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2015 Class. Quantum Grav. 32 045009

(http://iopscience.iop.org/0264-9381/32/4/045009)

IOP Publishing

Classical and Quantum Gravity

Class. Quantum Grav. **32** (2015) 045009 (21pp)

doi:10.1088/0264-9381/32/4/045009

Tidal deformations of spinning black holes in Bowen–York initial data

Miriam Cabero[1,2] **and Badri Krishnan**[1]

[1] Max Planck Institute for Gravitational Physics, Albert Einstein Institute, Callinstrasse 38, D-30167 Hannover, Germany

[2] Leibniz Universität Hannover, Appelstrasse 2, D-30167 Hannover, Germany

E-mail: badri.krishnan@aei.mpg.de

Received 7 August 2014, revised 11 November 2014
Accepted for publication 24 November 2014
Published 21 January 2015

Abstract

We study the tidal deformations of the shape of a spinning black hole horizon due to a binary companion in the Bowen–York initial data set. We use the framework of quasi-local horizons and identify a black hole by marginally outer trapped surfaces. The intrinsic horizon geometry is specified by a set of mass and angular-momentum multipole moments $\mathcal{M}_n$ and $\mathcal{J}_n$, respectively. The tidal deformations are described by the change in these multipole moments caused by an external perturbation. This leads us to define two sets of dimensionless numbers, the tidal coefficients for $\mathcal{M}_n$ and $\mathcal{J}_n$, which specify the deformations of a black hole with a binary companion. We compute these tidal coefficients in a specific model problem, namely the Bowen–York initial data set for binary black holes. We restrict ourselves to axisymmetric situations and to small spins. Within this approximation, we analytically compute the conformal factor, the location of the marginally trapped surfaces, and finally the multipole moments and the tidal coefficients.

Keywords: black holes, tidal deformations, initial data

1. Introduction

Among the fundamental properties of any classical or quantum mechanical physical system is its response to external perturbations. For example, the study of elasticity is the study of the deformation of a solid body subject to an external force; in quantum mechanics, an important property of atoms is the splitting of atomic spectral lines in the presence of external electric and magnetic fields. In gravitational physics, an important example is the deformation of a star due to the gravitational field of a binary companion. This paper studies the deformation of a black hole horizon subject to an external perturbation. One of the earliest discussions of tidal deformations in general relativity, the Love numbers, and their role in formulating the laws of motion is due to Damour [1]. Some recent studies of tidal deformations, Love numbers for neutron stars and their potential implications for gravitational wave observations, are [2–6]. Love numbers for non-spinning black holes are discussed in [7].

Both in Newtonian gravity and in general relativity, one could consider either (i) the deformations in the gravitational field of the object at large distances from it, or (ii) the change in the shape of the body itself. However, the relationship between the two calculations is yet to be fully understood in general relativity. The papers cited in the previous paragraph were, for the most part, concerned with the distortions of the body's asymptotic gravitational field. Deformations of the shape of black hole horizons (again in the non-spinning case) are discussed in [8, 9]. Some other papers which discuss the geometry of tidally distorted non-spinning black holes are [10–12].

Tidal deformations of spinning black holes are not as well understood. The tidal deformations of a Kerr black hole were studied by Hartle in a series of papers in the 1970s [13, 14] using the framework of black hole perturbation theory for small spins. Recent work by Hughes and O'Sullivan implements Hartle's formalism numerically [15] for larger spins. This paper studies the deformation of a spinning black hole and in particular, the deformation of its horizon shape in a very different framework, namely, in an initial data set and uses the isolated horizon multipole moments. We shall assume that the black hole angular momentum[3] is small and that the companion is far away (compared to the mass of the black hole). Furthermore, we shall specialize to the manifestly axisymmetric case when the black hole angular momentum and the separation vector between the black holes are parallel to each other.

The essential ingredients in our calculation are the invariant horizon multipole moments $\mathcal{M}_n$ and $\mathcal{J}_n$ for $n = 0, 1, 2, \ldots$. These moments fully characterize the intrinsic geometry of a black hole horizon and will be affected by the external field. We shall therefore begin with a brief introduction to these moments. Our goal will be to compute how these moments are affected when a binary companion is introduced. We shall work with a particular model binary system, namely black holes in the Bowen–York initial data set [16]. This is one of the simplest ways of studying a binary black hole system consisting of spinning components. The Bowen–York initial data construction assumes that the spatial three metric is conformally flat and that the extrinsic curvature tensor of the initial data surface is trace-free. With these assumptions, it provides a prescription for solving the Hamiltonian and momentum constraints for an arbitrary number of black holes including both angular momentum $\mathbf{J}$ and linear momentum $\mathbf{P}$ for each black hole. We shall solve for the conformal factor perturbatively assuming that both $\mathbf{J}$ and $\mathbf{P}$ are small in magnitude. This allows us to find the location of the marginally trapped surfaces perturbatively and to thereby calculate the black hole source multipole moments. We can then identify how the multipole moments are affected by the presence of the second black hole and therefore find a set of numbers which uniquely characterize how the moments are affected by an external perturbation. The calculation of these coefficients is the main result of this paper.

It is important to keep in mind an important caveat here. From a physical viewpoint, what we really want is to carry out a similar computation for two Kerr black holes rather than for Bowen–York black holes. It is known that the Kerr spacetime does not admit conformally flat spatial slices [17] and thus, the Bowen–York black hole horizon is expected *a priori* to be different from a Kerr horizon. In binary black hole numerical simulations which start with Bowen–York data, it is found that the initial deviations from Kerr are radiated away in the so called 'junk radiation' and the individual black hole horizons very quickly become

[3] In this paper by 'angular momentum' we shall always mean the intrinsic angular momentum of the black hole.

indistinguishable from a Kerr horizon. While this is not a problem for the numerical simulations which can ignore the initial burst of junk radiation, in our case this will be more important. The tidal properties of a Bowen–York black hole may well be quite different from a Kerr black hole. While it would be interesting to investigate this further, it is nevertheless useful since this would be the first such calculation for a spinning black hole. It also provides an interesting application of the horizon multipole moments which clearly quantify the deviations of a Bowen–York black hole from the Kerr horizon. Since the Bowen–York data set is commonly used as a starting point for numerical relativity calculations, this might be useful for numerical relativity applications. There are numerous suggestions for constructing initial data which resembles a system of two Kerr black holes more closely (see e.g. [18, 19]), but the Bowen–York data is the simplest example for a spinning black hole.

A similar comment applies in fact also to non-spinning black holes. In principle we would like to consider a spacetime consisting of two Schwarzschild black holes far away from each other and falling head-on towards each other. However, there are potentially different ways of approximating this physical situation. We could, as in [8], note that the spacetime must be axisymmetric and thus model it by a static Weyl metric. This will have the unphysical feature that the black holes will continue to remain in a static configuration. The two black holes are held apart by a 'strut' and it is not clear how one should separate the influence of this strut from the gravitational interaction between the two bodies. Alternatively, we could consider the Brill–Lindquist [20] or Misner [21] initial data sets for binary systems, both of which contain two black holes initially at rest. None of these choices are perhaps entirely unreasonable[4] but we cannot suppose that they will yield the same values of the horizon Love numbers or in general, the tidal coefficients.

The plan for the rest of this paper is as follows. In section 2 we shall review the definitions of the horizon multipole moments and describe how these moments change under the influence of an external perturbation. The Bowen–York initial data set is described briefly in section 3. Section 4 considers a single spinning Bowen–York black hole. Section 5 discusses a binary system and the deformation of the black hole multipole moments due to a binary companion. Finally section 6 discusses some implications of these results and directions for future work. We shall work in geometric units with $G = c = 1$. The spacetime metric, with signature $(- + + +)$, will be denoted by g_{ab} and ∇_a will be the derivative operator compatible with it. Our convention for the Riemann tensor R_{abcd} is $(\nabla_a \nabla_b - \nabla_b \nabla_a)\omega_c = R_{abc}{}^{d}\omega_d$.

2. The horizon multipole moments

2.1. The general framework

We shall use the framework of quasi-local horizons to describe black holes. This encompasses a wide range of physical situations: black holes in equilibrium are modeled by isolated horizons [22–27], and a black hole growing due to in-falling matter/radiation is modeled as a dynamical horizon [28, 29]. Both of these are closely related to the notion of trapping horizons introduced earlier by Hayward [30–33]. All these notions build on the idea of a marginally outer trapped surface. Let S be a closed two-dimensional surface. Let ℓ^a and n^a be

[4] Each of these are unphysical in their own way. The Weyl metric approach yields a static solution as mentioned previously, the Brill–Lindquist data is time symmetric, and the Misner data represents a wormhole connecting the two black holes. Thus, strictly speaking, none of these can represent two Schwarzschild black holes which were far away in the past and are coming closer to each other.

its outward and inward pointing null normals respectively. $\mathcal{S}$ is said to be a marginally trapped surface if the expansions of ℓ^a and n^a, denoted by $\Theta_{(\ell)}$ and $\Theta_{(n)}$ respectively, satisfy $\Theta_{(\ell)} = 0$ and $\Theta_{(n)} < 0$. $\mathcal{S}$ is said to be a marginally *outer* trapped surface if $\Theta_{(\ell)} = 0$ with no restriction on $\Theta_{(n)}$. In practice we will not check the condition on $\Theta_{(n)}$ (though, since we work with perturbed surfaces in this paper, we will not deal with highly distorted cases which might violate $\Theta_{(n)} < 0$). $\mathcal{S}$ will be assumed to have spherical topology. The time evolution of $\mathcal{S}$ has been shown to be well behaved (locally in time) provided it satisfies a suitable stability condition [34, 35]; the three-dimensional hypersurface generated by this time evolution is thus smooth. Isolated, dynamical and trapping horizons are all special cases of such three-dimensional hypersurfaces. We shall not go into the detailed definitions of these notions because we shall work only with marginally outer trapped surfaces $\mathcal{S}$ at a single instant of time.

We shall denote the spacelike two metric on $\mathcal{S}$ by $\tilde{q}_{ab}$, the covariant derivative compatible with $\tilde{q}_{ab}$ is $\widetilde{\mathcal{D}}_a$, the intrinsic scalar curvature of $\tilde{q}_{ab}$ is $\widetilde{\mathcal{R}}$, and the invariant volume two-form is $\tilde{\epsilon}_{ab}$. We shall work with $\mathcal{S}$ embedded in a spatial hypersurface Σ. The outgoing unit spacelike normal to $\mathcal{S}$ within Σ is denoted by r^a and the extrinsic curvature of Σ embedded within the spacetime manifold $\mathcal{M}$ is K_{ab}. Another important field is the one-form $\tilde{\omega}_a = K_{cb}\tilde{q}^c_a r^b$. We assume that $\mathcal{S}$ is axisymmetric, i.e. it admits a rotational symmetry φ^a which preserves $\tilde{q}_{ab}$ and $\tilde{\omega}_a$ [19, 36–39]. Let $A_{\mathcal{S}}$ be the area of $\mathcal{S}$ and $R_{\mathcal{S}} = \sqrt{A_{\mathcal{S}}/4\pi}$ its area radius. The angular momentum associated with $\mathcal{S}$ in vacuum general relativity is given by

$$J_{\mathcal{S}}^{(\varphi)} = \frac{1}{8\pi}\oint_{\mathcal{S}} \tilde{\omega}_a \varphi^a \,\mathrm{d}^2V = \frac{1}{8\pi}\oint_{\mathcal{S}} K_{ab}\varphi^a r^b \,\mathrm{d}^2V, \tag{1}$$

where d^2V is the invariant volume element on $\mathcal{S}$. We shall usually drop the superscript in $J_{\mathcal{S}}^{(\varphi)}$. The mass associated with $\mathcal{S}$ is

$$M_{\mathcal{S}} = \frac{1}{2R_{\mathcal{S}}}\sqrt{R_{\mathcal{S}}^4 + 4J_{\mathcal{S}}^2}. \tag{2}$$

We shall need higher order multipoles beyond the mass and angular momentum. Multipole moments for isolated horizons were introduced in [40]. A general procedure valid for dynamical black holes without assuming symmetries is given in [41]. However, this requires access to the time evolution of $\mathcal{S}$ which is beyond the scope of this paper. We shall therefore use a simpler and more limited method described in [42] which is a simple extension of [40]. See also [43] for numerical methods for computing multipole moments.

The starting point for this method is to construct a preferred coordinate system on $\mathcal{S}$ adapted to the axial symmetry: (ζ, ϕ), with $-1 \leqslant \zeta \leqslant 1$ and $0 \leqslant \phi < 2\pi$. We normalize φ^a so that it has affine length 2π. Then ϕ is the affine parameter along φ^a: $\varphi^a\widetilde{\mathcal{D}}_a\phi = 1$. The other coordinate ζ is defined by:

$$\widetilde{\mathcal{D}}_a\zeta = \frac{1}{R_{\mathcal{S}}^2}\varphi^b\tilde{\epsilon}_{ba}, \qquad \oint_{\mathcal{S}} \zeta\tilde{\epsilon} = 0. \tag{3}$$

It can then be shown that in these coordinates the metric $\tilde{q}_{ab}$ takes the form [40]:

$$\tilde{q}_{ab} = R_{\mathcal{S}}^2\left(f^{-1}\widetilde{\mathcal{D}}_a\zeta\widetilde{\mathcal{D}}_b\zeta + f\,\widetilde{\mathcal{D}}_a\phi\widetilde{\mathcal{D}}_b\phi\right), \tag{4}$$

where f is a function of ζ: $f = \varphi_a\varphi^a/R_{\mathcal{S}}^2$. On a round sphere in Euclidean space with the usual spherical coordinates (θ, ϕ), $\zeta = \cos\theta$. Regularity of $\tilde{q}_{ab}$ at the poles requires

$$\lim_{\zeta\to\pm1} f'(\zeta) = \mp 2. \tag{5}$$

It can also be shown that the scalar curvature is

$$\widetilde{\mathcal{R}} = -\frac{1}{R_S^2} f''(\zeta). \tag{6}$$

In these coordinates, the invariant volume element on $\mathcal{S}$ is independent of f, and is thus the same as on a round two-sphere where $f = \sin^2\theta = 1 - \zeta^2$. The normalization condition for spherical harmonics therefore works with the invariant volume element. The mass multipoles are:

$$\mathcal{M}_n = \frac{M_S R_S^n}{8\pi} \oint_S P_n(\zeta)\widetilde{\mathcal{R}}\, \mathrm{d}^2V, \tag{7}$$

and the angular momentum multipoles are

$$\mathcal{J}_n = \frac{R_S^{n+1}}{8\pi} \oint_S \tilde{\epsilon}^{ab}\widetilde{\mathcal{D}}_a P_n(\zeta) K_{bc} r^c\, \mathrm{d}^2V. \tag{8}$$

Here, $P_n(\zeta)$ are the Legendre polynomials. On general grounds, it follows that $\mathcal{M}_0 = M_S$ and $\mathcal{J}_0 = 0$. The first angular momentum multipole is just the angular momentum: $\mathcal{J}_1 = J_S$. Furthermore, since $P_n(-\zeta) = (-1)^n P_n(\zeta)$, if the horizon is axisymmetric and also symmetric under a reflection ($\zeta \to -\zeta$, $\phi \to \phi + \pi$), then $\mathcal{M}_n = 0$ for all odd n and $\mathcal{J}_n = 0$ for all even n.

2.2. The Kerr multipole moments

It is useful to illustrate these notions for a Kerr horizon parameterized by a mass m and spin parameter a. We shall later compare the Kerr multipole moments with the corresponding moments for a single spinning Bowen–York black hole.

We first note that the horizon mass and spin are respectively m and am as expected. In Boyer–Lindquist coordinates (t, r, θ, ϕ), the horizon is located at $r = r_+$ such that (see e.g. [44])

$$r_+ = m + \sqrt{m^2 - a^2}. \tag{9}$$

The two metric on a cross-section of the horizon is

$$\tilde{q}_{ab} = \rho_+^2 \nabla_a\theta \nabla_b\theta + \frac{\left(r_+^2 + a^2\right)^2}{\rho_+^2} \sin^2\theta \nabla_a\phi \nabla_b\phi, \tag{10}$$

where $\rho_+^2 = r_+^2 + a^2\cos^2\theta$. The volume two-form on $\mathcal{S}$ is

$$\tilde{\epsilon} = \left(r_+^2 + a^2\right)\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\phi. \tag{11}$$

The area of any closed cross-section of the horizon is $A = 4\pi(r_+^2 + a^2)$ and the area radius is $R = \sqrt{r_+^2 + a^2}$. The invariant coordinate ζ is, as for round two-spheres, $\zeta = \cos\theta$. The two metric can be written in the form given in equation (4) with

$$f(\zeta) = \frac{R^2\sin^2\theta}{\rho_+^2} = \frac{1 - \zeta^2}{1 - (a/R)^2\left(1 - \zeta^2\right)}. \tag{12}$$

It is easy to see that equation (5) is satisfied. The general calculation of the Kerr multipole moments is discussed in [40] and here we shall need the multipoles in the limit of small spins.

Keeping terms up to $\mathcal{O}(a^2)$:

$$f(\zeta) = 1 - \zeta^2 + \left(\frac{a}{R}\right)^2\left(1 - \zeta^2\right)^2 + \mathcal{O}\left(a^4\right), \tag{13}$$

which implies

$$\widetilde{\mathcal{R}}(\zeta) = -\frac{1}{R^2}f''(\zeta) = \frac{2}{R^2} - \frac{8a^2}{R^4}P_2(\zeta) + \mathcal{O}\left(a^4\right). \tag{14}$$

Apart from the mass and angular momentum, the only non-vanishing multipole moment at $\mathcal{O}(a^2)$ is

$$\mathcal{M}_2 = -\frac{4}{5}ma^2 = -\frac{4J^2}{5m}. \tag{15}$$

It is easier to calculate the $\mathcal{J}_n$ from the Weyl tensor component Ψ_2 (in Boyer–Lindquist coordinates):

$$\Psi_2 = -\frac{m}{(r - \mathrm{i}a\cos\theta)^3} = -\frac{m}{r^3}\left(1 + \frac{3\mathrm{i}a}{r}\cos\theta - \frac{6a^2}{r^2}\cos^2\theta + \mathcal{O}\left(a^3\right)\right). \tag{16}$$

For isolated horizons in vacuum general relativity, it can be shown that $\widetilde{\mathcal{R}} = -4\,\mathrm{Re}\,[\Psi_2]$ [24, 26]. Writing m and r in terms of R and a, equation (14) is recovered. The moments of the imaginary part of Ψ_2 yield the angular momentum multipole moments $\mathcal{J}_n$ [40]. It is again easy to see that $\mathcal{J}_0 = 0$ and $\mathcal{J}_1 = J = am$. All other moments vanish at this order of approximation.

2.3. Perturbations of the multipole moments

Consider a black hole with mass M_1, angular momentum J_1, and multipole moments $\mathcal{M}_n$, $\mathcal{J}_n$. We shall be concerned with how the multipole moments change under the influence of an external perturbation. Let $\delta\mathcal{M}_n$ and $\delta\mathcal{J}_n$ be the changes in $\mathcal{M}_n$ and $\mathcal{J}_n$ respectively. Consider an external perturbation caused by a non-spinning binary companion of mass M_2 placed at a distance d. We shall restrict ourselves to axisymmetric situations where the separation vector between the two black holes is parallel to the spin-vector of the first black hole. Let us define the dimensionless spin of the first black hole as $\chi = J_1/M_1^2$; χ can be shown to be restricted to $|\chi| < 1$ [45]. We assume that the full set of multipole moments is fully determined by the lowest non-vanishing moments, i.e. the mass and the angular momentum.

The small quantities in the problem are M_2/d and χ. On general grounds, $\delta\mathcal{M}_n$ and $\delta\mathcal{J}_n$ can be expanded as

$$\frac{\delta\mathcal{M}_n}{M_1^{n+1}} = \sum_{m,k=1;\, j=0}^{\infty} \alpha_{mkj}^{(n)}\frac{M_1^m M_2^k}{d^{m+k}}\chi^j, \tag{17}$$

$$\frac{\delta\mathcal{J}_n}{M_1^{n+1}} = \sum_{m,k=1;\, j=0}^{\infty} \beta_{mkj}^{(n)}\frac{M_1^m M_2^k}{d^{m+k}}\chi^j. \tag{18}$$

The dimensionless coefficients $\alpha^{(n)}$ and $\beta^{(n)}$ will be called tidal coefficients. The masses $M_{1,2}$ are the physical masses and will be combinations of the 'bare' parameters of the system which might include the bare masses $m_{1,2}$, d and the angular momenta; we shall see explicit examples of this later.

If $M_2 \to 0$ or $d \to \infty$, then the external perturbation vanishes and hence $(\delta\mathcal{M}_n, \delta\mathcal{J}_n)$ must also vanish. This means that in the above sums, we need only consider $k, m \geqslant 1$.

Similarly, we do not expect a divergence when $\chi \to 0$, which shows that $j \geqslant 1$. Thus, all the exponents (m, k, j) can take only positive values. We expect additional terms if the second black hole were also spinning, and if both black holes had non-zero linear momentum. In non-axisymmetric systems, when for example the angular and linear momenta, and the separation vector, are not aligned, we would have to consider moments $(\mathcal{M}_{\ell m}, \mathcal{J}_{\ell m})$ for $m \neq 0$ as well. These generalizations will be discussed in a forthcoming publication.

It is also useful to note that in the non-spinning case, the perturbations start to be non-vanishing only from $\mathcal{O}(M_2/d^3)$ onwards. Thus, the first term in, say $\mathcal{M}_2$, is proportional to $M_1^2 M_2/d^3$ and the corresponding tidal coefficient is $\alpha_{210}^{(2)}$. This coefficient will be proportional to the tidal Love number h_2 calculated in [8, 9].

3. The Brill–Lindquist and Bowen–York initial data sets

We work in a 3+1 split of spacetime where initial data are specified on a spacelike hypersurface Σ. The initial data consist of the positive-definite three metric h_{ab} on Σ, and the extrinsic curvature K_{ab} describing the embedding of Σ within the spacetime manifold $\mathcal{M}$. The initial data (Σ, h_{ab}, K_{ab}) satisfy the momentum and Hamiltonian constraint equations respectively:

$$D_a\left(K^{ab} - Kh^{ab}\right) = 0, \qquad {}^3R - K^{ab}K_{ab} + K^2 = 0. \tag{19}$$

Here ${}^3R$ is the Ricci scalar computed from h_{ab} and D_a is the covariant derivative compatible with h_{ab}.

We shall take h_{ab} to be conformally flat so that $h_{ab} = \psi^4 f_{ab}$, with f_{ab} being a flat metric. Furthermore, we shall take K_{ab} to be trace-free: $K = 0$, and the metric will be taken to be asymptotically flat at spatial infinity. With these choices, the constraint equations become

$$\partial_a \widetilde{K}^{ab} = 0. \qquad \Delta\psi = -\frac{1}{8}\psi^{-7}\widetilde{K}_{ab}\widetilde{K}^{ab}, \tag{20}$$

Here $\widetilde{K}_{ab} = \psi^2 K_{ab}$ is the re-scaled extrinsic curvature, $\Delta := \partial_a\partial^a$ is the flat-space Laplacian and ∂_{a} is the derivative operator compatible with the flat metric f_{ab}. Since the momentum constraint is now decoupled, we use an appropriate solution $\widetilde{K}_{ab}$ to the momentum constraint, plug it into the Hamiltonian constraint and solve the resulting elliptic equation for ψ. Furthermore, since the momentum constraint is seen to be linear, we can linearly superpose various solutions. The Hamiltonian constraint however is nonlinear and introduces various cross-terms between the different pieces included in $\widetilde{K}_{ab}$.

The simplest solutions are when the extrinsic curvature vanishes identically so that the data are time-symmetric. In this case the conformal factor satisfies the Laplace equation in flat space. Non-trivial solutions are obtained when we have 'point charges'. Thus, if we place N masses m_i at points $\mathbf{r}_i$ respectively $(i = 1\ldots N)$, then at any position $\mathbf{r}$ on Σ away from $\mathbf{r}_i$:

$$\psi_{\mathrm{BL}}(\mathbf{r}) = 1 + \sum_{i=1}^{N}\frac{m_i}{2\left|\mathbf{r} - \mathbf{r}_i\right|}. \tag{21}$$

This is the well known Brill–Lindquist solution [20]; see also [46, 47]. We shall consider the cases of a single black hole or a binary system so that the sum over i is either just a single term or the sum of two. The parameters m_i are the *bare* masses of the black holes. In the absence of any other black hole, these would be the physical horizon mass (and also the ADM mass). However, this is not the case if other black holes are present.

Linear momentum $\mathbf{P}$ and angular momentum $\mathbf{J}$ for a single black hole are handled by non-trivial choices for $\widetilde{K}_{ab}$, denoted by ${}^{P}\widetilde{K}_{ab}$ and ${}^{J}\widetilde{K}_{ab}$ [16]:

$$ {}^{P}\widetilde{K}_{ab} = \frac{3}{2r^2}\left[P_a n_b + P_b n_a - \left(f_{ab} - n_a n_b\right)P_c n^c\right], \tag{22} $$

$$ {}^{J}\widetilde{K}_{ab} = \frac{3}{r^3}\left[\epsilon_{acd}J^c n^d n_b + \epsilon_{bcd}J^c n^d n_a\right]. \tag{23} $$

These are the well known Bowen–York solutions to the momentum constraints. Here we have chosen standard spherical coordinates centered on the location of the black hole, with r as the radial coordinate and n^a the unit three-vector orthogonal to the spheres of constant r. Solutions with multiple black holes are obtained by superposing the different individual extrinsic curvatures. The solution for the conformal factor is determined, however, by a nonlinear combination of the extrinsic curvatures.

We study the effects of momentum, spin and presence of a binary companion on a black hole by considering perturbative solutions to the conformal factor [48]:

$$ \psi = \psi_{\rm BL} + u. \tag{24} $$

This is the so-called puncture ansatz, where $\psi_{\rm BL}$ contains all the singularities in the conformal factor and u is taken to be smooth everywhere and vanishing at spatial infinity. The equation for the conformal factor becomes:

$$ \tilde{\Delta} u = -\frac{1}{8}\left(\psi_{\rm BL} + u\right)^{-7}\widetilde{K}_{ab}\widetilde{K}^{ab}. \tag{25} $$

We shall keep terms up to $\mathcal{O}(P^2)$, $\mathcal{O}(J^2)$ and $\mathcal{O}(PJ)$. It is easy to see that at this level of accuracy (since $\widetilde{K}_{ab}\widetilde{K}^{ab}$ contains only terms of this order), the conformal factor satisfies a linear Poisson equation:

$$ \tilde{\Delta} u = -\frac{1}{8}\psi_{\rm BL}^{-7}\widetilde{K}_{ab}\widetilde{K}^{ab}. \tag{26} $$

Even with this simplification, the right-hand side of this equation is fairly complicated and it contains various cross terms between the spin and linear-momenta (of either black hole in the case of a binary system). Still, given its linearity, we can treat it analytically. We would get a linear equation if we keep terms linear in u on the right-hand side of equation (25). This case would still be amenable to an analytic treatment and would allow us to go to higher orders in P and J, but we shall restrict ourselves to dropping all u dependence within the source term.

4. A single spinning black hole

4.1. The conformal factor

The solution to the momentum constraint for a single spinning black hole at rest and placed at the origin is given by equation (23). A simple calculation shows

$$ \widetilde{K}_{ab}\widetilde{K}^{ab} = \frac{18J^2\sin^2\theta}{r^6}. \tag{27} $$

The angle θ is measured from $\mathbf{J}$. Note again that the parameters m and J are the bare mass and angular momentum respectively. The physical parameters (either at the horizon or at spatial infinity) will be determined below. With the puncture ansatz of equation (24), the Hamiltonian constraint becomes

$$\nabla^2 u = -\frac{\psi^{-7}}{8}\widetilde{K}_{ab}\widetilde{K}^{ab} = -\frac{288 r J^2 \sin^2\theta}{(m + 2r + 2ur)^7}$$
$$\approx -\frac{192 J^2 r}{(m + 2r)^7}\big(1 - P_2(\cos\theta)\big). \tag{28}$$

In the last step, as explained earlier, we have dropped the u dependent terms on the right-hand side, and the resulting Poisson equation is thus only valid up to $\mathcal{O}(J^2)$. We require that u is regular and $u \to 0$ when $r \to \infty$. Since we are working in spherical coordinates, regularity at the origin implies

$$\left.\frac{\partial u}{\partial r}\right|_{r=0} = 0. \tag{29}$$

The solution $u(r, \theta)$ will be of the form

$$u(r, \theta) = u_0(r)P_0(\cos\theta) + u_2(r)P_2(\cos\theta), \tag{30}$$

and the radial equations for $u_0(r)$ and $u_2(r)$ are:

$$u_0'' + \frac{2}{r}u_0' = -\frac{192 J^2 r}{(m + 2r)^7}, \tag{31}$$

$$u_2'' + \frac{2}{r}u_2' - \frac{6}{r^2}u_2 = \frac{192 J^2 r}{(m + 2r)^7}. \tag{32}$$

The solutions which are regular at $r = 0$ and asymptotically flat are:

$$u_0(r) = \frac{2J^2}{5m^3(m + 2r)^5}\Big(m^4 + 10m^3 r + 40m^2 r^2 + 40 m r^3 + 16 r^4\Big), \tag{33}$$

$$u_2(r) = -\frac{16 J^2 r^2}{5m(m + 2r)^5}. \tag{34}$$

We see that for large r, u_2 falls off as $1/r^3$ and

$$u_0(r) = \frac{2J^2}{5m^3}\frac{1}{2r} + \mathcal{O}\Big(1/r^2\Big). \tag{35}$$

Thus, the ADM mass is, ignoring higher powers in J,

$$m_{\mathrm{ADM}} = m + \frac{2J^2}{5m^3}. \tag{36}$$

The values of u_0 and u_2 at $r = m/2$ will be used later. These are:

$$u_0\big|_{r=m/2} = \frac{11}{40}\frac{J^2}{m^4}, \qquad \text{and} \qquad u_2\big|_{r=m/2} = -\frac{1}{40}\frac{J^2}{m^4}. \tag{37}$$

4.2. The location of the marginal surface

Let us now turn to the location of the marginal surface $\mathcal{S}$. We need to find closed surface (s) within Σ such that the outward null normal has vanishing expansion. If r^a is the outward spacelike unit-normal to $\mathcal{S}$ within Σ, and τ^a is the unit timelike normal to Σ, then all outward null normals are parallel to $\ell^a = \tau^a + r^a$. Thus, $\mathcal{S}$ is a marginally outer trapped surface if

$$\tilde{q}^{ab}\nabla_a\ell_b = \left(h^{ab} - r^a r^b\right)\nabla_a(\tau_b + r_b) = D_a r^a + K - K_{ab}r^a r^b = 0. \tag{38}$$

The general solution is given by $f(r, \theta) = r - h(\theta) = 0$. Cook and York have previously studied the horizon location for a single spinning and boosted Bowen–York black hole [49]. It is however useful for us to repeat some of the calculations for the zero-boost case.

The unit normal to $\mathcal{S}$ is

$$\mathbf{r} = \frac{\psi^{-2}}{\sqrt{1 + h_\theta^2/r^2}}\left(\partial_r - \frac{h_\theta}{r^2}\partial_\theta\right). \tag{39}$$

Here h_θ denotes the partial derivative $\partial h/\partial\theta$. It is easy to check that $K_{ab}r^a r^b = 0$. The horizon is thus a minimal surface and, more explicitly, it is obtained by solving

$$\frac{\partial}{\partial r}\left(\frac{r^2\psi^4}{\sqrt{1 + h_\theta^2/r^2}}\right) = \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\psi^4\sin\theta h_\theta}{\sqrt{1 + h_\theta^2/r^2}}\right). \tag{40}$$

We can solve this order by order in J. If all terms in J are ignored, equation (40) becomes $\partial_r(r^2\psi_{\rm BL}^4) = 0$ whose solution is $r = m/2$ [20]. Now keep terms linear in h_θ (this can be, at best, linear in J), and dropping all terms beyond $\mathcal{O}(J^2)$, equation (40) becomes

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\psi_{\rm BL}^4\right)\Big|_{r=h(\theta)} + \frac{\mathrm{d}}{\mathrm{d}r}\left(4r^2\psi_{\rm BL}^3 u\right)\Big|_{r=h(\theta)} = \frac{16}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}h}{\mathrm{d}\theta}\right). \tag{41}$$

In the second term on the left, the derivative can be evaluated at $r = m/2$ since u is already $\mathcal{O}(J^2)$. Using the solutions for u_0 and u_2 derived earlier, it turns out somewhat surprisingly that this term vanishes. As for the first term:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\psi_{\rm BL}\right)\Big|_{r=h(\theta)} &= \frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\psi_{\rm BL}\right)\Big|_{r=m/2} + \left(h - \frac{m}{2}\right)\frac{\mathrm{d}^2}{\mathrm{d}r^2}\left(r^2\psi_{\rm BL}\right)\Big|_{r=m/2} \\ &= 16\left(h - \frac{m}{2}\right).\end{aligned} \tag{42}$$

Putting it all together, it is easily seen that the solution to equation (41) is just $h = m/2$. A similar calculation shows that this holds also at $\mathcal{O}(J^2)$.

4.3. The area, angular momentum, and mass

With the location of the marginal surface $\mathcal{S}$ known, we can turn to its physical and geometrical properties. The first is simply its area. The induced metric on a surface given by $r = h(\theta)$ is

$$\mathrm{d}s_{\tilde{q}}^2 = \psi^4\left(\left(r^2 + h_\theta^2\right)\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2\right). \tag{43}$$

The invariant volume measure is

$$\sqrt{\det\tilde{q}} = r^2\psi^4\sin\theta\sqrt{1 + h_\theta^2/r^2}. \tag{44}$$

Specializing to the marginal surface $r = m/2$ found earlier, and keeping terms up to $\mathcal{O}(J^2)$, we get

$$\sqrt{\det\tilde{q}} \approx \frac{m^2}{4}\left(\psi_{\rm BL} + u\right)^4\Big|_{r=m/2}\sin\theta \approx 4m^2\left(1 + 2u\,|_{r=m/2}\right)\sin\theta. \tag{45}$$

The area is thus

$$A = 2\pi \int_0^\pi \sqrt{\det \tilde{q}}\ \mathrm{d}\theta \approx 8\pi m^2 \int_0^\pi \left(1 + 2u\,|_{r=m/2}\right) \sin\theta\ \mathrm{d}\theta \tag{46}$$

$$= 16\pi m^2 \left(1 + 2\ u_0\big|_{r=m/2}\right) = 16\pi m^2 \left(1 + \frac{11J^2}{20m^4}\right), \tag{47}$$

and the corresponding area radius is

$$R = \sqrt{\frac{A}{4\pi}} = 2m\left(1 + \frac{11J^2}{20m^4}\right)^{1/2} \approx 2m\left(1 + \frac{11J^2}{40m^4}\right). \tag{48}$$

The angular momentum turns out to be just the parameter J appearing in the extrinsic curvature. To see this, consider any surface $r = h(\theta)$ ($h(\theta)$ could be arbitrary, subject only to the condition that the surface is smooth and of spherical topology). Then, taking all the factors of ψ into account, we get

$$K_{ab} r^a \varphi^b = -\frac{3\psi^{-4} J \sin^2\theta}{r^2\sqrt{1 + h_\theta^2/r^2}}. \tag{49}$$

Using equation (1) and the volume element given by equation (44), it can be shown that the angular momentum associated with the marginal surface is just J. Similarly, this shows that the angular momentum associated with the sphere at spatial infinity is also J. This fact can also be seen by the balance law for angular momentum discussed in [50], obtained by integrating the momentum constraint over Σ after a contraction with φ^a. Using the fact that φ^a is a symmetry of h_{ab} then shows that the angular momentum for any closed spherical two-surface is J.

Using equations (2) and (48), the mass of the horizon is

$$M \approx \frac{R}{2}\left(1 + \frac{2J^2}{R^4}\right) = m\left(1 + \frac{2J^2}{5m^4}\right) + \mathcal{O}\left(J^4\right). \tag{50}$$

We have dropped the subscript $\mathcal{S}$ on M for simplicity. Henceforth, we shall usually use M for the horizon mass to distinguish it from the bare mass m. It is interesting to note that the value obtained here is the same as the ADM mass given in equation (36). We are now ready to turn to the higher multipole moments.

4.4. Higher multipole moments

In order to calculate the multipole moments ($\mathcal{M}_n$, $\mathcal{J}_n$) we first need to find the preferred coordinate system (ζ, ϕ) compatible with the axial symmetry. Starting with equation (43), keeping terms up to $\mathcal{O}(J^2)$, we see that the metric $\mathrm{d}s^2_{\tilde{q}}$ can be put in the form of equation (4) with

$$f = \frac{r^2\psi^4}{R^2}\sin^2\theta \qquad \text{and} \qquad \mathrm{d}\zeta = \frac{r^2\psi^4}{R^2}\sin\theta\ \mathrm{d}\theta. \tag{51}$$

It is useful to again note that at $r = m/2$, $r^2\psi^4 \approx 4m^2(1 + 2u)$. Setting $r = m/2$, using the values of u_0 and u_2 at $r = m/2$, and the result for R, it is not difficult to show that

$$\zeta = \cos\theta\left(1 + \frac{J^2 \sin^2\theta}{40m^4}\right) + \mathcal{O}\left(J^4\right). \tag{52}$$

As expected, $\zeta = 1$ and -1 at the north and south poles respectively. It is also easy to check that the condition of equation (5) is indeed satisfied. We use equation (6) to calculate the scalar curvature $\widetilde{\mathcal{R}}$. We begin with:

$$\frac{\mathrm{d}f}{\mathrm{d}\zeta} = \frac{\mathrm{d}f}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}\zeta} = \frac{(1 + 2u)2\sin\theta\cos\theta + 2u_\theta \sin^2\theta}{(1 + 2u)\sin\theta} \approx 2\cos\theta + 2u_\theta \sin\theta. \tag{53}$$

A similar further short calculation, utilizing also equation (37), yields the intrinsic scalar curvature of the horizon:

$$\widetilde{\mathcal{R}} = -\frac{1}{R^2}\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2} = \frac{2}{R^2} - \frac{J^2}{20\,M^6}P_2(\zeta) + \mathcal{O}\left(J^4\right). \tag{54}$$

This finally allows us to calculate the mass multipole moments. At the approximation that we are working in, the mass quadrupole moment is

$$\mathcal{M}_2 = -\frac{2J^2}{25\,M} + \mathcal{O}\left(J^4\right). \tag{55}$$

All higher moments $\mathcal{M}_n$ vanish. Similarly, it turns out that the only non-vanishing angular momentum multipole $\mathcal{J}_n$ within our approximation is the angular momentum $\mathcal{J}_1$. All other $\mathcal{J}_n$ vanish up to $\mathcal{O}(J^2)$.

It is interesting to compare these results with the corresponding moments for the Kerr black hole horizon. The only one we can compare is the mass-quadrupole $\mathcal{M}_2$. Comparing equations (55) and (15), we see that the Kerr value is exactly ten times larger; thus the Bowen–York black hole is in fact closer to the Schwarzschild black hole (with the same mass). This observation might shed some light on the initial junk radiation observed in numerical relativity studies involving Bowen–York data (see e.g. [51]). We are led to speculate that when we try to make a spinning black hole conformally flat, its mass quadrupole moment ends up closer to Schwarzschild than it should be. It would be useful to accurately monitor the multipole moments of the individual black holes in a binary black hole simulation during the initial junk radiation phase.

4.5. A single boosted and spinning black hole

Let us now consider a single Bowen–York black hole with non-vanishing boost, i.e. including the solution ${}^{P}\widetilde{K}_{ab}$ of the momentum constraint given in equation (22). For the moment, let us set the angular momentum to zero and consider a non-spinning boosted black hole. Let us align the z-axis with the linear momentum $\mathbf{P}$ and, for an arbitrary point P away from the puncture $r = 0$, let θ be the angle between the position vector $\mathbf{r}$ of P and the z-axis. Then, it is easy to show that

$$\widetilde{K}_{ab}\widetilde{K}^{ab} = \frac{9P^2}{r^4}\left(\frac{1}{2} + \cos^2\theta\right). \tag{56}$$

A perturbative solution to the Hamiltonian constraint for small P has been obtained previously [49, 52]. The calculations are very similar to what we have seen for the spinning case earlier and we shall not repeat all the steps here. As shown in [52], with the puncture ansatz, the correction term for the conformal factor is

$$u(r,\theta) = \epsilon_P^2\left(u_0(r)P_0(\cos\theta) + u_2(r)P_2(\cos\theta)\right) + \mathcal{O}\left(\epsilon_P^4\right). \tag{57}$$

Here $\epsilon_P := P/m$. The solutions for the radial functions u_0 and u_2 which are regular everywhere and asymptotically flat are given in equations (A8) and (A9) of [52]. The marginally trapped surface is located in equation (24) of [52]:

$$r = h(\theta) = \frac{m}{2} - \frac{P}{16}\cos\theta + \mathcal{O}\left(\epsilon_P^2\right). \tag{58}$$

We can easily calculate the multipole moments of this horizon. The horizon mass, which in this case is just the irreducible mass, has already been calculated in [52]:

$$M = m\left(1 + \frac{P^2}{8m^2}\right) + \mathcal{O}\left(P^4\right). \tag{59}$$

The angular momentum multipoles all vanish, and the mass quadrupole moment turns out to be:

$$\mathcal{M}_2 = \frac{MP^2}{200}(1871 - 2688\ln[2]) + \mathcal{O}\left(P^4\right). \tag{60}$$

Let us now combine these results with the results of the previous sections on the single spinning black hole. We shall restrict ourselves to the axisymmetric situation with $\mathbf{P}$ and $\mathbf{J}$ parallel to each other. Then, from the form of the spin and momentum contributions to the extrinsic curvature, a short calculation shows that

$$\widetilde{K}_{ab}\widetilde{K}^{ab} = \frac{9P^2}{r^4}\left(\frac{1}{2} + \cos^2\theta\right) + \frac{18}{r^6}J^2\sin^2\theta. \tag{61}$$

There are no cross terms between the angular and linear momenta; this would cease to hold if $\mathbf{P}$ and $\mathbf{J}$ were not aligned. We again look for solutions of the form given in equation (30). Solutions to the two radial equations which are regular and asymptotically flat are obtained by linearly superposition of equations (33)–(35) with the corresponding solutions given in [52]. The location of the horizon is still given by equation (58). The only non-vanishing multipole moment apart from the mass and the spin (at the approximation we are working in) is the mass quadrupole moment, which is the sum of the pure spin and boost values given in equations (55) and (60).

5. A spinning black hole with a non-spinning binary companion

We now place our spinning black hole in a binary system. We shall simplify our calculation in three ways. First, we shall ignore the effects of linear momentum. Second, we shall take the companion black hole to be non-spinning and finally, we shall take the separation vector between the two black holes to be parallel to the angular momentum vector $\mathbf{J}$; see figure 1. With these restrictions, the initial data is guaranteed to be axisymmetric. While not trivial, it is in fact not hard to relax these assumptions since we have a flat background metric available to us. However, breaking axial symmetry introduces complications in the definitions of the multipole moments and calls for a separate discussion. We shall address this in a forthcoming paper. Moreover, as in the earlier sections, we shall work in the limit of small angular momentum (including terms accurate to $\mathcal{O}(J^2)$); this restriction is however difficult to avoid in an analytic treatment and numerical calculations will be required for more accuracy.

When the spins vanish identically and we have time symmetry, the exact solution to the Hamiltonian constraint is given by the Brill–Lindquist solution (equation (21)):

$$\psi_{\mathrm{BL}}(\mathbf{r}) := \frac{1}{\alpha} = 1 + \frac{m_1}{2r_1} + \frac{m_2}{2r_2}. \tag{62}$$

With the puncture ansatz of equation (24), again ignoring u in the source term, we obtain:

$$\tilde{\Delta} u = -\frac{1}{8}\alpha^7 \widetilde{K}_{ab}\widetilde{K}^{ab}. \tag{63}$$

We take the second black hole to be non-spinning so that it does not have any contribution to the extrinsic curvature and all the dependence on m_2 and d is through α in the above equation. From the Bowen–York extrinsic curvature we get explicitly:

$$\tilde{\Delta} u = -\frac{9}{4r^6}\alpha^7 J^2 \sin^2\theta + \mathcal{O}\left(J^4\right). \tag{64}$$

We could, in principle, choose to keep terms up to any order in $1/d$ that we wish. Since tidal effects (in the absence of spin) are proportional to m_2/d^3, we shall keep our calculations accurate to $\mathcal{O}(1/d^3)$. We start by expanding α in terms of Legendre polynomials:

$$\begin{aligned} \frac{1}{\alpha} &= 1 + \frac{m_1}{2r} + \frac{m_2}{2\sqrt{r^2 + d^2 - 2dr\cos\theta}} \\ &= 1 + \frac{m_1}{2r} + \frac{m_2}{2d}\sum_{n=0}^{\infty} P_n(\cos\theta)\left(\frac{r}{d}\right)^n. \end{aligned} \tag{65}$$

We have chosen to expand in powers of $1/d$ because we are interested in the region near the first black hole, i.e. near the origin where r can be small. We should not expect the solutions we obtain using this approximation to be uniformly accurate for large r.

If we keep terms up to $\mathcal{O}(1/d^3)$, we see that α will include Legendre polynomials up to $P_2(\cos\theta)$. Since α^7 is multiplied by $\sin^2\theta$ in equation (64), it is clear that the source term in that equation will include terms up to $P_4(\cos\theta)$. We thus look for a solution of the form

$$u(r, \theta) = \sum_{n=0}^{4} u_n(r) P_n(\cos\theta). \tag{66}$$

Substituting this in equation (64) then leads to five linear ODEs for each of the five radial functions $u_n(r)$. We display explicitly the five differential equations (each equation is accurate up to correction terms, which are $\mathcal{O}(J^4)$ and $\mathcal{O}(1/d^4)$, and we define $\beta := m_1 + 2r$):

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u_0}{\mathrm{d}r}\right) = -\frac{192J^2r^3}{\beta^7}\left[1 - \frac{7m_2 r}{\beta d} + \frac{28m_2^2 r^2}{\beta^2 d^2} + \frac{7m_2 r^3}{5\beta^3 d^3}\left(\beta^2 - 60m_2^2\right)\right], \tag{67}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u_1}{\mathrm{d}r}\right) - 2u_1 = \frac{4032J^2 m_2 r^5}{5\beta^8}\left[\frac{1}{d^2} - \frac{8m_2 r}{\beta d^3}\right], \tag{68}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u_2}{\mathrm{d}r}\right) - 6u_2 = \frac{192J^2r^3}{\beta^7}\left[1 - \frac{7m_2 r}{\beta d} + \frac{28m_2^2 r^2}{\beta^2 d^2} + \frac{m_2 r^3}{\beta^3 d^3}\left(5\beta^2 - 84m_2^2\right)\right], \tag{69}$$

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u_3}{\mathrm{d}r}\right) - 12u_3 = -\frac{4032J^2 m_2 r^5}{5\beta^8}\left[\frac{1}{d^2} - \frac{8m_2 r}{\beta d^3}\right], \tag{70}$$

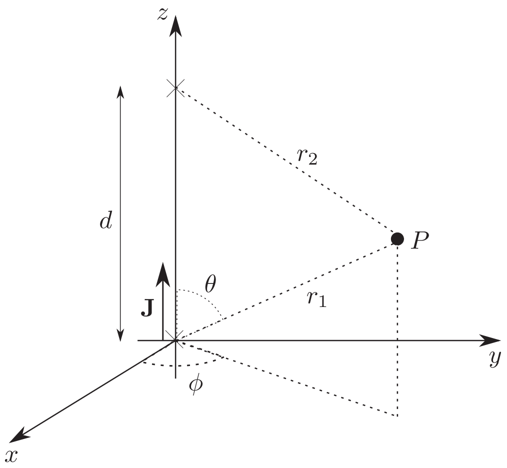

Figure 1. A depiction of the binary system. The first black hole is placed at the origin and it's angular momentum $\mathbf{J}$ is aligned with the z-axis. The second black hole is placed at a distance d on the z-axis. The distances of an arbitrary point P from the two black holes are r_1 and r_2, and the angular coordinates (θ, ϕ) of P are defined in the usual way.

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u_4}{\mathrm{d}r}\right) - 20u_4 = -\frac{3456J^2m_2r^6}{5\beta^8d^3}. \tag{71}$$

The solutions which are regular at the origin are given in appendix.

5.1. The marginal surface and multipole moments

The procedure for calculating the multipole moments is the same as before. First we locate the marginal surface, find the axially-symmetric geometry (i.e. the coordinate ζ), expand the scalar curvature and $\widetilde{\omega}_a$ in terms of the Legendre polynomials and read off the multipole moments. Since the methods employed for each of these steps are technically very similar to what was done in the previous section, we shall skip most of the intermediate details and mostly provide results. We shall start with the non-spinning case and include spin effects subsequently.

5.1.1. Non-spinning black holes. For two non-spinning black holes, the conformal factor is known exactly and is just the Brill–Lindquist result Ψ_{BL} given in equation (62). As before, we expand this in powers of $1/d$ given in equation (65) and keep terms up to $\mathcal{O}(1/d^3)$. The marginal surface is again found by solving equation (40). This time we proceed order-by-order in $1/d$; details are provided in appendix B of [53]. The location of the horizon is:

$$\begin{aligned} r = h(\theta) = \frac{m_1}{2}\Bigg[1 - \frac{m_2}{2d} + \frac{m_2}{4d^2}\big(m_2 - m_1 P_1(\cos\theta)\big) \\ - \frac{m_2}{8d^3}\left(m_2^2 - 3m_1m_2P_1(\cos\theta) + \frac{5}{7}m_1^2P_2(\cos\theta)\right)\Bigg] + \mathcal{O}\left(1/d^4\right). \end{aligned} \tag{72}$$

The angular dependence starts only from $1/d^2$ onwards. In order to find the area of the marginal surface and its geometric properties, we need to evaluate $r^2\psi^4$ at the horizon accurate to $\mathcal{O}(1/d^3)$:

$$\left(r^2\psi^4\right)_{r=h(\theta)} \approx 4m_1^2\left[\left(1+\frac{m_2}{2d}\right)^2 + \frac{m_1m_2}{2d^2}P_1(\cos\theta) + \frac{m_1^2m_2}{4d^3}P_2(\cos\theta)\right]. \tag{73}$$

The area is then

$$A = \int_0^\pi \sin\theta\, \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\left(r^2\psi^4\sqrt{1+h_\theta^2/r^2}\right)_{r=h(\theta)} \approx 2\pi\int_0^\pi \left(r^2\psi^4\right)_{r=h(\theta)} \sin\theta\, \mathrm{d}\theta. \tag{74}$$

Thus we obtain the area and the horizon mass

$$A = 16\pi m_1^2\left(1+\frac{m_2}{2d}\right)^2 + \mathcal{O}\left(1/d^4\right) \Rightarrow M_1 = m_1\left(1+\frac{m_2}{2d}\right) + \mathcal{O}\left(1/d^4\right). \tag{75}$$

The corresponding expression for M_2 is obtained by interchanging m_1 and m_2.

The intrinsic metric on the marginal surface is of the form given in equation (4) with f and the coordinate ζ given as in equation (51). A straightforward calculation then leads to

$$\widetilde{\mathcal{R}} = -\frac{1}{R^2}\frac{\mathrm{d}^2 f}{\mathrm{d}\zeta^2} = \frac{1}{2M_1^2} + \frac{M_2}{4d^3}P_2(\zeta) + \mathcal{O}\left(1/d^4\right). \tag{76}$$

We have expressed the result in terms of the physical horizon masses M_1 and M_2 rather than the bare parameters m_1 and m_2.

It is then interesting to compare this with the work of Damour and Lecian [8]. This result is to be compared with equation (32) of [8] which can be written in our notation as

$$\widetilde{\mathcal{R}} = \frac{1}{2M_1^2} + \frac{4M_2}{d^3}P_2(\zeta) + \dots . \tag{77}$$

This disagrees with the corresponding term of equation (76) containing $P_2(\zeta)$; the Brill–Lindquist black hole is less distorted. As discussed at towards the end of the introduction, a disagreement is not entirely surprising since the Brill–Lindquist data can be different from the Weyl ansatz used in [8]. Recent work by Landry and Poisson [9] reproduces the results of Damour and Lecian. However it uses a different formalism, and we have not carried out a detailed comparison just yet.

5.1.2. Incorporating spin effects. For the case of a spinning black hole with a binary companion, new terms appear in the higher orders of J and $1/d$ starting with $\mathcal{O}(J^2/d^2)$. We write $h(\theta) = h^{\mathrm{BL}}(\theta) + \tilde{h}(\theta)$, where h^{BL} denotes the Brill–Lindquist result of equation (72). Then, it can be shown that

$$\begin{aligned}\tilde{h}(\theta) = &-\frac{3m_2J^2}{200m_1^2d^2}\left(P_1(\cos\theta) - \frac{(4657-6720\ln[2])}{13}P_3(\cos\theta)\right)\\ &-\frac{m_2J^2}{16m_1^2d^3}\left(\frac{47m_1}{70} - \frac{21m_2}{25}P_1(\cos\theta) + \frac{(32567-47040\ln[2])m_1}{343}P_2(\cos\theta)\right.\\ &+\frac{21(4657-6720\ln[2])m_2}{325}P_3(\cos\theta)\\ &\left.-\frac{3(391259-564480\ln[2])m_1}{3430}P_4(\cos\theta)\right) + \mathcal{O}\left(J^4\right) + \mathcal{O}\left(\frac{1}{d^4}\right).\end{aligned} \tag{78}$$

Thus, unlike the case of a single spinning Bowen–York black hole studied earlier, the horizon is no longer reflection symmetric, and the location is not independent of J.

We can finally compute the physical quantities of the horizon. As before, we start with the area:

$$A = 16\pi m_1^2\left[\left(1+\frac{11J^2}{20m_1^4}\right)+\frac{m_2}{d}\left(1-\frac{11J^2}{20m_1^4}\right)+\frac{m_2^2}{d^2}\left(\frac{1}{4}+\frac{33J^2}{80m_1^4}\right)\right.$$
$$\left.+\frac{m_2}{d^3}\frac{J^2\left(39m_1^2-55m_2^2\right)}{200m_1^4}\right]+\mathcal{O}\left(J^4\right)+\mathcal{O}\left(\frac{1}{d^4}\right). \tag{79}$$

The angular momentum is just J and we can thus easily compute the horizon mass:

$$M_1 = m_1\left[1+\frac{m_2}{2d}+\frac{2}{5}\frac{J^2}{m_1^4}-\frac{3}{5}\frac{m_2J^2}{m_1^4 d}+\frac{3}{5}\frac{m_2^2J^2}{m_1^4 d^2}-\frac{1}{2}\frac{m_2^3J^2}{m_1^4 d^3}+\frac{39}{400}\frac{m_2J^2}{m_1^2 d^3}\right]. \tag{80}$$

It is easy to check that previous results are recovered for either a single spinning black hole ($d=\infty$), or for a non-spinning black hole with a binary companion ($J=0$).

Using the solution above for the conformal factor and the horizon location, as before, we follow the procedure of computing the preferred coordinate ζ and the scalar curvature $\widetilde{\mathcal{R}}$. We shall not show the intermediate results, but rather just move on to the main quantities of interest, i.e. the multipole moments.

Apart from the mass and angular momentum, the non-vanishing multipole moments are (as usual, all results ignore terms of $\mathcal{O}(1/d^4)$ or $\mathcal{O}(J^3)$ or higher):

$$\mathcal{M}_2 \approx \frac{2}{5}\frac{m_1^5 m_2}{d^3}-\frac{2}{25}\frac{J^2}{m_1}+\frac{1}{25}\frac{J^2}{m_1}\frac{m_2}{d}-\frac{1}{50}\frac{J^2}{m_1}\frac{m_2^2}{d^2}+\frac{1}{100}\frac{J^2}{m_1}\frac{m_2^3}{d^3}$$
$$+\frac{(-5294+7680\ln[2])}{100}\frac{m_1m_2}{d^3}J^2, \tag{81}$$

$$\mathcal{M}_3 \approx \frac{72m_1m_2J^2}{35d^2}(111-160\ln[2])-\frac{72m_1m_2^2J^2}{35d^3}(111-160\ln[2]), \tag{82}$$

$$\mathcal{J}_2 \approx -\frac{6m_1^2m_2J}{5d^2}+\frac{3m_1^2m_2^2J}{5d^3}, \tag{83}$$

$$\mathcal{J}_3 \approx -\frac{6m_1^4m_2J}{7d^3}. \tag{84}$$

One might suspect that these results would simplify by using the physical horizon masses instead of the bare masses m_1, m_2. This is indeed the case. For $\mathcal{M}_2$ we get

$$\mathcal{M}_2 \approx -\frac{2}{25}M_1^3\chi^2+\frac{2}{5}\frac{M_1^5M_2}{d^3}+k\frac{M_1^5M_2}{d^3}\chi^2. \tag{85}$$

Here, as defined earlier, $\chi := J/M_1^2$ and

$$k=\frac{-5294+7680\ln[2]}{100}-\frac{2}{25}\approx 0.2137. \tag{86}$$

The first two terms of $\mathcal{M}_2$ have been calculated earlier and the first effect of spin appears through the coefficient k. Viewing this as a perturbation of $\mathcal{M}_2$:

$$\frac{\delta\mathcal{M}_2}{M_1^3} = \frac{2}{5}\frac{M_1^2M_2}{d^3} + k\frac{M_1^2M_2}{d^3}\chi^2 + \dots . \tag{87}$$

Thus, we conclude that the tidal coefficients (defined in equation (17)) $\alpha_{210}^{(2)} = 2/5$ and $\alpha_{212}^{(2)} = k$ characterize the perturbations of the mass quadrupole moment.

Consider now the third mass moment $\mathcal{M}_3$. This is rewritten as:

$$\frac{\mathcal{M}_3}{M_1^4} \approx \frac{72\,h}{35}\frac{M_1M_2}{d^2}\chi^2 - \frac{108h}{35}\frac{M_1M_2^2}{d^3}\chi^2 - \frac{36h}{35}\frac{M_1^2M_2}{d^3}\chi^2 \dots , \tag{88}$$

where $h := 110 - 160\ln(2) \approx 0.0965$. This determines the tidal coefficients $\alpha_{112}^{(3)}$, $\alpha_{122}^{(3)}$ and $\alpha_{212}^{(3)}$.

Finally, turning to the angular momentum moments:

$$\frac{\delta\mathcal{J}_2}{M_1^3} \approx -\frac{6}{5}\frac{M_1M_2}{d^2}\chi + \frac{9}{5}\frac{M_1M_2^2}{d^3}\chi + \frac{3}{5}\frac{M_1^2M_2}{d^3}\chi \tag{89}$$

and

$$\frac{\delta\mathcal{J}_3}{M_1^4} \approx -\frac{6M_1^2M_2\chi}{7d^3} . \tag{90}$$

This determines the tidal coefficients

$$\beta_{111}^{(2)} = -\frac{6}{5}, \qquad \beta_{121}^{(2)} = \frac{9}{5}, \qquad \beta_{211}^{(2)} = \frac{3}{5}, \qquad \beta_{211}^{(3)} = -\frac{6}{7} . \tag{91}$$

These coefficients describe the distortion of the spinning black hole horizon to linear order in the perturbation and up to order J^2.

6. Conclusions

In this paper we have computed the tidal deformations of the horizon of a spinning black hole in a binary system using the Bowen–York initial data. We have defined a set of dimensionless numbers, the tidal coefficients, which characterize the deformations. We have seen that the effect of the tidal deformations appears already at $\mathcal{O}(1/d)$ for a spinning black hole. A number of immediate generalizations are possible even just for the Bowen–York data for small angular momentum as considered here. The first is the deviation from axisymmetry. This includes the case when the spin and the separation vector are not parallel, including linear momenta for both black holes, and finally, including spin in the second black hole. Since we have a flat background metric available to us, all of these cases can be dealt with. This will enable us to, for example, determine circular orbits, find the minimum energy circular orbit, and compare these results with expectations from post-Newtonian theory.

It would be interesting to compute the tidal coefficients during the course of a binary black hole numerical simulation. Assuming that the horizon geometry can be tracked with sufficient accuracy, this would give us a more accurate value of the tidal coefficients for the physical situation that we are interested in, namely two Kerr black holes orbiting each other and each being distorted by the gravitational field of the other.

Most importantly, an important missing piece in the literature on isolated horizons is the relation between the horizon multipole moments and the usual field moments at infinity. There should similarly be a relation between the tidal coefficients at the horizon and at infinity. From the viewpoint of isolated horizons, this is expected because the horizon geometry (plus the transverse radiation Ψ_4) determines the spacetime in the neighborhood of an isolated horizon [23]. Thus, we can expect that a knowledge of the tidal coefficients at the horizon should determine the tidal coefficients for the field moments. This is well known in Newtonian theory where the two sets of Love numbers are simply related to each other. Landry and Poisson [9] have determined this relationship in general relativity for non-spinning neutron stars and black holes, but similar results for horizons with general multipole moments and spins are still lacking.

Acknowledgments

We are grateful to Domenico Giulini, Scott Hughes and the referees for valuable comments and discussions.

Appendix. Radial functions for the binary system

We give here the radial functions $u_0 \dots u_4$ of equation (66) obtained by solving equations (67)–(71):

$$\begin{aligned}
u_0(r) = {} & \frac{2J^2}{5m_1^3\beta^5}\left(m_1^4 + 10m_1^3 r + 40m_1^2 r^2 + 40m_1 r^3 + 16r^4\right) \\
& - \frac{1}{d}\frac{m_2 J^2}{5m_1^3\beta^6}\Big(3m_1^5 + 36m_1^4 r + 180m_1^3 r^2 + 480m_1^2 r^3 \\
& + 384m_1 r^4 + 128r^5\Big) \\
& + \frac{1}{d^2}\frac{m_2^2 J^2}{5m_1^3\beta^7}\Big(3m_1^6 + 42m_1^5 r + 252m_1^4 r^2 + 840m_1^3 r^3 \\
& + 1680m_1^2 r^4 + 1120m_1 r^5 + 320r^6\Big) \\
& - \frac{1}{d^3}\frac{m_2 J^2}{10m_1^3\beta^8}\Big[5m_2^2\Big(m_1^7 + 16m_1^6 r + 112m_1^5 r^2 + 448m_1^4 r^3 \\
& + 1120m_1^3 r^4 + 1792m_1^2 r^5 + 1024m_1 r^6 + 256r^7\Big) \\
& - m_1^2\beta^2\left(m_1^5 + 12m_1^4 r + 60m_1^3 r^2 + 160m_1^2 r^3 + 240m_1 r^4 + 192r^5\right)\Big].
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
u_1(r) = {} & \frac{48m_2 r^5 J^2}{25m_1^4\beta^6}\left[\frac{1}{d^2}\left(15m_1^2 + 12m_1 r + 4r^2\right)\right. \\
& \left. - \frac{1}{d^3}\frac{4m_2 r\left(21m_1^2 + 14m_1 r + 4r^2\right)}{\beta}\right].
\end{aligned} \tag{A.2}$$

$$u_2(r) = -\frac{16J^2r^2}{5m_1\beta^5} + \frac{8J^2m_2r^2}{5m_1\beta^6}\frac{1}{d}\left(m_1 + 12r\right)$$
$$- \frac{4J^2m_2^2r^2}{5m_1\beta^7}\frac{1}{d^2}\left(m_1^2 + 14m_1r + 84r^2\right)$$
$$+ \frac{J^2m_2}{35m_1r^3\beta^8}\frac{1}{d^3}\left(14m_2^2r^5\left(m_1^3 + 16m_1^2r + 112m_1r^2 + 448r^3\right)\right.$$
$$- 5m_1r\beta^2\left(42m_1^6 + 462m_1^5r + 2072m_1^4r^2\right.$$
$$\left. + 4788m_1^3r^3 + 5847m_1^2r^4 + 3300m_1r^5 + 420r^6\right)$$
$$\left. - 105m_1^2\beta^8\ln\left[\frac{m_1}{\beta}\right]\right). \tag{A.3}$$

$$u_3(r) = \frac{3m_2J^2}{25r^4\beta^6}\frac{1}{d^2}\left(2r\left(15m_1^6 + 165m_1^5r + 740m_1^4r^2 + 1710m_1^3r^3\right.\right.$$
$$\left. + 2088m_1^2r^4 + 1176m_1r^5 + 140r^6\right)$$
$$\left. + 15m_1\beta^6\ln\left[\frac{m_1}{\beta}\right]\right)$$
$$+ \frac{3m_2^2J^2}{50r^4\beta^7}\frac{1}{d^3}\left(2r\left(135m_1^7 + 1755m_1^6r + 9630m_1^5r^2\right.\right.$$
$$+ 28710m_1^4r^3 + 49572m_1^3r^4 + 48168m_1^2r^5$$
$$\left. + 22408m_1r^6 + 2240r^7\right)$$
$$\left. + 135m_1\beta^7\ln\left[\frac{m_1}{\beta}\right]\right). \tag{A.4}$$

$$u_4(r) = -\frac{3m_2J^2}{70r^5\beta^6}\frac{1}{d^3}\left(2r\left(105m_1^8 + 1155m_1^7r + 5180m_1^6r^2\right.\right.$$
$$+ 11970m_1^5r^3 + 14616m_1^4r^4 + 8232m_1^3r^5$$
$$\left. + 960m_1^2r^6 - 240m_1r^7 + 112r^8\right)$$
$$\left. + 105m_1^3\beta^6\ln\left[\frac{m_1}{\beta}\right]\right). \tag{A.5}$$

We note that u_1 does not vanish at spatial infinity. This is connected to the fact that the approximation of expanding in powers of $1/d$ is valid only near the first black hole and should not be expected to be valid away from it.

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