stranded structure. The contraction of an edge  $e$  incident to  $v_1$  and  $v_2$  is performed by removing  $e$  and its end vertices and introduce another vertex containing all the remaining edges incident to  $v_1$  and  $v_2$  in such a way to conserve their stranded structure and incidence relations (see Fig.9). Starting from a colored graph, such an operation immediately leads to a non colored graph. However, the stranded structure and stranded bi-coloring are preserved. These are the important ingredients that we need in our next developments.

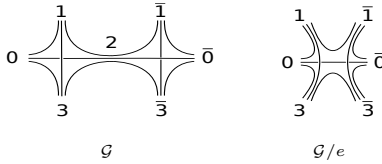


Figure 9: Contraction of a stranded line 2.

A colored graph does not have loop edges (a loop edge is incident to the same vertex). Thus our initial class of rank  $d$  colored tensor graphs does not generate any loops. However, after contractions of regular edges, it is easy to imagine that one might end up with configurations with loops from a generic graph. Since we will be interested in situations where such configurations arise and where we must further perform contractions, a definition of loop contraction is required. In [70], such a contraction has been defined in the case of a trivial loop.<sup>2</sup> We provide here a straightforward generalization of this definition which turns out to be useful for our following study. For simplicity, we restrict to the rank 3 colored case, and the general situation can easily be recovered from this point.

- The **contraction of a loop** stranded edge: Consider a loop edge  $e$ , and its bi-colored strands called  $i = 1, 2, 3$ . Call  $\alpha_i$  and  $\beta_i$ 's,  $1 \leq i \leq 3$ , the points where the strands connect other half-lines (or legs) of the vertex (see Fig.10). We write  $1 \leq i \leq 3$ , because it may happen that the strand  $i$  does not exit at another leg of the vertex but directly becomes a loop. Note that the  $\alpha_i$ 's (and  $\beta_i$ 's) are all pairwise distinct by definition of a bi-colored stranded vertex. The contraction of  $e$  incident to a vertex  $v$  in  $\mathcal{G}$  is performed by removing  $e$  and directly connecting all  $\alpha_i$  to  $\beta_i$  with the same color index. Several situations may occur. The graph might split if the resulting parts of the vertex form themselves vertices with their incident edges (see Fig.11). If there is a closed strand passing through  $e$  and  $v$  only, the resulting graph, by convention, contains a disc issued from this closed strand (see Fig.12). We will see that this procedure will extend the similar contraction in the case of

<sup>2</sup>After the contraction of a tree of regular edges, we always end up with a generalized stranded rosette graph. In ribbon graphs, a loop on a rosette is called trivial if it does not interlace with any other loops. In stranded graphs, one might impose further conditions categorized by possible consequences of the contraction of these loops before calling them trivial.

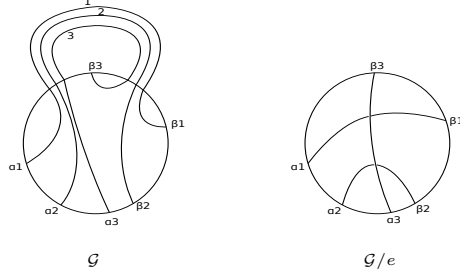


Figure 10: A loop in a graph  $\mathcal{G}$  and its bi-colored strands  $i = 1, 2, 3$ . After contraction, in the graph  $\mathcal{G}/e$ , the sectors  $\alpha_i$  are joined with the  $\beta_i$ 's in the resulting vertex.

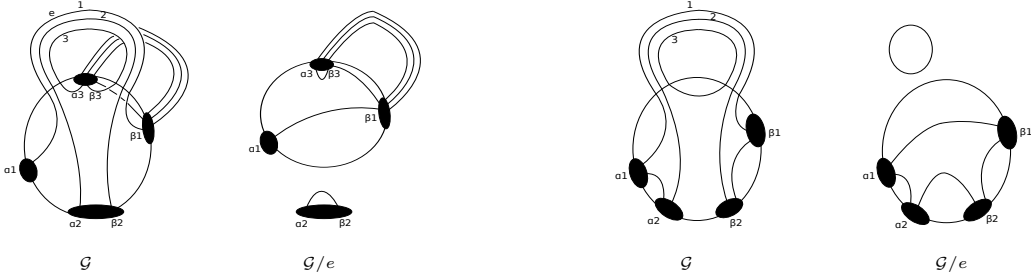


Figure 11: A loop contraction: black sectors represent some parts of the graph where the  $\alpha_i$  and  $\beta_i$  are connected. After contraction, the vertex splits.

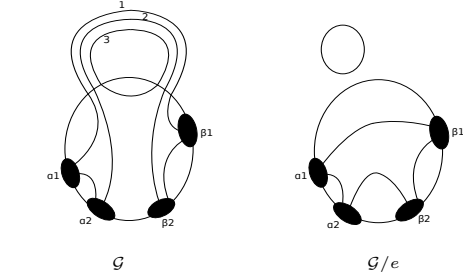


Figure 12: A loop contraction: the strand 3 is closed and does not pass by any other edges. After contraction,  $\mathcal{G}$  splits and  $\mathcal{G}/e$  contains a disc.

ribbon graphs.

The above contraction has been called “soft” in [70] as opposed to the so-called “hard” contraction. The hard contraction follows the same rules of the soft contraction but whenever a disc graph (without any edges) is generated during the procedure we remove it from the resulting graph. Note that hard contraction cannot be distinguished from soft contraction on non-loop edges and even on specific loops which do not contain these particular closed strands. Hard contraction is useful in the quantum field theory setting. However, during the study of invariant polynomials on graph structures, considering soft contraction which preserves the number of faces becomes capital to achieve all main results and recurrence relations.

## 2.2 Ribbon graphs

Let us define the type of graphs for the rank  $d = 2$  case that will retain our attention.

**Definition 1** (Ribbon graphs [71][69]). *A ribbon graph  $\mathcal{G}$  is a (not necessarily orientable) surface with boundary represented as the union of two sets of closed topological discs called vertices  $\mathcal{V}$  and edges  $\mathcal{E}$ . These sets satisfy the following:*

- *Vertices and edges intersect by disjoint line segment,*
- *each such line segment lies on the boundary of precisely one vertex and one edge,*
- *every edge contains exactly two such line segments.*

In the following, when no ambiguity can occur, we might simply call ribbon graphs as graphs.

Ribbon edges can be twisted or not and this induces consequences on the orientability and genus of the ribbon graph as a surface.

Defining the class of ribbon graphs, we take the point of view of Bollobàs and Riordan [71]. Arbitrary cyclic orientation (+ or -) signs on vertices are fixed, and then one assigns to each ribbon edge an orientation, + or -, according to the fact that the orientation of its end-vertices across the edge are consistent or not, respectively. Note that flipping a vertex (or reversing its cyclic ordering) has the effect of changing the orientation of all its incident edges except its “loops” (ribbon edges incident to the same vertex). Two ribbon graphs are isomorphic if there exist a series of vertex flips composed with isomorphisms of cyclic graphs [72] which transform one into the other. Now, according to the class of ribbon graphs, only the parity of the number of twists matters.

The notions of regular ribbon edges and bridges are direct (these can be also called non-loop edges). The notion of loop in ribbon graphs must be clarified. A loop is a ribbon edge incident to the same vertex. In particular [71], we say that a loop  $e$  at a vertex  $v$  of a ribbon graph  $\mathcal{G}$  is twisted if  $v \cup e$  forms a Möbius band as opposed to an annulus for an untwisted loop. A loop  $e$  is called trivial if there is no cycle in  $\mathcal{G}$  which can be contracted to form a loop  $e'$  interlaced with  $e$ .

An edge is called special if it is either a bridge or a loop. A ribbon graph is called a terminal form when it contains only special edges.

**Ribbon graph operations.** Let us first address the notion of contraction and deletion for ribbon edges: Let  $\mathcal{G}$  be a ribbon graph and  $e$  one of its edges.

- We call  $\mathcal{G} - e$  the ribbon graph obtained from  $\mathcal{G}$  by deleting  $e$ .

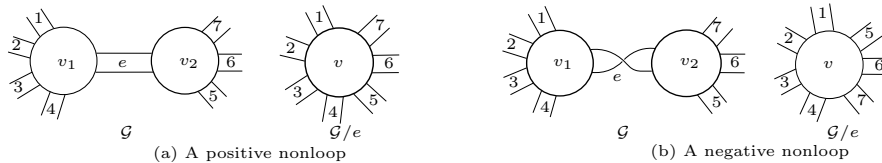


Figure 13: Non-loop edge contractions.

- If  $e$  is not a loop and is positive, consider its end-vertices  $v_1$  and  $v_2$ . The graph  $\mathcal{G}/e$  obtained by contracting  $e$  is defined from  $\mathcal{G}$  by replacing  $e$ ,  $v_1$  and  $v_2$  by a single vertex disc  $e \cup v_1 \cup v_2$  [75]. If  $e$  is a negative non-loop, then untwist it (by flipping one of its incident vertex) and contract. Both contractions are illustrated in Fig.13.

- If  $e$  is a trivial twisted loop, contraction is deletion:  $\mathcal{G} - e = \mathcal{G}/e$ . The contraction of a trivial untwisted loop  $e$  is the deletion of  $e$  and the addition of a new connected component vertex  $v_0$  to the graph  $\mathcal{G} - e$ . We write  $\mathcal{G}/e = (\mathcal{G} - e) \sqcup \{v_0\}$  (see Fig.14).

- If  $e$  is general loop (not necessarily trivial), the definition of a contraction becomes a little bit more involved. One way to address this can be done within the framework of arrow presentations [75]. In the end, the result can be simply described as follows:

- if the loop is positive (orientable), the vertex splits into two parts which were previously separated by the edge  $e$  in the vertex. Each new vertex has the same ribbons in the same cyclic order that they appeared before (see Fig.15a);

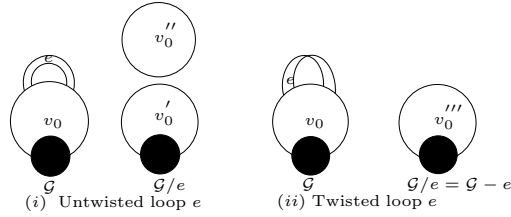


Figure 14: (i) The contraction of the untwisted trivial loop  $e$  generates two separate graphs one of which is a vertex. (ii) The contraction of the trivial twisted loop  $e$  in  $\mathcal{G}$  is the same as its deletion.

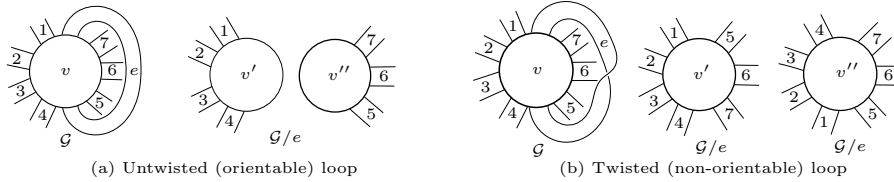


Figure 15: General loop contractions.

- if the loop is negative (non-orientable), then the vertex does not split. Consider the part  $\alpha$  and  $\beta$  on the vertex which are separated by the edge (see Fig.15b,  $\alpha = \{1, 2, 3, 4\}$  and  $\beta = \{5, 6, 7\}$ ). The result of the contraction is given by the graph obtained after removing  $e$  and drawing on a new vertex  $v'$  the part  $\alpha$  in the same cyclic order and the part  $\beta$  drawn in opposite cyclic order. Note that using a vertex flip on  $v'$ , one could achieve the equivalent vertex configuration  $v''$  obtained by reversing the role of  $\alpha$  and  $\beta$ .

In practice, we will be interested in generic situations listed in Fig.16.

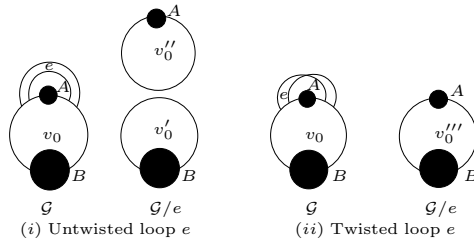


Figure 16: (i) Contraction of the untwisted  $e$  in  $\mathcal{G}$  generates two separate graphs. (ii) Contraction of the twisted  $e$  in  $\mathcal{G}$  generates one graph.

In this context of loop contraction, one can also introduce the concept of hard contraction removing extra discs generated. There exist other types of operations that are useful in ordinary graph theory and extends to ribbon graphs. In our developments, we will only need the disjoint union of graphs  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  which needs no comment.

**Definition 2** (Faces [71]). *A face is a component of a boundary of  $\mathcal{G}$  considered as a geometric ribbon graph, and hence as a surface with boundary.*

Note that vertex graph made with one disc has one face.

The notion of ribbon graphs being properly introduced, we can proceed further and define an extended class of ribbon graphs. The class in question is called the class of ribbon graphs

with half-ribbons. In the work by Krajewski et al. [69], the authors called these graphs ribbon graphs with flags.

**Definition 3** (Half-ribbons and half-edges). *A half-ribbon is a ribbon incident to a unique vertex by a unique segment and without forming a loop. (An illustration is given in Figure 17.)*

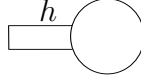


Figure 17: A half-ribbon  $h$  incident to one vertex disc.

As opposed to ribbon edges, we do not assign any orientation to half-ribbons.

**Definition 4** (Cut of a ribbon edge [69]). *Let  $\mathcal{G}$  be a ribbon graph and let  $e$  be one of its ribbon edge. The cut graph  $\mathcal{G} \vee e$ , is the graph obtained by removing  $e$  and let two half-ribbons attached at the end vertices of  $e$  (see Fig.18). If  $e$  is a loop, the two half-ribbons are on the same vertex.*



Figure 18: Cutting a ribbon edge.

- A half-ribbon generated by the cut of a ribbon edge is called a half-ribbon edge, but sometimes it will be simply referred to as half-edge.

- A ribbon graph with half-ribbons is a ribbon graph together with a set of half-ribbons attached to its discs.

- The set of half-ribbons is denoted by  $\mathcal{HR}$  (with cardinal  $HR$ ) and it includes the set of half-edges by  $\mathcal{HE}$  (with cardinal  $HE$ ). The rest of the half-ribbons will be called flags and denoted by  $\mathcal{FL}$  (with cardinal  $FL$ ). Thus  $\mathcal{HR} = \mathcal{HE} \cup \mathcal{FL}$ .

Precisions must be now given on the equivalence relation of ribbon graphs we will be working on. First, one must extend the notion of cyclic graphs to cyclic graphs with half-edges (the notion of “half-edge” in simple graph theory exists). Then two ribbon graphs with half-ribbons are isomorphic if there exist a series of vertex flips composed with isomorphisms of cyclic graphs with half-edges which transform one into the other.

The cut of a ribbon edge modifies the boundary faces of the ribbon graph. After the procedure, the new boundary faces follow the contour of the half-ribbons. It is always possible to introduce a distinction between this type of new faces and the initial ones. We will give a precision on this below.

As defined in Section 2.1, the notion of open and closed graphs and their constituents (forgetting the coloring) can be also addressed here. A closed ribbon graph does not have half-ribbons, otherwise it is called open. To harmonize our notations with Section 2.1 and make transparent the link with the above tensor models, we will explicitly draw half ribbons as two parallel strands, see Fig.19. We can now introduce a definition for closed or open face

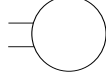


Figure 19: Stranded structure of a half-ribbon.

as simply closed or open strand, respectively. The notions of pinched and boundary graphs find equivalent notions in ribbon graphs. We will refrain to introduce more definitions at this point (Fig.20 illustrates an open ribbon graph, with open and closed faces, its pinched and boundary graph).

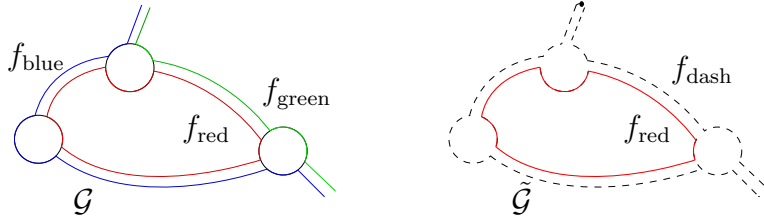


Figure 20: An open ribbon graph  $\mathcal{G}$  with a closed face  $f_{\text{red}}$  and open faces  $f_{\text{green,blue}}$  (with suggestive labels). The pinched graph  $\tilde{\mathcal{G}}$  and the boundary  $\partial\mathcal{G}$  of  $\mathcal{G}$  represented in dashed lines.

### 3 Parametric representation of amplitudes

We start by reviewing our notations for tensor models. From the following subsection, we present new results on the parametric form of the amplitudes of these models.

#### 3.1 Abelian rank $d$ models

Consider a rank  $d \geq 2$  complex field  $\varphi$  over the Lie group  $G_D = U(1)^D$ ,  $D \in \mathbb{N} \setminus \{0\}$ ,  $\varphi : (G_D)^d \rightarrow \mathbb{C}$ , decomposed in Fourier components as

$$\varphi(h_1, h_2, \dots, h_d) = \sum_{P_{I_s}} \tilde{\varphi}_{P_{I_1}, P_{I_2}, \dots, P_{I_d}} D^{P_{I_1}}(h_1) D^{P_{I_2}}(h_2) \dots D^{P_{I_d}}(h_d), \quad (2)$$

where  $h_s \in G_D$ . The sum is performed over all values of momenta  $P_{I_s}$ .  $P_{I_s}$  are labeled by multi-indices  $I_s$ , with  $s = 1, 2, \dots, d$ , where  $I_s$  defines the representation indices of the group element  $h_s$  in the momentum space.  $D^{P_{I_s}}(h_s)$  plays the role of the plane wave in that representation. More specifically, one has

$$h_s = (h_{s,1}, \dots, h_{s,D}) \in G_D, \quad h_{s,l} = e^{i\theta_{s,l}} \in U(1), \quad D^{P_{I_s}}(h_s) = \prod_{l=1}^D e^{ip_{s,l}\theta_{s,l}}, \quad p_{s,l} \in \mathbb{Z}$$

$$P_{I_s} = \{p_{s,1}, \dots, p_{s,D}\}, \quad I_s = \{(s, 1), \dots, (s, D)\}. \quad (3)$$

Concerning the tensor  $\tilde{\varphi}$ , we will simply use the notation  $\varphi_{[I]} := \tilde{\varphi}_{P_{I_1}, P_{I_2}, \dots, P_{I_d}}$ , where the super index  $[I]$  collects all momentum labels, i.e.  $[I] = \{I_1, I_2, \dots, I_d\}$ . Note that no symmetry

under permutation of the arguments is assumed for  $\varphi_{[I]}$ . We rewrite (2) in these shorthand notations as

$$\varphi(h_1, h_2, \dots, h_d) = \sum_{P_{[I]}} \varphi_{[I]} D^{I_1}(h_1) D^{I_2}(h_2) \dots D^{I_d}(h_d), \quad D^{I_s}(h_s) := D^{P_{I_s}}(h_s). \quad (4)$$

Restricting to  $d = 2$ ,  $\varphi_{I_1, I_2}$  will be referred to a matrix.

**Kinetic term.** Upon writing an action, we must define a kinetic term and, in the present higher rank models, several interactions. In the momentum space, we define as kinetic term for our model

$$S^{\text{kin}} = \sum_{P_{[I]}} \bar{\varphi}_{P_{[I]}} \left( \sum_{s=1}^d |P_{I_s}| + \mu \right) \varphi_{P_{[I]}}, \quad |P_{I_s}| := \sum_{l=1}^D |p_{s,l}|, \quad (5)$$

where the sum is performed over all values of the momenta  $p_{s,l} \in \mathbb{Z}$  and  $\mu \geq 0$  is a mass coupling constant.

In direct space formulation, the term (5) corresponds to a kinetic term defined by  $\sum_s |\Delta_s|^{\frac{1}{2}} + \mu$  and acts on the field  $\varphi$ . The non-integer power of the Laplacian can be motivated from several points of view:

- (i) With the exact power of momentum in the propagator, there exist rank  $d$  models that are renormalizable among which we have a rank 3 tensor model and several matrix models [27]. They will be the prototype models on which our following dimensional regularization procedure will be applied.
- (ii) From axiomatic quantum field theory, models with  $\Delta^a$ , where  $a \in (0, 1]$  are susceptible to be Osterwalder-Schrader positive [7, 9].
- (iii) To the above significant features, we add the fact that, with this power of the momenta, the parametric amplitudes of the models find a summable and tractable formula with interesting properties worth to be investigated in greater details.

Passing to the quantum realm, we introduce a Gaussian measure on the tensor fields as  $d\nu_C(\varphi, \bar{\varphi})$  with a covariance given by

$$C[\{P_{I_s}\}, \{\tilde{P}_{I_s}\}] = \left[ \prod_{s=1}^d \delta_{P_{I_s}, \tilde{P}_{I_s}} \right] \left( \sum_{s=1}^d |P_{I_s}| + \mu \right)^{-1}, \quad (6)$$

such that,  $\delta_{P_{I_s}, \tilde{P}_{I_s}} := \prod_{l=1}^D \delta_{p_{s,l}, \tilde{p}_{s,l}}$ . Using the Schwinger trick, the covariance can be recast as

$$C[\{P_{I_s}\}, \{\tilde{P}_{I_s}\}] = \left[ \prod_{s=1}^d \delta_{P_{I_s}, \tilde{P}_{I_s}} \right] \int_0^\infty d\alpha e^{-\alpha(\sum_{s=1}^d |P_{I_s}|^a + \mu^2)}. \quad (7)$$

The propagator is represented by a line made as a collection of  $d$  strands, see Fig.1.

**Interactions.** Depending on the rank  $d$ , two types of interactions dictated by the possible notions of invariance will be discussed.

- In rank  $d \geq 3$ : the interactions of the models considered are effective interactions obtained after integrating  $d$  colors in the rank  $d + 1$  colored tensor model [54] as discussed in Section 2.1 (for a complete discussion, we refer to [27]). The above field  $\varphi$  is nothing but the remaining field  $\varphi^0 = \varphi$ . An interaction term is defined from unsymmetrized tensors as unitary tensor invariant objects and built from the particular convolution of arguments of some set of tensors  $\varphi_{[I]}$  and  $\bar{\varphi}_{[I']}$ . Such a contraction is performed only between the  $s^{th}$  label of some  $\varphi_{[I]}$  to another  $s^{th}$  label of some  $\bar{\varphi}_{[I']}$ . It turns out that the total contraction of these tensors follows the pattern of a connected  $d$ -colored graphs called  $d$ -bubbles denoted  $\mathbf{b}$  (we recall that  $p$ -bubble were introduced in Section 2.1; see Fig.21).

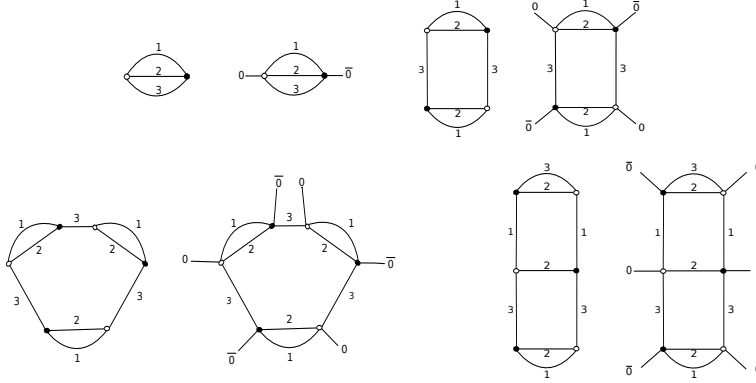


Figure 21: Colored 3-bubbles and their corresponding tensor invariants (in compact representation): The tensor fields are  $0$  and  $\bar{0}$  and are contracted according to the pattern of the 3-bubble they are associated with.

In rank  $d \geq 3$ , a general interaction can be written:

$$S^{\text{int}}(\varphi, \bar{\varphi}) = \sum_{\mathbf{b} \in \mathcal{B}} \lambda_{\mathbf{b}} I_{\mathbf{b}}(\varphi, \bar{\varphi}), \quad (8)$$

where the sum is over a finite set  $\mathcal{B}$  of rank  $d$  colored tensor bubble graphs and  $\lambda_{\mathbf{b}}$  is a coupling constant associated with that interaction. To each  $I_{\mathbf{b}}(\varphi, \bar{\varphi})$  corresponds a vertex operator identifying incoming and outgoing momenta and is of the form of a product of delta functions. In Fig.21, we have illustrated some of these tensor invariants in rank 3 models.

- In rank  $d = 2$  or matrix models, the interactions are simply trace invariants in the ordinary sense:

$$S^{\text{int}}(\varphi, \bar{\varphi}) = \sum_{p=2}^{p_{\text{max}}} \lambda_p S_p^{\text{int}}(\varphi, \bar{\varphi}), \quad S_p^{\text{int}}(\varphi, \bar{\varphi}) = \text{tr}[(\bar{\varphi}\varphi)^p], \quad (9)$$

where  $\lambda_p$  stands for a coupling constant. Graphically, each term in (9) is represented by a cyclic graph with  $p$  external legs, see Fig.22. One might wonder how the graphs obtained in matrix models relate to the ribbon graphs with flags explained earlier in Section 2.2. The answer to this is simple since one maps the vertices of matrix models to discs with half-ribbons (see Fig.23) whereas propagators are viewed as ribbon lines. In order to achieve the mapping, one must attach the vertex/propagator data to the abstract discs with half ribbons and ribbon lines.



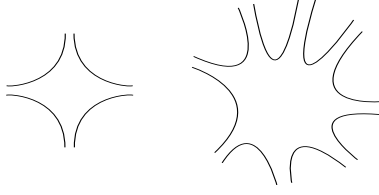


Figure 22: Examples of matrix model cyclic invariants:  $\text{tr}(\varphi^4)$  (left) and  $\text{tr}(\varphi^8)$  (right).

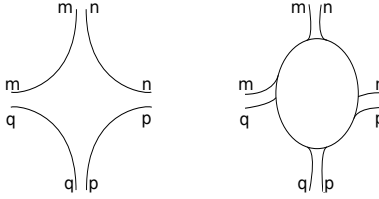


Figure 23: A  $\text{tr}(\varphi^4)$  vertex as a disc with half-ribbons and the set of data.

### 3.2 Parametric amplitudes

The partition function of any models described above is of the form

$$Z = \int d\nu_C(\varphi, \bar{\varphi}) e^{-S^{\text{int}}(\varphi, \bar{\varphi})}, \quad (10)$$

where  $C$  is given by (7) and  $S^{\text{int}}$  given either by (8) for rank  $d \geq 3$  or by (9) in the case  $d = 2$ .

As it is in the ordinary case, Feynman amplitudes are obtained from Wick's theorem. We compute for any connected graph  $\mathcal{G}$  made with the set  $\mathcal{L}$  of lines and the set  $\mathcal{V}$  of vertices, the amplitude

$$A_{\mathcal{G}} = \lambda_{\mathcal{G}} \sum_{P_{[I](v)}} \prod_{\ell \in \mathcal{L}} C_{\ell}[\{P_{I_s(\ell)}; v(\ell)\}, \{\tilde{P}_{I_s(\ell)}; v'(\ell)\}] \prod_{v \in \mathcal{V}; s} \delta_{P_{I_s}; v; P'_{I_s}; v}, \quad (11)$$

where  $\lambda_{\mathcal{G}}$  incorporates all coupling constants and the symmetry factors, and where the sum is performed over all values of the momenta  $P_{[I](v)}$  associated with vertices  $v$  on which the propagator lines are incident. The propagators  $C_{\ell}$  possess line labels  $\ell \in \mathcal{L}$ .

Due to the fact that vertex operators and propagators are product of delta's enforcing conservation of momenta along a strand, the amplitude (11) factorizes in terms of connected strand components (faces) of the graph. There exist two types of faces: open faces the set of which will be denoted by  $\mathcal{F}_{\text{ext}}$  (with cardinal  $F_{\text{ext}} = |\mathcal{F}_{\text{ext}}|$ ) and closed faces (or closed strands) the set of which will be denoted by  $\mathcal{F}_{\text{int}}$  (with cardinal  $F_{\text{int}} = |\mathcal{F}_{\text{int}}|$ ). Evaluating (11) using (7), one gets

$$A_{\mathcal{G}} = \lambda_{\mathcal{G}} \sum_{P_{I_f}} \int \left[ \prod_{\ell \in \mathcal{L}} d\alpha_{\ell} \right] \left\{ \prod_{f \in \mathcal{F}_{\text{ext}}} \left[ e^{-(\sum_{\ell \in f} \alpha_{\ell}) |P_{I_f}^{\text{ext}}|} \right] \prod_{f \in \mathcal{F}_{\text{int}}} \left[ e^{-(\sum_{\ell \in f} \alpha_{\ell}) |P_{I_f}|} \right] \right\}, \quad (12)$$

where  $P_{I_f}^{\text{ext}}$  are external momenta (not summed and labeled by external faces) and the sum is over all values of internal momenta  $P_{I_f}$  (indexed by internal faces).

It turns out that, from the linear dependency in momenta of the propagator, all momentum dependency in the amplitude can be summed. The following proposition holds:

**Proposition 1.** *Let  $\mathcal{G}$  be graph,  $\mathcal{L}_{\mathcal{G}}$  its set of lines,  $\mathcal{F}_{\text{int};\mathcal{G}}$  its set of internal faces,  $\mathcal{F}_{\text{ext};\mathcal{G}}$  its set of external faces, we denote the cardinal  $L_{\mathcal{G}} = |\mathcal{L}_{\mathcal{G}}|$ . Then, the amplitude  $A_{\mathcal{G}}$  of  $\mathcal{G}$  is given by*

$$A_{\mathcal{G}} = c \lambda_{\mathcal{G}} \int_{[0,1]^{L_{\mathcal{G}}}} \left[ \prod_{l \in \mathcal{L}_{\mathcal{G}}} dt_l \frac{(1-t_l)^{\mu-1}}{(1+t_l)^{\mu+1}} \right] \frac{W_{\mathcal{G}}(\{m_f\}; \{t_l\})}{[U_{\mathcal{G}}^{\text{od}}(\{t_l\})]^D}, \quad (13)$$

where  $c = 2^{L_{\mathcal{G}}}$  is an inessential factor, and

$$W_{\mathcal{G}}(\{m_f\}; \{t_l\}) = \tilde{W}_{\mathcal{G}}(\{m_f\}; \{t_l\}) \left[ U_{\mathcal{G}}^{\text{ev}}(\{t_l\}) \right]^D, \quad (14)$$

$$\begin{aligned} \tilde{W}_{\mathcal{G}}(\{m_f\}; \{t_l\}) &= \prod_{f \in \mathcal{F}_{\text{ext};\mathcal{G}}} \left( A_f^{\text{ext}} \right)^{|m_f|}, & A_f^{\text{ext}} &\equiv \prod_{l \in f} \frac{1-t_l}{1+t_l}, \\ U_{\mathcal{G}}^{\text{ev/od}}(\{t_l\}) &= \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}} A_f^{\text{ev/od}}, & A_f^{\text{ev/od}} &\equiv \sum_{A \subset f} \prod_{l \in A} t_l, \end{aligned} \quad (15)$$

|A| even/odd

where  $m_f$  is the external momentum associated with an open face  $f$ .

*Proof.* For any connected graph  $\mathcal{G}$ , we re-write the amplitude (12) as

$$A_{\mathcal{G}} = \lambda_{\mathcal{G}} \int \left[ \prod_{l \in \mathcal{L}_{\mathcal{G}}} d\alpha_l e^{-\alpha_l \mu} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext};\mathcal{G}}} e^{-(\sum_{l \in f} \alpha_l) |m_f|} \right] \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}} \left[ \frac{1 + e^{-(\sum_{l \in f} \alpha_l)}}{1 - e^{-(\sum_{l \in f} \alpha_l)}} \right]^D. \quad (16)$$

Now, we change variable as

$$t_l = \tanh \frac{\alpha_l}{2}, \quad (17)$$

and obtain

$$\begin{aligned} A_{\mathcal{G}} &= 2^{L_{\mathcal{G}}} \int \left[ \prod_{l \in \mathcal{L}_{\mathcal{G}}} dt_l \frac{(1-t_l)^{\mu-1}}{(1+t_l)^{\mu+1}} \right] \tilde{W}_{\mathcal{G}}(\{m_f\}; \{t_l\}) \frac{\left( \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}} \frac{\prod_{l \in f} (1+t_l) + \prod_{l \in f} (1-t_l)}{2} \right)^D}{\left( \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}} \frac{\prod_{l \in f} (1+t_l) - \prod_{l \in f} (1-t_l)}{2} \right)^D}, \\ \tilde{W}_{\mathcal{G}}(\{m_f\}; \{t_l\}) &= \prod_{f \in \mathcal{F}_{\text{ext};\mathcal{G}}} \prod_{l \in f} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|}. \end{aligned} \quad (18)$$

Using now the definitions (15), we can infer that the numerator in the amplitude is given by  $\tilde{W}_{\mathcal{G}}(U_{\mathcal{G}}^{\text{ev}})^D$ , in other words (14), and the denominator can be further expanded and yields  $U_{\mathcal{G}}^{\text{od}}$  (15). □

The formulas (13) and (16) provide, for any rank  $d$  model over  $G_D$  with a propagator linear in momentum, the parametric amplitude for a graph  $\mathcal{G}$ . The parametric form (13) appears more adapted to our following developments. For the reduced rank  $d = 2$ , the same parametric amplitudes do not fully coincide with the analog amplitudes of the GW model

[78, 79, 80] in the matrix basis neither in 2D nor in 4D [69]. The reason this occurs comes from the fact that the GW model in the matrix basis is described in terms of matrices  $M_{m,n}$  with indices  $n$  and  $m$  having values only in positive integers  $\mathbb{N}$  (2D) or  $\mathbb{N}^2$  (4D). In order to recover the amplitudes for the GW models from (13), one must replace in  $W_{\mathcal{G}}$ ,  $U_{\mathcal{G}}^{\text{ev}}(\{t_l\})$  by  $c' \prod_{f \in \mathcal{F}_{\text{int}; \mathcal{G}}} \prod_{l \in f} (1 + t_l)$  with  $c'$  a inessential factor  $2^{-D\mathcal{F}_{\text{int}; \mathcal{G}}}$  which should be combined with  $c = 2^{L_{\mathcal{G}}}$ .

The polynomials  $U^{\text{od/ev}}$  appear as a product over faces of some other polynomials. The following analysis rests strongly on this face structure.

**Definition 5** (Odd, even and external face polynomial). *Let  $f$  be an internal face in a tensor graph of the above models. We call  $A_f^{\text{od/ev}}$  (15) the odd/even face polynomial in the variables  $\{t_l\}_{l \in f}$  associated with  $f$ . If  $f$  is external, then we call  $A_f^{\text{ext}}$  the external face polynomial associated with  $f$  in the variable  $T_l = (1 - t_l)/(1 + t_l)$ .*

Some conventions must be set at this stage. For the empty graph  $\mathcal{G} = \emptyset$  (no vertex), we set  $U_{\mathcal{G}}^{\text{od}} = 1$  and  $U_{\mathcal{G}}^{\text{ev}} = \tilde{W}_{\mathcal{G}} = W_{\mathcal{G}} = 1$ . Consider the vertex as a simple disc. As a graph we will denote it by  $\mathcal{G} = o$ . It has one closed face  $f$  and, for such a graph, we set:

$$A_f^{\text{od}} = 0, \quad A_f^{\text{ev}} = 1. \quad (19)$$

As a result, for the vertex graph  $\mathcal{G} = o$ , we set  $U_{\mathcal{G}}^{\text{od}} = 0$ ,  $U_{\mathcal{G}}^{\text{ev}} = 1$ , and  $\tilde{W}_{\mathcal{G}} = 1 = W_{\mathcal{G}}$ . Furthermore, there exist open faces which do not have any lines. For these types of faces, we set

$$A_f^{\text{ext}} = 1. \quad (20)$$

Now, for a graph  $\mathcal{G}$  without any lines but external faces, we have  $U_{\mathcal{G}}^{\text{od}} = 1 = U_{\mathcal{G}}^{\text{ev}}$  and  $\tilde{W}_{\mathcal{G}} = 1 = W_{\mathcal{G}}$ .

Consider two distinct graphs  $\mathcal{G}_1$  and  $\mathcal{G}_2$ , we have

$$U_{\mathcal{G}_1 \sqcup \mathcal{G}_2}^{\text{od/ev}} = U_{\mathcal{G}_1}^{\text{od/ev}} U_{\mathcal{G}_2}^{\text{od/ev}}, \quad \tilde{W}_{\mathcal{G}_1 \sqcup \mathcal{G}_2} = \tilde{W}_{\mathcal{G}_1} \tilde{W}_{\mathcal{G}_2}. \quad (21)$$

From this rule, a drastic consequence follows: for any graph  $\mathcal{G}$ ,  $U_{\mathcal{G} \sqcup o}^{\text{od}} = U_{\mathcal{G}}^{\text{od}} U_o^{\text{od}} = 0$ . This means that to (soft) contract arbitrary edges in a graph might lead to vanishing polynomials on the resulting graphs. Thus, one can have severe implications on the amplitudes of contracted graphs that we will aim at studying in the following section. Nevertheless, this present convention makes transparent the analysis of polynomials undertaken in Section 5. In any case, there should exist a set of conventions (for e.g. setting  $U_o^{\text{od}} = 1$ ), under which the following amplitude analysis should be valid and the analysis of polynomials should be slightly re-adjusted. In the next section, we will use hard contractions on rank  $d$  graphs and these, by definition, do not generate discs to avoid any issues.

## 4 Dimensional regularization and renormalization

In this section, we start the investigation of the parametric amplitudes in view of a dimensional regularization and its associated renormalization procedure.

The idea of the subsequent procedure can be considered as a “classic” in the field [67, 81, 66]. It also proves to be powerful enough for nonlocal theories [82, 83] and can even lead to further applications in noncommutative field theory [84, 85]. Let us review quickly this method in the ordinary field theoretical formalism.

Using a parametric form of the quantum field amplitudes in a  $\mathbf{d}$  dimensional spacetime, the dimension  $\mathbf{d}$  appears as an explicit parameter in these amplitudes and, as such, can be complexified. First, one must show that there exists a complex domain in  $\mathbf{d}$  (which can be small) which guarantees the convergence of all amplitudes and their analytic structure. Then, one extends the domain and show that the only possible divergences occurring in the amplitudes are located at distinct values of  $\mathbf{d}$  involving only isolated poles. As functions of  $\mathbf{d}$  on this extended domain, amplitudes are therefore meromorphic. From this point, the so-called amplitude regularization can be undertaken by removing the problematic infinite contributions using a neat subtraction operator. This operator acts on the amplitudes and leads to finite and analytic integrals on the whole meromorphicity domain. The new amplitudes are called renormalized.

To be complete, it is noteworthy to signal that, in order to prove the meromorphic structure of the Feynman amplitudes, there are at least two known ways. One of the methods uses the so-called complete Mellin representation of the parametric amplitudes [86, 87, 88] (which can be applied to the context of noncommutative field theory [89]) and the other introduces the method of Hepp sectors [67, 81] and factorization techniques. The first approach in the present context leads to peculiarities which need to be understood. Using the second path, one discovers that the method is well defined and finds a non-trivial counterpart for, at least, some just-renormalizable tensor models. We, thereafter, focus on this second alternative.

## 4.1 Regularization using Hepp sectors

We now proceed with the dimensional regularization scheme. Using Hepp sectors (or a meaningful subgraphs’ decomposition) of the amplitude, one can identify the singular part of any diverging amplitude. The singular part is expressed in terms of the complexified dimension  $D$ . It is important to first study the factorization properties of the amplitudes in terms of divergent subgraphs.

Our main concern is the regularization of the integral (13) when  $t_l \rightarrow 0$  corresponding to the UV (ultraviolet) limit of the model. One notices that when  $t_l \rightarrow 1$ , the integral is divergent when the mass  $\mu$  is bounded as  $0 \leq \mu < 1$  and if all external momenta  $|m_f|$  are equally put to 0. For a massive field theory, one can assume the mass to be strictly larger than 1 with no loss of generality, and for a massless field theory, one can define fields without 0-momentum modes. In the direct space formalism [14], the same limit  $t_l \rightarrow 1$  corresponds to an IR (infrared) limit, and the amplitude turns out to be bounded simply because of the compactness of  $U(1)^D$ . Given these reasons and since we discuss UV divergences, we will only investigate  $t_l \rightarrow 0$ .

In the following, we are interested in Abelian models (i.e.  $G_D = U(1)^D$ ) with a kinetic term of the form  $\sum_s |P_s| + \mu$ . A generic model will be written as  ${}_D\Phi_d^{k_{\max}}$  where  $D$  refers to the dimension of the group  $G_D$ ,  $k_{\max}$  to the maximal valence of the vertices, and  $d$  to the theory rank. According to the analysis [27], only the following models respect these conditions and

are perturbatively renormalizable (at all orders):

$$\begin{aligned}
& {}_1\Phi_3^4, \quad G_D = U(1) \quad (\text{Just-renormalizable [20]}); \\
& {}_2\Phi_2^4, \quad G_D = U(1)^2 \quad (\text{Just-renormalizable}); \\
& \forall n \geq 2, \quad {}_1\Phi_2^{2n}, \quad G_D = U(1) \quad (\text{Super-renormalizable}).
\end{aligned} \tag{22}$$

We refer the last family of models  ${}_1\Phi_2^{2n}$  to a tower of models parametrized by the maximal valence of its vertices  $k_{\max} = 2n$ . The matrix interactions are, as discussed in the previous section, single trace invariants. For the model  ${}_1\Phi_3^4$ , the type of tensor invariant interactions that one considers are constructed with 4 tensors contracted according to the pattern of a 3-bubble colored graph made with 4 vertices (2 white and 2 black, see Fig.21). There are 3 colored symmetric connected invariants of this type. Fully expanded, one of these invariants is drawn in Fig.24. The rest of the invariants participating to the interaction of  ${}_1\Phi_3^4$  can be obtained by color symmetry.

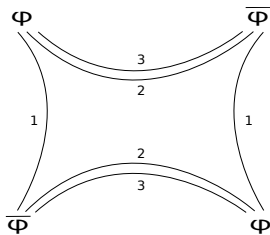


Figure 24: A tensor invariant  $\varphi^4$ .

The graph amplitudes in rank  $d \geq 2$  TGFTs were studied using multi-scale analysis in [27]. In this work, we provide a new and independent way of regularizing these divergent graphs using now the particular form of their parametric amplitude representation and their underlying meromorphic structure.

#### 4.1.1 Factorization of the amplitudes

A particular factorization property of the parametric amplitudes is now investigated. Such a factorization is necessary for undertaking the subsequent renormalization procedure of the models (22).

The key idea is the following: we assign scales to propagators in a graph  $\mathcal{G}$  and define the corresponding Hepp sectors. Choose a subgraph  $S$  in  $\mathcal{G}$  and contract all its lines to give  $\mathcal{G}/S$ . The interesting case is when  $S$  is primitively divergent (determined by a set of conditions on the graph  $S$ ). Roughly speaking, one must prove that the amplitude  $A_{\mathcal{G}}$  factorizes in two contributions: one determined by  $A_S$  and the other  $A_{\mathcal{G}/S}$  such that by replacing  $S$  by a (counter) term of the Lagrangian, the integral  $A_{\mathcal{G}/S}$  becomes finite. This factorization plays a crucial role in the definition of a co-product for the Connes-Kreimer Hopf algebra structure intimately associated with the renormalization of the model (see [90, 91] for seminal works). How this applies to tensor models can be found in [28]. For recent approaches in the framework of noncommutative field theory, one can consult [92, 93]).

We shall need some information about the scaling of the polynomials  $U^{\text{od/ev}}$ . A specific terminology and more notations are now introduced:

- We strengthen the notations  $\mathcal{L}_{\mathcal{G}} = \mathcal{L}(\mathcal{G})$  and  $\mathcal{F}_{\text{int};\mathcal{G}} = \mathcal{F}_{\text{int}}(\mathcal{G})$  making explicit the dependence on the graph  $\mathcal{G}$ .

- A subgraph  $S$  of  $\mathcal{G}$  is defined by a subset  $\mathcal{L}(S)$  of lines of  $\mathcal{G}$  and their incident vertices and cutting all remaining lines incident to these vertices. Thus, from the field theory point of view, we will always consider a “subgraph” as a “cutting subgraph”.

- We call a divergent subgraph  $S$  of  $\mathcal{G}$  a subgraph of  $\mathcal{G}$ , such that  $A_S$  is divergent. There is a set on conditions under which it occurs. We will come back on these in a subsequent section.

- We recall the following operations on subgraphs: Consider a subgraph  $S$  of  $\mathcal{G}$ .

• “Contraction” always refers in this section to hard contraction unless otherwise explicitly stated.

• Let  $\mathcal{G}$  be a graph and  $e$  be one of its edges (lines). The graph  $\mathcal{G}/e$  is defined as in Section 2 and is called the graph obtained after contraction of  $e$ .

• For connected  $S$ , the contracted graph  $\mathcal{G}/S$  is a graph obtained from the full contraction of the lines in  $S$  (see an illustration in Fig.25). If  $S$  is non connected, one must apply the same procedure to each connected component.

- Consider  $S \subset \mathcal{G}$ , strictly speaking,  $\mathcal{G}/S$  is not a subgraph of  $\mathcal{G}$ . The only point which prevents to regard  $\mathcal{G}/S$  as a subgraph of  $\mathcal{G}$  is the fact that it might contain one or several vertices which are not included in  $\mathcal{G}$ . These vertices come from the contraction of  $S$ . One notices that, by definition,  $\mathcal{L}(\mathcal{G}/S) = \mathcal{L}(\mathcal{G}) \setminus \mathcal{L}(S)$ .

Let us introduce notations for subsets of  $\mathcal{F}_{\bullet}(\mathcal{G})$ ,  $\bullet = \text{int}, \text{ext}$ .

**Definition 6** (Sets of faces). For all  $S \subset \mathcal{G}$ ,

-  $\mathcal{F}_{\bullet}^*(S)$  is the set of  $\bullet$ -faces in  $S$  having all their lines lying only in  $S$ , i.e.  $\forall f \in \mathcal{F}_{\bullet}^*(S)$ ,  $\forall l \in f$ ,  $l \in \mathcal{L}(S)$ .

-  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)$  is the subset of  $\bullet$ -faces of  $\mathcal{G}$  passing through at least one line of  $S$  and also through at least one line or vertex in  $\mathcal{G}/S$ . We have for this category of faces,  $\forall f \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)$ ,  $\exists (l, l') \in f \times f$  such that  $l \in \mathcal{L}(\mathcal{G}/S)$  and  $l' \in \mathcal{L}(S)$ ;

-  $\mathcal{F}''_{\bullet}(\mathcal{G}, S) = \mathcal{F}_{\bullet}(\mathcal{G}) \setminus (\mathcal{F}_{\bullet}^*(S) \cup \mathcal{F}'_{\bullet}(\mathcal{G}, S))$ .

-  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)/S$  denotes the set of  $\bullet$ -faces in  $\mathcal{G}$ , also in  $\mathcal{G}/S$ , coming from  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)$  and which are shortened after the contraction of  $S$ .

-  $\mathcal{F}_{\text{ext}}^*(S)/S$  is the set of external faces in  $\mathcal{G}$ , also in  $\mathcal{G}/S$ , resulting from  $\mathcal{F}_{\text{ext}}^*(S)$  after the contraction of  $S$ .

- Given  $e \in f$ , we denote  $f/e$  (resp.  $f - e$ ) the face resulting from  $f$  after the contraction (resp. the deletion) of  $e$  in  $\mathcal{G}$  yielding  $\mathcal{G}/e$  (resp.  $\mathcal{G} - e$ ). Given a subgraph  $S \subset \mathcal{G}$ , we denote  $f/S$  the face resulting from  $f$  in  $\mathcal{G}$  after successive contractions of all edges of  $S$ .

Some sets of faces as defined above for a ribbon graph  $\mathcal{G}$  and one of its subgraph  $S$  have been illustrated in Fig.25.

Few remarks can be spelled out:

- It is true that  $\mathcal{F}_{\text{int}}^*(S) = \mathcal{F}_{\text{int}}(S)$ , however,  $\mathcal{F}_{\text{ext}}^*(S) \neq \mathcal{F}_{\text{ext}}(S)$  as a general external face in  $S$  might have other lines in the larger graph  $\mathcal{G}$  or might even close in  $\mathcal{G}$ . Moreover, there are external faces which do not contain any lines. These are generated by strands in vertices which are not connected to any lines. For this type of faces, we impose  $f \in \mathcal{F}_{\text{ext}}^*(S)$  if the vertex attached to  $f$  is in  $\mathcal{V}(S)$ .

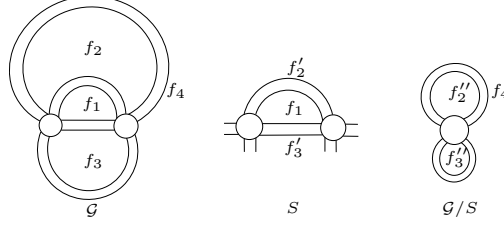


Figure 25: A ribbon graph  $\mathcal{G}$  and one of its subgraph  $S$ :  $\mathcal{F}_{\text{int}}(S) = \{f_1\}$ ,  $\mathcal{F}'_{\text{int}}(\mathcal{G}, S) = \{f_2, f_3\}$ , and  $\mathcal{F}''_{\text{int}}(\mathcal{G}, S) = \{f_4\}$ ;  $f'_{2,3} \in \mathcal{F}_{\text{ext}}(S)$ ;  $\mathcal{F}'_{\text{int}}(\mathcal{G}, S)/S = \{f''_2, f''_3\}$  and  $\mathcal{F}''_{\text{int}}(\mathcal{G}, S)/S = \{f_4\}$ . For  $\mathcal{G}/S$ ,  $\mathcal{F}_{\text{int}}(\mathcal{G}/S) = \{f''_2, f''_3, f_4\}$ .

- If the external face  $f$  does not pass through any lines, we say that  $f \in \mathcal{F}''_{\text{ext}}(\mathcal{G}, S)$  if the vertex touching  $f$  belongs to  $\mathcal{V}(\mathcal{G}) \setminus \mathcal{V}(S)$ .

- We define  $\mathcal{F}_{\bullet}(\mathcal{G})/S = \mathcal{F}_{\bullet}(\mathcal{G}/S)$ .

The following statement will be useful (the symbol  $\equiv$  below means “one-to-one”).

**Lemma 1** (Sets of faces decomposition). *Consider a subgraph  $S$  of a graph  $\mathcal{G}$ . We have*

$$\mathcal{F}_{\bullet}(\mathcal{G}) = \mathcal{F}_{\bullet}^*(S) \cup \mathcal{F}'_{\bullet}(\mathcal{G}, S) \cup \mathcal{F}''_{\bullet}(\mathcal{G}, S), \quad \bullet = \text{int, ext} . \quad (23)$$

The subsets  $\mathcal{F}_{\bullet}^*(S)$ ,  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)$  and  $\mathcal{F}''_{\bullet}(\mathcal{G}, S)$  are pairwise disjoint. Furthermore,

$$\mathcal{F}'_{\bullet}(\mathcal{G}, S) \equiv \mathcal{F}'_{\bullet}(\mathcal{G}, S)/S, \quad (24)$$

$$\mathcal{F}_{\text{int}}(\mathcal{G}/S) = (\mathcal{F}'_{\text{int}}(\mathcal{G}, S)/S) \cup \mathcal{F}''_{\text{int}}(\mathcal{G}, S), \quad (25)$$

$$\mathcal{F}_{\text{ext}}(\mathcal{G}/S) = (\mathcal{F}^*_{\text{ext}}(S)/S) \cup (\mathcal{F}'_{\text{ext}}(\mathcal{G}, S)/S) \cup \mathcal{F}''_{\text{ext}}(\mathcal{G}, S). \quad (26)$$

*Proof.* The soft contraction of a line in  $S$  only shortens faces. No faces can be created or destroyed by such a move. The number of faces must be conserved at the end of the soft contraction of all lines in  $S$ . Moreover, the “internal” or “external” nature of faces is preserved during the procedure. The result of a hard contraction can be inferred from this point.

We will focus on (24) and on (25), since the rest of the relations falls quite from the definitions.

- To prove (24), one must notice that we can associate with each element  $f \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)$  a line  $l_f$  in  $\mathcal{G}/S$  which is not touched by the (hard) contraction of  $S$ . This line ensures the one-to-one correspondence between an element in  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)$  and an element in  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)/S$  after (hard) contraction. Indeed, take  $f \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)$ , and  $\exists l_f \in \mathcal{L}(\mathcal{G}/S)$  such that  $l_f \in f$ . Then (hard) contract  $S$ , then  $l_f \in f/S$  and  $f/S \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)/S$ . Reciprocally, take  $f \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)/S$ , then, by definition  $\exists f_0 \in \mathcal{F}'_{\bullet}(\mathcal{G}, S)$  such that  $f_0/S = f$  and  $f_0$  is not empty, since by definition there must exist  $l \in \mathcal{L}(S)$  and  $l \in f_0$ . Note also that (24) does not depend on the type of contraction.

- To achieve (25), one notes that, after the complete hard contraction of all lines in  $S$ ,  $\mathcal{F}_{\text{int}}(S)$  is mapped to the empty set. Indeed, a closed face  $f$  in  $S$  either becomes shorter and shorter after (hard or soft) contraction whenever there still exists a line  $l \in f$ . At some point,  $f$  reaches a stage where it must generate a disc after soft contraction of its last line. Using hard contraction, this disc does not occur.

□

We focus now on the scaling properties of the polynomials  $U^{\text{od}/\text{ev}}$  and  $W$ . Consider  $S$  a subgraph of  $\mathcal{G}$ . Rescaling by  $\rho$  all variables  $t_l$  such that  $l \in \mathcal{L}(S)$ , one gets from  $U_{\mathcal{G}}^{\text{od}/\text{ev}}$  a new polynomial in  $\rho$ . We call  $U_{\mathcal{G}}^{\text{od}/\text{ev};\ell}$  the sum of terms with minimal degree in the expansion of  $U_{\mathcal{G}}^{\text{od}/\text{ev}}$ , and  $U_{\mathcal{G}}^{\text{od}/\text{ev};\ell(\rho)}$  the analogue sum for the minimal degree in  $\rho$  in the rescaled polynomial. Note that it is immediate to realize that

$$U_S^{\text{od};\ell(\rho)} = \rho^{F_{\text{int}}(S)} U_S^{\text{od};\ell}, \quad U_S^{\text{ev};\ell(\rho)} = 1 = U_S^{\text{ev};\ell}. \quad (27)$$

The following statement holds.

**Lemma 2** (Factorization of leading polynomials). *Consider a graph  $\mathcal{G}$  and a subgraph  $S$  of  $\mathcal{G}$ . Under rescaling  $t_l \rightarrow \rho t_l, \forall l \in \mathcal{L}(S)$ , we have*

$$U_{\mathcal{G}}^{\text{od};\ell(\rho)} = U_S^{\text{od};\ell(\rho)} U_{\mathcal{G}/S}^{\text{od}}, \quad (28)$$

$$U_{\mathcal{G}}^{\text{ev};\ell(\rho)} = U_{\mathcal{G}/S}^{\text{ev}}. \quad (29)$$

Performing a Taylor expansion in  $\rho$  around 0 of  $W_{\mathcal{G}}(\{m_f\}; \{\rho t_l\}_{l \in \mathcal{L}(S)}; \{t_l\}_{l \in \mathcal{L}(\mathcal{G}) \setminus \mathcal{L}(S)})$  and taking  $W_{\mathcal{G}}^{\ell(\rho)}$  as the lowest order in  $\rho$ , we have

$$W_{\mathcal{G}}^{\ell(\rho)}(\{m_f\}; \{t_l\}) = W_{\mathcal{G}/S}(\{m_f\}; \{t_l\}). \quad (30)$$

*Proof.* Computing the amplitude of  $\mathcal{G}/S$ , one must simply put to 0 some of the variables  $\alpha_l$   $l \in \mathcal{L}(S)$  in (12) and do not integrate over them. This expansion involves  $U_{\mathcal{G}/S}^{\text{od}}$  defined with  $\mathcal{F}_{\text{int}}(\mathcal{G}/S)$  as given by (26) in Lemma 1.

On the other hand, using (23) in Lemma 1, we can write the following expression for a partially rescaled polynomial  $U_{\mathcal{G}}^{\text{od}}$ ,

$$\begin{aligned} U_{\mathcal{G}}^{\text{od}}(\{\rho t_l\}_{l \in \mathcal{L}(S)}; \{t_l\}_{l \in \mathcal{L}(\mathcal{G}) \setminus \mathcal{L}(S)}) &= \left[ \prod_{f \in \mathcal{F}_{\text{int}}(S)} \dots \right] \left[ \prod_{f \in \mathcal{F}'_{\text{int}}} \dots \right] \left[ \prod_{f \in \mathcal{F}''_{\text{int}}} \dots \right] \\ &= \left[ \prod_{f \in \mathcal{F}_{\text{int}}(S)} (\rho \sum_{l \in f} t_l + \dots) \right] \left[ \prod_{f \in \mathcal{F}'_{\text{int}}} \left\{ \left( \sum_{l \in f \cap \mathcal{L}(\mathcal{G}/S)} t_l + \rho \sum_{l \in f \cap \mathcal{L}(S)} t_l \right) + \dots \right\} \right] \\ &\quad \times \left[ \prod_{f \in \mathcal{F}''_{\text{int}}} \dots \right]. \end{aligned} \quad (31)$$

At the smallest order in  $\rho$ , we collect from the first bracket  $U_S^{\text{od};\ell(\rho)}$  and from the two remaining brackets, after putting  $\rho = 0$  (this is similar to put  $\alpha_l = 0$ , for  $l \in \mathcal{L}(S)$ , in (12)) the polynomial  $U_{\mathcal{G}/S}^{\text{od}}$ . Thus (28) holds.

In order to find the second equality for  $U_{\mathcal{G}}^{\text{ev};\ell(\rho)}$  (29), we use the same decomposition (23) of Lemma 1 and (27).

We now perform a Taylor expansion around  $\rho = 0$  of the following expression (in suggestive though loose notations):

$$W_{\mathcal{G}}(\{m_f\}; \{\rho t_l\}_{l \in \mathcal{L}(S)}; \{t_l\}_{l \in \mathcal{L}(\mathcal{G}) \setminus \mathcal{L}(S)}) = \left[ \prod_{f \in \mathcal{F}_{\text{ext}}^*(S)} \dots \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}^L} \dots \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}''} \dots \right] (U_{\mathcal{G}}^{\text{ev}})^D$$



$$= \left[ \prod_{f \in \mathcal{F}_{\text{ext}}^*(S)} (1 + \rho \dots) \right] \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}} \{(1 + \rho \dots)\} \prod_{l \in f / l \in \mathcal{L}(\mathcal{G}/S)} \left( \frac{1 - t_l}{1 + t_l} \right)^{|m_f|} \right] \left[ \prod_{f \in \mathcal{F}''_{\text{ext}}} \dots \right] (U_{\mathcal{G}}^{\text{ev}})^D, \quad (32)$$

where we used (23) in Lemma 1. Now at minimal degree in  $\rho$ , we infer

$$W_{\mathcal{G}}^{\ell(\rho)}(\{m_f\}; \{t_l\}) = \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l \in f / l \in \mathcal{L}(\mathcal{G}/S)} \left( \frac{1 - t_l}{1 + t_l} \right)^{|m_f|} \right] \left[ \prod_{f \in \mathcal{F}''_{\text{ext}}} \dots \right] (U_{\mathcal{G}}^{\text{ev}; \ell(\rho)})^D \quad (33)$$

and one concludes using: (a) (29) to map  $U_{\mathcal{G}}^{\text{ev}; \ell(\rho)}$  onto  $U_{\mathcal{G}/S}^{\text{ev}}$ , (b)  $\mathcal{F}_{\text{ext}}(\mathcal{G}/S)$  from (26) in Lemma 1, and finally (c) observe that  $\mathcal{F}_{\text{ext}}^*(S)/S \subset \mathcal{F}_{\text{ext}}(\mathcal{G}/S)$  are external faces in the contracted graph  $\mathcal{G}/S$  which do not pass through any lines and by convention  $A_f^{\text{ext}} = 1$  (20).  $\square$

The preliminary factorization properties addressed in Lemma 2 will allow us to understand the most diverging part of the amplitude. However, in some cases, there exist sub-leading divergences which need to be renormalized as well. In particular, these kinds of divergences occur in the two-point function and the factorization must be extended up to higher orders in the scale parameter  $\rho$ . This is our next goal.

Consider a diverging subgraph  $S$  with internal lines with parameters  $t_l \in \mathcal{L}(S)$  which should be such that  $t_l \ll t'_l$  for any  $t'_l \in \mathcal{L}(\mathcal{G}/S)$ . This condition simply suggests that any internal propagator line is of higher energy than any external lines, as required in ordinary renormalization procedure. In this way, from the point of view of external legs the internal subgraph appears local. We will then perform a Taylor expansion on the variables  $t_l \in \mathcal{L}(S)$  but only on the faces which are in  $\mathcal{F}'_{\bullet}(\mathcal{G}, S)$  (that we simply denote henceforth  $\mathcal{F}'_{\bullet}$ ) which links the subgraph  $S$  to the rest of the graph and we will prove that, at each order, the result factorizes for small  $t_l$ .

Consider the notations:  $U_{\mathcal{G};S}^{\text{mod/ev}} = \prod_{f \in \mathcal{F}'_{\text{int}}} A_f^{\text{od/ev}}$  and  $U_{\mathcal{G};S}^{\text{od/ev}} = \prod_{f \in \mathcal{F}'_{\text{int}}} A_f^{\text{od/ev}}$ .

**Lemma 3** (Factorization of a  $N$ -point subgraph). *Consider a graph  $\mathcal{G}$  of a rank  $d$  model and a subgraph  $S$  of  $\mathcal{G}$  with external legs. For small  $t_l, \forall l \in \mathcal{L}(S)$ , we have*

$$\begin{aligned} A_{\mathcal{G}}(\{m_f\}; D) &= c\lambda_{\mathcal{G}} \int \left[ \prod_{l' \in \mathcal{L}(\mathcal{G}/S)} dt'_{l'} \frac{(1 - t'_{l'})^{\mu-1}}{(1 + t'_{l'})^{\mu+1}} \right] \tilde{W}_{\mathcal{G}/S}(\{t_{l'}\}) \left[ \frac{U_{\mathcal{G}/S}^{\text{ev}}}{U_{\mathcal{G}/S}^{\text{od}}}(\{t_{l'}\}) \right]^D \\ &\int \left[ \prod_{l \in \mathcal{L}(S)} dt_l \frac{(1 - t_l)^{\mu-1}}{(1 + t_l)^{\mu+1}} \right] \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}(S)} \prod_{l \in f} \left( \frac{1 - t_l}{1 + t_l} \right)^{|m_f|} \right] \left[ \frac{U_S^{\text{ev}}}{U_S^{\text{od}}}(\{t_l\}) \right]^D \\ &\left[ 1 - 2 \sum_{f \in \mathcal{F}'_{\text{ext}}} |m_f| R_{S,f}(\{t_l\}) + D \sum_{f \in \mathcal{F}'_{\text{int}}} M_f(\{t_{l'}\}) R_{S,f}(\{t_l\}) + O(t_l^2) \right], \\ R_{S,f}(\{t_l\}) &= \sum_{l \in f \cap \mathcal{L}(S)} t_l, \quad M_f(\{t_l\}_{l \in \mathcal{L}(\mathcal{G}/S)}) = \frac{(A_{f/S}^{\text{od}})^2 - (A_{f/S}^{\text{ev}})^2}{A_{f/S}^{\text{ev}} A_{f/S}^{\text{od}}}(\{t_l\}), \quad (34) \end{aligned}$$

where  $O(t_l^2)$  is a big- $O$  function of all possible products  $t_l t_{l'}$ , for  $l, l' \in \mathcal{L}(S)$ .

*Proof.* Consider a graph  $\mathcal{G}$  and fix one of its  $N$ -point subgraphs  $S$ . We write:

$$\begin{aligned}
A_{\mathcal{G}} &= c\lambda_{\mathcal{G}} \int \left[ \prod_{l' \in \mathcal{L}(\mathcal{G}/S)} dt_{l'} \frac{(1-t_{l'})^{\mu-1}}{(1+t_{l'})^{\mu+1}} \right] \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l' \in f} \left( \frac{1-t_{l'}}{1+t_{l'}} \right)^{|m_f|} \right] \left( \frac{U_{\mathcal{G};S}^{\text{ev}}}{U_{\mathcal{G};S}^{\text{od}}}(\{t_{l'}\}) \right)^D \\
&\int \left[ \prod_{l \in \mathcal{L}(S)} dt_l \frac{(1-t_l)^{\mu-1}}{(1+t_l)^{\mu+1}} \right] \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}(S)} \prod_{l \in f} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|} \right] \left[ \prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l \in f} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|} \right] \\
&\left( \frac{U_{\mathcal{G};S}^{\text{ev}}}{U_{\mathcal{G};S}^{\text{od}}}(\{t_l\}) \right)^D \left( \frac{U_S^{\text{ev}}}{U_S^{\text{od}}}(\{t_l\}) \right)^D. \tag{35}
\end{aligned}$$

We used  $l$  for a generic line label in  $\mathcal{L}(\mathcal{G})$ . Now, we perform a Taylor expansion on part of the factor  $\prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l \in f} (\dots)$  for small  $t_l$  only if  $l \in \mathcal{L}(S)$ . We obtain the contribution:

$$\begin{aligned}
\prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l \in f \cap \mathcal{L}(S)} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|} &= 1 - 2 \sum_{f \in \mathcal{F}'_{\text{ext}}} |m_f| R_{S,f}(t_l) + O(t_l^2), \\
R_{S,f}(t_l) &= \sum_{l \in f \cap \mathcal{L}(S)} t_l, \tag{36}
\end{aligned}$$

where  $l$  in  $O(t_l^2)$  only refers to the lines in  $\mathcal{L}(S)$ . Note that the remaining factors compile to

$$\prod_{f \in \mathcal{F}'_{\text{ext}}} \prod_{l \in f \cap \mathcal{L}(\mathcal{G}/S)} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|} = \prod_{f \in \mathcal{F}'_{\text{ext}}/S} \prod_{l \in f} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|}. \tag{37}$$

Focusing on the factor  $U^{\text{ev}}/U^{\text{od}}$ , we have

$$\begin{aligned}
U_{\mathcal{G};S}^{\text{ev/od}} &= \prod_{f \in \mathcal{F}'_{\text{int}}} (A_{f/S}^{\text{ev/od}} + \sum_{l \in f \cap \mathcal{L}(S)} t_l A_{f/S}^{\text{od/ev}} + O(t_l^2)) \\
&= \prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{ev/od}} + \sum_{f \in \mathcal{F}'_{\text{int}}} \left[ \prod_{f' \in \mathcal{F}'_{\text{int}}/f' \neq f} A_{f'/S}^{\text{ev/od}} \right] A_{f/S}^{\text{od/ev}} R_{S,f}(t_l) + O(t_l^2). \tag{38}
\end{aligned}$$

Thus, the ratio behaves like

$$\begin{aligned}
\frac{U_{\mathcal{G};S}^{\text{ev}}}{U_{\mathcal{G};S}^{\text{od}}} &= \frac{1}{\prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{od}}} \left( \prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{ev}} - \frac{\prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{ev}}}{\prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{od}}} \sum_{f \in \mathcal{F}'_{\text{int}}} \left[ \prod_{f' \in \mathcal{F}'_{\text{int}}/f' \neq f} A_{f'/S}^{\text{od}} \right] A_{f/S}^{\text{ev}} R_{S,f}(t_l) \right. \\
&\quad \left. + \sum_{f \in \mathcal{F}'_{\text{int}}} \left[ \prod_{f' \in \mathcal{F}'_{\text{int}}/f' \neq f} A_{f'/S}^{\text{ev}} \right] A_{f/S}^{\text{od}} R_{S,f}(t_l) + O(t_l^2) \right) \\
&= \frac{U_{\mathcal{G};S}^{\text{ev}}}{U_{\mathcal{G};S}^{\text{od}}} \left( 1 + \sum_{f \in \mathcal{F}'_{\text{int}}} \left[ \frac{(A_{f/S}^{\text{od}})^2 - (A_{f/S}^{\text{ev}})^2}{A_{f/S}^{\text{ev}} A_{f/S}^{\text{od}}} \right] R_{S,f}(t_l) + O(t_l^2) \right), \tag{39}
\end{aligned}$$

where we define  $U_{\mathcal{G};S}^{\text{od/ev}} := \prod_{f \in \mathcal{F}'_{\text{int}}} A_{f/S}^{\text{od/ev}}$ . One must use the bijection relation (24) in Lemma 1 to map  $\mathcal{F}'_{\text{int}}(\mathcal{G};S)$  to  $\mathcal{F}'_{\text{int}}(\mathcal{G};S)/S$  and we can write  $U_{\mathcal{G};S}^{\text{od/ev}} := \prod_{f \in \mathcal{F}'_{\text{int}}/S} A_f^{\text{od/ev}}$ . We insert (36), (37) and (39) in (35), and get the expansion

$$A_{\mathcal{G}} =$$

$$\begin{aligned}
& c\lambda_{\mathcal{G}} \int \left[ \prod_{l' \in \mathcal{L}(\mathcal{G}/S)} dt_{l'} \frac{(1-t_{l'})^{\mu-1}}{(1+t_{l'})^{\mu+1}} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}'' \cup \mathcal{F}'_{\text{ext}}/S} \prod_{l' \in f} \left( \frac{1-t_{l'}}{1+t_{l'}} \right)^{|m_f|} \right] \left[ \frac{U_{\mathcal{G};\mathcal{G}/S}^{\text{ev}} U_{\mathcal{G};S}^{\text{ev}}}{U_{\mathcal{G};\mathcal{G}/S}^{\text{od}} U_{\mathcal{G};S}^{\text{od}}}(\{t_{l'}\}) \right]^D \\
& \int \left[ \prod_{l \in \mathcal{L}(S)} dt_l \frac{(1-t_l)^{\mu-1}}{(1+t_l)^{\mu+1}} \right] \left[ \prod_{f \in \mathcal{F}_{\text{ext}}^*(S)} \prod_{l \in f} \left( \frac{1-t_l}{1+t_l} \right)^{|m_f|} \right] \left( \frac{U_S^{\text{ev}}}{U_S^{\text{od}}}(\{t_l\}) \right)^D \\
& \left[ 1 - 2 \sum_{f \in \mathcal{F}_{\text{ext}}^l} |m_f| R_{S,f}(t_l) + D \sum_{f \in \mathcal{F}'_{\text{int}}} \left[ \frac{(A_{f/S}^{\text{od}})^2 - (A_{f/S}^{\text{ev}})^2}{A_{f/S}^{\text{ev}} A_{f/S}^{\text{od}}} \right] R_{S,f}(t_l) + O(t_l^2) \right]. \tag{40}
\end{aligned}$$

Now using (26) in Lemma 1, we see that the complementary of  $\mathcal{F}_{\text{ext}}'' \cup \mathcal{F}'_{\text{ext}}/S$  in  $\mathcal{F}_{\text{ext}}(\mathcal{G}/S)$  is  $\mathcal{F}_{\text{ext}}^*/S$ . But for any  $f \in \mathcal{F}_{\text{ext}}^*/S$ ,  $A_f^{\text{ext}} = 1$ , so (34) becomes immediate.  $\square$

We can now interpret Lemma 3:

- At the leading 0th-order the amplitude  $A_{\mathcal{G}}(\{m_f\})$  factorizes as (in loose notations)

$$A_{\mathcal{G}}^0(\{m_f\}) = \left( \int [dt_l]_{\mathcal{L}(S)} \tilde{A}_S(\{t_l\}, \{m_f\}) \right) A_{\mathcal{G}/S}(\{m_f\}) \tag{41}$$

where  $\tilde{A}_S(\{t_l\}, \{m_f\})$  is not exactly the integrand of the amplitude of  $S$ , namely  $A_S(\{m_f\})$ . Their set of external faces and set of external momenta do not always match. However, the diverging or converging behavior of  $\int \tilde{A}_S$  and  $A_S$  are identical. Thus (41) means that, at this order of perturbation, the amplitude  $A_{\mathcal{G}} \simeq A_{\mathcal{G}}^0$  can be computed by evaluating the amplitude of  $\mathcal{G}/S$ , where we insert a counter-term (generalized) vertex obtained after the contraction of  $S$ , times the amplitude  $\int \tilde{A}_S$  of the divergent subgraph  $S$ .

- Up to the first order of perturbation, focusing on the internal variables associated with  $t_l$ ,  $l \in \mathcal{L}(S)$ , we re-express the types of contributions appearing in (34) as:

$$\begin{aligned}
A_{\mathcal{G}}^1(\{m_f\}) &= \left[ \int [dt_l]_{\mathcal{L}(S)} \tilde{A}_S(\{t_l\}; \{m_f\}) \left( 1 + B_S(\{t_l\}) \mathcal{O}_S(\{m_f\}) \right) \right] A_{\mathcal{G}/S}(\{m_f\}) \tag{42} \\
&+ \sum_{f \in \mathcal{F}'_{\text{int}}/S} \left( \int [dt_l]_{\mathcal{L}(S)} \tilde{A}_S(\{t_l\}; \{m_f\}) C_{S,f}(\{t_l\}) \right) \\
&\times \int [dt_{l'}]_{\mathcal{L}(\mathcal{G}/S)} M_f(\{t_{l'}\}) \tilde{A}_{\mathcal{G}/S}(\{m_f\}, \{t_{l'}\}),
\end{aligned}$$

where  $\tilde{A}_{\mathcal{G}/S}$  is the integrand of  $A_{\mathcal{G}/S}$ .

$\sim$  For a 2-point subgraph  $S$ , the term  $\int \tilde{A}_S B_S \mathcal{O}_S$  of the form  $\int \tilde{A}_S (\sum_f |m_f| \cdot R_{S,f})$  contributes to a wave function renormalization counter-term. In general, it is well-known that this counter-term  $|m_f| \int \tilde{A}_S R_{S,f}$  might have a different value for different external faces  $f$ . Therefore, it is not always true that  $\int \tilde{A}_S (\sum_f |m_f| \cdot R_{S,f}) = (\sum_f |m_f|) (\int \tilde{A}_S \cdot R_{S,f})$  where the last factor should be independent of  $f$ . In order to achieve a final wave function renormalization, we carefully sum symmetric graph contributions.

$\sim$  The remaining term  $\sum_f \int \tilde{A}_S C_{S,f} \times \int M_f \tilde{A}_{\mathcal{G}/S}$  will be subleading compared to  $\int \tilde{A}_S \times A_{\mathcal{G}/S}$ . Still in the case of  $N = 2$  and for the just-renormalizable models (22), this term may lead to a mass sub-leading divergence.

At this order of perturbation  $A_{\mathcal{G}} \simeq A_{\mathcal{G}}^1$  and its expansion organizes as follows. For a  $N$ -point divergent subgraph  $S$ , one inserts in  $\mathcal{G}/S$ , two types of counter-terms or operators:

a generalized “wave-function” vertex with amplitude  $|m_f| \int \tilde{A}_S R_{S,f}$  associated with each external face  $f$  of this vertex, and the vertex with amplitude  $\int \tilde{A}_S (1 + \sum_f C_{S,f}) \sim \int \tilde{A}_S$ .

- At higher order terms, the contributions will be sub-leading in the same way as explained above. For the remaining analysis, the study of higher order terms will only be sketched.

#### 4.1.2 Meromorphic structure of the regularized amplitudes

In this section, we consider a fixed graph  $\mathcal{G}$  and some of its subgraphs. We simplify notations and omit the dependency in the largest graph  $\mathcal{G}$  in integrands and several expressions when no confusion might occur, such that  $L = L(\mathcal{G})$ ,  $F_{\text{int}} = F_{\text{int}}(\mathcal{G})$  and so forth.

Take a Hepp sector  $\sigma$  such that

$$0 \leq t_1 \leq t_2 \leq \dots \leq t_L, \quad (43)$$

and perform the following change of variables

$$\forall l = 1, \dots, L, \quad t_l = \prod_{k=l}^L x_k. \quad (44)$$

Consider the subgraph  $G_i$  of  $\mathcal{G}$  defined by the lines associated with the variables  $t_j$ ,  $j = 1, \dots, i$ . We denote  $L(G_i) = i$ ,  $F_{\text{int}}(G_i)$  the number of lines and internal faces, respectively, of  $G_i$ . The amplitude (13) of Proposition 1 in the sector  $\sigma$  in terms of the variables  $x_l$  finds a new form;

$$\begin{aligned} A_{\mathcal{G}}^{\sigma}(\{m_f\}; D) &= \lambda_{c,\mathcal{G}} \int_{[0,1]^L} \left[ \prod_{l=1}^L dx_l \frac{(1 - \prod_{k=l}^L x_k)^{\mu-1}}{(1 + \prod_{k=l}^L x_k)^{\mu+1}} \right] \prod_{f \in \mathcal{F}_{\text{ext}}} \prod_{l \in f} \left( \frac{1 - \prod_{k=l}^L x_k}{1 + \prod_{k=l}^L x_k} \right)^{|m_f|} \\ &\times \left[ \prod_{i=1}^L x_i^{L(G_i)-1} \right] \left[ \frac{\tilde{U}_{\mathcal{G}}^{\text{ev}}(\{x_l\})}{\tilde{U}_{\mathcal{G}}^{\text{od}}(\{x_l\})} \right]^D, \end{aligned} \quad (45)$$

where  $\tilde{U}^{\text{od/ev}}$  are new polynomials obtained from  $U^{\text{od/ev}}$  after the substitution of Eq. (44). For the moment,  $D$  is real positive. In order to recover the full amplitude  $A_{\mathcal{G}}$  (13), one sums over all possible Hepp assignments:  $A_{\mathcal{G}} = \sum_{\sigma} A_{\mathcal{G}}^{\sigma}$ . Hereunder, we will focus on  $A_{\mathcal{G}}^{\sigma}$  and the last sum over  $\sigma$  will be addressed later.

Focusing on the denominator of the last line of (45), we want to extract the term of minimal degree in  $x_l$  in the polynomial. The term of minimal degree in  $t_l$  any face amplitude  $A_f^{\text{od}}$  is nothing but  $\sum_{l \in f} t_l$ . However, this term is not yet the term with minimal degree in  $x_k$ 's. To obtain the monomial of minimal degree in  $x_k$ , one picks  $t_f^0 = t_{l_f}^0$  with  $l_f^0 = \max_{l \in f} l$ . We have

$$\tilde{U}_{\mathcal{G}}^{\text{od}}(\{x_l\}) = \prod_{f \in \mathcal{F}_{\text{int}}} A_f^{\text{od}}(\{x_l\}) = \prod_{f \in \mathcal{F}_{\text{int}}} \left[ \prod_{\alpha=l_f^0}^L x_{\alpha} \right] \left( 1 + A_f^{\text{r}}(\{x_k\}) \right), \quad (46)$$

where  $A_f^{\text{r}}$  is the rest of the face amplitude after the factorization. Focusing on the first factor, it recasts as

$$\prod_{f \in \mathcal{F}_{\text{int}}} \left[ \prod_{\alpha=l_f^0}^L x_{\alpha} \right] = \prod_{\alpha=1}^L \prod_{f \in \mathcal{F}_{\text{int}} / \alpha \geq l_f^0} x_{\alpha} = \prod_{\alpha=1}^L x_{\alpha}^{|\{f \in \mathcal{F}_{\text{int}} / \alpha \geq l_f^0\}|}. \quad (47)$$

An internal face  $f$  in  $\mathcal{F}_{\text{int}}(G_i)$  is an internal face of  $\mathcal{G}$  such that its most higher index  $l_f^0$  among  $l \in f$  must be lower than  $i$ . We can conclude that  $|\{f \in \mathcal{F}_{\text{int}} / i \geq l_f^0\}| = F_{\text{int}}(G_i)$  and it is direct to get:

$$A_{\mathcal{G}}^{\sigma}(\{m_f\}, D) = \lambda_{c, \mathcal{G}} \int_{[0,1]^L} \left[ \prod_{l=1}^L dx_l \frac{(1 - \prod_{k=l}^L x_k)^{\mu-1}}{(1 + \prod_{k=l}^L x_k)^{\mu+1}} \right] \prod_{f \in \mathcal{F}_{\text{ext}}} \prod_{l \in f} \left( \frac{1 - \prod_{k=l}^L x_k}{1 + \prod_{k=l}^L x_k} \right)^{|m_f|} \\ \times \left[ \prod_{i=1}^L x_i^{L(G_i)-1-D F_{\text{int}}(G_i)} \right] \left[ \frac{\tilde{U}_{\mathcal{G}}^{\text{ev}}(\{x_l\})}{(1 + U'_{\mathcal{G}}(\{x_l\}))} \right]^D, \quad (48)$$

with  $U'_{\mathcal{G}}$  readily obtained from (46). The quantity

$$\omega_d(\mathcal{G}) = (L - D F_{\text{int}})(\mathcal{G}) \quad (49)$$

is called the convergence degree of the graph amplitude.

Before considering complex valued variables involved in this object, we will discuss better its constituents. In particular, the number of internal faces  $F_{\text{int}}(\mathcal{G})$  must be dissected. This number  $F_{\text{int}}(\mathcal{G})$  of a connected graph  $\mathcal{G}$  has been worked out in [24]. We have for a connected graph  $\mathcal{G}$  and in our case:

- In rank  $d \geq 3$ , introducing  $d^- = d - 1$ ,

$$F_{\text{int}}(\mathcal{G}) = -\frac{2}{(d^-)!} (\omega(\mathcal{G}_{\text{color}}) - \omega(\partial\mathcal{G})) - (C_{\partial\mathcal{G}} - 1) - \frac{d^-}{2} N_{\text{ext}} + d^- - \frac{d^-}{4} (4 - 2n) \cdot V, \quad (50)$$

where  $\mathcal{G}_{\text{color}}$  the so-called colored extension of  $\mathcal{G}$  in the sense of Subsection 2.1,  $\partial\mathcal{G}$  its boundary with number  $C_{\partial\mathcal{G}}$  of connected components,  $V_k$  its number of vertices of coordination  $k$ ,  $V = \sum_k V_k$  its total number of vertices,  $n \cdot V = \sum_k k V_k$  its number of half-lines exiting from vertices,  $N_{\text{ext}}$  its number of external legs. We call  $\omega(\mathcal{G}_{\text{color}}) = \sum_J g_{\tilde{J}}$  the degree of  $\mathcal{G}_{\text{color}}$ ,  $\tilde{J}$  is the pinched jacket associated with  $J$  a jacket of  $\mathcal{G}_{\text{color}}$ ,  $\omega(\partial\mathcal{G}) = \sum_{J_{\partial}} g_{J_{\partial}}$  is the degree of  $\partial\mathcal{G}$ . Specifically, in rank  $d = 3$ ,  $\omega(\partial\mathcal{G}) = g_{\partial\mathcal{G}}$ , since the boundary graph is a ribbon graph.

- In rank  $d = 2$ , using the Euler characteristics, the following holds

$$F_{\text{int}}(\mathcal{G}) = -2g_{\tilde{\mathcal{G}}} - (C_{\partial\mathcal{G}} - 1) - \frac{1}{2} (N_{\text{ext}} - 2) - \frac{1}{2} (2 - n) \cdot V, \quad (51)$$

where  $\tilde{\mathcal{G}}$  is the closed (pinched) graph associated with  $\mathcal{G}$  and we used the relation  $2L = n \cdot V - N_{\text{ext}}$ .

Thus, one can write both (50) and (51) under the form

$$F_{\text{int}}(\mathcal{G}) = -\frac{2}{(d^-)!} \mathbf{\Omega}(\mathcal{G}) - (C_{\partial\mathcal{G}} - 1) + d^- \tilde{F}_{\text{int}}(\mathcal{G}), \quad \tilde{F}_{\text{int}}(\mathcal{G}) = \frac{1}{2} (2 - N_{\text{ext}} + (n - 2) \cdot V), \quad (52)$$

where  $\mathbf{\Omega}(\mathcal{G}) = \omega(\mathcal{G}_{\text{color}}) - \omega(\partial\mathcal{G})$  if  $d = 3$ , and  $\mathbf{\Omega}(\mathcal{G}) = g_{\tilde{\mathcal{G}}}$  if  $d = 2$ .

Note that the number of internal faces does not depend on  $D$  but only on the combinatorics of the graph itself. From (52), a formula for  $\omega_d(\mathcal{G})$  can be easily obtained after substituting this expression in (49). However, in the following we are interested only in

bounds involving directly the degree of convergence. It is a non-trivial fact that, for any graph in this category of models one has (see [14] and its addendum [15]),

$$\text{either } \Omega(\mathcal{G}) = 0, \quad \text{or } \frac{2}{(d^-)!} \Omega(\mathcal{G}) \geq d - 2 \geq 0. \quad (53)$$

In a renormalization program, we are mainly interested in graphs with external legs. These are graphs with boundary, in other words graphs satisfying  $C_{\partial\mathcal{G}} \geq 1$ . Therefore, for any connected graph, the following is true for  $d^- \geq 1$ ,

$$F_{\text{int}}(\mathcal{G}) \leq d^- \tilde{F}_{\text{int}}(\mathcal{G}). \quad (54)$$

It is also a known fact that in any rank  $d \geq 3$ , the so-called melonic graphs defined such that  $\omega(\mathcal{G}_{\text{color}}) = 0$  with a melonic boundary, i.e.  $\omega(\partial\mathcal{G}) = 0$ , and with a unique connected component on the boundary saturate this bound. Therefore, the melonic graphs have a dominant power counting and this shows that (54) is an optimal bound. Matrix models are similar. The dominant amplitudes in power counting are those with a maximal number of internal faces. These are planar graphs with  $g_{\tilde{\mathcal{G}}} = 0$  and  $C_{\partial\mathcal{G}} = 1$ . Hence (54) is again saturated.

We now discuss possible interesting complexifications of the amplitude  $A_{\mathcal{G}}(\{m_f\})$ . So far, we have two parameters which are the dimension  $D$  of the group and the theory rank  $d$ . A priori, from (48), we can define a complex integral  $A_{\mathcal{G}}(\{m_f\}, D, d)$ , for  $D, d \in \mathbb{C}$ . However, the non trivial dependency in  $d$  in the amplitude makes the study of this function drastically complicated. We will only achieve a complexification in the standard way, i.e. by considering a complex dimension  $A_{\mathcal{G}}(\{m_f\}, D)$  for  $D \in \mathbb{C}$ , and will undertake the dimensional regularization of an arbitrary amplitude in this variable.

**Domain of analyticity.** The analysis of  $A_{\mathcal{G}}(\{m_f\}, D)$  can be undertaken as follows. From the fact that  $\tilde{U}_{\mathcal{G}}^{\text{ev}} = 1 + U_{\mathcal{G}}''$ , the last factor  $\tilde{U}_{\mathcal{G}}^{\text{ev}}/(1 + U_{\mathcal{G}}'(\{x_l\}))$  in (48) can be bounded by a constant  $k_{\mathcal{G}} = \tilde{U}_{\mathcal{G}}^{\text{ev}}(\{x_k = 1\})$  depending on the graph. It is immediate to infer from (48) that, in the UV regime  $x_i \rightarrow 0$ ,

$$\begin{aligned} \text{if } \forall i = 1, \dots, L, \quad \Re(\omega_d(G_i)) > 0, & \quad \text{then the amplitude converges;} \\ \text{if } \exists i = 1, \dots, L, \quad \Re(\omega_d(G_i)) \leq 0, & \quad \text{then the amplitude diverges.} \end{aligned} \quad (55)$$

Consider the subgraphs  $G_i$  associated with Hepp sectors, with positive numbers of vertices  $V(G_i) \geq 1$  and lines  $L(G_i) \geq 1$ . The convergence of the amplitude is then guaranteed (sufficient condition) if we have

$$\Re(D) < D_0^\sigma = \inf_i \frac{L(G_i)}{F_{\text{int}}(G_i)}. \quad (56)$$

Note that if  $L = 0$ , the graph is actually either empty or formed by disconnected vertices and so,  $F_{\text{int}} = 0$  and there are no divergences. On the other hand, setting  $F_{\text{int}} = 0$  means already that we have no divergences. We are led to the following bound,  $\forall i$ ,

$$\Re(D) < \frac{1}{d^-} \leq \frac{2L}{d^-(2L + 2(1 - V))} (G_i) \leq \frac{L(G_i)}{d^- \tilde{F}_{\text{int}}(G_i)} \leq \frac{L(G_i)}{F_{\text{int}}(G_i)} \quad (57)$$

where uses have been made of (54), and the fact that either  $V(G_i) = 1$  or  $V(G_i) > 1$  and so  $\frac{2L}{d^-(2L+2(1-V))}(G_i) > \frac{2L}{d^-(2L)}(G_i)$ .

We infer that, the amplitude  $A_{\mathcal{G}}(\{m_f\}, D)$  is convergent and analytic in  $D$  in the strip

$$\mathcal{D}^\sigma = \left\{ D \in \mathbb{C} \mid 0 < \Re(D) < \frac{1}{d^-} \right\}. \quad (58)$$

At the first sight, increasing the rank of the theory induces a reduction of the analyticity strip of the amplitude. Also, as a recurrent feature, this analyticity domain is again restricted to a strip the real part of which is bounded by half of the dimension of the group manifold.

Now, we proceed further and extend  $A_{\mathcal{G}}(\{m_f\}, D)$  to a complex function of  $D$  in the strip  $1/d^- \leq \Re(D) \leq \delta$ , where  $\delta$  plays the role of the initial dimension of the group that is either  $\delta = 1$  for the models  ${}_1\Phi_3^4$  and  ${}_1\Phi_2^{2k}$ , or  $\delta = 2$  for  ${}_2\Phi_2^4$ .

**Theorem 1** (Extended domain of analyticity). *Consider a tensor model  $\delta\Phi_d^{k_{\max}}$ ,  $(\delta, d, k_{\max}) \in \{(1, 3, 4), (2, 2, 4), (1, 2, 2n)\}$ , for  $n \geq 2$ . Let  $\mathcal{G}$  be one of its graphs and define  $\sigma$  an associated Hepp sector of  $\mathcal{G}$ . For  $\delta\Phi_d^{k_{\max}}$ , if one of the following conditions is fulfilled,*

- (a)  $\forall i, N_{\text{ext}}(G_i) > k_{\max}$ ,
  - (b1) for  $d = 2$ ,  $\forall i, \Omega(G_i) > 0$ ,
  - (b2) for  $d = 3$ ,  $\forall i, \{N_{\text{ext}}(G_i) > 2, \Omega(G_i) > 0\}$  or  $\{N_{\text{ext}}(G_i) > 0, \Omega(G_i) > 1\}$ ,
  - (c)  $\forall i, C_{\partial G_i} > 1$ ,
- (59)

and, specifically for  ${}_1\Phi_2^{2n}$ ,  $n \geq 2$ , if

- (d)  $\forall i, V(G_i) > 1$ ,
  - (e)  $\forall i, N_{\text{ext}}(G_i) = k_{\max}$ ,
- (60)

then  $A_{\mathcal{G}}(\{m_f\}, D)$  converges and is analytic in the strip

$$\tilde{\mathcal{D}}^\sigma = \{D \in \mathbb{C} \mid 0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}\}, \quad (61)$$

for  $\varepsilon_{\mathcal{G}}$  a small positive constant depending on the graph.

Let us comment that although in the following proof of Theorem 1, the main variables  $k_{\max}$ ,  $d^-$  are always fixed and are expressed simply, we first perform general calculations and then sometimes use the specific values of  $k_{\max}$  and  $d^-$ . Foreseeing the generic dimensional regularization of the other tensor models in higher ranks, the method used and the several relations generated will remain valid. As such, these are worth to be listed.

*Proof of Theorem 1.* • We shall start by the models  ${}_1\Phi_3^4$  and  ${}_2\Phi_2^4$ . First, one notices that the following relations hold:

$$\delta d^- - 1 > 0, \quad (\delta d^- - 1)k_{\max} - 2\delta d^- = 0, \quad (62)$$

where  $k_{\max} = 4$  stands for the maximal valence allowed for vertices in these models.

Consider the subgraphs  $G_i$  associated with Hepp sectors, with positive numbers of vertices  $V(G_i) \geq 1$  and lines  $L(G_i) \geq 1$  such that (a) holds, i.e., for all  $i$ ,  $N_{\text{ext}}(G_i) > k_{\max}$  holds. We

define  $q(G_i) = N_{\text{ext}}(G_i) - k_{\text{max}} > 0$ ,  $V_{<} = \sum_{k=2}^{k_{\text{max}}-2} V_k = V_2$  and  $n \cdot V_{<} = \sum_{k=2}^{k_{\text{max}}-2} k V_k = 2V_2$  and write

$$\begin{aligned} \frac{L(G_i)}{F_{\text{int}}(G_i)} &\geq \frac{L(G_i)}{d^- \tilde{F}_{\text{int}}(G_i)} \geq \frac{(n \cdot V - N_{\text{ext}})(G_i)}{d^- (2 - N_{\text{ext}} + (n-2)V)(G_i)} \\ &\geq \frac{(k_{\text{max}}(V_{k_{\text{max}}-1}) + n \cdot V_{<} - q)(G_i)}{d^- (2 - k_{\text{max}} - q + (k_{\text{max}}-2)V_{k_{\text{max}}})(G_i)} \geq \frac{(k_{\text{max}}(V_{k_{\text{max}}-1}) - q)(G_i)}{d^- ((k_{\text{max}}-2)(V_{k_{\text{max}}-1}) - q)(G_i)}, \end{aligned} \quad (63)$$

where we used  $(n-2)V_{<} = 0$  in an intermediate step. Two cases may occur: (A) if  $V_{k_{\text{max}}-1} = 0$ , this means that the graph is formed with a unique vertex (forgetting mass vertices). This is a tadpole graph and certainly,  $N_{\text{ext}} \leq k_{\text{max}}$ , which is inconsistent with our initial assumption. (B) We consider then  $V_{k_{\text{max}}-1} > 0$  and obtain

$$\begin{aligned} \frac{L(G_i)}{F_{\text{int}}(G_i)} &> \frac{1}{d^-} \left( 1 + \frac{2}{k_{\text{max}}-2} + \tilde{\varepsilon}_{\mathcal{G}} \right) \geq \frac{1}{d^-} \left( \frac{k_{\text{max}}}{k_{\text{max}}-2} + \tilde{\varepsilon}_{\mathcal{G}} \right) > \boldsymbol{\delta} \geq \Re(D), \\ 0 &< \tilde{\varepsilon}_{\mathcal{G}} < \inf_i \frac{2q(G_i)}{[(k_{\text{max}}-2)(V_{k_{\text{max}}-1}) - q](k_{\text{max}}-2)(G_i)}, \end{aligned} \quad (64)$$

where (62) has been used to get  $\boldsymbol{\delta}$ . We can then introduce  $\varepsilon_{\mathcal{G}} = \tilde{\varepsilon}_{\mathcal{G}}/d^-$  so that (61) holds under the condition (a).

Now, we focus on the conditions (b1)-(b2). Assume  $\boldsymbol{\Omega}(G_i) > 0$  for all  $i$ . It immediately implies that  $2\boldsymbol{\Omega}(G_i)/(d^-)! \geq d^- - 1$  (53). However, this bound is quite loose. For  $d = 2$ , clearly  $2\boldsymbol{\Omega}(G_i) = 2g_{\tilde{\mathcal{G}}} > 1$  while  $d^- - 1 = 0$ . Meanwhile, for  $d = 3$ , it is still a good bound as  $2\boldsymbol{\Omega}(G_i)/(d^-)! = \boldsymbol{\Omega}(G_i) \geq d^- - 1 = 1$ . Hence for generic  $d = 2, 3$ , assuming  $\boldsymbol{\Omega}(G_i) > 0$ , we shall use  $2\boldsymbol{\Omega}(G_i)/(d^-)! \geq 1$ . One finds a bound on the number of internal faces as

$$F_{\text{int}}(G_i) \leq -1 + d^- \tilde{F}_{\text{int}}(\mathcal{G}). \quad (65)$$

We then obtain new bounds on the ratio

$$\frac{L(G_i)}{F_{\text{int}}(G_i)} \geq \frac{\frac{1}{2}(n \cdot V - N_{\text{ext}})(G_i)}{-1 + d^- \tilde{F}_{\text{int}}(G_i)} \geq \frac{\frac{1}{2}(n \cdot V - N_{\text{ext}})(G_i)}{-1 + \frac{d^-}{2}(2 - N_{\text{ext}} + (n-2) \cdot V)(G_i)}. \quad (66)$$

The last expression exhibits a different behavior for  $d = 2$  and  $d = 3$ . We discuss them separately.

- If  $d = 2$ , for all  $N_{\text{ext}}(G_i) > 0$  and  $\boldsymbol{\Omega}(G_i) > 0$ , we get the bound

$$\begin{aligned} \frac{L(G_i)}{F_{\text{int}}(G_i)} &\geq \frac{(n \cdot V - N_{\text{ext}})(G_i)}{((n-2) \cdot V - N_{\text{ext}})(G_i)} \geq 1 + \frac{2V}{((n-2) \cdot V - N_{\text{ext}})(G_i)} \\ &\geq 1 + \frac{2V_{k_{\text{max}}}}{((k_{\text{max}}-2)V_{k_{\text{max}}} - N_{\text{ext}})(G_i)} > 1 + \frac{2}{k_{\text{max}}-2} + \tilde{\varepsilon}_{\mathcal{G}} \geq 1 + 1 + \tilde{\varepsilon}_{\mathcal{G}} > \boldsymbol{\delta} \geq \Re(D), \\ 0 &< \tilde{\varepsilon}_{\mathcal{G}} < \inf_i \frac{2N_{\text{ext}}(G_i)}{[(k_{\text{max}}-2)V_{k_{\text{max}}} - N_{\text{ext}}](k_{\text{max}}-2)(G_i)}. \end{aligned} \quad (67)$$

- If  $d = 3$ , assume that  $N_{\text{ext}}(G_i) - 2 > 0$  and  $\boldsymbol{\Omega}(G_i) > 0$ , and the following bound is instead valid

$$\begin{aligned} \frac{L(G_i)}{F_{\text{int}}(G_i)} &\geq \frac{(n \cdot V - N_{\text{ext}})(G_i)}{d^- [(n-2) \cdot V - N_{\text{ext}}](G_i) + 1} \geq \frac{1}{d^-} \left( 1 + \frac{2V(G_i) - 1}{((n-2) \cdot V - N_{\text{ext}})(G_i) + 1} \right) \\ &\geq \frac{1}{d^-} \left( 1 + \frac{2V_{k_{\text{max}}}(G_i) - 1}{((k_{\text{max}}-2)V_{k_{\text{max}}} - (N_{\text{ext}} - 2))(G_i) - 1} \right) > \frac{1}{d^-} \left( 1 + \frac{2V_{k_{\text{max}}}(G_i) - 1}{((k_{\text{max}}-2)V_{k_{\text{max}}})(G_i) - 1} + \tilde{\varepsilon}_{\mathcal{G}} \right) \end{aligned}$$



$$\begin{aligned}
&> \frac{1}{d^-} (1 + 1 + \tilde{\varepsilon}_{\mathcal{G}}) > \boldsymbol{\delta} \geq \Re(D), \\
0 < \tilde{\varepsilon}_{\mathcal{G}} < \inf_i \frac{[(2V_{k_{\max}}-1)(N_{\text{ext}}-2)](G_i)}{[(k_{\max}-2)V_{k_{\max}}-1][(k_{\max}-2)V_{k_{\max}}-(N_{\text{ext}}-2)](G_i)}. \tag{68}
\end{aligned}$$

- For the last case of the rank  $d = 3$  imposing that  $N_{\text{ext}}(G_i) > 0$  and  $\boldsymbol{\Omega}(G_i) > 1$ , one has a better bound  $F_{\text{int}}(G_i) \leq -2 + d^- \tilde{F}_{\text{int}}(\mathcal{G})$ , and the rest of the proof is similar to (67).

Hence, setting  $\varepsilon_{\mathcal{G}} = \tilde{\varepsilon}_{\mathcal{G}}/d^-$ , one recovers (61) under the statements (b1)-(b2).

Now, we assume that (c) holds i.e.  $C_{\partial G_i} > 1$  for all  $i$ . The number of internal faces is again bounded in the same way as (65). For the rank  $d = 2$ , the analysis can be redone in the same way as (b1) and allows us to conclude. For the rank  $d = 3$ , we have another piece of information on the number of external legs:  $N_{\text{ext}}(G_i) \geq 2C_{\partial G_i} > 2$  (an important fact to notice is that each component of the boundary passes through necessarily at least two external legs in a complex model). Thus the analysis of the above case (b2) applies once again and leads to the same conclusion. This achieves the proof of the analyticity domain of amplitudes in the models  $\delta_{=2}\Phi_2^4$  and  $\delta_{=1}\Phi_3^4$ .

• Let us discuss the tower of matrix models  ${}_1\Phi_2^{2n}$ . We shall first prove that the analyticity domain extends when (d)  $V(G_i) > 1$  holds. It is direct to achieve this by noting

$$\begin{aligned}
\frac{L(G_i)}{F_{\text{int}}(G_i)} &\geq \frac{(n \cdot V - N_{\text{ext}})(G_i)}{(2 - N_{\text{ext}} + (n-2) \cdot V)(G_i)} \geq 1 + \frac{2(V-1)(G_i)}{(2 - N_{\text{ext}} + (n-2) \cdot V)(G_i)} > 1 + \varepsilon_{\mathcal{G}} > \boldsymbol{\delta} \geq \Re(D), \\
0 < \varepsilon_{\mathcal{G}} < \inf_i \frac{2(V-1)(G_i)}{(2 - N_{\text{ext}} + (n-2) \cdot V)(G_i)}. \tag{69}
\end{aligned}$$

Then the analyticity domain of  $A_{\mathcal{G}}$  extends to (61).

The fact that the domain extends under the assumption (d) has a consequence for the same study now under the conditions (a) and (e). Meanwhile, the reason that under (b1) and (c) the domain extends as well is derived simply from the similar situation of the model  ${}_2\Phi_2^4$ .

- Consider all graphs  $G_i$  such that (a) or (e) holds then  $N_{\text{ext}}(G_i) \geq k_{\max}$ . It implies that either we are using a number of vertices larger than 1 or a unique vertex with the maximal valency  $k_{\max} = N_{\text{ext}}(G_i)$  and no lines are present in the graph. Clearly, both situations lead to a convergent amplitude.

- Consider now (b1). For all  $i$ ,  $\boldsymbol{\Omega}(G_i) > 0$  such that (65) holds now. The calculations are similar to the previous case (b1). We have, for all  $N_{\text{ext}}(G_i) > 0$ ,

$$\begin{aligned}
\frac{L(G_i)}{F_{\text{int}}(G_i)} &\geq \frac{(n \cdot V - N_{\text{ext}})(G_i)}{((n-2) \cdot V - N_{\text{ext}})(G_i)} \geq 1 + \frac{2V}{((n-2) \cdot V - N_{\text{ext}})(G_i)} > 1 + \varepsilon_{\mathcal{G}} > \boldsymbol{\delta} \geq \Re(D), \\
0 < \varepsilon_{\mathcal{G}} < \inf_i \frac{2V(G_i)}{((n-2) \cdot V - N_{\text{ext}})(G_i)}. \tag{70}
\end{aligned}$$

- Finally, assuming (c) so that  $C_{\partial G_i} > 1$ , we use again (70) to complete the proof. □

**Meromorphic structure.** The next task is to prove the meromorphic structure of the amplitudes  $A_{\mathcal{G}}(\{m_f\}, D)$  on the domain  $\tilde{\mathcal{D}}^\sigma$  when all conditions listed in Theorem 1 are dropped.

From Theorem 1, the only cases which lead to divergent amplitudes can be listed as follows:

- (di) In the  ${}_1\Phi_3^4$  model, graphs with  $N_{\text{ext}}(\mathcal{G}) \leq 4 = k_{\text{max}}$  with  $C_{\partial\mathcal{G}} = 1$  and
- a-  $\Omega(\mathcal{G}) = 0$  (melonic with melonic boundary) and  $N_{\text{ext}}(\mathcal{G}) = 4$ ;
  - b-  $\Omega(\mathcal{G}) = 0$  (melonic with melonic boundary) and  $N_{\text{ext}}(\mathcal{G}) = 2$ ;
  - c-  $\Omega(\mathcal{G}) = 1$  (non melonic with melonic boundary) and  $N_{\text{ext}}(\mathcal{G}) = 2$ .
- (dii) In the  ${}_2\Phi_2^4$  model, graphs with  $N_{\text{ext}}(\mathcal{G}) \leq 4 = k_{\text{max}}$  with  $C_{\partial\mathcal{G}} = 1$  and
- a-  $\Omega(\mathcal{G}) = 0$  (planar) and  $N_{\text{ext}}(\mathcal{G}) = 4$ ;
  - b-  $\Omega(\mathcal{G}) = 0$  (planar) and  $N_{\text{ext}}(\mathcal{G}) = 2$ .
- (diii) In the  ${}_1\Phi_2^{2n}$  model, graphs with  $V = V_{2k} = 1$ ,  $k \leq n = k_{\text{max}}/2$ ,  $N_{\text{ext}}(\mathcal{G}) < 2k$  with  $C_{\partial\mathcal{G}} = 1$  and  $\Omega(\mathcal{G}) = 0$  (planar).

The above list of primitively divergent graphs matches with the one issued in [27].

We come back to the integrand of the amplitude (48) and focus on the following function:

$$I_{\mathcal{G}}^{\sigma}(\{x_l\}, \{m_f\}, D) = \left[ \prod_{l=1}^L \frac{(1 - \prod_{k=l}^L x_k)^{\mu-1}}{(1 + \prod_{k=l}^L x_k)^{\mu+1}} \right] \prod_{f \in \mathcal{F}_{\text{ext}}} \prod_{l \in f} \left( \frac{1 - \prod_{k=l}^L x_k}{1 + \prod_{k=l}^L x_k} \right)^{|m_f|} \left[ \frac{\tilde{U}_{\mathcal{G}}^{\text{ev}}(\{x_l\})}{(1 + U'_{\mathcal{G}}(\{x_l\}))} \right]^D. \quad (71)$$

Since all  $x_l$  are positive,  $I_{\mathcal{G}}^{\sigma}$  is a continuous function in the  $x_l$  variables and admits a simultaneous Taylor expansion in the  $x_l$ 's around  $x_l = 0$ .

At this point, we use a different strategy from the one introduced in [67]. We do not perform the generic Taylor expansion in all the  $x_l$ 's before integrating all  $x_l$ 's and getting the poles and meromorphicity conditions on the amplitude using diverging subgraphs. The reason motivating our present study is that, given a divergent primitively graph  $S$  (with characteristics listed above) of a graph  $\mathcal{G}$ , we know exactly on which variables we must perform the Taylor expansion for extracting the divergence. Precisely, by Lemma 3, we know that the variables in which one must perform a Taylor series are the ones which ‘‘tie’’  $S$  and  $\mathcal{G}/S$ . One must then prove that, the first order of the Taylor expansion is large enough to ensure analyticity at a higher order.

Consider then a primitively divergent subgraph  $S \subset \mathcal{G}$ , and a decomposition in Hepp sectors  $\sigma'$ ,  $t_1 \leq t_2 \leq \dots \leq t_{L(S)}$  of the lines of  $S$  and introduce the usual change of variables in  $x_k$  as in (44). Because  $S$  is primitively divergent, for all subgraphs  $G_i$  except  $S = G_{L(S)}$ , we have  $\Re(\omega_d(G_i)) > 0$  and  $\Re(\omega_d(G_{L(S)})) \leq 0$ . Lemma 3 ensures, using (34) and writing  $\mathcal{F}'_{\bullet}(\mathcal{G}, S) = \mathcal{F}'_{\bullet}$ , that

$$\begin{aligned} A_{\mathcal{G}} &= \int [dt_l]_{\mathcal{L}(S)} \tilde{A}_S \left[ 1 + \sum_{f \in \mathcal{F}'_{\text{ext}}} R_f(t_l) \mathcal{O}_S \right] A_{\mathcal{G}/S} \\ &\quad + \int [dt_l]_{\mathcal{L}(S)} \tilde{A}_S \left[ \int [dt_{l'}]_{\mathcal{L}(\mathcal{G}/S)} \sum_{f \in \mathcal{F}'_{\text{int}}} R_f(t_l) \mathcal{O}'_S(t_{l'}) + O(t_l^2) \right] \tilde{A}_{\mathcal{G}/S} \\ &= \sum_{\sigma'} A_{\mathcal{G}}^{\sigma'}, \end{aligned} \quad (72)$$

with some appropriate operator insertions  $\mathcal{O}_S$  and  $\mathcal{O}'_S$ . Now, we perform the change of variables  $t_l \rightarrow x_k$ , and, after denoting  $x_{L(S)} = x$  and singling out the integration in  $x$ , one gets

$$\begin{aligned}
A_{\mathcal{G}}^{\sigma'} &= \int dx x^{\omega_d(G_{L(S)})-1} \left( A_{\mathcal{G}/S} \mathcal{J}_S^{\sigma'} + x \left\{ \mathcal{R}_S(x, \{m_f\}) A_{\mathcal{G}/S} \right. \right. \\
&\quad \left. \left. + \int [dt'_l]_{l \in \mathcal{L}(\mathcal{G}/S)} \mathcal{R}'_S(x, \{m_f\}, \{t'_l\}) \tilde{A}_{\mathcal{G}/S} \right\} + O(x^2) \right), \quad (73) \\
\mathcal{J}_S^{\sigma'} &= \int \left[ \prod_{k \neq L(S)} dx_k \right] \tilde{I}_S^{\sigma'}(\{x_l\}, \{m_f\}), \\
x \mathcal{R}_S(x, \{m_f\}) &= \int \left[ \prod_{k \neq L(S)} dx_k \right] \tilde{I}_S^{\sigma'}(\{x_l\}, \{m_f\}) \sum_{f \in \mathcal{F}'_{\text{ext}}} \tilde{R}_f(x_l) \mathcal{O}_S, \\
x \mathcal{R}'_S(x, \{m_f\}, \{t'_l\}) &= \int \left[ \prod_{k \neq L(S)} dx_k \right] \tilde{I}_S^{\sigma'}(\{x_l\}, \{m_f\}) \sum_{f \in \mathcal{F}'_{\text{int}}} \tilde{R}_f(x_l) \mathcal{O}'_S(t'_l),
\end{aligned}$$

where  $\tilde{R}_f(x_l)$  are obtained from  $R_f$  after the change of variables and where  $\tilde{I}_S^{\sigma'}$  is of form  $(\prod_{k \neq L(S)} x_k^{\omega_d(G_k)-1}) I_S^{\sigma'}$  and  $I_S^{\sigma'}$  is given by (71) except that the product over external faces must be restricted to  $\mathcal{F}'_{\text{ext}}$ . Interestingly, one notes that we have traded a big-O function in  $t'_l$  for the same function in  $x^2$  without having lost any terms.

The Taylor expansions of  $\mathcal{R}_S(x, \{m_f\})$  and of  $\mathcal{R}'_S(x, \{m_f\}, \{t'_l\})$  around  $x = 0$  involve the expansions of  $\tilde{I}_S^{\sigma'}(\{x_l\}, \{m_f\})$  and of  $R_f(x_l)$  (as a simple polynomial). This yields extra  $x^p$  factors,  $p \geq 0$ . Thus, the convergence of the integral of each term in the expansion is expected to improve for the simple reason that  $\Re(\omega_d(G_k)) + p \geq \Re(\omega_d(G_k))$ , for  $p \geq 0$ .

Assume that the subgraph  $S$  of  $\mathcal{G}$  obeys one of the conditions (di)–(diii). The following cases might occur:

- For a 4-point subgraph under conditions (dia) or (diia), or a  $2k'$ -point subgraph obeying (diii), the integration over  $x$  yields the divergent part of  $A_{\mathcal{G}}^{\sigma'}$  like the first term of (73):

$$A_{4\text{pt}/2k'}(D) = \frac{c_\lambda}{\omega_d(G_{L(S)})} = \frac{c_\lambda}{L - DF_{\text{int}}(S)}, \quad (74)$$

for some constant  $c_\lambda$  which incorporates the couplings and other constant factors. We get a pole at

$$D_0 = \boldsymbol{\delta}, \quad (75)$$

with the further condition that, in the  $\delta_{=1}\Phi_3^4$  and  $\delta_{=2}\Phi_2^4$  models,  $S$  does not contain mass vertices  $V_2(S) = 0$ . If  $V_2(S) \neq 0$ , in these models, we discover poles at the rational values

$$D_1 = \frac{2}{d^-} \left[ 1 + \frac{V_2}{2(V_4 - 1)} \right] = \boldsymbol{\delta} + \frac{V_2}{d^-(V_4 - 1)}, \quad V_4 - 1 > 0. \quad (76)$$

The last condition on  $V_4$  is imposed since we want  $N_{\text{ext}} = 4 = k_{\text{max}}$  (as previously discussed, this is only possible if there is a number of 4-valent vertices strictly greater than 1). Note also that (76) means that for a fixed graph  $S$ ,  $V_2$  and  $V_4$  are certainly fixed, thus  $V_2/[d^-(V_4 - 1)]$  cannot be very large otherwise  $D_1$  would lie outside the strip  $0 < \Re(D) < \boldsymbol{\delta} + \varepsilon_S$ . Therefore, this gives us an estimate like  $0 < \varepsilon_S < \frac{V_2}{d^-(V_4 - 1)}$  to get a convergent integral whenever  $V_2 \neq 0$ .

Now we inspect a generic term in the Taylor expansion. This situation is similar to the case when  $V_2 \neq 0$ . Under the current conditions, let us assume that we integrate another term  $\int x^{\omega_d(G_{L(S)})+p-1}$ ,  $p > 0$ , in the Taylor series. We get a pole at  $D'_1(p) = \delta + (V_2 + p)/[d^-(V_4 - 1)]$ , for the models  $\delta\Phi_3^4$  and  $\delta\Phi_2^4$ . For  $\delta\Phi_2^{2n}$ , one obtains a pole at  $D'_1(p) = \delta + p/[d^-(k - k')]$ , where  $2k$  is the vertex valence and  $N_{\text{ext}} = 2k' < 2k$ . Using the same previous argument, this also means that  $p + V_2$  (resp.  $p$ ) cannot be very large otherwise  $D'_1(p)$  (resp.  $D''_1(p)$ ) would lie outside the strip  $0 < \Re(D) < \delta + \varepsilon_S$ . It is immediate that this term certainly converges for a given estimate of  $\varepsilon_S$  that one can choose as  $0 < \varepsilon_S < \frac{(V_2+1)}{d^-(V_4-1)}$  (resp.  $0 < \varepsilon_S < \frac{1}{d^-(k-k')}$ ). If we allow  $D$  to explore in the whole complex plane (and wander outside of the strip  $0 < \Re(D) < \delta + \varepsilon_S$ ), then we discover that the function is meromorphic with poles at  $D_0$ ,  $D_1$ ,  $D'_1(p)$  and  $D''_1(p)$  which are rationals.

- For a 2-point subgraph of  $\delta_{=1}\Phi_3^4$  under assumption (dic), such that  $V_2 = 0$ , one obtains again a pole at  $D = \delta$ . However, when  $V_2 \neq 0$ , the poles shift to the rational values

$$D_1''' = \delta + \frac{V_2}{2V_4 - 1}. \quad (77)$$

The fact that the remaining Taylor terms converge in a precise strip follows from the same arguments invoked above.

- For a 2-point function respecting conditions (dib) or (diib), we get the following contributions to the amplitude of  $A_{\mathcal{G}}^{\sigma'}$

$$A_{2\text{pt}}(D) = \frac{c_{1;\lambda}}{\omega_d(G_{L(S)})} + \frac{c_{2;\lambda}}{\omega_d(G_{L(S)}) + 1}, \quad (78)$$

for some constants  $c_{1;\lambda}$  and  $c_{2;\lambda}$ . For  $V_2 = 0$ , we have two types of poles located at rational values

$$D_0 = \delta, \quad D_2 = \delta - \frac{1}{d^-V_4}, \quad V_4 > 0, \quad (79)$$

where the last condition  $V_4 > 0$  refers to the trivial fact that a 2-pt subgraph with  $V_4 = 0$  cannot diverge. If  $V_2 \neq 0$ , the poles shift to

$$D'_0 = \delta + \frac{V_2}{d^-V_4}, \quad D'_2 = \delta + \frac{V_2 - 1}{d^-V_4}, \quad V_4 > 0. \quad (80)$$

As performed previously, the case  $V_2 \neq 0$  and the integration of higher order Taylor terms at  $p \geq 2$  with poles like  $D_3(p) = \delta + (V_2 - 1 + p)/[d^-V_4]$ , lead to a similar conclusion as above.

One remark must be made at this stage. In the above pole equations including the variable  $V_2$ , the value of  $V_2$  cannot be large as compared to  $V_4$ . Indeed, if  $V_2$  grows faster than  $V_4$ , most of the above equations yield  $\Re(\omega_d(S)) > 0$ , immediately leading to convergence. One must also remember that the analyticity domain is only extended up to a small enough parameter  $\varepsilon_S$  such that  $0 < \Re(D) < \delta + \varepsilon_S$ . The appearance of  $V_2$  if not trivial must be simply considered as a mild modification of the case when  $V_2 = 0$ . The later case simplifies the formalism and one finds only two types of poles  $D_0 = \delta$  and  $D_2 = \delta - 1/d^-V_4$  (79). They certainly lie in the strip  $0 < \Re(D) < \delta + \varepsilon_S$  and are quite reminiscent of the poles found in [83].

We have listed all poles of the amplitude  $A_{\mathcal{G}}^{\sigma'}$  related to a primitively divergent subgraph  $S$  of  $\mathcal{G}$ . These poles are always located at rational values.

We are in position to achieve the main statement on the meromorphic structure of  $A_{\mathcal{G}}$ . Let us now come back to the beginning and consider  $\sigma$  a Hepp sector of  $\mathcal{G}$  and its associated amplitude expansion  $A_{\mathcal{G}} = \sum_{\sigma} A_{\mathcal{G}}^{\sigma}$ . Consider all primitively divergent subgraphs  $S$  of  $\mathcal{G}$ . Each subgraph  $S$  must appear at least in one sector  $\sigma$ , and therefore  $S$  must be included in some  $G_i$  with  $i \geq L(S)$ . We can conclude that the amplitude  $A_{\mathcal{G}}$  is convergent, defines an analytic function in  $D$  in the strip  $\mathcal{D}$  (58) and admits an analytic continuation as a meromorphic function in the strip  $\tilde{\mathcal{D}}^{\sigma} = \{D \in \mathbb{C} \mid 0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}\}$ , with

$$0 < \varepsilon_{\mathcal{G}} < \inf_{S \subset \mathcal{G}; S \text{ primitively divergent}} \varepsilon_S. \quad (81)$$

In fact, the above bound on  $\varepsilon_{\mathcal{G}}$  can be improved in a more useful way using Hepp sectors. A crucial observation is that the set of the  $G_i$ 's is totally ordered under inclusion,  $G_i \subset G_{i+1}$ . Precisely, the set of all of their connected components  $G_i^{(k)}$  of all  $G_i$ 's is partially ordered and forms an abstract tree with nodes the  $G_i^{(k)}$ 's. This is the Gallavotti-Nicolò tree [94]. The set  $\{G_i^{(k)}\}_{k;i}$  also defines the set of quasi local (or high) subgraphs in the formulation of [13]. The introduction of such tree becomes extremely useful for the treatment of the so-called overlapping divergences appearing in ordinary renormalized expansion in the coupling constants. The point is that divergences in some sector  $A_{\mathcal{G}}^{\sigma}$  are now indexed by disconnected subgraphs organized into a forest.

Thus, given two primitively divergent subgraphs  $S$  and  $S'$  of  $\mathcal{G}$ , the only relevant case is when  $S \cap S' = \emptyset$ . They form connected and disjoint subgraphs in the same Hepp sector  $\sigma$ ,  $t_1 \leq \dots \leq t_{L(\mathcal{G})}$ . Let us assume, without loss of generality, that the lines of  $S$  are indexed from 1 to  $L(S)$  and those of  $S'$  indexed from  $p+1$  to  $p+L(S')$ , with  $p > L(S)$ . In other words, consider the ordering of lines of  $\mathcal{G}$  like

$$t_1 \leq \dots \leq t_{L(S)} \leq \dots \leq t_p \leq t_{p+1} \leq \dots \leq t_{p+L(S')} \leq \dots \leq t_{L(\mathcal{G})}. \quad (82)$$

Then there exist two independent variables  $x_S = x_{L(S)}$  and  $x_{S'} = x_{p+L(S')}$  that we can use to perform distinct integrals of the same form as (73) with this time the final factor as  $\tilde{A}_{(\mathcal{G}/S)/S'}$ . The above reasoning applies (note that the order of the integrations does not matter either if we solve the most nested divergence which corresponds to  $S$  and then the second associated with  $S'$  or the other way around). We proceed by induction on the rest of primitively divergent graphs in this sector. In the end, the amplitude  $A_{\mathcal{G}}^{\sigma}$  is meromorphic in the strip  $0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}^{\sigma}$  with poles at rationals, where

$$0 < \varepsilon_{\mathcal{G}}^{\sigma} < \inf_{S \subset \mathfrak{F}^{\sigma}} \varepsilon_S \quad (83)$$

where  $\mathfrak{F}^{\sigma}$  is the forest of connected primitively divergent subgraphs of  $\mathcal{G}$  related to the Hepp sector  $\sigma$ . Summing over all Hepp sectors  $\sigma$ , we simply have to infer that  $A_{\mathcal{G}}$  defines a convergent integral and a meromorphic function in the strip  $0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}$  with

$$0 < \varepsilon_{\mathcal{G}} < \inf_{\sigma} \varepsilon_{\mathcal{G}}^{\sigma}. \quad (84)$$

We have finally achieved the following statement:

**Theorem 2** (Meromorphic structure of the amplitudes). *The amplitude  $A_{\mathcal{G}}(\{m_f\}, D)$  of the model  $\delta\Phi_d^{k_{\max}}$  (listed above) is a meromorphic function in  $D$  on the strip*

$$\tilde{\mathcal{D}}^\sigma = \{D \in \mathbb{C} \mid 0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}\}, \quad (85)$$

for  $\varepsilon_{\mathcal{G}}$  a small positive quantity depending on the graph  $\mathcal{G}$ .

## 4.2 Renormalization

From this point, the standard definition of the subtraction operator [67] can be applied. The discrepancy between the present study and the formalism therein is that we are considering a radically different set of primitively divergent subgraphs. We will only sketch the definition of the subtraction operator (details can be found in the above reference).

We introduce the operator  $\tau$  as the generalized Taylor operator defined as follows: let  $f(x)$  be a function such that  $x^{-\nu}f(x)$  is infinitely differentiable at  $x = 0$ ,  $\nu$  might be complex. One defines

$$\tau_x^n f(x) = x^{-\xi-\epsilon} T_x^{n+\xi}(x^{\xi+\epsilon} f(x)), \quad T_x^{m \geq 0}(f) = \sum_{k=0}^m \frac{x^k}{k!} f^{(k)}(0), \quad T_x^{m < 0}(f) = 0, \quad (86)$$

where  $\xi$  is an integer obeying  $\xi \geq -E(\nu)$ ,  $E(\nu)$  is the smallest integer  $\geq \Re(\nu)$ , and  $\epsilon = E(\nu) - \nu$ .

Consider a subgraph  $S \subset \mathcal{G}$  and a function  $f(\{t_l\})$  on the graph  $\mathcal{G}$ ,  $l \in \mathcal{L}(\mathcal{G})$ . We associate  $S$  with the following operator:

$$\tau_S^n f(\{t_l\}) = \left[ \tau_\rho^n f(\{\rho t_l\}_{l \in \mathcal{L}(S)}; \{t_l\}_{l \in \mathcal{L}(\mathcal{G}/S)}) \right] \Big|_{\rho=1}. \quad (87)$$

Finally, one defines the subtraction operator acting on amplitudes as

$$R = 1 + \sum_{\mathfrak{F}} \prod_{S \in \mathfrak{F}} (-\tau_S^{-L(S)})_{\triangleright}, \quad (88)$$

where the sum is performed over the set  $\mathfrak{F}$  of all forests of primitively divergent subgraphs, and the symbol  $\triangleright$  refers to the fact that the operator  $R$  must act on the integrand of a given amplitude. Another important remark is that any subtraction operator is defined by two actions: (1) the action of a Taylor operator on the amplitude integrand targeting specifically the variables which are associated with a primitively divergent subgraph  $S$  (using in particular scaling properties of Lemma 2) and (2) the subtraction of the pole induced by the diverging part for each term in the Taylor expansion. The procedure is completely standard at this stage for our models (22). Thus,

$$RA_{\mathcal{G}} = A_{\mathcal{G}}^{\text{ren}} \quad (89)$$

is a finite integral and an analytic function in  $D$  on the strip  $\tilde{\mathcal{D}}^\sigma$ .

Finally, let us mention that written in the way (88), the operator  $R$  might not seem to be related to a Hepp decomposition. It is simply a subtraction operator acting on the amplitude which will prove to lead to a convergent integral. There is of course a way to make this operator compatible with Hepp sectors and has been defined in the context of TGFT models [8].

## 5 Polynomial invariants

We study now in details the polynomials obtained in the parametric amplitudes (13).  $U_{\mathcal{G}}^{\text{od/ev}}$  will be referred to the first Symanzik polynomial associated with the model amplitude. Since  $W_{\mathcal{G}}$  is not a polynomial, it cannot be directly called the second Symanzik polynomial. Nevertheless, we can study its properties as well. It is worth emphasizing that the following analysis does not specially focus on the models listed in (22). The amplitude (13) is completely general for a generic rank  $d \geq 2$  Abelian model with particular linear kinetic term. Therefore, the following study is valid for any model of this kind using tensor invariant vertices (in the sense of definition (8)) and rank  $d$  stranded lines. Furthermore, as one can realize in a straightforward manner, the definitions of the polynomial  $U_{\mathcal{G}}^{\text{od/ev}}$  and function  $W_{\mathcal{G}}$  can be extended to the larger class of rank  $d$  colored tensor graphs as defined in Subsection 2.1. This means that these generalized polynomials should appear in the parametric amplitudes of dynamical Abelian colored tensor models [95]. One must simply observe that, in the definition of the polynomials  $U_{\mathcal{G}}^{\text{od/ev}}$  and  $W_{\mathcal{G}}$ , the factorization in faces and the bi-coloring of strands play the main roles. The following analysis only relies on these ingredients which are in both models (the unitary tensor invariant and rank  $d$  colored models). In the following, we will not distinguish the study between these frameworks. Any graphs which might come from these models are simply referred to rank  $d$  color tensor graphs. Finally, the particular case of  $d = 2$  might generate some peculiarities that we will often address in a separate discussion. For higher rank  $d > 2$  illustrations, we will restrict ourselves to  $d = 3$  which is already not trivial. The higher rank case can be deduced from the  $d = 3$  case.

The usual Symanzik polynomials must satisfy some invariance properties under specific operations on their graphs. In scalar quantum field theory, it is well-known that such polynomials satisfy a contraction/deletion rule, hence, by a famous universality theorem, define Tutte polynomials [68]. For the GW model in 4D, the polynomials on ribbon graphs discovered in the parametric representation of this model [82] were deformed versions of the Bollobàs-Riordan polynomial [73][72]. The recurrence relation obeyed by these invariants is however much more involved [69] (a four-term recurrence using Chmutov partial duality [74]). Our remaining task is to investigate the types of relations which are satisfied by the identified functions  $U_{\mathcal{G}}^{\text{od/ev}}$  and  $\tilde{W}_{\mathcal{G}}$  ( $W_{\mathcal{G}}$  will satisfy relations which can be inferred from these points).

The rest of the section is divided into three parts. The first part focuses on the study of  $U_{\mathcal{G}}^{\text{od/ev}}$  and  $\tilde{W}_{\mathcal{G}}$  and the type of modified relation that they satisfy. In rank 2, a connection with the work by Krajewski et al. [69] is rigorously established in the second point. Motivated by the two initial discussions, in the third part of this section, we identify a polynomial that we call of the second kind,  $\mathcal{U}_{\mathcal{G}}$ , which is stable under a contraction reduction. To the best of our knowledge, it is for the first time that such a rule without referring to the deletion operation can be defined on a graph polynomial invariant. As an intriguing object worth to be exemplified, we list its properties and include several illustrations. The definition of the new polynomial is however totally abstract and, of course, it remains an open question if there exists a quantum field theory having such a polynomial appearing in its parametric amplitudes.

Few remarks must be made at this stage. The cut of an edge in a tensor invariant theory

is performed in the same way as is done in the colored case as discussed in Subsection 2.1 (see Fig.8). However, the contraction of an edge in a graph in an invariant tensor model must be understood as the contraction of a stranded line of color 0 with the same rule explained in Subsection 2.1 (see Fig.9). It turns out that our final statements are always independent on the type of models either tensor invariant or colored. Finally, in the following a rank  $d$  graph can either be a ribbon graph with half-ribbons or a rank  $d > 2$  colored tensor graph (either in the sense of Section 2 or coming from the gluing of rank  $d$  unitary tensor invariants).

## 5.1 Polynomials of the first kind

The objects of interest are the polynomials  $U_{\mathcal{G}}^{\text{od/ev}}$  and  $\tilde{W}_{\mathcal{G}}$ . These polynomials are called of the first kind.

Ordinary operations of contraction and deletion of edges of a graph  $\mathcal{G}$  have been defined in Section 2. We recall some terminology and give precisions:

- Given an edge  $e$  and a face  $f$ , we write  $e \in f$  when the face  $f$  passes through  $e$  (we also say that “ $e$  belongs to  $f$ ”). If  $f$  passes through  $e$  exactly  $\alpha$  times, we denote as  $e^\alpha \in f$ . Note that  $0 \leq \alpha \leq 2$ . From now,  $e^1 \in f \equiv e \in f$ .

- In the rank  $d > 2$ , the theory is colored and always  $e^\alpha \in f$ ,  $\alpha$  is necessarily 1.

- In this section, “contraction” always refers to soft contraction.

- Given  $e^\alpha \in f$ , we denote  $f/e$  (resp.  $f - e$ ,  $f \vee e$ ) the face resulting from  $f$  after the contraction (resp. the deletion, the cut) of  $e$  in  $\mathcal{G}$  yielding  $\mathcal{G}/e$  (resp.  $\mathcal{G} - e$ ,  $\mathcal{G} \vee e$ ).

The following statement holds.

**Lemma 4** (Face contraction). *Let  $e$  be an edge of  $\mathcal{G}$  a rank  $d \geq 2$  graph, which is not a loop and consider  $e^\alpha \in f$ ,  $f \in \mathcal{F}_{\text{int}}$ . We have*

(i) *If  $\alpha = 1$ , then*

$$A_f^{\text{od}} = t_e A_{f/e}^{\text{ev}} + A_{f/e}^{\text{od}}, \quad A_f^{\text{ev}} = t_e A_{f/e}^{\text{od}} + A_{f/e}^{\text{ev}}, \quad (90)$$

(ii) *If  $\alpha = 2$ , then*

$$A_f^{\text{od}} = 2t_e A_{f/e}^{\text{ev}} + (t_e^2 + 1)A_{f/e}^{\text{od}}, \quad A_f^{\text{ev}} = 2t_e A_{f/e}^{\text{od}} + (t_e^2 + 1)A_{f/e}^{\text{ev}}. \quad (91)$$

*When  $e$  is a trivial untwisted (resp. twisted) loop, then one can only have  $e \in f$  (resp.  $e^2 \in f$ ) and (i) (resp. (ii)) holds.*

*Proof.* Clearly, the even face polynomial and odd face polynomial play a symmetric role. We shall prove the claims for the odd case, from this, the even case can simply be inferred.

Let us assume that  $e \in f$ . This means that the factor  $t_e$  appears just once in  $A_f^{\text{od}}$  so that

$$A_f^{\text{od}}(\{t_l\}) = \left[ \sum_{\substack{ACf \\ |A| \text{ odd}; e \in A}} + \sum_{\substack{ACf \\ |A| \text{ odd}; e \notin A}} \right] \prod_{l \in A} t_l. \quad (92)$$

The subsets  $A \subset f$  such that  $e \notin A$  correspond exactly to subsets  $A' \subset f/e$ . This shows that  $\sum_{\substack{ACf \\ |A| \text{ odd}; e \notin A}} \prod_{l \in A} t_l = A_{f/e}^{\text{od}}$ . Meanwhile, the subsets  $A \subset f$  such that  $e \in A$  have a common factor  $t_e$ . This simply factorizes and yields the even monomials generated by  $A \subset f/e$ .



Assume now that  $e^2 \in f$ . The terms  $(1 + t_e^2)$  and  $2t_e$  must appear in  $A_f^{\text{od}}$ . We have

$$A_f^{\text{od}}(\{t_l\}) = \left[ \sum_{\substack{A \subset f \\ |A| \text{ odd}; e \in A}} + \sum_{\substack{A \subset f \\ |A| \text{ odd}; e^2 \in A}} + \sum_{\substack{A \subset f \\ |A| \text{ odd}; e \notin A}} \right] \prod_{l \in A} t_l. \quad (93)$$

One notices that factoring out  $t_e^2$  common in all monomials in the middle sum, the odd monomials generated by  $A \subset f$ , such that  $e^2 \in A$  and  $l \in A, l \neq e$ , are precisely those generated by  $A \subset f$  such that  $e \notin A$ . Moreover, the last sum coincides with  $A_{f/e}^{\text{od}}$  for the same reason invoked above (in the case  $e^1 \in f$ ). Then the two last sums are nothing but  $(t_e^2 + 1)A_{f/e}^{\text{od}}$ . In the first sum in (93), after factoring out  $2t_e$ , for the same reason as previously stated, we obtain exactly the even monomials generated by the contracted face  $f/e$ .

The last point on trivial loops can be inferred in the similar way.  $\square$

We are in position to investigate the recurrence rules obeyed by  $U_{\mathcal{G}}^{\text{od/ev}}$  in rank 2.

**Proposition 2** (Broken recurrence rules for  $U^{\text{od/ev}}$  in rank 2). *Let  $\mathcal{G}$  be a ribbon graph with half-ribbons,  $\mathcal{F}_{\text{int};\mathcal{G}}$  and  $\mathcal{F}_{\text{ext};\mathcal{G}}$  be its sets of internal and external faces, respectively, and  $e$  be a regular edge of  $\mathcal{G}$ . Then:*

(i) *If  $e$  belongs only to external faces then*

$$U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}/e}^{\text{od/ev}}. \quad (94)$$

*Furthermore, if the deletion of the edge  $e$  does not generate a new internal face  $U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}/e}^{\text{od/ev}} = U_{\mathcal{G}-e}^{\text{od/ev}}$ . If it generates a new internal face  $f$ , then  $A_f^{\text{od/ev}} U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}-e}^{\text{od/ev}}$ .*

(ii) *If  $e \in f$  and  $e \in f'$ ,  $f \in \mathcal{F}_{\text{int};\mathcal{G}}$  and  $f' \in \mathcal{F}_{\text{ext};\mathcal{G}}$ , we have  $U_{\mathcal{G} \vee e}^{\text{od/ev}} = U_{\mathcal{G}-e}^{\text{od/ev}}$  and*

$$U_{\mathcal{G}}^{\text{od/ev}} = t_e A_{f/e}^{\text{ev/od}} U_{\mathcal{G}-e}^{\text{od/ev}} + U_{\mathcal{G}/e}^{\text{od/ev}}. \quad (95)$$

(iii) *If  $e \in f$  and  $e \in f'$ ,  $f \neq f'$ , and  $f, f' \in \mathcal{F}_{\text{int};\mathcal{G}}$ , we get*

$$U_{\mathcal{G}}^{\text{od/ev}} = t_e U_{\mathcal{G}-e}^{\text{od}} + U_{\mathcal{G}/e}^{\text{od/ev}} + t_e^2 A_{f/e}^{\text{ev/od}} A_{f'/e}^{\text{ev/od}} U_{\mathcal{G} \vee e}^{\text{od/ev}}. \quad (96)$$

(iv) *If  $e^2 \in f$ ,  $f \in \mathcal{F}_{\text{int};\mathcal{G}}$ , then two cases occur:*

(a) *the deletion of  $e$  yields two internal faces  $f_1$  and  $f_2$ , then*

$$\begin{cases} U_{\mathcal{G}}^{\text{od}} \\ U_{\mathcal{G}}^{\text{ev}} \end{cases} = \begin{cases} (1 + t_e^2) U_{\mathcal{G}/e}^{\text{od}} + 2t_e U_{\mathcal{G}-e}^{\text{od}} + 2t_e A_{f_1}^{\text{ev}} A_{f_2}^{\text{ev}} U_{\mathcal{G} \vee e}^{\text{od}} \\ (1 + t_e^2) U_{\mathcal{G}/e}^{\text{ev}} + 2t_e (A_{f_1}^{\text{ev}} A_{f_2}^{\text{od}} + A_{f_1}^{\text{od}} A_{f_2}^{\text{ev}}) U_{\mathcal{G} \vee e}^{\text{ev}} \end{cases}. \quad (97)$$

(b) *the deletion of  $e$  yields one internal face  $f_{12}$ ,*

$$U_{\mathcal{G}}^{\text{od/ev}} = (1 + t_e^2) U_{\mathcal{G}/e}^{\text{od/ev}} + 2t_e A_{f_{12}}^{\text{ev/od}} U_{\mathcal{G} \vee e}^{\text{od/ev}}. \quad (98)$$

*Proof.* See Appendix A. □

For special edges, the above proposition is still valid but simplifies drastically:

- if  $e$  is a bridge, then under condition (i), (94) and the following relations are all valid and, under condition (iv.a), (97) holds. These are the only possibilities for a bridge.

- if  $e$  is a trivial untwisted loop, under condition (i), (94) is valid and since the contraction of a such a loop cannot create a new internal face, we always have  $U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}/e}^{\text{od/ev}} = U_{\mathcal{G}-e}^{\text{od/ev}}$ . Assuming (ii), (95) holds as well. Now, under (iii), one proves that (96) is true. We do not have any further choices.

- if  $e$  is a trivial twisted loop, assuming (i) holds, (94) is valid and  $U_{\mathcal{G}/e}^{\text{od/ev}} = U_{\mathcal{G}-e}^{\text{od/ev}}$  by definition. Under (iv.b), (98) holds. These are the only cases valid for a trivial twisted loop.

**Proposition 3** (Broken recurrence rules for  $U^{\text{od/ev}}$  in rank  $d > 2$ ). *Let  $\mathcal{G}$  be a rank  $d > 2$  colored tensor model graph. Let  $e$  be an edge of  $\mathcal{G}$  and  $N \in [0, d]$  be the number of internal faces that pass through the edge  $e$ . Then*

$$U_{\mathcal{G}}^{\text{od/ev}} = \begin{cases} U_{\mathcal{G}/e}^{\text{od/ev}} + U_{\mathcal{G}\vee e}^{\text{od/ev}} \sum_{K \in [1, N]/K \neq \emptyset} \left[ t_e^K \left( \prod_{i \in K} A_{f_i/e}^{\text{ev/od}} \right) \left( \prod_{i \in K^c} A_{f_i/e}^{\text{od/ev}} \right) \right] & \text{for } N \geq 1, \\ U_{\mathcal{G}/e}^{\text{od/ev}} & \text{for } N = 0. \end{cases} \quad (99)$$

*Proof.* Let us assume that  $N = 0$ , no internal faces pass through  $e$ . The result  $U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}/e}^{\text{od/ev}}$  is direct. Let us call  $f_i, i = 1, \dots, N$  the internal face passing through  $e$ . We start by writing, using Lemma 4,

$$\begin{aligned} U_{\mathcal{G}}^{\text{od/ev}} &= \prod_{f_i \in \mathcal{F}_{\text{int}}} A_{f_i}^{\text{od/ev}} \prod_{\substack{f \in \mathcal{F}_{\text{int}} \\ f \neq f_i}} A_f^{\text{od/ev}} = \prod_{i=1}^N (t_e A_{f_i/e}^{\text{ev/od}} + A_{f_i/e}^{\text{od/ev}}) \prod_{\substack{f \in \mathcal{F}_{\text{int}} \\ f \neq f_i}} A_f^{\text{od/ev}} \\ &= \left[ \sum_{K \subset [1, N]} \prod_{i \in K} (t_e A_{f_i/e}^{\text{ev/od}}) \prod_{i \in K^c} A_{f_i/e}^{\text{od/ev}} \right] \prod_{\substack{f \in \mathcal{F}_{\text{int}} \\ f \neq f_i}} A_f^{\text{od/ev}} \\ &= U_{\mathcal{G}\vee e}^{\text{od/ev}} \sum_{K \subset [1, N]/K \neq \emptyset} \left[ t_e^K \left( \prod_{i \in K} A_{f_i/e}^{\text{ev/od}} \right) \left( \prod_{i \in K^c} A_{f_i/e}^{\text{od/ev}} \right) \right] + U_{\mathcal{G}/e}^{\text{od/ev}}. \end{aligned} \quad (100)$$

□

One notices that Proposition 3 generalizes Proposition 2 in rank  $d = 2$  if the internal faces do not pass more than once through  $e$ . In particular, (94), (95) and (96) can be recovered from (99). Meanwhile, for trivial tensor loop edges, the result again holds. For a bridge, in a colored or invariant tensor model, all faces are necessarily external [70], and we have  $U_{\mathcal{G}}^{\text{od/ev}} = U_{\mathcal{G}/e}^{\text{od/ev}}$ .

**Remark 1.** *We notice that the polynomials  $U_{\mathcal{G}}^{\text{od/ev}}$  do not obey stable contraction/deletion/cut rules on ribbon graphs with flags like the Tutte and Bollobàs-Riordan polynomials. The insisting appearance of the face polynomials  $A_f^{\text{ev/od}}$ , in the above broken recurrence relations, suggests the existence of a more general polynomial. We will introduce an extended version of  $U^{\text{od/ev}}$  in Subsection 5.3.*

**Proposition 4** (Modified recurrence rules for  $\tilde{W}$  in rank 2). *Let  $\mathcal{G}$  be a ribbon graph with half-ribbons,  $\mathcal{F}_{\text{int};\mathcal{G}}$  and  $\mathcal{F}_{\text{ext};\mathcal{G}}$  be its sets of internal and external faces, respectively, and  $e$  be a regular edge of  $\mathcal{G}$ . We write  $T_{l,f} = \left(\frac{1-t_l}{1+t_l}\right)^{|m_f|}$ .*

(i) *Consider  $e$  belongs only to external faces,  $f$  and  $f'$ . Then*

$$\tilde{W}_{\mathcal{G}} = T_{e,f}T_{e,f'}\tilde{W}_{\mathcal{G}/e}. \quad (101)$$

Furthermore,

(a) *if either  $\{f \neq f'\}$  or  $\{f = f' (e^2 \in f)$  and the deletion of  $e$  does not generate any new internal faces*, then

$$\tilde{W}_{\mathcal{G}/e} = \tilde{W}_{\mathcal{G}\vee e} = \tilde{W}_{\mathcal{G}-e}, \quad (102)$$

(b)  *$f = f' (e^2 \in f)$  and the deletion of  $e$  generates a new internal face  $f''$ ,*

$$\tilde{W}_{\mathcal{G}/e} = \tilde{W}_{\mathcal{G}\vee e} = \left(\prod_{l \in f''} T_{l,f''}\right)\tilde{W}_{\mathcal{G}-e}. \quad (103)$$

(iii) *If  $e \in f$  and  $e \in f'$ ,  $f \in \mathcal{F}_{\text{int};\mathcal{G}}$  and  $f' \in \mathcal{F}_{\text{ext};\mathcal{G}}$ ,*

$$\tilde{W}_{\mathcal{G}} = T_{e,f}\tilde{W}_{\mathcal{G}/e}, \quad T_{e,f'}\tilde{W}_{\mathcal{G}\vee e} = T_{e,f'}\tilde{W}_{\mathcal{G}-e} = \left(\prod_{\substack{l \in f \\ l \neq e}} T_{l,f}\right)\tilde{W}_{\mathcal{G}}. \quad (104)$$

(iv) *If  $e \in f$  and  $e \in f'$ , and  $f, f' \in \mathcal{F}_{\text{int};\mathcal{G}}$*

$$\tilde{W}_{\mathcal{G}} = \tilde{W}_{\mathcal{G}/e} = \tilde{W}_{\mathcal{G}-e}. \quad (105)$$

(a) *Furthermore, if  $f \neq f'$ ,*

$$\tilde{W}_{\mathcal{G}\vee e} = \left(\prod_{\substack{l \in f \\ l \neq e}} T_{l,f}\right)\left(\prod_{\substack{l \in f' \\ l \neq e}} T_{l,f'}\right)\tilde{W}_{\mathcal{G}}. \quad (106)$$

(b) *If  $f = f'$ , ( $e^2 \in f$ )*

$$\tilde{W}_{\mathcal{G}\vee e} = \left(\prod_{\substack{l \in f \\ l \neq e}} T_{l,f}\right)\tilde{W}_{\mathcal{G}}. \quad (107)$$

*Proof.* We will concentrate on the cases which can only occur in rank  $d = 2$ . These cases include  $e^2 \in f$ , for some face  $f$ , or when the deletion  $\mathcal{G} - e$  can be performed. All the remaining relations will be a corollary of the next Proposition 5.

By cutting an external face ( $f \vee e$ ), or by contracting it ( $f/e$ ), then  $\prod_{l \in f \vee e} T_{l,f \vee e} = \prod_{l \in f/e} T_{l,f/e}$ . Using this, one proves that in (102) and (103),  $\tilde{W}_{\mathcal{G}/e} = \tilde{W}_{\mathcal{G}\vee e}$ .

Proving  $\tilde{W}_{\mathcal{G} \vee e} = \tilde{W}_{\mathcal{G} - e}$  (102), one must observe that  $\prod_{l \in f \vee e} T_{l, f \vee e} = \prod_{l \in f - e} T_{l, f - e}$ , where  $f - e$  is the external face resulting from  $f$  in  $\mathcal{G} - e$ .

Focusing on (103), the cut graph  $\mathcal{G} \vee e$  contains an additional external face compared to  $\mathcal{G} - e$  (in fact, this additional external face becomes a closed face in  $\mathcal{G} - e$ ). The same external face of  $\mathcal{G} \vee e$  generates the additional factor.

Now (104) holds for almost the same reasons mentioned above: cutting  $e$  or removing it, from the graph  $\mathcal{G}$  cannot be distinguished by  $\tilde{W}$ . The set of lines in  $f - e$  union the set of lines in  $f' - e$  is one-to-one with the set of lines in  $f \vee e$  union the set of lines  $f' \vee e$ .

Concerning (105), one must pay attention that, either in  $\mathcal{G} - e$  or in  $\mathcal{G}/e$ , the faces passing through  $e$  are internal after the operation.

We focus on (107) and note that the set of lines in  $f$  subtracted by  $e$  coincides with the set of lines of  $f \vee e$  in  $\mathcal{G} \vee e$ . Thus, after cutting  $e$  in  $\mathcal{G}$ ,  $\tilde{W}_{\mathcal{G} \vee e}$  possesses an extra factor coming from the set of lines resulting from the cut of  $f$ . □

**Proposition 5** (Modified recurrence rules for  $\tilde{W}$  for rank  $d > 2$ ). *Let  $\mathcal{G}$  be a colored tensor graph of rank  $d > 2$ ,  $\mathcal{F}_{\text{int}; \mathcal{G}}$  and  $\mathcal{F}_{\text{ext}; \mathcal{G}}$  be its sets of internal and external faces, respectively, and  $e$  be a regular edge of  $\mathcal{G}$ . Let  $\mathcal{F}_{\text{ext}; e}$  (resp.  $\mathcal{F}_{\text{int}; e}$ ) be the set of external (resp. internal) faces going through  $e$ . Then,*

$$\tilde{W}_{\mathcal{G}} = \begin{cases} \tilde{W}_{\mathcal{G}/e} & \text{for } \mathcal{F}_{\text{ext}; e} = \emptyset, \\ (\prod_{f \in \mathcal{F}_{\text{ext}; e}} T_{e, f}) \tilde{W}_{\mathcal{G}/e} & \text{for } \mathcal{F}_{\text{ext}; e} \neq \emptyset. \end{cases} \quad (108)$$

and

$$\left( \prod_{f \in \mathcal{F}_{\text{ext}; e}} T_{e, f} \right) \tilde{W}_{\mathcal{G} \vee e} = \left( \prod_{f \in \mathcal{F}_{\text{int}; e}} \prod_{l \in f / l \neq e} T_{l, f} \right) \tilde{W}_{\mathcal{G}}. \quad (109)$$

*Proof.* Noting that the operation of contraction preserves the number of external (resp. internal) faces in  $\mathcal{G}$ , then in  $\mathcal{G}/e$ , we only lose the variables associated with  $e$ . Then, (108) follows.

For the cut operation, one must pay attention to the fact that the internal faces in  $\mathcal{G}$  become external faces in  $\mathcal{G} \vee e$ , whereas external faces in  $\mathcal{G}$  generate only more external faces in  $\mathcal{G} \vee e$ . Then, (109) follows. □

Some comments are in order:

- One can check now that all statements except those involving  $e^2 \in f$  or  $\mathcal{G} - e$  in Proposition 4 can be recovered from Proposition 5.

- Notice that  $\tilde{W}_{\mathcal{G}}$  is a polynomial in  $\{T_{l, f}\}$  which always satisfies a well defined recurrence relation under contraction operation. To be clearer,  $\tilde{W}_{\mathcal{G}}$  is stable under contraction or cut rule.

- Discussing special edges (bridges, trivial loops), one can check that the above propositions specialize but are still valid.

## 5.2 Relations to other polynomials

The type of graphs we are treating here have been discussed in several works. However, the only polynomial that we find related to  $U_{\mathcal{G}}^{\text{od}}$  is provided by Krajewski et al. [69] in

the context of ribbon graphs with flags. We do not see, at this stage, any relationships between the polynomial on rank 3 colored graphs as worked out by Avohou et al. [70] and the polynomials of the present work.

In this section, we will concentrate on the relationship between the polynomial  $U_{\mathcal{G}}^{\text{od}}$  and polynomials discussed in [69]. As an outcome of this discussion, we will motivate the introduction of a new invariant  $\mathcal{U}_{\mathcal{G}}$  in the next section. We mention that this section is devoted exclusively to matrix model case or ribbon graphs. Henceforth, when there is no possible confusion, we simply refer ribbon graphs (possibly with flags) to graphs.

First, one must clarify the setting in which two (Hyperbolic) polynomials  $\text{HU}_{\mathcal{G}}(t)$  and  $\widetilde{\text{HU}}_{\mathcal{G}}(t)$  by Krajewski et al. are found. The model considered is the GW model in  $D$  dimensions. The corresponding parametric amplitudes have been computed and give, as expected, generalized Symanzik polynomials. The first Symanzik polynomial is  $\text{HU}_{\mathcal{G}}(\Omega, t)$ . Such an object has two kinds of variables  $t = \{t_l\}$  and  $\Omega = \{\Omega_l\}$  which are line or edge variables ( $\Omega_l$  is a new parameter important for ensuring renormalizability through the cure of the so-called UV/IR mixing).

The key relation that  $\text{HU}_{\mathcal{G}}^3$  satisfies is a four-term recurrence relation of the form (omitting boundary conditions i.e. vertices with only flags and terminal forms), for a regular edge  $e$ ,

$$\text{HU}_{\mathcal{G}} = t_e \text{HU}_{\mathcal{G}-e} + t_e \Omega^2 \text{HU}_{\mathcal{G}\vee e} + \Omega_e \text{HU}_{\mathcal{G}^e-e} + \Omega_e t_e^2 \text{HU}_{\mathcal{G}^e\vee e}, \quad (110)$$

where  $\mathcal{G}^e$  stands for the so-called Chmutov partial dual [74] of  $\mathcal{G}$  with respect to the edge  $e$ . This operation can simply be explained as follows: one must cut all lines in  $\mathcal{G}$  except  $e$ , then perform a dual on the pinched graph  $\widetilde{\mathcal{G}}$ , and glue back all edges previously cut. The interest of introducing such partial dual reflects on the contraction operation:  $\mathcal{G}/e = \mathcal{G}^e - e$ .

It turns out that the GW model can be expressed as well as a matrix model [79]. Moreover, at the limit when  $\Omega \rightarrow 1$ , the amplitudes of this model are of the form (13). To be precise, the rank  $d$  must be fixed to 2, and since the summation over the matrix indices in the GW model are performed over  $\mathbb{N}^2$ , one obtains a modified definition of  $W_{\mathcal{G}} = \widetilde{W}_{\mathcal{G}}$ . Finally, after this re-adjustment, we have the same amplitude up to a constant (a power of 2) depending on the graph. This constant has been incorporated in the definition of the polynomial but for the ensuing discussion this factor is totally inessential.

The problem as raised by authors, to the best of our understanding, is how to relate the new first Symanzik polynomial  $\widetilde{\text{HU}}_{\mathcal{G}}(t)$ <sup>4</sup> obtained in this matrix base and the limit  $\text{HU}_{\mathcal{G}}(1, t)$ . We emphasize a series of subtleties in the comparison procedure which will make clear our next point:

- First, the polynomial  $\widetilde{\text{HU}}_{\mathcal{G}}(t)$  was computed in an amplitude involving a closed graph i.e. a ribbon graph without flags. In fact, it directly extends to the case of a ribbon graph with flags provided one still performs a product over closed faces. Hence,  $\widetilde{\text{HU}}_{\mathcal{G}} = U_{\mathcal{G}}^{\text{od}}$ , up to a constant, on ribbon graphs with flags.

- Second, in order to relate  $\widetilde{\text{HU}}_{\mathcal{G}}(t)$  and  $\text{HU}_{\mathcal{G}}$  the authors introduce another polynomial

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<sup>3</sup>The expression of  $\text{HU}_{\mathcal{G}}$  can be found in [69]. For the rest of the discussion, we only need the recurrence relation that this polynomial satisfies.

<sup>4</sup>Note that in [69],  $\widetilde{\text{HU}}$  is denoted HU again, see Eq.(6.8) therein. For avoiding confusion, we use a different notation here.

called  $U_{\mathcal{G}}$  (Eq.(6.12), p. 532). This polynomial is defined as

$$U_{\mathcal{G}}(t) := \sum_{g \in \underline{\text{Odd}}(\mathcal{G}^*)} \prod_{l \in \mathcal{HE}(g)} t_l, \quad (111)$$

where  $\mathcal{G}^*$  is the dual of  $\mathcal{G}$ ,  $\underline{\text{Odd}}(\mathcal{G})$  is the set odd (colored) cutting spanning subgraphs of a graph  $\mathcal{G}$ . In the previous sentence, we put in parenthesis colored because precisely, the coloring refers to the bi-coloring of vertices. It has the effect of introducing a prefactor  $2^{v(\mathcal{G})}$  which is inessential in our discussion. An odd graph is a graph with all degrees of its vertices of odd parity. An odd cutting spanning subgraph  $g \in \underline{\text{Odd}}(\mathcal{G}^*)$  is a spanning subgraph of a graph  $\mathcal{G}$  (having all its vertices), obtained by choosing  $\mathcal{HR}(g) \subset \mathcal{HR}(\mathcal{G})$  and  $\mathcal{FL}(\mathcal{G}) \subset \mathcal{HR}(g)$  and such that  $g$  is odd.

The issue is that  $U_{\mathcal{G}}(t)$  is defined on open and closed graphs. And, as proved in the above reference, this quantity always coincides with  $\text{HU}_{\mathcal{G}}(1, t)$  and so satisfies the same four-term recurrence rule (110). On closed graphs,  $U_{\mathcal{G}}(t) = \widetilde{\text{HU}}_{\mathcal{G}}(t)$  and so matches with  $U_{\mathcal{G}}^{\text{od}}$ . However, on open graphs it is not true that  $U_{\mathcal{G}}(t)$  is equal to  $\widetilde{\text{HU}}_{\mathcal{G}}(t) = U_{\mathcal{G}}^{\text{od}}$ . The reason why there certainly is a discrepancy is because  $U_{\mathcal{G}}^{\text{od}}$  meets another formula. Indeed, since a closed face in  $\mathcal{G}$  corresponds to a vertex in  $\mathcal{G}^*$  which does not have any flags, we partition the vertices of  $\mathcal{G}^*$  in two distinct subsets:  $\mathcal{V}(\mathcal{G}^*) = \mathcal{V}'(\mathcal{G}^*) \cup \mathcal{V}''(\mathcal{G}^*)$ , where  $v \in \mathcal{V}'(\mathcal{G}^*)$  is without flags. Now considering the cutting subgraph  $S(\mathcal{G}^*)$  of  $\mathcal{G}^*$  having a set of vertices  $\mathcal{V}(S(\mathcal{G}^*)) = \mathcal{V}'(\mathcal{G}^*)$  and a set of edges  $\mathcal{E}(S(\mathcal{G}^*))$  containing all edges from  $\mathcal{V}'(\mathcal{G}^*)$  to  $\mathcal{V}''(\mathcal{G}^*)$  and cutting all edges from  $\mathcal{V}''(\mathcal{G}^*)$  to  $\mathcal{V}''(\mathcal{G}^*)$ . We write

$$U_{\mathcal{G}}^{\text{od}} = \prod_{v^* \in \mathcal{V}'(\mathcal{G}^*)} \left\{ \sum_{\substack{A \subset \mathcal{HE}(v^*) \\ |A| \text{ odd}}} \prod_{l \in A} t_l \right\} = \sum_{g \in \text{Odd}^{\natural}(\mathcal{G}^*)} \prod_{l \in \mathcal{HR}(g)} t_l = U_{S(\mathcal{G}^*)}^{\natural}, \quad (112)$$

where  $\mathcal{HE}(v^*)$  stands for the set of half-edges incident to  $v^*$ , and in  $\mathcal{HR}(g)$ , flags are labeled with the same label of the edges where they come from.  $\text{Odd}^{\natural}(\mathcal{G})$  is set of odd cutting spanning subgraphs of a second kind:  $g \in \text{Odd}^{\natural}(\mathcal{G})$  is defined such that  $\mathcal{HR}(g) \subset \mathcal{HR}(\mathcal{G})$ . Hence,  $U_{\mathcal{G}}^{\text{od}} \neq U_{\mathcal{G}}$  and  $U_{S(\mathcal{G}^*)}^{\natural}$  is the closest expression that we have found related to  $U_{\mathcal{G}}$ .

We fully illustrate now the above discussion by examples.

**Example 1: Triangle with flags.** Consider the graph  $\mathcal{G}$  as a triangle with one flag on each

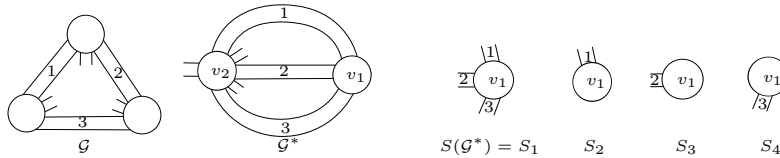


Figure 26: Triangle graph with flags  $\mathcal{G}$ ,  $\mathcal{G}^*$ ,  $S_1$ ,  $S_2$ ,  $S_3$ , and  $S_4$  which define  $\text{Odd}^{\natural}(S(\mathcal{G}^*))$ .

vertex, all in the same face (see Fig.26). In [69],  $\text{HU}_{\mathcal{G}}(1, t) = U_{\mathcal{G}}(t)$  was already computed and it gives

$$\text{HU}_{\mathcal{G}}(1, t) = 4(t_1 + t_2 + t_3 + t_1 t_2 t_3)(1 + t_1 t_2 + t_1 t_3 + t_2 t_3). \quad (113)$$

Computing  $U_{\mathcal{G}}^{\text{od}}$  directly from the face amplitude formula, one has:

$$U_{\mathcal{G}}^{\text{od}}(t) = t_1 + t_2 + t_3 + t_1 t_2 t_3. \quad (114)$$

Clearly, for open graphs the polynomials do not agree.

Let us now explain the expansion (112). Consider the dual  $\mathcal{G}^*$  of  $\mathcal{G}$  in Fig.26. First  $\mathcal{V}'(\mathcal{G}^*) = \{v_1\}$  (the vertex without flags),  $S(\mathcal{G}^*)$  is the graph made with  $v_1$  with three flags labeled by 1, 2, 3 in the same way of the lines  $l_1$ ,  $l_2$  and  $l_3$  and are associated with variables  $t_1, t_2$  and  $t_3$ . We obtain four cutting spanning subgraphs in

$$\text{Odd}^{\text{h}}(S(\mathcal{G}^*)) = \{\{l_1\}, \{l_2\}, \{l_3\}, \{l_1, l_2, l_3\}\} \quad (115)$$

as in Fig.26 with contributions  $t_1$ ,  $t_2$ ,  $t_3$  and  $t_1 t_2 t_3$ , respectively. On the other hand,

$$\begin{aligned} \check{\text{O}}\text{dd}(\mathcal{G}^*) = & \{\{l_1; l_1, l_2\}, \{l_1; l_1, l_3\}, \{l_1; l_2, l_3\}, \{l_2; l_2, l_1\}, \{l_2; l_2, l_3\}, \{l_2; l_1, l_3\}, \{l_3; l_3, l_1\}, \\ & \{l_3; l_3, l_2\}, \{l_3; l_1, l_2\}, \{l_1, l_2, l_3; l_1, l_2\}, \{l_1, l_2, l_3; l_1, l_3\}, \{l_1, l_2, l_3; l_2, l_3\}, \\ & \{l_1; \emptyset\}, \{l_2; \emptyset\}, \{l_3; \emptyset\}, \{l_1, l_2, l_3; \emptyset\}\}, \end{aligned} \quad (116)$$

where on each side the semi-colon in the brackets, we collect half-edges on each vertex  $v_1$  and  $v_2$ .

**Example 2: Pretzel without flags.** Consider the graph  $\mathcal{G}$  drawn in Fig.27. We also

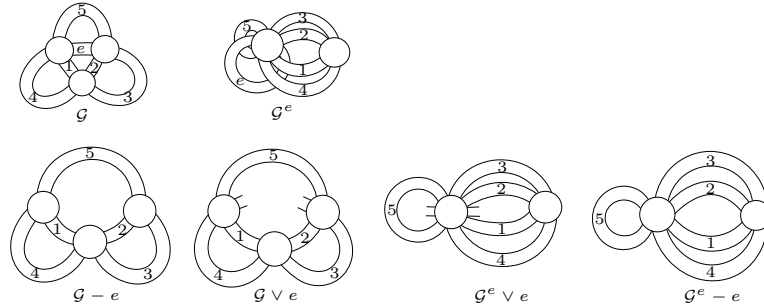


Figure 27: The pretzel graph  $\mathcal{G}$ , its partial dual with respect to  $e$ ,  $\mathcal{G}^e$ , and its operated graphs.

illustrate  $\mathcal{G}^e$ ,  $\mathcal{G} - e$ ,  $\mathcal{G} \vee e$ ,  $\mathcal{G}^e \vee e$ , and  $\mathcal{G}^e - e$  in that picture. We call  $C(t) = (t_2 + t_3)(t_1 + t_4)(t_2 + t_3)(t_3 + t_4 + t_5 + t_3 t_4 t_5)$ , and we evaluate

$$\begin{aligned} U_{\mathcal{G}}^{\text{od}} &= (t_e + t_1 + t_2 + t_e t_1 t_2)(t_e + t_5)C(t) \\ &= [t_e^2 + t_e(t_1 + t_2 + t_5 + t_1 t_2 t_5) + t_e t_1 t_2 + t_1 t_5 + t_2 t_5]C(t), \\ U_{\mathcal{G} - e}^{\text{od}} &= [t_1 + t_2 + t_5 + t_1 t_2 t_5]C(t), \\ U_{\mathcal{G} \vee e}^{\text{od}} &= C(t) = U_{\mathcal{G}^e \vee e}^{\text{od}}, \\ U_{\mathcal{G}^e - e}^{\text{od}} &= t_5(t_1 + t_2)C(t) = (t_1 t_5 + t_2 t_5)C(t). \end{aligned} \quad (117)$$

From this point, by observing the term  $t_e t_1 t_2$  in  $U_{\mathcal{G}}^{\text{od}}$ , one can readily check that there exist no polynomials function  $p_i(t_e)$  in  $t_e$  variable such that a relation of the type

$$p_1(t_e)U_{\mathcal{G}}^{\text{od}} = p_2(t_e)U_{\mathcal{G} - e}^{\text{od}} + p_3(t_e)U_{\mathcal{G} \vee e}^{\text{od}} + p_4(t_e)U_{\mathcal{G}^e - e}^{\text{od}} + p_5(t_e)U_{\mathcal{G}^e \vee e}^{\text{od}} \quad (118)$$

is satisfied. Thus  $U_{\mathcal{G}}^{\text{od}}$  does not obey the same relation as  $\text{HU}_{\mathcal{G}}$ . Therefore, it is not a  $\mathcal{Q}_{\mathcal{G}}$  polynomial in the general sense of Krajewski et al.

We understand now that  $U_{\mathcal{G}}^{\text{od}}$  on the class of open graphs does not obey any known recurrence relations. In the tensor situation, things become worse: we do not have any clear notion of duality at the level of graphs and the notion of Chmutov dual is still not defined. This urges us to find another path to understand this object  $U_{\mathcal{G}}^{\text{od/ev}}$ . A natural route that we will investigate is the understanding of the notion of face amplitude that we observe to be at the heart of this theory. We will introduce an extended framework, where a generalized version of  $U_{\mathcal{G}}^{\text{od/ev}}$  makes sense and turns out to satisfy a proper invariance rule. This is the purpose of the next section.

### 5.3 Polynomial of the second kind

First recognizing that the polynomials are sensitive to the properties of faces, we will exploit this face-structure by defining a new polynomial  $\mathcal{U}_{\mathcal{G}}$ . This object generalizes  $U_{\mathcal{G}}^{\text{od/ev}}$  and obeys a novel recurrence relations based only on contraction operation. We call it of the second kind. An extension of  $\tilde{W}_{\mathcal{G}}$  will not be discussed for two main reasons: first,  $\tilde{W}_{\mathcal{G}}$  is already stable under contraction and, second, the notion of parity in  $U_{\mathcal{G}}^{\text{od/ev}}$  which is at the core of the next developments does not appear at all in  $\tilde{W}_{\mathcal{G}}$ . Finally, most of the ingredients used in the following have been introduced in Subsection 5.1.

Let  $\mathcal{G}^*$  be the set of isomorphism classes of rank  $d$  tensor graphs (including ribbon graphs with half-ribbons)  $\{\text{od}, \text{ev}\}$  be the set of parities (in obvious notations). Let  $\mathcal{G} \in \mathcal{G}^*$  with a set of internal faces  $\mathcal{F}_{\text{int};\mathcal{G}}$  and  $\mathcal{P}_{\text{int};\mathcal{G}}$  be the power set of  $\mathcal{F}_{\text{int};\mathcal{G}}$ .

**Definition 7** (Generalized polynomial). *Consider an element  $(\mathcal{G}, \mathcal{F}, \overline{\mathcal{F}}, \epsilon, \epsilon') \in \mathcal{G}^* \times (\mathcal{P}_{\text{int};\mathcal{G}})^{\times 2} \times \{\text{od}, \text{ev}\}^{\times 2}$  such that  $\mathcal{F} \cup \overline{\mathcal{F}} = \mathcal{F}_{\text{int};\mathcal{G}}$  and  $\mathcal{F} \cap \overline{\mathcal{F}} = \emptyset$ . We define a generalized polynomial associated with  $(\mathcal{G}, \mathcal{F}, \overline{\mathcal{F}}, \epsilon, \epsilon')$  as*

$$\mathcal{U}_{\mathcal{G};(\mathcal{F}, \overline{\mathcal{F}})}^{\epsilon, \epsilon'}(\{t_l\}) = \left[ \prod_{f \in \mathcal{F}} A_f^{\epsilon}(\{t_l\}_{l \in f}) \right] \left[ \prod_{f \in \overline{\mathcal{F}}} A_f^{\epsilon'}(\{t_l\}_{l \in f}) \right]. \quad (119)$$

Note that from the definition of  $U_{\mathcal{G}}$  (15), it is immediate to have (when using subscripts, we write  $Q_{\mathcal{F}_{\text{int};\mathcal{G}}} = Q_{\mathcal{F}_{\text{int}}}$ , for any quantity  $Q$ )

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F}_{\text{int}}, \emptyset)}^{\epsilon, \epsilon'}(\{t_l\}) &= \mathcal{U}_{\mathcal{G};(\emptyset, \mathcal{F}_{\text{int}})}^{\epsilon', \epsilon}(\{t_l\}) = U_{\mathcal{G}}^{\epsilon}(\{t_l\}), & \epsilon &= \text{od}, \text{ev}; \\ \forall \mathcal{F} \in \mathcal{P}_{\text{int};\mathcal{G}}, \quad \mathcal{U}_{\mathcal{G};(\mathcal{F}, \overline{\mathcal{F}})}^{\epsilon, \epsilon}(\{t_l\}) &= U_{\mathcal{G}}^{\epsilon}(\{t_l\}), & \epsilon &= \text{od}, \text{ev}, \end{aligned} \quad (120)$$

for any value of  $\epsilon'$ . Furthermore  $\mathcal{U}_{\mathcal{G}}$  is symmetric under the flips:

$$\mathcal{U}_{\mathcal{G};(\mathcal{F}, \overline{\mathcal{F}})}^{\epsilon, \epsilon'}(\{t_l\}) = \mathcal{U}_{\mathcal{G};(\overline{\mathcal{F}}, \mathcal{F})}^{\epsilon', \epsilon}(\{t_l\}). \quad (121)$$

From these properties, the only case of interest is of  $\mathcal{U}_{(\cdot)}^{\text{od, ev}}$ .

As a convention, for the empty graph  $\mathcal{G} = \emptyset$ ,

$$\mathcal{U}_{\emptyset;(\emptyset, \emptyset)}^{\text{od, ev}} = 1 \quad (122)$$



and, on the bare vertex graph  $\mathcal{G} = o$  with a unique closed face  $f$ , according to (19), we have

$$\mathcal{U}_{o;(\{f\},\emptyset)}^{\text{od,ev}} = 0, \quad \mathcal{U}_{o;(\emptyset,\{f\})}^{\text{od,ev}} = 1. \quad (123)$$

Now, if it occurs that  $\mathcal{G} \neq \emptyset$  and  $\mathcal{F}_{\text{int};\mathcal{G}} = \emptyset$ , then  $\mathcal{F} = \overline{\mathcal{F}} = \emptyset$ , so that

$$\mathcal{U}_{\mathcal{G};(\emptyset,\emptyset)}^{\text{od,ev}} = 1. \quad (124)$$

The following proposition follows from definitions.

**Proposition 6** (Disjoint union operations). *Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two rank  $d$  graphs, let  $\mathcal{G}_1 \sqcup \mathcal{G}_2$  be their disjoint union. Then,*

$$\mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2;(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\epsilon'} = \mathcal{U}_{\mathcal{G}_1;(\mathcal{F}_1,\overline{\mathcal{F}}_1)}^{\epsilon,\epsilon'} \mathcal{U}_{\mathcal{G}_2;(\mathcal{F}_2,\overline{\mathcal{F}}_2)}^{\epsilon,\epsilon'} \quad (125)$$

where  $\mathcal{F}_1 \cup \mathcal{F}_2 = \mathcal{F}$  and  $\overline{\mathcal{F}}_1 \cup \overline{\mathcal{F}}_2 = \overline{\mathcal{F}}$ .

### 5.3.1 A new recurrence rule: Regular edges

We shall drop the subscript  $\mathcal{G}$  in the subsequent notations for sets. For instance,  $\mathcal{L}$  and  $\mathcal{F}_{\text{int}/\text{ext}}$  will denote the set of lines and set of faces of  $\mathcal{G}$ .

- We introduce the following definition: Given a subset  $\mathcal{F}$  of internal faces, we define  $\mathcal{F}/e$  to be the subset of faces corresponding to  $\mathcal{F}$  after the contraction of  $e$  in  $\mathcal{G}$ .

The following statement holds.

**Theorem 3** (Generalized contraction rule for  $\mathcal{U}^{\epsilon,\bar{\epsilon}}$  in rank  $d = 2$ ). *Let  $\mathcal{G}$  be a ribbon graph with half-ribbons with  $\mathcal{L}$  set of lines,  $\mathcal{F}_{\text{int}}$  set of internal faces,  $(\mathcal{F},\overline{\mathcal{F}})$  a pair of disjoint subsets of  $\mathcal{F}_{\text{int}}$  with  $\mathcal{F} \cup \overline{\mathcal{F}} = \mathcal{F}_{\text{int}}$ .*

*Let  $e$  be a regular edge of  $\mathcal{G}$ , we have four disjoint cases:*

(0) *If  $e$  passes through only external faces, then*

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \mathcal{U}_{\mathcal{G}/e;(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}, \quad (126)$$

where  $(\mathcal{F},\overline{\mathcal{F}}) = (\mathcal{F}/e,\overline{\mathcal{F}}/e)$ .

(i) *If  $e \in f$ , for a unique internal face  $f \in \mathcal{F}$  (the other strand of  $e$  is external), then*

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + t_e \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\setminus\{f/e\},\overline{\mathcal{F}}\cup\{f/e\})}^{\epsilon,\bar{\epsilon}}, \quad (127)$$

where  $\overline{\mathcal{F}} = \overline{\mathcal{F}}/e$ .

(ii) *If  $e^2 \in f$  with  $f \in \mathcal{F}$ , then*

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = (1 + t_e^2) \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + 2t_e \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\setminus\{f/e\},\overline{\mathcal{F}}\cup\{f/e\})}^{\epsilon,\bar{\epsilon}}, \quad (128)$$

where  $\overline{\mathcal{F}} = \overline{\mathcal{F}}/e$ .

(iii) *If  $e \in f_1$  and  $e \in f_2$ ,  $f_1 \neq f_2$ , then*

(a) if  $f_i \in \mathcal{F}$ , then

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} \\ &+ t_e \left( \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\setminus\{f_1/e\},\overline{\mathcal{F}}\cup\{f_1/e\})}^{\epsilon,\bar{\epsilon}} + (1 \leftrightarrow 2) \right) \\ &+ t_e^2 \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\setminus\{f_1/e,f_2/e\},\overline{\mathcal{F}}\cup\{f_1/e,f_2/e\})}^{\epsilon,\bar{\epsilon}}, \end{aligned} \quad (129)$$

where  $\overline{\mathcal{F}} = \overline{\mathcal{F}}/e$ ;

(b) if  $f_1 \in \mathcal{F}$  and  $f_2 \in \overline{\mathcal{F}}$ , then

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}}/e)}^{\epsilon,\bar{\epsilon}} \\ &+ t_e \left( \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\setminus\{f_1/e\},\overline{\mathcal{F}}/e\cup\{f_1/e\})}^{\epsilon,\bar{\epsilon}} + \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e)\cup\{f_2/e\},(\overline{\mathcal{F}}/e)\setminus\{f_2/e\})}^{\epsilon,\bar{\epsilon}} \right) \\ &+ t_e^2 \mathcal{U}_{\mathcal{G}/e;(((\mathcal{F}/e)\setminus\{f_1/e\})\cup\{f_2/e\},[(\overline{\mathcal{F}}/e)\setminus\{f_2/e\}]\cup\{f_1/e\})}^{\epsilon,\bar{\epsilon}}. \end{aligned} \quad (130)$$

*Proof.* See Appendix B. □

Theorem 3 expresses the reduction of the polynomial  $\mathcal{U}_{\mathcal{G}}$  only in terms of edge contractions. It is a new feature of an polynomial invariant on a graph. As a function depending on a partition of the set of internal faces, one must pay attention that in each expression involving  $\mathcal{U}_{\mathcal{G}/e;(\mathcal{F},\overline{\mathcal{F}})}^{(\cdot)}$  in the r.h.s of the equations (127)-(130),  $\mathcal{F}$  and  $\overline{\mathcal{F}}$  always define a partition of the set  $\mathcal{F}_{\text{int}}$  of internal faces of  $\mathcal{G}/e$ . The invariant  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}$  is a multivariate polynomial distinct from the Bollobàs-Riordan polynomial [71].

In rank  $d = 2$ , seeking a state sum formula for  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\text{od,ev}}$ , we have using (112),

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\text{od,ev}} = \sum_{(g_1,g_2) \in \text{Odd}^{\natural}(S_1(\mathcal{G}^*)) \times \text{Even}^{\natural}(S_2(\mathcal{G}^*))} \left[ \prod_{l \in \mathcal{HR}(g_1)} t_l \right] \left[ \prod_{l \in \mathcal{HR}(g_2)} t_l \right] \quad (131)$$

where the definition of  $\text{Even}^{\natural}(\cdot)$  can be deduced from  $\text{Odd}^{\natural}(\cdot)$  by replacing “odd” by “even”,  $S_1(\mathcal{G}^*) \cup S_2(\mathcal{G}^*)$  are defined through a partition of the vertices of the subgraph  $S(\mathcal{G}^*)$ .

Let us comment now special edges. Considering first the bridge case, relations (0) and (ii) in the above theorem are valid. For the trivial untwisted loop, (0), (i) and (iii) are true. Finally, for the trivial twisted loop (0) and (ii) hold. Thus, once again special edges are evaluated from the same theorem. This brings the following important question: “Can we find a closed formula for any polynomial  $\mathcal{U}_{\mathcal{G}}^{\epsilon,\bar{\epsilon}}$  on any graph  $\mathcal{G}$  using only the recurrence relation and a finite list of boundary conditions?”. In other words, given a graph, its number of internal and external lines, its number of bridges, loops, etc., is there a unique polynomial solution of the above recurrence relations expressed simply as a function of these numbers? If the answer to this question is yes, then the above polynomial will prove to be a very neat and computable invariant simpler than the Bollobàs-Riordan polynomial. However, a notion captured by the Bollobàs-Riordan polynomial is the genus of the ribbon graph and of its spanning subgraphs. The polynomial  $\mathcal{U}_{\mathcal{G}}^{\epsilon,\bar{\epsilon}}$ , in its present form, does not explicitly exhibit this genus. It would be interesting to investigate if  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}$  could be provided with another variable associated with the genus of the ribbon graph. For the moment, as a naive example,

if we consider a closed graph  $\mathcal{G}$ , and add a new set of variables  $\{x_f\}$  associated with the faces, we can define

$$\tilde{\mathcal{U}}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\epsilon'}(\{t_l\};\{x_f\}) = \left[ \prod_{f \in \mathcal{F}} x_f A_f^\epsilon(\{t_l\}_{l \in f}) \right] \left[ \prod_{f \in \bar{\mathcal{F}}} x_f A_f^{\epsilon'}(\{t_l\}_{l \in f}) \right]. \quad (132)$$

Thus this polynomial computes to  $\tilde{\mathcal{U}}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\epsilon'} = (\prod_{f \in \mathcal{F}_{\text{int}}} x_f) \cdot \mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\epsilon'}$  and should obey modified contraction rules from (126)-(130). Under the rescaling  $x_f \rightarrow \rho x_f$ , we have

$$\tilde{\mathcal{U}}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\epsilon'}(\{t_l\};\{\rho x_f\}) = \rho^{F_{\text{int}}} \tilde{\mathcal{U}}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\epsilon'}(\{t_l\};\{x_f\}). \quad (133)$$

Then certainly,  $\tilde{\mathcal{U}}_{\mathcal{G}}$  knows about the (generalized) genus  $\kappa$  of the closed ribbon graph since  $F_{\text{int}} = 2 - \kappa - (V - E)$ . Maybe to have a better picture and a good starting point for extracting information about the genus of the subgraphs, one can consider the expression (131). This problem is left to a subsequent work.

Before addressing the tensor case, let us recall the definition of a trivial loop in rank  $d$ . These have been called in [70]  $p$ -inner self-loops,  $p = 1, 2, 3$ , in the context  $d = 3$ ; this definition extends in any  $d$ . A trivial loop is an edge in a rank  $d$  colored tensor graph such that after its contraction the number of connected components is always  $d$ .

**Theorem 4** (Recurrence relation for  $\mathcal{U}^{\epsilon,\bar{\epsilon}}$  for rank  $d > 2$ ). *Let  $\mathcal{G}$  be a rank  $d$  colored tensor graph and  $e$  one of its regular edges or trivial loops. Let  $\mathcal{F}_e$  be the set of internal faces passing through  $e$  and denote  $\mathcal{F}_e^\epsilon = \mathcal{F}_e \cap \mathcal{F}$  and  $\mathcal{F}_e^{\bar{\epsilon}} = \mathcal{F}_e \cap \bar{\mathcal{F}}$ . We have*

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \sum_{K \times LC(\mathcal{F}_e^\epsilon/e) \times (\mathcal{F}_e^{\bar{\epsilon}}/e)} t_e^{K+|L|} \mathcal{U}_{\mathcal{G}/e;((\mathcal{F}/e) \setminus (\mathcal{F}_e^\epsilon/e)) \cup K^c \cup L, ((\bar{\mathcal{F}}/e) \setminus (\mathcal{F}_e^{\bar{\epsilon}}/e)) \cup K \cup L^c}, \quad (134)$$

in particular, for  $\mathcal{F}_e = \emptyset$

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\bar{\mathcal{F}}/e)}^{\epsilon,\bar{\epsilon}}. \quad (135)$$

*Proof.* Consider  $\mathcal{G}$ , a rank  $d \geq 3$  colored tensor graph and  $\mathcal{F}, \bar{\mathcal{F}} \subset \mathcal{F}_{\text{int}}$  which satisfy the ordinary conditions for defining  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}$ .

Let us assume that  $\mathcal{F}_e = \emptyset$ , namely there are no internal faces pass through  $e$ . The result  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\bar{\mathcal{F}}/e)}^{\epsilon,\bar{\epsilon}}$  is obvious. Now, we assume that  $\mathcal{F}_e \neq \emptyset$ . Using Lemma 4, one writes

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\bar{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \prod_{f \in \mathcal{F}_e^\epsilon} A_f^\epsilon \prod_{f \in \mathcal{F}_e^{\bar{\epsilon}}} A_f^{\bar{\epsilon}} \prod_{f \in \mathcal{F} \setminus \mathcal{F}_e^\epsilon} A_f^\epsilon \prod_{f \in \bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}} A_f^{\bar{\epsilon}} \\ &= \prod_{f \in \mathcal{F}_e^\epsilon} (t_e A_{f/e}^{\bar{\epsilon}} + A_{f/e}^\epsilon) \prod_{f \in \mathcal{F}_e^{\bar{\epsilon}}} (t_e A_{f/e}^\epsilon + A_{f/e}^{\bar{\epsilon}}) \prod_{f \in \mathcal{F} \setminus \mathcal{F}_e^\epsilon} A_f^\epsilon \prod_{f \in \bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}} A_f^{\bar{\epsilon}} \\ &= \left[ \sum_{K \subset \mathcal{F}_e^\epsilon} \prod_{f \in K} (t_e A_{f/e}^{\bar{\epsilon}}) \prod_{f \in K^c} A_{f/e}^\epsilon \right] \left[ \sum_{L \subset \mathcal{F}_e^{\bar{\epsilon}}} \prod_{f \in L} (t_e A_{f/e}^\epsilon) \prod_{f \in L^c} A_{f/e}^{\bar{\epsilon}} \right] \prod_{f \in \mathcal{F} \setminus \mathcal{F}_e^\epsilon} A_f^\epsilon \prod_{f \in \bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}} A_f^{\bar{\epsilon}} \\ &= \sum_{K \times LC \mathcal{F}_e^\epsilon \times \mathcal{F}_e^{\bar{\epsilon}}} t_e^{K+|L|} \left[ \prod_{f \in \mathcal{F} \setminus \mathcal{F}_e^\epsilon} A_f^\epsilon \prod_{f \in K^c} A_{f/e}^\epsilon \prod_{f \in L} A_{f/e}^{\bar{\epsilon}} \right] \left[ \prod_{f \in \bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}} A_f^{\bar{\epsilon}} \prod_{f \in L^c} A_{f/e}^{\bar{\epsilon}} \prod_{f \in K} A_{f/e}^\epsilon \right] \end{aligned}$$

$$\begin{aligned}
&= \sum_{K \times L \subset \mathcal{F}_e^\epsilon \times \mathcal{F}_e^{\bar{\epsilon}}} t_e^{|K|+|L|} \mathcal{U}_{\mathcal{G}/e; ((\mathcal{F} \setminus \mathcal{F}_e^\epsilon) \cup [K^c \cup L])/e, (\bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}) \cup [L^c \cup K])/e}^{\epsilon, \bar{\epsilon}} \\
&= \sum_{K \times L \subset \mathcal{F}_e^\epsilon/e \times \mathcal{F}_e^{\bar{\epsilon}}/e} t_e^{|K|+|L|} \mathcal{U}_{\mathcal{G}/e; (((\mathcal{F}/e) \setminus (\mathcal{F}_e^\epsilon/e)) \cup K^c \cup L, [(\bar{\mathcal{F}}/e) \setminus (\mathcal{F}_e^{\bar{\epsilon}}/e)] \cup L^c \cup K)}^{\epsilon, \bar{\epsilon}}, \tag{136}
\end{aligned}$$

where we used  $\mathcal{F} \setminus \mathcal{F}_e^\epsilon = (\mathcal{F}/e) \setminus (\mathcal{F}_e^\epsilon/e)$ .

Now, given  $K \times L \subset \mathcal{F}_e^\epsilon/e \times \mathcal{F}_e^{\bar{\epsilon}}/e$ , one must prove that  $\tilde{\mathcal{F}} = (\mathcal{F} \setminus \mathcal{F}_e^\epsilon) \cup K^c \cup L$  and  $\tilde{\bar{\mathcal{F}}} = (\bar{\mathcal{F}} \setminus \mathcal{F}_e^{\bar{\epsilon}}) \cup L^c \cup K$ , are such that (1)  $\tilde{\mathcal{F}} \cup \tilde{\bar{\mathcal{F}}} = \mathcal{F}_{\text{int}}(\mathcal{G}/e)$  and (2)  $\tilde{\mathcal{F}} \cap \tilde{\bar{\mathcal{F}}} = \emptyset$ .

With a moment of thoughts one sees that the statement (2) is true. Now the former is proved. First, one recognizes that  $\mathcal{F}_{\text{int}}(\mathcal{G}/e) = \mathcal{F}_{\text{int}}(\mathcal{G})/e = \{[(\mathcal{F} \setminus \mathcal{F}_e^\epsilon) \cup \mathcal{F}_e^\epsilon]/e\} \cup (\mathcal{F} \leftrightarrow \bar{\mathcal{F}})$ . Furthermore,  $(\mathcal{F} \setminus \mathcal{F}_e^\epsilon)/e = \mathcal{F} \setminus \mathcal{F}_e^\epsilon$ , and  $\mathcal{F}_e^\epsilon/e = K \cup K^c$ ,  $\forall K \subset \mathcal{F}_e^\epsilon$ , and the same is true for  $\mathcal{F}_e^{\bar{\epsilon}}/e = L \cup L^c$ ,  $\forall L \subset \mathcal{F}_e^{\bar{\epsilon}}$ . We can conclude to the equality at this point.  $\square$

Again, we note here that Theorem 4 is consistent with Theorem 3 for rank  $d = 2$  models if we exclude the cases where the same face goes through the same edge more than once.

It appears possible to further precise some relations and to introduce rules involving the deletion in the case of ribbon graphs. This question will be addressed now. In particular, the interesting cases correspond to (0), (ii) and (iii.a) of Theorem 3. Note that, in the ribbon graph case and for a given subset of internal faces  $\mathcal{F}$ , the notation  $\mathcal{F} - e$  might not always make sense. We define  $\mathcal{F} - e$  as a set of internal faces in  $\mathcal{G} - e$  as follows:

- a)  $\mathcal{F} - e = \mathcal{F}$  if the removal of  $e$  do not affect the faces in  $\mathcal{F}$ ;
- b)  $\mathcal{F} - e = \mathcal{F} \setminus \{f\}$ ,
- b1) if  $e \in f$ ,  $f \in \mathcal{F}$ , and if  $\mathcal{F}$  loses the internal face  $f$  passing through  $e$  and  $f$  merges with an external face;
- b2) if the face  $f \in \mathcal{F}$  is such that  $e^2 \in f$  and  $f$  does not split into two internal faces after the removal of  $e$ ; then  $f - e$  makes sense as a unique internal face;
- b3) if the face  $f \in \mathcal{F}$  splits into two faces  $f_1$  and  $f_2$  both internal after the removal of  $e$ , and in this case  $\{f - e\} = \{f_1, f_2\}$ ;
- c) if  $e$  passes through two different internal lines  $f_1$  and  $f_2$ ,  $f_{1,2}$  are in  $\mathcal{F}_{1,2}$  and the removal of  $e$  merges these two lines in one, then  $\mathcal{F}_i - e = \mathcal{F}_i \setminus \{f_1, f_2\}$ .

Cases a), b2), b3) and c) are the ones under which we can recast some polynomials  $\mathcal{U}_{\mathcal{G}/e}^{\epsilon, \bar{\epsilon}}$  in terms of the deleted graph  $\mathcal{G} - e$ . The following statement holds.

**Proposition 7** (Deletion relations). *Let  $\mathcal{G}$  be a ribbon graph with half-ribbons and  $e$  be one of its edges.*

- (0) *If  $e$  belongs only to a unique external face, and if it does not generate any new internal faces after the deletion of  $e$  in  $\mathcal{G}$ , then*

$$\mathcal{U}_{\mathcal{G}; (\mathcal{F}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} = \mathcal{U}_{\mathcal{G}/e; (\mathcal{F}/e, \bar{\mathcal{F}}/e)}^{\epsilon, \bar{\epsilon}} = \mathcal{U}_{\mathcal{G}-e; (\mathcal{F}-e, \bar{\mathcal{F}}-e)}^{\epsilon, \bar{\epsilon}}, \tag{137}$$

with  $\mathcal{F} - e = \mathcal{F}$  and  $\bar{\mathcal{F}} - e = \bar{\mathcal{F}}$ .

(i) If  $e^2 \in f$  with  $f \in \mathcal{F}$ ,

(a) and if the removal of  $e$  will result in one unique internal face  $f - e$  from  $f$ , then

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = (1 + t^2)\mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e)\cup\{f-e\},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + 2t_e\mathcal{U}_{\mathcal{G}-e;(\mathcal{F}-e,\overline{\mathcal{F}}\cup\{f-e\})}^{\epsilon,\bar{\epsilon}}, \quad (138)$$

where  $\mathcal{F} - e = \mathcal{F} \setminus \{f\}$  and  $\overline{\mathcal{F}} - e = \overline{\mathcal{F}}$ ;

(b) if the removal of  $e$  produces two internal faces  $f_1$  and  $f_2$  from  $f$ , then

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \rho_{\bar{\epsilon},\epsilon} \left( \mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e)\cup\{f-e\},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + \mathcal{U}_{\mathcal{G}-e;(\mathcal{F}-e,\overline{\mathcal{F}}\cup\{f-e\})}^{\epsilon,\bar{\epsilon}} \right) \\ &+ \rho_{\epsilon,\bar{\epsilon}} \left( \mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e)\cup\{f_1\},\overline{\mathcal{F}}\cup\{f_2\})}^{\epsilon,\bar{\epsilon}} + (1 \leftrightarrow 2) \right), \\ \rho_{\text{od,ev}}(t_e) &:= 1 + t_e^2, \quad \rho_{\text{ev,od}}(t_e) := 2t_e, \end{aligned} \quad (139)$$

where, we denote  $\{f_1, f_2\} := \{f - e\}$ ,  $\mathcal{F} - e = \mathcal{F} \setminus \{f\}$  and  $\overline{\mathcal{F}} - e = \overline{\mathcal{F}}$ .

(ii) If  $e \in f_1$  and  $e \in f_2$ ,  $f_1 \neq f_2$ ,

(a) and if  $f_{1,2} \in \mathcal{F}$ ,

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}}/e)}^{\epsilon,\bar{\epsilon}} + t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f_1/e, f_2/e\}, \overline{\mathcal{F}} \cup \{f_1/e, f_2/e\})}^{\epsilon,\bar{\epsilon}} \\ &+ t_e \left( \delta_{\epsilon,\text{od}} \mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e)\cup\{f\},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + \delta_{\bar{\epsilon},\text{od}} \mathcal{U}_{\mathcal{G}-e;(\mathcal{F}-e,\overline{\mathcal{F}}\cup\{f\})}^{\epsilon,\bar{\epsilon}} \right), \end{aligned} \quad (140)$$

where we denote  $f$  the unique resulting internal face in  $\mathcal{G} - e$  coming from the faces  $f_1$  and  $f_2$ , and where we note that  $\mathcal{F} - e = \mathcal{F} \setminus \{f_1, f_2\}$ , and  $\overline{\mathcal{F}} - e = \overline{\mathcal{F}} = \overline{\mathcal{F}}/e$ ;

(b) if  $f_1 \in \mathcal{F}$  and  $f_2 \in \overline{\mathcal{F}}$ , then

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}}/e)}^{\epsilon,\bar{\epsilon}} + t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f_1/e\} \cup \{f_2/e\}, \overline{\mathcal{F}}/e \setminus \{f_2/e\} \cup \{f_1/e\})}^{\epsilon,\bar{\epsilon}} \\ &+ t_e \left( \delta_{\epsilon,\text{od}} \mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e), (\overline{\mathcal{F}}-e) \cup \{f\})}^{\epsilon,\bar{\epsilon}} + \delta_{\bar{\epsilon},\text{od}} \mathcal{U}_{\mathcal{G}-e;((\mathcal{F}-e) \cup \{f\}, (\overline{\mathcal{F}}-e))}^{\epsilon,\bar{\epsilon}} \right), \end{aligned} \quad (141)$$

where we denote  $f$  the unique resulting internal face in  $\mathcal{G} - e$  coming from the faces  $f_1$  and  $f_2$ , with  $\mathcal{F} - e = \mathcal{F} \setminus \{f_1\}$ , and  $\overline{\mathcal{F}} - e = \overline{\mathcal{F}} \setminus \{f_2\}$ .

*Proof.* The first relation does not cause any trouble. We focus on (ia) and expand the polynomial, for  $e^2 \in f$  with  $f \in \mathcal{F}$ , and get

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = ((t_e^2 + 1)A_{f/e}^\epsilon + 2t_e A_{f/e}^{\bar{\epsilon}}) \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f}} A_f^\epsilon \right) \left( \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right). \quad (142)$$

Because all the edges contained in  $f/e$  and  $f - e$  are the same, we can write:

$$A_{f/e}^\epsilon = A_{f-e}^\epsilon, \quad \forall \epsilon. \quad (143)$$

By definition  $\mathcal{F} - e = \mathcal{F} \setminus \{f\}$ , we can conclude (ia).

One proves (ib) by first observing that

$$\begin{cases} A_{f/e}^{\text{od}} = A_{f_1}^{\text{od}} A_{f_2}^{\text{ev}} + A_{f_1}^{\text{ev}} A_{f_2}^{\text{od}} \\ A_{f/e}^{\text{ev}} = A_{f_1}^{\text{od}} A_{f_2}^{\text{od}} + A_{f_1}^{\text{ev}} A_{f_2}^{\text{ev}}, \end{cases} \quad (144)$$

where  $f_1$  and  $f_2$  are generated by the deletion of  $e$ . We insert (144) in (142), and using the definition  $\mathcal{F} - e = \mathcal{F} \setminus \{f\}$  and  $\{f - e\} = \{f_1, f_2\}$ , we arrive at the desired relations.

Let us now prove (ii). One starts from the expansion of  $\mathcal{U}_{\mathcal{G}; (\mathcal{F}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}}$  focusing on the two amplitudes of faces  $f_i$  sharing  $e$ . From Theorem 3, in particular (129) and (130), know that the two contraction terms present in (140) and (141), respectively, have been shown true. We focus on the additional terms in (129) and (130) and prove that they involve contraction terms. For (iia), the key relation is

$$A_{f_1/e}^{\bar{\epsilon}} A_{f_2/e}^{\epsilon} + A_{f_1/e}^{\epsilon} A_{f_2/e}^{\bar{\epsilon}} = A_f^{\text{od}}, \quad (145)$$

where  $f$  is the face formed from  $f_{1,2}$  after the deletion of  $e$ . This leads us to choose the parities of each sector  $\mathcal{F}$  and  $\bar{\mathcal{F}}$ . Using this and the definition  $\mathcal{F} \setminus \{f_1, f_2\} = \mathcal{F} - e$ , we write

$$A_f^{\text{od}} \left( \prod_{f \in \mathcal{F} - e} A_f^{\epsilon} \right) \left( \prod_{f \in \bar{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) = \begin{cases} \mathcal{U}_{\mathcal{G} - e; ((\mathcal{F} - e) \cup \{f\}, \bar{\mathcal{F}})}^{\epsilon = \text{od}, \bar{\epsilon} = \text{ev}} \\ \mathcal{U}_{\mathcal{G} - e; (\mathcal{F} - e, \bar{\mathcal{F}} \cup \{f\})}^{\epsilon = \text{ev}, \bar{\epsilon} = \text{od}} \end{cases} \quad (146)$$

leading to (140). Now if  $f_1 \in \mathcal{F}$  and  $f_2 \in \bar{\mathcal{F}}$ , a counterpart relation of (145) is

$$A_{f_1/e}^{\text{ev}} A_{f_2/e}^{\text{ev}} + A_{f_1/e}^{\text{od}} A_{f_2/e}^{\text{od}} = A_f^{\text{ev}}, \quad (147)$$

and one concludes (141) with the definitions  $\mathcal{F} - e = \mathcal{F} \setminus \{f_1\}$  and  $\bar{\mathcal{F}} - e = \bar{\mathcal{F}} \setminus \{f_2\}$ .  $\square$

Let us comment that in the above statement, in the cases (ia), (ib) and (iia), we assumed that the face  $f$  or faces  $f_i$  passing through  $e$  are in  $\mathcal{F}$ . It is simple to infer what happens if they all belong to the other set  $\bar{\mathcal{F}}$ . As an illustration of some of the configurations involved in Proposition 7, we provide Figures 28 and 29.

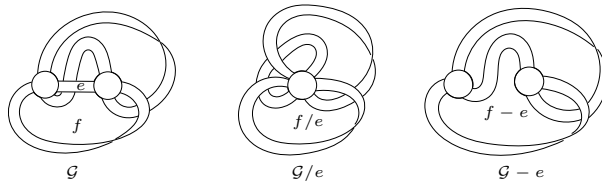


Figure 28: A graph  $\mathcal{G}$  obeying condition (ia) of Proposition 7.

Proposition 7 establishes that some terms appearing in the recurrence relations of  $\mathcal{U}_{\cdot; (\cdot, \cdot)}^{\epsilon, \bar{\epsilon}}$  as stated in Theorem 3 may be re-expressed in terms of the polynomials involving a deletion of  $e$ . After such reductions in terms of contraction/deletion of an edge, the reader may wonder if the polynomial  $\mathcal{U}_{\mathcal{G}}$  may be expressed in term of the Tutte polynomial (the sole universal invariant satisfying the contraction/deletion rule on a graph) or Bollobàs-Riordan polynomial on ribbon graphs. The answer to that question is definitely no because there exist several cases for which the present rule fails to be a proper contraction/deletion relation with exactly two terms:  $\mathcal{U}_{\mathcal{G}} \neq \sigma_e \mathcal{U}_{\mathcal{G} - e} + \tau_e \mathcal{U}_{\mathcal{G}/e}$ , for all  $e$  regular, with  $\sigma_e$  and  $\tau_e$  functions of  $t_e$ . Thus  $\mathcal{U}_{\mathcal{G}}$  is certainly not a Tutte polynomial and therefore defines a new kind of invariant on its enlarged space.

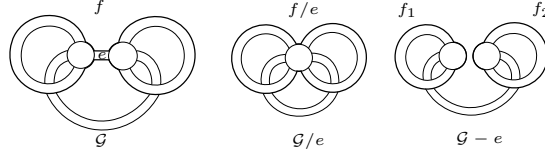


Figure 29: An example of graph  $\mathcal{G}$  satisfying condition (ib) of Proposition 7.

### 5.3.2 Special edges

We give a treatment of some of the special edges (or terminal forms) when evaluating  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}$ . Terminal forms are crucial because they specify the boundary conditions of the recurrence relations. Hence, the following study may help for the evaluation of the polynomial, when after a sequence of reductions (contraction/deletions), the graph reaches some cases listed below.

As commented after Theorems 3 and 4, terminal forms in any rank  $d$  also satisfy special relations listed therein. Now the issue addressed in the present section is to show that, under particular circumstances, these relations reduce and, sometimes, yield neat factorizations.

**Matrix case.** We show that using the disjoint union operation, some recurrence relations when applied to special edges lead to further simplification in terms of subgraphs within the larger graph.

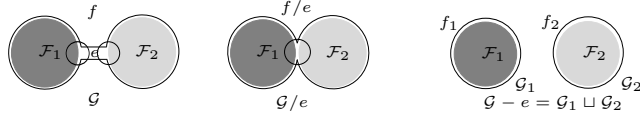


Figure 30: A bridge  $e$ : two blobs  $\mathcal{F}_1$  and  $\mathcal{F}_2$  which are collections of faces in subgraphs on each side of the bridge. Circles denote the end-vertices of  $e$ .

- (1) We consider a graph  $\mathcal{G}$  with a bridge  $e$  (Fig.30). We are interested in the nontrivial configuration when  $e$  belongs to a unique internal face  $f \in \mathcal{F}_{\text{int}}$  which corresponds to Theorem 3 (ii). Furthermore, we take  $f \in \mathcal{F}$ . Call  $\mathcal{G}_1$  and  $\mathcal{G}_2$  the two disconnected subgraphs resulting from the deletion of  $e$ , namely  $\mathcal{G} - e = \mathcal{G}_1 \sqcup \mathcal{G}_2$ . Call  $\mathcal{F}_i$  the set of internal faces in  $\mathcal{G}_i$ . Taking a partition  $\mathcal{F} \cup \overline{\mathcal{F}}$  of the set of internal faces of  $\mathcal{G}$ , four distinct cases can occur: (1)  $\mathcal{F}_1 \subset \mathcal{F}$  and  $\mathcal{F}_2 \subset \overline{\mathcal{F}}$ , (2)  $\mathcal{F}_i \subset \mathcal{F}$ , (3)  $\mathcal{F}_i \subset \overline{\mathcal{F}}$ , and (4)  $\mathcal{F}_1 \subset \overline{\mathcal{F}}$  and  $\mathcal{F}_2 \subset \mathcal{F}$ . Only the case (1) will be discussed here, as the other ones can be derived in a similar manner. We identify for the bridge graph,

$$\mathcal{G} - e = \mathcal{G}_1 \sqcup \mathcal{G}_2. \quad (148)$$

The result of Proposition 7 (ib) still holds. Then, assuming  $\mathcal{F}_1 \subset \mathcal{F} = \mathcal{F}_1 \cup \{f\}$  and  $\overline{\mathcal{F}} = \mathcal{F}_2$ , and noting that  $\mathcal{F} - e = \mathcal{F}_1$  and  $\overline{\mathcal{F}} = \overline{\mathcal{F}} - e = \mathcal{F}_2$ , we start from (139). Apply repeatedly Proposition 6 and get:

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} = \rho_{\bar{\epsilon},\epsilon} \left( \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2;(\mathcal{F}_1 \cup \{f_1, f_2\}, \mathcal{F}_2)}^{\epsilon,\bar{\epsilon}} + \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2;(\mathcal{F}_1, \mathcal{F}_2 \cup \{f_1, f_2\})}^{\epsilon,\bar{\epsilon}} \right)$$

$$\begin{aligned}
& + \rho_{\epsilon, \bar{\epsilon}} \left( \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2; \mathcal{F}_1 \cup \{f_1\}, \mathcal{F}_2 \cup \{f_2\}}^{\epsilon, \bar{\epsilon}} + \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2; \mathcal{F}_1 \cup \{f_2\}, \mathcal{F}_2 \cup \{f_1\}}^{\epsilon, \bar{\epsilon}} \right) \\
& = \rho_{\bar{\epsilon}, \epsilon} \left( \mathcal{U}_{\mathcal{G}_1; (\mathcal{F}_1 \cup \{f_1\}, \emptyset)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\{f_2\}, \mathcal{F}_2)}^{\epsilon, \bar{\epsilon}} + \mathcal{U}_{\mathcal{G}_1; (\mathcal{F}_1, \{f_1\})}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\emptyset, \mathcal{F}_2 \cup \{f_2\})}^{\epsilon, \bar{\epsilon}} \right) \\
& + \rho_{\epsilon, \bar{\epsilon}} \left( \mathcal{U}_{\mathcal{G}_1; (\mathcal{F}_1 \cup \{f_1\}, \emptyset)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\emptyset, \mathcal{F}_2 \cup \{f_2\})}^{\epsilon, \bar{\epsilon}} + \mathcal{U}_{\mathcal{G}_1; (\mathcal{F}_1, \{f_1\})}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\{f_2\}, \mathcal{F}_2)}^{\epsilon, \bar{\epsilon}} \right) \quad (149)
\end{aligned}$$

and it partially factorizes.

- (2) We now consider a trivial untwisted loop. This configuration divides into nontrivial cases where  $e$  is shared between two internal faces (see Fig.31) or between one internal and one external faces. The first case subdivides into two subcases determined by the fact that the faces passing through  $e$  may or may not belong to the same parity when the polynomial will be evaluated. We focus on the situation described by the condition (iii) of Proposition 7 while the same technique can be applied for all the remaining cases. A basic relation is

$$\mathcal{G}/e = \mathcal{G}_1 \sqcup \mathcal{G}_2. \quad (150)$$

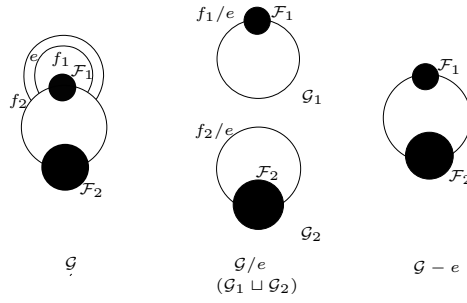


Figure 31: A trivial untwisted loop and the internal faces  $f_i$  through  $e$ .  $\mathcal{F}_i \cup \{f_i/e\}$  are the sets internal faces of the subgraphs  $\mathcal{G}_i$  obtained by contraction of  $e$ .

Consider a partition of the set of internal faces of  $\mathcal{G}$  as  $\mathcal{F} \cup \bar{\mathcal{F}}$ . In Fig.31, consider that the internal faces passing through  $e$  are such that  $f_1 \in \mathcal{F}$  and  $f_2 \in \bar{\mathcal{F}}$  (and  $f_1 \neq f_2$ ). We contract  $e$  in the original graph  $\mathcal{G}$  and call the resulting graphs as  $\mathcal{G}_1$  and  $\mathcal{G}_2$ .  $\mathcal{G}_1$  (resp.  $\mathcal{G}_2$ ) contains the set of internal faces  $\mathcal{H}_1^* = \mathcal{F}_1 \cup \{f_1/e\}$  (resp.  $\mathcal{H}_2^* = \mathcal{F}_2 \cup \{f_2/e\}$ ). Let us denote  $\mathcal{H}_i = \mathcal{F} \cap \mathcal{H}_i^*$  and  $\bar{\mathcal{H}}_i = \bar{\mathcal{F}} \cap \mathcal{H}_i^*$ . We define  $f$  the face obtained from  $f_{1,2}$  after the deletion of  $e$ .

From (140), one writes, using the intermediate step (146),

$$\begin{aligned}
\mathcal{U}_{\mathcal{G}; (\mathcal{F}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} & = \mathcal{U}_{\mathcal{G}/e; (\mathcal{F}/e, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} + t_e^2 \mathcal{U}_{\mathcal{G}/e; (\mathcal{F}/e \setminus \{f_1/e, f_2/e\}, \bar{\mathcal{F}} \cup \{f_1/e, f_2/e\})}^{\epsilon, \bar{\epsilon}} \\
& + t_e (A_{f_1/e}^\epsilon A_{f_2/e}^{\bar{\epsilon}} + A_{f_1/e}^{\bar{\epsilon}} A_{f_2/e}^\epsilon) \left( \prod_{f \in \mathcal{F}; f \neq f_i} A_f^\epsilon \right) \left( \prod_{f \in \bar{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) \\
& = \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2; (\mathcal{H}_1 \cup \mathcal{H}_2, \bar{\mathcal{H}}_1 \cup \bar{\mathcal{H}}_2)}^{\epsilon, \bar{\epsilon}} + t_e^2 \mathcal{U}_{\mathcal{G}_1 \sqcup \mathcal{G}_2; (\mathcal{H}_1 \cup \mathcal{H}_2 \setminus \{f_1/e, f_2/e\}, \bar{\mathcal{H}}_1 \cup \bar{\mathcal{H}}_2 \cup \{f_1/e, f_2/e\})}^{\epsilon, \bar{\epsilon}}
\end{aligned}$$



$$+ t_e(A_{f_1/e}^\epsilon A_{f_2/e}^{\bar{\epsilon}} + A_{f_1/e}^{\bar{\epsilon}} A_{f_2/e}^\epsilon) \left[ \prod_{f \in \mathcal{H}_1 \cup \mathcal{H}_2 \setminus \{f_1/e, f_2/e\}} A_f^\epsilon \right] \left[ \prod_{f \in \bar{\mathcal{H}}_1 \cup \bar{\mathcal{H}}_2} A_f^{\bar{\epsilon}} \right], \quad (151)$$

where we recall that

$$\begin{aligned} \mathcal{F}/e &= (\mathcal{F} \setminus \{f_1, f_2\}) \cup \{f_1/e, f_2/e\} = \mathcal{H}_1 \cup \mathcal{H}_2, & \mathcal{F} - e &= \mathcal{F} \setminus \{f_1, f_2\}, \\ \bar{\mathcal{F}}/e &= \bar{\mathcal{F}} - e = \bar{\mathcal{F}} = \bar{\mathcal{H}}_1 \cup \bar{\mathcal{H}}_2. \end{aligned} \quad (152)$$

We arrive at

$$\begin{aligned} \mathcal{U}_{\mathcal{G}; (\mathcal{F}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} &= \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1, \bar{\mathcal{H}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2, \bar{\mathcal{H}}_2)}^{\epsilon, \bar{\epsilon}} \\ &+ t_e^2 \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1 \setminus \{f_1/e\}, \bar{\mathcal{H}}_1 \cup \{f_1/e\})}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2 \setminus \{f_2/e\}, \bar{\mathcal{H}}_2 \cup \{f_2/e\})}^{\epsilon, \bar{\epsilon}} \\ &+ t_e \left( \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1, \bar{\mathcal{H}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2 \setminus \{f_2/e\}, \bar{\mathcal{H}}_2 \cup \{f_2/e\})}^{\epsilon, \bar{\epsilon}} \right. \\ &+ \left. \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1 \setminus \{f_1/e\}, \bar{\mathcal{H}}_1 \cup \{f_1/e\})}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2, \bar{\mathcal{H}}_2)}^{\epsilon, \bar{\epsilon}} \right) \\ &= \left( \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1, \bar{\mathcal{H}}_1)}^{\epsilon, \bar{\epsilon}} + t_e \mathcal{U}_{\mathcal{G}_1; (\mathcal{H}_1 \setminus \{f_1/e\}, \bar{\mathcal{H}}_1 \cup \{f_1/e\})}^{\epsilon, \bar{\epsilon}} \right) \\ &\quad \times \left( \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2, \bar{\mathcal{H}}_2)}^{\epsilon, \bar{\epsilon}} + t_e \mathcal{U}_{\mathcal{G}_2; (\mathcal{H}_2 \setminus \{f_2/e\}, \bar{\mathcal{H}}_2 \cup \{f_2/e\})}^{\epsilon, \bar{\epsilon}} \right) \end{aligned} \quad (153)$$

which is a factorized polynomial.

- (3) We now consider a graph with a trivial twisted loop as in Fig.32. This necessarily leads to unique face passing through the edge  $e$ . This case has already been computed in (128) in Theorem 3 and (138) in Proposition 7. No factorization occurs and we have the relations:

$$\begin{aligned} \mathcal{U}_{\mathcal{G}; (\mathcal{F}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} &= 2t_e \mathcal{U}_{\mathcal{G}/e; (\mathcal{F}/e \setminus \{f/e\}, \bar{\mathcal{F}} \cup \{f/e\})}^{\epsilon, \bar{\epsilon}} + (t_e^2 + 1) \mathcal{U}_{\mathcal{G}/e; (\mathcal{F}/e, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} \\ &= 2t_e \mathcal{U}_{\mathcal{G}-e; (\mathcal{F}-e, \bar{\mathcal{F}} \cup \{f-e\})}^{\epsilon, \bar{\epsilon}} + (1 + t_e^2) \mathcal{U}_{\mathcal{G}-e; ((\mathcal{F}-e) \cup \{f-e\}, \bar{\mathcal{F}})}^{\epsilon, \bar{\epsilon}}. \end{aligned} \quad (154)$$

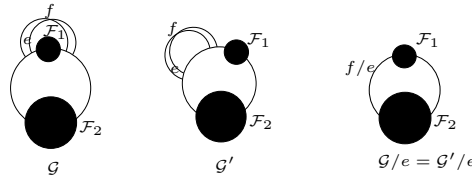


Figure 32: A trivial twisted loop  $e$  and  $f$  the internal face passing through  $e$ .  $\mathcal{G}$  and  $\mathcal{G}'$  possess the same polynomial. After contracting or deleting  $e$ ,  $\mathcal{G}/e = \mathcal{G}'/e = \mathcal{G} - e = \mathcal{G}' - e$ .

**Rank 3 colored tensor models.** Theorem 4 is also valid in the case of the terminal forms. Particular classes of terminal forms have been discussed in [70]. We will use one of these terminal forms depicted in Fig.33 (illustrated for rank  $d = 3$ , but the idea generalizes easily).

This graph is a higher rank generalization of a trivial untwisted loop in the ribbon case. Each blob appearing in black is a subgraph of  $\mathcal{G}$  which is not connected (by any strand) to any other blobs. After contraction of the edge, the graph splits in  $d$  disjoint subgraphs.

Let us now restrict to  $d = 3$  and call the sets of internal faces contained in each blob  $\mathcal{F}_i$ . Assume the internal faces  $f_i$  passing through the edge  $e$  obey  $f_i \notin \mathcal{F}_i$ . Given a partition  $\mathcal{F} \cup \overline{\mathcal{F}}$  of the set of internal faces of  $\mathcal{G}$ , we shall use the assumption that this set decomposes as  $\mathcal{F} = \{f_1, f_2, f_3\} \cup (\cup_j \widetilde{\mathcal{F}}_j)$ ,  $\widetilde{\mathcal{F}}_j = \mathcal{F} \cap \mathcal{F}_j$ , and the complementary set of faces  $\overline{\mathcal{F}} = \cup_k \widetilde{\overline{\mathcal{F}}}_k$ , with  $\widetilde{\overline{\mathcal{F}}}_k = \overline{\mathcal{F}} \cap \mathcal{F}_k$ .

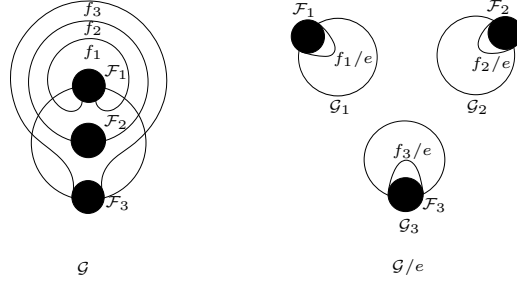


Figure 33: A terminal form for rank 3 colored tensor model.  $\mathcal{G}$  is a trivial loop with the set  $\mathcal{F}_i$  of internal faces of the blob-subgraphs. Contracting  $e$  gives three graphs  $\mathcal{G}_i$  with the set of faces  $\mathcal{F}_i \cup \{f_i/e\}$ .

Since we chose all  $f_i \in \mathcal{F}$ , we have  $\mathcal{F}/e = \{f_1/e, f_2/e, f_3/e\} \cup (\cup_j \widetilde{\mathcal{F}}_j)$  and  $\overline{\mathcal{F}}/e = \cup_k \widetilde{\overline{\mathcal{F}}}_k$ . Using Theorem 4 and after some algebras, we obtain

$$\begin{aligned}
\mathcal{U}_{\mathcal{G}; (\{f_1, f_2, f_3\} \cup (\cup_j \widetilde{\mathcal{F}}_j), \cup_k \widetilde{\overline{\mathcal{F}}}_k)}^{\epsilon, \bar{\epsilon}} &= t_e^3 \mathcal{U}_{\mathcal{G}_1; (\widetilde{\mathcal{F}}_1, \{f_1/e\} \cup \widetilde{\overline{\mathcal{F}}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\widetilde{\mathcal{F}}_2, \{f_2/e\} \cup \widetilde{\overline{\mathcal{F}}}_2)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_3; (\widetilde{\mathcal{F}}_3, \{f_3/e\} \cup \widetilde{\overline{\mathcal{F}}}_3)}^{\epsilon, \bar{\epsilon}} \\
&+ t_e^2 \left( \mathcal{U}_{\mathcal{G}_1; (\widetilde{\mathcal{F}}_1, \{f_1/e\} \cup \widetilde{\overline{\mathcal{F}}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\widetilde{\mathcal{F}}_2, \{f_2/e\} \cup \widetilde{\overline{\mathcal{F}}}_2)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_3; (\widetilde{\mathcal{F}}_3 \cup \{f_3/e\}, \widetilde{\overline{\mathcal{F}}}_3)}^{\epsilon, \bar{\epsilon}} + (1 \rightarrow 2 \rightarrow 3) \right) \\
&+ t_e \left( \mathcal{U}_{\mathcal{G}_1; (\widetilde{\mathcal{F}}_1, \{f_1/e\} \cup \widetilde{\overline{\mathcal{F}}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\widetilde{\mathcal{F}}_2 \cup \{f_2/e\}, \widetilde{\overline{\mathcal{F}}}_2)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_3; (\widetilde{\mathcal{F}}_3 \cup \{f_3/e\}, \widetilde{\overline{\mathcal{F}}}_3)}^{\epsilon, \bar{\epsilon}} + (1 \rightarrow 2 \rightarrow 3) \right) \\
&+ \mathcal{U}_{\mathcal{G}_1; (\widetilde{\mathcal{F}}_1 \cup \{f_1/e\}, \widetilde{\overline{\mathcal{F}}}_1)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_2; (\widetilde{\mathcal{F}}_2 \cup \{f_2/e\}, \widetilde{\overline{\mathcal{F}}}_2)}^{\epsilon, \bar{\epsilon}} \mathcal{U}_{\mathcal{G}_3; (\widetilde{\mathcal{F}}_3 \cup \{f_3/e\}, \widetilde{\overline{\mathcal{F}}}_3)}^{\epsilon, \bar{\epsilon}}, \tag{155}
\end{aligned}$$

where  $(1 \rightarrow 2 \rightarrow 3)$  simply refers to a permutation over the three labels which make the contribution symmetric in 1, 2, and 3.

Other choices of the parities of the  $f_i$ 's can also be made. The calculation becomes a little bit involved but the idea and techniques used above remain the same.

Assuming again that  $f_i \in \mathcal{F}$  and  $\mathcal{F}_1 = \mathcal{F}_2 = \mathcal{F}_3 = \emptyset$ , further noting that  $\{f_1/e\} = \{f_2/e\} = \{f_3/e\} = o$  are all bare vertices, and using our conventions (123), we have from (155)

$$\mathcal{U}_{\mathcal{G}; (\{f_1, f_2, f_3\}, \emptyset)}^{\epsilon, \bar{\epsilon}} = \begin{cases} t_e^3, & \text{for } \epsilon = \text{od}, \\ 1, & \text{for } \epsilon = \text{ev}. \end{cases} \tag{156}$$

These are the values of the polynomial  $\mathcal{U}^{\epsilon, \bar{\epsilon}}$  for the simple tensor graph made with one vertex (with two half-edges) and one edge.

## 5.4 Illustrations

We provide examples in order to check the recurrence relations using the polynomial of the second kind  $\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\overline{\epsilon}}$  on some particular nontrivial cases.

**Matrix case.** Consider the ribbon graph  $\mathcal{G}$  given in Fig.34. We distribute its closed faces

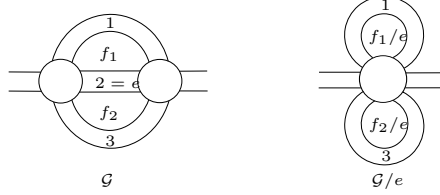


Figure 34: A ribbon graph  $\mathcal{G}$  with an ribbon edge  $e$  which passes through two different internal faces  $f_1$  and  $f_2$ .  $\mathcal{G}/e$  includes  $f_1/e$  and  $f_2/e$ .

as  $\mathcal{F} = \{f_1\}$  and  $\overline{\mathcal{F}} = \{f_2\}$ . From direct evaluation, using (119), we obtain

$$\mathcal{U}_{\mathcal{G};(\{f_1\},\{f_2\})}^{\text{od,ev}} = (t_1 + t_2)(1 + t_2 t_3). \quad (157)$$

Now, we compute the same polynomial using the recurrence relation. Pick the edge 2 which is shared by the internal faces  $f_1 \neq f_2$ . We use (iii.b) in Theorem 3, noting also (124), to write

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\text{od,ev}} &= \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e,\overline{\mathcal{F}}/e)}^{\text{od,ev}} + t_e \left( \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \cup \{f_2/e\},\overline{\mathcal{F}} \setminus \{f_2\})}^{\text{od,ev}} + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\},\overline{\mathcal{F}}/e \cup \{f_1/e\})}^{\text{od,ev}} \right) \\ &+ t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\} \cup \{f_2/e\},\overline{\mathcal{F}} \setminus \{f_2\} \cup \{f_1/e\})}^{\text{od,ev}} \\ &= t_1 + t_2(t_1 t_3 + 1) + t_2^2 t_3. \end{aligned} \quad (158)$$

Expanding (157), one gets of course (158).

**Rank 3 colored tensor models.** Consider the rank 3 graph in Fig.35. We pick the edge  $e$  which is shared by two internal faces  $f \neq f'$ , and  $f, f' \in \mathcal{F}$ . We also choose that the remaining closed face  $h \in \mathcal{F}$ . Therefore,  $\mathcal{F} = \{f, f', h\}$  and  $\overline{\mathcal{F}} = \emptyset$  form a partition of internal faces of  $\mathcal{G}$ . From direct computation, i.e. using (119), we obtain

$$\mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\text{od,ev}} = (t_e + t_1)^2(t_1 + t_2). \quad (159)$$

Using the recurrence relations given in Theorem 4, one has

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\text{od,ev}} &= t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f/e, f'/e\}, \overline{\mathcal{F}} \cup \{f/e, f'/e\})}^{\text{od,ev}} + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f/e, f'/e\} \cup \{f/e, f'/e\}, \overline{\mathcal{F}})}^{\text{od,ev}} \\ &+ t_e \left( \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f/e, f'/e\} \cup \{f'/e\}, \overline{\mathcal{F}} \cup \{f/e\})}^{\text{od,ev}} + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F}/e \setminus \{f/e, f'/e\} \cup \{f/e\}, \overline{\mathcal{F}}/e \cup \{f'/e\})}^{\text{od,ev}} \right) \\ &= t_e^2 \mathcal{U}_{\mathcal{G}/e;(\{h\}, \{f/e, f'/e\})}^{\text{od,ev}} + \mathcal{U}_{\mathcal{G}/e;(\{f/e, f'/e, h\}, \emptyset)}^{\text{od,ev}} \\ &+ t_e \left( \mathcal{U}_{\mathcal{G}/e;(\{f'/e, h\}, \{f/e\})}^{\text{od,ev}} + \mathcal{U}_{\mathcal{G}/e;(\{f/e, h\}, \{f'/e\})}^{\text{od,ev}} \right) \end{aligned} \quad (160)$$

and

$$\begin{aligned} \mathcal{U}_{\mathcal{G}/e;(\{h\}, \{f/e, f'/e\})}^{\text{od,ev}} &= t_1 + t_2, & \mathcal{U}_{\mathcal{G}/e;(\{f/e, f'/e, h\}, \emptyset)}^{\text{od,ev}} &= t_1^2(t_1 + t_2), \\ \mathcal{U}_{\mathcal{G}/e;(\{f'/e, h\}, \{f/e\})}^{\text{od,ev}} &= t_1(t_1 + t_2), & \mathcal{U}_{\mathcal{G}/e;(\{f/e, h\}, \{f'/e\})}^{\text{od,ev}} &= t_1(t_1 + t_2). \end{aligned} \quad (161)$$

Thus, the last equations are consistent with (159).

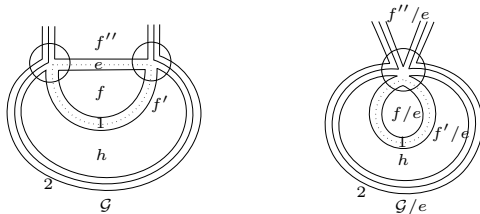


Figure 35: A rank 3 colored tensor graph  $\mathcal{G}$  with internal faces  $f$ ,  $f'$  (dotted), and  $h$ . In  $\mathcal{G}/e$ ,  $f/e$  and  $f'/e$  (dotted) pass through the edge 1.

## 6 Conclusion

The parametric representation of tensor models over the Abelian group  $U(1)^D$  with a kinetic term linear in momenta has been investigated in this work. We have first introduced a dimensional regularization scheme and perform the ensuing renormalization procedure on amplitudes of specific tensor models. An important fact revealed by this work is that these well-known procedures can be made compatible with the Feynman amplitudes depending on stranded graph structures. We have also shown that the amplitudes define analytic functions in the complex  $D$ , for  $\Re(D)$  small enough. These graph amplitudes  $A_{\mathcal{G}}$  can be extended in meromorphic functions in the strip  $0 < \Re(D) < \delta + \varepsilon_{\mathcal{G}}$ , where  $\delta$  is a given dimension of the group in the model considered and  $\varepsilon_{\mathcal{G}}$  is a small positive quantity depending on the graph  $\mathcal{G}$ . Due to the presence of another independent parameter in this class of models, namely the theory rank  $d$ , it also seems possible to define a new “rank regularization” procedure of the amplitudes by complexifying the parameter  $d$ . This deserves to be fully investigated.

In a second part, we have thoroughly investigated and extended the Symanzik polynomials yielded by the parametric representation of generic Abelian models. The ordinary contraction/deletion rules satisfied by Symanzik polynomials are now clearly broken by the stranded graph structure. We have introduced an abstract class of polynomials which depends both on the graph  $\mathcal{G}$  but also on a peculiar decomposition of its set  $\mathcal{F}_{\text{int};\mathcal{G}}$  of faces. Then, we prove that these new polynomials satisfy (only) contraction rules. We have also provided some terminal form recurrence rules and several illustrations. Let us emphasize that the fact that one might incorporate more information in graph polynomials which depend not only on the graph but also on the sets of its constituents opens an avenue of new investigations. To be clearer, the Tutte polynomial  $T_{\mathcal{G}}$  is defined by a state sum over the set  $\mathcal{P}(\mathcal{G})$  of spanning subgraphs of  $\mathcal{G}$ . Using insights of the present work, the question is whether or not  $T_{\mathcal{G}}$  could have been identified as a function of  $\mathcal{G}$  and  $\mathcal{P}(\mathcal{G})$  itself. If the answer to this question is positive then it will prove that the Tutte polynomial can be read differently. All of its consequences and its ramification in higher dimensions, like the Bollobàs-Riordan polynomial, might find a different representation which might lead to a richer interpretation. This must also be investigated elsewhere.

Finally, the present work has addressed the simplest setting that one could envisage using tensor models. There exists a  $\Phi^6$  model defined with rank 4 tensors over  $U(1)^4$  generating 4D simplicial topologies [14]. This model is endowed with a kinetic term including a quadratic dependence in momenta:  $\sum_s p_s^2 + \mu$ . Finding a complete parametric representation of its amplitudes will be a true challenge. The present work might be helpful for understanding a

way to perform a dimensional regularization for this model and for studying the polynomials which will arise from such a representation. This will be addressed in the forthcoming work.

## Acknowledgements

Discussions with Vincent Rivasseau and Thomas Krajewski are gratefully acknowledged. The authors are thankful to Perimeter Institute for Theoretical Physics, Waterloo, Canada, for having initially fostered this collaboration.

## Appendix

### A Proof of Proposition 2

In this section we provide the proof of Proposition 2. Consider  $\mathcal{G}$  a ribbon graph with sets  $\mathcal{F}_{\text{int};\mathcal{G}}$  and  $\mathcal{F}_{\text{ext};\mathcal{G}}$  of internal and external faces, respectively, and  $e$  an edge of  $\mathcal{G}$ .

- (i) The polynomial  $U_{\mathcal{G}}^{\text{od/ev}}$  (15) only takes into consideration internal faces. If  $e$  only belongs to external faces, then a contraction of  $e$  will not affect  $U_{\mathcal{G}}^{\text{od/ev}}$ . This proves (94). The points about the deletion of  $e$  and the creation or not of a new internal face are also direct by definition.
- (ii) Let us assume now that  $e \in f$ ,  $f \in \mathcal{F}_{\text{int};\mathcal{G}}$  and  $e$  belongs to another external face  $f'$ . Then, we decompose  $U_{\mathcal{G}}^{\text{od/ev}}$  using Lemma 4 as follows

$$\begin{aligned} U_{\mathcal{G}}^{\text{od/ev}} &= (t_e A_{f/e}^{\text{ev/od}} + A_{f/e}^{\text{od/ev}}) \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}, f \neq f} A_f^{\text{od/ev}} \\ &= t_e A_{f/e}^{\text{ev/od}} U_{\mathcal{G}-e}^{\text{od/ev}} + U_{\mathcal{G}/e}^{\text{od/ev}}, \end{aligned} \quad (\text{A.1})$$

where we used the fact that the set of internal faces of  $\mathcal{G}/e$  is given by  $\{f/e\} \cup \mathcal{F}_{\text{int};\mathcal{G}} \setminus \{f\}$  and, after removing  $e$  in  $\mathcal{G}$ , the face  $f$  merges to the external face  $f'$ . As a result, the set of internal faces of  $\mathcal{G} - e$  coincides with  $\mathcal{F}_{\text{int};\mathcal{G}} \setminus \{f\}$ . Finally, one observes that either cutting or deleting  $e$  has the same effect on the set of internal faces of  $\mathcal{G} \vee e$  and  $\mathcal{G} - e$  (these both loose  $f$ ). This achieves the proof of (95).

- (iii) Consider that  $e \in f$  and  $e \in f'$ ,  $f \neq f'$  and both internal. Still by Lemma 4, we expand  $U_{\mathcal{G}}$  as

$$\begin{aligned} U_{\mathcal{G}}^{\text{od/ev}} &= (t_e A_{f/e}^{\text{ev/od}} + A_{f/e}^{\text{od/ev}})(t_e A_{f'/e}^{\text{ev/od}} + A_{f'/e}^{\text{od/ev}}) \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}, f \neq f, f'} A_f^{\text{od/ev}} \\ &= t_e^2 A_{f/e}^{\text{ev/od}} A_{f'/e}^{\text{ev/od}} \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}, f \neq f, f'} A_f^{\text{od/ev}} \\ &\quad + t_e [A_{f/e}^{\text{od/ev}} A_{f'/e}^{\text{ev/od}} + (f \leftrightarrow f')] \prod_{f \in \mathcal{F}_{\text{int};\mathcal{G}}, f \neq f, f'} A_f^{\text{od/ev}} + U_{\mathcal{G}/e}^{\text{od/ev}} \end{aligned}$$

$$= t_e^2 A_{f/e}^{\text{ev/od}} A_{f'/e}^{\text{ev/od}} U_{\mathcal{G} \vee e}^{\text{od/ev}} + t_e U_{\mathcal{G} - e}^{\text{od}} + U_{\mathcal{G}/e}^{\text{od/ev}}, \quad (\text{A.2})$$

where, clearly, by cutting  $e$  in  $\mathcal{G}$ , one loses  $f$  and  $f'$  so that  $\mathcal{F}_{\text{int}; \mathcal{G} \vee e} = \mathcal{F}_{\text{int}; \mathcal{G}} \setminus \{f, f'\}$ , and where  $\mathcal{F}_{\text{int}; \mathcal{G}/e} = \{f/e, f'/e\} \cup \mathcal{F}_{\text{int}; \mathcal{G}} \setminus \{f, f'\}$ . The middle term is more subtle. The removal of  $e$  merges  $f$  and  $f'$  into a unique internal face. The complete odd face polynomial for this new face is given by summing over odd subsets in  $f/e \cup f'/e$ . To get an odd subset, one must take an odd part from one and an even part from the other. In the end the new face polynomial exactly corresponds to  $[A_{f/e}^{\text{od}} A_{f'/e}^{\text{ev}} + (f \leftrightarrow f')]$ . This achieves (96).

(iv) We have  $e^2 \in f$ ,  $f \in \mathcal{F}_{\text{int}; \mathcal{G}}$ .

(a) Let us assume that the deletion of  $e$  gives rise to two distinct internal faces  $f_1$  and  $f_2$ . Lemma 4 helps us to write

$$\begin{aligned} U_{\mathcal{G}}^{\text{od/ev}} &= (2t_e A_{f/e}^{\text{ev/od}} + (t_e^2 + 1) A_{f/e}^{\text{od/ev}}) \prod_{f \in \mathcal{F}_{\text{int}; \mathcal{G}}, f \neq f} A_f^{\text{od/ev}} \\ &= (1 + t_e^2) U_{\mathcal{G}/e}^{\text{od/ev}} + 2t_e \left( A_{f_1}^{\text{od/ev}} A_{f_2}^{\text{od}} + A_{f_1}^{\text{ev/od}} A_{f_2}^{\text{ev}} \right) \prod_{f \in \mathcal{F}_{\text{int}; \mathcal{G}}, f \neq f} A_f^{\text{od/ev}} \\ \begin{cases} U_{\mathcal{G}}^{\text{od}} \\ U_{\mathcal{G}}^{\text{ev}} \end{cases} &= \begin{cases} (1 + t_e^2) U_{\mathcal{G}/e}^{\text{od}} + 2t_e U_{\mathcal{G} - e}^{\text{od}} + 2t_e A_{f_1}^{\text{ev}} A_{f_2}^{\text{ev}} U_{\mathcal{G} \vee e}^{\text{od}} \\ (1 + t_e^2) U_{\mathcal{G}/e}^{\text{ev}} + 2t_e (A_{f_1}^{\text{ev}} A_{f_2}^{\text{od}} + A_{f_1}^{\text{od}} A_{f_2}^{\text{ev}}) U_{\mathcal{G} \vee e}^{\text{ev}} \end{cases}. \end{aligned} \quad (\text{A.3})$$

The fact that we have  $U_{\mathcal{G}/e}^{\text{od/ev}}$  goes by the same argument as before. We have split  $A_{f/e}^{\text{ev/od}}$  into two types of contributions which come from the face polynomials associated with  $f_1$  and  $f_2$ . The set of internal faces of  $\mathcal{G} - e$  are readily obtained from  $\mathcal{F}_{\text{int}; \mathcal{G}} \setminus \{f\} \cup \{f_1, f_2\}$  whereas  $\mathcal{F}_{\text{int}; \mathcal{G}} \setminus \{f\}$  coincides again with the set of faces of  $\mathcal{G} \vee e$ . We get (97).

(b) Finally, we consider that the removal of  $e$  generates one internal face  $f_{12}$ . The first line of (A.3) remains the same. We identify  $f/e$  with  $f_{12}$ , and the rest follows:

$$U_{\mathcal{G}}^{\text{od/ev}} = (1 + t_e^2) U_{\mathcal{G}/e}^{\text{od/ev}} + 2t_e A_{f_{12}}^{\text{ev/od}} U_{\mathcal{G} \vee e}^{\text{od/ev}}. \quad (\text{A.4})$$

## B Proof of Theorem 3

In this section, we give the proof of Theorem 3. Let  $\mathcal{G}$  be a ribbon graph with half-ribbons,  $\mathcal{F}_{\text{int}}$  being its set of internal faces. Let  $\mathcal{F}$  and  $\overline{\mathcal{F}}$  be subsets of  $\mathcal{F}_{\text{int}}$  as stated in the theorem. In the following, the face polynomial expansions are always performed using Lemma 4.

(0) External faces under contraction remain external and do not affect  $\mathcal{U}^{\epsilon, \bar{\epsilon}}$ . This is also why  $\mathcal{F}/e = \mathcal{F}$  and  $\overline{\mathcal{F}}/e = \overline{\mathcal{F}}$ .

(i) Consider  $e$  which belongs to an external face and an internal face denoted by  $f \in \mathcal{F} \subset \mathcal{F}^{\text{int}}$ . We have

$$\begin{aligned} \mathcal{U}_{\mathcal{G}; (\mathcal{F}, \overline{\mathcal{F}})}^{\epsilon, \bar{\epsilon}} &= (t_e A_{f/e}^{\bar{\epsilon}} + A_{f/e}^{\epsilon}) \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f}} A_f^{\epsilon} \right) \left( \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) \\ &= t_e \mathcal{U}_{\mathcal{G}/e; (\mathcal{F} \setminus \{f\}, \overline{\mathcal{F}} \cup \{f/e\})}^{\epsilon, \bar{\epsilon}} + \mathcal{U}_{\mathcal{G}/e; (\mathcal{F} \setminus \{f\} \cup \{f/e\}, \overline{\mathcal{F}})}^{\epsilon, \bar{\epsilon}}. \end{aligned} \quad (\text{B.5})$$

One notices that  $(\mathcal{F} \setminus \{f\}) \cup \{f/e\}$  coincides with  $\mathcal{F}/e$  which is the subset of faces corresponding to  $\mathcal{F}$  in the graph  $\mathcal{G}/e$ . We have also  $\mathcal{F} \setminus \{f\} = (\mathcal{F}/e) \setminus \{f/e\}$  and  $\overline{\mathcal{F}}/e = \overline{\mathcal{F}}$  as  $e$  does not belong to any faces in  $\overline{\mathcal{F}}$ . We get (127).

(ii) If  $e^2 \in f$ ,  $f \in \mathcal{F}$ ,

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= A_f^\epsilon \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f}} A_f^\epsilon \right) \left( \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) \\ &= ((t_e^2 + 1) A_{f/e}^\epsilon + 2t_e A_{f/e}^{\bar{\epsilon}}) \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f}} A_f^\epsilon \right) \left( \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) \\ &= (t_e^2 + 1) \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f\} \cup \{f/e\}, \overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} + 2t_e \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f\}, \overline{\mathcal{F}} \cup \{f/e\})}^{\epsilon,\bar{\epsilon}}. \end{aligned} \quad (\text{B.6})$$

Finally, to get (128), we apply the same identities as in (i).

(iii.a) If  $e \in f_i$ ,  $f_1 \neq f_2$ , with  $f_i \in \mathcal{F}$ , then

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= A_{f_1}^\epsilon A_{f_2}^\epsilon \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f_1, f_2}} A_f^\epsilon \right) \left( \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right) \\ &= (t_e^2 A_{f_1/e}^{\bar{\epsilon}} A_{f_2/e}^{\bar{\epsilon}} + t_e A_{f_1/e}^\epsilon A_{f_2/e}^{\bar{\epsilon}} + t_e A_{f_2/e}^\epsilon A_{f_1/e}^{\bar{\epsilon}} + A_{f_1/e}^\epsilon A_{f_2/e}^\epsilon) \left[ \prod_{\substack{f \in \mathcal{F} \\ f \neq f_1, f_2}} A_f^\epsilon \right] \left[ \prod_{f \in \overline{\mathcal{F}}} A_f^{\bar{\epsilon}} \right] \\ &= t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1, f_2\}, \overline{\mathcal{F}} \cup \{f_1/e, f_2/e\})}^{\epsilon,\bar{\epsilon}} \\ &\quad + t_e \left( \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1, f_2\} \cup \{f_1/e\}, \overline{\mathcal{F}} \cup \{f_2/e\})}^{\epsilon,\bar{\epsilon}} + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1, f_2\} \cup \{f_2/e\}, \overline{\mathcal{F}} \cup \{f_1/e\})}^{\epsilon,\bar{\epsilon}} \right) \\ &\quad + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1, f_2\} \cup \{f_1/e, f_2/e\}, \overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}}, \end{aligned} \quad (\text{B.7})$$

where  $(\mathcal{F} \setminus \{f_1, f_2\}) \cup \{f_1/e, f_2/e\} = \mathcal{F}/e$ ,  $(\mathcal{F} \setminus \{f_1, f_2\}) \cup \{f_1/e\} = (\mathcal{F}/e) \setminus \{f_2/e\}$ ,  $\mathcal{F} \setminus \{f_1, f_2\} = (\mathcal{F}/e) \setminus \{f_1/e, f_2/e\}$ , and  $\overline{\mathcal{F}}/e = \overline{\mathcal{F}}$ . One gets (129).

(iii.b) If  $e \in f_i$ ,  $f_1 \in \mathcal{F}$  and  $f_2 \in \overline{\mathcal{F}}$

$$\begin{aligned} \mathcal{U}_{\mathcal{G};(\mathcal{F},\overline{\mathcal{F}})}^{\epsilon,\bar{\epsilon}} &= A_{f_1}^\epsilon A_{f_2}^{\bar{\epsilon}} \left( \prod_{\substack{f \in \mathcal{F} \\ f \neq f_1}} A_f^\epsilon \right) \left( \prod_{\substack{f \in \overline{\mathcal{F}} \\ f \neq f_2}} A_f^{\bar{\epsilon}} \right) \\ &= (t_e^2 A_{f_2/e}^\epsilon A_{f_1/e}^{\bar{\epsilon}} + t_e A_{f_1/e}^\epsilon A_{f_2/e}^\epsilon + t_e A_{f_1/e}^{\bar{\epsilon}} A_{f_2/e}^{\bar{\epsilon}} + A_{f_1/e}^\epsilon A_{f_2/e}^{\bar{\epsilon}}) \left[ \prod_{\substack{f \in \mathcal{F} \\ f \neq f_1}} A_f^\epsilon \right] \left[ \prod_{\substack{f \in \overline{\mathcal{F}} \\ f \neq f_2}} A_f^{\bar{\epsilon}} \right] \\ &= t_e^2 \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\} \cup \{f_2/e\}, \overline{\mathcal{F}} \setminus \{f_2\} \cup \{f_1/e\})}^{\epsilon,\bar{\epsilon}} \\ &\quad + t_e \left( \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\} \cup \{f_1/e, f_2/e\}, \overline{\mathcal{F}} \setminus \{f_2\})}^{\epsilon,\bar{\epsilon}} + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\}, \overline{\mathcal{F}} \setminus \{f_2\} \cup \{f_1/e, f_2/e\})}^{\epsilon,\bar{\epsilon}} \right) \\ &\quad + \mathcal{U}_{\mathcal{G}/e;(\mathcal{F} \setminus \{f_1\} \cup \{f_1/e\}, \overline{\mathcal{F}} \setminus \{f_2\} \cup \{f_2/e\})}^{\epsilon,\bar{\epsilon}}. \end{aligned} \quad (\text{B.8})$$

We conclude to (130) after identifying  $(\mathcal{F} \setminus \{f_1\}) \cup \{f_1/e\} = \mathcal{F}/e$  and  $(\overline{\mathcal{F}} \setminus \{f_2\}) \cup \{f_2/e\} = \overline{\mathcal{F}}/e$ ,  $(\mathcal{F} \setminus \{f_1\}) \cup \{f_1/e, f_2/e\} = (\mathcal{F}/e) \cup \{f_2/e\}$  and  $\overline{\mathcal{F}} \setminus \{f_2\} = (\overline{\mathcal{F}}/e) \setminus \{f_2/e\}$ , and  $(\mathcal{F} \setminus \{f_1\}) \cup \{f_2/e\} = ((\mathcal{F}/e) \setminus \{f_1/e\}) \cup \{f_2/e\}$  and  $(\overline{\mathcal{F}} \setminus \{f_2\}) \cup \{f_1/e\} = ((\overline{\mathcal{F}}/e) \cup \{f_1/e\}) \setminus \{f_2/e\}$ . The rest of the equalities are obtained in a similar way.

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