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Quantum Gravity and Cosmological Density Perturbations

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ABSTRACT

We explore possible cosmological consequences of a running Newton's constant $G(\Box)$, as suggested by the non-trivial ultraviolet fixed point scenario for Einstein gravity with a cosmological constant term. Here we examine what possible effects a scale-dependent coupling might have on large scale cosmological density perturbations. Starting from a set of manifestly covariant effective field equations, we develop the linear theory of density perturbations for a non-relativistic perfect fluid. The result is a modified equation for the matter density contrast, which can be solved and thus provides an estimate for the corrections to the growth index parameter γ.

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1 Introduction

Recent years have seen the development of a bewildering variety of alternative theories of gravity, in addition to the more traditional ones such as scalar-tensor, higher derivative, and dilaton gravities, just to mention a few examples. Many of these theories eventually predict some level of deviation from classical gravity, which is often parametrized either by a suitable set post-Newtonian parameters, or more recently by the introduction of a slip function [1, 2]. The latter has been quite useful in describing deviations from classical GR, and specifically from the standard ΛCMD model, when analyzing the latest cosmological CMB, weak lensing, supernovae and galaxy clustering data.

In this paper we will focus on the systematic analysis of departures from GR in the growth history of matter perturbations arising from a quantum running of G, within the narrow context of the non-trivial ultraviolet fixed point scenario for Einstein gravity with a cosmological term. Thus instead of looking at deviations from GR at very short distances, due to new interactions such as the ones suggested by string theories [3], we will be considering here infrared effects, which could therefore become manifest at very large distances. We will argue here that such effects are in principle calculable, and could therefore be confronted with present and future astrophysical observation. The classical theory of small density perturbations is by now well developed in standard textbooks, and the resulting theoretical predictions for the growth exponents are simple to state, and well understood. Except possibly on the very largest scales, where the data so far is still rather limited, the predictions agree quite well with current astrophysical observations. Here we will be interested in computing and predicting possible small deviations in the growth history of matter perturbations, and specifically in the values of the growth exponents, arising from a very specific scenario, namely a weakly scale-dependent gravitational coupling, whose value very gradually increases with distance. The specific nature of the scenario we will be investigating here is motivated by the treatment of field-theoretic models of quantum gravity, based on the Einstein action with a bare cosmological term. Its long distance scaling properties are derived from the existence of a non-trivial ultraviolet fixed point of the renormalization group in Newton's constant G [4, 5, 6, 7, 8, 9].

The first step in analyzing the consequences of a running of G is thus to re-write the expression for $G(k^2)$ in a coordinate-independent way, either by the use of a non-local Vilkovisky-type effective gravity action [10, 11], or by the use of a set of consistent effective field equations. In going from momentum to position space one employs $k^2 \to -\Box$, which gives for the quantum-mechanical running of the gravitational coupling the replacement $G \rightarrow G(\Box)$. One then finds that the running of G is given, in the vicinity of the UV fixed point, by

$$G(\Box) \;=\; G_0 \left[\, 1 \, + \, c_0 \left(\frac{1}{\xi^2 \Box} \right)^{1/2\nu} \, + \, \dots \right] \, , \tag{1.1}$$

where $\Box \equiv g^{\mu\nu} \nabla_\mu \nabla_\nu$ is the covariant d'Alembertian, and the dots represent higher order terms in an expansion in $1/(\xi^2 \Box)$. Current evidence from Euclidean lattice quantum gravity points toward $c_0 > 0$ (implying infrared growth) and $\nu \simeq \frac{1}{3}$ [7].

Within the quantum-field-theoretic renormalization group treatment, the quantity ξ arises as an integration constant of the Callan-Symanzik renormalization group equations. One challenging issue, and of great relevance to the physical interpretation of the results, is a correct identification of the renormalization group invariant scale ξ. A number of arguments can be given in support of the suggestion that the infrared scale ξ (analogous to the $\Lambda_{\overline{MS}}$ of QCD) can in fact be very large, even cosmological, in the gravity case (see for ex. [5] and references therein). From these arguments one would then first infer that the constant G_0 can, to a very close approximation, be identified with the laboratory value of Newton's constant, $\sqrt{G_0} \sim 1.6 \times 10^{-33}$cm. The appearance of the d'Alembertian $\Box$ in the running of G then naturally leads to a set of non-local field equations; instead of the ordinary Einstein field equations with constant G one is now lead to consider the modified effective field equations

$$R_{\mu\nu} \, - \, \tfrac{1}{2} \, g_{\mu\nu} \, R \, + \, \lambda \, g_{\mu\nu} \;=\; 8 \pi \, G(\Box) \, T_{\mu\nu} \tag{1.2}$$

with a new non-local term due to the $G(\Box)$. By being manifestly covariant they still satisfy some of the basic requirements for a set of consistent field equations incorporating the running of G. Not unexpectedly though, the new nonlocal equations are much harder to solve than the original classical field equations for constant G.

As stated above, physically it would seem at first, based on the perturbative treatment alone [4, 8], that the non-perturbative scale ξ could take any value (including perhaps a very small one), which could then possibly preclude any observable quantum effects in the foreseeable future. In perturbation theory the reason for this is that the non-perturbative scale ξ appears, as in gauge theories, as an integration constant of the renormalization group equations, and is therefore not fixed by perturbation theory alone. But a number of recent non-perturbative results for the gravitational Wilson loop on the Euclidean lattice at strong coupling, giving an area law, and their subsequent interpretation in light of the observed large scale semiclassical curvature [14, 5], would suggest otherwise: namely that the non-perturbative scale ξ appears in fact to be related to macroscopic *curvature.* From astrophysical observation the average curvature on very large scales, or, stated in somewhat better terms, the measured cosmological constant λ, is very small. This would then suggest that the new scale ξ can be very large, even cosmological, and comparable to the Hubble scale, $1/\xi^2 \simeq \lambda/3$. This would then give a more concrete semi-quantitative estimate for the scale in the $G(\Box)$ of Eq. (1.1), namely $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$cm. It is this option that we want to explore in this paper.

A scale dependent Newton's constant is already expected to lead to small modifications of the standard cosmological solutions to the Einstein field equations. The starting point are the quantum effective field equations of Eq. (1.2), with $G(\Box)$ defined in Eq. (1.1). In the Friedmann-Lemaître-Robertson-Walker (FLRW) framework these are applied to the standard homogeneous isotropic metric. In the following we will mainly consider the case $k = 0$ (spatially flat universe). The next step therefore is a systematic examination of the nature of the solutions to the full effective field equations, with $G(\Box)$ involving the relevant covariant d'Alembertian operator $\Box = g^{\mu\nu} \nabla_\mu \nabla_\nu$. To start the process, one assumes for example that the matter $T_{\mu\nu}$ has the perfect fluid form,

$$T_{\mu\nu} = [\, p(t) + \rho(t) \,]\, u_\mu \, u_\nu + g_{\mu\nu} \, p(t) \qquad (1.3)$$

for which one needs to compute the action of $\Box^n$ on $T_{\mu\nu}$, and then analytically continues the answer to negative fractional values of $n = -1/2\nu$. Even in the simplest case, with $G(\Box)$ acting on a *scalar* such as the trace of the energy-momentum tensor $T_\lambda{}^\lambda$, one finds a rather unwieldy expression.

A more general calculation [12] shows that a non-vanishing pressure contribution is generated in the effective field equations, even if one initially assumes a pressureless fluid, $p(t) = 0$. After a somewhat lengthy derivation one obtains for a universe filled with non-relativistic matter (p=0) a set of effective Friedmann equations incorporating the running of G. It was also noted in [12] that the effective field equations with a running G can be recast in an equivalent, but slightly more appealing, form by defining a vacuum polarization pressure p_{vac} and density ρ_{vac}, such that for the FLRW background one has

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0} \, \rho(t) \qquad\qquad p_{vac}(t) = \frac{1}{3} \, \frac{\delta G(t)}{G_0} \, \rho(t) \; . \qquad (1.4)$$

with $G(t)$ given by

$$G(t) \; \equiv \; G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \; = \; G_0 \left[1 + c_t \left(\frac{t}{t_0} \right)^{1/\nu} \; + \; \ldots \right] \; . \qquad (1.5)$$

The explicit computations also shows that c_t is of the same order as c_0 in Eq. (1.1), and $t_0 = \xi$ [12]; in the quoted reference it was estimated $c_t = 0.450 \; c_0$ for the tensor box operator.

Then the source term in the effective tt field equation can be regarded as a combination of the two density terms $\rho(t) + \rho_{vac}(t)$, while the effective rr equation involves the new vacuum polarization pressure term $p_{vac}(t)$. Just as one introduces the parameter w, describing the matter equation of state, $p(t) = w\,\rho(t)$, with $w = 0$ for non-relativistic for matter, one can do the same for the remaining contribution by setting $p_{vac}(t) = w_{vac}\;\rho_{vac}(t)$. This more compact notation allows one to finally re-write the field equations for the FLRW background (and $k = 0$) as

$$\begin{aligned} 3\,\frac{\dot{a}^2(t)}{a^2(t)} &= 8\pi\,G_0\left(1 + \frac{\delta G(t)}{G_0}\right)\bar{\rho}(t) + \lambda \\ \frac{\dot{a}^2(t)}{a^2(t)} + 2\,\frac{\ddot{a}(t)}{a(t)} &= -8\pi\,G_0\left(w + w_{vac}\frac{\delta G(t)}{G_0}\right)\bar{\rho}(t) + \lambda\;. \end{aligned} \tag{1.6}$$

2 Relativistic Treatment of Matter Density Perturbations

Besides the modified cosmic scale factor evolution just discussed, the running of $G(\Box)$ given in Eq. (1.1) also affects the nature of matter density perturbations on very large scales. In computing these effects, it is customary to introduce a perturbed metric of the form

$$d\tau^2 = dt^2 - a^2\,(\delta_{ij} + h_{ij})\,dx^i dx^j\;, \tag{2.1}$$

with $a(t)$ the unperturbed scale factor and $h_{ij}(\mathbf{x}, t)$ a small metric perturbation, and $h_{00} = h_{i0} = 0$ by choice of coordinates. As will become clear later, we will mostly be concerned here with the trace mode $h_{ii} \equiv h$, which determines the nature of matter density perturbations. After decomposing the matter fields into background and fluctuation contribution, $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, and $\mathbf{v} = \bar{\mathbf{v}} + \delta\mathbf{v}$, it is customary in these treatments to expand the density, pressure and metric trace perturbation modes in spatial Fourier modes, $\delta\rho(\mathbf{x}, t) = \delta\rho_{\mathbf{q}}(t)\,e^{i\,\mathbf{q}\cdot\mathbf{x}}$ and similarly for $\delta p(\mathbf{x}, t)$, $\delta\mathbf{v}(\mathbf{x}, t)$ and $h_{ij}(\mathbf{x}, t)$ with $\mathbf{q}$ the comoving wavenumber.

The first equation one obtains is the zeroth (in the fluctuations) order energy conservation in the presence of $G(\Box)$, which reads

$$3\,\frac{\dot{a}(t)}{a(t)}\left[(1 + w) + (1 + w_{vac})\,\frac{\delta G(t)}{G_0}\right]\bar{\rho}(t) + \frac{\delta\dot{G}(t)}{G_0}\,\bar{\rho}(t) + \left(1 + \frac{\delta G(t)}{G_0}\right)\dot{\bar{\rho}}(t) = 0\;. \tag{2.2}$$

It will be convenient in the following to solve the energy conservation equation not for $\bar{\rho}(t)$, but instead for $\bar{\rho}(a)$. This requires that, instead of using the expression for $G(t)$ in Eq. (1.5), one uses the equivalent expression for $G(a)$

$$G(a) = G_0 \left(1 + \frac{\delta G(a)}{G_0} \right) , \qquad \text{with } \frac{\delta G(a)}{G_0} \equiv c_a \left(\frac{a}{a_0} \right)^{\gamma_\nu} + \dots \, . \tag{2.3}$$

In this last expression the power is $\gamma_\nu = 3/2\nu$, since from Eq. (1.5) one has for non-relativistic matter $a(t)/a_0 \approx (t/t_0)^{2/3}$ in the absence of a running G. In the following we will almost exclusively consider the case $\nu = \frac{1}{3}$ [7] for which therefore $\gamma_\nu = 9/2$. Then in the above expression $c_a \approx c_t$ if a_0 is identified with a scale factor appropriate for a universe of size ξ; to a good approximation this should correspond to the universe "today", with the relative scale factor customarily normalized at such a time to $a/a_0 = 1$. Consequently, and with the above proviso, the constant c_a in Eq. (2.3) can safely be taken to be of the same order as the constant c_0 appearing in the original expressions for $G(\Box)$ in Eq. (1.1). The solution to Eq. (2.2) for $w_{vac} = \frac{1}{3}$ can then be written as

$$\bar{\rho}(a) = \bar{\rho}_0 \left(\frac{a_0}{a} \right)^3 \left(\frac{1 + c_a}{1 + c_a \left(\frac{a}{a_0} \right)^{\gamma_\nu}} \right)^{(1+\gamma_\nu)/\gamma_\nu} \tag{2.4}$$

with $\bar{\rho}(a)$ normalized so that $\bar{\rho}(a = a_0) = \bar{\rho}_0$. For $c_a = 0$ the above expression reduces of course to the usual result for non-relativistic matter.

The zeroth order field equations with the running of G included were already given in Eq. (1.6). The next step consists in obtaining the equations which govern the effects of small field perturbations. These equations will involve, apart from the metric perturbation h_{ij}, the matter and vacuum polarization contributions. The latter arise from

$$\left(1 + \frac{\delta G(\Box)}{G_0} \right) T_{\mu\nu} \; = T_{\mu\nu} + T_{\mu\nu}^{vac} \tag{2.5}$$

with a nonlocal $T_{\mu\nu}^{vac}$. Fortunately to zeroth order in the fluctuations the results of Ref. [12] indicated that the modifications from the nonlocal vacuum polarization term could simply be accounted for by the substitution $\bar{\rho}(t) \rightarrow \; \bar{\rho}(t) + \bar{\rho}_{vac}(t)$ and $\bar{p}(t) \rightarrow \; \bar{p}(t) + \bar{p}_{vac}(t)$. Here we will apply this last result to the small field fluctuations as well, and set

$$\delta\rho_{\mathbf{q}}(t) \rightarrow \; \delta\rho_{\mathbf{q}}(t) + \delta\rho_{\mathbf{q}\, vac}(t) \qquad \qquad \delta p_{\mathbf{q}}(t) \rightarrow \; \delta p_{\mathbf{q}}(t) + \delta p_{\mathbf{q}\, vac}(t) \; . \tag{2.6}$$

The underlying assumptions is of course that the equation of state for the vacuum fluid still remains roughly correct when a small perturbation is added. Furthermore, just like we had $\bar{p}(t) = w \, \bar{\rho}(t)$ and $\bar{p}_{vac}(t) = w_{vac} \, \bar{\rho}_{vac}(t)$ with $w_{vac} = \frac{1}{3}$, we now write for the fluctuations

$$\delta p_{\mathbf{q}}(t) = w \, \delta\rho_{\mathbf{q}}(t) \qquad \quad \delta p_{\mathbf{q}\, vac}(t) = w_{vac} \, \delta\rho_{\mathbf{q}\, vac}(t) \; , \tag{2.7}$$

at least to leading order in the long wavelength limit, $\mathbf{q} \to 0$. In this limit we then have simply

$$\delta p(t) = w\, \delta\rho(t) \qquad \delta p_{vac}(t) = w_{vac}\, \delta\rho_{vac}(t) \equiv w_{vac}\, \frac{\delta G(t)}{G_0} \delta\rho(t) \ , \tag{2.8}$$

with $G(t)$ given in Eq. (1.5), and we have used Eq. (1.4), now applied to the fluctuation $\delta\rho_{vac}(t)$,

$$\delta\rho_{vac}(t) \ = \ \frac{\delta G(t)}{G_0}\, \delta\rho(t) + \dots \tag{2.9}$$

where the dots indicate possible additional $O(h)$ contributions. Indeed a bit of thought reveals that the above treatment is incomplete, since $G(\Box)$ in the effective field equation of Eq. (1.2) contains, for the perturbed RW metric of Eq. (2.1), terms of order h_{ij}, which need to be accounted for in the effective $T^{\mu\nu}_{vac}$. Consequently the covariant d'Alembertian has to be Taylor expanded in the small field perturbation h_{ij}, $\Box(g) \ = \ \Box^{(0)} + \Box^{(1)}(h) + O(h^2)$, and similarly for $G(\Box)$

$$G(\Box) \ = \ G_0 \left[1 + \frac{c_0}{\xi^{1/\nu}} \left(\frac{1}{\Box^{(0)} + \Box^{(1)}(h) + O(h^2)} \right)^{1/2\nu} + \dots \right] \ . \tag{2.10}$$

To compute the correction of $O(h)$ to $\delta\rho_{vac}(t)$ one needs to consider the relevant term in the expansion of $(1 + \delta G(\Box)/G_0)\, T_{\mu\nu}$, which we write as

$$-\frac{1}{2\,\nu}\, \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot \frac{\delta G(\Box^{(0)})}{G_0} \cdot T_{\mu\nu} \ . \tag{2.11}$$

This last form allows us to use the results obtained previously for the FLRW case in [12], namely

$$\frac{\delta G(\Box^{(0)})}{G_0}\, T_{\mu\nu} \ = \ T^{vac}_{\mu\nu} \tag{2.12}$$

with here $T^{vac}_{\mu\nu} \ = \ [p_{vac}(t) + \rho_{vac}(t)]\, u_\mu\, u_\nu + g_{\mu\nu}\, p_{vac}(t)$. and (see Eq. (1.4)), to zeroth order in h,

$$\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\, \bar\rho(t) \qquad\qquad p_{vac}(t) = w_{vac}\, \frac{\delta G(t)}{G_0}\, \bar\rho(t) \ . \tag{2.13}$$

with $w_{vac} \ = \ \frac{1}{3}$. Therefore, in light of the results of Ref. [12], the problem has been dramatically reduced to just computing the much more tractable expression

$$-\frac{1}{2\,\nu}\, \frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h) \cdot T^{vac}_{\mu\nu} \ . \tag{2.14}$$

Still, in general the resulting expression for $\frac{1}{\Box^{(0)}} \cdot \Box^{(1)}(h)$ is rather complicated if evaluated for arbitrary functions. Here we will resort, for lack of better insights, to a treatment where one assumes a harmonic time dependence for the metric trace fluctuation $h(t) = h_0\, e^{i\omega t}$, and similarly for $a(t) = a_0\, e^{i\Gamma t}$ and $\rho(t) = \rho_0\, e^{i\Gamma t}$. In the limit $\omega \gg \Gamma$, corresponding to $\dot h/h \gg \dot a/a$, one finds for the fluctuation $\delta\rho_{vac}(t)$

$$\delta\rho_{vac}(t) = \frac{\delta G(t)}{G_0}\, \delta\rho(t) + \frac{1}{2\,\nu}\, c_h\, \frac{\delta G(t)}{G_0}\, h(t)\, \bar\rho(t) \ . \tag{2.15}$$

The $O(h)$ correction factor c_h for the tensor box is then found to be

$$c_h = \frac{11}{3} \frac{\dot{a}}{a} \frac{h}{\dot{h}} , \tag{2.16}$$

with all other off-diagonal matrix elements vanishing. Furthermore one finds to this order, but only for the specific choice $w_{vac} = \frac{1}{3}$ in the zeroth order $T_{\mu\nu}^{vac}$, $\delta p_{vac}(t) = \frac{1}{3} \delta\rho_{vac}(t)$, *i.e.* the $O(h)$ correction preserves the original result $w_{vac} = \frac{1}{3}$.

As far as the magnitude of the correction c_h in Eq. (2.16) one can argue that from Eq. (2.17) one can relate the combination $(\dot{h}/h)(a/\dot{a})$ to the growth index $f(a)$,

$$\frac{\dot{h}}{h} \frac{a}{\dot{a}} = \frac{\partial \log h(a)}{\partial \log a} = \frac{\partial \log \delta(a)}{\partial \log a} \equiv f(a) , \tag{2.17}$$

where $\delta(a)$ is the matter density contrast, and $f(a)$ the known density growth index [15]. Then, in the absence of a running G (which is all that is needed here, to the order one is working), an explicit form for $f(a)$ is known in terms of suitable derivatives of a Gauss hypergeometric function. These can then be inserted into Eq. (2.16). Alternatively, one can make use again of the fact that for a scale factor referring to "today" $a/a_0 \approx 1$, and for a matter fraction $\Omega \approx 0.25$, one knows that $f(a = a_0) \simeq 0.4625$, and thus in Eq. (2.15) $c_h \simeq (11/3) \times 2.1621 = +7.927$. Furthermore, as an exercise one can redo the whole calculation in the much simpler scalar box acting on ${T_\lambda}^\lambda$ case, where one finds the smaller value $c_h \simeq +2.162$,

Finally one can do the same analysis in the opposite, but less physical, limit $\omega \ll \Gamma$ or $\dot{h}/h \ll \dot{a}/a$. But this second limit is in our opinion less physical, because of the fact that now the background is assumed to be varying more rapidly in time than the metric perturbation itself, $\dot{a}/a \gg \dot{h}/h$. Furthermore, one disturbing but not entirely surprising general aspect of the whole calculation in this second $\omega \ll \Gamma$ limit, is its extreme sensitivity as far as magnitudes and signs of the results are concerned, to the set of assumptions initially made about the time development of the background. For the reasons mentioned, in the following we will no longer consider this limit of rapid background fluctuations any further.

To summarize, the results for a scalar box and for a very slowly varying background, $\dot{h}/h \gg \dot{a}/a$, give the $O(h)$ corrected expression for $\delta\rho_{vac}(t)$ in Eq. (2.15) and $\delta p_{vac}(t) = w_{vac} \, \delta\rho_{vac}(t)$ with $c_h \simeq +2.162$, while the tensor box calculation, under essentially the same assumptions, gives the somewhat larger result $c_h \simeq +7.927$. From now on, these will be the only two choices we shall consider here.

The next step in the analysis involves the derivation of the energy-momentum conservation to first order in the fluctuations, and a derivation of the relevant field equations to the same order.

After that, energy conservation is used to eliminate the h field entirely, and thus obtain a single equation for the matter density fluctuation δ. First we will look here at the implications of energy-momentum conservation, $\nabla^\mu \left(T_{\mu\nu} + T_{\mu\nu}^{vac} \right) = 0$, to first order in the fluctuations. After defining the matter density contrast $\delta(t)$ as the ratio $\delta(t) \equiv \delta\rho(t)/\bar{\rho}(t)$, the energy conservation equation to first order in the perturbations is found to be

$$\left[-\frac{1}{2} \left((1+w) + (1+w_{vac}) \, \frac{\delta G(t)}{G_0} \right) - \frac{1}{2\nu} \, c_h \, \frac{\delta G(t)}{G_0} \right] \quad \dot{h}(t) \\ + \quad \left[\frac{1}{2\nu} \, c_h \, \left(3 \, (w - w_{vac}) \, \frac{\dot{a}(t)}{a(t)} \, \frac{\delta G(t)}{G_0} - \frac{\delta \dot{G}(t)}{G_0} \right) \right] \quad h(t) \quad = \quad \left[1 + \frac{\delta G(t)}{G_0} \right] \, \dot{\delta}(t) \; . \tag{2.18}$$

In the absence of a running G ($\delta G(t) = 0$) this reduces simply to $-\frac{1}{2} \, (1+w) \, h(t) = \delta(t)$. This last result then allows us to solve explicitly, at the given order, *i.e.* to first order in the fluctuations and to first order in δG, for the metric perturbation $\dot{h}(t)$ in terms of the matter density fluctuation $\delta(t)$ and $\dot{\delta}(t)$.

Also, to first order in the perturbations, the tt and ii effective field equations become, respectively,

$$\frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) - 8\pi \, G_0 \, \frac{1}{2\nu} \, c_h \, \frac{\delta G(t)}{G_0} \, \bar{\rho}(t) \, h(t) = 8\pi \, G_0 \left(1 + \frac{\delta G(t)}{G_0} \right) \, \bar{\rho}(t) \, \delta(t) \tag{2.19}$$

and

$$\ddot{h}(t) \, + \, 3 \, \frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) \, + \, 24\pi \, G_0 \, \frac{1}{2\nu} \, c_h \, w_{vac} \, \frac{\delta G(t)}{G_0} \, \bar{\rho}(t) \, h(t) = - \, 24\pi \, G_0 \left(w + w_{vac} \, \frac{\delta G(t)}{G_0} \right) \, \bar{\rho}(t) \, \delta(t) \tag{2.20}$$

In the second ii equation, the zeroth order ii field equation of Eq. (1.6) has been used to achieve some simplification. It is easy to check the overall consistency of the first order energy conservation equation of Eq. (2.18), and of the two field equations given in Eqs. (2.19) and (2.20).

To obtain an equation for the matter density contrast $\delta(t) = \delta\rho(t)/\bar{\rho}(t)$ one needs to eliminate the metric trace field $h(t)$ from the field equations. This is first done by taking a suitable linear combination of the two field equations in Eqs. (2.19) and (2.20), to get the equivalent equation

$$\ddot{h}(t) + 2 \, \frac{\dot{a}(t)}{a(t)} \, \dot{h}(t) \quad + \quad 8\pi \, G_0 \, \frac{1}{2\nu} \, c_h \, (1 + 3 \, w_{vac}) \, \frac{\delta G(t)}{G_0} \, \bar{\rho}(t) \, h(t) \\ = \quad - \, 8\pi \, G_0 \left[(1 + 3 \, w) + (1 + 3 \, w_{vac}) \, \frac{\delta G(t)}{G_0} \right] \bar{\rho}(t) \, \delta(t) \; . \tag{2.21}$$

Then the first order energy conservation equations to zeroth and first order in δG allow one to completely eliminate the h, $\dot{h}$ and $\ddot{h}$ field in terms of the matter density perturbation $\delta(t)$ and its derivatives. The resulting equation reads, for $w = 0$ and $w_{vac} = \frac{1}{3}$,

$$\ddot{\delta}(t) \qquad + \left[\left(2 \, \frac{\dot{a}(t)}{a(t)} - \frac{1}{3} \, \frac{\delta \dot{G}(t)}{G_0} \right) - \frac{1}{2\nu} \cdot 2 \, c_h \cdot \left(\frac{\dot{a}(t)}{a(t)} \, \frac{\delta G(t)}{G_0} + 2 \, \frac{\delta \dot{G}(t)}{G_0} \right) \right] \dot{\delta}(t)$$

$$
+\left[-4\pi\,G_0\left(1+\frac{7}{3}\,\frac{\delta G(t)}{G_0}-\frac{1}{2\nu}\cdot 2\,c_h\cdot\frac{\delta G(t)}{G_0}\right)\bar\rho(t)\right.
$$
$$
\left.-\frac{1}{2\nu}\cdot 2\,c_h\cdot\left(\frac{\dot a^2(t)}{a^2(t)}\,\frac{\delta G(t)}{G_0}+3\,\frac{\dot a(t)}{a(t)}\,\frac{\delta\dot G(t)}{G_0}+\frac{\ddot a(t)}{a(t)}\,\frac{\delta G(t)}{G_0}+\frac{\delta\ddot G(t)}{G_0}\right)\right]\delta(t)=0\;. \tag{2.22}
$$

This last equation then describes matter density perturbations to linear order, taking into account the running of $G(\Box)$, and is therefore the main result of this paper. The terms proportional to c_h, which can be clearly identified in the above equation, describe the feedback of the metric fluctuations h on the vaccum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations.

The above equation can now be compared with the corresponding, much simpler, equation obtained for constant G, *i.e.*, for $G\to G_0$ and still $w=0$ (see for example [16] and [15])

$$
\ddot\delta(t)+2\,\frac{\dot a}{a}\,\dot\delta(t)-4\pi\,G_0\,\bar\rho(t)\,\delta(t)=0\;. \tag{2.23}
$$

It is common practice at this point to write an equation for the density contrast $\delta(a)$ as a function not of t, but of the scale factor $a(t)$. This is done by utilizing simple derivative identities to relate derivatives with respect to t to derivatives with respect to $a(t)$, with $H\equiv\dot a(t)/a(t)$ the Hubble constant. This last quantity can be obtained from the zeroth order tt field equation, sometimes written in terms of current density fractions,

$$
H^2(a)\equiv\left(\frac{\dot a}{a}\right)^2=\left(\frac{\dot z}{1+z}\right)^2=H_0^2\left[\Omega\;(1+z)^3+\Omega_R\;(1+z)^2+\Omega_\lambda\right] \tag{2.24}
$$

with $a/a_0=1/(1+z)$ where z is the red shift, and a_0 the scale factor "today". Then H_0 is the Hubble constant evaluated today, Ω the (baryonic and dark) matter density, Ω_R the space curvature contribution corresponding to a curvature k term, and Ω_λ the dark energy or cosmological constant part, all again measured *today*. In the absence of spatial curvature $k=0$ one has today

$$
\Omega_\lambda\equiv\frac{\lambda}{3\,H_0^2}\qquad\Omega\equiv\frac{8\,\pi\,G_0\,\bar\rho_0}{3\,H_0^2}\qquad\Omega+\Omega_\lambda=1\;. \tag{2.25}
$$

It is convenient at this stage to introduce a parameter θ describing the cosmological constant fraction as measured today, $\theta\equiv\Omega_\lambda/\Omega$. While the following discussion will continue with some level of generality, in practice one is mostly interested in the observationally favored case of a current matter fraction $\Omega\approx 0.25$, for which $\theta\approx 3$. In terms of the parameter θ the growing solution to the differential equation for the density contrast $\delta(a)$ for constant G is

$$
\delta_0(a)\propto a\cdot{}_2F_1\left(\frac{1}{3},1;\frac{11}{6};-a^3\,\theta\right) \tag{2.26}
$$

where ${}_2F_1$ is the Gauss hypergeometric function. The subscript 0 in $\delta_0(a)$ is to remind us that this solution is appropriate for the case of constant $G = G_0$. To evaluate the correction to $\delta_0(a)$ coming from the terms proportional to c_a one sets

$$\delta(a) \propto \delta_0(a) \; [\, 1 + c_a \, \mathcal{F}(a) \,] \ , \tag{2.27}$$

and inserts the resulting expression in Eq. (2.22), written now as a differential equation in $a(t)$. One only needs to determine the differential equations for density perturbations δ up to first order in the fluctuations, so it will be sufficient to obtain an expression for Hubble constant H from the tt component of the effective field equation to zeroth order in the fluctuations, namely the first of Eqs. (1.6). One has

$$H(a) = \sqrt{\frac{8\pi}{3} \, G_0 \left(1 + \frac{\delta G(a)}{G_0} \right) \bar{\rho}(a) + \frac{\lambda}{3}} \tag{2.28}$$

with $G(a)$ giben in Eq. (2.3) and $\bar{\rho}(a)$ in Eq. (2.4). In this last expression the exponent is $\gamma_\nu = 3/2\nu \simeq 9/2$ for a matter dominated background universe, although more general choices, such as $\gamma_\nu = 3(1+w)/2\nu$ are possible and should be explored (see discussion later). Also, $c_a \approx c_t$ if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe "today", with the relative scale factor customarily normalized at that time to $a/a_0 = 1$. In [12] it was found that in Eq. (1.5) $c_t \simeq 0.785 \, c_0$ in the scalar box case, and $c_t \simeq 0.450 \, c_0$ in the tensor box case; in the following we will use the average of the two values.

After the various substitutions and insertions have been performed, one obtains, after expanding to linear order in a_0, a second order linear differential equation for the correction $\mathcal{F}(a)$ to $\delta(a)$, as defined in Eq. (2.27). Since this equation looks rather complicated for general $\delta G(a)$ it won't be recorded here, but it is easily obtained from Eq. (2.22) by a sequence of straightforward substitutions and expansions. The resulting equation can then be solved for $\mathcal{F}(a)$, giving the desired density contrast $\delta(a)$ as a function of the parameter Ω.

To obtain an explicit solution to the $\delta(a)$ equation one needs to know the coefficient c_a and the exponent γ_ν in Eq. (2.3), whose likely values are discussed above and right after the quoted expression for $G(a)$. For the exponent ν one has $\nu \simeq \frac{1}{3}$, whereas for the value for c_h one finds, according to the discussion in the previous section, $c_h \simeq 7.927$ for the tensor box case. Furthermore one needs at some point to insert a value for the matter density fraction parameter θ, which based on current observation is close to $\theta = (1 - \Omega)/\Omega \simeq 3$.

3 Relativistic Growth Index with G(□)

When discussing the growth of density perturbations in classical General Relativity it is customary at this point to introduce a scale-factor-dependent *growth index* $f(a)$ defined as

$$f(a) \equiv \frac{\partial \ln \delta(a)}{\partial \ln a} \; , \tag{3.29}$$

which is in principle obtained from the differential equation for any scale factor $a(t)$. Nevertheless, here one is mainly interested in the neighborhood of the present era, $a(t) \approx a_0$. One therefore introduces today's *growth index parameter* γ via

$$f(a = a_0) \equiv \left. \frac{\partial \ln \delta(a)}{\partial \ln a} \right|_{a=a_0} \equiv \; \Omega^{\gamma} \; . \tag{3.30}$$

The solution of the above differential equation for $\delta(a)$ then determines an explicit value for the growth index γ parameter, for any value of the current matter fraction Ω. In the end, because of observational constraints, one is mostly interested in the range $\Omega \approx 0.25$, so the following discussion will be limited to this case only, although from the original differential equation for $\delta(a)$ one can in principle obtain a solution for any sensible Ω.

It is known that in the absence of a running Newton's constant G ($G \to G_0$, thus $c_a = 0$) one has $f(a = a_0) = 0.4625$ and $\gamma = 0.5562$ for the standard ΛCDM scenario with $\Omega = 0.25$ [15]. On the other hand, when the running of $G(\Box)$ is taken into account, one finds from the solution to Eq. (2.22) for the growth index parameter γ at $\Omega = 0.25$ the following set of results. For γ one has

$$\gamma = 0.5562 - (0.703 + 25.04 \, c_h) \, c_a + O(c_a^2) \; . \tag{3.31}$$

with $c_h = (11/3) \times 2.1621 = 7.927$ in the tensor box case (see Eq. (2.15)), and $c_h = 2.1621$ in the scalar box case. In the Newtonian (non-relativistic) treatment one finds the much smaller correction

$$\gamma = 0.5562 - 0.0142 \, c_a + O(c_a^2) \; . \tag{3.32}$$

Among these last expressions, the tensor box case is supposed to give ultimately the correct answer; the scalar box case only serves as a qualitative comparison. The c_h term is responsible for the feedback of the metric fluctuations h on the vacuum density $\delta\rho_{vac}$ and pressure δp_{vac} fluctuations.

It should be emphasized here that all of the above results have been obtained by solving the differential equation for $\delta(a)$ with $G(a)$ given in Eq. (2.3), and exponent $\gamma_\nu = 3/2\nu \simeq 9/2$ relevant for a matter dominated background universe. But it is this last choice that needs to be critically analyzed, as it might give rise to a definite bias. Our value for γ_ν so far reflects our choice of a matter dominated background. More general choices, such as an "effective" $\gamma_\nu = 3(1+w)/2\nu$ with and "effective" w, are in principle possible. Then, although Eq. (2.22) for $\delta(t)$ remains unchanged, the differential equation for $\delta(a)$ would have to be solved with new parameters. Therefore in a little bit we will discuss a number of options which should allow one to increase on the accuracy of the above result, and in particular correct the possible shortcomings coming so far from the specific choice of the exponent γ_ν.

To quantitatively estimate the actual size of the correction in the above expressions for the growth index parameter γ, and make some preliminary comparison to astrophysical observations, some additional information is needed.

The first item is the coefficient $c_0 \approx 33.3$ in Eq. (1.1) as obtained from lattice gravity calculations of invariant correlation functions at fixed geodesic distance [17]. We have re-analyzed the results of [17] which involve rather large uncertainties for this particular quantity, nevertheless it would seem difficult to accommodate values for c_0 that are more than an order of magnitude smaller than the quoted value.

The next item that is needed here is a quantitative estimate for the magnitude of the coefficient c_a in Eq. (2.3) in terms of c_t in Eq. (1.5), and therefore in terms of c_0 in the original Eq. (1.1). First of all one has $c_a \approx c_t$, if a_0 is identified with a scale factor corresponding to a universe of size ξ; to a good approximation this corresponds to the universe "today", with the relative scale factor customarily normalized at that time to $a/a_0 = 1$, although some large conversion factor might be hidden in this perhaps naive identification (see below).

Regarding the numerical value of the coefficient c_t itself, it was found in [12] that in Eq. (1.5) $c_t \simeq 0.785\, c_0$ in the scalar box case, and $c_t \simeq 0.450\, c_0$ in the tensor box case. In both cases these estimates refer to values obtained from the zeroth order covariant effective field equations. In the following we will take for concreteness the average of the two values, thus $c_t \approx 0.618\, c_0$. Then for all three covariant calculations recorded above $c_a \approx 0.618 \times 33.3 \approx 20.6$, a rather large coefficient.

From all of these considerations one would tend to get estimates for the growth parameter γ with rather large corrections! For example, in the tensor box case the corrections would add up to $-199.\, c_a = -199. \times 0.618 \times 33.3 = -4095.$.

It would seem though that one should account somewhere for the fact that the largest galaxy clusters and superclusters studied today up to redshifts $z \simeq 1$ extend for only about, at the very most, 1/20 the overall size of the visible universe. This would suggest then that the corresponding scale for the running coupling $G(t)$ or $G(a)$ in Eqs. (1.5) and (2.3) respectively, should be reduced by a suitable ratio of the two relevant length scales, one for the largest observed galaxy clusters or superclusters, and the second for the very large, cosmological scale $\xi \sim 1/\sqrt{\lambda/3} \sim 1.51 \times 10^{28}$cm entering the expression for $\delta G(\Box)$ in Eqs. (1.2) and (1.1). This would dramatically reduce the magnitude of the quantum correction by as much as a factor of the order of $(1/20)^{\gamma_\nu} = (1/20)^{4.5} \approx 1.398 \times 10^{-6}$.

A second possibility we will pursue here briefly is to consider a shortcoming, mentioned previously, in the use of $a(t) \sim a_0 (t/t_0)^{2/3}$ in relating $G(a)$ in Eq. (2.3) to $G(t)$ in Eq. (1.5). In general, if w is not small, one should use the more general equation relating the variable t to $a(t)$. The problem here is that, loosely speaking, for $w \neq 0$ at least two w's are involved, $w = 0$ (matter) and $w = -1$ (λ term). Unfortunately, this issue complicates considerably the problem of relating $\delta G(t)$ to $\delta G(a)$, and therefore the solution to the resulting differential equation for $\delta(a)$. As a tractable approximation though, one should set instead $a(t) \sim a_0 (t/t_0)^{2/3(1+w)}$, and then use an "effective" value of $w \approx -7/9$, which would seem more appropriate for the final target value of $\Omega \approx 0.25$. For this choice one then obtains a significantly reduced power in Eq. (2.3), namely $\gamma_\nu = 3(1 + w)/2\nu = 1$. Furthermore, the resulting differential equation for $\delta(a)$ is still relatively easy to solve, by the same methods used in the previous section. One now finds

$$\gamma = 0.5562 - (0.92 + 7.70\, c_h)\, c_a + O(c_a^2) \; . \tag{3.33}$$

which should be compared to the previous result of Eq. (3.31). In particular for the tensor box case one still has $c_h = 7.927$. Thus by reducing the value of γ_ν by about a factor of four, the c_a coefficient in the above expression has been reduced by about a factor of three, a significant change.

After using this improved value for the power γ_ν, the problem of correcting for relative scales needs to be addressed again, in light of the corrected estimate for the growth exponent parameter of Eq. (3.33). Given this new choice for $\gamma_\nu = 1$, on can now consider, for example, the types of galaxy clusters studied recently in [18, 19, 20], which typically involve comoving radii of $\sim 8.5 Mpc$ and viral radii of $\sim 1.4 Mpc$. For these one would obtain an approximate overall scale reduction factor of $(1.4/4890)^1 \approx 2.9 \times 10^{-4}$. Note that in these units ($Mpcs$) the reference scale appearing in $G(\Box)$ is of the order of $\xi \simeq 4890 Mpc$. This would give for the tensor box ($c_h = 7.927$) correction to the growth index γ in Eq. (3.33) the more reasonable order of magnitude estimate $-62. \times 20.6 \times 2.9 \times 10^{-4} \approx -0.37$, and for γ itself the reduced value would end up at ≈ 0.19. Clearly at this point these should only be considered as rough order of magnitude estimates.

Nevertheless this last case is suggestive of a trend, quite independently of the specific value of c_h and therefore of the overall numerical coefficient of the correction in Eq. (3.33): namely that the correction to the growth index parameter will increase close to linearly (for γ_ν close to one, as we have argued) in the size of the cluster. Consequently one expects that the deviations will increase tenfold in going from a cluster size of $1 Mpc$ to one of $10 Mpc$, and a hundredfold in going from $1 Mpc$ to $100 Mpc$.

Finally we note the effects discussed in this paper are only relevant for very large scales, much bigger than those usually considered, and well constrained, by laboratory, solar or galactic dynamics tests [1, 21, 22, 23]. Furthermore the effects we have described here are quite different from what one would expect in $f(R)$ theories [24, 25], which also tend to predict some level of deviation from classical GR in the growth exponents [26, 27, 28]. Future more accurate astrophysical observations might make it possible to see the difference in the predictions of various models [29, 30, 31, 32, 33, 34, 35]. In conclusion let us summarize that we have attempted here to systematically analyze the effects on matter density perturbations of a running $G(\Box)$ appearing in the original effective, non-local covariant field equations of Eq. (1.2). The specific form of $G(\Box)$ in Eq. (1.1) is inspired by the non-perturbative treatment of covariant path integral quantum gravity, and follows from the existence of a non-trivial fixed point in G of the renormalization group in four dimensions. The resulting effective field equations are manifestly covariant, and in principle besides the non-perturbative scale ξ there are no adjustable parameters, since the coefficients (c_0) and scaling dimensions (ν) entering $G(\Box)$ are, again in principle, calculable by systematic field theory and lattice methods (see e.g. [5], and references therein). The main body of this paper has then be devoted to determining what effects the relevant equations can have on structure growth and the growth indices.

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