

Leading order finite size effects with spins for inspiralling compact binaries

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ABSTRACT: The leading order finite size effects due to spin, namely that of the cubic and quartic spin interaction, are derived for the first time for generic compact binaries via an Effective Field Theory approach. These corrections enter at the third and a half and fourth post-Newtonian orders, respectively, for rapidly rotating compact objects. Hence, we complete the leading order finite size effects with spin up to the fourth post-Newtonian accuracy. We arrive at this by augmenting the effective action with new higher dimensional nonminimal coupling worldline operators, involving higher-order derivatives of the field, and introducing new Wilson coefficients, corresponding to constants, which describe the octupole and hexadecapole deformations of the object due to spin. These Wilson coefficients are matched to unity in the black hole case. The nonminimal coupling worldline operators enter the action with the electric and magnetic, even and odd parity type, components of the Weyl tensor coupled to the even and odd worldline spins, respectively. Moreover, the non relativistic gravitational field decomposition, which we employ, demonstrates a coupling hierarchy of the gravito-magnetic vector and the Newtonian scalar, to the odd and even in spin operators, respectively, which extends that of the minimal coupling case. This observation is useful for the construction of the Feynman diagrams, and provides an instructive analogy between the leading order spin-orbit and cubic spin interactions, and between the leading order quadratic spin and quartic spin interactions.

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1 Introduction

Second-generation ground-based interferometers, such as Advanced LIGO [1], Advanced Virgo [2], and KAGRA [3], will start to operate in the next few years, making the anticipated direct detection of gravitational waves (GWs) a realistic prospect. Among the most promising sources in the accessible frequency band of such experiments are inspiralling binaries of compact objects, which can be treated analytically in terms of the post-Newtonian (PN) approximation of General Relativity [4]. It turns out that even relative high order corrections beyond Newtonian gravity, such as the fourth PN (4PN) order, are crucial to obtain a successful detection from such sources, and furthermore to gain information about the inner structure of the constituents of the binary [5]. Moreover, such objects are expected to have large spins [6], thus PN corrections involving spins should be completed to similar high orders as in the non spinning case, which was recently completed to 4PN order [7].

In particular, finite size effects involving spins should also be taken into account in order to obtain the required 4PN accuracy. The leading order (LO) finite size spin effects at the quadrupole level, i.e. of the LO spin-squared interaction, were first derived for black holes in [8, 9]. Generic quadrupoles were included already in [10], and the proportionality of the quadrupole to spin-squared was introduced in [11]. The LO spin-squared interaction enters already at the 2PN order for rapidly rotating generic compact objects. Yet, the next-to-leading order (NLO) spin-squared interaction at 3PN order was treated much later in the following series of works [12–15]. Finally, the LO cubic and quartic in spin interaction Hamiltonians for black hole binaries were computed in parts in [14, 16]. These corrections

enter formally at the 2PN order, and for rapidly rotating compact objects at the 3.5PN and 4PN orders, respectively. However, these results were found to be incomplete in the test particle limit for the quartic in spin sector [17].

In this paper we derive for the first time the LO cubic and quartic spin interaction potentials for generic compact binaries via an Effective Field Theory (EFT) approach. Hence, we complete the LO finite size effects with spin up to the 4PN accuracy. The novel EFT approach for the binary inspiral problem was introduced in [18, 19] with its extension to spinning objects in [20]. This approach provides a systematic methodology to construct the action to arbitrarily high accuracy, in terms of operators ordered by their relevance and their Wilson coefficients, which is invaluable for the obtainment of finite size effects. Moreover, the EFT approach also applies the efficient standard tools of Quantum Field Theory, such as Feynman diagrams. Indeed, we arrive at our results by augmenting the effective action with new higher dimensional nonminimal coupling worldline operators, involving higher-order derivatives of the field, and introducing new Wilson coefficients, corresponding to constants, which describe the octupole and hexadecapole deformations of the object due to spin. These Wilson coefficients are matched to unity in the black hole case. The nonminimal coupling worldline operators enter the action with the electric and magnetic, even and odd parity type, components of the Weyl tensor, coupled to the even and odd worldline spins, respectively.

Moreover, another advantageous practice in the EFT approach is the use of the non relativistic gravitational (NRG) fields, which were introduced in [21]. The NRG decomposition of spacetime is essentially a reduction over the time dimension, and therefore it is the sensible decomposition for the PN limit [22]. Indeed, the NRG field decomposition, which we employ, demonstrates a coupling hierarchy of the gravito-magnetic vector and the Newtonian scalar to the odd and even in spin operators, respectively, which extends that of the minimal coupling case for spin interactions, as was illustrated in [23–25]. This observation is very useful for the construction of the Feynman diagrams, and provides an instructive analogy between the LO spin-orbit and cubic spin interactions, and between the LO quadratic spin and quartic spin interactions.

The outline of the paper is as follows. In section 2 we review and present the new ingredients in the EFT formulation for finite size effects with spins to the order required for this work. In section 3 we derive the LO cubic spin interaction potential for generic compact objects through the relevant Feynman diagrams and their evaluation, and we compare to the ADM Hamiltonian result for a black hole binary. In section 4 we similarly derive the LO quartic spin interaction potential for generic compact objects, and correct the corresponding ADM Hamiltonian result for the black hole binary case. Finally, in section 5 we summarize our main conclusions.

Throughout this paper we use $c \equiv 1$, and $\eta_{\mu\nu} \equiv \text{Diag}[1, -1, -1, -1]$. Greek letters denote indices in the global coordinate frame. The spin variables are always considered in the local Lorentz frame. Spatial tensor indices are denoted with lowercase Latin letters from the middle of the alphabet. Lowercase Latin letters from the middle of the alphabet denote particle labels. The scalar triple product appears here with no brackets, i.e. $\vec{a} \times \vec{b} \cdot \vec{c} \equiv (\vec{a} \times \vec{b}) \cdot \vec{c}$.

2 Finite size effects with spins via Effective Field Theory

In this section we will present and augment the effective action, that removes the scale of the compact objects via the EFT approach, which is required in order to take into account finite size effects with spins. We build on the works in [12, 18–20, 24, 26] for the construction of the effective action, and employ the NRG fields [21, 22] for the Feynman rules. The NRG fields also continue to play a central role in the construction of the Feynman diagrams in spin interactions, as will be seen in the next section. This is an extension beyond the minimal coupling part of the action, which was already illustrated in [23–25].

We recall that the action describing the binary system is given by

$$S = S_g + \sum_{a=1}^2 S_{a(p)}, \quad (2.1)$$

where S_g is the pure gravitational action, and $S_{a(p)}$ is the worldline particle action for each of the two particles in the binary. The gravitational action is the usual Einstein-Hilbert action plus a gauge-fixing term, that we choose as the fully harmonic gauge, such that we have

$$S_g = S_{\text{EH}} + S_{\text{GF}} = -\frac{1}{16\pi G} \int d^4x \sqrt{g} R + \frac{1}{32\pi G} \int d^4x \sqrt{g} g_{\mu\nu} \Gamma^\mu \Gamma^\nu, \quad (2.2)$$

where $\Gamma^\mu \equiv \Gamma_{\rho\sigma}^\mu g^{\rho\sigma}$.

In terms of the NRG fields the metric reads

$$g_{\mu\nu} = \begin{pmatrix} e^{2\phi} & -e^{2\phi} A_j \\ -e^{2\phi} A_i & -e^{-2\phi} \gamma_{ij} + e^{2\phi} A_i A_j \end{pmatrix} \simeq \begin{pmatrix} 1 + 2\phi & -A_j \\ -A_i & -\delta_{ij} + 2\phi \delta_{ij} \end{pmatrix}, \quad (2.3)$$

where we have written the approximation for the metric in the weak-field limit up to linear order in the fields as required for this work.

The NRG scalar and vector field propagators in the harmonic gauge are given by

$$\text{—————} = \langle \phi(x_1) \phi(x_2) \rangle = 4\pi G \int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^2} \delta(t_1 - t_2), \quad (2.4)$$

$$\text{-----} = \langle A_i(x_1) A_j(x_2) \rangle = -16\pi G \delta_{ij} \int_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}_1-\mathbf{x}_2)}}{\mathbf{k}^2} \delta(t_1 - t_2). \quad (2.5)$$

Next, we recall that the minimal coupling part of the effective action of each of the particles with spins [20, 24, 27] is given by

$$L_{(p)} = m\sqrt{u^2} + \frac{1}{2} S_{\mu\nu} \Omega^{\mu\nu}. \quad (2.6)$$

Considering this particle action in eq. (2.1), the LO monopole-monopole interaction, that is the Newtonian interaction, which involves no spin, and the LO dipole-monopole interaction, that is the linear in spin LO spin-orbit interaction, are derived. These are obtained through the following Feynman rules, which are also those required to the order that we are considering in this work. For the one-graviton couplings to the worldline mass, we have

$$\begin{array}{c} | \\ \bullet \text{---} \end{array} = -m \int dt \phi [1 + \dots], \quad (2.7)$$

$$\bullet \text{---} = m \int dt A_i v^i [1 + \dots], \quad (2.8)$$

where the heavy solid lines represent the worldlines, and the spherical black blobs represent the masses on the worldline. The ellipsis denotes higher orders in v , beyond the order considered here.

The Feynman rules for the one-graviton couplings to the worldline spin are:

$$\text{---} \text{---} = \int dt \frac{1}{2} (S^{ij} \partial_i A_j), \quad (2.9)$$

$$\text{---} \text{---} = \int dt (S^{ij} \partial_j \phi v^i + S^{0i} \partial_i \phi), \quad (2.10)$$

where the gray oval blobs represent the spins on the worldlines.

In order to take into account finite size effects with spins the effective action should be extended beyond minimal coupling. Higher dimensional operators should be introduced into the action, constructed with the Riemann tensor, which is equivalent to the Weyl tensor in vacuum, using its even and odd parity components. For LO effects only linear in Riemann terms should be considered. These operators will be constructed according to the symmetries that the effective action should satisfy. For the LO spin-squared finite size effects the finite size operator, which is added to the action [12], is given by

$$L_{\text{ES}^2} = \frac{C_{\text{ES}^2}}{2m} R_{\mu\alpha\nu\beta} \frac{u^\alpha u^\beta}{\sqrt{u^2}} S^\mu{}_\gamma S^{\gamma\nu} = \frac{C_{\text{ES}^2}}{2m} \frac{E_{\mu\nu}}{\sqrt{u^2}} S^\mu{}_\gamma S^{\gamma\nu}, \quad (2.11)$$

where $E_{\mu\nu}$ is the electric component of the Weyl tensor, that is

$$E_{\mu\nu} = R_{\mu\alpha\nu\beta} u^\alpha u^\beta. \quad (2.12)$$

C_{ES^2} is the Wilson coefficient corresponding to the quadrupole deformation of the object due to spin, which was introduced in [11], where the proportionality factor is called a instead of C_{ES^2} .

Considering this addition to the particle action in eq. (2.6) and eq. (2.1), the LO quadrupole-monopole interaction, that is the LO spin-squared, is derived, using the following Feynman rules, required to the order that we are considering in this paper. The Feynman rules for the one-graviton couplings to the worldline spin-squared are given by

$$\blacksquare \text{---} = \int dt \left[-\frac{C_{\text{ES}^2}}{2m} S^{ik} S^{jk} \partial_i \partial_j \phi \right], \quad (2.13)$$

$$\blacksquare \text{---} = \int dt \left[\frac{C_{\text{ES}^2}}{2m} S^{ik} S^{jk} \partial_i \partial_j A_l v^l \right], \quad (2.14)$$

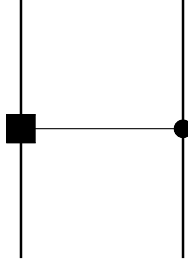


Figure 1. LO spin-squared interaction Feynman diagram. This diagram should be included together with its mirror image. This is a quadrupole-monopole interaction. Note the analogy with the Newtonian interaction.

where the black square boxes represent the ES^2 spin operators on the worldlines. Using the leading coupling of the spin-squared to the Newtonian scalar in eq. (2.13), contracted with the corresponding leading mass coupling, we obtain the single Feynman diagram, which makes up the LO spin-squared interaction, shown in figure 1.

The value of the Feynman diagram for the LO spin-squared interaction is given by

$$Fig. 1 = \frac{1}{2} C_{1(ES^2)} \frac{Gm_2}{m_1 r^3} \left[S_1^2 - 3 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right], \quad (2.15)$$

and the LO spin-squared potential is just $V_{S_1^2}^{LO} = -Fig. 1$. Notice that this is a purely Newtonian effect, and that the worldline spin-squared acts just like a generic mass quadrupole.

In this paper we want to complete the LO octupole and hexadecapole level in the spins. For that, the action should be extended to LO cubic and quartic order in the spin. We do this by adding covariant derivatives on the field. Operators including covariant derivatives of the worldline velocity can be eliminated through a shift of the worldline coordinate using the LO EOM, and get absorbed into the Riemann dependent finite size operators, see also e.g. [28], namely into their Wilson coefficients.

The cubic in spin operator, that should be added here, is

$$L_{BS^3} = \frac{C_{BS^3}}{6m^2} \frac{B_{\mu\nu;\lambda}}{u^2} S^{\mu\kappa} S_\kappa^\nu S^\lambda = -\frac{C_{BS^3}}{12m^2} R_{\mu\nu\alpha\beta;\gamma} u^\alpha S^{\mu\nu} S^\beta_\rho S^{\rho\gamma}, \quad (2.16)$$

where $\epsilon_{\alpha\beta\gamma\mu}$ is the Levi-Civita tensor, $B_{\mu\nu}$ is the magnetic component of the Weyl tensor, that is

$$B_{\mu\nu} \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\mu} R^{\alpha\beta}_{\delta\nu} u^\gamma u^\delta, \quad (2.17)$$

and the indices μ and ν should be symmetrized. The index of the covariant derivative is also symmetrized with respect to them, so that the Bianchi identity is automatically taken into account, and the symmetry factor of the operator is fixed accordingly. The spin pseudovector S^λ is defined by

$$S^\lambda \equiv \frac{1}{2} \epsilon_{\alpha\beta\gamma\lambda} S^{\alpha\beta} u^\gamma. \quad (2.18)$$

We have introduced here C_{BS^3} , which is the Wilson coefficient, or constant describing the octupole deformation due to spin.

The Feynman rules for the one-graviton couplings to the worldline cubic spin are then given by

$$\begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \text{---} = \int dt \left[-\frac{C_{BS^3}}{12m^2} S^i S^j S^{kl} \partial_i \partial_j \partial_k A_l \right], \quad (2.19)$$

$$\begin{array}{c} | \\ | \\ \text{---} \end{array} \text{---} = \int dt \left[\frac{C_{BS^3}}{3m^2} S^i S^j S^{kl} \partial_i \partial_j \partial_k \phi v^l \right], \quad (2.20)$$

where the gray rectangular boxes represent the BS^3 cubic spin operators on the worldlines. Note that here it is the gravito-magnetic vector, which is the leading one in the hierarchy of coupling of the magnetic component of the Weyl tensor in the worldline cubic spin operator.

The quartic in spin operator, that should be added here, is

$$L_{ES^4} = \frac{C_{ES^4}}{24m^3} \frac{E_{\mu\nu;\lambda\kappa}}{\sqrt{u^2}} S^{\mu\gamma} S_\gamma^\nu S^{\lambda\delta} S_\delta^\kappa, \quad (2.21)$$

where again the covariant derivatives are symmetrized with respect to the curvature tensor indices. We have introduced here C_{ES^4} , which is the Wilson coefficient, or constant describing the hexadecapole deformation due to spin.

Then, the Feynman rule for the one-graviton coupling to the worldline quartic spin is given by

$$\begin{array}{c} | \\ \text{---} \\ | \end{array} \text{---} = \int dt \left[-\frac{C_{ES^4}}{24m^3} S^{ik} S^{jk} S^{ln} S^{mn} \partial_i \partial_j \partial_l \partial_m \phi \right], \quad (2.22)$$

where the black crossed box represents the ES^4 quartic spin operators on the worldline. Note that here it is again the Newtonian scalar, which is the leading one in the hierarchy of coupling of the electric component of the Weyl tensor in the worldline quartic spin operator.

3 Leading order cubic spin interaction

The LO cubic and quartic spin interaction Hamiltonians for the black hole binary case were approached in parts in [14, 16]. These corrections enter formally at the 2PN order, and at the 3.5PN and 4PN orders, respectively, for rapidly rotating compact objects. In this section and the next we derive these interaction potentials for any generic compact binary from an EFT action approach, where we construct these interactions in a direct and instructive manner.

3.1 Feynman diagrams

The cubic spin interaction contains two kinds of interaction: a quadrupole-dipole interaction, and an octupole-monopole one. Each of these two interactions is analogous to the

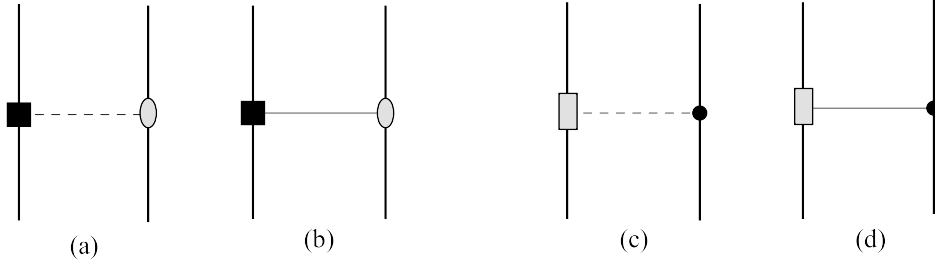


Figure 2. LO cubic spin interaction Feynman diagrams. These diagrams should be included together with their mirror images. On the left pair we have the quadrupole-dipole interaction, and on the right pair we have the octupole-monopole one. Note the analogy of each pair with the LO spin-orbit interaction in figure 1 of [24].

LO spin-orbit interaction, which is a dipole-monopole interaction. The correspondence is between even and odd parity multipole moments of the spin, such that the quadrupole and octupole moments correspond to the monopole (mass) and dipole (spin), respectively. We recall from figure 1 in section IV of [24], that the LO spin-orbit interaction contains two contributing Feynman diagrams, mediated by one-graviton exchanges of the gravito-magnetic vector and the Newtonian scalar of the NRG fields. Therefore, we expect to have here four contributing Feynman diagrams, two for each of the two kinds of interaction, that make up the cubic spin interaction.

Indeed, the four contributing Feynman diagrams can be seen here in figure 2, where on the left diagrams (a) and (b), we have the quadrupole-dipole interaction, and on the right diagrams (c) and (d), we have the octupole-monopole interaction. These diagrams are obtained by the following contractions: in figure 2(a) the LO worldline spin coupling to the gravito-magnetic vector from eq. (2.9) is contracted with the corresponding quadrupole coupling in eq. (2.14); in figure 2(b) we contract the LO worldline spin quadrupole coupling to the Newtonian scalar from eq. (2.13) with the corresponding spin coupling in eq. (2.10); in figure 2(c) the LO worldline spin octupole coupling to the gravito-magnetic vector from eq. (2.19) is contracted with the corresponding mass coupling in (2.8); finally, in figure 2(d) the LO worldline mass coupling to the Newtonian scalar from eq. (2.7) is contracted with the corresponding spin octupole coupling in eq. (2.20).

Hence, the values of the Feynman diagrams of the LO cubic spin interaction are given by the following:

$$Fig. 2(a) = -3 \frac{C_{1(ES^2)} G}{m_1 r^4} \left[S_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_1 + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_1 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_1 \right], \quad (3.1)$$

$$Fig. 2(b) = 3 \frac{C_{1(ES^2)} G}{m_1 r^4} \left[S_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 \right], \quad (3.2)$$

$$Fig. 2(c) = C_{1(BS^3)} \frac{G m_2}{m_1^2 r^4} \vec{S}_1 \cdot \vec{n} \times \vec{v}_2 \left[S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right], \quad (3.3)$$

$$\text{Fig. 2(d)} = -C_{1(\text{BS}^3)} \frac{Gm_2}{m_1^2 r^4} \vec{S}_1 \cdot \vec{n} \times \vec{v}_1 \left[S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right]. \quad (3.4)$$

The evaluation of the diagrams here is straightforward. Yet, note that the value of diagram 2(b) is SSC dependent, and following the procedure from [29], we insert here the covariant SSC, to be supplemented with an addition from the extra potential in eq. (4.7) there.

3.2 Effective potential and Hamiltonian

As we noted in the end of the previous section, we recall that we have an addition from the extra potential of [29, 30], coming from the insertion of spin gauge constraints in the rotational kinetic term, which enters first at the LO spin-orbit sector. This addition contributes here only the kinetic piece, which is noted in eq. (72) of [24] as

$$L_{\text{S}^3 \text{extra}}^{\text{LO}} = \frac{1}{2} \vec{S}_1 \cdot \vec{v}_1 \times \dot{\vec{a}}_1 + 1 \leftrightarrow 2, \quad (3.5)$$

and is acceleration dependent. We proceed then to eliminate the acceleration in this extra piece by substituting the EOM, as explained in [29, 31], which come from the LO quadratic spin sectors, and given by

$$\begin{aligned} m_1 \vec{a}_{1(\text{S}^2)} = & -\frac{3}{2} C_{1(\text{ES}^2)} \frac{Gm_2}{m_1 r^4} \left[\left(S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right) \vec{n} + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_1 \right] + 1 \leftrightarrow 2 \\ & - \frac{3G}{r^4} \left[\left(\vec{S}_1 \cdot \vec{S}_2 - 5 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right) \vec{n} + \vec{S}_1 \cdot \vec{n} \vec{S}_2 + \vec{S}_2 \cdot \vec{n} \vec{S}_1 \right]. \end{aligned} \quad (3.6)$$

We obtain then the following addition:

$$\begin{aligned} L_{\text{S}^3 \text{extra}}^{\text{LO}} = & \frac{3}{4} C_{1(\text{ES}^2)} \frac{Gm_2}{m_1^2 r^4} \left[\left(\vec{S}_1 \cdot \vec{n} \times \vec{v}_1 - \frac{m_1}{m_2} \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 \right) \left(S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right) \right. \\ & \left. - 2 \frac{m_1}{m_2} \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_2 \right] + \frac{3}{2} \frac{G}{m_1 r^4} \left[\left(\vec{S}_1 \cdot \vec{S}_2 - 5 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right) \vec{S}_1 \cdot \vec{n} \times \vec{v}_1 \right. \\ & \left. - \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_1 \right] + 1 \leftrightarrow 2. \end{aligned} \quad (3.7)$$

Summing all diagrams in figure 2, and the extra addition in eq. (3.7), we get the following effective potential for the LO cubic spin interaction:

$$\begin{aligned} V_{\text{S}^3}^{\text{LO}} = & C_{1(\text{BS}^3)} \frac{Gm_2}{m_1^2 r^4} \left(\vec{S}_1 \cdot \vec{n} \times \vec{v}_1 - \vec{S}_1 \cdot \vec{n} \times \vec{v}_2 \right) \left(S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right) \\ & + 3 \frac{C_{1(\text{ES}^2)} G}{m_1 r^4} \left[\left(S_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_1 + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_1 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_1 \right) \right. \\ & \quad \left. - \left(S_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 \right) \right] \\ & - \frac{1}{4} \left[\left(\frac{m_2}{m_1} \vec{S}_1 \cdot \vec{n} \times \vec{v}_1 - \vec{S}_2 \cdot \vec{n} \times \vec{v}_2 \right) \left(S_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right) - 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_2 \right] \\ & - \frac{3}{2} \frac{G}{m_1 r^4} \left[\left(\vec{S}_1 \cdot \vec{S}_2 - 5 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right) \vec{S}_1 \cdot \vec{n} \times \vec{v}_1 - \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{v}_1 \right] \\ & + 1 \longleftrightarrow 2. \end{aligned} \quad (3.8)$$

The new Wilson coefficient C_{BS^3} should be matched for the black hole case. The binding energy can be used for a gauge invariant matching of all the Wilson coefficients encountered in this paper. A comparison of the gauge invariant binding energy, derived from our potential, with the one for a test-particle in the Kerr geometry [17], leads to $C_{\text{BS}^3} = 1$ for black holes. At the same time, this provides a check for our result against the small mass ratio case.

We would like to compare our effective potential to the ADM Hamiltonian results for a black hole binary derived in parts in [14, 16]. Collecting the pieces from eq. (144) in [16], and eqs. (7.1), (7.2) in [14] (notice that eq. (2.13) in [14] has a typo), we obtain

$$\begin{aligned}
H_{\text{S}^3}^{\text{LO}} = & \frac{G}{m_1^2 r^4} \left[\frac{3}{2} \left(\vec{S}_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{p}_1 + \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{p}_1 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{p}_1 \right. \right. \\
& + \vec{n} \cdot \vec{S}_1 \times \vec{S}_2 \left(\vec{S}_1 \cdot \vec{p}_1 - 5 \vec{S}_1 \cdot \vec{n} \vec{p}_1 \cdot \vec{n} \right) \\
& - \left. \frac{3m_1}{2m_2} \left(\vec{S}_1^2 \vec{S}_2 \cdot \vec{n} \times \vec{p}_2 + 2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{S}_1 \times \vec{p}_2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \vec{S}_2 \cdot \vec{n} \times \vec{p}_2 \right) \right) \\
& - \left. \vec{S}_1 \times \vec{n} \cdot \left(\vec{p}_2 - \frac{m_2}{4m_1} \vec{p}_1 \right) \left(\vec{S}_1^2 - 5 \left(\vec{S}_1 \cdot \vec{n} \right)^2 \right) \right] \\
& + 1 \longleftrightarrow 2. \tag{3.9}
\end{aligned}$$

We note that the Legendre transform of the effective potentials at LO is trivial. The velocities are expressed in terms of the momenta, which at the LO level is just the Newtonian relation, i.e. $v = p/m$. Also we can set in eq. (3.8) the Wilson coefficients $C_{\text{ES}^2} = C_{\text{BS}^3} = 1$ for the black hole case.

Then we find for the difference between the LO cubic spin potentials, that is

$$\Delta V_{\text{S}^3}^{\text{BHB}} \equiv V_{\text{S}^3}^{\text{ADM}} - V_{\text{S}^3}^{\text{EFT}}, \tag{3.10}$$

which originates in our potential from diagram 2(a), and the extra addition, that it vanishes by virtue of the following vector identity:

$$\vec{N}[\vec{A}, \vec{B}, \vec{C}, \vec{D}] \equiv \vec{A} \vec{B} \cdot \vec{C} \times \vec{D} - \vec{B} \vec{C} \cdot \vec{D} \times \vec{A} + \vec{C} \vec{D} \cdot \vec{A} \times \vec{B} - \vec{D} \vec{A} \cdot \vec{B} \times \vec{C} \equiv \vec{0}, \tag{3.11}$$

where \vec{N} is a null vector. More specifically, we have

$$\Delta V_{\text{S}^3}^{\text{BHB}} = \frac{3}{2} \frac{G}{m_1 r^4} \left(\vec{S}_1 \cdot \vec{N}[\vec{v}_1, \vec{n}, \vec{S}_1, \vec{S}_2] - 5 \vec{S}_1 \cdot \vec{n} \vec{n} \cdot \vec{N}[\vec{S}_2, \vec{n}, \vec{v}_1, \vec{S}_1] \right) + 1 \leftrightarrow 2 \equiv 0. \tag{3.12}$$

Therefore, our potential agrees with [14, 16] for the case of binary black holes. One can also use this comparison to conclude that $C_{\text{BS}^3} = 1$ for black holes.

4 Leading order quartic spin interaction

4.1 Feynman diagrams

The quartic spin interaction contains three kinds of interaction: a quadrupole-quadrupole interaction, an octupole-dipole one, and a hexadecapole-monopole one. The octupole-dipole interaction is analogous to the LO spin1-spin2, which is a dipole-dipole interaction,

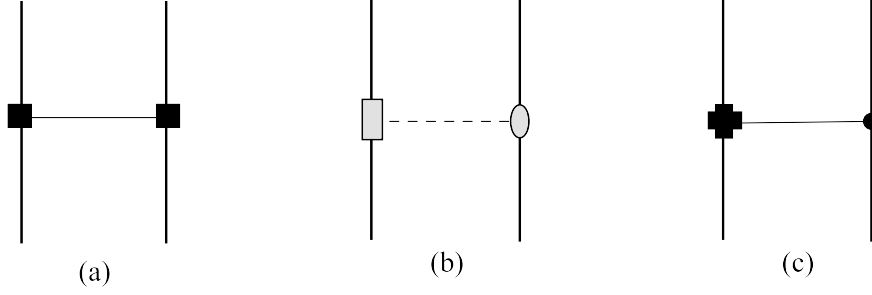


Figure 3. LO quartic spin interaction Feynman diagrams. Diagrams b and c here should be included together with their mirror images. On the left and right we have the quadrupole-quadrupole and hexadecapole-monopole interactions, each of which is analogous to the LO spin-squared interaction in figure 1 here. On the middle we have the octupole-dipole interaction analogous to the LO spin1-spin2 interaction in figure 1 of [23].

and as we noted the octupole moment corresponds to the dipole due to its odd parity. Then, the quadrupole-quadrupole and hexadecapole-monopole interactions are analogous each to the LO spin-squared interaction, which is a quadrupole-monopole interaction as we noted in section 2, since the quadrupole and hexadecapole moments correspond to the monopole due to their even parity. We recall from figure 1 in [23], that the LO spin1-spin2 interaction contains a single Feynman diagram, mediated by a one-graviton exchange of the gravito-magnetic vector. Moreover, we saw in figure 1 in section 2 here, that the LO spin-squared interaction also contains a single Feynman diagram mediated by a one-graviton exchange of the Newtonian scalar. Therefore, all in all we expect to have here three contributing Feynman diagrams, one for each of the three kinds of interaction, that make up the quartic spin interaction.

Indeed, the 3 contributing Feynman diagrams are shown here in figure 3, where on the left and right diagrams, (a) and (c), we have the quadrupole-quadrupole and hexadecapole-monopole interactions, and on the middle diagram, (b), we have the octupole-dipole interaction. These diagrams are obtained by the following contractions: in figure 3(a) the LO worldline spin quadrupole coupling to the Newtonian scalar from eq. (2.13) is contracted with itself; in figure 3(b) we contract the LO worldline spin octupole coupling to the gravito-magnetic vector from eq. (2.19) with the LO spin coupling in (2.9); finally, in figure 3(c) the LO worldline spin hexadecapole coupling to the Newtonian scalar from eq. (2.22) is contracted with the LO mass coupling in eq. (2.7).

Hence, the values of the Feynman diagrams of the LO quartic spin interaction are given by the following:

$$\begin{aligned}
 \text{Fig. 3(a)} = \frac{3}{4} C_{1(\text{ES}^2)} C_{2(\text{ES}^2)} \frac{G}{m_1 m_2 r^5} & \left[S_1^2 S_2^2 + 2 (\vec{S}_1 \cdot \vec{S}_2)^2 - 5 \left(S_1^2 (\vec{S}_2 \cdot \vec{n})^2 \right. \right. \\
 & \left. \left. + S_2^2 (\vec{S}_1 \cdot \vec{n})^2 + 4 \vec{S}_1 \cdot \vec{S}_2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} - 7 (\vec{S}_1 \cdot \vec{n})^2 (\vec{S}_2 \cdot \vec{n})^2 \right) \right],
 \end{aligned}
 \tag{4.1}$$

$$\text{Fig. 3(b)} = \frac{3}{2}C_{1(\text{BS}^3)}\frac{G}{m_1^2r^5}\left[S_1^2\vec{S}_1\cdot\vec{S}_2 - 5S_1^2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n} - 5\vec{S}_1\cdot\vec{S}_2\left(\vec{S}_1\cdot\vec{n}\right)^2 + \frac{35}{3}\vec{S}_2\cdot\vec{n}\left(\vec{S}_1\cdot\vec{n}\right)^3\right], \quad (4.2)$$

$$\text{Fig. 3(c)} = \frac{3}{8}C_{1(\text{ES}^4)}\frac{Gm_2}{m_1^3r^5}\left[S_1^4 - 10S_1^2\left(\vec{S}_1\cdot\vec{n}\right)^2 + \frac{35}{3}\left(\vec{S}_1\cdot\vec{n}\right)^4\right]. \quad (4.3)$$

Here too the evaluation of the diagrams is straightforward, and we should have no additions to the effective potential.

4.2 Effective potential and Hamiltonian

Summing all diagrams in figure 3, we get the following effective potential for the LO quartic spin interaction:

$$\begin{aligned} V_{\text{S}^4}^{\text{LO}} = & -\frac{3}{4}C_{1(\text{ES}^2)}C_{2(\text{ES}^2)}\frac{G}{m_1m_2r^5}\left[S_1^2S_2^2 + 2\left(\vec{S}_1\cdot\vec{S}_2\right)^2 - 5\left(S_1^2\left(\vec{S}_2\cdot\vec{n}\right)^2 + S_2^2\left(\vec{S}_1\cdot\vec{n}\right)^2 + 4\vec{S}_1\cdot\vec{S}_2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n} - 7\left(\vec{S}_1\cdot\vec{n}\right)^2\left(\vec{S}_2\cdot\vec{n}\right)^2\right] \\ & -\frac{3}{2}C_{1(\text{BS}^3)}\frac{G}{m_1^2r^5}\left[S_1^2\vec{S}_1\cdot\vec{S}_2 - 5S_1^2\vec{S}_1\cdot\vec{n}\vec{S}_2\cdot\vec{n} - 5\vec{S}_1\cdot\vec{S}_2\left(\vec{S}_1\cdot\vec{n}\right)^2 + \frac{35}{3}\vec{S}_2\cdot\vec{n}\left(\vec{S}_1\cdot\vec{n}\right)^3\right] + 1 \longleftrightarrow 2 \\ & -\frac{3}{8}C_{1(\text{ES}^4)}\frac{Gm_2}{m_1^3r^5}\left[S_1^4 - 10S_1^2\left(\vec{S}_1\cdot\vec{n}\right)^2 + \frac{35}{3}\left(\vec{S}_1\cdot\vec{n}\right)^4\right] + 1 \longleftrightarrow 2. \end{aligned} \quad (4.4)$$

Here too, for black holes the new Wilson coefficient C_{ES^4} can be matched against the gauge invariant binding energy in [17], which also checks our result in the small mass ratio case. From this, we find that $C_{\text{ES}^4} = 1$ for black holes.

We proceed to compare our effective potential with the ADM Hamiltonian results for a black hole binary in [14, 16]. However, it was found in [17], that the black hole binary Hamiltonian at quartic order in each of the spins, which was derived in [14], must be incomplete. Indeed, at leading order the source part of the Hamilton constraint $\mathcal{H}^{\text{matter}}$ is the source of the Newtonian potential, which corresponds to the NRG scalar field ϕ . From eq. (2.22) we therefore expect a contribution to the Hamilton constraint of the form:

$$\mathcal{H}_{\text{hexadecapole}}^{\text{matter}} = \frac{C_{\text{ES}^4}}{24m^3}S^{ik}S^{jk}S^{ln}S^{mn}\partial_i\partial_j\partial_l\partial_m\delta. \quad (4.5)$$

Indeed, this term was not considered in [14]. The resulting contribution to the Hamiltonian is identical to the value of figure 3(c) up to an overall sign. Hence, we see that the conclusions of section VI and in particular eq. (6.5) in [14] are incorrect, as was already noted in [17].

Collecting the pieces from eqs. (124), (131) in [16], and the correction of eq. (6.5) in [14], that we just noted, coming from the hexadecapole-monopole interaction in fig. 3(c) here (for $C_{\text{ES}^4} = 1$), we obtain the correct binary black hole Hamiltonian:

$$H_{\text{S}^4}^{\text{LO}} = -\frac{3}{2}\frac{G}{m_1m_2r^5}\left[\frac{1}{2}\vec{S}_1^2\vec{S}_2^2 + \left(\vec{S}_1\cdot\vec{S}_2\right)^2 - \frac{5}{2}\left(\vec{S}_1^2\left(\vec{S}_2\cdot\vec{n}\right)^2 + \vec{S}_2^2\left(\vec{S}_1\cdot\vec{n}\right)^2\right)\right]$$

$$\begin{aligned}
& -10\vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \left(\vec{S}_1 \cdot \vec{S}_2 - \frac{7}{4} \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right) \Big] \\
& - \frac{3}{2} \frac{G}{m_1^2 r^5} \left[\vec{S}_1^2 \vec{S}_1 \cdot \vec{S}_2 - 5\vec{S}_1 \cdot \vec{S}_2 (\vec{S}_1 \cdot \vec{n})^2 - 5\vec{S}_1^2 \vec{S}_1 \cdot \vec{n} \vec{S}_2 \cdot \vec{n} \right. \\
& \quad \left. + \frac{35}{3} \vec{S}_2 \cdot \vec{n} (\vec{S}_1 \cdot \vec{n})^3 \right] + 1 \longleftrightarrow 2 \\
& - \frac{3Gm_2}{8m_1^3 r^5} \left[S_1^4 - 10S_1^2 (\vec{S}_1 \cdot \vec{n})^2 + \frac{35}{3} (\vec{S}_1 \cdot \vec{n})^4 \right] + 1 \longleftrightarrow 2. \tag{4.6}
\end{aligned}$$

With this correction included we find full agreement of our result with the black hole binary Hamiltonian in [14, 16]. Again, this comparison can also be used to match $C_{\text{BS}^3} = 1$ for the black hole case.

5 Conclusions

In this paper we derived for the first time the LO cubic and quartic spin interaction potentials for generic compact binaries via an Effective Field Theory approach. These corrections, which enter at the 3.5PN and 4PN orders, respectively, for rapidly rotating compact objects, complete the LO finite size effects with spin up to the 4PN accuracy.

We arrive at these results by augmenting the effective action with new higher dimensional nonminimal coupling worldline operators, involving higher-order derivatives of the field, corresponding to the higher-order multipole moments with spins, and introducing new Wilson coefficients, corresponding to constants, which describe the octupole and hexadecapole deformations of the object due to spin. These Wilson coefficients are matched to unity in the black hole case via comparisons with the gauge invariant binding energy in the test particle limit and with the ADM Hamiltonian. We also see that the ADM Hamiltonian result for the quartic spin interaction potential for a black hole binary, which was derived in [14], is incorrect, and we complete this result.

The nonminimal coupling worldline operators enter the action with the electric and magnetic, or even and odd parity type, components of the Weyl tensor, coupled to the even and odd worldline spins, respectively. Moreover, the NRG field decomposition, which we employ, demonstrates a coupling hierarchy of the gravito-magnetic vector and the Newtonian scalar to the odd and even in spin operators, respectively, which extends that of the minimal coupling case. Therefore the NRG fields are found to be very useful for the treatment of interactions involving spins, since they also facilitate the construction of the Feynman diagrams, and provide instructive analogies between the LO spin-orbit and cubic spin interactions, and between the LO quadratic spin and quartic spin interactions. These analogies are based on the correspondence between the even and odd parity multipole moments with spin.

Finally, we note that we see, that beyond the LO finite size effect with spin, that is the LO spin-squared interaction, which is a purely Newtonian effect, all LO finite size effects with spin are relativistic ones.

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References

- [1] “LIGO webpage.” <http://www.ligo.caltech.edu>.
- [2] “Virgo webpage.” <https://www.cascina.virgo.infn.it>.
- [3] “KAGRA webpage.” <http://gwcenter.icrr.u-tokyo.ac.jp/en>.
- [4] L. Blanchet, *Gravitational Radiation from Post-Newtonian Sources and Inspiralling Compact Binaries*, *Living Rev.Rel.* **17** (2014) 2, [[arXiv:1310.1528](#)].
- [5] K. Yagi and N. Yunes, *Love can be Tough to Measure*, *Phys.Rev.* **D89** (2014) 021303, [[arXiv:1310.8358](#)].
- [6] J. E. McClintock, R. Narayan, S. W. Davis, L. Gou, A. Kulkarni, et al., *Measuring the Spins of Accreting Black Holes*, *Class.Quant.Grav.* **28** (2011) 114009, [[arXiv:1101.0811](#)].
- [7] T. Damour, P. Jaranowski, and G. Schäfer, *Nonlocal-in-time action for the fourth post-Newtonian conservative dynamics of two-body systems*, *Phys.Rev.* **D89** (2014) 064058, [[arXiv:1401.4548](#)].
- [8] P. D. D’Eath, *Interaction of two black holes in the slow-motion limit*, *Phys.Rev.* **D12** (1975) 2183–2199.
- [9] K. S. Thorne and J. B. Hartle, *Laws of motion and precession for black holes and other bodies*, *Phys.Rev.* **D31** (1985) 1815–1837.
- [10] B. Barker and R. O’Connell, *Gravitational Two-Body Problem with Arbitrary Masses, Spins, and Quadrupole Moments*, *Phys.Rev.* **D12** (1975) 329–335.
- [11] E. Poisson, *Gravitational waves from inspiraling compact binaries: The Quadrupole moment term*, *Phys.Rev.* **D57** (1998) 5287–5290, [[gr-qc/9709032](#)].
- [12] R. A. Porto and I. Z. Rothstein, *Next to Leading Order Spin(1)Spin(1) Effects in the Motion of Inspiralling Compact Binaries*, *Phys.Rev.* **D78** (2008) 044013, [*Erratum-ibid.* **D81** (2010) 029905], [[arXiv:0804.0260](#)].
- [13] J. Steinhoff, S. Hergt, and G. Schäfer, *Spin-squared Hamiltonian of next-to-leading order gravitational interaction*, *Phys.Rev.* **D78** (2008) 101503, [[arXiv:0809.2200](#)].
- [14] S. Hergt and G. Schäfer, *Higher-order-in-spin interaction Hamiltonians for binary black holes from Poincare invariance*, *Phys.Rev.* **D78** (2008) 124004, [[arXiv:0809.2208](#)].
- [15] S. Hergt, J. Steinhoff, and G. Schäfer, *Reduced Hamiltonian for next-to-leading order Spin-Squared Dynamics of General Compact Binaries*, *Class.Quant.Grav.* **27** (2010) 135007, [[arXiv:1002.2093](#)].
- [16] S. Hergt and G. Schaefer, *Higher-order-in-spin interaction Hamiltonians for binary black holes from source terms of Kerr geometry in approximate ADM coordinates*, *Phys.Rev.* **D77** (2008) 104001, [[arXiv:0712.1515](#)].
- [17] J. Steinhoff and D. Puetzfeld, *Influence of internal structure on the motion of test bodies in extreme mass ratio situations*, *Phys.Rev.* **D86** (2012) 044033, [[arXiv:1205.3926](#)].

- [18] W. D. Goldberger and I. Z. Rothstein, *An Effective field theory of gravity for extended objects*, *Phys.Rev.* **D73** (2006) 104029, [[hep-th/0409156](#)].
- [19] W. D. Goldberger, *Les Houches lectures on effective field theories and gravitational radiation*, [hep-ph/0701129](#).
- [20] R. A. Porto, *Post-Newtonian corrections to the motion of spinning bodies in NRGR*, *Phys.Rev.* **D73** (2006) 104031, [[gr-qc/0511061](#)].
- [21] B. Kol and M. Smolkin, *Non-Relativistic Gravitation: From Newton to Einstein and Back*, *Class.Quant.Grav.* **25** (2008) 145011, [[arXiv:0712.4116](#)].
- [22] B. Kol, M. Levi, and M. Smolkin, *Comparing space+time decompositions in the post-Newtonian limit*, *Class.Quant.Grav.* **28** (2011) 145021, [[arXiv:1011.6024](#)].
- [23] M. Levi, *Next to Leading Order gravitational Spin1-Spin2 coupling with Kaluza-Klein reduction*, *Phys.Rev.* **D82** (2010) 064029, [[arXiv:0802.1508](#)].
- [24] M. Levi, *Next to Leading Order gravitational Spin-Orbit coupling in an Effective Field Theory approach*, *Phys.Rev.* **D82** (2010) 104004, [[arXiv:1006.4139](#)].
- [25] M. Levi, *Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order*, *Phys.Rev.* **D85** (2012) 064043, [[arXiv:1107.4322](#)].
- [26] D. Bini, T. Damour, and G. Faye, *Effective action approach to higher-order relativistic tidal interactions in binary systems and their effective one body description*, *Phys.Rev.* **D85** (2012) 124034, [[arXiv:1202.3565](#)].
- [27] A. J. Hanson and T. Regge, *The Relativistic Spherical Top*, *Annals Phys.* **87** (1974) 498.
- [28] C. R. Galley, A. K. Leibovich, and I. Z. Rothstein, *Reply to 'Comment on 'Finite size corrections to the radiation reaction force in classical electrodynamics''*, *Phys.Rev.Lett.* **109** (2012) 029502, [[arXiv:1206.4773](#)].
- [29] M. Levi and J. Steinhoff, *Equivalence of ADM Hamiltonian and Effective Field Theory approaches at fourth post-Newtonian order for binary inspirals with spins*, [arXiv:1408.5762](#).
- [30] S. Hergt, J. Steinhoff, and G. Schäfer, *Elimination of the spin supplementary condition in the effective field theory approach to the post-Newtonian approximation*, *Annals Phys.* **327** (2012) 1494–1537, [[arXiv:1110.2094](#)].
- [31] T. Damour and G. Schäfer, *Lagrangians for n point masses at the second post-Newtonian approximation of general relativity*, *Gen.Rel.Grav.* **17** (1985) 879–905.