A NUMERICAL ALGORITHM FOR ROBUST STABILIZATION OF SYSTEMS WITH SECTOR-BOUNDED NONLINEARITY

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Abstract: We consider the robust stabilization problem for systems with a nonlinear, sector-bounded uncertainty. A solution for this problem can be obtained via dynamical output feedback if a Lyapunov function of Lur'e-Postnikov type is known. Computationally, this involves the solution of an algebraic Riccati equation of H_∞-type. We show how to compute the robustly stabilizing output feedback solving a generalized eigenproblem of Hamiltonian/skew-Hamiltonian structure, thereby avoiding the numerically hazardous formation of the coefficients in the algebraic Riccati equation which may lead to large errors in the computed Riccati solution and thus in the output feedback.

Keywords: robust stabilization, absolute stability, Lyapunov function, Hamiltonian/skew-Hamiltonian pencil

1. INTRODUCTION

We consider time-invariant systems with uncertain scalar nonlinearity of the following form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1 u(t) + B_2 \Phi(C_1 x(t)), \\ y(t) &= C_2 x(t), \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m}$, $B_2 \in \mathbb{R}^{n}$, $C_1 \in \mathbb{R}^{1\times n}$, $C_2 \in \mathbb{R}^{p\times n}$, $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input, and $y(t) \in \mathbb{R}^p$ is the measured output. The uncertain nonlinearity is given by a function $\Phi : \mathbb{R} \mapsto \mathbb{R}$. Our aim is to find a dynamic output feedback controller

$$\begin{aligned} \dot{\hat{x}}(t) &= \hat{A}\hat{x}(t) + \hat{B}y(t), \\ u(t) &= \hat{C}\hat{x}(t) + \hat{D}y(t), \end{aligned} \tag{2}$$

such that the resulting closed-loop system is absolutely stable. Thus, we are looking for a controller robustly stabilizing the given uncertain system. Robustness follows from the fact that absolute stabilization can be shown to be equivalent to H_∞ control, see (Petersen *et al.*, 2000, Section 3.5.2).

In the following we will assume $m = 1$. That is, the system (1) is a *single-input system.* Note that $p > 1$ is allowed, though, so that multiple output channels are possible. Furthermore, the (uncertain) nonlinearity Φ is assumed to be *sector-bounded* in the following sense:

$$0 \le \frac{\Phi(\zeta)}{\zeta} \le \delta, \quad \zeta \in \mathbb{R}, \tag{3}$$

see Figure 1 for an example.

Recently, the well-known absolute stability criterion for systems containing an uncertain sector-bounded nonlinearity derived by Popov in the early Sixties was extended to the problem of ro-

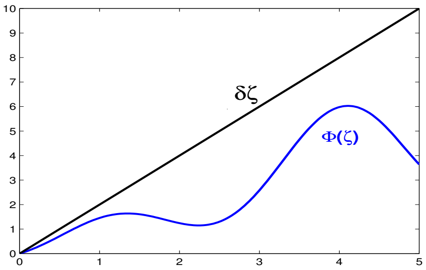

Fig. 1. Example for a sector-bounded function Φ.

bust stabilization (Petersen *et al.*, 2000, Section 7.2). For this purpose, systems which are absolutely stabilizable by Lyapunov functions of Lur'e-Postnikov form are considered. This allows to extend Popov's absolute stability notion to the stabilizability concept for output-feedback control. Using these concepts, it is possible to show that uncertain systems with sector-bounded nonlinearity can be robustly stabilized using a linear output feedback controller.

The method proposed in (Petersen *et al.*, 2000) for computing the robustly stabilizing controller is based on solving two algebraic Riccati equations (AREs) of H_∞-type and thus has severe limitations regarding its *numerical robustness* similar to the construction of an optimal H_∞ output feedback controller. We will review this approach in Section 2. Numerical difficulties of the approach based on two AREs for solving H_∞ control problems have been observed since the state-space solution of the optimal H_∞ control was formulated in (Doyle and Glover, 1988; Doyle *et al.*, 1989). Concepts for circumventing these problems in H_∞ (sub-)optimal control design for linear systems are discussed, e.g., in (Benner *et al.*, 1999*b*; Benner *et al.*, 2004*b*; Copeland and Safonov, 1992; Gahinet and Laub, 1997; Safonov *et al.*, 1989). Here we employ concepts analogous to those in (Benner *et al.*, 2004*b*; Benner *et al.*, 2004*a*) to derive a numerically reliable method to robustly stabilize nonlinear systems in the considered class. We show how to compute the robustly stabilizing output feedback solving a generalized eigenproblem of Hamiltonian/skew-Hamiltonian structure, thereby avoiding the numerically hazardous formation of the coefficients in the AREs which may lead to large errors in the computed Riccati solution and thus in the output feedback. A new characterization of the absolute stabilizability of (1) based on the Hamiltonian/skew-Hamiltonian eigenproblem will be given in Section 3. Thus, a numerically backward stable method for Hamiltonian/skew-Hamiltonian eigenproblems suggested in (Benner *et al.*, 1999*a*) allows to compute a robustly stabilizing output feedback controller in a reliable way.

We would like to emphasize that the robust stabilization problem considered here can also be tackled via an LMI approach; see (Balas *et al.*, 2005) and the references therein. That is, alternatively to the condition based on the two AREs (or Hamiltonian/skew-Hamiltonian pencils as discussed here), the absolute stabilizability by a Lyapunov functions of Lur'e-Postnikov form can be characterized by an LMI. Hence, in such an approach an LMI in $\mathcal{O}(n^2)$ variables needs to be solved which in general results in a complexity of $\mathcal{O}(n^6)$. Despite recent progress in reducing this complexity based on exploiting duality in the related semidefinite programs (Vandenberghe *et al.*, 2005), the best complexity achievable is $\mathcal{O}(n^4)$ as compared to the $\mathcal{O}(n^3)$ cost of the procedure discussed here. It should be noted, though, that in contrast to our approach, the LMI approach is not restricted to the single-input case.

In the following, I_n will denote the identity matrix of order n, $M > 0$ ($M \geq 0$) will be used for positive-(semi)definite matrices. With $\|x\|$ we mean the L_2-norm of a function $x(t)$ while $\|x(t)\|$ denotes the Euclidean (vector) norm of $x(t) \in \mathbb{R}^n$.

2. ROBUST STABILIZATION OF SYSTEMS WITH NONLINEAR SECTOR-BOUNDED UNCERTAINTY

In this section we review the necessary background from (Petersen *et al.*, 2000, Section 7.2) for deriving a robustly stabilizing controller for the problem stated in the introduction.

Defining the state vector of the closed-loop system resulting from (1) and (2) by $z := \left[\begin{smallmatrix} x \\ \hat{x} \end{smallmatrix}\right]$, we can define a special class of Lyapunov functions.

Definition 2.1. A *Lyapunov function of Lur'e-Postnikov form* for the closed-loop system resulting from (1) and (2) has the form

$$V(z) = z^T M z + \beta \int_0^\sigma \Phi(\zeta)\, d\zeta, \qquad (4)$$

where $M = M^T > 0$ is a given positive definite matrix, $\beta > 0$ is a constant, and $\sigma = C_1 x$.

With this definition, the stabilization concept used here can be formulated.

Definition 2.2. The uncertain system (1) with sector-bounded nonlinearity as in (3) is called *absolutely stabilizable with a Lyapunov function of Lur'e-Postnikov form* if there exist a linear output feedback controller as in (2), a matrix $M = M^T > 0$, and constants $\beta > 0$, $\varepsilon > 0$, such that for V as in (4), the derivative $\dot{V}$ along solution trajectories of the closed-loop system satisfies the inequality

$$\dot{V}(z) \leq -\varepsilon \left(\|z\|^2 + \Phi^2(\sigma)\right). \quad (5)$$

Note that the existence of a Lyapunov function of Lur'e-Postnikov form satisfying (5) implies absolute stability for the uncertain closed-loop system resulting from (1), (3), and (2). In turn, this implies exponential stability, i.e., there exist constants $\mu, \nu > 0$ with $\|z(t)\| \leq \mu\|z(0)\|e^{-\nu t}$ for all solutions of the closed-loop system.

Now let $\tau \geq 0$ be a given constant so that

$$\alpha := \frac{\tau}{\delta} - C_1 B_2 \neq 0. \quad (6)$$

We then further define

$$\begin{aligned} C_\tau &:= \frac{1}{2\alpha} C_1(\tau I_n + A), \quad D_\tau := \frac{1}{2\alpha} C_1 B_1, \\ A_\tau &:= A + B_2 C_\tau, \quad B_\tau := B_1 + B_2 D_\tau. \end{aligned} \quad (7)$$

Assuming that $R := D_\tau^T D_\tau > 0$, we can define the following two parameter-dependent AREs for a given constant $\gamma > 0$ and $\tilde{A}_\tau := A_\tau - B_\tau R^{-1} D_\tau^T C_\tau$:

$$\begin{aligned} 0 &= C_\tau^T C_\tau - C_\tau^T D_\tau R^{-1} D_\tau^T C_\tau + \tilde{A}_\tau^T X + X \tilde{A}_\tau \\ &\quad + X(B_2 B_2^T - B_\tau R^{-1} B_\tau^T) X, \quad (8) \\ 0 &= \gamma I_n + B_2 B_2^T + A_\tau Y + Y A_\tau^T \\ &\quad + Y(C_\tau^T C_\tau - \frac{1}{\gamma} C_2^T C_2) Y. \quad (9) \end{aligned}$$

Having set the stage, the main result about robust stabilization with Lyapunov functions of Lur'e-Postnikov form can be stated.

Theorem 2.3. (Petersen *et al.*, 2000, Theorem 7.2.1) Consider the uncertain system (1) with sector-bounded nonlinearity as in (3) and assume that (A, B_2) is controllable, (A, C_1) is observable and $C_1 B_1 \neq 0$. Then the following statements are equivalent.

a) The uncertain system (1) with sector-bounded nonlinearity as in (3) is absolutely stabilizable with a Lyapunov function of Lur'e-Postnikov form.
b) There exist constants $\tau \geq 0$, $\gamma > 0$ such that in (7), $\alpha > 0$, (A, C_τ) is observable and the AREs (8) and (9) have stabilizing solutions $\hat{X} > 0$ and $\hat{Y} > 0$ satisfying the spectral radius condition $\rho(\hat{X}\hat{Y}) < 1$.

Furthermore, if condition b) is satisfied, then a robustly stabilizing linear output feedback controller for (1) of the form (2) is given by

$$\begin{aligned} \hat{D} &:= 0, \quad \hat{C} := -R^{-1}(B_\tau^T \hat{X} + D_\tau^T C_\tau), \\ \hat{B} &:= \frac{1}{\gamma}(I - \hat{Y}\hat{X})^{-1} \hat{Y} C_2^T, \\ \hat{A} &:= A_\tau + B_\tau \hat{C} - \hat{B} C_2 + B_2 B_2^T \hat{X} + \gamma \hat{X}. \end{aligned} \quad (10)$$

The above theorem resembles the classical theorem about suboptimal H_∞ control (Doyle *et al.*, 1989; Zhou *et al.*, 1996). Thus the same numerical problems with a computational procedure based on the two AREs (8) and (9) can be expected. That is, tiny errors in forming the coefficients may lead to large errors in the solutions. Ill-conditioned or diverging Riccati solutions make it difficult or impossible to check the spectral radius condition numerically. Frequently, the closed-loop spectrum associated to either (8) or (9) will approach the imaginary axis if τ approaches a situation where condition b) is violated. Most numerical methods for solving AREs face severe problems in this situation; particularly if the symmetry properties of the associated Hamiltonian eigenproblems are not respected. But even if this difficulty is not encountered, already rounding errors and cancellation effects resulting from computing the coefficients of the AREs may cause such a procedure to deliver erroneous results. For the H_∞ case, remedies for these problems are suggested, e.g., in (Benner *et al.*, 2004*b*; Gahinet and Laub, 1997; Safonov *et al.*, 1989; Zhou *et al.*, 1996) by partially or completely circumventing the AREs (8) and (9). This also amounts to replacing the spectral radius condition by an equivalent condition for which the explicit ARE solutions are no longer required. In the next section we show how this can be achieved in the situation faced here.

3. MAIN RESULT

The first observation that will be used is the well-known fact (Anderson and Moore, 1979; Zhou *et al.*, 1996) that the two AREs (8) and (9) have stabilizing solutions if and only if the corresponding Hamiltonian matrices H_X, H_Y shown in Table 1 have unique stable invariant subspaces. (By a *stable invariant subspace* we mean the invariant subspace corresponding to the eigenvalues in the left half complex plane.) Suppose these are given by the relations

$$H_X \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} = \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} T_X,, \quad (15)$$

$$H_Y \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} T_Y, \quad (16)$$

where $U_j, V_j \in \mathbb{R}^{n \times n}$, $j = 1, 2$, and T_X and T_Y contain the stable eigenvalues of H_X and H_Y, respectively, then U_1, V_1 are invertible and

$$\hat{X} = U_2 U_1^{-1}, \quad \hat{Y} = V_2 V_1^{-1}.$$

Table 1. Hamiltonian matrices and corresponding Hamiltonian/skew-Hamiltonian pencils

$$H_X := \begin{bmatrix} A_\tau - B_\tau R^{-1} D_\tau^T C_\tau & B_2 B_2^T - B_\tau R^{-1} B_\tau^T \\ C_\tau^T D_\tau R^{-1} D_\tau^T C_\tau - C_\tau^T C_\tau & -(A_\tau - B_\tau R^{-1} D_\tau^T C_\tau)^T \end{bmatrix}, \tag{11}$$

$$H_Y := \begin{bmatrix} A_\tau^T & C_\tau^T C_\tau - \frac{1}{\gamma} C_2^T C_2 \\ -(\gamma I_n + B_2 B_2^T) & -A_\tau \end{bmatrix}, \tag{12}$$

$$M_X - \lambda L_X := \begin{bmatrix} A_\tau & \tilde{B}_2 & 0 & -\tilde{B}_1 \\ S_1^T & 0 & -\tilde{B}_1^T & -R_1 \\ 0 & S_2 & -A_\tau^T & -S_1 \\ S_2^T & I_2 & -\tilde{B}_2^T & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{13}$$

$$M_Y - \lambda L_Y := \begin{bmatrix} A_\tau^T & \tilde{C}_2 & 0 & -\tilde{C}_1 \\ T_1^T & 0 & -\tilde{C}_1^T & -\tilde{R}_1 \\ 0 & T_2 & -A_\tau & -T_1 \\ T_2^T & \tilde{R}_2 & -\tilde{C}_2^T & 0 \end{bmatrix} - \lambda \begin{bmatrix} I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

Several authors have proposed matrix pencil formulations for the Hamiltonian eigenproblems arising in classical H_∞ control. Using these matrix pencils, the stable invariant subspaces can be computed without the necessity to invert R (Safonov *et al.*, 1989; Copeland and Safonov, 1992; Gahinet and Laub, 1997; Benner *et al.*, 2004*b*; Benner *et al.*, 2004*a*). Analogous considerations lead us to the following matrix pencils:

$$\hat{M}_X - \lambda \hat{L}_X := \begin{bmatrix} A_\tau & 0 & B_\tau & B_2 & 0 \\ 0 & -A_\tau^T & -C_\tau^T D_\tau & 0 & C_\tau^T \\ -D_\tau^T C_\tau & -B_\tau^T & -R & 0 & 0 \\ 0 & -B_2^T & 0 & 1 & 0 \\ C_\tau & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$-\lambda \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{17}$$

$$\hat{M}_Y - \lambda \hat{L}_Y := \begin{bmatrix} A_\tau^T & 0 & \frac{1}{\sqrt{\gamma}} C_2^T & C_\tau & 0 \\ -\gamma I & -A_\tau & 0 & 0 & B_2 \\ 0 & -\frac{1}{\sqrt{\gamma}} C_2 & -I_p & 0 & 0 \\ 0 & -C_\tau & 0 & 1 & 0 \\ B_2^T & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$-\lambda \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

We then have the following relations between the Hamiltonian matrices (11), (12) and the matrix pencils (17), (18).

Lemma 3.1. The Hamiltonian matrices H_X and H_Y have stable, n-dimensional invariant subspaces as in (15) and (16), respectively, if and only if the matrix pencils $\hat{M}_X - \lambda \hat{L}_X$ and $\hat{M}_Y - \lambda \hat{L}_Y$ in (17) and (18), respectively, have n-dimensional stable deflating subspaces given by the columns of $\hat{U} \in \mathbb{R}^{2n+3\times n}$, $\hat{V} \in \mathbb{R}^{2n+p+2\times n}$ with

$$\hat{U} = \left[\, U_1^T \;\, U_2^T \;\, \hat{U}_3^T \;\, \hat{U}_4^T \;\, \hat{U}_5^T \,\right]^T,$$

$$\hat{V} = \left[\, V_1^T \;\, V_2^T \;\, \hat{V}_3^T \;\, \hat{V}_4^T \;\, \hat{V}_5^T \,\right]^T.$$

Proof: The two matrix pencils in (17) and (18) are both regular and have exactly $2n$ finite eigenvalues. This follows from the fact that in both cases, the lower right-hand corner of $\hat{M}_X, \hat{M}_Y$ is invertible. Thus, we can perform a block elimination (Schur complement) to deflate the infinite part of the spectrum yielding the reduced matrix pencils

$$-\lambda \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} A_\tau & 0 \\ 0 & -A_\tau^T \end{bmatrix}$$
$$+ \begin{bmatrix} B_\tau^T & -D_\tau^T C_\tau \\ B_2^T & 0 \\ 0 & C_\tau \end{bmatrix}^T \begin{bmatrix} -R & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} D_\tau^T C_\tau & B_\tau^T \\ 0 & B_2^T \\ C_\tau & 0 \end{bmatrix},$$
$$-\lambda \begin{bmatrix} I_n & 0 \\ 0 & I_n \end{bmatrix} + \begin{bmatrix} A_\tau & 0 \\ -\gamma I_n & -A_\tau^T \end{bmatrix}$$
$$- \begin{bmatrix} \frac{1}{\sqrt{\gamma}} C_2 & 0 \\ C_\tau^T & 0 \\ 0 & B_2^T \end{bmatrix}^T \begin{bmatrix} -I_p & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 & -\frac{1}{\sqrt{\gamma}} C_2 \\ 0 & -C_\tau \\ B_2^T & 0 \end{bmatrix}.$$

But these matrix pencils equal $H_X - \lambda I_{2n}$ and $H_Y - \lambda I_{2n}$, respectively, with H_X, H_Y as in (11), (12). Thus the $2n$ finite eigenvalues of the matrix pencils in (17) and (18) are exactly those of the Hamiltonian matrices H_X, H_Y. Moreover, the block elimination process corresponds to an equivalence transformation using only a transformation matrix from the left. Hence, the right deflating subspaces are not changed so that the upper $2n \times n$ parts of $\hat{U}, \hat{V}$ coincide with U, V from (15) and (16), respectively. □

Lemma 3.1 shows that in order to check the conditions on the existence of the ARE solution in part b) of Theorem 2.3 it is not necessary to solve the AREs explicitely and all inversions needed to form its coefficients can be circumvented by using the matrix pencils $\hat{M}_X - \lambda \hat{L}_Y$, $\hat{M}_Y - \lambda \hat{L}_Y$. Still, the spectral radius condition involves the explicit solutions $\hat{X}, \hat{Y}$. For the H_∞-problem considered in

(Benner *et al.*, 2004*b*), a condition based on the stable invariant subspaces of the corresponding Hamiltonian matrices is derived. We can use this result in the following way. Let

$$Z := \begin{bmatrix} U_2^T U_1 & U_2^T V_2 \\ V_2^T U_2 & V_2^T V_1 \end{bmatrix} \tag{19}$$

with U_j, V_j as in (15), (16). Then the spectral radius condition can be checked by inspecting Z:

Lemma 3.2. The matrix Z in (19) is positive definite if and only if stabilizing solutions $\hat{X} > 0$, $\hat{Y} > 0$ of (8), (9) exist and satisfy $\rho(\hat{X}\hat{Y}) < 1$.

Proof: As $\operatorname{rank}\left(\hat{X}\right) = \operatorname{rank}\left(\hat{Y}\right) = n$ if these ARE solutions exist, the assertion follows from Lemma 5.3 in (Benner *et al.*, 2004*b*). □

With the results derived so far it is possible to re-formulate Theorem 2.3 avoiding AREs completely. For the purpose of numerical computation, there is still a problem to be solved. The Hamiltonian matrices H_X, H_Y corresponding to the AREs (8), (9) have the well-known spectral symmetry: if λ is an eigenvalue of H_X, then so is $-\bar{\lambda}$. As we have seen, the finite eigenvalues of $\hat{M}_X - \lambda\hat{L}_X$, $\hat{M}_Y - \lambda\hat{L}_Y$ coincide with the spectra of the Hamiltonian matrices H_X, H_Y and thus inherit this symmetry property. Unfortunately, standard numerical algorithms like the QZ algorithm (Golub and Van Loan, 1996) to compute eigenvalues and deflating subspaces do not respect this property. This may lead to unwanted effects as roundoff errors can cause computed eigenvalues to cross the imaginary axis. In that situation, the numerical computation of an n-dimensional stable deflating subspace is no longer possible. The structure of the matrix pencils in (17), (18) does not allow to use numerical algorithms for problems with Hamiltonian spectral symmetry as suggested in (Benner *et al.*, 1998; Benner *et al.*, 1999*a*; Benner *et al.*, 2002). We thus go one step further in order to transform the matrix pencils in (17), (18) to matrix pencils with Hamiltonian/skew-Hamiltonian structure, that is, one matrix will be Hamiltonian, the other one skew-Hamiltonian. For matrix pencils of this kind, structure-preserving numerically backward stable methods for computing eigenvalues and deflating subspace are suggested in (Benner *et al.*, 1999*a*; Benner *et al.*, 2002).

First we consider the matrix pencil $\hat{M}_X - \lambda\hat{L}_X$ which is of odd dimension. Thus, we have to add an infinite eigenvalue without changing the top $2n \times n$ part of the matrix representing the stable deflating subspace. This can simply be achieved by adding a zero row and column to $\hat{M}_X$ and $\hat{L}_X$ and setting $(\hat{M}_X)_{2n+4,2n+4} = 1$. Then we define

$$\begin{aligned} \begin{bmatrix} \tilde{B}_1 & \tilde{B}_2 \end{bmatrix} &:= \begin{bmatrix} B_\tau & B_2 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{n\times 4}, \\ \begin{bmatrix} S_1 & S_2 \end{bmatrix} &:= \begin{bmatrix} -C_\tau^T D_\tau & 0 & C_\tau^T & 0 \end{bmatrix} \in \mathbb{R}^{n\times 4}, \\ R_1 &:= \begin{bmatrix} -R & 0 \\ 0 & 1 \end{bmatrix}, \end{aligned}$$

where $\tilde{B}_j, S_j \in \mathbb{R}^{n\times 2}$, and form the Hamiltonian/skew-Hamiltonian pencil $M_X - \lambda L_X$ shown in (13) in Table 1. We then have the following obvious relation to $\hat{M}_X - \lambda\hat{L}_X$ which can be obtained by simple row and column permutations.

Lemma 3.3. The Hamiltonian/skew-Hamiltonian pencil $M_X - \lambda L_X$ from (13) has a stable, n-dimensional deflating subspace if and only if $\hat{M}_X - \lambda\hat{L}_X$ has one. Moreover, if such a deflating subspace of $M_X - \lambda L_X$ exists, it can be represented by the columns of $\begin{bmatrix} U_1^T & U_3^T & U_2^T & U_4^T \end{bmatrix}^T$ with U_1, U_2 as in (15).

For the second ARE (9) the situation is slightly more complicated depending on the number p of measured outputs. We will concentrate here on p even; the case p odd can again be treated by adding an infinite eigenvalue. For even p we define

$$\begin{aligned} \begin{bmatrix} \tilde{C}_1 & \tilde{C}_2 \end{bmatrix} &:= \begin{bmatrix} \tfrac{1}{\sqrt{\gamma}} C_2^T & C_\tau & 0 \end{bmatrix} \in \mathbb{R}^{n\times p+2}, \\ \begin{bmatrix} T_1 & T_2 \end{bmatrix} &:= \begin{bmatrix} 0 & 0 & B_2 \end{bmatrix} \in \mathbb{R}^{n\times p+2}, \\ \begin{bmatrix} \tilde{R}_1 & 0 \\ 0 & \tilde{R}_2 \end{bmatrix} &:= \begin{bmatrix} -I_p & 0 \\ 0 & I_2 \end{bmatrix}, \end{aligned}$$

where $\tilde{C}_j, T_j \in \mathbb{R}^{n\times \frac{p}{2}+1}$, $\tilde{R}_j \in \mathbb{R}^{\frac{p}{2}+1\times\frac{p}{2}+1}$, and form the Hamiltonian/skew-Hamiltonian pencil $M_Y - \lambda L_Y$ shown in (14) in Table 1. We then have the following obvious relation to $\hat{M}_Y - \lambda\hat{L}_Y$.

Lemma 3.4. The Hamiltonian/skew-Hamiltonian pencil $M_Y - \lambda L_Y$ from (14) has a stable, n-dimensional deflating subspace if and only if $\hat{M}_Y - \lambda\hat{L}_Y$ has one. Moreover, if such a deflating subspace of $M_Y - \lambda L_Y$ exists, it can be represented by the columns of $\begin{bmatrix} V_1^T & V_3^T & V_2^T & V_4^T \end{bmatrix}^T$ with V_1, V_2 as in (16).

Summarizing the results obtained in this section, we have the following algorithm for checking condition b) of Theorem 2.3 which can be embedded in a one-parameter search with respect to τ as suggested in (Petersen *et al.*, 2000, Remark 7.2.1).

(1) Choose $\gamma > 0$ and $\tau \neq \delta C_1 B_2$.
(2) Form $A_\tau, B_\tau, C_\tau, D_\tau, R$ as in (7).
(3) Form the Hamiltonian/skew-Hamiltonian pencils (13) and (14).
(4) Compute the stable deflating subspaces of $M_X - \lambda L_X$ and $M_Y - \lambda L_Y$. If these are not n-dimensional, then STOP.

(5) Check whether U_1, V_1 are invertible using an SVD. If at least one of them is rank-deficient, then STOP.
(6) Form Z as in (19) and check its definiteness using a Cholesky decomposition.

This procedure consists of numerically stable steps, exploiting the structure and relevant symmetries as much as possible. This suggests a better numerical reliability than a procedure based on the AREs (8), (9) in analogy to the observations in (Benner *et al.*, 2004*b*; Gahinet and Laub, 1997; Safonov *et al.*, 1989). The verification of this claim through numerical experiments is under current investigation and will be reported elsewhere.

4. CONCLUSIONS

We have presented an alternative characterization for the absolute stabilizability with a Lyapunov function of Lur'e-Postnikov type of uncertain systems with sector-bounded nonlinearity. The new characterization avoids the explicit solution of algebraic Riccati equations. It is based on stable deflating subspaces for Hamiltonian/skew-Hamiltonian pencils and a positive definiteness condition for a symmetric matrix. The suggested approach can also be used to formulate a new algorithm for state feedback control with guaranteed cost for the kind of nonlinear systems considered here. A numerical procedure based on the new characterization is also suggested. Future work will include the implementation of this approach and testing it for practical examples.

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